PATTERNS

GLIMEUXE

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About

The live file is available for free at `http://github.com/glimeuxe/writings`, where updates are made on an occasional basis. I thank all those who have made suggestions to improve the text.

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Theories

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Classical Logic

2.1 Prologue

What could it mean for something to be *true*? Throughout history, various definitions for the word "truth" have been proposed.

Definition 1 (Correspondence Theory of Truth)**.**
Truth is that which corresponds to reality.

Definition 2 (Coherence Theory of Truth)**.**
Truth is that which coheres with every other truth.

Ought mathematics concern itself with the universe we inhabit? In any case, the practice of mathematics seems to be based on something like the coherence theory of truth; this presents a *seeming* trilemma.

Definition 3 (Münchhausen Trilemma)**.**
Every proof is completed by circularity, infinite regress, or assumption.

Reasoning by coherence appears to obtain truth by circularity, infinite regress, or assumption. It is no secret that mathematical theories admit assumptions. Mathematical theories are also said to build upon, and thus derive from, each other. A theory from which other theories can be derived is said to be *foundational*.

Broadly speaking, a logic can be thought of as a language for reasoning about truth — that is, a system which prescribes symbols, and ways of interchanging those symbols. This text studies propositional and predicate logics, in particular.

With any human language, definitions may be said to originate in *pre-linguistic* thought. Writers sometimes choose their starting points based on what feels the most intuitive to them. For this text, I have decided to use words and phrases like "not", "and", "if. . . then", "either. . . or", "otherwise", "every", "same", in addition to a few other mathematical symbols, in my appeal to the intuition of the reader.

2.2 Language

2.2.1 Propositional Logic

Definition 4 (Propositional Formula)**.**
Let $p_\circ, \ldots, p_\bullet$ be propositional variables.

1. If p is a propositional variable, then p is a propositional formula.
2. If φ is a propositional formula, then $(\neg\varphi)$ is a propositional formula.
   If φ and φ' are propositional formulas, then $(\varphi \wedge \varphi')$ is a propositional formula.

Definition 5 (Truth Value of Propositional Formula)**.**

1. Every propositional variable is either true or false.
2. If φ is true, then $(\neg\varphi)$ is false. Otherwise, $(\neg\varphi)$ is true.
   If φ and φ' are true, then $(\varphi \wedge \varphi')$ is true. Otherwise, $(\varphi \wedge \varphi')$ is false.

2.2.2 Predicate Logic

Definition 6 (Formula)**.**
Let $x_\circ, \ldots, x_\bullet$ be variables.

1. If x is a variable, and x' is a variable, then $(x = x')$ is a formula.
   If x is a variable, and x' is a variable, then $(x \in x')$ is a formula.
2. If φ is a formula, then $(\neg\varphi)$ is a formula.
   If φ is a formula, and φ' is a formula, then $(\varphi \wedge \varphi')$ is a formula.
   If x is a variable, and φ is a formula, then $\forall x(\varphi)$ is a formula.

Definition 7 (Free Variable)**.**

1. If $(x = x')$ is a formula, then x and x' are free variables in the formula.
   If $(x \in x')$ is a formula, then x and x' are free variables in the formula.
2. If x is a free variable in φ, then x is a free variable in $(\neg\varphi)$.
   If x is a free variable in φ and φ', then x is a free variable in $(\varphi \wedge \varphi')$.
   If x is a free variable in φ, and $(x \neq x')$, then x is a free variable in $\forall x'(\varphi)$.

Notation 1 (Naïve Variable Substitution)**.**
$(x = x)\{^{\tau}_{x} \lessdot (\tau = \tau)$
$(x = x')\{^{\tau}_{x} \lessdot (\tau = x')$
$(x' = x)\{^{\tau}_{x} \lessdot (x' = \tau)$
$(x' = x')\{^{\tau}_{x} \lessdot (x' = x')$
$(x \in x)\{^{\tau}_{x} \lessdot (\tau \in \tau)$
$(x \in x')\{^{\tau}_{x} \lessdot (\tau \in x')$
$(x' \in x)\{^{\tau}_{x} \lessdot (x' \in \tau)$
$(x' \in x')\{^{\tau}_{x} \lessdot (x' \in x')$
$(\neg\varphi)\{^{\tau}_{x} \lessdot (\neg\varphi\{^{\tau}_{x})$
$(\varphi \wedge \varphi')\{^{\tau}_{x} \lessdot (\varphi\{^{\tau}_{x} \wedge \varphi'\{^{\tau}_{x})$
$\forall x(\varphi)\{^{\tau}_{x} \lessdot \forall \tau(\varphi\{^{\tau}_{x})$

Definition 8 (Truth Value of Formula)**.**

1. Every variable is a set.
2. If x is the same set as x', then $(x = x')$ is true. Otherwise, $(x = x')$ is false.
   If φ is true, then $(\neg\varphi)$ is false. Otherwise, $(\neg\varphi)$ is true.
   If φ and φ' are true, then $(\varphi \wedge \varphi')$ is true. Otherwise, $(\varphi \wedge \varphi')$ is false.
   If φ is true for every possible x, then $\forall x(\varphi)$ is true. Otherwise, $\forall x(\varphi)$ is false.

2.3 Interlogue

Brackets are used to guarantee uniqueness for every reading of logical syntax. In the interest of brevity, we shall omit outermost pairs of brackets.

Notation 2.
$(x) \lessdot ((x))$

Notation 3 (Connective of Disjunction)**.**
$(\varphi \vee \varphi') \lessdot (\neg((\neg\varphi) \wedge (\neg\varphi')))$

Notation 4 (Connective of Implication)**.**
$(\varphi \Rightarrow \varphi') \lessdot (\neg(\varphi \wedge (\neg\varphi')))$

Notation 5 (Connective of Equivalence)**.**
$(\varphi \Leftrightarrow \varphi') \lessdot ((\varphi \Rightarrow \varphi') \wedge (\varphi' \Rightarrow \varphi))$

Notation 6 (Connective of Exclusive Disjunction)**.**
$(\varphi \not\Leftrightarrow \varphi') \lessdot (\neg(\varphi \Leftrightarrow \varphi'))$

Definition 9.
Every operator symbol is either a connective symbol or a function symbol.

Notation 7.
Let $\circ$ be a binary operator symbol.

$(a \circ \ldots \circ y \circ z) \lessdot ((a \circ \ldots \circ y) \circ z)$

Notation 8.
Let R be a relation symbol.

$\forall x \mathrel{\mathsf{R}} x'(\varphi) \lessdot \forall x((x \mathrel{\mathsf{R}} x') \Rightarrow \varphi)$

Notation 9.
Let $\mathsf{Q}_\circ, \ldots, \mathsf{Q}_\bullet$ be quantifier symbols.

$\mathsf{Q}_\circ ¿_\circ, \ldots, \mathsf{Q}_\bullet ¿_\bullet(\varphi) \lessdot \mathsf{Q}_\circ ¿_\circ(\ldots(\mathsf{Q}_\bullet ¿_\bullet(\varphi)))$

Notation 10 (Quantifier of Existence)**.**
$\exists ¿(\varphi) \lessdot (\neg\forall ¿(\neg\varphi))$

Notation 11 (Quantifier of Unique Existence)**.**
$\exists_! ¿(\varphi) \lessdot \exists ¿, \forall ¿'(\varphi\{^{¿'}_{¿} \Leftrightarrow (¿ = ¿'))$

Notation 12 (Quantifier of Dichotomous Existence)**.**
$\reflectbox{$\exists$} ¿(\varphi) \lessdot ((\neg\exists ¿(\varphi)) \not\Leftrightarrow \exists_! ¿(\varphi))$

Notation 13 (Reflectability of Relation)**.**
Let R be a relation symbol.

