# CLASSICAL LOGIC

GREGORY LIM

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# Preface

*The unexamined life is not worth living.*

— SOCRATES

What could it mean for something to be *true*? Throughout history, various definitions for the word "truth" have been proposed.

**DEFINITION I** — CORRESPONDENCE THEORY OF TRUTH
Truth is that which corresponds to reality.

**DEFINITION II** — COHERENCE THEORY OF TRUTH
Truth is that which coheres with every other truth.

Ought mathematics concern itself with the universe we inhabit? In any case, the practice of mathematics seems to be based on something like **DEFINITION II**; this presents a *seeming* trilemma.

**DEFINITION III** — MÜNCHHAUSEN TRILEMMA
Every proof is completed by circularity, infinite regress, and/or assumption.

Reasoning by coherence appears to obtain truth by circularity, infinite regress, and/or assumption. It is no secret that mathematical theories admit assumptions. Mathematical theories are also said to build upon, and thus derive from, each other. A theory from which other theories can be derived is said to be *foundational*.

Broadly speaking, a logic can be thought of as a language for reasoning about truth — that is, a system which prescribes symbols, and ways of interchanging those symbols. This text studies propositional and predicate logics, in particular.

With any human language, definitions may be said to originate in *pre-linguistic* thought. Writers sometimes choose their starting points based on what feels the most intuitive to them. For this text, I have decided to use words and phrases like "not", "and", "if... then", "either... or", "otherwise", "every", "same", in addition to a few other mathematical symbols, in my appeal to the intuition of the reader.

# 2

# Language

*There are three rules for writing a novel. Unfortunately, no one knows what they are.*

— SOMERSET MAUGHAM

A propositional logic is sometimes called a *zeroth-order calculus*, and a predicate logic is sometimes called a *first-order calculus*. A predicate logic can be said to "extend" a propositional logic.

Propositional variables and (predicate) variables are known as *logical variables*. Similarly, propositional formulas and (predicate) formulas are known as *logical formulas*. Logical formulas can be recursively defined from (their) logical variables.

## 2.1 Propositional Logic

### 2.1.1 Syntax

**DEFINITION IV** — PROPOSITIONAL FORMULA

Let $p_\circ, \ldots, p_\bullet$ be propositional variables.

1. If $p$ is a propositional variable, then $p$ is a propositional formula.
2. If $\varphi$ is a propositional formula, then $(\neg\varphi)$ is a propositional formula.
   If $\varphi$ and $\varphi'$ are propositional formulas, then $(\varphi \wedge \varphi')$ is a propositional formula.

### 2.1.2 Semantics

**DEFINITION V** — TRUTH VALUE OF PROPOSITIONAL FORMULA

1. Every propositional variable is either assigned true, or assigned false.
2. If $\varphi$ is assigned true, then $(\neg\varphi)$ is assigned false.
   Otherwise, $(\neg\varphi)$ is assigned true.
   If $\varphi$ and $\varphi'$ are assigned true, then $(\varphi \land \varphi')$ is assigned true.
   Otherwise, $(\varphi \land \varphi')$ is assigned false.

## 2.2 Predicate Logic

### 2.2.1 Syntax

**DEFINITION VI** — FORMULA

Let $x_\circ, \ldots, x_\bullet$ be variables.

1. If $x$ is a variable, and $x'$ is a variable, then $(x = x')$ is a formula.
   If $x$ is a variable, and $x'$ is a variable, then $(x \in x')$ is a formula.
2. If $\varphi$ is a formula, then $(\neg\varphi)$ is a formula.
   If $\varphi$ is a formula, and $\varphi'$ is a formula, then $(\varphi \wedge \varphi')$ is a formula.
   If $x$ is a variable, and $\varphi$ is a formula, then $\forall x(\varphi)$ is a formula.

**DEFINITION VII** — FREE VARIABLE

1. If $(x = x')$ is a formula, then $x$ and $x'$ are free variables in the formula.
   If $(x \in x')$ is a formula, then $x$ and $x'$ are free variables in the formula.
2. If $x$ is a free variable in $\varphi$, then $x$ is a free variable in $(\neg\varphi)$.
   If $x$ is a free variable in $\varphi$ and $\varphi'$, then $x$ is a free variable in $(\varphi \wedge \varphi')$.
   If $x$ is a free variable in $\varphi$, and $(x \neq x')$, then $x$ is a free variable in $\forall x'(\varphi)$.

