PATTERNS

GLIMEUXE

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About

The live file is available for free at `http://github.com/glimeuxe/writings`, where updates are made on an occasional basis. I thank all those who have made suggestions to improve the text.

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Theories

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Classical Logic

2.1 Prologue

What could it mean for something to be *true*? Throughout history, various definitions for the word "truth" have been proposed.

Definition 1 (Correspondence Theory of Truth)**.**
Truth is that which corresponds to reality.

Definition 2 (Coherence Theory of Truth)**.**
Truth is that which coheres with every other truth.

Ought mathematics concern itself with the universe we inhabit? In any case, the practice of mathematics seems to be based on something like the coherence theory of truth; this presents a *seeming* trilemma.

Definition 3 (Münchhausen Trilemma)**.**
Every proof is completed by circularity, infinite regress, or assumption.

Reasoning by coherence appears to obtain truth by circularity, infinite regress, or assumption. It is no secret that mathematical theories admit assumptions. Mathematical theories are also said to build upon, and thus derive from, each other. A theory from which other theories can be derived is said to be *foundational*.

Broadly speaking, a logic can be thought of as a language for reasoning about truth — that is, a system which prescribes symbols, and ways of interchanging those symbols. This text studies propositional and predicate logics, in particular.

With any human language, definitions may be said to originate in *pre-linguistic* thought. Writers sometimes choose their starting points based on what feels the most intuitive to them. For this text, I have decided to use words and phrases like "not", "and", "if…then", "either…or", "otherwise", "every", "same", in addition to a few other mathematical symbols, in my appeal to the intuition of the reader.

2.2 Language

2.2.1 Propositional Logic

Definition 4 (Propositional Formula)**.**
Let $p_\circ, \ldots, p_\bullet$ be propositional variables.

1. If p is a propositional variable, then p is a propositional formula.
2. If φ is a propositional formula, then $(\neg\varphi)$ is a propositional formula.
   If φ and φ' are propositional formulas, then $(\varphi \wedge \varphi')$ is a propositional formula.

Definition 5 (Truth Value of Propositional Formula)**.**

1. Every propositional variable is either true or false.
2. If φ is true, then $(\neg\varphi)$ is false. Otherwise, $(\neg\varphi)$ is true.
   If φ and φ' are true, then $(\varphi \wedge \varphi')$ is true. Otherwise, $(\varphi \wedge \varphi')$ is false.

2.2.2 Predicate Logic

Definition 6 (Formula)**.**
Let $x_\circ, \ldots, x_\bullet$ be variables.

1. If x is a variable, and x' is a variable, then $(x = x')$ is a formula.
   If x is a variable, and x' is a variable, then $(x \in x')$ is a formula.
2. If φ is a formula, then $(\neg\varphi)$ is a formula.
   If φ is a formula, and φ' is a formula, then $(\varphi \wedge \varphi')$ is a formula.
   If x is a variable, and φ is a formula, then $\forall x(\varphi)$ is a formula.

Definition 7 (Free Variable)**.**

1. If $(x = x')$ is a formula, then x and x' are free variables in the formula.
   If $(x \in x')$ is a formula, then x and x' are free variables in the formula.
2. If x is a free variable in φ, then x is a free variable in $(\neg\varphi)$.
   If x is a free variable in φ and φ', then x is a free variable in $(\varphi \wedge \varphi')$.
   If x is a free variable in φ, and $(x \neq x')$, then x is a free variable in $\forall x'(\varphi)$.

Notation 1 (Naïve Variable Substitution)**.**
$(x = x)\{^{\tau}_{x} \lessdot (\tau = \tau)$
$(x = x')\{^{\tau}_{x} \lessdot (\tau = x')$
$(x' = x)\{^{\tau}_{x} \lessdot (x' = \tau)$
$(x' = x')\{^{\tau}_{x} \lessdot (x' = x')$
$(x \in x)\{^{\tau}_{x} \lessdot (\tau \in \tau)$
$(x \in x')\{^{\tau}_{x} \lessdot (\tau \in x')$
$(x' \in x)\{^{\tau}_{x} \lessdot (x' \in \tau)$
$(x' \in x')\{^{\tau}_{x} \lessdot (x' \in x')$
$(\neg\varphi)\{^{\tau}_{x} \lessdot (\neg\varphi\{^{\tau}_{x})$
$(\varphi \wedge \varphi')\{^{\tau}_{x} \lessdot (\varphi\{^{\tau}_{x} \wedge \varphi'\{^{\tau}_{x})$
$\forall x(\varphi)\{^{\tau}_{x} \lessdot \forall\tau(\varphi\{^{\tau}_{x})$

Definition 8 (Truth Value of Formula)**.**

1. Every variable is a set.
2. If x is the same set as x', then $(x = x')$ is true. Otherwise, $(x = x')$ is false.
   If φ is true, then $(\neg\varphi)$ is false. Otherwise, $(\neg\varphi)$ is true.
   If φ and φ' are true, then $(\varphi \wedge \varphi')$ is true. Otherwise, $(\varphi \wedge \varphi')$ is false.
   If φ is true for every possible x, then $\forall x(\varphi)$ is true. Otherwise, $\forall x(\varphi)$ is false.

2.3 Interlogue

Brackets are used to guarantee uniqueness for every reading of logical syntax. In the interest of brevity, we shall omit outermost pairs of brackets.

Notation 2.
$(x) \lessdot ((x))$

Notation 3.
$(a \circledast \cdots \circledast y \circledast z) \lessdot ((a \circledast \cdots \circledast y) \circledast z)$

Notation 4 (Connective of Disjunction)**.**
$(\varphi \vee \varphi') \lessdot (\neg((\neg\varphi) \wedge (\neg\varphi')))$

Notation 5 (Connective of Implication)**.**
$(\varphi \Rightarrow \varphi') \lessdot (\neg(\varphi \wedge (\neg\varphi')))$

Notation 6 (Connective of Equivalence)**.**
$(\varphi \Leftrightarrow \varphi') \lessdot ((\varphi \Rightarrow \varphi') \wedge (\varphi' \Rightarrow \varphi))$

Notation 7 (Connective of Exclusive Disjunction)**.**
$(\varphi \nLeftrightarrow \varphi') \lessdot (\neg(\varphi \Leftrightarrow \varphi'))$

Notation 8.
$(x_\circ, \ldots, x_\bullet \simeq X) \lessdot ((x_\circ \simeq X) \wedge \cdots \wedge (x_\bullet \simeq X))$

Notation 9.
$\forall x_\circ, \ldots, x_\bullet \simeq X(\varphi) \lessdot \forall x_\circ((x_\circ \simeq X) \Rightarrow \cdots \Rightarrow \forall x_\bullet((x_\bullet \simeq X) \Rightarrow \varphi))$

