CLASSICAL LOGIC

GREGORY LIM

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Preface

The unexamined life is not worth living.
— SOCRATES

What could it mean for something to be *true*? Throughout history, various definitions for the word "truth" have been proposed.

DEFINITION I — CORRESPONDENCE THEORY OF TRUTH
Truth is that which corresponds to reality.

DEFINITION II — COHERENCE THEORY OF TRUTH
Truth is that which coheres with every other truth.

Ought mathematics concern itself with the universe we inhabit? In any case, the practice of mathematics seems to be based on something like **DEFINITION II**; this presents a *seeming* trilemma.

DEFINITION III — MÜNCHHAUSEN TRILEMMA
Every proof is completed by circularity, infinite regress, and/or assumption.

Reasoning by coherence appears to obtain truth by circularity, infinite regress, and/or assumption. It is no secret that mathematical theories admit assumptions. Mathematical theories are also said to build upon, and thus derive from, each other. A theory from which other theories can be derived is said to be *foundational*.

Broadly speaking, a logic can be thought of as a language for reasoning about truth — that is, a system which prescribes symbols, and ways of interchanging those symbols. This text studies propositional and predicate logics, in particular.

With any human language, definitions may be said to originate in *pre-linguistic* thought. Writers sometimes choose their starting points based on what feels the most intuitive to them. For this text, I have decided to use words and phrases like "not", "and", "if... then", "either... or", "otherwise", "every", "same", in addition to a few other mathematical symbols, in my appeal to the intuition of the reader.

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Language

There are three rules for writing a novel. Unfortunately, no one knows what they are.
— SOMERSET MAUGHAM

A propositional logic is sometimes called a *zeroth-order calculus*, and a predicate logic is sometimes called a *first-order calculus*. A predicate logic can be said to "extend" a propositional logic.

Propositional variables and (predicate) variables are known as *logical variables*. Similarly, propositional formulas and (predicate) formulas are known as *logical formulas*. Logical formulas can be recursively defined from (their) logical variables.

2.1 Propositional Logic

2.1.1 Syntax

DEFINITION IV — PROPOSITIONAL FORMULA

Let $p_\circ, \ldots, p_\bullet$ be propositional variables.

1. If p is a propositional variable, then p is a propositional formula.
2. If φ is a propositional formula, then $(\neg\varphi)$ is a propositional formula.
   If φ and φ' are propositional formulas, then $(\varphi \wedge \varphi')$ is a propositional formula.

2.1.2 Semantics

DEFINITION V — TRUTH VALUE OF PROPOSITIONAL FORMULA

1. Every propositional variable is either assigned true, or assigned false.
2. If φ is assigned true, then $(\neg\varphi)$ is assigned false.
   Otherwise, $(\neg\varphi)$ is assigned true.
   If φ and φ' are assigned true, then $(\varphi \wedge \varphi')$ is assigned true.
   Otherwise, $(\varphi \wedge \varphi')$ is assigned false.

2.2 Predicate Logic

2.2.1 Syntax

DEFINITION VI — FORMULA

Let $x_\circ, \dots, x_\bullet$ be variables.

1. If x is a variable, and x' is a variable, then $(x = x')$ is a formula.
   If x is a variable, and x' is a variable, then $(x \in x')$ is a formula.
2. If φ is a formula, then $(\neg\varphi)$ is a formula.
   If φ is a formula, and φ' is a formula, then $(\varphi \wedge \varphi')$ is a formula.
   If x is a variable, and φ is a formula, then $\forall x(\varphi)$ is a formula.

DEFINITION VII — FREE VARIABLE

1. If $(x = x')$ is a formula, then x and x' are free variables in the formula.
   If $(x \in x')$ is a formula, then x and x' are free variables in the formula.
2. If x is a free variable in φ, then x is a free variable in $(\neg\varphi)$.
   If x is a free variable in φ and φ', then x is a free variable in $(\varphi \wedge \varphi')$.
   If x is a free variable in φ, and $(x \neq x')$, then x is a free variable in $\forall x'(\varphi)$.

NOTATION I — NAÏVE SUBSTITUTION

$(x = x)\|_{x\mapsto\tau} \lessdot (\tau = \tau)$

$(x = x')\|_{x\mapsto\tau} \lessdot (\tau = x')$

$(x' = x)\|_{x\mapsto\tau} \lessdot (x' = \tau)$

$(x' = x')\|_{x\mapsto\tau} \lessdot (x' = x')$

$(x \in x)\|_{x\mapsto\tau} \lessdot (\tau \in \tau)$

$(x \in x')\|_{x\mapsto\tau} \lessdot (\tau \in x')$

$(x' \in x)\|_{x\mapsto\tau} \lessdot (x' \in \tau)$

$(x' \in x')\|_{x\mapsto\tau} \lessdot (x' \in x')$

$(\neg\varphi)\|_{x\mapsto\tau} \lessdot (\neg\varphi\|_{x\mapsto\tau})$

$(\varphi \wedge \varphi')\|_{x\mapsto\tau} \lessdot (\varphi\|_{x\mapsto\tau} \wedge \varphi'\|_{x\mapsto\tau})$

$\forall x(\varphi)\|_{x\mapsto\tau} \lessdot \forall\tau(\varphi\|_{x\mapsto\tau})$

NOTATION II

$(x \not\circledast x') \lessdot (\neg(x \circledast x'))$

NOTATION III

$(\varphi \Rightarrow \varphi') \lessdot (\neg(\varphi \wedge (\neg\varphi')))$

NOTATION IV

$(\varphi \Leftrightarrow \varphi') \lessdot ((\varphi \Rightarrow \varphi') \wedge (\varphi' \Rightarrow \varphi))$

NOTATION V

$(\varphi \oplus \varphi') \lessdot ((\varphi \vee \varphi') \wedge \neg((\varphi \wedge \varphi')))$

NOTATION VI

$\exists x(\varphi) \lessdot (\neg\forall x((\neg\varphi)))$

NOTATION VII

$\exists_! x(\varphi) \lessdot \exists x(\forall x'((\varphi\|_x^{x'} \Leftrightarrow (x = x'))))$

NOTATION VIII

$\mathsf{E} x(\varphi) \lessdot (\nexists x(\varphi) \oplus \exists_! x(\varphi))$

NOTATION IX

$\forall x_\circ, \ldots, x_\bullet \circledast X(\varphi) \lessdot \forall x_\circ(((x_\circ \circledast X) \Rightarrow \cdots \Rightarrow \forall x_\bullet(((x_\bullet \in X) \Rightarrow \varphi))))$

NOTATION X

$\exists x_\circ, \ldots, x_\bullet \circledast X(\varphi) \lessdot \exists x_\circ(((x_\circ \circledast X) \wedge \cdots \wedge \exists x_\bullet(((x_\bullet \in X) \wedge \varphi))))$

NOTATION XI

$\exists_! x_\circ, \ldots, x_\bullet \circledast X(\varphi) \lessdot \exists_! x_\circ(((x_\circ \circledast X) \wedge \cdots \wedge \exists_! x_\bullet(((x_\bullet \in X) \wedge \varphi))))$

