CLASSICAL LOGIC

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Contents

The unexamined life is not worth living.

SOCRATES

1

Truth

What could it mean for something to be *true*? Throughout history, various definitions for the word "truth" have been proposed.

DEFINITION I — CORRESPONDENCE THEORY OF TRUTH

Truth is that which corresponds to reality.

DEFINITION II — COHERENCE THEORY OF TRUTH

Truth is that which coheres with every other truth.

Ought mathematics concern itself with the universe we inhabit? In any case, the practice of mathematics seems to be based on something like **DEFINITION II**; this presents a *seeming* trilemma.

DEFINITION III — MÜNCHHAUSEN TRILEMMA

Every proof is completed by circularity, infinite regress, and/or assumption.

Reasoning by coherence appears to obtain truth by circularity, infinite regress, and/or assumption. It is no secret that mathematical theories admit assumptions. Mathematical theories are also said to build upon, and thus derive from, each other. A theory from which other theories can be derived is described as *foundational*.

Broadly speaking, a logic can be thought of as a language for reasoning about truth. A propositional logic is sometimes called a *zeroth-order calculus*, and a predicate logic is sometimes called a *first-order calculus*. A predicate logic can be said to "extend" a propositional logic.

With any human language, definitions may be said to originate in *pre-linguistic* thought. Writers sometimes choose their starting points based on what feels the most intuitive to them. For this text, I have decided to use words and phrases like "not", "and", "if. . . then", "either. . . or", "otherwise", "every", "same", in addition to a few other mathematical symbols, in my appeal to the intuition of the reader.

There are three rules for writing a novel.
Unfortunately, no one knows what they are.

SOMERSET MAUGHAM

2

Language

Propositional variables and (predicate) variables are known as *logical variables*. Similarly, propositional formulas and (predicate) formulas are known as *logical formulas*. Logical formulas can be recursively defined from their logical variables.

2.1 Propositional Logic

2.1.1 Syntax

DEFINITION IV — PROPOSITIONAL FORMULA

Let $p_\circ, \ldots, p_\bullet$ be propositional variables.

1. If p is a propositional variable, then p is a propositional formula.
2. If φ is a propositional formula, then $(\neg\varphi)$ is a propositional formula.
   If φ and φ' are propositional formulas, then $(\varphi \wedge \varphi')$ is a propositional formula.

2.1.2 Semantics

DEFINITION V — TRUTH VALUE OF PROPOSITIONAL FORMULA

1. Every propositional variable is either assigned true, or assigned false.
2. If φ is assigned true, then $(\neg\varphi)$ is assigned false.
   Otherwise, $(\neg\varphi)$ is assigned true.
   If φ and φ' are assigned true, then $(\varphi \land \varphi')$ is assigned true.
   Otherwise, $(\varphi \land \varphi')$ is assigned false.

2.2 Predicate Logic

2.2.1 Syntax

DEFINITION VI — FORMULA

Let $x_\circ, \ldots, x_\bullet$ be variables.

1. If x is a variable, and x' is a variable, then $(x = x')$ is a formula.
   If x is a variable, and x' is a variable, then $(x \in x')$ is a formula.
2. If φ is a formula, then $(\neg\varphi)$ is a formula.
   If φ is a formula, and φ' is a formula, then $(\varphi \wedge \varphi')$ is a formula.
   If x is a variable, and φ is a formula, then $\forall x(\varphi)$ is a formula.

DEFINITION VII — FREE VARIABLE

1. If $(x = x')$ is a formula, then x and x' are free variables in the formula.
   If $(x \in x')$ is a formula, then x and x' are free variables in the formula.
2. If x is a free variable in φ, then x is a free variable in $(\neg\varphi)$.
   If x is a free variable in φ and φ', then x is a free variable in $(\varphi \wedge \varphi')$.
   If x is a free variable in φ, and $(x \neq x')$, then x is a free variable in $\forall x'(\varphi)$.

NOTATION I — NAÏVE SUBSTITUTION

$(x = x)\|_{x\mapsto\tau} \lessdot (\tau = \tau)$

$(x = x')\|_{x\mapsto\tau} \lessdot (\tau = x')$

$(x' = x)\|_{x\mapsto\tau} \lessdot (x' = \tau)$

$(x' = x')\|_{x\mapsto\tau} \lessdot (x' = x')$

$(x \in x)\|_{x\mapsto\tau} \lessdot (\tau \in \tau)$

$(x \in x')\|_{x\mapsto\tau} \lessdot (\tau \in x')$

$(x' \in x)\|_{x\mapsto\tau} \lessdot (x' \in \tau)$

$(x' \in x')\|_{x\mapsto\tau} \lessdot (x' \in x')$

$(\neg\varphi)\|_{x\mapsto\tau} \lessdot (\neg\varphi\|_{x\mapsto\tau})$

$(\varphi \wedge \varphi')\|_{x\mapsto\tau} \lessdot (\varphi\|_{x\mapsto\tau} \wedge \varphi'\|_{x\mapsto\tau})$

$\forall x(\varphi)\|_{x\mapsto\tau} \lessdot \forall s(\varphi\|_{x\mapsto\tau})$

NOTATION II

$(x \not\circledast x') \lessdot (\neg(x \circledast x'))$

NOTATION III

$(\varphi \Rightarrow \varphi') \lessdot (\neg(\varphi \wedge (\neg\varphi')))$

NOTATION IV

$(\varphi \Leftrightarrow \varphi') \lessdot ((\varphi \Rightarrow \varphi') \wedge (\varphi' \Rightarrow \varphi))$

NOTATION V

$(\varphi \oplus \varphi') \lessdot ((\varphi \vee \varphi') \wedge \neg((\varphi \wedge \varphi')))$

NOTATION VI

$\exists x(\varphi) \lessdot (\neg\forall x((\neg\varphi)))$

NOTATION VII

$\exists_! x(\varphi) \lessdot \exists x(\forall x'((\varphi\|_x^{x'} \Leftrightarrow (x = x'))))$

NOTATION VIII

$\mathsf{E}x(\varphi) \lessdot (\nexists x(\varphi) \oplus \exists_! x(\varphi))$

NOTATION IX

$(x_\circ, \dots, x_\bullet \circledast X) \lessdot ((x_\circ \circledast X) \wedge \cdots \wedge (x_\bullet \circledast X))$

NOTATION X

$\forall x_\circ, \dots, x_\bullet \circledast X(\varphi) \lessdot \forall x_\circ(((x_\circ \circledast X) \Rightarrow \cdots \Rightarrow \forall x_\bullet(((x_\bullet \in X) \Rightarrow \varphi))))$

NOTATION XI

$\exists x_\circ, \dots, x_\bullet \circledast X(\varphi) \lessdot \exists x_\circ(((x_\circ \circledast X) \wedge \cdots \wedge \exists x_\bullet(((x_\bullet \in X) \wedge \varphi))))$