Notation 10.
$Q_\circ x_\circ \simeq X_\circ, \ldots, Q_\bullet x_\bullet \simeq X_\bullet(\varphi) \lessdot Q_\circ x_\circ \simeq X_\circ(\ldots(Q_\bullet x_\bullet \simeq X_\bullet(\varphi)))$

Notation 11 (Quantifier of Existence)**.**
$\exists x(\varphi) \lessdot (\neg\forall x(\neg\varphi))$

Notation 12 (Quantifier of Unique Existence)**.**
$\exists_! x(\varphi) \lessdot \exists x, \forall x'(\varphi\{^{x'}_{x} \Leftrightarrow (x = x'))$

Notation 13 (Quantifier of Dichotomous Existence)**.**
$\mathsf{E}x(\varphi) \lessdot ((\neg\exists x(\varphi)) \nLeftrightarrow \exists_! x(\varphi))$

Notation 14 (Reflectability of Relation)**.**
$(a \simeq b) \lessdot (b \backsimeq a)$

Notation 15 (Negatability of Relation)**.**
$(a \not\simeq b) \lessdot (\neg(a \simeq b))$

Notation 16 (Equatability of Relation)**.**
$(a \sim b) \lessdot ((a \simeq b) \nLeftrightarrow (a \neq b))$

Definition 9 (Reflexive Relation)**.**
$\mathrm{RefRel}(\simeq) \Leftrightarrow (a \simeq a)$

Definition 10 (Symmetric Relation)**.**
$\mathrm{SymRel}(\simeq) \Leftrightarrow ((a \simeq b) \Leftrightarrow (b \simeq a))$

Definition 11 (Transitive Relation)**.**
$\mathrm{TranRel}(\simeq) \Leftrightarrow (((a \simeq b) \wedge (b \simeq c)) \Rightarrow (a \simeq c))$

Definition 12 (Equivalence Relation)**.**
$\mathrm{EqvRel}(\simeq) \Leftrightarrow (\mathrm{RefRel}(\simeq) \wedge \mathrm{SymRel}(\simeq) \wedge \mathrm{TranRel}(\simeq))$

Proposition 1.
$\mathrm{EqvRel}(=)$

2.4 Zermelo–Frænkel Set Theory with Choice

Around the 1920s, an axiomatic set theory was proposed by Ernst Zermelo and Abraham Frænkel: this Zermelo–Frænkel set theory (ZF), when paired with the axiom of choice (AC), came to be known as Zermelo–Frænkel set theory with choice (ZFC). ZFC is commonly used as a foundational theory, and is conventionally written in a classical predicate logic.

It has been shown that if ZFC is consistent, then every formulation of ZFC must include at least one axiom schema. For brevity, we shall examine a particular formulation of ZFC which includes the axiom schema of separation, and no other axiom schema.

With the admission of ZF, AC is equivalent to various theorems, including the theorem that every vector space has a basis, and the theorem that every surjective function has a right inverse. Famously, AC in ZF also implies the Banach–Tarski paradox.

Axiom 1 (Empty Set).
$\exists N, \forall x(x \notin N)$

Definition 13 (Empty Set).
$(N = \varnothing) \Leftrightarrow \forall x(x \notin N)$

Axiom 2 (Extensionality).
$\forall X, Y(\forall m((m \in X) \Leftrightarrow (m \in Y)) \Rightarrow (X = Y))$

Definition 14 (Subset).
$(X \subseteq Y) \Leftrightarrow \forall m((m \in X) \Rightarrow (m \in Y))$

Axiom 3 (Pairing).
$\forall c, c', \exists C((c \in C) \wedge (c' \in C))$

Definition 15 (Couple Set).
$(C = \{c, c'\}) \Leftrightarrow ((c \in C) \wedge (c' \in C))$

Definition 16 (Singleton Set).
$\{c\} = \{c, c\}$

Axiom 4 (Union).
$\forall W, \exists U, \forall w \in W, \forall u \in w(u \in U)$

Definition 17 (Unary Union Function).
$(U = \bigcup W) \Leftrightarrow \forall w \in W, \forall u \in w(u \in U)$

Definition 18 (Binary Union Function).
$(x \cup x') = \bigcup\{x, x'\}$

Definition 19 (Space of Free Variables).
Let $x_\circ, \ldots, x_\bullet$ be the free variables in φ.
$\text{free}(\varphi) = \bigcup\{\{x_\circ\}, \ldots, \{x_\bullet\}\}$

Axiom 5 (Power Set).
$\forall X, \exists P, \forall x((x \subseteq X) \Rightarrow (x \in P))$

Definition 20 (Power Set).
$(x \in \mathscr{P}(X)) \Leftrightarrow (x \subseteq X)$

Axiom Schema 1 (Separation).
Let φ be a formula, and $\text{free}(\varphi) \subseteq \{D, f\}$.
$\forall D, \exists F, \forall f(((f \in D) \wedge \varphi) \Rightarrow (f \in F))$

Definition 21.
Let φ be a formula, and $\text{free}(\varphi) \subseteq \{D, f\}$.
$(F = \{(f \in D) \mid \varphi\}) \Leftrightarrow (((f \in D) \wedge \varphi) \Rightarrow (f \in F))$

Definition 22.
Let φ be a formula, and $\text{free}(\varphi) \subseteq \{D, f\}$.
$(F = \{f \mid \varphi\}) \Leftrightarrow (\varphi \Rightarrow (f \in F))$

Definition 23 (Unary Intersection Function).
$\bigcap W = \{(u \in \bigcup W) \mid \forall w \in W(u \in w)\}$

Definition 24 (Binary Intersection Function).
$(x \cap x') = \bigcap\{x, x'\}$

Definition 25 (Successor Function).
$\mathscr{S}(n) = (n \cup \{n\})$

Axiom 6 (Infinity).
$\exists R((\varnothing \in R) \wedge \forall r \in R(\mathscr{S}(r) \in R))$

Axiom 7 (Regularity).
$\forall O \neq \varnothing, \exists o \in O((o \cap O) = \varnothing)$

Definition 26 (Set of Pairwise Disjoint Sets).
$\text{PDis}(X) \Leftrightarrow \forall x, x' \in X((x \neq x') \Rightarrow ((x \cap x') = \varnothing))$

Axiom 8 (Choice).
$\forall B((\text{PDis}(B) \wedge \forall S \in B(S \neq \varnothing)) \Rightarrow \exists B', \forall S \in B, \exists_! s \in S(s \in B'))$

2.5 Interlogue

Proposition 2.
$\mathrm{ParOrdRel}(\subseteq)$

Definition 27 (Difference)**.**
$(X \setminus Y) = \{x \mid ((x \in X) \wedge (x \notin Y))\}$

Definition 28 (Symmetric Difference)**.**
$(X \triangledown Y) = ((X \setminus Y) \cup (Y \setminus X))$

Definition 29 (Complement)**.**
Let $x \subseteq X$.
$x^{\complement} = (X \setminus x)$