**NOTATION I** — NAÏVE SUBSTITUTION

$((x = x))_{x \rightsquigarrow \tau} \lessdot (\tau = \tau)$
$((x = x'))_{x \rightsquigarrow \tau} \lessdot (\tau = x')$
$((x' = x))_{x \rightsquigarrow \tau} \lessdot (x' = \tau)$
$((x' = x'))_{x \rightsquigarrow \tau} \lessdot (x' = x')$
$((x \in x))_{x \rightsquigarrow \tau} \lessdot (\tau \in \tau)$
$((x \in x'))_{x \rightsquigarrow \tau} \lessdot (\tau \in x')$
$((x' \in x))_{x \rightsquigarrow \tau} \lessdot (x' \in \tau)$
$((x' \in x'))_{x \rightsquigarrow \tau} \lessdot (x' \in x')$
$((\neg\varphi))_{x \rightsquigarrow \tau} \lessdot (\neg(\varphi)_{x \rightsquigarrow \tau})$
$((\varphi \wedge \varphi'))_{x \rightsquigarrow \tau} \lessdot ((\varphi)_{x \rightsquigarrow \tau} \wedge (\varphi')_{x \rightsquigarrow \tau})$
$(\forall x(\varphi))_{x \rightsquigarrow \tau} \lessdot \forall \tau((\varphi)_{x \rightsquigarrow \tau})$

**NOTATION II**

$(x \not\circledast x') \lessdot (\neg(x \circledast x'))$

**NOTATION III**

$(\varphi \Rightarrow \varphi') \lessdot (\neg(\varphi \wedge (\neg\varphi')))$

**NOTATION IV**

$(\varphi \Leftrightarrow \varphi') \lessdot ((\varphi \Rightarrow \varphi') \wedge (\varphi' \Rightarrow \varphi))$

**NOTATION V**

$(\varphi \oplus \varphi') \lessdot ((\varphi \vee \varphi') \wedge \neg((\varphi \wedge \varphi')))$

**NOTATION VI**

$\exists x(\varphi) \lessdot (\neg\forall x((\neg\varphi)))$

**NOTATION VII**

$\exists_{!} x(\varphi) \lessdot \exists x(\forall x'(((\varphi)_{x}^{x'} \Leftrightarrow (x = x'))))$

**NOTATION VIII**

$\mathsf{E} x(\varphi) \lessdot (\nexists x(\varphi) \oplus \exists_{!} x(\varphi))$

**NOTATION IX**

$\forall x_{\circ}, \ldots, x_{\bullet} \circledast X(\varphi) \lessdot \forall x_{\circ}(((x_{\circ} \circledast X) \Rightarrow \cdots \Rightarrow \forall x_{\bullet}(((x_{\bullet} \in X) \Rightarrow \varphi))))$

**NOTATION X**

$\exists x_{\circ}, \ldots, x_{\bullet} \circledast X(\varphi) \lessdot \exists x_{\circ}(((x_{\circ} \circledast X) \wedge \cdots \wedge \exists x_{\bullet}(((x_{\bullet} \in X) \wedge \varphi))))$

**NOTATION XI**

$\exists_{!} x_{\circ}, \ldots, x_{\bullet} \circledast X(\varphi) \lessdot \exists_{!} x_{\circ}(((x_{\circ} \circledast X) \wedge \cdots \wedge \exists_{!} x_{\bullet}(((x_{\bullet} \in X) \wedge \varphi))))$

**NOTATION XII**

$\mathsf{E} x_{\circ}, \ldots, x_{\bullet} \circledast X(\varphi) \lessdot \mathsf{E} x_{\circ}(((x_{\circ} \circledast X) \wedge \cdots \wedge \mathsf{E} x_{\bullet}(((x_{\bullet} \in X) \wedge \varphi))))$

**NOTATION XIII**

$(x_{\circ}, \ldots, x_{\bullet} \circledast X) \lessdot ((x_{\circ} \circledast X) \wedge \cdots \wedge (x_{\bullet} \circledast X))$

### 2.2.2 Semantics

**DEFINITION VIII** — TRUTH VALUE OF FORMULA

1. Every variable is assigned a set.
2. If $x$ is assigned the same set as $x'$, then $(x = x')$ is assigned true.
   Otherwise, $(x = x')$ is assigned false.
   If $\varphi$ is assigned true, then $(\neg\varphi)$ is assigned false.
   Otherwise, $(\neg\varphi)$ is assigned true.
   If $\varphi$ and $\varphi'$ are assigned true, then $(\varphi \wedge \varphi')$ is assigned true.
   Otherwise, $(\varphi \wedge \varphi')$ is assigned false.
   If $\varphi$ is assigned true for every possible $x$, then $\forall x(\varphi)$ is assigned true.
   Otherwise, $\forall x(\varphi)$ is assigned false.

# 3

# Zermelo–Frænkel Set Theory with Choice

*A set is a 'many' that allows itself to be thought of as a 'one'.*

— GEORG CANTOR

Previously, we used brackets to guarantee uniqueness for every reading of logical syntax. In the interest of brevity, we shall, henceforth, omit outermost pairs of brackets.

Around the 1920s, an axiomatic set theory was proposed by Ernst Zermelo and Abraham Frænkel: this Zermelo–Frænkel set theory (ZF), when paired with the axiom of choice (AC), came to be known as Zermelo–Frænkel set theory with choice (ZFC). ZFC is commonly used as a foundational theory, and is conventionally written in a classical predicate logic. With the admission of ZF, AC is equivalent to various theorems, including the theorem that every vector space has a basis, and the theorem that every surjective function has a right inverse. Famously, AC in ZF also implies the Banach–Tarski paradox.