NOTATION XII

$\exists_! x_\circ, \dots, x_\bullet \circledast X(\varphi) \lessdot \exists_! x_\circ(((x_\circ \circledast X) \wedge \cdots \wedge \exists_! x_\bullet(((x_\bullet \in X) \wedge \varphi))))$

NOTATION XIII

$\mathsf{E}x_\circ, \dots, x_\bullet \circledast X(\varphi) \lessdot \mathsf{E}x_\circ(((x_\circ \circledast X) \wedge \cdots \wedge \mathsf{E}x_\bullet(((x_\bullet \in X) \wedge \varphi))))$

2.2.2 Semantics

DEFINITION VIII — TRUTH VALUE OF FORMULA

1. Every variable is assigned a set.
2. If x is assigned the same set as x', then $(x = x')$ is assigned true.
   Otherwise, $(x = x')$ is assigned false.
   If φ is assigned true, then $(\neg\varphi)$ is assigned false.
   Otherwise, $(\neg\varphi)$ is assigned true.
   If φ and φ' are assigned true, then $(\varphi \wedge \varphi')$ is assigned true.
   Otherwise, $(\varphi \wedge \varphi')$ is assigned false.
   If φ is assigned true for every possible x, then $\forall x(\varphi)$ is assigned true.
   Otherwise, $\forall x(\varphi)$ is assigned false.

3

Zermelo–Frænkel Set Theory with Choice

Previously, we used brackets to guarantee uniqueness for every reading of logical syntax. In the interest of brevity, we shall, henceforth, omit outermost pairs of brackets.

Around the 1920s, an axiomatic set theory was proposed by Ernst Zermelo and Abraham Frænkel: this Zermelo–Frænkel set theory (ZF), when paired with the axiom of choice (AC), came to be known as Zermelo–Frænkel set theory with choice (ZFC). ZFC is commonly used as a foundational theory, and is conventionally written in a classical predicate logic. With the admission of ZF, AC is equivalent to various theorems, including the theorem that every vector space has a basis, and the theorem that every surjective function has a right inverse. Famously, AC in ZF also implies the Banach–Tarski paradox.

AXIOM I — EMPTY SET

$\exists N(\forall x((x \notin N)))$

DEFINITION IX — EMPTY SET

$((N = \varnothing) = \{\}) \Leftrightarrow \forall x((x \notin N))$

AXIOM II — EXTENSIONALITY

$\forall X, Y((\forall m(((m \in X) \Leftrightarrow (m \in Y)))) \Rightarrow (X = Y)))$

DEFINITION X — SUBSET

$(X \subseteq Y) \Leftrightarrow \forall m(((m \in X) \Rightarrow (m \in Y)))$

AXIOM III — PAIRING

$\forall c, c'(\exists C(((c \in C) \wedge (c' \in C))))$

DEFINITION XI

$(C = \{c, c'\}) \Leftrightarrow ((c \in C) \wedge (c' \in C))$

DEFINITION XII — SINGLETON SET

$\{c\} = \{c, c\}$

AXIOM IV — UNION

$\forall X(\exists U(\forall u(((u \in U) \Leftrightarrow \exists x \in X((u \in x))))))$

DEFINITION XIII — UNARY UNION FUNCTION

$(U = \bigcup X) \Leftrightarrow \forall u(((u \in U) \Leftrightarrow \exists x \in X((u \in x))))$

DEFINITION XIV — BINARY UNION FUNCTION

$(x \cup x') = \bigcup\{x, x'\}$

DEFINITION XV — SET OF FREE VARIABLES

Let $x_\circ, \ldots, x_\bullet$ be the free variables in φ.

$\text{free}(\varphi) = \bigcup\{\{x_\circ\}, \ldots, \{x_\bullet\}\}$

AXIOM V — POWER SET

$\forall X(\exists P(\forall p(((p \subseteq X) \Rightarrow (p \in P)))))$

AXIOM SCHEMA I — REPLACEMENT

Let $\text{free}(\varphi) \subseteq \{D, d, i\}$.

$\forall D((\forall d \in D(\exists_! i(\varphi)) \Rightarrow \exists I(\forall i(((i \in I) \Leftrightarrow \exists d \in D(\varphi))))))$

AXIOM SCHEMA II — SEPARATION

Let $\text{free}(\varphi) \subseteq \{D, a_\circ, \ldots, a_\bullet, f\}$.

$\forall D, a_\circ, \ldots, a_\bullet(\exists F(\forall f(((f \in F) \Leftrightarrow ((f \in D) \wedge \varphi)))))$

DEFINITION XVI

Let $\text{free}(\varphi) \subseteq \{D, f\}$.

$(F = \{(d \in D) \mid \varphi\}) \Leftrightarrow \forall f(((f \in F) \Leftrightarrow ((f \in D) \wedge \varphi)))$

DEFINITION XVII

Let $\text{free}(\varphi) \subseteq \{f\}$.

$(F = \{x \mid \varphi\}) \Leftrightarrow \forall f(((f \in F) \Leftrightarrow \varphi))$

DEFINITION XVIII — UNARY INTERSECTION FUNCTION

$$\bigcap X = \{(u \in \bigcup X) \mid \forall x \in X((u \in x))\}$$

DEFINITION XIX — BINARY INTERSECTION FUNCTION

$$(x \cap x') = \bigcap\{x, x'\}$$

AXIOM VI — INFINITY

$$\exists R(((\varnothing \in R) \land \forall r \in R(((r \cup \{r\}) \in R))))$$

AXIOM VII — REGULARITY

$$\forall O(((O \neq \varnothing) \Rightarrow \exists o \in O(((o \cap O) = \varnothing))))$$

DEFINITION XX — SET OF PAIRWISE DISJOINT SETS

$$\bigcirc\!\!\!\!\!\not\cap(X) \Leftrightarrow \forall x, x' \in X((x \neq x') \Rightarrow ((x \cap x') = \varnothing))$$

AXIOM VIII — CHOICE

$$\forall B(((\forall S \in B((S \neq \varnothing)) \land \bigcirc\!\!\!\!\!\not\cap(B)) \Rightarrow \exists B'(\forall S \in B(\exists! s \in S((s \in B'))))))$$

4

Interlude I

DEFINITION XXI — POWER SET

$\mathscr{P}(X) = \{x \mid (x \subseteq X)\}$

DEFINITION XXII — RELATIVE COMPLEMENT

$(X \setminus Y) = \{(d \in X) \mid (d \notin Y)\}$

DEFINITION XXIII — SUCCESSOR

$i(x) = (x \cup \{x\})$

DEFINITION XXIV — SPACE OF NATURAL NUMBERS

$(\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \dots\}) = \bigcap\{R \mid ((\varnothing \in R) \land \forall r \in R((i(r) \in R)))\}$

DEFINITION XXV

Let $n \in \mathbb{N}$.