NOTATION XII

$\mathsf{E} x_\circ, \ldots, x_\bullet \circledast X(\varphi) \lessdot \mathsf{E} x_\circ(((x_\circ \circledast X) \wedge \cdots \wedge \mathsf{E} x_\bullet(((x_\bullet \in X) \wedge \varphi))))$

2.2.2 Semantics

DEFINITION VIII — TRUTH VALUE OF FORMULA

1. Every variable is assigned a set.
2. If x is assigned the same set as x', then $(x = x')$ is assigned true.
   Otherwise, $(x = x')$ is assigned false.
   If φ is assigned true, then $(\neg\varphi)$ is assigned false.
   Otherwise, $(\neg\varphi)$ is assigned true.
   If φ and φ' are assigned true, then $(\varphi \wedge \varphi')$ is assigned true.
   Otherwise, $(\varphi \wedge \varphi')$ is assigned false.
   If φ is assigned true for every possible x, then $\forall x(\varphi)$ is assigned true.
   Otherwise, $\forall x(\varphi)$ is assigned false.

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Zermelo–Frænkel Set Theory with Choice

A set is a 'many' that allows itself to be thought of as a 'one'.
— GEORG CANTOR

Previously, we used brackets to guarantee uniqueness for every reading of logical syntax. In the interest of brevity, we shall, henceforth, omit outermost pairs of brackets.

Around the 1920s, an axiomatic set theory was proposed by Ernst Zermelo and Abraham Frænkel: this Zermelo–Frænkel set theory (ZF), when paired with the axiom of choice (AC), came to be known as Zermelo–Frænkel set theory with choice (ZFC). ZFC is commonly used as a foundational theory, and is conventionally written in a classical predicate logic. With the admission of ZF, AC is equivalent to various theorems, including the theorem that every vector space has a basis, and the theorem that every surjective function has a right inverse. Famously, AC in ZF also implies the Banach–Tarski paradox.

AXIOM I — EMPTY SET
$\exists N(\forall x((x \notin N)))$

DEFINITION IX — EMPTY SET
$((N = \varnothing) = \{\}) \Leftrightarrow \forall x((x \notin N))$

AXIOM II — EXTENSIONALITY
$\forall X, Y((\forall m(((m \in X) \Leftrightarrow (m \in Y)))) \Rightarrow (X = Y)))$

DEFINITION X — SUBSET
$(X \subseteq Y) \Leftrightarrow \forall m(((m \in X) \Rightarrow (m \in Y)))$

AXIOM III — PAIRING
$\forall c, c'(\exists C(((c \in C) \wedge (c' \in C))))$

DEFINITION XI
$(C = \{c, c'\}) \Leftrightarrow ((c \in C) \wedge (c' \in C))$

DEFINITION XII — SINGLETON SET

$\{c\} = \{c, c\}$

AXIOM IV — UNION

$\forall X(\exists U(\forall u(((u \in U) \Leftrightarrow \exists x \in X((u \in x))))))$

DEFINITION XIII — UNARY UNION FUNCTION

$(U = \bigcup X) \Leftrightarrow \forall u(((u \in U) \Leftrightarrow \exists x \in X((u \in x))))$

DEFINITION XIV — BINARY UNION FUNCTION

$(x \cup x') = \bigcup\{x, x'\}$

DEFINITION XV — SET OF FREE VARIABLES

Let $x_\circ, \ldots, x_\bullet$ be the free variables in φ.

$\text{free}(\varphi) = \bigcup\{\{x_\circ\}, \ldots, \{x_\bullet\}\}$

AXIOM V — POWER SET

$\forall X(\exists P(\forall p(((p \subseteq X) \Rightarrow (p \in P)))))$

AXIOM SCHEMA I — REPLACEMENT

Let $\text{free}(\varphi) \subseteq \{D, d, i\}$.

$\forall D((\forall d \in D(\exists_! i(\varphi)) \Rightarrow \exists I(\forall i(((i \in I) \Leftrightarrow \exists d \in D(\varphi))))))$

AXIOM SCHEMA II — SEPARATION

Let $\text{free}(\varphi) \subseteq \{D, a_\circ, \ldots, a_\bullet, f\}$.

$\forall D, a_\circ, \ldots, a_\bullet(\exists F(\forall f(((f \in F) \Leftrightarrow ((f \in D) \wedge \varphi)))))$

DEFINITION XVI

Let $\text{free}(\varphi) \subseteq \{D, f\}$.

$(F = \{(d \in D) \mid \varphi\}) \Leftrightarrow \forall f(((f \in F) \Leftrightarrow ((f \in D) \wedge \varphi)))$

DEFINITION XVII

Let $\text{free}(\varphi) \subseteq \{f\}$.

$(F = \{x \mid \varphi\}) \Leftrightarrow \forall f(((f \in F) \Leftrightarrow \varphi))$

DEFINITION XVIII — UNARY INTERSECTION FUNCTION

$\bigcap X = \{(u \in \bigcup X) \mid \forall x \in X((u \in x))\}$

DEFINITION XIX — BINARY INTERSECTION FUNCTION

$(x \cap x') = \bigcap\{x, x'\}$

AXIOM VI — INFINITY

$\exists R(((\varnothing \in R) \wedge \forall r \in R(((r \cup \{r\}) \in R))))$

AXIOM VII — REGULARITY

$\forall O(((O \neq \varnothing) \Rightarrow \exists o \in O(((o \cap O) = \varnothing))))$

DEFINITION XX — SET OF PAIRWISE DISJOINT SETS

$\textcircled{$\not\cap$}(X) \Leftrightarrow \forall x, x' \in X((x \neq x') \Rightarrow ((x \cap x') = \varnothing))$

AXIOM VIII — CHOICE

$\forall B(((\forall S \in B((S \neq \varnothing))) \wedge \textcircled{$\not\cap$}(B)) \Rightarrow \exists B'(\forall S \in B(\exists! s \in S((s \in B')))))$

4

Interlude I

DEFINITION XXI

$(x_\circ, \dots, x_\bullet \in X) \Leftrightarrow ((x_\circ \in X) \wedge \cdots \wedge (x_\bullet \in X))$

DEFINITION XXII

$(x_\circ, \dots, x_\bullet \subseteq X) \Leftrightarrow ((x_\circ \subseteq X) \wedge \cdots \wedge (x_\bullet \subseteq X))$

DEFINITION XXIII — POWER SET

$\mathscr{P}(X) = \{x \mid (x \subseteq X)\}$

DEFINITION XXIV — RELATIVE COMPLEMENT

$(X \setminus Y) = \{(d \in X) \mid (d \notin Y)\}$

DEFINITION XXV — SUCCESSOR

$i(x) = (x \cup \{x\})$

DEFINITION XXVI — SPACE OF NATURAL NUMBERS

$(\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \dots\}) = \bigcap\{R \mid ((\varnothing \in R) \wedge \forall r \in R((i(r) \in R)))\}$

DEFINITION XXVII

Let $n \in \mathbb{N}$.