Notation 17 (Antisymmetric Relation)**.**
$\mathrm{AntiSymRel}(\simeq) \lessdot ((a \simeq b) \Leftrightarrow (b \simeq a))$

Notation 18 (Partial-Ordering Relation)**.**
$\mathrm{ParOrdRel}(\simeq) \lessdot (\mathrm{RefRel}(\simeq) \wedge \mathrm{AntiSymRel}(\simeq) \wedge \mathrm{Tran}(\simeq))$

Definition 30.
$\mathrm{ParOrdRel}(() \preceq)$

Notation 19 (Total Relation)**.**
$\mathrm{TotRel}(\simeq) \lessdot ((a \simeq b) \vee (b \simeq a))$

Notation 20 (Total-Ordering Relation)**.**
$\mathrm{TotOrdRel}(\simeq) \lessdot (\mathrm{ParOrdRel}(\simeq) \wedge \mathrm{TotRel}(\simeq))$

Definition 31.
$\mathrm{TotOrdRel}(\leq)$

Notation 21.
$(a \simeq b \simeq c) \lessdot ((a \simeq b) \wedge (b \simeq c))$

Definition 32 (Interval)**.**
$X_{[a,b]} = \{(x \in X) \mid \exists a, b \in X(a \leq x \leq b)\}$
$X_{[a,b)} = \{(x \in X) \mid \exists a, b \in X(a \leq x < b)\}$
$X_{(a,b]} = \{(x \in X) \mid \exists a, b \in X(a < x \leq b)\}$
$X_{(a,b)} = \{(x \in X) \mid \exists a, b \in X(a < x < b)\}$
$X_{\geq a} = X_{[a,\infty)} = \{(x \in X) \mid \exists a \in X(x \geq a)\}$
$X_{>a} = X_{(a,\infty)} = \{(x \in X) \mid \exists a \in X(x > a)\}$
$X_{\leq b} = X_{(-\infty,b]} = \{(x \in X) \mid \exists b \in X(x \leq b)\}$
$X_{<b} = X_{(-\infty,b)} = \{(x \in X) \mid \exists b \in X(x < b)\}$

Definition 33 (Space of Natural Numbers)**.**
$\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \dots\} = \bigcap\{R \mid ((\varnothing \in R) \wedge \forall r \in R(\mathcal{S}(r) \in R))\}$

Definition 34 (Order of Natural Numbers)**.**
Let $a, b \in \mathbb{N}$.
$(a \leq b) \Leftrightarrow ((a \in b) \not\Leftrightarrow (a = b))$

Definition 35 (Tuple)**.**
Let $n \in \mathbb{N}_{\geq 1}$.
$\langle\rangle = \varnothing$
$\langle x_1 \rangle = \{\{\langle\rangle\}, \{\langle\rangle, x_1\}\}$
$\langle x_1, \ldots, x_n, x_{\mathcal{S}(n)} \rangle = \{\{\langle x_1, \ldots, x_n \rangle\}, \{\langle x_1, \ldots, x_n \rangle, x_{\mathcal{S}(n)}\}\}$

Definition 36.
Let $n \in \mathbb{N}_{\geq 1}$.
$\Diamond(\langle\rangle) = \varnothing$
$\Diamond(\langle x_1 \rangle) = \{x_1\}$
$\Diamond(\langle x_1, \ldots, x_n, x_{\mathcal{S}(n)} \rangle) = (\Diamond(\langle x_1, \ldots, x_n \rangle) \cup \{x_{\mathcal{S}(n)}\})$

Definition 37.
Let $n \in \mathbb{N}_{\geq 1}$, and $k \in \mathbb{N}_{[1,n]}$.
$\langle x_1, \ldots, x_n \rangle_{[k]} = x_k$

Definition 38 (n-ary Cartesian Product)**.**
Let $n \in \mathbb{N}_{\geq 1}$.
$(X_1 \times \cdots \times X_n) = \{\langle x_1, \ldots, x_n \rangle \mid ((x_n \in X_1) \wedge \cdots \wedge (x_n \in X_n))\}$

Notation 22.
Let $n \in \mathbb{N}_{\geq 1}$, and $A_1 = \cdots = A_n$.
$A^{\circledast(n)} \lessdot (A_1 \circledast \cdots \circledast A_n)$

Definition 39.
$A^* = \bigcup\{A^{\times(i)} \mid (i \in \mathbb{N})\}$

Definition 40 (Space of Functions)**.**
$\{X \to Y\} = \{(f \subseteq (X \times Y)) \mid \forall x \in X, \exists_! y \in Y(\langle x, y \rangle \in f)\}$

Definition 41 (Function)**.**
$(f : X \to Y) \Leftrightarrow (f \in \{X \to Y\})$
$(f(x) = y) \Leftrightarrow \forall x \in X(\langle x, y \rangle \in f)$

Notation 23.
Let $f : (X_1 \times \cdots \times X_n) \to Y$.
$f(x_1, \ldots, x_n) \lessdot f(\langle x_1, \ldots, x_n \rangle)$

Notation 24.
$([.] : X \to Y) \lessdot (._{[]} : X \to Y)$
$([x] = y) \lessdot (._{[]}(x) = y)$

Definition 42 (Identity Function)**.**
$\mathrm{id}(x) = x$

Definition 43 (String).
Let $n \in \mathbb{N}_{\geq 1}$.
$"" = \langle\rangle$
$"x_1 \ldots x_n" = \langle "x_1", \ldots, "x_n" \rangle$

Definition 44 (Mutually Exclusive Set of Formulas).
$\not\wedge(\Phi) \Leftrightarrow \forall "\varphi", "\varphi'" \in \Phi((("\varphi" \neq "\varphi'") \Rightarrow (\neg(\varphi \wedge \varphi'))))$

Definition 45 (Collectively Exhaustive Set of Formulas).
$\vee(\{"\varphi_1", \ldots, "\varphi_n"\}) \Leftrightarrow (\varphi_1 \vee \cdots \vee \varphi_n)$

Definition 46 (Logical Partition).
$\veebar(\Phi) \Leftrightarrow (\not\emptyset(\Phi) \wedge \not\wedge(\Phi) \wedge \vee(\Phi))$

Definition 47 (Piecewise Function).
Let $n \in \mathbb{N}_{\geq 1}$, and $f : X \to Y$.

$$\left(f(x) = \begin{cases} y_1, & \varphi_1 \\ \vdots & \vdots \\ y_n, & \varphi_n \\ y_{\mathcal{S}(n)}, & \varphi_{\mathcal{S}(n)} \end{cases} \right) \Leftrightarrow (\forall x \in X, \forall i \in \mathbb{N}_{[1,\mathcal{S}(n)]}(\varphi_i \Rightarrow ((\langle x, y_i \rangle \in f)) \wedge \veebar(\{\varphi_1, \ldots, \varphi_{\mathcal{S}(n)}\}))$$