$\mathbb{N}_{\geq n} = \bigcap\{R \mid ((n \in R) \land \forall r \in R((i(r) \in R)))\}$

DEFINITION XXVI

Let $n \in \mathbb{N}$.

$\mathbb{N}_{\leq n} = (\{n\} \cup \{p \mid (p \notin \mathbb{N}_{\geq n})\})$

DEFINITION XXVII

Let $m, n \in \mathbb{N}$.

$\mathbb{N}_{[m,n]} = (\mathbb{N}_{\geq m} \cap \mathbb{N}_{\leq n})$

DEFINITION XXVIII — n-TUPLE

Let $n \in \mathbb{N}_{\geq 1}$.

$\langle x_1, x_2 \rangle = \{\{x_1\}, \{x_1, x_2\}\}$

$\langle x_1, \ldots, x_{i(n)} \rangle = \langle \langle x_1, \ldots, x_n \rangle, x_{i(n)} \rangle$

DEFINITION XXIX

Let $k, n \in \mathbb{N}_{\geq 1}$.

$\langle x_1, \ldots, x_k \rangle_n = x_n$

DEFINITION XXX

Let $n \in \mathbb{N}_{\geq 1}$.

$\lozenge(\langle x_1, \ldots, x_n \rangle) = \{x_1, \ldots, x_n\}$

DEFINITION XXXI — CARTESIAN PRODUCT

$(X \times Y) = \{\langle x, y \rangle \mid ((x \in X) \wedge (y \in Y))\}$

DEFINITION XXXII — FUNCTION SPACE

${}^X Y = \{(f \subseteq (X \times Y)) \mid \forall x \in X(\exists_! y \in Y((\langle x, y \rangle \in f)))\}$

DEFINITION XXXIII — FUNCTION

$(f : X \to Y) \Leftrightarrow (f \in {}^X Y)$

$(f(x) = y) \Leftrightarrow (\langle x, y \rangle \in f)$

DEFINITION XXXIV — IDENTITY FUNCTION

$\mathrm{id} : X \to X$

$\mathrm{id}(x) = x$

DEFINITION XXXV — SET OF MUTUALLY EXCLUSIVE FORMULAS

Let $\Phi \in \mathcal{F}_1$.

$\bigcirc\!\!\!\!\!\not\curlywedge\,(\Phi) \Leftrightarrow \forall ``\varphi", ``\varphi'" \in \Phi(((``\varphi" \neq ``\varphi'") \Rightarrow (\varphi \not\curlywedge \varphi')))$

DEFINITION XXXVI — PIECEWISE FUNCTION

Let $n \in \mathbb{N}$, and $\bigcirc\!\!\!\!\!\not\curlywedge\,(\{\varphi_1, \ldots, \varphi_n\})$.

$$\left(f(x) = \begin{cases} y_1, & \varphi_1 \\ \vdots & \vdots \\ y_n, & \varphi_n \end{cases}\right) \Leftrightarrow \left(f = \bigcup\{\{\langle x, y_i \rangle \mid \varphi_i\} \mid (i \in \mathbb{N}_{[1,n]})\}\right)$$

DEFINITION XXXVII — n-LENGTH STRING

Let $n \in \mathbb{N}$.

$"x_1 \dots x_n" = \langle "x_1", \dots, "x_n" \rangle$

DEFINITION XXXVIII — SPACE OF n-LENGTH STRINGS

Let $n \in \mathbb{N}$.

$A^n = (\underbrace{A \times \cdots \times A}_{n \text{ times}})$

DEFINITION XXXIX — SPACE OF FINITE-LENGTH STRINGS

$A^* = \bigcup\{A^n \mid (n \in \mathbb{N})\}$

NOTATION XIV

Let $k, n \in \mathbb{N}_{\geq 1}$.

$\forall(x_1, \dots, x_k) \in (X_1, \dots, X_n)(\varphi) \lessdot \forall x_{1,1}, \dots, x_{1,k} \in X_1(\cdots \forall x_{n,1}, \dots, x_{n,k} \in X_n(\varphi))$

DEFINITION XL — $|k \times n|$-ARY SET CLOSURE

Let $k, n \in \mathbb{N}_{\geq 1}$.

$\mathscr{C}_n^k(X_1, \dots, X_n, F) \Leftrightarrow \forall(x_1, \dots, x_k) \in (X_1, \dots, X_n)(\forall f \in F((f(x_{1,1}, \dots, x_{n,k}) \in X_n)))$

If a book is worth writing, it is worth re-writing.

PAUL ATTEWELL

5

Metalanguage

Previously, we defined propositional and predicate logics. In this chapter, we shall define propositional and predicate metalogics that mimic, and thus help to more formally define, their logical counterparts. To avoid circularity, metalogic ought to be distinguished from logic. However, in the interest of brevity, we shall, henceforth, not always make this distinction as apparently as one might hope.

5.1 Propositional Logic

5.1.1 Syntax

DEFINITION XLI — SPACE OF PROPOSITIONAL VARIABLES

$\mathcal{X}_0 = \bigcup\{\text{"}p_1\text{"}, \ldots, \text{"}p_n\text{"} \mid (n \in \mathbb{N})\}$

DEFINITION XLII — CONCATENATION FUNCTION OF NEGATION

$\mathrm{neg} : \Phi \to \Phi$

$\mathrm{neg}(\text{"}\varphi\text{"}) = \text{"}(\neg\varphi)\text{"}$

DEFINITION XLIII — CONCATENATION FUNCTION OF CONJUNCTION

$\mathrm{conj} : \Phi^2 \to \Phi$

$\mathrm{conj}(\text{"}\varphi\text{"}, \text{"}\varphi'\text{"}) = \text{"}(\varphi \wedge \varphi')\text{"}$

DEFINITION XLIV — SPACE OF PROPOSITIONAL CHARACTERS

$\mathcal{C}_0 = (\mathcal{X}_0 \cup \{\text{"}\neg\text{"}, \text{"}\wedge\text{"}, \text{"}(\text{"}, \text{"})\text{"}\})$

DEFINITION XLV — SPACE OF PROPOSITIONAL FORMULAS

$\mathcal{F}_0 = \bigcap\{(\Phi \subseteq \mathcal{C}_0^*) \mid (((\mathcal{X}_0 \subseteq \Phi) \wedge \mathscr{C}_1^1(\Phi, \{\mathrm{neg}\})) \wedge \mathscr{C}_1^2(\Phi, \{\mathrm{conj}\}))\}$

5.1.2 Semantics

DEFINITION XLVI — TRUTH FUNCTION OF NEGATION

$\mathrm{not} : \{\mathbf{T}, \mathbf{F}\} \to \{\mathbf{T}, \mathbf{F}\}$

$\mathrm{not}(\mathbf{T}) = \mathbf{F}$

$\mathrm{not}(\mathbf{F}) = \mathbf{T}$

DEFINITION XLVII — TRUTH FUNCTION OF CONJUNCTION

$\mathrm{and} : \{\mathbf{T}, \mathbf{F}\}^2 \to \{\mathbf{T}, \mathbf{F}\}$

$\mathrm{and}(\mathbf{T}, \mathbf{T}) = \mathbf{T}$

$\mathrm{and}(\mathbf{T}, \mathbf{F}) = \mathbf{F}$

$\mathrm{and}(\mathbf{F}, \mathbf{T}) = \mathbf{F}$

$\mathrm{and}(\mathbf{F}, \mathbf{F}) = \mathbf{F}$

DEFINITION XLVIII — SPACE OF TRUTH ASSIGNMENTS

$\mathscr{T}_0 = {}^{\mathscr{X}_0}\{\mathbf{T}, \mathbf{F}\}$

DEFINITION XLIX — PROPOSITIONAL FORMULA EVALUATION FUNCTION

Let $t \in \mathscr{T}_0$.