$\mathbb{N}_{\geq n} = \bigcap\{R \mid ((n \in R) \wedge \forall r \in R((i(r) \in R)))\}$

DEFINITION XXVIII

Let $n \in \mathbb{N}$.

$\mathbb{N}_{\leq n} = (\{n\} \cup \{p \mid (p \notin \mathbb{N}_{\geq n})\})$

DEFINITION XXIX

Let $m, n \in \mathbb{N}$.

$\mathbb{N}_{[m,n]} = (\mathbb{N}_{\geq m} \cap \mathbb{N}_{\leq n})$

DEFINITION XXX — n-TUPLE

Let $n \in \mathbb{N}_{\geq 1}$.

$\langle x_1, x_2 \rangle = \{\{x_1\}, \{x_1, x_2\}\}$

$\langle x_1, \ldots, x_{i(n)} \rangle = \langle \langle x_1, \ldots, x_n \rangle, x_{i(n)} \rangle$

DEFINITION XXXI

Let $k, n \in \mathbb{N}_{\geq 1}$.

$\langle x_1, \ldots, x_k \rangle_n = x_n$

DEFINITION XXXII

Let $n \in \mathbb{N}_{\geq 1}$.

$\Diamond(\langle x_1, \ldots, x_n \rangle) = \{x_1, \ldots, x_n\}$

DEFINITION XXXIII — CARTESIAN PRODUCT

$(X \times Y) = \{\langle x, y \rangle \mid ((x \in X) \wedge (y \in Y))\}$

DEFINITION XXXIV — FUNCTION SPACE

${}^{X}Y = \{(f \subseteq (X \times Y)) \mid \forall x \in X(\exists_! y \in Y((\langle x, y \rangle \in f)))\}$

DEFINITION XXXV — FUNCTION

$(f : X \to Y) \Leftrightarrow (f \in {}^{X}Y)$

$(f(x) = y) \Leftrightarrow (\langle x, y \rangle \in f)$

DEFINITION XXXVI — IDENTITY FUNCTION

$\mathrm{id} : X \to X$

$\mathrm{id}(x) = x$

DEFINITION XXXVII — SET OF MUTUALLY EXCLUSIVE FORMULAS

Let $\Phi \subseteq \mathcal{F}_1$.

$\not\wedge(\Phi) \Leftrightarrow \forall "\varphi", "\varphi'" \in \Phi((("\varphi" \neq "\varphi'") \Rightarrow (\varphi \not\wedge \varphi')))$

DEFINITION XXXVIII — PIECEWISE FUNCTION

Let $n \in \mathbb{N}$, and $\not\wedge(\{\varphi_1, \ldots, \varphi_n\})$.

$$\left(f(x) = \begin{cases} y_1, & \varphi_1 \\ \vdots & \vdots \\ y_n, & \varphi_n \end{cases}\right) \Leftrightarrow \left(f = \bigcup\{\{\langle x, y_i \rangle \mid \varphi_i\} \mid (i \in \mathbb{N}_{[1,n]})\}\right)$$

DEFINITION XXXIX — n-LENGTH STRING

Let $n \in \mathbb{N}$.

$"x_1 \ldots x_n" = \langle "x_1", \ldots, "x_n" \rangle$

DEFINITION XL — SPACE OF n-LENGTH STRINGS

Let $n \in \mathbb{N}$.

$A^n = (\underbrace{A \times \cdots \times A}_{n \text{ times}})$

DEFINITION XLI — SPACE OF FINITE-LENGTH STRINGS

$A^* = \bigcup\{A^n \mid (n \in \mathbb{N})\}$

NOTATION XIII

Let $k, n \in \mathbb{N}_{\geq 1}$.

$\forall(x_1, \ldots, x_k) \in (X_1, \ldots, X_n)(\varphi) \lessdot \forall x_{1,1}, \ldots, x_{1,k} \in X_1(\cdots \forall x_{n,1}, \ldots, x_{n,k} \in X_n(\varphi))$

DEFINITION XLII — $|k \times n|$-ARY SET CLOSURE

Let $k, n \in \mathbb{N}_{\geq 1}$.

$\mathscr{C}_n^k(X_1, \ldots, X_n, F) \Leftrightarrow \forall(x_1, \ldots, x_k) \in (X_1, \ldots, X_n)(\forall f \in F((f(x_{1,1}, \ldots, x_{n,k}) \in X_n)))$

5

Metalanguage

If a book is worth writing, it is worth re-writing.

— PAUL ATTEWELL

Previously, we defined propositional and predicate logics. In this chapter, we shall define propositional and predicate metalogics that mimic, and thus help to more formally define, their logical counterparts. To avoid circularity, metalogic ought to be distinguished from logic. However, in the interest of brevity, we shall, henceforth, not always make this distinction as apparently as one might hope.

5.1 Propositional Logic

5.1.1 Syntax

DEFINITION XLIII — SPACE OF PROPOSITIONAL VARIABLES

$\mathcal{X}_0 = \bigcup\{ "p_1", \ldots, "p_n" \mid (n \in \mathbb{N})\}$

DEFINITION XLIV — CONCATENATION FUNCTION OF NEGATION

$\text{neg} : \Phi \to \Phi$

$\text{neg}("\varphi") = "(\neg\varphi)"$

DEFINITION XLV — CONCATENATION FUNCTION OF CONJUNCTION

$\text{conj} : \Phi^2 \to \Phi$

$\text{conj}("\varphi", "\varphi'") = "(\varphi \wedge \varphi')"$

DEFINITION XLVI — SPACE OF PROPOSITIONAL CHARACTERS

$\mathcal{C}_0 = (\mathcal{X}_0 \cup \{ "\neg", "\wedge", "(", ")"\})$

DEFINITION XLVII — SPACE OF PROPOSITIONAL FORMULAS

$\mathcal{F}_0 = \bigcap\{(\Phi \subseteq \mathcal{C}_0^*) \mid (((\mathcal{X}_0 \subseteq \Phi) \wedge \mathscr{C}_1^1(\Phi, \{\text{neg}\})) \wedge \mathscr{C}_1^2(\Phi, \{\text{conj}\}))\}$

5.1.2 Semantics

DEFINITION XLVIII — TRUTH FUNCTION OF NEGATION

$\mathrm{not} : \{\mathbf{T}, \mathbf{F}\} \to \{\mathbf{T}, \mathbf{F}\}$

$\mathrm{not}(\mathbf{T}) = \mathbf{F}$

$\mathrm{not}(\mathbf{F}) = \mathbf{T}$

DEFINITION XLIX — TRUTH FUNCTION OF CONJUNCTION

$\mathrm{and} : \{\mathbf{T}, \mathbf{F}\}^2 \to \{\mathbf{T}, \mathbf{F}\}$

$\mathrm{and}(\mathbf{T}, \mathbf{T}) = \mathbf{T}$

$\mathrm{and}(\mathbf{T}, \mathbf{F}) = \mathbf{F}$

$\mathrm{and}(\mathbf{F}, \mathbf{T}) = \mathbf{F}$

$\mathrm{and}(\mathbf{F}, \mathbf{F}) = \mathbf{F}$

DEFINITION L — SPACE OF TRUTH ASSIGNMENTS

$\mathcal{T}_0 = {}^{\mathcal{X}_0}\{\mathbf{T}, \mathbf{F}\}$

DEFINITION LI — PROPOSITIONAL FORMULA EVALUATION FUNCTION

Let $t \in \mathcal{T}_0$.