$(\text{o.w.} = \varphi_{\mathcal{S}(n)}) \Leftrightarrow (\varphi_{\mathcal{S}(n)} \Leftrightarrow (\neg(\varphi_1 \vee \cdots \vee \varphi_n)))$

Definition 48.
Let $n, s \in \mathbb{N}_{\geq 1}$.
$\forall x_{i,j} \overset{i \in \langle 1, \ldots, n \rangle}{\underset{j \in \langle 1, \ldots, s \rangle}{\in}} X_i(\varphi) \Leftrightarrow \forall x_{1,1}, \ldots, x_{1,s} \in X_1, \ldots, \forall x_{n,1}, \ldots, x_{n,s} \in X_n(\varphi)$

Definition 49.
Let $n, s \in \mathbb{N}_{\geq 1}$.
$\$_k(X_1, \ldots, X_n, F) \Leftrightarrow \forall x_{i,j} \overset{i \in \langle 1, \ldots, n \rangle}{\underset{j \in \langle 1, \ldots, s \rangle}{\in}} X_i, \forall f \in F(f(x_{1,1}, \ldots, x_{n,s}) \in X_n)$

Definition 50.
Let $k, n \in \mathbb{N}_{\geq 1}$.
$\widehat{\$}_k(X_1, \ldots, X_n, F) \Leftrightarrow \forall x_{i,j} \overset{i \in \langle 1, \ldots, n \rangle}{\underset{j \in \langle 1, \ldots, s \rangle}{\in}} X_i, \forall f \in F(\langle x_{1,1}, \ldots, x_{n,s}, f(x_{1,1}, \ldots, x_{n,s}) \rangle \in X_n)$

2.6 Metalanguage

Previously, we defined propositional and predicate logics. Now, we shall define propositional and predicate metalogics that mimic, and thus help to more formally define, their logical counterparts. To avoid circularity, a metalogic ought to be distinguished from its logic. However, in the interest of brevity, we shall not always make such a distinction.

2.6.1 Propositional Logic

Definition 51 (Space of Propositional Variables).
$\mathcal{X}_0 = \bigcup\{\{\texttt{"}p_1\texttt{"}, \ldots, \texttt{"}p_n\texttt{"}\} \mid (n \in \mathbb{N})\}$

Definition 52 (Propositional Alphabet).
$\text{🔡}_0 = (\mathcal{X}_0 \cup \{\texttt{"}\neg\texttt{"}, \texttt{"}\wedge\texttt{"}, \texttt{"}(\texttt{"}, \texttt{"})\texttt{"}\})$

Definition 53 (Concatenation Function of Negation).
$\mathrm{neg}(\texttt{"}\varphi\texttt{"}) = \texttt{"}(\neg\varphi)\texttt{"}$

Definition 54 (Concatenation Function of Conjunction).
$\mathrm{conj}(\texttt{"}\varphi\texttt{"}, \texttt{"}\varphi'\texttt{"}) = \texttt{"}(\varphi \wedge \varphi')\texttt{"}$

Definition 55 (Space of Propositional Formulas).
$\mathcal{F}_0 = \bigcap\{(\Phi \subseteq \text{🔡}_0^*) \mid ((\mathcal{X}_0 \subseteq \Phi) \wedge \$_1(\Phi, \{\mathrm{neg}\}) \wedge \$_2(\Phi, \{\mathrm{conj}\}))\}$

Definition 56 (Space of Truth Values).
$2 = \{0, 1\}$

Definition 57 (Truth Function of Negation).
$\mathrm{not} : 2 \to 2$
$\mathrm{not}(1) = 0$
$\mathrm{not}(0) = 1$

Definition 58 (Truth Function of Conjunction).
$\mathrm{and} : 2^{\times(2)} \to 2$
$\mathrm{and}(1, 1) = 1$
$\mathrm{and}(1, 0) = 0$
$\mathrm{and}(0, 1) = 0$
$\mathrm{and}(0, 0) = 0$

Definition 59 (Space of Truth Assignments).
$\mathcal{T}_0 = \{\mathcal{X}_0 \to 2\}$

Definition 60 (Valuation of Propositional Formula).
Let $t \in \mathcal{T}_0$.
$\mathfrak{v}_0^t : \mathcal{F}_0 \to 2$

$$\mathfrak{v}_0^t(\varphi) = \begin{cases} t(p), & \exists p \in \mathcal{X}_0(\varphi = p) \\ \mathrm{not}(\mathfrak{v}_0^t(\psi)), & \exists \psi \in \mathcal{F}_0(\varphi = \mathrm{neg}(\psi)) \\ \mathrm{and}(\mathfrak{v}_0^t(\psi), \mathfrak{v}_0^t(\psi')), & \exists \psi, \psi' \in \mathcal{F}_0(\varphi = \mathrm{conj}(\psi, \psi')) \end{cases}$$

2.6.2 Predicate Logic

Definition 61 (Space of Variables).
$\mathscr{X}_1 = \bigcup\{\{"x_1", \ldots, "x_n"\} \mid (n \in \mathbb{N})\}$

Definition 62 (Predicate Alphabet).
$\text{🔡}_1 = (\mathscr{X}_1 \cup \{"=", "\in", "\neg", "\wedge", "(", ")"\})$

Definition 63 (Concatenation Function of Equality).
$\mathrm{eq}("x", "x'") = "(x = x')"$

Definition 64 (Concatenation Function of Membership).
$\mathrm{in}("x", "x'") = "(x \in x')"$

Definition 65 (Concatenation Function of Universal Quantification).
$\mathrm{fa}("x", "\varphi") = "\forall x(\varphi)"$

Definition 66 (Space of Formulas).
$\mathscr{F}_{1,1} = \bigcap\{(\Phi \subseteq \text{🔡}_1^*) \mid ((\mathscr{X}_1 \subseteq \Phi) \wedge \$_2(\Phi, \{\mathrm{eq}, \mathrm{in}\}))\}$
$\mathscr{F}_1 = \bigcap\{(\Phi \subseteq \text{🔡}_1^*) \mid (\mathscr{F}_{1,1} \subseteq \Phi \wedge \$_1(\Phi, \{\mathrm{neg}\}) \wedge \$_2(\Phi, \{\mathrm{conj}\}) \wedge \$_1(\mathscr{X}_1, \Phi, \{\mathrm{fa}\}))\}$

Definition 67 (Space of Free Variables).
$\mathrm{free} : \mathscr{F}_1 \to \mathscr{P}(\mathscr{X}_1)$

$$\mathrm{free}(\varphi) = \begin{cases} \{x, x'\}, & \exists x, x' \in \mathscr{X}_1(\varphi = \mathrm{eq}(x, x')) \\ \{x, x'\}, & \exists x, x' \in \mathscr{X}_1(\varphi = \mathrm{in}(x, x')) \\ \mathrm{free}(\psi), & \exists \psi \in \mathscr{F}_1(\varphi = \mathrm{neg}(\psi)) \\ (\mathrm{free}(\psi) \cup \mathrm{free}(\psi')), & \exists \psi, \psi' \in \mathscr{F}_1(\varphi = \mathrm{conj}(\psi, \psi')) \\ (\mathrm{free}(\psi) \setminus \{x\}), & \exists x \in \mathscr{X}_1, \exists \psi \in \mathscr{F}_1(\varphi = \mathrm{fa}(x, \psi)) \end{cases}$$