$\mathscr{v}_0^t : \mathscr{F}_0 \to \{\mathbf{T}, \mathbf{F}\}$

$\mathscr{v}_0^t(p) = t(p)$

$\mathscr{v}_0^t(\mathrm{neg}(\varphi)) = \mathrm{not}(\mathscr{v}_0^t(\varphi))$

$\mathscr{v}_0^t(\mathrm{conj}(\varphi, \varphi')) = \mathrm{and}(\mathscr{v}_0^t(\varphi), \mathscr{v}_0^t(\varphi'))$

5.2 Predicate Logic

5.2.1 Syntax

DEFINITION L — SPACE OF VARIABLES

$$\mathcal{X}_1 = \bigcup\{\,\text{"}x_1\text{"}, \ldots, \text{"}x_n\text{"} \mid (n \in \mathbb{N})\}$$

DEFINITION LI — CONCATENATION FUNCTION OF EQUALITY

$$\text{eq} : X^2 \to \Phi$$

$$\text{eq}(\text{"}x\text{"}, \text{"}x'\text{"}) = \text{"}(x = x')\text{"}$$

DEFINITION LII — CONCATENATION FUNCTION OF MEMBERSHIP

$$\text{in} : X^2 \to \Phi$$

$$\text{in}(\text{"}x\text{"}, \text{"}x'\text{"}) = \text{"}(x \in x')\text{"}$$

DEFINITION LIII — CONCATENATION FUNCTION OF UNIVERSAL QUANTIFICATION

$$\text{forall} : (X \times \Phi) \to \Phi$$

$$\text{forall}(\text{"}x\text{"}, \text{"}\varphi\text{"}) = \text{"}\forall x(\varphi)\text{"}$$

DEFINITION LIV — SPACE OF PREDICATE CHARACTERS

$$\mathcal{C}_1 = (\mathcal{X}_1 \cup \{\text{"}=\text{"}, \text{"}\in\text{"}, \text{"}\neg\text{"}, \text{"}\wedge\text{"}, \text{"}(\text{"}, \text{"})\text{"}\})$$

DEFINITION LV — SPACE OF ATOMIC FORMULAS

$$\mathcal{F}_1^{\odot} = \bigcap\{(\Phi \subseteq \mathcal{C}_1^*) \mid ((\mathcal{X}_1 \subseteq \Phi) \wedge \mathscr{C}_1^2(\Phi, \{\text{eq}, \text{in}\}))\}$$

DEFINITION LVI — SPACE OF FORMULAS

$$\mathcal{F}_1 = \bigcap\{(\Phi \subseteq \mathcal{C}_1^*) \mid ((((\mathcal{F}_1^{\odot} \subseteq \Phi) \wedge \mathscr{C}_1^1(\Phi, \{\text{neg}\})) \wedge \mathscr{C}_1^2(\Phi, \{\text{conj}\})) \wedge \mathscr{C}_2^1(\mathcal{X}_1, \Phi, \{\text{forall}\}))\}$$

DEFINITION LVII — SPACE OF FREE VARIABLES

$$\text{free} : \mathcal{F}_1 \to \mathscr{P}(\mathcal{X}_1)$$

$$\text{free}(\text{eq}(x, x')) = \{x, x'\}$$

$$\text{free}(\text{in}(x, x')) = \{x, x'\}$$

$$\text{free}(\text{neg}(\varphi)) = \text{free}(\varphi)$$

$$\text{free}(\text{conj}(\varphi, \varphi')) = (\text{free}(\varphi) \cup \text{free}(\varphi'))$$

$$\text{free}(\text{forall}(x, \varphi)) = (\text{free}(\varphi) \setminus \{x\})$$

DEFINITION LVIII — SPACE OF SENTENCES

$$\mathcal{S} = \{(\varphi \in \mathcal{F}_1) \mid (\text{free}(\varphi) = \varnothing)\}$$

DEFINITION LIX — SUBSTITUTION

$$\mathrm{eq}(x,x')|_{m\mapsto s} = \begin{cases} \mathrm{eq}(s,s), & ((x=m)\wedge(x'=m)) \\ \mathrm{eq}(s,x'), & ((x=m)\wedge(x'\neq m)) \\ \mathrm{eq}(x,s), & ((x\neq m)\wedge(x'=m)) \\ \mathrm{eq}(x,x'), & ((x\neq m)\wedge(x'\neq m)) \end{cases}$$

$$\mathrm{in}(x,x')|_{m\mapsto s} = \begin{cases} \mathrm{in}(s,s), & ((x=m)\wedge(x'=m)) \\ \mathrm{in}(s,x'), & ((x=m)\wedge(x'\neq m)) \\ \mathrm{in}(x,s), & ((x\neq m)\wedge(x'=m)) \\ \mathrm{in}(x,x'), & ((x\neq m)\wedge(x'\neq m)) \end{cases}$$

$$\mathrm{neg}(\varphi)|_{m\mapsto s} = \mathrm{neg}(\varphi|_{m\mapsto s})$$

$$\mathrm{conj}(\varphi,\varphi')|_{m\mapsto s} = \mathrm{conj}(\varphi|_{m\mapsto s},\varphi'|_{m\mapsto s})$$

$$\mathrm{forall}(x,\varphi)|_{m\mapsto s} = \begin{cases} \mathrm{forall}(x,\varphi), & (x=m) \\ \mathrm{forall}(x,\varphi|_{m\mapsto s}), & ((x\neq m)\wedge(x\notin\Diamond(s))) \\ \mathrm{forall}(x_\ast,\varphi|_{x\mapsto x_\ast}|_{m\mapsto s}), & (((x\neq m)\wedge(x\in\Diamond(s)))\wedge(x_\ast\notin(\Diamond(s)\cup\Diamond(\varphi)))) \end{cases}$$