$\mathfrak{v}_0^t : \mathcal{F}_0 \to \{\mathbf{T}, \mathbf{F}\}$

$\mathfrak{v}_0^t(p) = t(p)$

$\mathfrak{v}_0^t(\mathrm{neg}(\varphi)) = \mathrm{not}(\mathfrak{v}_0^t(\varphi))$

$\mathfrak{v}_0^t(\mathrm{conj}(\varphi, \varphi')) = \mathrm{and}(\mathfrak{v}_0^t(\varphi), \mathfrak{v}_0^t(\varphi'))$

5.2 Predicate Logic

5.2.1 Syntax

DEFINITION LII — SPACE OF VARIABLES

$\mathcal{X}_1 = \bigcup\{"x_1", \ldots, "x_n" \mid (n \in \mathbb{N})\}$

DEFINITION LIII — CONCATENATION FUNCTION OF EQUALITY

$\text{eq} : X^2 \to \Phi$

$\text{eq}("x", "x'") = "(x = x')"$

DEFINITION LIV — CONCATENATION FUNCTION OF MEMBERSHIP

$\text{in} : X^2 \to \Phi$

$\text{in}("x", "x'") = "(x \in x')"$

DEFINITION LV — CONCATENATION FUNCTION OF UNIVERSAL QUANTIFICATION

$\text{forall} : (X \times \Phi) \to \Phi$

$\text{forall}("x", "\varphi") = "\forall x(\varphi)"$

DEFINITION LVI — SPACE OF PREDICATE CHARACTERS

$\mathcal{C}_1 = (\mathcal{X}_1 \cup \{"=", "\in", "\neg", "\wedge", "(", ")"\})$

DEFINITION LVII — SPACE OF ATOMIC FORMULAS

$\mathcal{F}_1^{\odot} = \bigcap\{(\Phi \subseteq \mathcal{C}_1^*) \mid ((\mathcal{X}_1 \subseteq \Phi) \wedge \mathscr{C}_1^2(\Phi, \{\text{eq}, \text{in}\}))\}$

DEFINITION LVIII — SPACE OF FORMULAS

$\mathcal{F}_1 = \bigcap\{(\Phi \subseteq \mathcal{C}_1^*) \mid ((((\mathcal{F}_1^{\odot} \subseteq \Phi) \wedge \mathscr{C}_1^1(\Phi, \{\text{neg}\})) \wedge \mathscr{C}_1^2(\Phi, \{\text{conj}\})) \wedge \mathscr{C}_2^1(\mathcal{X}_1, \Phi, \{\text{forall}\}))\}$

DEFINITION LIX — SPACE OF FREE VARIABLES

$\text{free} : \mathcal{F}_1 \to \mathscr{P}(\mathcal{X}_1)$

$\text{free}(\text{eq}(x, x')) = \{x, x'\}$

$\text{free}(\text{in}(x, x')) = \{x, x'\}$

$\text{free}(\text{neg}(\varphi)) = \text{free}(\varphi)$

$\text{free}(\text{conj}(\varphi, \varphi')) = (\text{free}(\varphi) \cup \text{free}(\varphi'))$

$\text{free}(\text{forall}(x, \varphi)) = (\text{free}(\varphi) \setminus \{x\})$

DEFINITION LX — SPACE OF SENTENCES

$\mathcal{S} = \{(\varphi \in \mathcal{F}_1) \mid (\text{free}(\varphi) = \varnothing)\}$

DEFINITION LXI — CAPTURE-AVOIDING SUBSTITUTION

$$\mathrm{eq}(x,x')|_{m\mapsto s} = \begin{cases} \mathrm{eq}(s,s), & ((x=m)\wedge(x'=m)) \\ \mathrm{eq}(s,x'), & ((x=m)\wedge(x'\neq m)) \\ \mathrm{eq}(x,s), & ((x\neq m)\wedge(x'=m)) \\ \mathrm{eq}(x,x'), & ((x\neq m)\wedge(x'\neq m)) \end{cases}$$

$$\mathrm{in}(x,x')|_{m\mapsto s} = \begin{cases} \mathrm{in}(s,s), & ((x=m)\wedge(x'=m)) \\ \mathrm{in}(s,x'), & ((x=m)\wedge(x'\neq m)) \\ \mathrm{in}(x,s), & ((x\neq m)\wedge(x'=m)) \\ \mathrm{in}(x,x'), & ((x\neq m)\wedge(x'\neq m)) \end{cases}$$

$$\mathrm{neg}(\varphi)|_{m\mapsto s} = \mathrm{neg}(\varphi|_{m\mapsto s})$$

$$\mathrm{conj}(\varphi,\varphi')|_{m\mapsto s} = \mathrm{conj}(\varphi|_{m\mapsto s},\varphi'|_{m\mapsto s})$$

$$\mathrm{forall}(x,\varphi)|_{m\mapsto s} = \begin{cases} \mathrm{forall}(x,\varphi), & (x=m) \\ \mathrm{forall}(x,\varphi|_{m\mapsto s}), & ((x\neq m)\wedge(x\notin\Diamond(s))) \\ \mathrm{forall}(x_\ast,\varphi|_{x\mapsto x_\ast}|_{m\mapsto s}), & (((x\neq m)\wedge(x\in\Diamond(s)))\wedge(x_\ast\notin(\Diamond(s)\cup\Diamond(\varphi)))) \end{cases}$$

5.2.2 Semantics

DEFINITION LXII — SPACE OF ARITY FUNCTIONS

$\mathcal{R}(F, R) = {}^{(F \cup R)}\mathbb{N}$

DEFINITION LXIII — SPACE OF INTERPRETATION FUNCTIONS

$\mathcal{I}'(U, F, R) = (\bigcup\{{}^{U^{a(f)}}U \mid (f \in F)\} \cup \bigcup\{\mathscr{P}(U^{a(r)}) \mid (r \in R)\})$

$\mathcal{I}(U, F, R) = {}^{(F \cup R)}\mathcal{I}'(U, F, R)$

DEFINITION LXIV — SPACE OF STRUCTURES

$\mathcal{M} = \{\langle U, \langle (F \cup R), a\rangle, i\rangle \mid ((((U \neq \varnothing) \wedge ((F \cap R) = \varnothing)) \wedge (a \in \mathcal{R}(F, R))) \wedge (i \in \mathcal{I}(U, F, R)))\}$

DEFINITION LXV — STRUCTURE OF PREDICATE LOGIC

$(\mathfrak{L} = \langle U_{\mathfrak{L}}, \langle (\varnothing \cup \{\text{"}\in\text{"}\}), a_{\mathfrak{L}}\rangle, i_{\mathfrak{L}}\rangle) \in \mathcal{M}$

$a_{\mathfrak{L}}(\text{"}\in\text{"}) = 2$

$i_{\mathfrak{L}}(\text{"}\in\text{"}) = \{\langle x, X\rangle \mid (x \in X)\}$

DEFINITION LXVI — SPACE OF VARIABLE ASSIGNMENTS

Let $m \in \mathcal{M}$.