**AXIOM I** — EMPTY SET

$\exists N(\forall x((x \notin N)))$

**DEFINITION IX** — EMPTY SET

$((N = \varnothing) = \{\}) \Leftrightarrow \forall x((x \notin N))$

**AXIOM II** — EXTENSIONALITY

$\forall X, Y((\forall m(((m \in X) \Leftrightarrow (m \in Y)))) \Rightarrow (X = Y)))$

**DEFINITION X** — SUBSET

$(X \subseteq Y) \Leftrightarrow \forall m(((m \in X) \Rightarrow (m \in Y)))$

**AXIOM III** — PAIRING

$\forall c, c'(\exists C(((c \in C) \wedge (c' \in C))))$

**DEFINITION XI**

$(C = \{c, c'\}) \Leftrightarrow ((c \in C) \wedge (c' \in C))$

**DEFINITION XII** — SINGLETON SET

$\{c\} = \{c, c\}$

**AXIOM IV** — UNION

$\forall X(\exists U(\forall u(((u \in U) \Leftrightarrow \exists x \in X((u \in x))))))$

**DEFINITION XIII** — UNARY UNION FUNCTION

$(U = \bigcup X) \Leftrightarrow \forall u(((u \in U) \Leftrightarrow \exists x \in X((u \in x))))$

**DEFINITION XIV** — BINARY UNION FUNCTION

$(x \cup x') = \bigcup\{x, x'\}$

**DEFINITION XV** — SET OF FREE VARIABLES

Let $x_\circ, \ldots, x_\bullet$ be the free variables in $\varphi$.

$\mathrm{free}(\varphi) = \bigcup\{\{x_\circ\}, \ldots, \{x_\bullet\}\}$

**AXIOM V** — POWER SET

$\forall X(\exists P(\forall p(((p \subseteq X) \Rightarrow (p \in P)))))$

**AXIOM SCHEMA I** — REPLACEMENT

Let $\mathrm{free}(\varphi) \subseteq \{D, d, i\}$.

$\forall D((\forall d \in D(\exists_! i(\varphi))) \Rightarrow \exists I(\forall i(((i \in I) \Leftrightarrow \exists d \in D(\varphi)))))$

**AXIOM SCHEMA II** — SEPARATION

Let $\mathrm{free}(\varphi) \subseteq \{D, a_\circ, \ldots, a_\bullet, f\}$.

$\forall D, a_\circ, \ldots, a_\bullet(\exists F(\forall f(((f \in F) \Leftrightarrow ((f \in D) \wedge \varphi)))))$

**DEFINITION XVI**

Let $\mathrm{free}(\varphi) \subseteq \{D, f\}$.

$(F = \{(d \in D) \mid \varphi\}) \Leftrightarrow \forall f(((f \in F) \Leftrightarrow ((f \in D) \wedge \varphi)))$

**DEFINITION XVII**

Let $\mathrm{free}(\varphi) \subseteq \{f\}$.

$(F = \{x \mid \varphi\}) \Leftrightarrow \forall f(((f \in F) \Leftrightarrow \varphi))$

**DEFINITION XVIII** — UNARY INTERSECTION FUNCTION

$\bigcap X = \{(u \in \bigcup X) \mid \forall x \in X((u \in x))\}$

**DEFINITION XIX** — BINARY INTERSECTION FUNCTION

$(x \cap x') = \bigcap\{x, x'\}$

**AXIOM VI** — INFINITY

$\exists R(((\varnothing \in R) \wedge \forall r \in R(((r \cup \{r\}) \in R))))$

**AXIOM VII** — REGULARITY

$\forall O(((O \neq \varnothing) \Rightarrow \exists o \in O(((o \cap O) = \varnothing))))$

**DEFINITION XX** — SET OF PAIRWISE DISJOINT SETS

$\textcircled{$\not\cap$}(X) \Leftrightarrow \forall x, x' \in X((x \neq x') \Rightarrow ((x \cap x') = \varnothing))$

**AXIOM VIII** — CHOICE

$\forall B(((\forall S \in B((S \neq \varnothing))) \wedge \textcircled{$\not\cap$}(B)) \Rightarrow \exists B'(\forall S \in B(\exists ! s \in S((s \in B')))))$

# 4

# Interlude I

**DEFINITION XXI** — POWER SET

$\mathscr{P}(X) = \{x \mid (x \subseteq X)\}$

**DEFINITION XXII** — RELATIVE COMPLEMENT

$(X \setminus Y) = \{(d \in X) \mid (d \notin Y)\}$

**DEFINITION XXIII** — SUCCESSOR

$i(x) = (x \cup \{x\})$

**DEFINITION XXIV** — SPACE OF NATURAL NUMBERS

$(\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \dots\}) = \bigcap\{R \mid ((\varnothing \in R) \wedge \forall r \in R((i(r) \in R)))\}$

**DEFINITION XXV**

Let $n \in \mathbb{N}$.