5.2.2 Semantics

DEFINITION LX — CODOMAIN OF INTERPRETATION FUNCTION

$\mathcal{I}(U, F, R) = \left(\bigcup\{^{U^{a(f)}}U \mid (f \in F)\} \cup \bigcup\{\mathscr{P}(U^{a(r)}) \mid (r \in R)\}\right)$

DEFINITION LXI — SPACE OF STRUCTURES

$\mathcal{M} = \{\langle U, \langle (F \cup R), a\rangle, i\rangle \mid ((((U \neq \varnothing) \wedge ((F \cap R) = \varnothing)) \wedge (a \in {}^{(F \cup R)}\mathbb{N})) \wedge (i \in {}^{(F \cup R)}\mathcal{I}(U, F, R)))\}$

DEFINITION LXII — STRUCTURE OF PREDICATE LOGIC

$(\mathfrak{L} = \langle U_{\mathfrak{L}}, \langle (\varnothing \cup \{\text{"}\in\text{"}\}), a_{\mathfrak{L}}\rangle, i_{\mathfrak{L}}\rangle) \in \mathcal{M}$

$a_{\mathfrak{L}}(\text{"}\in\text{"}) = 2$

$i_{\mathfrak{L}}(\text{"}\in\text{"}) = \{\langle x, X\rangle \mid (x \in X)\}$

DEFINITION LXIII — SPACE OF VARIABLE ASSIGNMENTS

Let $m \in \mathcal{M}$.

$\mathcal{T}_1^m = \left(\bigcup\{^{\mathrm{free}(\varphi)}(m_1) \mid (\varphi \in \mathcal{F}_1)\} \cup \{\mathrm{id}\}\right)$

DEFINITION LXIV — FORMULA EVALUATION FUNCTION

Let $m \in \mathcal{M}$, and $t \in \mathcal{T}_1^m$.

$\mathcal{v}_1^{m,t} : \mathcal{F}_1 \to \{\mathbf{T}, \mathbf{F}\}$

$$\mathcal{v}_1^{m,t}(\mathrm{eq}(x, x')) = \begin{cases} \mathbf{T}, & (v(x) = v(x')) \\ \mathbf{F}, & (v(x) \neq v(x')) \end{cases}$$

$$\mathcal{v}_1^{m,t}(\mathrm{in}(x, x')) = \begin{cases} \mathbf{T}, & (v(x) \in v(x')) \\ \mathbf{F}, & (v(x) \notin v(x')) \end{cases}$$

$\mathcal{v}_1^{m,t}(\mathrm{neg}(\varphi)) = \mathrm{not}(\mathcal{v}_1^{m,t}(\varphi))$

$\mathcal{v}_1^{m,t}(\mathrm{conj}(\varphi, \varphi')) = \mathrm{and}(\mathcal{v}_1^{m,t}(\varphi), \mathcal{v}_1^{m,t}(\varphi'))$

$$\mathcal{v}_1^{m,t}(\mathrm{forall}(x, \varphi)) = \begin{cases} \mathbf{T}, & \forall s \in m_1((\mathcal{v}_1^{m,t}(\varphi|_{x \mapsto s}) = \mathbf{T})) \\ \mathbf{F}, & (\neg\forall s \in m_1((\mathcal{v}_1^{m,t}(\varphi|_{x \mapsto s}) = \mathbf{T}))) \end{cases}$$

6

Interlude II

DEFINITION LXV — FUNCTION COMPOSITION

Let $f : X \to Y$, and $g : Y \to Z$.

$(g \circ f) = \{\langle x, z\rangle \mid ((\langle x, y\rangle \in f) \land (\langle y, z\rangle \in g))\}$

DEFINITION LXVI

Let $c : X \to X$.

$c^{[n]} = (\underbrace{c \circ \cdots \circ c}_{n \text{ times}})$

DEFINITION LXVII — INJECTIVE FUNCTION

Let $f : X \to Y$.

$\text{injective}(f) \Leftrightarrow \forall y \in Y(\mathrm{E}x \in X((f(x) = y)))$

DEFINITION LXVIII — SURJECTIVE FUNCTION

Let $f : X \to Y$.

$\text{surjective}(f) \Leftrightarrow \forall y \in Y(\exists x \in X((f(x) = y)))$

DEFINITION LXIX — BIJECTIVE FUNCTION

Let $f : X \to Y$.

$\text{bijective}(f) \Leftrightarrow (\text{injective}(f) \land \text{surjective}(f))$

DEFINITION LXX — FINITE SET

$(|X| < \infty) \Leftrightarrow \exists n \in \mathbb{N}(\exists f \in {}^{X}\mathbb{N}_{\leq n}(\text{bijective}(f)))$

7

Adequacy

Adequacy is also known as *functional completeness*.

THEOREM I — ADEQUACY OF NEGATION WITH CONJUNCTION

$\forall f \in \bigcup\{{}^{\{\mathbf{T},\mathbf{F}\}^n}\{\mathbf{T},\mathbf{F}\} \mid (n \in \mathbb{N})\}(\exists g \in \bigcap\{X \mid (((\{\mathbf{T},\mathbf{F}\} \subseteq X) \land \mathscr{C}_1^1(X, \mathrm{not})) \land \mathscr{C}_1^2(X, \mathrm{and}))\}((f = g)))$

8

Satisfiability and Definability

A logical formula is said to be *tautological* only if it is "always true", *satisfiable* only if it is "sometimes true", and *contradictory* only if it is "never true". The property of being tautological can be seen as *opposite* to the property of being contradictory, while the property of being satisfiable can be seen as *complementary* to the property of being contradictory. A set is said to be *definable* only if there exists a logical formula whose truth is equivalent to existence of the set.

8.1 Propositional Logic

DEFINITION LXXI — TAUTOLOGICAL SET OF PROPOSITIONAL FORMULAS

Let $\Phi \subseteq \mathscr{F}_0$.

$\forall t \in \mathscr{T}_0(\forall \varphi \in \Phi((\mathfrak{v}_0^t(\varphi) = \mathbf{T})))$

DEFINITION LXXII — SATISFIABLE SET OF PROPOSITIONAL FORMULAS

Let $\Phi \subseteq \mathscr{F}_0$.

$\exists t \in \mathscr{T}_0(\forall \varphi \in \Phi((\mathfrak{v}_0^t(\varphi) = \mathbf{T})))$

DEFINITION LXXIII — CONTRADICTORY SET OF PROPOSITIONAL FORMULAS

Let $\varphi \subseteq \mathscr{F}_0$.

$\forall t \in \mathscr{T}_0(\forall \varphi \in \Phi((\mathfrak{v}_0^t(\varphi) = \mathbf{F})))$

DEFINITION LXXIV — DEFINABLE SET OF TRUTH ASSIGNMENTS

Let $T \subseteq \mathscr{T}_0$.