$\mathcal{T}_1^m = (\bigcup\{{}^{\mathrm{free}(\varphi)}(m_1) \mid (\varphi \in \mathcal{F}_1)\} \cup \{\mathrm{id}\})$

DEFINITION LXVII — FORMULA EVALUATION FUNCTION

Let $m \in \mathcal{M}$, and $t \in \mathcal{T}_1^m$.

$\mathscr{v}_1^{m,t} : \mathcal{F}_1 \to \{\mathbf{T}, \mathbf{F}\}$

$\mathscr{v}_1^{m,t}(\mathrm{eq}(x, x')) = \begin{cases} \mathbf{T}, & (v(x) = v(x')) \\ \mathbf{F}, & (v(x) \neq v(x')) \end{cases}$

$\mathscr{v}_1^{m,t}(\mathrm{in}(x, x')) = \begin{cases} \mathbf{T}, & (v(x) \in v(x')) \\ \mathbf{F}, & (v(x) \notin v(x')) \end{cases}$

$\mathscr{v}_1^{m,t}(\mathrm{neg}(\varphi)) = \mathrm{not}(\mathscr{v}_1^{m,t}(\varphi))$

$\mathscr{v}_1^{m,t}(\mathrm{conj}(\varphi, \varphi')) = \mathrm{and}(\mathscr{v}_1^{m,t}(\varphi), \mathscr{v}_1^{m,t}(\varphi'))$

$\mathscr{v}_1^{m,t}(\mathrm{forall}(x, \varphi)) = \begin{cases} \mathbf{T}, & \forall s \in m_1((\mathscr{v}_1^{m,t}(\varphi|_{x \mapsto s}) = \mathbf{T})) \\ \mathbf{F}, & (\neg\forall s \in m_1((\mathscr{v}_1^{m,t}(\varphi|_{x \mapsto s}) = \mathbf{T}))) \end{cases}$

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Interlude II

DEFINITION LXVIII — FUNCTION COMPOSITION

Let $f : X \to Y$, and $g : Y \to Z$.

$(g \circ f) = \{\langle x, z\rangle \mid ((\langle x, y\rangle \in f) \land (\langle y, z\rangle \in g))\}$

DEFINITION LXIX

Let $c : X \to X$.

$c^{[n]} = (\underbrace{c \circ \cdots \circ c}_{n \text{ times}})$

DEFINITION LXX — INJECTIVE FUNCTION

Let $f : X \to Y$.

$\text{injective}(f) \Leftrightarrow \forall y \in Y(\mathsf{E}x \in X((f(x) = y)))$

DEFINITION LXXI — SURJECTIVE FUNCTION

Let $f : X \to Y$.

$\text{surjective}(f) \Leftrightarrow \forall y \in Y(\exists x \in X((f(x) = y)))$

DEFINITION LXXII — BIJECTIVE FUNCTION

Let $f : X \to Y$.

$\text{bijective}(f) \Leftrightarrow (\text{injective}(f) \land \text{surjective}(f))$

DEFINITION LXXIII — FINITE SET

$(|X| < \infty) \Leftrightarrow \exists n \in \mathbb{N}(\exists f \in {}^{X}\mathbb{N}_{\leq n}(\text{bijective}(f)))$

7

Adequacy

Adequacy is also known as *functional completeness*.

DEFINITION LXXIV — SPACE OF n-ARY TRUTH FUNCTIONS
$\mathscr{B} = \bigcup\{{\{\mathbf{T},\mathbf{F}\}^n}\{\mathbf{T},\mathbf{F}\} \mid (n \in \mathbb{N})\}$

THEOREM I — ADEQUACY OF NEGATION WITH CONJUNCTION
$\forall f \in \mathscr{B}(\exists g \in \bigcap\{X \mid (((\{\mathbf{T},\mathbf{F}\} \subseteq X) \wedge \mathscr{C}_1^1(X, \mathrm{not})) \wedge \mathscr{C}_1^2(X, \mathrm{and}))\}((f = g)))$

8

Satisfiability and Definability

A logical formula is said to be *tautological* only if it is “always true”, *satisfiable* only if it is “sometimes true”, and *contradictory* only if it is “never true”. The property of being tautological can be seen as *opposite* to the property of being contradictory, while the property of being satisfiable can be seen as *complementary* to the property of being contradictory. A set is said to be *definable* only if there exists a logical formula whose truth is equivalent to existence of the set.

8.1 Propositional Logic

DEFINITION LXXV — TAUTOLOGICAL SET OF PROPOSITIONAL FORMULAS

Let $\Phi \subseteq \mathscr{F}_0$.

$\forall t \in \mathscr{T}_0(\forall \varphi \in \Phi((\mathfrak{v}_0^t(\varphi) = \mathbf{T})))$

DEFINITION LXXVI — SATISFIABLE SET OF PROPOSITIONAL FORMULAS

Let $\Phi \subseteq \mathscr{F}_0$.

$\exists t \in \mathscr{T}_0(\forall \varphi \in \Phi((\mathfrak{v}_0^t(\varphi) = \mathbf{T})))$

DEFINITION LXXVII — CONTRADICTORY SET OF PROPOSITIONAL FORMULAS

Let $\Phi \subseteq \mathscr{F}_0$.

$\forall t \in \mathscr{T}_0(\forall \varphi \in \Phi((\mathfrak{v}_0^t(\varphi) = \mathbf{F})))$

DEFINITION LXXVIII — DEFINABLE SET OF TRUTH ASSIGNMENTS

Let $T \subseteq \mathscr{T}_0$.

$\exists \varphi \in \mathscr{F}_0(\forall t \in T((\mathfrak{v}_0^t(\varphi) = \mathbf{T})))$

DEFINITION LXXIX — SUBJECT OF SET OF PROPOSITIONAL FORMULAS

Let $\Phi \in \mathscr{F}_0$.

$\mathrm{subject}_0(\Phi) = \{(t \in \mathscr{T}_0) \mid \forall \varphi \in \Phi((\mathfrak{v}_0^t(\varphi) = \mathbf{T}))\}$

DEFINITION LXXX — THEORY OF SET OF TRUTH ASSIGNMENTS

Let $T \in \mathscr{T}_0$.