$\mathbb{N}_{\geq n} = \bigcap\{R \mid ((n \in R) \wedge \forall r \in R((i(r) \in R)))\}$

**DEFINITION XXVI**

Let $n \in \mathbb{N}$.

$\mathbb{N}_{\leq n} = (\{n\} \cup \{p \mid (p \notin \mathbb{N}_{\geq n})\})$

**DEFINITION XXVII**

Let $m, n \in \mathbb{N}$.

$\mathbb{N}_{[m,n]} = (\mathbb{N}_{\geq m} \cap \mathbb{N}_{\leq n})$

**DEFINITION XXVIII** — $n$-TUPLE

Let $n \in \mathbb{N}_{\geq 1}$.

$\langle x_1, x_2 \rangle = \{\{x_1\}, \{x_1, x_2\}\}$

$\langle x_1, \dots, x_{i(n)} \rangle = \langle \langle x_1, \dots, x_n \rangle, x_{i(n)} \rangle$

**DEFINITION XXIX**

Let $k, n \in \mathbb{N}_{\geq 1}$.

$\langle x_1, \dots, x_k \rangle_n = x_n$

**DEFINITION XXX**

Let $n \in \mathbb{N}_{\geq 1}$.

$\lozenge(\langle x_1, \dots, x_n \rangle) = \{x_1, \dots, x_n\}$

**DEFINITION XXXI** — $n$-ARY CARTESIAN PRODUCT

Let $n \in \mathbb{N}_{\geq 1}$.

$(X_1 \times \cdots \times X_n) = \{\langle x_1, \dots, x_n \rangle \mid ((x_n \in X_1) \wedge \cdots \wedge (x_n \in X_n))\}$

**DEFINITION XXXII** — FUNCTION SPACE

${}^{X}Y = \{(f \subseteq (X \times Y)) \mid \forall x \in X(\exists_! y \in Y(((\langle x, y \rangle \in f))))\}$

**DEFINITION XXXIII** — FUNCTION

$(f : X \to Y) \Leftrightarrow (f \in {}^{X}Y)$

$(f(x) = y) \Leftrightarrow (\langle x, y \rangle \in f)$

**DEFINITION XXXIV** — IDENTITY FUNCTION

$\mathrm{id} : X \to X$

$\mathrm{id}(x) = x$

**DEFINITION XXXV** — SET OF MUTUALLY EXCLUSIVE FORMULAS

Let $\Phi \subseteq \mathcal{F}_1$.

$\textcircled{\not\wedge}(\Phi) \Leftrightarrow \forall "\varphi", "\varphi'" \in \Phi((("\varphi" \neq "\varphi'") \Rightarrow (\varphi \not\wedge \varphi')))$

**DEFINITION XXXVI** — PIECEWISE FUNCTION

Let $n \in \mathbb{N}$, and $\textcircled{\not\wedge}(\{\varphi_1, \dots, \varphi_n\})$.

$$\left(f(x) = \begin{cases} y_1, & \varphi_1 \\ \vdots & \vdots \\ y_n, & \varphi_n \end{cases}\right) \Leftrightarrow \left(f = \bigcup\{\{\langle x, y_i \rangle \mid \varphi_i\} \mid (i \in \mathbb{N}_{[1,n]})\}\right)$$

**DEFINITION XXXVII** — $n$-LENGTH STRING

Let $n \in \mathbb{N}_{\geq 1}$.

$"x_1 \dots x_n" = \langle "x_1", \dots, "x_n" \rangle$

**DEFINITION XXXVIII** — SPACE OF $n$-LENGTH STRINGS

Let $n \in \mathbb{N}_{\geq 1}$.

$A^n = \{\langle a_1, \dots, a_n \rangle \mid ((a_1 \in A) \wedge \cdots \wedge (a_n \in A))\}$

**DEFINITION XXXIX** — SPACE OF FINITE-LENGTH STRINGS

$A^* = \bigcup\{A^n \mid (n \in \mathbb{N})\}$

**NOTATION XIV**

Let $k, n \in \mathbb{N}_{\geq 1}$.

$\forall(x_1, \dots, x_k) \in (X_1, \dots, X_n)(\varphi) \lessdot \forall x_{1,1}, \dots, x_{1,k} \in X_1(\cdots \forall x_{n,1}, \dots, x_{n,k} \in X_n(\varphi))$

**DEFINITION XL** — $|k \times n|$-ARY SET CLOSURE

Let $k, n \in \mathbb{N}_{\geq 1}$.

$\mathscr{C}_k(X_1, \ldots, X_n, F) \Leftrightarrow \forall (x_1, \ldots, x_k) \in (X_1, \ldots, X_n) \left( \forall f \in F \left( \left( f(x_{1,1}, \ldots, x_{n,k}) \in X_n \right) \right) \right)$

# 5

# Metalanguage

*If a book is worth writing, it is worth re-writing.*

— PAUL ATTEWELL

Previously, we defined propositional and predicate logics. In this chapter, we shall define propositional and predicate metalogics that mimic, and thus help to more formally define, their logical counterparts. To avoid circularity, metalogic ought to be distinguished from logic. However, in the interest of brevity, we shall, henceforth, not always make this distinction as apparently as one might hope.