Definition 68 (Space of Sentences).
$\mathscr{F}_1^{\bowtie} = \{(\varphi \in \mathscr{F}_1) \mid (\mathrm{free}(\varphi) = \varnothing)\}$

Definition 69 (Fresh Variable).
Let $n \in \mathbb{N}_{\geq 1}$.
$\ast(S_1, \ldots, S_n) = (\mathscr{X}_1 \setminus (\Diamond(S_1) \cup \cdots \cup \Diamond(S_n)))$

Definition 70 (Variable Substitution)**.**
Let $s, s' \in \mathcal{X}_1$.
$.|_s^{s'} : \mathcal{F}_1 \to \mathcal{F}_1$

$$\varphi|_s^{s'} = \begin{cases} \mathrm{eq}(x,x')|_s^{s'} = \begin{cases} \mathrm{eq}(s',s'), & (x, x' = s) \\ \mathrm{eq}(s',x'), & ((x = s) \wedge (x' \neq s)) \\ \mathrm{eq}(x,s'), & ((x \neq s) \wedge (x' = s)) \\ \mathrm{eq}(x,x'), & \text{o.w.} \end{cases}, & \exists x, x' \in \mathcal{X}_1(\varphi = \mathrm{eq}(x,x')) \\ \mathrm{in}(x,x')|_s^{s'} = \begin{cases} \mathrm{in}(s',s'), & (x, x' = s) \\ \mathrm{in}(s',x'), & ((x = s) \wedge (x' \neq s)) \\ \mathrm{in}(x,s'), & ((x \neq s) \wedge (x' = s)) \\ \mathrm{in}(x,x'), & \text{o.w.} \end{cases}, & \exists x, x' \in \mathcal{X}_1(\varphi = \mathrm{in}(x,x')) \\ \mathrm{neg}(\psi)|_s^{s'} = \mathrm{neg}(\psi|_s^{s'}), & \exists \psi \in \mathcal{F}_1(\varphi = \mathrm{neg}(\psi)) \\ \mathrm{conj}(\psi,\psi')|_s^{s'} = \mathrm{conj}(\psi|_s^{s'}, \psi'|_s^{s'}), & \exists \psi, \psi' \in \mathcal{F}_1(\varphi = \mathrm{conj}(\psi,\psi')) \\ \mathrm{fa}(x,\psi) = \begin{cases} \mathrm{fa}(x,\psi), & (x = s) \\ \mathrm{fa}(x,\psi|_s^{s'}), & ((x \neq s) \wedge (x \notin \Diamond(s'))) \\ \mathrm{fa}(\ast(s',\psi), \psi|_x^{\ast(s',\psi)}|_s^{s'}), & \text{o.w.} \end{cases}, & \exists x \in \mathcal{X}_1, \exists \psi \in \mathcal{F}_1(\varphi = \mathrm{fa}(x,\psi)) \end{cases}$$

Definition 71 (Space of Structures)**.**
$n_{F,R} = \{(F \cup R) \to \mathbb{N}\}$
$i_{U,F,R,a} = \{(F \cup R) \to (\bigcup\{\{U^{a(f)} \to U\} \mid (f \in F)\} \cup \bigcup\{\mathscr{P}(U^{a(r)}) \mid (r \in R)\})\}$
$\mathcal{M} = \{\langle U, \langle F, R, a\rangle, i\rangle \mid ((U \neq \varnothing) \wedge ((F \cap R) = \varnothing) \wedge (a \in n_{F,R}) \wedge (i \in i_{U,F,R,a}))\}$

Definition 72 (Structure of Predicate Logic)**.**
$\mathbb{s} = \langle U_{\mathbb{s}}, \langle \varnothing, \{"\in"\}, a_{\mathbb{s}}\rangle, i_{\mathbb{s}}\rangle$
$\mathbb{s} \in \mathcal{M}$
$a_{\mathbb{s}}("\in") = 2$
$i_{\mathbb{s}}("\in") = \{\langle x, X\rangle \mid (x \in X)\}$

Definition 73 (Space of Variable Assignments)**.**
Let $m \in \mathcal{M}$.
$\mathcal{T}_1^m = (\bigcup\{\{\mathrm{free}(\varphi) \to m_{[1]}\} \mid (\varphi \in \mathcal{F}_1)\} \cup \{\mathrm{id}\})$

Definition 74 (Valuation of Formula)**.**
Let $m \in \mathcal{M}$, and $t \in \mathcal{T}_1^m$.
$\upsilon_1^{m,t} : \mathcal{F}_1 \to 2$

$$\upsilon_1^{m,t}(\varphi) = \begin{cases} \upsilon_1^{m,t}(\mathrm{eq}(x,x')) = \begin{cases} 1, & t(x) = t(x') \\ 0, & \text{o.w.} \end{cases}, & \exists x, x' \in \mathcal{X}_1(\varphi = \mathrm{eq}(x,x')) \\ \upsilon_1^{m,t}(\mathrm{in}(x,x')) = \begin{cases} 1, & t(x) \in t(x') \\ 0, & \text{o.w.} \end{cases}, & \exists x, x' \in \mathcal{X}_1(\varphi = \mathrm{in}(x,x')) \\ \mathrm{not}(\upsilon_1^{m,t}(\psi)), & \exists \psi \in \mathcal{F}_1(\varphi = \mathrm{neg}(\psi)) \\ \mathrm{and}(\upsilon_1^{m,t}(\psi), \upsilon_1^{m,t}(\psi')), & \exists \psi, \psi' \in \mathcal{F}_1(\varphi = \mathrm{conj}(\psi,\psi')) \\ \upsilon_1^{m,t}(\mathrm{fa}(x,\psi)) = \begin{cases} 1, & \forall s \in m_{[1]}(\upsilon_1^{m,t}(\psi|_s^x) = 1) \\ 0, & \text{o.w.} \end{cases}, & \exists x \in \mathcal{X}_1, \psi \in \mathcal{F}_1(\varphi = \mathrm{fa}(x,\psi)) \end{cases}$$

2.7 Interlogue

Definition 75 (Composition)**.**
Let $f : X \to Y$, and $g : Y \to Z$.
$(g \circ f) = \{\langle x, z\rangle \mid ((\langle x, y\rangle \in f) \wedge (\langle y, z\rangle \in g))\}$