$\mathrm{theory}_0(T) = \{(\varphi \in \mathscr{F}_0) \mid \forall t \in T((\mathfrak{v}_0^t(\varphi) = \mathbf{T}))\}$

THEOREM II — EXISTENCE OF UNSATISFIABLE SET OF PROPOSITIONAL FORMULAS

$\exists \Phi \subseteq \mathscr{F}_0((\ \mathrm{subject}_0(\Phi) = \varnothing))$

THEOREM III — EXISTENCE OF UNDEFINABLE SET OF TRUTH ASSIGNMENTS

$\exists T \subseteq \mathscr{T}_0((\ \mathrm{theory}_0(T) = \varnothing))$

THEOREM IV

$\forall \Phi \subseteq \mathscr{F}_0(\forall T \subseteq \mathscr{T}_0((\Phi \subseteq \mathrm{theory}_0(T)) \Leftrightarrow (T \subseteq \mathrm{subject}_0(\Phi))))$

8.2 Predicate Logic

DEFINITION LXXXI — TAUTOLOGICAL SET OF FORMULAS

Let $\Phi \subseteq \mathcal{F}_1$.

$\forall m \in \mathcal{M}(\forall t \in \mathcal{T}_1^m(\forall \varphi \in \Phi((\mathfrak{v}_1^{m,t}(\varphi) = \mathbf{T}))))$

DEFINITION LXXXII — SATISFIABLE SET OF SENTENCES

Let $\Phi \subseteq \mathcal{S}$.

$\exists m \in \mathcal{M}(\forall \varphi \in \Phi((\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = \mathbf{T})))$

DEFINITION LXXXIII — SATISFIABLE SET OF FORMULAS

Let $m \in \mathcal{M}$, and $\Phi \subseteq \mathcal{F}_1$.

$\exists t \in \mathcal{T}_1^m(\forall \varphi \in \Phi((\mathfrak{v}_1^{m,t}(\varphi) = \mathbf{T})))$

DEFINITION LXXXIV — CONTRADICTORY SET OF FORMULAS

Let $\Phi \subseteq \mathcal{F}_1$.

$\forall m \in \mathcal{M}(\forall t \in \mathcal{T}_1^m(\forall \varphi \in \Phi((\mathfrak{v}_1^{m,t}(\varphi) = \mathbf{F}))))$

DEFINITION LXXXV — DEFINABLE SET OF STRUCTURES

Let $M \subseteq \mathcal{M}$.

$\exists \varphi \in \mathcal{S}(\forall m \in M((\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = \mathbf{T})))$

DEFINITION LXXXVI — DEFINABLE SET OF VARIABLE ASSIGNMENTS

Let $m \in \mathcal{M}$, and $T \subseteq \mathcal{T}_1^m$.

$\exists \varphi \in \mathcal{F}_1(\forall t \in T((\mathfrak{v}_1^{m,t}(\varphi) = \mathbf{T})))$

DEFINITION LXXXVII — SUBJECT OF SET OF SENTENCES

Let $\Phi \subseteq \mathcal{S}$.

$\mathrm{subject}_1(\Phi) = \{(m \in \mathcal{M}) \mid \forall \varphi \in \Phi((\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = \mathbf{T}))\}$

DEFINITION LXXXVIII — SUBJECT OF SET OF FORMULAS

Let $m \in \mathcal{M}$, and $\Phi \subseteq \mathcal{F}_1$.

$\mathrm{subject}_1^m(\Phi) = \{(t \in \mathcal{T}_1^m) \mid \forall \varphi \in \Phi((\mathfrak{v}_1^{m,t}(\varphi) = \mathbf{T}))\}$

DEFINITION LXXXIX — THEORY OF SET OF STRUCTURES

Let $M \subseteq \mathcal{M}$.

$\mathrm{theory}_1(M) = \{(\varphi \in \mathcal{S}) \mid \forall m \in M((\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = \mathbf{T}))\}$

DEFINITION XC — THEORY OF SET OF VARIABLE ASSIGNMENTS

Let $m \in \mathcal{M}$, and $T \subseteq \mathcal{T}_1^m$.

$\mathrm{theory}_1^m(T) = \{(\varphi \in \mathcal{F}_1) \mid \forall t \in T((\mathfrak{v}_1^{m,t}(\varphi) = \mathbf{T}))\}$

THEOREM V — EXISTENCE OF UNSATISFIABLE SET OF SENTENCES

$\exists \Phi \subseteq \mathcal{S}((\mathrm{subject}_1(\Phi) = \varnothing))$

THEOREM VI — EXISTENCE OF UNDEFINABLE SET OF STRUCTURES

$\exists M \subseteq \mathcal{M}((\mathrm{theory}_1(M) = \varnothing))$

THEOREM VII — EXISTENCE OF UNSATISFIABLE SET OF FORMULAS

$\forall m \in \mathcal{M}(\exists \Phi \subseteq \mathcal{F}_1((\text{subject}_1^m(\Phi) = \varnothing)))$

THEOREM VIII — EXISTENCE OF UNDEFINABLE SET OF VARIABLE ASSIGNMENTS

$\forall m \in \mathcal{M}(\exists T \subseteq \mathcal{T}_1^m((\text{theory}_1^m(T) = \varnothing)))$

THEOREM IX

$\forall \Phi \subseteq \mathcal{S}(\forall M \subseteq \mathcal{M}((\Phi \subseteq \text{theory}_1(M)) \Leftrightarrow (M \subseteq \text{subject}_1(\Phi))))$

THEOREM X

$\forall m \in \mathcal{M}(\forall \Phi \subseteq \mathcal{F}_1(\forall T \subseteq \mathcal{T}_1^m((\Phi \subseteq \text{theory}_1^m(T)) \Leftrightarrow (T \subseteq \text{subject}_1^m(\Phi)))))$

9

Soundness and Completeness

For a logic, soundness and completeness concern the truth of every proof, and the proof of every truth, respectively. Soundness can be seen as the property that every proof has truth, while completeness can be seen as the property that every truth has proof. Taken together, soundness and completeness establish a correspondence between notions of proof, which are *syntactic*, and notions of truth, which are *semantic*.

DEFINITION XCI — MODUS PONENS INFERENCE FUNCTION

$\text{ponens} : \Phi^2 \to \Phi$

$\text{ponens}(\text{"}\varphi\text{"}, \text{"}(\varphi \Rightarrow \varphi')\text{"}) = \text{"}\varphi'\text{"}$

9.1 Propositional Logic

DEFINITION XCII — SPACE OF AXIOMS FOR PROPOSITIONAL LOGIC

$\mathcal{A}_0' = \{\text{"}(\varphi \Rightarrow (\psi \Rightarrow \varphi))\text{"} \mid (\varphi, \psi \in \mathcal{F}_0)\}$

$\mathcal{A}_0'' = \{\text{"}((\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi)))\text{"} \mid (\varphi, \psi, \chi \in \mathcal{F}_0)\}$

$\mathcal{A}_0''' = \{\text{"}(((\neg\varphi) \Rightarrow (\neg\psi)) \Rightarrow (\psi \Rightarrow \varphi))\text{"} \mid (\varphi, \psi \in \mathcal{F}_0)\}$

$\mathcal{A}_0 = \bigcup\{\mathcal{A}_0', \mathcal{A}_0'', \mathcal{A}_0'''\}$

DEFINITION XCIII — PROOF SYSTEM FOR PROPOSITIONAL LOGIC

Let $\Gamma \subseteq \mathcal{F}_0$.