## 5.1 Propositional Logic

### 5.1.1 Syntax

**DEFINITION XLI** — SPACE OF PROPOSITIONAL VARIABLES

$\mathcal{X}_0 = \bigcup\{\text{"}p_1\text{"}, \ldots, \text{"}p_n\text{"} \mid (n \in \mathbb{N})\}$

**DEFINITION XLII** — CONCATENATION FUNCTION OF NEGATION

$\mathrm{neg} : \Phi \to \Phi$

$\mathrm{neg}(\text{"}\varphi\text{"}) = \text{"}(\neg\varphi)\text{"}$

**DEFINITION XLIII** — CONCATENATION FUNCTION OF CONJUNCTION

$\mathrm{conj} : \Phi^2 \to \Phi$

$\mathrm{conj}(\text{"}\varphi\text{"}, \text{"}\varphi'\text{"}) = \text{"}(\varphi \wedge \varphi')\text{"}$

**DEFINITION XLIV** — SPACE OF PROPOSITIONAL CHARACTERS

$\mathcal{C}_0 = (\mathcal{X}_0 \cup \{\text{"}\neg\text{"}, \text{"}\wedge\text{"}, \text{"}(\text{"}, \text{"})\text{"}\})$

**DEFINITION XLV** — SPACE OF PROPOSITIONAL FORMULAS

$\mathcal{F}_0 = \bigcap\{(\Phi \subseteq \mathcal{C}_0^*) \mid (((\mathcal{X}_0 \subseteq \Phi) \wedge \mathscr{C}_1(\Phi, \{\mathrm{neg}\})) \wedge \mathscr{C}_2(\Phi, \{\mathrm{conj}\}))\}$

### 5.1.2 Semantics

**DEFINITION XLVI** — TRUTH FUNCTION OF NEGATION

$\mathrm{not} : \{\mathbf{T}, \mathbf{F}\} \to \{\mathbf{T}, \mathbf{F}\}$

$\mathrm{not}(\mathbf{T}) = \mathbf{F}$

$\mathrm{not}(\mathbf{F}) = \mathbf{T}$

**DEFINITION XLVII** — TRUTH FUNCTION OF CONJUNCTION

$\mathrm{and} : \{\mathbf{T}, \mathbf{F}\}^2 \to \{\mathbf{T}, \mathbf{F}\}$

$\mathrm{and}(\mathbf{T}, \mathbf{T}) = \mathbf{T}$

$\mathrm{and}(\mathbf{T}, \mathbf{F}) = \mathbf{F}$

$\mathrm{and}(\mathbf{F}, \mathbf{T}) = \mathbf{F}$

$\mathrm{and}(\mathbf{F}, \mathbf{F}) = \mathbf{F}$

**DEFINITION XLVIII** — SPACE OF TRUTH ASSIGNMENTS

$\mathcal{T}_0 = {}^{\mathcal{X}_0}\{\mathbf{T}, \mathbf{F}\}$

**DEFINITION XLIX** — PROPOSITIONAL FORMULA EVALUATION FUNCTION

Let $t \in \mathcal{T}_0$.

$\mathfrak{v}_0^t : \mathcal{F}_0 \to \{\mathbf{T}, \mathbf{F}\}$

$\mathfrak{v}_0^t(p) = t(p)$

$\mathfrak{v}_0^t(\mathrm{neg}(\varphi)) = \mathrm{not}(\mathfrak{v}_0^t(\varphi))$

$\mathfrak{v}_0^t(\mathrm{conj}(\varphi, \varphi')) = \mathrm{and}(\mathfrak{v}_0^t(\varphi), \mathfrak{v}_0^t(\varphi'))$

## 5.2 Predicate Logic

### 5.2.1 Syntax

**DEFINITION L** — SPACE OF VARIABLES

$\mathcal{X}_1 = \bigcup\{\text{"}x_1\text{"}, \dots, \text{"}x_n\text{"} \mid (n \in \mathbb{N})\}$

**DEFINITION LI** — CONCATENATION FUNCTION OF EQUALITY

$\mathrm{eq} : X^2 \to \Phi$

$\mathrm{eq}(\text{"}x\text{"}, \text{"}x'\text{"}) = \text{"}(x = x')\text{"}$

**DEFINITION LII** — CONCATENATION FUNCTION OF MEMBERSHIP

$\mathrm{in} : X^2 \to \Phi$

$\mathrm{in}(\text{"}x\text{"}, \text{"}x'\text{"}) = \text{"}(x \in x')\text{"}$

**DEFINITION LIII** — CONCATENATION FUNCTION OF UNIVERSAL QUANTIFICATION

$\mathrm{forall} : (X \times \Phi) \to \Phi$

$\mathrm{forall}(\text{"}x\text{"}, \text{"}\varphi\text{"}) = \text{"}\forall x(\varphi)\text{"}$