$\exists \varphi \in \mathscr{F}_0(\forall t \in T((\mathfrak{v}_0^t(\varphi) = \mathbf{T})))$

DEFINITION LXXV — SUBJECT OF SET OF PROPOSITIONAL FORMULAS

Let $\Phi \in \mathscr{F}_0$.

$\text{subject}_0(\Phi) = \{(t \in \mathscr{T}_0) \mid \forall \varphi \in \Phi((\mathfrak{v}_0^t(\varphi) = \mathbf{T}))\}$

DEFINITION LXXVI — THEORY OF SET OF TRUTH ASSIGNMENTS

Let $T \in \mathscr{T}_0$.

$\text{theory}_0(T) = \{(\varphi \in \mathscr{F}_0) \mid \forall t \in T((\mathfrak{v}_0^t(\varphi) = \mathbf{T}))\}$

THEOREM II — EXISTENCE OF UNSATISFIABLE SET OF PROPOSITIONAL FORMULAS

$\exists \Phi \subseteq \mathscr{F}_0((\text{subject}_0(\Phi) = \varnothing))$

THEOREM III — EXISTENCE OF UNDEFINABLE SET OF TRUTH ASSIGNMENTS

$\exists T \subseteq \mathscr{T}_0((\text{theory}_0(T) = \varnothing))$

THEOREM IV

$\forall \Phi \subseteq \mathscr{F}_0(\forall T \subseteq \mathscr{T}_0((\Phi \subseteq \text{theory}_0(T)) \Leftrightarrow (T \subseteq \text{subject}_0(\Phi))))$

8.2 Predicate Logic

DEFINITION LXXVII — TAUTOLOGICAL SET OF FORMULAS

Let $\Phi \subseteq \mathcal{F}_1$.

$\forall m \in \mathcal{M}(\forall t \in \mathcal{T}_1^m(\forall \varphi \in \Phi((\mathfrak{v}_1^{m,t}(\varphi) = \mathbf{T}))))$

DEFINITION LXXVIII — SATISFIABLE SET OF SENTENCES

Let $\Phi \subseteq \mathcal{S}$.

$\exists m \in \mathcal{M}(\forall \varphi \in \Phi((\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = \mathbf{T})))$

DEFINITION LXXIX — SATISFIABLE SET OF FORMULAS

Let $m \in \mathcal{M}$, and $\Phi \subseteq \mathcal{F}_1$.

$\exists t \in \mathcal{T}_1^m(\forall \varphi \in \Phi((\mathfrak{v}_1^{m,t}(\varphi) = \mathbf{T})))$

DEFINITION LXXX — CONTRADICTORY SET OF FORMULAS

Let $\Phi \subseteq \mathcal{F}_1$.

$\forall m \in \mathcal{M}(\forall t \in \mathcal{T}_1^m(\forall \varphi \in \Phi((\mathfrak{v}_1^{m,t}(\varphi) = \mathbf{F}))))$

DEFINITION LXXXI — DEFINABLE SET OF STRUCTURES

Let $M \subseteq \mathcal{M}$.

$\exists \varphi \in \mathcal{S}(\forall m \in M((\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = \mathbf{T})))$

DEFINITION LXXXII — DEFINABLE SET OF VARIABLE ASSIGNMENTS

Let $m \in \mathcal{M}$, and $T \subseteq \mathcal{T}_1^m$.

$\exists \varphi \in \mathcal{F}_1(\forall t \in T((\mathfrak{v}_1^{m,t}(\varphi) = \mathbf{T})))$

DEFINITION LXXXIII — SUBJECT OF SET OF SENTENCES

Let $\Phi \subseteq \mathcal{S}$.

$\mathrm{subject}_1(\Phi) = \{(m \in \mathcal{M}) \mid \forall \varphi \in \Phi((\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = \mathbf{T}))\}$

DEFINITION LXXXIV — SUBJECT OF SET OF FORMULAS

Let $m \in \mathcal{M}$, and $\Phi \subseteq \mathcal{F}_1$.

$\mathrm{subject}_1^m(\Phi) = \{(t \in \mathcal{T}_1^m) \mid \forall \varphi \in \Phi((\mathfrak{v}_1^{m,t}(\varphi) = \mathbf{T}))\}$

DEFINITION LXXXV — THEORY OF SET OF STRUCTURES

Let $M \subseteq \mathcal{M}$.

$\mathrm{theory}_1(M) = \{(\varphi \in \mathcal{S}) \mid \forall m \in M((\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = \mathbf{T}))\}$

DEFINITION LXXXVI — THEORY OF SET OF VARIABLE ASSIGNMENTS

Let $m \in \mathcal{M}$, and $T \subseteq \mathcal{T}_1^m$.

$\text{theory}_1^m(T) = \{(\varphi \in \mathcal{F}_1) \mid \forall t \in T((v_1^{m,t}(\varphi) = \mathbf{T}))\}$

THEOREM V — EXISTENCE OF UNSATISFIABLE SET OF SENTENCES

$\exists \Phi \subseteq \mathcal{S}((\text{subject}_1(\Phi) = \varnothing))$

THEOREM VI — EXISTENCE OF UNDEFINABLE SET OF STRUCTURES

$\exists M \subseteq \mathcal{M}((\text{theory}_1(M) = \varnothing))$

THEOREM VII — EXISTENCE OF UNSATISFIABLE SET OF FORMULAS

$\forall m \in \mathcal{M}(\exists \Phi \subseteq \mathcal{F}_1((\text{subject}_1^m(\Phi) = \varnothing)))$

THEOREM VIII — EXISTENCE OF UNDEFINABLE SET OF VARIABLE ASSIGNMENTS

$\forall m \in \mathcal{M}(\exists T \subseteq \mathcal{T}_1^m((\text{theory}_1^m(T) = \varnothing)))$

THEOREM IX

$\forall \Phi \subseteq \mathcal{S}(\forall M \subseteq \mathcal{M}((\Phi \subseteq \text{theory}_1(M)) \Leftrightarrow (M \subseteq \text{subject}_1(\Phi))))$

THEOREM X

$\forall m \in \mathcal{M}(\forall \Phi \subseteq \mathcal{F}_1(\forall T \subseteq \mathcal{T}_1^m((\Phi \subseteq \text{theory}_1^m(T)) \Leftrightarrow (T \subseteq \text{subject}_1^m(\Phi)))))$

9

Soundness and Completeness

For a logic, soundness and completeness concern the truth of every proof, and the proof of every truth, respectively. Soundness can be seen as the property that every proof has truth, while completeness can be seen as the property that every truth has proof. Taken together, soundness and completeness establish a correspondence between notions of proof, which are *syntactic*, and notions of truth, which are *semantic*.