$(x \mathrel{\reflectbox{$\mathsf{R}$}} y) \lessdot (y \mathrel{\mathsf{R}} x)$

Notation 14 (Negatability of Relation)**.**
Let R be a relation symbol.

$(x \mathrel{\not\mathsf{R}} y) \lessdot (\neg(x \mathrel{\mathsf{R}} y))$

Notation 15 (Equatability of Relation)**.**
Let $\underline{\mathsf{R}}$ be a relation symbol.
$(x \mathrel{\mathsf{R}} y) \lessdot ((x \mathrel{\underline{\mathsf{R}}} y) \not\Leftrightarrow (x \neq y))$

Definition 10 (Reflexive Relation)**.**
Let R be a relation symbol.
$\mathrm{RefRel}(\mathsf{R}) \Leftrightarrow \forall x(x \mathrel{\mathsf{R}} x)$

Definition 11 (Symmetric Relation)**.**
Let R be a relation symbol.
$\mathrm{SymRel}(\mathsf{R}) \Leftrightarrow \forall x, y((x \mathrel{\mathsf{R}} y) \Leftrightarrow (y \mathrel{\mathsf{R}} x))$

Definition 12 (Transitive Relation)**.**
Let R be a relation symbol.
$\mathrm{TranRel}(\mathsf{R}) \Leftrightarrow \forall x, y, z(((x \mathrel{\mathsf{R}} y) \wedge (y \mathrel{\mathsf{R}} z)) \Rightarrow (x \mathrel{\mathsf{R}} z))$

Definition 13 (Equivalence Relation)**.**
Let R be a relation symbol.
$\mathrm{EqvRel}(\mathsf{R}) \Leftrightarrow (\mathrm{RefRel}(\mathsf{R}) \wedge \mathrm{SymRel}(\mathsf{R}) \wedge \mathrm{TranRel}(\mathsf{R}))$

Proposition 1.
$\mathrm{EqvRel}(=)$

2.4 Zermelo–Frænkel Set Theory with Choice

Around the 1920s, an axiomatic set theory was proposed by Ernst Zermelo and Abraham Frænkel: this Zermelo–Frænkel set theory (ZF), when paired with the axiom of choice (AC), came to be known as Zermelo–Frænkel set theory with choice (ZFC). ZFC is commonly used as a foundational theory, and is conventionally written in a classical predicate logic.

It has been shown that if ZFC is consistent, then every formulation of ZFC must include at least one axiom schema. For brevity, we shall examine a particular formulation of ZFC which includes the axiom schema of separation, and no other axiom schema.

With the admission of ZF, AC is equivalent to various theorems, including the theorem that every vector space has a basis, and the theorem that every surjective function has a right inverse. Famously, AC in ZF also implies the Banach–Tarski paradox.

Axiom 1 (Empty Set)**.**
$\exists N, \forall x(x \notin N)$

Definition 14 (Empty Set)**.**
$(N = \varnothing) \Leftrightarrow \forall x(x \notin N)$

Axiom 2 (Extensionality)**.**
$\forall X, Y(\forall m((m \in X) \Leftrightarrow (m \in Y)) \Rightarrow (X = Y))$

Definition 15 (Subset)**.**
$(X \subseteq Y) \Leftrightarrow \forall m \in X((m \in X) \Rightarrow (m \in Y))$

Axiom 3 (Pairing)**.**
$\forall c, c', \exists C((c \in C) \wedge (c' \in C))$

Definition 16 (Couple Set)**.**
$(C = \{c, c'\}) \Leftrightarrow ((c \in C) \wedge (c' \in C))$

Definition 17 (Singleton Set)**.**
$\{c\} = \{c, c\}$

Axiom 4 (Union)**.**
$\forall W, \exists U, \forall w \in W, \forall u \in w(u \in U)$

Definition 18 (Unary Union Function)**.**
$(U = \bigcup W) \Leftrightarrow \forall w \in W, \forall u \in w(u \in U)$

Definition 19 (Binary Union Function)**.**
$(x \cup x') = \bigcup\{x, x'\}$

Definition 20 (Space of Free Variables)**.**
Let $x_\circ, \ldots, x_\bullet$ be the free variables in φ.
$\text{free}(\varphi) = \bigcup\{\{x_\circ\}, \ldots, \{x_\bullet\}\}$

Axiom 5 (Power Set)**.**
$\forall X, \exists P, \forall x((x \subseteq X) \Rightarrow (x \in P))$

Definition 21 (Power Set).
$(x \in \mathscr{P}(X)) \Leftrightarrow (x \subseteq X)$

Axiom Schema 1 (Separation).
Let φ be a formula, and $\text{free}(\varphi) \subseteq \{D, f\}$.
$\forall D, \exists F, \forall f(((f \in D) \wedge \varphi) \Rightarrow (f \in F))$

Definition 22.
Let φ be a formula, and $\text{free}(\varphi) \subseteq \{D, f\}$.
$(F = \{(f \in D) \mid \varphi\}) \Leftrightarrow (((f \in D) \wedge \varphi) \Rightarrow (f \in F))$

Definition 23.
Let φ be a formula, and $\text{free}(\varphi) \subseteq \{D, f\}$.
$(F = \{f \mid \varphi\}) \Leftrightarrow (\varphi \Rightarrow (f \in F))$

Definition 24 (Unary Intersection Function).
$\bigcap W = \{(u \in \bigcup W) \mid \forall w \in W(u \in w)\}$

Definition 25 (Binary Intersection Function).
$(x \cap x') = \bigcap\{x, x'\}$

Definition 26 (Successor Function).
$\mathscr{S}(n) = (n \cup \{n\})$

Axiom 6 (Infinity).
$\exists R((\varnothing \in R) \wedge \forall r \in R(\mathscr{S}(r) \in R))$

Axiom 7 (Regularity).
$\forall O \neq \varnothing, \exists o \in O((o \cap O) = \varnothing)$

Definition 27 (Set of Pairwise Disjoint Sets).
$\text{PDis}(X) \Leftrightarrow \forall x, x' \in X((x \neq x') \Rightarrow ((x \cap x') = \varnothing))$

Axiom 8 (Choice).
$\forall B((\text{PDis}(B) \wedge \forall S \in B(S \neq \varnothing)) \Rightarrow \exists B', \forall S \in B, \exists_! s \in S(s \in B'))$

2.5 Interlogue

Proposition 2.
$\mathrm{ParOrdRel}(\subseteq)$

Definition 28 (Difference)**.**
$(X \setminus Y) = \{ (x \in X) \mid (x \notin Y) \}$

Definition 29 (Symmetric Difference)**.**
$(X \triangledown Y) = ((X \setminus Y) \cup (Y \setminus X))$

Definition 30 (Complement)**.**
Let $x \subseteq X$.
$x^{\complement} = (X \setminus x)$

Notation 16 (Antisymmetric Relation)**.**
Let R be a relation symbol.
$\mathrm{AntiSymRel}(\mathsf{R}) \lessdot ((a \mathrel{\mathsf{R}} b) \Leftrightarrow (b \mathrel{\text{Я}} a))$

Notation 17 (Partial-Ordering Relation)**.**
Let R be a relation symbol.
$\mathrm{ParOrdRel}(\mathsf{R}) \lessdot (\mathrm{RefRel}(\mathsf{R}) \land \mathrm{AntiSymRel}(\mathsf{R}) \land \mathrm{TranRel}(\mathsf{R}))$

Definition 31.
$\mathrm{ParOrdRel}(\preceq)$

Notation 18 (Total Relation)**.**
Let R be a relation symbol.
$\mathrm{TotRel}(\mathsf{R}) \lessdot ((a \mathrel{\mathsf{R}} b) \lor (b \mathrel{\mathsf{R}} a))$

Notation 19 (Total-Ordering Relation)**.**
Let R be a relation symbol.
$\mathrm{TotOrdRel}(\mathsf{R}) \lessdot (\mathrm{ParOrdRel}(\mathsf{R}) \land \mathrm{TotRel}(\mathsf{R}))$