Definition 76 (Injective Function)**.**
Let $f : X \to Y$.
$\mathrm{Inj}(f) \Leftrightarrow \forall y \in Y, \exists x \in X(f(x) = y)$

Definition 77 (Surjective Function)**.**
Let $f : X \to Y$.
$\mathrm{Surj}(f) \Leftrightarrow \forall y \in Y, \exists x \in X(f(x) = y)$

Definition 78 (Bijective Function)**.**
Let $f : X \to Y$.
$\mathrm{Bij}(f) \Leftrightarrow (\mathrm{Inj}(f) \wedge \mathrm{Surj}(f))$

Definition 79.
$\mathrm{PropDual}(\langle p, X, f\rangle, \langle p', X', f'\rangle) \Leftarrow (\forall x \subseteq X(p(x) \Leftrightarrow p'(f(x))) \wedge \forall x' \subseteq X'(p'(x') \Leftrightarrow p(f'(x'))))$

Definition 80 (Galois-Connected Set of Tuples)**.**
$\mathrm{GalConn}(\langle X, f\rangle, \langle X', f'\rangle) \Leftarrow \forall x \subseteq X, \forall x' \subseteq X'(x \subseteq f'(x') \Leftrightarrow x' \subseteq f(x))$

Definition 81 (Finite Set)**.**
$\mathrm{Fin}(X) \Leftrightarrow \exists n \in \mathbb{N}, \exists f \in \{X \to \mathbb{N}_{\leq n}\}(\mathrm{Bij}(f))$

2.8 Adequacy

Adequacy is also known as *functional completeness.* A set of truth functions is said to be adequate only if every other truth function is "expressible" as a composition of those in the set.

Theorem 1 (Adequacy of Truth Functions of Negation and Conjunction)**.**
Let $b \in \bigcup\{\{2^{\times(n)} \to 2\} \mid (n \in \mathbb{N})\}$, and $k \in \mathbb{N}_{\geq 1}$.
$b \subseteq \bigcup\{(T \supseteq 2) \mid (\widehat{\$}_1(T, \text{not}) \wedge \cdots \wedge \widehat{\$}_2(T, \text{and}))\}$

2.9 Satisfiability and Definability

A logical formula is said to be *tautological* only if it is "always true", *satisfiable* only if it is "sometimes true", and *contradictory* only if it is "never true". The property of being tautological can be seen as *opposite* to the property of being contradictory, while the property of being satisfiable can be seen as *complementary* to the property of being contradictory. A set is said to be *definable* only if there exists a logical formula whose truth is equivalent to existence of the set.

2.9.1 Propositional Logic

Definition 82 (Satisfiable Set of Propositional Formulas)**.**
Let $\Phi \subseteq \mathcal{F}_0$.
$\mathrm{Sat}(\Phi) \Leftrightarrow \exists t \in \mathcal{T}_0, \forall \varphi \in \Phi(v_0^t(\varphi) = 1)$

Definition 83 (Definable Set of Truth Assignments)**.**
Let $T \subseteq \mathcal{T}_0$.
$\mathrm{Def}(T) \Leftrightarrow \exists \varphi \in \mathcal{F}_0, \forall t \in T(v_0^t(\varphi) = 1)$

Definition 84 (Subject of Set of Propositional Formulas)**.**
Let $\Phi \subseteq \mathcal{F}_0$.
$\mathrm{subj}_0(\Phi) = \{(t \in \mathcal{T}_0) \mid \forall \varphi \in \Phi(v_0^t(\varphi) = 1)\}$

Definition 85 (Theory of Set of Truth Assignments)**.**
Let $T \subseteq \mathcal{T}_0$.
$\mathrm{th}_0(T) = \{(\varphi \in \mathcal{F}_0) \mid \forall t \in T(v_0^t(\varphi) = 1)\}$

Theorem 2.
$\mathrm{PropDual}(\langle \mathrm{Sat}, \mathcal{F}_0, \mathrm{subj}_0 \rangle, \langle \mathrm{Def}, \mathcal{T}_0, \mathrm{th}_0 \rangle)$

Theorem 3.
$\mathrm{GalConn}(\langle \mathcal{F}_0, \mathrm{subj}_0 \rangle, \langle \mathcal{T}_0, \mathrm{th}_0 \rangle)$

Theorem 4.
$\exists \Phi \subseteq \mathcal{F}_0(\neg\, \mathrm{Sat}(\Phi))$

Theorem 5.
$\exists T \subseteq \mathcal{T}_0(\neg\, \mathrm{Def}(T))$

2.9.2 Predicate Logic

Definition 86 (Satisfiable Set of Sentences)**.**
Let $\Phi \subseteq \mathcal{F}_1^{\bowtie}$.
$\mathrm{Sat}(\Phi) \Leftrightarrow \exists m \in \mathcal{M}, \forall \varphi \in \Phi(v_1^{m,\mathrm{id}}(\varphi) = 1)$

Definition 87 (Definable Set of Structures)**.**
Let $M \subseteq \mathcal{M}$.
$\mathrm{Def}(M) \Leftrightarrow \exists \varphi \in \mathcal{F}_1^{\bowtie}, \forall m \in M(v_1^{m,\mathrm{id}}(\varphi) = 1)$

Definition 88 (Subject of Set of Sentences)**.**
Let $\Phi \subseteq \mathcal{F}_1^{\bowtie}$.
$\mathrm{subj}_1(\Phi) = \{(m \in \mathcal{M}) \mid \forall \varphi \in \Phi(v_1^{m,\mathrm{id}}(\varphi) = 1)\}$

Definition 89 (Theory of Set of Structures)**.**
Let $M \subseteq \mathcal{M}$.
$\mathrm{th}_1(M) = \{(\varphi \in \mathcal{F}_1^{\bowtie}) \mid \forall m \in M(\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = 1)\}$

Theorem 6.
$\mathrm{PropDual}(\langle \mathrm{Sat}, \mathcal{F}_1^{\bowtie}, \mathrm{subj}_1 \rangle, \langle \mathrm{Def}, \mathcal{M}, \mathrm{th}_1 \rangle)$

Theorem 7.
$\mathrm{GalConn}(\langle \mathcal{F}_1^{\bowtie}, \mathrm{subj}_1 \rangle, \langle \mathcal{M}, \mathrm{th}_1 \rangle)$

Theorem 8.
$\exists \Phi \subseteq \mathcal{F}_1^{\bowtie}(\neg \mathrm{Sat}(\Phi))$

Theorem 9.
$\exists M \subseteq \mathcal{M}(\neg \mathrm{Def}(M))$

2.10 Soundness and Completeness

For a logic, soundness can be seen as the property that “every proof has truth”, and completeness can be seen as the property that “every truth has proof”.