**DEFINITION LIV** — SPACE OF PREDICATE CHARACTERS

$\mathcal{C}_1 = (\mathcal{X}_1 \cup \{\text{"}=\text{"}, \text{"}\in\text{"}, \text{"}\neg\text{"}, \text{"}\wedge\text{"}, \text{"}(\text{"}, \text{"})\text{"}\})$

**DEFINITION LV** — SPACE OF ATOMIC FORMULAS

$\mathcal{F}_1^{\odot} = \bigcap\{(\Phi \subseteq \mathcal{C}_1^*) \mid ((\mathcal{X}_1 \subseteq \Phi) \wedge \mathscr{C}_2(\Phi, \{\mathrm{eq}, \mathrm{in}\}))\}$

**DEFINITION LVI** — SPACE OF FORMULAS

$\mathcal{F}_1 = \bigcap\{(\Phi \subseteq \mathcal{C}_1^*) \mid ((((\mathcal{F}_1^{\odot} \subseteq \Phi) \wedge \mathscr{C}_1(\Phi, \{\mathrm{neg}\})) \wedge \mathscr{C}_2(\Phi, \{\mathrm{conj}\})) \wedge \mathscr{C}_1(\mathcal{X}_1, \Phi, \{\mathrm{forall}\}))\}$

**DEFINITION LVII** — SPACE OF FREE VARIABLES

$\mathrm{free} : \mathcal{F}_1 \to \mathscr{P}(\mathcal{X}_1)$

$\mathrm{free}(\mathrm{eq}(x, x')) = \{x, x'\}$

$\mathrm{free}(\mathrm{in}(x, x')) = \{x, x'\}$

$\mathrm{free}(\mathrm{neg}(\varphi)) = \mathrm{free}(\varphi)$

$\mathrm{free}(\mathrm{conj}(\varphi, \varphi')) = (\mathrm{free}(\varphi) \cup \mathrm{free}(\varphi'))$

$\mathrm{free}(\mathrm{forall}(x, \varphi)) = (\mathrm{free}(\varphi) \setminus \{x\})$

**DEFINITION LVIII** — SPACE OF SENTENCES

$\mathcal{S} = \{(\varphi \in \mathcal{F}_1) \mid (\mathrm{free}(\varphi) = \varnothing)\}$

**DEFINITION LIX** — CAPTURE-AVOIDING SUBSTITUTION

$$(\mathrm{eq}(x,x'))_{m\mapsto s} = \begin{cases} \mathrm{eq}(s,s), & ((x = m) \land (x' = m)) \\ \mathrm{eq}(s,x'), & ((x = m) \land (x' \neq m)) \\ \mathrm{eq}(x,s), & ((x \neq m) \land (x' = m)) \\ \mathrm{eq}(x,x'), & ((x \neq m) \land (x' \neq m)) \end{cases}$$

$$(\mathrm{in}(x,x'))_{m\mapsto s} = \begin{cases} \mathrm{in}(s,s), & ((x = m) \land (x' = m)) \\ \mathrm{in}(s,x'), & ((x = m) \land (x' \neq m)) \\ \mathrm{in}(x,s), & ((x \neq m) \land (x' = m)) \\ \mathrm{in}(x,x'), & ((x \neq m) \land (x' \neq m)) \end{cases}$$

$$(\mathrm{neg}(\varphi))_{m\mapsto s} = \mathrm{neg}((\varphi)_{m\mapsto s})$$

$$(\mathrm{conj}(\varphi,\varphi'))_{m\mapsto s} = \mathrm{conj}((\varphi)_{m\mapsto s}, (\varphi')_{m\mapsto s})$$

$$(\mathrm{forall}(x,\varphi))_{m\mapsto s} = \begin{cases} \mathrm{forall}(x,\varphi), & (x = m) \\ \mathrm{forall}(x,(\varphi)_{m\mapsto s}), & ((x \neq m) \land (x \notin \Diamond(s))) \\ \mathrm{forall}(x_\ast, ((\varphi)_{x\mapsto x_\ast})_{m\mapsto s}), & (((x \neq m) \land (x \in \Diamond(s))) \land (x_\ast \notin (\Diamond(s) \cup \Diamond(\varphi)))) \end{cases}$$

### 5.2.2 Semantics

**DEFINITION LX** — SPACE OF ARITY FUNCTIONS

$\mathscr{R}(F, R) = {}^{(F \cup R)}\mathbb{N}$

**DEFINITION LXI** — SPACE OF INTERPRETATION FUNCTIONS

$\mathscr{I}'(U, F, R) = \left(\bigcup\{{}^{U^{a(f)}}U \mid (f \in F)\} \cup \bigcup\{\mathscr{P}(U^{a(r)}) \mid (r \in R)\}\right)$

$\mathscr{I}(U, F, R) = {}^{(F \cup R)}\mathscr{I}'(U, F, R)$

**DEFINITION LXII** — SPACE OF STRUCTURES

$\mathcal{M} = \{\langle U, \langle (F \cup R), a\rangle, i\rangle \mid ((((U \neq \varnothing) \wedge ((F \cap R) = \varnothing)) \wedge (a \in \mathscr{R}(F, R))) \wedge (i \in \mathscr{I}(U, F, R)))\}$