Definition 32.
$\mathrm{TotOrdRel}(\leq)$

Notation 20.
Let R be a relation symbol.
$(a \mathrel{\mathsf{R}} \cdots \mathrel{\mathsf{R}} y \mathrel{\mathsf{R}} z) \lessdot ((a \mathrel{\mathsf{R}} \cdots \mathrel{\mathsf{R}} y) \land (y \mathrel{\mathsf{R}} z))$

Definition 33 (Interval)**.**
$X_{[a,b]} = \{ (x \in X) \mid \exists a, b \in X(a \leq x \leq b)\}$
$X_{[a,b)} = \{ (x \in X) \mid \exists a, b \in X(a \leq x < b)\}$
$X_{(a,b]} = \{ (x \in X) \mid \exists a, b \in X(a < x \leq b)\}$
$X_{(a,b)} = \{ (x \in X) \mid \exists a, b \in X(a < x < b)\}$
$X_{\geq a} = X_{[a,\infty)} = \{ (x \in X) \mid \exists a \in X(x \geq a)\}$
$X_{>a} = X_{(a,\infty)} = \{ (x \in X) \mid \exists a \in X(x > a)\}$
$X_{\leq b} = X_{(-\infty,b]} = \{ (x \in X) \mid \exists b \in X(x \leq b)\}$
$X_{<b} = X_{(-\infty,b)} = \{ (x \in X) \mid \exists b \in X(x < b)\}$

Definition 34 (Space of Natural Numbers)**.**
$\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \dots\} = \bigcap\{R \mid ((\varnothing \in R) \wedge \forall r \in R(\mathcal{S}(r) \in R))\}$

Definition 35 (Order of Natural Numbers)**.**
Let $a, b \in \mathbb{N}$.
$(a \leq b) \Leftrightarrow ((a \in b) \not\Leftrightarrow (a = b))$

Definition 36 (Tuple)**.**
Let $n \in \mathbb{N}_{\geq 1}$.
$\langle\rangle = \varnothing$
$\langle x_1 \rangle = \{\{\langle\rangle\}, \{\langle\rangle, x_1\}\}$
$\langle x_1, \dots, x_n, x_{\mathcal{S}(n)} \rangle = \{\{\langle x_1, \dots, x_n \rangle\}, \{\langle x_1, \dots, x_n \rangle, x_{\mathcal{S}(n)}\}\}$

Definition 37.
Let $n \in \mathbb{N}_{\geq 1}$.
$\Diamond(\langle\rangle) = \varnothing$
$\Diamond(\langle x_1 \rangle) = \{x_1\}$
$\Diamond(\langle x_1, \dots, x_n, x_{\mathcal{S}(n)} \rangle) = (\Diamond(\langle x_1, \dots, x_n \rangle) \cup \{x_{\mathcal{S}(n)}\})$

Definition 38.
Let $n \in \mathbb{N}_{\geq 1}$, and $k \in \mathbb{N}_{[1,n]}$.
$\langle x_1, \dots, x_n \rangle_{[k]} = x_k$

Definition 39 (n-ary Cartesian Product)**.**
Let $n \in \mathbb{N}_{\geq 1}$.
$(X_1 \times \cdots \times X_n) = \{\langle x_1, \dots, x_n \rangle \mid ((x_n \in X_1) \wedge \cdots \wedge (x_n \in X_n))\}$

Notation 21.
Let $n \in \mathbb{N}_{\geq 1}$, and $A_1 = \cdots = A_n$.
$A^{\circledast(n)} \lessdot (A_1 \circledast \cdots \circledast A_n)$

Definition 40.
$A^* = \bigcup\{A^{\times(i)} \mid (i \in \mathbb{N})\}$

Definition 41 (Space of Functions)**.**
$\{X \to Y\} = \{(f \subseteq (X \times Y)) \mid \forall x \in X, \exists_! y \in Y(\langle x, y \rangle \in f)\}$

Definition 42 (Function)**.**
$(f : X \to Y) \Leftrightarrow (f \in \{X \to Y\})$
$(f(x) = y) \Leftrightarrow \forall x \in X(\langle x, y \rangle \in f)$

Notation 22.
Let $f : (X_1 \times \cdots \times X_n) \to Y$.
$f(x_1, \dots, x_n) \lessdot f(\langle x_1, \dots, x_n \rangle)$

Notation 23.
$([.] : X \to Y) \lessdot (._{[]} : X \to Y)$
$([x] = y) \lessdot (._{[]}(x) = y)$

Definition 43 (Identity Function)**.**
$\mathrm{id}(x) = x$

Definition 44 (String).
Let $n \in \mathbb{N}_{\geq 1}$.
$"" = \langle\rangle$
$"x_1 \ldots x_n" = \langle "x_1", \ldots, "x_n" \rangle$

Definition 45 (Mutually Exclusive Set of Formulas).
$\not\wedge(\Phi) \Leftrightarrow \forall "\varphi", "\varphi'" \in \Phi(("\varphi" \neq "\varphi'") \Rightarrow (\neg(\varphi \wedge \varphi')))$

Definition 46 (Collectively Exhaustive Set of Formulas).
$\vee(\{"\varphi_1", \ldots, "\varphi_n"\}) \Leftrightarrow (\varphi_1 \vee \cdots \vee \varphi_n)$

Definition 47 (Logical Partition).
$\veebar(\Phi) \Leftrightarrow (\not\emptyset(\Phi) \wedge \not\wedge(\Phi) \wedge \vee(\Phi))$

Definition 48 (Piecewise Function).
Let $n \in \mathbb{N}_{\geq 1}$, and $f : X \to Y$.

$$\left(f(x) = \begin{cases} y_1, & \varphi_1 \\ \vdots & \vdots \\ y_n, & \varphi_n \\ y_{\mathcal{S}(n)}, & \varphi_{\mathcal{S}(n)} \end{cases} \right) \Leftrightarrow (\forall x \in X, \forall i \in \mathbb{N}_{[1,\mathcal{S}(n)]}(\varphi_i \Rightarrow (\langle x, y_i \rangle \in f)) \wedge \veebar(\{\varphi_1, \ldots, \varphi_{\mathcal{S}(n)}\}))$$

$(\text{o.w.} = \varphi_{\mathcal{S}(n)}) \Leftrightarrow (\varphi_{\mathcal{S}(n)} \Leftrightarrow (\neg(\varphi_1 \vee \cdots \vee \varphi_n)))$

Definition 49.
Let $n, s \in \mathbb{N}_{\geq 1}$.
$\forall x_{i,j} \overset{i \in \langle 1,\ldots,n \rangle}{\underset{j \in \langle 1,\ldots,s \rangle}{\in}} X_i(\varphi) \Leftrightarrow \forall x_{1,1}, \ldots, x_{1,s} \in X_1, \ldots, \forall x_{n,1}, \ldots, x_{n,s} \in X_n(\varphi)$

Definition 50.
Let $n, s \in \mathbb{N}_{\geq 1}$.
$\$_k(X_1, \ldots, X_n, F) \Leftrightarrow \forall x_{i,j} \overset{i \in \langle 1,\ldots,n \rangle}{\underset{j \in \langle 1,\ldots,s \rangle}{\in}} X_i, \forall f \in F(f(x_{1,1}, \ldots, x_{n,s}) \in X_n)$

Definition 51.
Let $k, n \in \mathbb{N}_{\geq 1}$.
$\widehat{\$}_k(X_1, \ldots, X_n, F) \Leftrightarrow \forall x_{i,j} \overset{i \in \langle 1,\ldots,n \rangle}{\underset{j \in \langle 1,\ldots,s \rangle}{\in}} X_i, \forall f \in F(\langle x_{1,1}, \ldots, x_{n,s}, f(x_{1,1}, \ldots, x_{n,s}) \rangle \in X_n)$

2.6 Metalanguage

Previously, we defined propositional and predicate logics. Now, we shall define propositional and predicate metalogics that mimic, and thus help to more formally define, their logical counterparts. To avoid circularity, a metalogic ought to be distinguished from its logic. However, in the interest of brevity, we shall not always make such a distinction.