2.10.1 Propositional Logic

Definition 90 (Modus Ponens Function).
$\text{pon}(\text{"}\varphi\text{"}, \text{"}(\varphi \Rightarrow \varphi')\text{"}) = \text{"}\varphi'\text{"}$

Definition 91 (Space of Propositional Axioms).
$\mathcal{A}_{0,1} = \{\text{"}(\varphi \Rightarrow (\psi \Rightarrow \varphi))\text{"} \mid (\varphi, \psi \in \mathcal{F}_0)\}$
$\mathcal{A}_{0,2} = \{\text{"}((\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi)))\text{"} \mid (\varphi, \psi, \chi \in \mathcal{F}_0)\}$
$\mathcal{A}_{0,3} = \{\text{"}(((\neg\varphi) \Rightarrow (\neg\psi)) \Rightarrow (\psi \Rightarrow \varphi))\text{"} \mid (\varphi, \psi \in \mathcal{F}_0)\}$
$\mathcal{A}_0 = \bigcup\{\mathcal{A}_{0,1}, \mathcal{A}_{0,2}, \mathcal{A}_{0,3}\}$

Definition 92 (Propositional Proof System).
Let $\Gamma \subseteq \mathcal{F}_0$.
$\mathcal{D}_0(\Gamma) = \bigcap\{(\Phi \subseteq \text{🔡}_0^*) \mid (((\mathcal{A}_0 \cup \Gamma) \subseteq \Phi) \wedge \$_1(\Phi, \{\text{pon}\}))\}$

Definition 93 (Propositional Syntactic Entailment).
Let $\Gamma \subseteq \mathcal{F}_0$, and $\varphi \in \mathcal{F}_0$.
$(\Gamma \vdash \varphi) \Leftrightarrow (\varphi \in \mathcal{D}_0(\Gamma))$

Definition 94 (Propositional Semantic Entailment).
Let $\Gamma \subseteq \mathcal{F}_0$, and $\varphi \in \mathcal{F}_0$.
$(\Gamma \vDash \varphi) \Leftrightarrow \forall t \in \mathcal{T}_0(\forall \gamma \in \Gamma(v_0^t(\gamma) = 1) \Rightarrow (v_0^t(\varphi) = 1))$

Theorem 10 (Finitaryness of Propositional Proof System).
Let $\Gamma \subseteq \mathcal{F}_0$, and $\varphi \in \mathcal{D}_0(\Gamma)$.
$\exists n \in \mathbb{N}_{\geq 1}, \exists\langle \psi_1, \ldots, \psi_n\rangle \in \bigcap\{(\Phi \subseteq \text{🔡}_0^*) \mid (((\mathcal{A}_0 \cup \Gamma) \subseteq \Phi) \wedge \widehat{\$}_1(\Phi, \{\text{pon}\}))\}(\psi_n = \varphi)$

Theorem 11 (Soundness of Propositional Proof System).
Let $\Gamma \subseteq \mathcal{F}_0$, and $\varphi \in \mathcal{F}_0$.
$(\Gamma \vdash \varphi) \Rightarrow (\Gamma \vDash \varphi)$

Theorem 12 (Completeness of Propositional Proof System).
Let $\Gamma \subseteq \mathcal{F}_0$, and $\varphi \in \mathcal{F}_0$.
$(\Gamma \vDash \varphi) \Rightarrow (\Gamma \vdash \varphi)$

Definition 95 (Consistent Set of Propositional Formulas).
Let $\Gamma \subseteq \mathcal{F}_0$.
$\text{Cons}(\Gamma) \Leftrightarrow \forall \varphi \in \mathcal{F}_0(\neg((\Gamma \vdash \varphi) \wedge (\Gamma \nvdash \varphi)))$

Definition 96 (Satisfiable Set of Propositional Formulas).
Let $\Gamma \subseteq \mathcal{F}_0$.
$\text{Sat}(\Gamma) \Leftrightarrow \forall \varphi \in \mathcal{F}_0(\neg((\Gamma \vDash \varphi) \wedge (\Gamma \nvDash \varphi)))$

Theorem 13 (Propositional Consistency–Satisfiability Equivalence).
Let $\Gamma \subseteq \mathcal{F}_0$.
$\text{Cons}(\Gamma) \Leftrightarrow \text{Sat}(\Gamma)$

2.10.2 Predicate Logic

Definition 97 (Space of Predicate Axioms).
$\mathcal{A}_{1,1} = \{\tau|^{\varphi}_{p} \mid ((\tau \in \mathcal{F}_0) \wedge (p \in \mathcal{X}_0) \wedge (\varphi \in \mathcal{F}_1) \wedge (\varnothing \vDash \tau))\}$
$\mathcal{A}_{1,2} = \{\text{"}(\forall x(\varphi) \Rightarrow \varphi|^{x'}_{x})\text{"} \mid ((x, x' \in \mathcal{X}_1) \wedge (\varphi \in \mathcal{F}_1))\}$
$\mathcal{A}_{1,3} = \{\text{"}(\varphi \Rightarrow \forall x(\varphi|^{x'}_{x}))\text{"} \mid ((x, x' \in \mathcal{X}_1) \wedge (\varphi \in \mathcal{F}_1))\}$
$\mathcal{A}_1 = \bigcup\{\mathcal{A}_{1,1}, \mathcal{A}_{1,2}, \mathcal{A}_{1,3}\}$

Definition 98 (Proof System).
Let $\Gamma \subseteq \mathcal{F}_1$.
$\mathcal{D}_1(\Gamma) = \bigcap\{(\Phi \subseteq \text{🔡}_1^*) \mid (((\mathcal{A}_1 \cup \Gamma) \subseteq \Phi) \wedge \mathscr{S}_1(\Phi, \{\mathrm{pon}\}))\}$

Definition 99 (Predicate Syntactic Entailment).
Let $\Gamma \subseteq \mathcal{F}_1$, and $\varphi \in \mathcal{F}_1$.
$(\Gamma \vdash \varphi) \Leftrightarrow (\varphi \in \mathcal{D}_1(\Gamma))$

Definition 100 (Predicate Semantic Entailment).
Let $\Gamma \subseteq \mathcal{F}_1$, and $\varphi \in \mathcal{F}_1$.
$(\langle M, \Gamma\rangle \vDash \varphi) \Leftrightarrow \forall m \in M, \forall t \in \mathcal{T}_1^m(\forall \gamma \in \Gamma(\mathfrak{v}_1^{m,t}(\gamma) = 1) \Rightarrow (\mathfrak{v}_1^{m,t}(\varphi) = 1))$

Theorem 14 (Finitaryness of Proof System).
Let $\Gamma \subseteq \mathcal{F}_1$, and $\varphi \in \mathcal{D}_1(\Gamma)$.
$\exists n \in \mathbb{N}_{\geq 1}, \exists\langle\psi_1, \ldots, \psi_n\rangle \in \bigcap\{(\Phi \subseteq \text{🔡}_1^*) \mid (((\mathcal{A}_1 \cup \Gamma) \subseteq \Phi) \wedge \widehat{\mathscr{S}}_1(\Phi, \{\mathrm{pon}\}))\}(\psi_n = \varphi)$