$\mathscr{P}_0(\Gamma) = \bigcap\{(\Phi \subseteq C_0^*) \mid (((\mathcal{A}_0 \cup \Gamma) \subseteq \Phi) \wedge \mathscr{C}_1^2(\Phi, \{\text{ponens}\}))\}$

DEFINITION XCIV

Let $\Gamma, \Phi \subseteq \mathcal{F}_0$.

$(\Gamma \vdash_0 \Phi) \Leftrightarrow \forall\varphi \in \Phi((\varphi \in \mathscr{P}_0(\Gamma)))$

DEFINITION XCV

Let $\Gamma, \Phi \subseteq \mathcal{F}_0$.

$(\Gamma \vDash_0 \Phi) \Leftrightarrow \forall t \in \mathcal{T}_0(\forall\gamma \in \Gamma(\forall\varphi \in \Phi(((\mathcal{v}_0^t(\gamma) = \mathbf{T}) \Rightarrow (\mathcal{v}_0^t(\varphi) = \mathbf{T})))))$

THEOREM XI — FINITARYNESS OF PROOF SYSTEM FOR PROPOSITIONAL LOGIC

$\forall\Gamma \subseteq \mathcal{F}_0(\forall\varphi \in \mathscr{P}_0(\Gamma)(\exists\Psi \subseteq \mathscr{P}_0(\Gamma)(((|\Psi| < \infty) \wedge \forall\psi \in \Psi(((\psi = \varphi) \vee (\psi \in (\mathcal{A}_0 \cup \Gamma))))))))$

THEOREM XII — SOUNDNESS OF PROOF SYSTEM FOR PROPOSITIONAL LOGIC

$\forall\Gamma, \Phi \subseteq \mathcal{F}_0((\Gamma \vdash_0 \Phi) \Rightarrow (\Gamma \vDash_0 \Phi))$

THEOREM XIII — COMPLETENESS OF PROOF SYSTEM FOR PROPOSITIONAL LOGIC

$\forall\Gamma, \Phi \subseteq \mathcal{F}_0((\Gamma \vDash_0 \Phi) \Rightarrow (\Gamma \vdash_0 \Phi))$

DEFINITION XCVI — CONSISTENT SET OF PROPOSITIONAL FORMULAS

Let $\Gamma \subseteq \mathcal{F}_0$.

$\text{consistent}_0(\Gamma) \Leftrightarrow \forall\varphi \in \mathcal{F}_0(\neg(((\Gamma \vdash_0 \varphi) \wedge (\Gamma \nvdash_0 \varphi))))$

DEFINITION XCVII — SATISFIABLE SET OF PROPOSITIONAL FORMULAS

Let $\Gamma \subseteq \mathcal{F}_0$.

$\text{satisfiable}_0(\Gamma) \Leftrightarrow \forall\varphi \in \mathcal{F}_0(\neg(((\Gamma \vDash_0 \varphi) \wedge (\Gamma \nvDash_0 \varphi))))$

THEOREM XIV

$\forall\Gamma \subseteq \mathcal{F}_0((\text{consistent}_0(\Gamma) \Leftrightarrow \text{satisfiable}_0(\Gamma)))$

9.2 Predicate Logic

DEFINITION XCVIII — SPACE OF AXIOMS FOR PREDICATE LOGIC

$\mathcal{A}_1' = \{\tau|_{p\mapsto\varphi} \mid (((\varnothing \vDash_0 \tau) \wedge (p \in \mathcal{X}_0)) \wedge (\varphi \in \mathcal{F}_1))\}$

$\mathcal{A}_1'' = \{\text{"}(\forall x(\varphi) \Rightarrow \varphi|_{x\mapsto x'})\text{"} \mid ((x, x' \in \mathcal{X}_1) \wedge \varphi \in \mathcal{F}_1)\}$

$\mathcal{A}_1''' = \{\text{"}(\varphi \Rightarrow \forall x(\varphi|_{x\mapsto x'}))\text{"} \mid ((x, x' \in \mathcal{X}_1) \wedge \varphi \in \mathcal{F}_1)\}$

$\mathcal{A}_1 = \bigcup\{\mathcal{A}_1', \mathcal{A}_1'', \mathcal{A}_1'''\}$

DEFINITION XCIX — PROOF SYSTEM FOR PREDICATE LOGIC

Let $\Gamma \subseteq \mathcal{F}_1$.

$\mathcal{P}_1(\Gamma) = \bigcap\{(\Phi \subseteq \mathcal{C}_1^*) \mid (((\mathcal{A}_1 \cup \Gamma) \subseteq \Phi) \wedge \mathscr{C}_1^2(\Phi, \{\text{ponens}\}))\}$

DEFINITION C

Let $\Gamma, \Phi \subseteq \mathcal{F}_1$.

$(\Gamma \vdash_1 \Phi) \Leftrightarrow \forall\varphi \in \Phi((\varphi \in \mathcal{P}_1(\Gamma)))$

DEFINITION CI

Let $\Gamma, \Phi \subseteq \mathcal{F}_1$.

$(\Gamma \vDash_1 \Phi) \Leftrightarrow \forall m \in \mathcal{M}(\forall v \in \mathcal{T}_1^m(\forall\gamma \in \Gamma(\forall\varphi \in \Phi((\mathfrak{v}_1^{m,v}(\gamma) = \mathbf{T}) \Rightarrow (\mathfrak{v}_1^{m,v}(\varphi) = \mathbf{T})))))$

THEOREM XV — FINITARYNESS OF PROOF SYSTEM FOR PREDICATE LOGIC

$\forall\Gamma \subseteq \mathcal{F}_1(\forall\varphi \in \mathcal{P}_1(\Gamma)(\exists\Psi \subseteq \mathcal{P}_1(\Gamma)(((|\Psi| < \infty) \wedge \forall\psi \in \Psi(((\psi = \varphi) \vee (\psi \in (\mathcal{A}_1 \cup \Gamma))))))))$

THEOREM XVI — SOUNDNESS OF PROOF SYSTEM FOR PREDICATE LOGIC

$\forall\Gamma, \Phi \subseteq \mathcal{F}_1((\Gamma \vdash_1 \Phi) \Rightarrow (\Gamma \vDash_1 \Phi))$

THEOREM XVII — COMPLETENESS OF PROOF SYSTEM FOR PREDICATE LOGIC

$\forall\Gamma, \Phi \subseteq \mathcal{F}_1((\Gamma \vDash_1 \Phi) \Rightarrow (\Gamma \vdash_1 \Phi))$

DEFINITION CII — CONSISTENT SET OF FORMULAS

Let $\Gamma \subseteq \mathcal{F}_1$.