2.6.1 Propositional Logic

Definition 52 (Space of Propositional Variables).
$\mathscr{X}_0 = \bigcup\{\{\texttt{"}p_1\texttt{"}, \dots, \texttt{"}p_n\texttt{"}\} \mid (n \in \mathbb{N})\}$

Definition 53 (Propositional Alphabet).
$🔤_0 = (\mathscr{X}_0 \cup \{\texttt{"}\neg\texttt{"}, \texttt{"}\wedge\texttt{"}, \texttt{"}(\texttt{"}, \texttt{"})\texttt{"}\})$

Definition 54 (Concatenation Function of Negation).
$\mathrm{neg}(\texttt{"}\varphi\texttt{"}) = \texttt{"}(\neg\varphi)\texttt{"}$

Definition 55 (Concatenation Function of Conjunction).
$\mathrm{conj}(\texttt{"}\varphi\texttt{"}, \texttt{"}\varphi'\texttt{"}) = \texttt{"}(\varphi \wedge \varphi')\texttt{"}$

Definition 56 (Space of Propositional Formulas).
$\mathscr{F}_0 = \bigcap\{(\Phi \subseteq 🔤_0^*) \mid ((\mathscr{X}_0 \subseteq \Phi) \wedge \$_1(\Phi, \{\mathrm{neg}\}) \wedge \$_2(\Phi, \{\mathrm{conj}\}))\}$

Definition 57 (Space of Truth Values).
$2 = \{0, 1\}$

Definition 58 (Truth Function of Negation).
$\mathrm{not} : 2 \to 2$
$\mathrm{not}(1) = 0$
$\mathrm{not}(0) = 1$

Definition 59 (Truth Function of Conjunction).
$\mathrm{and} : 2^{\times(2)} \to 2$
$\mathrm{and}(1, 1) = 1$
$\mathrm{and}(1, 0) = 0$
$\mathrm{and}(0, 1) = 0$
$\mathrm{and}(0, 0) = 0$

Definition 60 (Space of Truth Assignments).
$\mathscr{T}_0 = \{\mathscr{X}_0 \to 2\}$

Definition 61 (Valuation of Propositional Formula).
Let $t \in \mathscr{T}_0$.
$\mathfrak{v}_0^t : \mathscr{F}_0 \to 2$

$$\mathfrak{v}_0^t(\varphi) = \begin{cases} t(p), & \exists p \in \mathscr{X}_0(\varphi = p) \\ \mathrm{not}(\mathfrak{v}_0^t(\psi)), & \exists \psi \in \mathscr{F}_0(\varphi = \mathrm{neg}(\psi)) \\ \mathrm{and}(\mathfrak{v}_0^t(\psi), \mathfrak{v}_0^t(\psi')), & \exists \psi, \psi' \in \mathscr{F}_0(\varphi = \mathrm{conj}(\psi, \psi')) \end{cases}$$

2.6.2 Predicate Logic

Definition 62 (Space of Variables).
$\mathscr{X}_1 = \bigcup\{(\{\text{"}x_1\text{"}, \dots, \text{"}x_n\text{"}\} \mid (n \in \mathbb{N})\}$

Definition 63 (Predicate Alphabet).
$🔤_1 = (\mathscr{X}_1 \cup \{\text{"="}, \text{"}\in\text{"}, \text{"}\neg\text{"}, \text{"}\wedge\text{"}, \text{"("}, \text{")"}\})$

Definition 64 (Concatenation Function of Equality).
$\mathrm{eq}(\text{"}x\text{"}, \text{"}x'\text{"}) = \text{"}(x = x')\text{"}$

Definition 65 (Concatenation Function of Membership).
$\mathrm{in}(\text{"}x\text{"}, \text{"}x'\text{"}) = \text{"}(x \in x')\text{"}$

Definition 66 (Concatenation Function of Universal Quantification).
$\mathrm{fa}(\text{"}x\text{"}, \text{"}\varphi\text{"}) = \text{"}\forall x(\varphi)\text{"}$

Definition 67 (Space of Formulas).
$\mathscr{F}_{1,1} = \bigcap\{(\Phi \subseteq 🔤_1^*) \mid ((\mathscr{X}_1 \subseteq \Phi) \wedge \$_2(\Phi, \{\mathrm{eq}, \mathrm{in}\}))\}$
$\mathscr{F}_1 = \bigcap\{(\Phi \subseteq 🔤_1^*) \mid (\mathscr{F}_{1,1} \subseteq \Phi \wedge \$_1(\Phi, \{\mathrm{neg}\}) \wedge \$_2(\Phi, \{\mathrm{conj}\}) \wedge \$_1(\mathscr{X}_1, \Phi, \{\mathrm{fa}\}))\}$

Definition 68 (Space of Free Variables).
$\mathrm{free} : \mathscr{F}_1 \to \mathscr{P}(\mathscr{X}_1)$

$$\mathrm{free}(\varphi) = \begin{cases} \{x, x'\}, & \exists x, x' \in \mathscr{X}_1(\varphi = \mathrm{eq}(x, x')) \\ \{x, x'\}, & \exists x, x' \in \mathscr{X}_1(\varphi = \mathrm{in}(x, x')) \\ \mathrm{free}(\psi), & \exists \psi \in \mathscr{F}_1(\varphi = \mathrm{neg}(\psi)) \\ (\mathrm{free}(\psi) \cup \mathrm{free}(\psi')), & \exists \psi, \psi' \in \mathscr{F}_1(\varphi = \mathrm{conj}(\psi, \psi')) \\ (\mathrm{free}(\psi) \setminus \{x\}), & \exists x \in \mathscr{X}_1, \exists \psi \in \mathscr{F}_1(\varphi = \mathrm{fa}(x, \psi)) \end{cases}$$

Definition 69 (Space of Sentences).
$\mathscr{F}_1^{\bowtie} = \{(\varphi \in \mathscr{F}_1) \mid (\mathrm{free}(\varphi) = \varnothing)\}$

Definition 70 (Fresh Variable).
Let $n \in \mathbb{N}_{\geq 1}$.
$🆕(S_1, \dots, S_n) = (\mathscr{X}_1 \setminus (\Diamond(S_1) \cup \cdots \cup \Diamond(S_n)))$

Definition 71 (Variable Substitution).
Let $s, s' \in \mathcal{X}_1$.
$.|_s^{s'} : \mathcal{F}_1 \to \mathcal{F}_1$

$$\varphi|_s^{s'} = \begin{cases} \mathrm{eq}(x,x')|_s^{s'} = \begin{cases} \mathrm{eq}(s',s'), & (x, x' = s) \\ \mathrm{eq}(s',x'), & ((x = s) \wedge (x' \neq s)) \\ \mathrm{eq}(x,s'), & ((x \neq s) \wedge (x' = s)) \\ \mathrm{eq}(x,x'), & \text{o.w.} \end{cases}, & \exists x, x' \in \mathcal{X}_1(\varphi = \mathrm{eq}(x,x')) \\ \mathrm{in}(x,x')|_s^{s'} = \begin{cases} \mathrm{in}(s',s'), & (x, x' = s) \\ \mathrm{in}(s',x'), & ((x = s) \wedge (x' \neq s)) \\ \mathrm{in}(x,s'), & ((x \neq s) \wedge (x' = s)) \\ \mathrm{in}(x,x'), & \text{o.w.} \end{cases}, & \exists x, x' \in \mathcal{X}_1(\varphi = \mathrm{in}(x,x')) \\ \mathrm{neg}(\psi)|_s^{s'} = \mathrm{neg}(\psi|_s^{s'}), & \exists \psi \in \mathcal{F}_1(\varphi = \mathrm{neg}(\psi)) \\ \mathrm{conj}(\psi,\psi')|_s^{s'} = \mathrm{conj}(\psi|_s^{s'}, \psi'|_s^{s'}), & \exists \psi, \psi' \in \mathcal{F}_1(\varphi = \mathrm{conj}(\psi,\psi')) \\ \mathrm{fa}(x,\psi) = \begin{cases} \mathrm{fa}(x,\psi), & (x = s) \\ \mathrm{fa}(x,\psi|_s^{s'}), & ((x \neq s) \wedge (x \notin \lozenge(s'))) \\ \mathrm{fa}(\text{🆕}(s',\psi), \psi|_x^{\text{🆕}(s',\psi)}|_s^{s'}), & \text{o.w.} \end{cases}, & \exists x \in \mathcal{X}_1, \exists \psi \in \mathcal{F}_1(\varphi = \mathrm{fa}(x,\psi)) \end{cases}$$