**DEFINITION LXIII** — STRUCTURE OF PREDICATE LOGIC

$(\mathfrak{L} = \langle U_{\mathfrak{L}}, \langle (\varnothing \cup \{\text{"}\in\text{"}\}), a_{\mathfrak{L}}\rangle, i_{\mathfrak{L}}\rangle) \in \mathcal{M}$

$a_{\mathfrak{L}}(\text{"}\in\text{"}) = 2$

$i_{\mathfrak{L}}(\text{"}\in\text{"}) = \{\langle x, X\rangle \mid (x \in X)\}$

**DEFINITION LXIV** — SPACE OF VARIABLE ASSIGNMENTS

Let $m \in \mathcal{M}$.

$\mathscr{T}_1^m = \left(\bigcup\{{}^{\mathrm{free}(\varphi)}(m_1) \mid (\varphi \in \mathscr{F}_1)\} \cup \{\mathrm{id}\}\right)$

**DEFINITION LXV** — FORMULA EVALUATION FUNCTION

Let $m \in \mathcal{M}$, and $t \in \mathscr{T}_1^m$.

$\mathfrak{v}_1^{m,t} : \mathscr{F}_1 \to \{\mathbf{T}, \mathbf{F}\}$

$\mathfrak{v}_1^{m,t}(\mathrm{eq}(x, x')) = \begin{cases} \mathbf{T}, & (v(x) = v(x')) \\ \mathbf{F}, & (v(x) \neq v(x')) \end{cases}$

$\mathfrak{v}_1^{m,t}(\mathrm{in}(x, x')) = \begin{cases} \mathbf{T}, & (v(x) \in v(x')) \\ \mathbf{F}, & (v(x) \notin v(x')) \end{cases}$

$\mathfrak{v}_1^{m,t}(\mathrm{neg}(\varphi)) = \mathrm{not}(\mathfrak{v}_1^{m,t}(\varphi))$

$\mathfrak{v}_1^{m,t}(\mathrm{conj}(\varphi, \varphi')) = \mathrm{and}(\mathfrak{v}_1^{m,t}(\varphi), \mathfrak{v}_1^{m,t}(\varphi'))$

$\mathfrak{v}_1^{m,t}(\mathrm{forall}(x, \varphi)) = \begin{cases} \mathbf{T}, & \forall s \in m_1((\mathfrak{v}_1^{m,t}((\varphi)_{x \mapsto s}) = \mathbf{T})) \\ \mathbf{F}, & (\neg\forall s \in m_1((\mathfrak{v}_1^{m,t}((\varphi)_{x \mapsto s}) = \mathbf{T}))) \end{cases}$

# 6

# Interlude II

**DEFINITION LXVI** — FUNCTION COMPOSITION

$(g \circ f) = \{\langle x, z \rangle \mid ((\langle x, y \rangle \in f) \wedge (\langle y, z \rangle \in g))\}$

**DEFINITION LXVII**

Let $i \in \mathbb{N}$.

$x_{[i]} = x$

**DEFINITION LXVIII**

Let $n \in \mathbb{N}_{\geq 1}$, and $f : X \to X$.

$f^n = (f_{[1]} \circ \cdots \circ f_{[n]})$

**DEFINITION LXIX** — INJECTIVE FUNCTION

Let $f : X \to Y$.

$\text{injective}(f) \Leftrightarrow \forall y \in Y (\mathsf{E} x \in X((f(x) = y)))$

**DEFINITION LXX** — SURJECTIVE FUNCTION

Let $f : X \to Y$.

$\text{surjective}(f) \Leftrightarrow \forall y \in Y (\exists x \in X((f(x) = y)))$

**DEFINITION LXXI** — BIJECTIVE FUNCTION

Let $f : X \to Y$.

$\text{bijective}(f) \Leftrightarrow (\text{injective}(f) \wedge \text{surjective}(f))$

**DEFINITION LXXII** — FINITE SET

$(|X| < \infty) \Leftrightarrow \exists n \in \mathbb{N} (\exists f \in {}^{X}\mathbb{N}_{\leq n}(\text{bijective}(f)))$

# 7

# Adequacy

Adequacy is also known as *functional completeness*.

**DEFINITION LXXIII** — SPACE OF TRUTH FUNCTIONS
$\mathscr{B} = \bigcup\{{\{\mathbf{T},\mathbf{F}\}^n}\{\mathbf{T},\mathbf{F}\} \mid (n \in \mathbb{N})\}$

**THEOREM I** — ADEQUACY OF NEGATION WITH CONJUNCTION
$\forall f \in \mathscr{B}(\exists g \in \bigcap\{X \mid (((\{\mathbf{T},\mathbf{F}\} \subseteq X) \wedge \mathscr{C}_1(X, \mathrm{not})) \wedge \mathscr{C}_2(X, \mathrm{and}))\}((f = g)))$

# 8

# Satisfiability and Definability

A logical formula is said to be *tautological* only if it is “always true”, *satisfiable* only if it is “sometimes true”, and *contradictory* only if it is “never true”. The property of being tautological can be seen as *opposite* to the property of being contradictory, while the property of being satisfiable can be seen as *complementary* to the property of being contradictory. A set is said to be *definable* only if there exists a logical formula whose truth is equivalent to existence of the set.