DEFINITION LXXXVII — MODUS PONENS INFERENCE FUNCTION

$\text{ponens} : \Phi^2 \to \Phi$

$\text{ponens}(\text{"}\varphi\text{"}, \text{"}(\varphi \Rightarrow \varphi')\text{"}) = \text{"}\varphi'\text{"}$

9.1 Propositional Logic

DEFINITION LXXXVIII — SPACE OF AXIOMS FOR PROPOSITIONAL LOGIC

$\mathcal{A}_0' = \{\text{"}(\varphi \Rightarrow (\psi \Rightarrow \varphi))\text{"} \mid (\varphi, \psi \in \mathcal{F}_0)\}$

$\mathcal{A}_0'' = \{\text{"}((\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi)))\text{"} \mid (\varphi, \psi, \chi \in \mathcal{F}_0)\}$

$\mathcal{A}_0''' = \{\text{"}(((\neg\varphi) \Rightarrow (\neg\psi)) \Rightarrow (\psi \Rightarrow \varphi))\text{"} \mid (\varphi, \psi \in \mathcal{F}_0)\}$

$\mathcal{A}_0 = \bigcup\{\mathcal{A}_0', \mathcal{A}_0'', \mathcal{A}_0'''\}$

DEFINITION LXXXIX — PROOF SYSTEM FOR PROPOSITIONAL LOGIC

Let $\Gamma \subseteq \mathcal{F}_0$.

$\mathcal{P}_0(\Gamma) = \bigcap\{(\Phi \subseteq C_0^*) \mid (((\mathcal{A}_0 \cup \Gamma) \subseteq \Phi) \wedge \mathcal{C}_1^2(\Phi, \{\text{ponens}\}))\}$

DEFINITION XC

Let $\Gamma, \Phi \subseteq \mathcal{F}_0$.

$(\Gamma \vdash_0 \Phi) \Leftrightarrow \forall\varphi \in \Phi((\varphi \in \mathcal{P}_0(\Gamma)))$

DEFINITION XCI

Let $\Gamma, \Phi \subseteq \mathcal{F}_0$.

$(\Gamma \vDash_0 \Phi) \Leftrightarrow \forall t \in \mathcal{T}_0(\forall\gamma \in \Gamma(\forall\varphi \in \Phi(((\mathfrak{v}_0^t(\gamma) = \mathbf{T}) \Rightarrow (\mathfrak{v}_0^t(\varphi) = \mathbf{T})))))$

THEOREM XI — FINITARYNESS OF PROOF SYSTEM FOR PROPOSITIONAL LOGIC

$\forall\Gamma \subseteq \mathcal{F}_0(\forall\varphi \in \mathcal{P}_0(\Gamma)(\exists\Psi \subseteq \mathcal{P}_0(\Gamma)(((|\Psi| < \infty) \wedge \forall\psi \in \Psi(((\psi = \varphi) \vee (\psi \in (\mathcal{A}_0 \cup \Gamma))))))))$

THEOREM XII — SOUNDNESS OF PROOF SYSTEM FOR PROPOSITIONAL LOGIC

$\forall\Gamma, \Phi \subseteq \mathcal{F}_0((\Gamma \vdash_0 \Phi) \Rightarrow (\Gamma \vDash_0 \Phi))$

THEOREM XIII — COMPLETENESS OF PROOF SYSTEM FOR PROPOSITIONAL LOGIC

$\forall\Gamma, \Phi \subseteq \mathcal{F}_0((\Gamma \vDash_0 \Phi) \Rightarrow (\Gamma \vdash_0 \Phi))$

DEFINITION XCII — CONSISTENT SET OF PROPOSITIONAL FORMULAS

Let $\Gamma \subseteq \mathcal{F}_0$.

$\text{consistent}_0(\Gamma) \Leftrightarrow \forall\varphi \in \mathcal{F}_0(\neg(((\Gamma \vdash_0 \varphi) \wedge (\Gamma \nvdash_0 \varphi))))$

DEFINITION XCIII — SATISFIABLE SET OF PROPOSITIONAL FORMULAS

Let $\Gamma \subseteq \mathcal{F}_0$.

$\text{satisfiable}_0(\Gamma) \Leftrightarrow \forall\varphi \in \mathcal{F}_0(\neg(((\Gamma \vDash_0 \varphi) \wedge (\Gamma \nvDash_0 \varphi))))$

THEOREM XIV

$\forall\Gamma \subseteq \mathcal{F}_0((\text{consistent}_0(\Gamma) \Leftrightarrow \text{satisfiable}_0(\Gamma)))$

9.2 Predicate Logic

DEFINITION XCIV — SPACE OF AXIOMS FOR PREDICATE LOGIC

$\mathcal{A}_1' = \{\tau|_{p\mapsto\varphi} \mid (((\varnothing \vDash_0 \tau) \wedge (p \in \mathcal{X}_0)) \wedge (\varphi \in \mathcal{F}_1))\}$

$\mathcal{A}_1'' = \{"(\forall x(\varphi) \Rightarrow \varphi|_{x\mapsto x'})" \mid ((x, x' \in \mathcal{X}_1) \wedge \varphi \in \mathcal{F}_1)\}$

$\mathcal{A}_1''' = \{"(\varphi \Rightarrow \forall x(\varphi|_{x\mapsto x'}))" \mid ((x, x' \in \mathcal{X}_1) \wedge \varphi \in \mathcal{F}_1)\}$

$\mathcal{A}_1 = \bigcup\{\mathcal{A}_1', \mathcal{A}_1'', \mathcal{A}_1'''\}$

DEFINITION XCV — PROOF SYSTEM FOR PREDICATE LOGIC

Let $\Gamma \subseteq \mathcal{F}_1$.

$\mathcal{P}_1(\Gamma) = \bigcap\{(\Phi \subseteq C_1^*) \mid (((\mathcal{A}_1 \cup \Gamma) \subseteq \Phi) \wedge \mathscr{C}_1^2(\Phi, \{\text{ponens}\}))\}$

DEFINITION XCVI

Let $\Gamma, \Phi \subseteq \mathcal{F}_1$.

$(\Gamma \vdash_1 \Phi) \Leftrightarrow \forall\varphi \in \Phi((\varphi \in \mathcal{P}_1(\Gamma)))$

DEFINITION XCVII

Let $\Gamma, \Phi \subseteq \mathcal{F}_1$.