Definition 72 (Space of Structures).
$n_{F,R} = \{(F \cup R) \to \mathbb{N}\}$
$i_{U,F,R,a} = \{(F \cup R) \to (\bigcup\{\{U^{a(f)} \to U\} \mid (f \in F)\} \cup \bigcup\{\mathscr{P}(U^{a(r)}) \mid (r \in R)\})\}$
$\mathcal{M} = \{\langle U, \langle F, R, a\rangle, i\rangle \mid ((U \neq \varnothing) \wedge ((F \cap R) = \varnothing) \wedge (a \in n_{F,R}) \wedge (i \in i_{U,F,R,a}))\}$

Definition 73 (Structure of Predicate Logic).
$\mathbb{s} = \langle U_{\mathbb{s}}, \langle \varnothing, \{"\in"\}, a_{\mathbb{s}}\rangle, i_{\mathbb{s}}\rangle$
$\mathbb{s} \in \mathcal{M}$
$a_{\mathbb{s}}("\in") = 2$
$i_{\mathbb{s}}("\in") = \{\langle x, X\rangle \mid (x \in X)\}$

Definition 74 (Space of Variable Assignments).
Let $m \in \mathcal{M}$.
$\mathcal{T}_1^m = (\bigcup\{\{\mathrm{free}(\varphi) \to m_{[1]}\} \mid (\varphi \in \mathcal{F}_1)\} \cup \{\mathrm{id}\})$

Definition 75 (Valuation of Formula).
Let $m \in \mathcal{M}$, and $t \in \mathcal{T}_1^m$.
$\mathfrak{v}_1^{m,t} : \mathcal{F}_1 \to 2$

$$\mathfrak{v}_1^{m,t}(\varphi) = \begin{cases} \mathfrak{v}_1^{m,t}(\mathrm{eq}(x,x')) = \begin{cases} 1, & t(x) = t(x') \\ 0, & \text{o.w.} \end{cases}, & \exists x, x' \in \mathcal{X}_1(\varphi = \mathrm{eq}(x,x')) \\ \mathfrak{v}_1^{m,t}(\mathrm{in}(x,x')) = \begin{cases} 1, & t(x) \in t(x') \\ 0, & \text{o.w.} \end{cases}, & \exists x, x' \in \mathcal{X}_1(\varphi = \mathrm{in}(x,x')) \\ \mathrm{not}(\mathfrak{v}_1^{m,t}(\psi)), & \exists \psi \in \mathcal{F}_1(\varphi = \mathrm{neg}(\psi)) \\ \mathrm{and}(\mathfrak{v}_1^{m,t}(\psi), \mathfrak{v}_1^{m,t}(\psi')), & \exists \psi, \psi' \in \mathcal{F}_1(\varphi = \mathrm{conj}(\psi,\psi')) \\ \mathfrak{v}_1^{m,t}(\mathrm{fa}(x,\psi)) = \begin{cases} 1, & \forall s \in m_{[1]}(\mathfrak{v}_1^{m,t}(\psi|_s^x) = 1) \\ 0, & \text{o.w.} \end{cases}, & \exists x \in \mathcal{X}_1, \psi \in \mathcal{F}_1(\varphi = \mathrm{fa}(x,\psi)) \end{cases}$$

2.7 Adequacy

Adequacy is also known as *functional completeness*. A set of truth functions is said to be adequate only if every other truth function is “expressible” as a composition of those in the set.

Theorem 1 (Adequacy of Truth Functions of Negation and Conjunction)**.**
Let $b \in \bigcup\{\{2^{\times(n)} \to 2\} \mid (n \in \mathbb{N})\}$, and $k \in \mathbb{N}_{\geq 1}$.
$b \subseteq \bigcup\{(T \supseteq 2) \mid (\widehat{\$}_1(T, \text{not}) \wedge \cdots \wedge \widehat{\$}_2(T, \text{and}))\}$

2.8 Interlogue

Definition 76.
$\mathrm{PropDual}(\langle p, X, f\rangle, \langle p', X', f'\rangle) \Leftarrow (\forall x \subseteq X(p(x) \Leftrightarrow p'(f(x))) \wedge \forall x' \subseteq X'(p'(x') \Leftrightarrow p(f'(x'))))$

Definition 77.
$\mathrm{GalConn}(\langle X, f\rangle, \langle X', f'\rangle) \Leftarrow \forall x \subseteq X, \forall x' \subseteq X'(x \subseteq f'(x') \Leftrightarrow x' \subseteq f(x))$

2.9 Satisfiability and Definability

A logical formula is said to be *tautological* only if it is "always true", *satisfiable* only if it is "sometimes true", and *contradictory* only if it is "never true". The property of being tautological can be seen as *opposite* to the property of being contradictory, while the property of being satisfiable can be seen as *complementary* to the property of being contradictory. A set is said to be *definable* only if there exists a logical formula whose truth is equivalent to existence of the set.

2.9.1 Propositional Logic

Definition 78 (Satisfiable Set of Propositional Formulas).
Let $\Phi \subseteq \mathscr{F}_0$.
$\mathrm{Sat}(\Phi) \Leftrightarrow \exists t \in \mathscr{T}_0, \forall \varphi \in \Phi(\mathfrak{v}_0^t(\varphi) = 1)$

Definition 79 (Definable Set of Truth Assignments).
Let $T \subseteq \mathscr{T}_0$.
$\mathrm{Def}(T) \Leftrightarrow \exists \varphi \in \mathscr{F}_0, \forall t \in T(\mathfrak{v}_0^t(\varphi) = 1)$

Definition 80 (Subject of Set of Propositional Formulas).
Let $\Phi \subseteq \mathscr{F}_0$.
$\mathrm{subj}_0(\Phi) = \{ t \in \mathscr{T}_0 \mid \forall \varphi \in \Phi(\mathfrak{v}_0^t(\varphi) = 1)\}$

Definition 81 (Theory of Set of Truth Assignments).
Let $T \subseteq \mathscr{T}_0$.
$\mathrm{th}_0(T) = \{ \varphi \in \mathscr{F}_0 \mid \forall t \in T(\mathfrak{v}_0^t(\varphi) = 1)\}$

Theorem 2.
$\mathrm{PropDual}(\langle \mathrm{Sat}, \mathscr{F}_0, \mathrm{subj}_0\rangle, \langle \mathrm{Def}, \mathscr{T}_0, \mathrm{th}_0\rangle)$

Theorem 3.
$\mathrm{GalConn}(\langle \mathscr{F}_0, \mathrm{subj}_0\rangle, \langle \mathscr{T}_0, \mathrm{th}_0\rangle)$