Theorem 15 (Soundness of Proof System).
Let $\Gamma \subseteq \mathcal{F}_1$, and $\varphi \in \mathcal{F}_1$.
$(\Gamma \vdash \varphi) \Rightarrow (\langle \mathcal{M}, \Gamma\rangle \vDash \varphi)$

Theorem 16 (Completeness of Proof System).
Let $\Gamma \subseteq \mathcal{F}_1$, and $\varphi \in \mathcal{F}_1$.
$(\langle \mathcal{M}, \Gamma\rangle \vDash \varphi) \Rightarrow (\Gamma \vdash \varphi)$

Definition 101 (Consistent Set of Formulas).
Let $\Gamma \subseteq \mathcal{F}_1$.
$\mathrm{Cons}(\Gamma) \Leftrightarrow \forall \varphi \in \mathcal{F}_1(\neg((\Gamma \vdash \varphi) \wedge (\Gamma \nvdash \varphi)))$

Definition 102 (Satisfiable Set of Formulas).
Let $\Gamma \subseteq \mathcal{F}_1$.
$\mathrm{Sat}(\Gamma) \Leftrightarrow \forall \varphi \in \mathcal{F}_1(\neg((\langle \mathcal{M}, \Gamma\rangle \vDash \varphi) \wedge (\langle \mathcal{M}, \Gamma\rangle \nvDash \varphi)))$

Theorem 17 (Predicate Consistency–Satisfiability Equivalence).
Let $\Gamma \subseteq \mathcal{F}_1$.
$\mathrm{Cons}(\Gamma) \Leftrightarrow \mathrm{Sat}(\Gamma)$

2.11 Compactness and Maximality

Previously, we established equivalence between consistency and satisfiability. In the interest of brevity, we shall occasionally omit discussion of consistency in favor of satisfiability, keeping in mind that every property which applies to the latter, also applies to the former, in equivalent fashion.

2.11.1 Propositional Logic

Theorem 18 (Propositional Compactness)**.**
Let $\Gamma \subseteq \mathscr{F}_0$.
$\mathrm{Sat}(\Gamma) \Leftrightarrow \forall \Gamma' \subseteq \Gamma(\mathrm{Fin}(\Gamma') \Rightarrow \mathrm{Sat}(\Gamma'))$

Definition 103 (Maximal Set of Propositional Formulas)**.**
Let $\Gamma \subseteq \mathscr{F}_0$.
$\mathrm{Max}(\Gamma) \Leftrightarrow \forall \varphi \in \mathscr{F}_0((\Gamma \vDash \varphi) \vee (\Gamma \nvDash \varphi))$

Theorem 19 (Propositional Lindenbaum)**.**
Let $\Gamma \subseteq \mathscr{F}_0$.
$\mathrm{Sat}(\Gamma) \Leftrightarrow \exists \Gamma' \supseteq \Gamma(\mathrm{Sat}(\Gamma') \wedge \mathrm{Max}(\Gamma'))$

2.11.2 Predicate Logic

Theorem 20 (Predicate Compactness)**.**
Let $\Gamma \subseteq \mathscr{F}_1$.
$\mathrm{Sat}(\Gamma) \Leftrightarrow \forall \Gamma' \subseteq \Gamma(\mathrm{Fin}(\Gamma') \Rightarrow \mathrm{Sat}(\Gamma'))$

Definition 104 (Maximal Set of Formulas)**.**
Let $\Gamma \subseteq \mathscr{F}_1$.
$\mathrm{Max}(\Gamma) \Leftrightarrow \forall \varphi \in \mathscr{F}_1((\langle \mathcal{M}, \Gamma \rangle \vDash \varphi) \vee (\langle \mathcal{M}, \Gamma \rangle \nvDash \varphi))$

Theorem 21 (Predicate Lindenbaum)**.**
Let $\Gamma \subseteq \mathscr{F}_1$.
$\mathrm{Sat}(\Gamma) \Leftrightarrow \exists \Gamma' \supseteq \Gamma(\mathrm{Sat}(\Gamma') \wedge \mathrm{Max}(\Gamma'))$

2.12 Gödel Incompleteness

A space of sentences which are true for a structure of natural numbers is known as a *theory of natural arithmetic*. A theory is said to be "sufficiently strong" only if it contains "sufficiently many" sentences.

Previously, we defined a proof system which is finitary, sound, and complete with respect to a theory of *predicate logic*. In the 1920s, it was wondered whether one could develop a proof system which is verifiable, sound, and complete with respect to theories of *mathematics*. Around the 1930s, Kurt Gödel demonstrated that, if "verifiable" is taken to mean "recursively axiomatisable", then no such proof system exists for "sufficiently strong" theories of natural arithmetic. Gödel also showed that no consistent proof system in which such theories of natural arithmetic are derivable can prove its *own* consistency. These theorems have come to be known as the *Gödel incompleteness theorems*. For the sake of concreteness, we shall state these theorems for a particular theory of natural arithmetic.

Famously, Gödel's incompleteness theorems have been shown to apply to a theory of natural arithmetic known as *Peano arithmetic* (PA). The strengthened finite Ramsey theorem, which can be seen as a sentence about the natural numbers, is provable in ZFC, but not in PA. Additionally, ZFC can prove PA consistent, but PA cannot prove *itself* consistent.

Definition 105.
$\mathbb{n} = \langle \mathbb{N}, \langle \{\texttt{"0"}, \texttt{"}\mathcal{S}\texttt{"}\}, \varnothing, a_{\mathbb{n}} \rangle, i_{\mathbb{n}} \rangle$
$\mathbb{n} \in \mathcal{M}$
$a_{\mathbb{n}}(\texttt{"0"}) = 0$
$a_{\mathbb{n}}(\texttt{"}\mathcal{S}\texttt{"}) = 1$
$i_{\mathbb{n}}(\texttt{"0"}) = 0$
$i_{\mathbb{n}}(\texttt{"}\mathcal{S}\texttt{"}) = \mathcal{S}$

Theorem 22 (First Gödel Incompleteness)**.**
$\nexists \Gamma \subseteq \mathcal{F}_1, \forall \varphi \in \mathrm{th}(\{\mathbb{n}\})((\langle \{\mathbb{n}\}, \Gamma \rangle \vDash \varphi) \Rightarrow (\Gamma \vdash \varphi))$

Theorem 23 (Second Gödel Incompleteness)**.**
Let $\Gamma \subseteq \mathcal{F}_1$, and $\mathrm{th}(\{\mathbb{n}\}) \subseteq \mathcal{D}_1(\Gamma)$.
$\mathrm{Cons}(\mathcal{D}_1(\Gamma)) \Leftrightarrow (\Gamma \nvdash \texttt{"}\mathrm{Cons}(\mathcal{D}_1(\Gamma))\texttt{"})$