$(\Gamma \vDash_1 \Phi) \Leftrightarrow \forall m \in \mathcal{M}(\forall v \in \mathcal{T}_1^m(\forall\gamma \in \Gamma(\forall\varphi \in \Phi((\mathfrak{v}_1^{m,v}(\gamma) = \mathbf{T}) \Rightarrow (\mathfrak{v}_1^{m,v}(\varphi) = \mathbf{T})))))$

THEOREM XV — FINITARYNESS OF PROOF SYSTEM FOR PREDICATE LOGIC

$\forall\Gamma \subseteq \mathcal{F}_1(\forall\varphi \in \mathcal{P}_1(\Gamma)(\exists\Psi \subseteq \mathcal{P}_1(\Gamma)(((|\Psi| < \infty) \wedge \forall\psi \in \Psi(((\psi = \varphi) \vee (\psi \in (\mathcal{A}_1 \cup \Gamma))))))))$

THEOREM XVI — SOUNDNESS OF PROOF SYSTEM FOR PREDICATE LOGIC

$\forall\Gamma, \Phi \subseteq \mathcal{F}_1((\Gamma \vdash_1 \Phi) \Rightarrow (\Gamma \vDash_1 \Phi))$

THEOREM XVII — COMPLETENESS OF PROOF SYSTEM FOR PREDICATE LOGIC

$\forall\Gamma, \Phi \subseteq \mathcal{F}_1((\Gamma \vDash_1 \Phi) \Rightarrow (\Gamma \vdash_1 \Phi))$

DEFINITION XCVIII — CONSISTENT SET OF FORMULAS

Let $\Gamma \subseteq \mathcal{F}_1$.

$\text{consistent}_1(\Gamma) \Leftrightarrow \forall\varphi \in \mathcal{F}_1(\neg(((\Gamma \vdash_1 \varphi) \wedge (\Gamma \nvdash_1 \varphi))))$

DEFINITION XCIX — SATISFIABLE SET OF FORMULAS

Let $\Gamma \subseteq \mathcal{F}_1$.

$\text{satisfiable}_1(\Gamma) \Leftrightarrow \forall\varphi \in \mathcal{F}_1(\neg(((\Gamma \vDash_1 \varphi) \wedge (\Gamma \nvDash_1 \varphi))))$

THEOREM XVIII

$\forall\Gamma \subseteq \mathcal{F}_1((\text{consistent}_1(\Gamma) \Leftrightarrow \text{satisfiable}_1(\Gamma)))$

10

Compactness and Maximisability

Previously, we established equivalence between consistency and satisfiability. In the interest of brevity, we shall, henceforth, occasionally omit discussion of consistency in favor of satisfiability, keeping in mind that every property which applies to the latter, also applies to the former, in equivalent fashion.

10.1 Propositional Logic

THEOREM XIX — PROPOSITIONAL COMPACTNESS

Let $\Gamma \subseteq \mathscr{F}_0$.

$\mathrm{satisfiable}_0(\Gamma) \Leftrightarrow \forall \Gamma' \subseteq \Gamma(((|\Gamma'| < \infty) \wedge \mathrm{satisfiable}_0(\Gamma')))$

DEFINITION C — MAXIMAL SET OF PROPOSITIONAL FORMULAS

Let $\Gamma \subseteq \mathscr{F}_0$.

$\mathrm{maximal}_0(\Gamma) \Leftrightarrow \forall \varphi \in \mathscr{F}_0(((\Gamma \vDash_0 \varphi) \vee (\Gamma \nvDash_0 \varphi)))$

THEOREM XX — PROPOSITIONAL LINDENBAUM

$\forall \Gamma \subseteq \mathscr{F}_0((\mathrm{satisfiable}_0(\Gamma) \Leftrightarrow \exists \Gamma' \supseteq \Gamma((\mathrm{satisfiable}_0(\Gamma') \wedge \mathrm{maximal}_0(\Gamma')))))$

10.2 Predicate Logic

THEOREM XXI — PREDICATE COMPACTNESS

$\forall \Gamma \subseteq \mathcal{F}_1((\text{satisfiable}_1(\Gamma) \Leftrightarrow \forall \Gamma' \subseteq \Gamma(((|\Gamma'| < \infty) \land \text{satisfiable}_1(\Gamma')))))$

DEFINITION CI — MAXIMAL SET OF FORMULAS

Let $\Gamma \subseteq \mathcal{F}_1$.

$\text{maximal}_1(\Gamma) \Leftrightarrow \forall \varphi \in \mathcal{F}_1(((\Gamma \vDash_1 \varphi) \lor (\Gamma \not\vDash_1 \varphi)))$

THEOREM XXII — PREDICATE LINDENBAUM

$\forall \Gamma \subseteq \mathcal{F}_1((\text{satisfiable}_1(\Gamma) \Leftrightarrow \exists \Gamma' \supseteq \Gamma((\text{maximal}_1(\Gamma) \land \text{satisfiable}_1(\Gamma)))))$

Everything fits in the number.

IAMBLICHUS

11

Gödel Incompleteness

The space of sentences which are true for the natural numbers, is known as the *theory of arithmetic*, or *number theory*. Historically, the theory of arithmetic has been regarded as a "staple" of mathematics. Previously, we defined a proof system which is finitary, sound, and complete with respect to the space of tautological formulas. In the 1920s, there was an interest in founding mathematics upon formal methods of proof. In particular, it was wondered whether one could develop a proof system which is verifiable, sound, and complete with respect to theories of mathematics. Around the 1930s, Kurt Gödel showed that, if "verifiable" is taken to mean "recursively axiomatisable", then no such proof system exists for the theory of arithmetic. He also demonstrated that no consistent proof system can simultaneously prove its *own* consistency, and derive the theory of arithmetic.

DEFINITION CII — STRUCTURE OF NATURAL NUMBERS

$(\mathfrak{N} = \langle \mathbb{N}, \langle (\{"0", "1", "+", "\times"\} \cup \varnothing), a_{\mathfrak{N}} \rangle, i_{\mathfrak{N}} \rangle) \in \mathcal{M}$

$a_{\mathfrak{N}}("0") = 0$

$a_{\mathfrak{N}}("1") = 0$

$a_{\mathfrak{N}}("+") = 2$

$a_{\mathfrak{N}}("\times") = 2$

$i_{\mathfrak{N}}("0") = 0$

$i_{\mathfrak{N}}("1") = 1$

$i_{\mathfrak{N}}("+") = \{\langle x, x', i^{[x']}(x) \rangle \mid (x, x', y \in \mathbb{N})\}$

$i_{\mathfrak{N}}("\times") = \{\langle x, x', (i^{[x']})^{[x']}(x) \rangle \mid (x, x', y \in \mathbb{N})\}$

THEOREM XXIII — FIRST GÖDEL INCOMPLETENESS

$\nexists \Gamma \subseteq \mathcal{F}_1 (\forall \Phi \subseteq \mathrm{th}_1(\{\mathfrak{N}\})(((\Gamma \vDash_1 \Phi) \Rightarrow (\Gamma \vdash_1 \Phi))))$

THEOREM XXIV — SECOND GÖDEL INCOMPLETENESS

$\forall \Gamma \subseteq \mathcal{F}_1(((\mathrm{th}_1(\{\mathfrak{N}\}) \subseteq \mathcal{P}_1(\Gamma)) \Rightarrow (\mathrm{cons}_1(\mathcal{P}_1(\Gamma)) \Leftrightarrow ("\mathrm{cons}_1(\mathcal{P}_1(\Gamma))" \notin \mathcal{P}_1(\Gamma)))))$