Theorem 4.
$\exists \Phi \subseteq \mathscr{F}_0(\neg \mathrm{Sat}(\Phi))$

Theorem 5.
$\exists T \subseteq \mathscr{T}_0(\neg \mathrm{Def}(T))$

2.9.2 Predicate Logic

Definition 82 (Satisfiable Set of Sentences).
Let $\Phi \subseteq \mathscr{F}_1^{\bowtie}$.
$\mathrm{Sat}(\Phi) \Leftrightarrow \exists m \in \mathcal{M}, \forall \varphi \in \Phi(\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = 1)$

Definition 83 (Definable Set of Structures).
Let $M \subseteq \mathcal{M}$.
$\mathrm{Def}(M) \Leftrightarrow \exists \varphi \in \mathscr{F}_1^{\bowtie}, \forall m \in M(\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = 1)$

Definition 84 (Subject of Set of Sentences).
Let $\Phi \subseteq \mathscr{F}_1^{\bowtie}$.
$\mathrm{subj}_1(\Phi) = \{ m \in \mathcal{M} \mid \forall \varphi \in \Phi(\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = 1)\}$

Definition 85 (Theory of Set of Structures)**.**
Let $M \subseteq \mathcal{M}$.
$\mathrm{th}_1(M) = \{ \varphi \in \mathcal{F}_1^{\bowtie} \mid \forall m \in M(\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = 1)\}$

Theorem 6.
$\mathrm{PropDual}(\langle \mathrm{Sat}, \mathcal{F}_1^{\bowtie}, \mathrm{subj}_1\rangle, \langle \mathrm{Def}, \mathcal{M}, \mathrm{th}_1\rangle)$

Theorem 7.
$\mathrm{GalConn}(\langle \mathcal{F}_1^{\bowtie}, \mathrm{subj}_1\rangle, \langle \mathcal{M}, \mathrm{th}_1\rangle)$

Theorem 8.
$\exists \Phi \subseteq \mathcal{F}_1^{\bowtie}(\neg\,\mathrm{Sat}(\Phi))$

Theorem 9.
$\exists M \subseteq \mathcal{M}(\neg\,\mathrm{Def}(M))$

2.10 Soundness and Completeness

For a logic, soundness can be seen as the property that "every proof has truth", and completeness can be seen as the property that "every truth has proof".

2.10.1 Propositional Logic

Definition 86 (Modus Ponens Function).
$\text{pon}(\texttt{"}\varphi\texttt{"}, \texttt{"}(\varphi \Rightarrow \varphi')\texttt{"}) = \texttt{"}\varphi'\texttt{"}$

Definition 87 (Space of Propositional Axioms).
$\mathcal{A}_{0,1} = \{\texttt{"}(\varphi \Rightarrow (\psi \Rightarrow \varphi))\texttt{"} \mid (\varphi, \psi \in \mathcal{F}_0)\}$
$\mathcal{A}_{0,2} = \{\texttt{"}((\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi)))\texttt{"} \mid (\varphi, \psi, \chi \in \mathcal{F}_0)\}$
$\mathcal{A}_{0,3} = \{\texttt{"}(((\neg\varphi) \Rightarrow (\neg\psi)) \Rightarrow (\psi \Rightarrow \varphi))\texttt{"} \mid (\varphi, \psi \in \mathcal{F}_0)\}$
$\mathcal{A}_0 = \bigcup\{\mathcal{A}_{0,1}, \mathcal{A}_{0,2}, \mathcal{A}_{0,3}\}$

Definition 88 (Propositional Proof System).
Let $\Gamma \subseteq \mathcal{F}_0$.
$\mathcal{D}_0(\Gamma) = \bigcap\{(\Phi \subseteq \text{🔤}_0^*) \mid (((\mathcal{A}_0 \cup \Gamma) \subseteq \Phi) \wedge \$_1(\Phi, \{\text{pon}\}))\}$

Definition 89 (Propositional Syntactic Entailment).
Let $\Gamma \subseteq \mathcal{F}_0$, and $\varphi \in \mathcal{F}_0$.
$(\Gamma \vdash \varphi) \Leftrightarrow (\varphi \in \mathcal{D}_0(\Gamma))$

Definition 90 (Propositional Semantic Entailment).
Let $\Gamma \subseteq \mathcal{F}_0$, and $\varphi \in \mathcal{F}_0$.
$(\Gamma \vDash \varphi) \Leftrightarrow \forall t \in \mathcal{T}_0(\forall \gamma \in \Gamma(v_0^t(\gamma) = 1) \Rightarrow (v_0^t(\varphi) = 1))$

Theorem 10 (Finitaryness of Propositional Proof System).
Let $\Gamma \subseteq \mathcal{F}_0$, and $\varphi \in \mathcal{D}_0(\Gamma)$.
$\exists n \in \mathbb{N}_{\geq 1}, \exists\langle\psi_1, \ldots, \psi_n\rangle \in \bigcap\{(\Phi \subseteq \text{🔤}_0^*) \mid (((\mathcal{A}_0 \cup \Gamma) \subseteq \Phi) \wedge \widehat{\$}_1(\Phi, \{\text{pon}\}))\}(\psi_n = \varphi)$

Theorem 11 (Soundness of Propositional Proof System).
Let $\Gamma \subseteq \mathcal{F}_0$, and $\varphi \in \mathcal{F}_0$.
$(\Gamma \vdash \varphi) \Rightarrow (\Gamma \vDash \varphi)$

Theorem 12 (Completeness of Propositional Proof System).
Let $\Gamma \subseteq \mathcal{F}_0$, and $\varphi \in \mathcal{F}_0$.
$(\Gamma \vDash \varphi) \Rightarrow (\Gamma \vdash \varphi)$

Definition 91 (Consistent Set of Propositional Formulas).
Let $\Gamma \subseteq \mathcal{F}_0$.
$\text{Cons}(\Gamma) \Leftrightarrow \forall\varphi \in \mathcal{F}_0(\neg((\Gamma \vdash \varphi) \wedge (\Gamma \nvdash \varphi)))$

Definition 92 (Satisfiable Set of Propositional Formulas).
Let $\Gamma \subseteq \mathcal{F}_0$.
$\text{Sat}(\Gamma) \Leftrightarrow \forall\varphi \in \mathcal{F}_0(\neg((\Gamma \vDash \varphi) \wedge (\Gamma \nvDash \varphi)))$

Theorem 13 (Propositional Consistency–Satisfiability Equivalence).
Let $\Gamma \subseteq \mathcal{F}_0$.
$\text{Cons}(\Gamma) \Leftrightarrow \text{Sat}(\Gamma)$

2.10.2 Predicate Logic

Definition 93 (Space of Predicate Axioms)**.**
$\mathcal{A}_{1,1} = \{\tau|_p^{\varphi} \mid ((\tau \in \mathcal{F}_0) \wedge (p \in \mathcal{X}_0) \wedge (\varphi \in \mathcal{F}_1) \wedge (\varnothing \vDash \tau))\}$
$\mathcal{A}_{1,2} = \{\text{"}(\forall x(\varphi) \Rightarrow \varphi|_x^{x'})\text{"} \mid ((x, x' \in \mathcal{X}_1) \wedge (\varphi \in \mathcal{F}_1))\}$
$\mathcal{A}_{1,3} = \{\text{"}(\varphi \Rightarrow \forall x(\varphi|_x^{x'}))\text{"} \mid ((x, x' \in \mathcal{X}_1) \wedge (\varphi \in \mathcal{F}_1))\}$
$\mathcal{A}_1 = \bigcup\{\mathcal{A}_{1,1}, \mathcal{A}_{1,2}, \mathcal{A}_{1,3}\}$

Definition 94 (Proof System)**.**
Let $\Gamma \subseteq \mathcal{F}_1$.
$\mathcal{D}_1(\Gamma) = \bigcap\{(\Phi \subseteq 🔤_1^*) \mid (((\mathcal{A}_1 \cup \Gamma) \subseteq \Phi) \wedge \$_1(\Phi, \{\text{pon}\}))\}$