$\text{consistent}_1(\Gamma) \Leftrightarrow \forall\varphi \in \mathcal{F}_1(\neg(((\Gamma \vdash_1 \varphi) \wedge (\Gamma \nvdash_1 \varphi))))$

DEFINITION CIII — SATISFIABLE SET OF FORMULAS

Let $\Gamma \subseteq \mathcal{F}_1$.

$\text{satisfiable}_1(\Gamma) \Leftrightarrow \forall\varphi \in \mathcal{F}_1(\neg(((\Gamma \vDash_1 \varphi) \wedge (\Gamma \nvDash_1 \varphi))))$

THEOREM XVIII

$\forall\Gamma \subseteq \mathcal{F}_1((\text{consistent}_1(\Gamma) \Leftrightarrow \text{satisfiable}_1(\Gamma)))$

10

Compactness and Maximisability

Previously, we established equivalence between consistency and satisfiability. In the interest of brevity, we shall, henceforth, occasionally omit discussion of consistency in favor of satisfiability, keeping in mind that every property which applies to the latter, also applies to the former, in equivalent fashion.

10.1 Propositional Logic

THEOREM XIX — PROPOSITIONAL COMPACTNESS

Let $\Gamma \subseteq \mathscr{F}_0$.

$\text{satisfiable}_0(\Gamma) \Leftrightarrow \forall \Gamma' \subseteq \Gamma(((|\Gamma'| < \infty) \wedge \text{satisfiable}_0(\Gamma')))$

DEFINITION CIV — MAXIMAL SET OF PROPOSITIONAL FORMULAS

Let $\Gamma \subseteq \mathscr{F}_0$.

$\text{maximal}_0(\Gamma) \Leftrightarrow \forall \varphi \in \mathscr{F}_0(((\Gamma \vDash_0 \varphi) \vee (\Gamma \nvDash_0 \varphi)))$

THEOREM XX — PROPOSITIONAL LINDENBAUM

$\forall \Gamma \subseteq \mathscr{F}_0((\text{satisfiable}_0(\Gamma) \Leftrightarrow \exists \Gamma' \supseteq \Gamma((\text{satisfiable}_0(\Gamma') \wedge \text{maximal}_0(\Gamma')))))$

10.2 Predicate Logic

THEOREM XXI — PREDICATE COMPACTNESS

$\forall\Gamma \subseteq \mathcal{F}_1((\text{satisfiable}_1(\Gamma) \Leftrightarrow \forall\Gamma' \subseteq \Gamma(((|\Gamma'| < \infty) \wedge \text{satisfiable}_1(\Gamma')))))$

DEFINITION CV — MAXIMAL SET OF FORMULAS

Let $\Gamma \subseteq \mathcal{F}_1$.

$\text{maximal}_1(\Gamma) \Leftrightarrow \forall\varphi \in \mathcal{F}_1(((\Gamma \vDash_1 \varphi) \vee (\Gamma \nvDash_1 \varphi)))$

THEOREM XXII — PREDICATE LINDENBAUM

$\forall\Gamma \subseteq \mathcal{F}_1((\text{satisfiable}_1(\Gamma) \Leftrightarrow \exists\Gamma' \supseteq \Gamma((\text{maximal}_1(\Gamma) \wedge \text{satisfiable}_1(\Gamma)))))$

11

Gödel Incompleteness

Everything fits in the number.
— IAMBLICHUS

The natural numbers have, historically, been seen as a “staple” of mathematics. Indeed, various historical accounts suggest that the ancient Pythagoreans viewed the natural numbers as “fundamental to reality”. Famously, Carl Friedrich Gauss dubbed number theory the “queen of mathematics”. A space of sentences which are true for a structure of natural numbers is known as a *theory of arithmetic*.

A proof system . Previously, we defined a proof system which is finitary, sound, and complete with respect to the space of tautological formulas. In the 1920s, there was an interest in founding mathematics upon formal methods of proof. In particular, it was wondered whether one could develop a proof system which is verifiable, sound, and complete with respect to *theories of mathematics*. Around the 1930s, Kurt Gödel demonstrated that, if “verifiable” is taken to mean “recursively axiomatisable”, then no such proof system exists for “sufficiently strong” theories of arithmetic. Gödel also showed that no consistent proof system can simultaneously prove its *own* consistency, and derive such theories of arithmetic. These theorems have come to be known as Gödel’s incompleteness theorems.

A theory is said to be “sufficiently strong” only if it proves “sufficiently many” sentences about the objects in its universe. Gödel’s incompleteness theorems have been shown to apply to Peano arithmetic and Robinson arithmetic. In this section, for the sake of concreteness, we shall state the incompleteness theorems using Robinson arithmetic.

DEFINITION CVI — STRUCTURE OF NATURAL NUMBERS

$(\mathfrak{N} = \langle \mathbb{N}, \langle (\{"0", "1", "i", "+", "\times"\} \cup \varnothing), a_{\mathfrak{N}} \rangle, i_{\mathfrak{N}} \rangle) \in \mathcal{M}$

$a_{\mathfrak{N}}("0") = 0$

$a_{\mathfrak{N}}("1") = 0$

$a_{\mathfrak{N}}("i") = 1$

$a_{\mathfrak{N}}("+") = 2$

$a_{\mathfrak{N}}("\times") = 2$

$i_{\mathfrak{N}}("0") = 0$

$i_{\mathfrak{N}}("1") = 1$

$i_{\mathfrak{N}}("i") = i$

$i_{\mathfrak{N}}("+") = \{\langle x, x', i^{[x']}(x) \rangle \mid (x, x', y \in \mathbb{N})\}$

$i_{\mathfrak{N}}("\times") = \{\langle x, x', (i^{[x']})^{[x']}(x) \rangle \mid (x, x', y \in \mathbb{N})\}$

THEOREM XXIII — FIRST GÖDEL INCOMPLETENESS

$\nexists \Gamma \subseteq \mathscr{F}_1 (\forall \Phi \subseteq \mathrm{th}_1(\{\mathfrak{N}\})(((\Gamma \vDash_1 \Phi) \Rightarrow (\Gamma \vdash_1 \Phi))))$

THEOREM XXIV — SECOND GÖDEL INCOMPLETENESS

$\forall \Gamma \subseteq \mathscr{F}_1 (((\mathrm{th}_1(\{\mathfrak{N}\}) \subseteq \mathscr{P}_1(\Gamma)) \Rightarrow (\mathrm{cons}_1(\mathscr{P}_1(\Gamma)) \Leftrightarrow (\text{"}\,\mathrm{cons}_1(\mathscr{P}_1(\Gamma))\,\text{"} \notin \mathscr{P}_1(\Gamma)))))$

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Bibliography

1. University of Waterloo, *CS245 Logic and Computation*.
   Available at `http://student.cs.uwaterloo.ca/~cs245`.
2. Lorenz Halbeisen, *The Axioms of Set Theory ZFC*.
   Available at `http://people.math.ethz.ch/~halorenz/4students/LogikGT/Ch13.pdf`.
3. Theoretical Cookies, *Set Theory | Lesson 9: Axiom of Choice [CC]*.
   Available at `http://youtube.com/watch?v=ActkCEU3eCQ`.