Definition 95 (Predicate Syntactic Entailment)**.**
Let $\Gamma \subseteq \mathcal{F}_1$, and $\varphi \in \mathcal{F}_1$.
$(\Gamma \vdash \varphi) \Leftrightarrow (\varphi \in \mathcal{D}_1(\Gamma))$

Definition 96 (Predicate Semantic Entailment)**.**
Let $\Gamma \subseteq \mathcal{F}_1$, and $\varphi \in \mathcal{F}_1$.
$(\langle M, \Gamma\rangle \vDash \varphi) \Leftrightarrow \forall m \in M, \forall t \in \mathcal{T}_1^m(\forall \gamma \in \Gamma(\mathfrak{v}_1^{m,t}(\gamma) = 1) \Rightarrow (\mathfrak{v}_1^{m,t}(\varphi) = 1))$

Theorem 14 (Finitaryness of Proof System)**.**
Let $\Gamma \subseteq \mathcal{F}_1$, and $\varphi \in \mathcal{D}_1(\Gamma)$.
$\exists n \in \mathbb{N}_{\geq 1}, \exists\langle\psi_1, \ldots, \psi_n\rangle \in \bigcap\{(\Phi \subseteq 🔤_1^*) \mid (((\mathcal{A}_1 \cup \Gamma) \subseteq \Phi) \wedge \widehat{\$}_1(\Phi, \{\text{pon}\}))\}(\psi_n = \varphi)$

Theorem 15 (Soundness of Proof System)**.**
Let $\Gamma \subseteq \mathcal{F}_1$, and $\varphi \in \mathcal{F}_1$.
$(\Gamma \vdash \varphi) \Rightarrow (\langle\mathcal{M}, \Gamma\rangle \vDash \varphi)$

Theorem 16 (Completeness of Proof System)**.**
Let $\Gamma \subseteq \mathcal{F}_1$, and $\varphi \in \mathcal{F}_1$.
$(\langle\mathcal{M}, \Gamma\rangle \vDash \varphi) \Rightarrow (\Gamma \vdash \varphi)$

Definition 97 (Consistent Set of Formulas)**.**
Let $\Gamma \subseteq \mathcal{F}_1$.
$\text{Cons}(\Gamma) \Leftrightarrow \forall\varphi \in \mathcal{F}_1(\neg((\Gamma \vdash \varphi) \wedge (\Gamma \nvdash \varphi)))$

Definition 98 (Satisfiable Set of Formulas)**.**
Let $\Gamma \subseteq \mathcal{F}_1$.
$\text{Sat}(\Gamma) \Leftrightarrow \forall\varphi \in \mathcal{F}_1(\neg((\langle\mathcal{M}, \Gamma\rangle \vDash \varphi) \wedge (\langle\mathcal{M}, \Gamma\rangle \nvDash \varphi)))$

Theorem 17 (Predicate Consistency–Satisfiability Equivalence)**.**
Let $\Gamma \subseteq \mathcal{F}_1$.
$\text{Cons}(\Gamma) \Leftrightarrow \text{Sat}(\Gamma)$

2.11 Interlogue

Definition 99 (Injective Function)**.**
Let $f : X \to Y$.
$\text{Inj}(f) \Leftrightarrow \forall y \in Y, \text{E}x \in X(f(x) = y)$

Definition 100 (Surjective Function)**.**
Let $f : X \to Y$.
$\text{Surj}(f) \Leftrightarrow \forall y \in Y, \exists x \in X(f(x) = y)$

Definition 101 (Bijective Function)**.**
Let $f : X \to Y$.
$\text{Bij}(f) \Leftrightarrow (\text{Inj}(f) \land \text{Surj}(f))$

Definition 102 (Finite Set)**.**
$\text{Fin}(X) \Leftrightarrow \exists n \in \mathbb{N}, \exists f \in \{X \to \mathbb{N}_{\leq n}\}(\text{Bij}(f))$

2.12 Compactness and Maximality

Previously, we established equivalence between consistency and satisfiability. In the interest of brevity, we shall occasionally omit discussion of consistency in favor of satisfiability, keeping in mind that every property which applies to the latter, also applies to the former, in equivalent fashion.

2.12.1 Propositional Logic

Theorem 18 (Propositional Compactness).
Let $\Gamma \subseteq \mathcal{F}_0$.
$\mathrm{Sat}(\Gamma) \Leftrightarrow \forall \Gamma' \subseteq \Gamma(\mathrm{Fin}(\Gamma') \Rightarrow \mathrm{Sat}(\Gamma'))$

Definition 103 (Maximal Set of Propositional Formulas).
Let $\Gamma \subseteq \mathcal{F}_0$.
$\mathrm{Max}(\Gamma) \Leftrightarrow \forall \varphi \in \mathcal{F}_0((\Gamma \vDash \varphi) \vee (\Gamma \nvDash \varphi))$

Theorem 19 (Propositional Lindenbaum).
Let $\Gamma \subseteq \mathcal{F}_0$.
$\mathrm{Sat}(\Gamma) \Leftrightarrow \exists \Gamma' \supseteq \Gamma(\mathrm{Sat}(\Gamma') \wedge \mathrm{Max}(\Gamma'))$

2.12.2 Predicate Logic

Theorem 20 (Predicate Compactness).
Let $\Gamma \subseteq \mathcal{F}_1$.
$\mathrm{Sat}(\Gamma) \Leftrightarrow \forall \Gamma' \subseteq \Gamma(\mathrm{Fin}(\Gamma') \Rightarrow \mathrm{Sat}(\Gamma'))$

Definition 104 (Maximal Set of Formulas).
Let $\Gamma \subseteq \mathcal{F}_1$.
$\mathrm{Max}(\Gamma) \Leftrightarrow \forall \varphi \in \mathcal{F}_1((\langle \mathcal{M}, \Gamma \rangle \vDash \varphi) \vee (\langle \mathcal{M}, \Gamma \rangle \nvDash \varphi))$

Theorem 21 (Predicate Lindenbaum).
Let $\Gamma \subseteq \mathcal{F}_1$.
$\mathrm{Sat}(\Gamma) \Leftrightarrow \exists \Gamma' \supseteq \Gamma(\mathrm{Sat}(\Gamma') \wedge \mathrm{Max}(\Gamma'))$

2.13 Gödel Incompleteness

A space of sentences which are true for a structure of natural numbers is known as a *theory of natural arithmetic*. A theory is said to be "sufficiently strong" only if it contains "sufficiently many" sentences.

Previously, we defined a proof system which is finitary, sound, and complete with respect to a theory of *predicate logic*. In the 1920s, it was wondered whether one could develop a proof system which is verifiable, sound, and complete with respect to theories of *mathematics*. Around the 1930s, Kurt Gödel demonstrated that, if "verifiable" is taken to mean "recursively axiomatisable", then no such proof system exists for "sufficiently strong" theories of natural arithmetic. Gödel also showed that no consistent proof system in which such theories of natural arithmetic are derivable can prove its *own* consistency. These theorems have come to be known as the *Gödel incompleteness theorems*. For the sake of concreteness, we shall state these theorems for a particular theory of natural arithmetic.

Famously, Gödel's incompleteness theorems have been shown to apply to a theory of natural arithmetic known as *Peano arithmetic* (PA). The strengthened finite Ramsey theorem, which can be seen as a sentence about the natural numbers, is provable in ZFC, but not in PA. Additionally, ZFC can prove PA consistent, but PA cannot prove *itself* consistent.

Definition 105.
$\mathbb{n} = \langle \mathbb{N}, \langle \{\texttt{"0"}, \texttt{"}\mathscr{S}\texttt{"}\}, \varnothing, a_{\mathbb{n}} \rangle, i_{\mathbb{n}} \rangle$
$\mathbb{n} \in \mathcal{M}$
$a_{\mathbb{n}}(\texttt{"0"}) = 0$
$a_{\mathbb{n}}(\texttt{"}\mathscr{S}\texttt{"}) = 1$
$i_{\mathbb{n}}(\texttt{"0"}) = 0$
$i_{\mathbb{n}}(\texttt{"}\mathscr{S}\texttt{"}) = \mathscr{S}$

Theorem 22 (First Gödel Incompleteness)**.**
$\neg \exists \Gamma \subseteq \mathscr{F}_1, \forall \varphi \in \mathrm{th}(\{\mathbb{n}\})\left(\left(\langle \{\mathbb{n}\}, \Gamma \rangle \vDash \varphi \right) \Rightarrow \left(\Gamma \vdash \varphi \right) \right)$

Theorem 23 (Second Gödel Incompleteness)**.**
Let $\Gamma \subseteq \mathscr{F}_1$, and $\mathrm{th}(\{\mathbb{n}\}) \subseteq \mathscr{D}_1(\Gamma)$.
$\mathrm{Cons}(\mathscr{D}_1(\Gamma)) \Leftrightarrow \left(\Gamma \nvdash \texttt{"}\,\mathrm{Cons}(\mathscr{D}_1(\Gamma))\texttt{"} \right)$

3

Elementary Number Theory

3.1 Prologue

Definition 106 (Functional Composition)**.**
Let $f : X \to Y$, and $g : Y \to Z$.
$(g \circ f) = \{\langle x, z\rangle \mid ((\langle x, y\rangle \in f) \wedge (\langle y, z\rangle \in g))\}$

Definition 107.
Let $f : X \to Y$.
$(x \sim_f y) \Leftrightarrow (f(x) = f(y))$

3.2 Natural, Integer, and Rational Number Systems

Definition 108 (Successor Function)**.**
$\text{suc} : \mathbb{N} \to \mathbb{N}$
$\text{suc}(n) = (n \cup \{n\})$

Definition 109 (Natural Addition Function)**.**
Let $a, b \in \mathbb{N}$.
$+ : \mathbb{N} \to \mathbb{N}$
$+(a, b) = \text{suc}^{\circ(b)}(a)$

Definition 110 (Natural Multiplication Function)**.**
Let $a, b \in \mathbb{N}$.
$* : \mathbb{N} \to \mathbb{N}$
$*(a, b) = \text{suc}^{\circ(b)^{\circ(b)}}(a)$

Definition 111 (Prototype of Integer)**.**
$[\![\langle a, b\rangle]\!]_{\mathbb{Z}} = \{ (\langle c, d\rangle \in \mathbb{N}^2) \mid \exists a, b \in \mathbb{N}(\langle a, c\rangle \sim_+ \langle b, d\rangle)\}$

Definition 112 (Space of Integers)**.**
$\mathbb{Z} = \{[\![\langle a, b\rangle]\!]_{\mathbb{Z}} \mid (a, b \in \mathbb{N})\}$

Definition 113 (Integer Identity of Natural Number)**.**
Let $n \in \mathbb{N}$.
$n_{[\mathbb{Z}]} = [\![\langle n, 0\rangle]\!]_{\mathbb{Z}}$

Definition 114 (Order of Integers)**.**
Let $a, b, c, d \in \mathbb{N}$, and $[\![\langle a, b\rangle]\!]_{\mathbb{Z}}, [\![\langle c, d\rangle]\!]_{\mathbb{Z}} \in \mathbb{Z}$.
$([\![\langle a, b\rangle]\!]_{\mathbb{Z}} \leq [\![\langle c, d\rangle]\!]_{\mathbb{Z}}) \Leftrightarrow ((a + d) \leq (b + c))$

Definition 115 (Addition of Integers)**.**
Let $a, b, c, d \in \mathbb{N}$, and $[\![\langle a, b\rangle]\!]_{\mathbb{Z}}, [\![\langle c, d\rangle]\!]_{\mathbb{Z}} \in \mathbb{Z}$.
$([\![\langle a, b\rangle]\!]_{\mathbb{Z}} + [\![\langle c, d\rangle]\!]_{\mathbb{Z}}) = [\![\langle (a + c), (b + d)\rangle]\!]_{\mathbb{Z}}$

Definition 116 (Multiplication of Integers)**.**
Let $a, b, c, d \in \mathbb{N}$, and $[\![\langle a, b\rangle]\!]_{\mathbb{Z}}, [\![\langle c, d\rangle]\!]_{\mathbb{Z}} \in \mathbb{Z}$.
$([\![\langle a, b\rangle]\!]_{\mathbb{Z}} * [\![\langle c, d\rangle]\!]_{\mathbb{Z}}) = [\![\langle ((a * c) + (b * d)), ((a * d) + (b * c))\rangle]\!]_{\mathbb{Z}}$

Definition 117 (Subtraction of Integers)**.**
Let $a, b, c \in \mathbb{Z}$.
$((a - b) = c) \Leftrightarrow (a = (b + c))$

Definition 118 (Prototype of Rational Number)**.**
Let $a, b \in \mathbb{Z}$.
$[\![\langle a, b\rangle]\!]_{\mathbb{Q}} = \{ (\langle c, d\rangle \in (\mathbb{Z} \times (\mathbb{Z} \setminus \{0_{[\mathbb{Z}]}\}))) \mid ((a * d) = (b * c))\}$

Definition 119 (Space of Rational Numbers)**.**
$\mathbb{Q} = \{[\![\langle a, b\rangle]\!]_{\mathbb{Q}} \mid ((a \in \mathbb{Z}) \wedge (b \in (\mathbb{Z} \setminus \{0_{[\mathbb{Z}]}\})))\}$

Definition 120 (Rational Identity of Integer)**.**
Let $n \in \mathbb{Z}$.
$n_{[\mathbb{Q}]} = [\![\langle n, 1_{[\mathbb{Z}]} \rangle]\!]_{\mathbb{Q}}$

Definition 121 (Order of Rational Numbers)**.**
Let $a, b, c, d \in \mathbb{Z}$, and $[\![\langle a, b \rangle]\!]_{\mathbb{Q}}, [\![\langle c, d \rangle]\!]_{\mathbb{Q}} \in \mathbb{Q}$.
$([\![\langle a, b \rangle]\!]_{\mathbb{Q}} \leq [\![\langle c, d \rangle]\!]_{\mathbb{Q}}) \Leftrightarrow ((a * d) \leq (b * c))$

Definition 122 (Addition of Rational Numbers)**.**
Let $a, b, c, d \in \mathbb{Z}$, and $[\![\langle a, b \rangle]\!]_{\mathbb{Q}}, [\![\langle c, d \rangle]\!]_{\mathbb{Q}} \in \mathbb{Q}$.
$([\![\langle a, b \rangle]\!]_{\mathbb{Q}} + [\![\langle c, d \rangle]\!]_{\mathbb{Q}}) = [\![\langle ((a * d) + (b * c)), (b * d) \rangle]\!]_{\mathbb{Q}}$

Definition 123 (Multiplication of Rational Numbers)**.**
Let $a, b, c, d \in \mathbb{Z}$, and $[\![\langle a, b \rangle]\!]_{\mathbb{Q}}, [\![\langle c, d \rangle]\!]_{\mathbb{Q}} \in \mathbb{Q}$.
$([\![\langle a, b \rangle]\!]_{\mathbb{Q}} * [\![\langle c, d \rangle]\!]_{\mathbb{Q}}) = [\![\langle (a * c), (b * d) \rangle]\!]_{\mathbb{Q}}$

Definition 124 (Subtraction of Rational Numbers)**.**
Let $a, b, c \in \mathbb{Q}$.
$((a - b) = c) \Leftrightarrow (a = (b + c))$

Definition 125 (Division of Rational Numbers)**.**
Let $a, b \in \mathbb{Q}$, and $b \neq 0$.
$(\frac{a}{b} = c) \Leftrightarrow (a = (b * c))$