## 8.1 Propositional Logic

**DEFINITION LXXIV** — TAUTOLOGICAL SET OF PROPOSITIONAL FORMULAS

Let $\Phi \subseteq \mathcal{F}_0$.

$\forall t \in \mathcal{T}_0(\forall \varphi \in \Phi((\mathfrak{v}_0^t(\varphi) = \mathbf{T})))$

**DEFINITION LXXV** — SATISFIABLE SET OF PROPOSITIONAL FORMULAS

Let $\Phi \subseteq \mathcal{F}_0$.

$\exists t \in \mathcal{T}_0(\forall \varphi \in \Phi((\mathfrak{v}_0^t(\varphi) = \mathbf{T})))$

**DEFINITION LXXVI** — CONTRADICTORY SET OF PROPOSITIONAL FORMULAS

Let $\Phi \subseteq \mathcal{F}_0$.

$\forall t \in \mathcal{T}_0(\forall \varphi \in \Phi((\mathfrak{v}_0^t(\varphi) = \mathbf{F})))$

**DEFINITION LXXVII** — DEFINABLE SET OF TRUTH ASSIGNMENTS

Let $T \subseteq \mathcal{T}_0$.

$\exists \varphi \in \mathcal{F}_0(\forall t \in T((\mathfrak{v}_0^t(\varphi) = \mathbf{T})))$

**DEFINITION LXXVIII** — SUBJECT OF SET OF PROPOSITIONAL FORMULAS

Let $\Phi \in \mathcal{F}_0$.

$\mathrm{subject}_0(\Phi) = \{(t \in \mathcal{T}_0) \mid \forall \varphi \in \Phi((\mathfrak{v}_0^t(\varphi) = \mathbf{T}))\}$

**DEFINITION LXXIX** — THEORY OF SET OF TRUTH ASSIGNMENTS

Let $T \in \mathcal{T}_0$.

$\mathrm{theory}_0(T) = \{(\varphi \in \mathcal{F}_0) \mid \forall t \in T((\mathfrak{v}_0^t(\varphi) = \mathbf{T}))\}$

**THEOREM II** — EXISTENCE OF UNSATISFIABLE SET OF PROPOSITIONAL FORMULAS

$\exists \Phi \subseteq \mathcal{F}_0((\mathrm{subject}_0(\Phi) = \varnothing))$

**THEOREM III** — EXISTENCE OF UNDEFINABLE SET OF TRUTH ASSIGNMENTS

$\exists T \subseteq \mathcal{T}_0((\mathrm{theory}_0(T) = \varnothing))$

**THEOREM IV**

$\forall \Phi \subseteq \mathcal{F}_0(\forall T \subseteq \mathcal{T}_0((\Phi \subseteq \mathrm{theory}_0(T)) \Leftrightarrow (T \subseteq \mathrm{subject}_0(\Phi))))$

## 8.2 Predicate Logic

**DEFINITION LXXX** — TAUTOLOGICAL SET OF FORMULAS

Let $\Phi \subseteq \mathcal{F}_1$.

$\forall m \in \mathcal{M}(\forall t \in \mathcal{T}_1^m(\forall \varphi \in \Phi(\mathfrak{v}_1^{m,t}(\varphi) = \mathbf{T})))$

**DEFINITION LXXXI** — SATISFIABLE SET OF SENTENCES

Let $\Phi \subseteq \mathcal{S}$.

$\exists m \in \mathcal{M}(\forall \varphi \in \Phi(\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = \mathbf{T}))$

**DEFINITION LXXXII** — SATISFIABLE SET OF FORMULAS

Let $m \in \mathcal{M}$, and $\Phi \subseteq \mathcal{F}_1$.

$\exists t \in \mathcal{T}_1^m(\forall \varphi \in \Phi(\mathfrak{v}_1^{m,t}(\varphi) = \mathbf{T}))$

**DEFINITION LXXXIII** — CONTRADICTORY SET OF FORMULAS

Let $\Phi \subseteq \mathcal{F}_1$.

$\forall m \in \mathcal{M}(\forall t \in \mathcal{T}_1^m(\forall \varphi \in \Phi(\mathfrak{v}_1^{m,t}(\varphi) = \mathbf{F})))$

**DEFINITION LXXXIV** — DEFINABLE SET OF STRUCTURES

Let $M \subseteq \mathcal{M}$.

$\exists \varphi \in \mathcal{S}(\forall m \in M(\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = \mathbf{T}))$

**DEFINITION LXXXV** — DEFINABLE SET OF VARIABLE ASSIGNMENTS

Let $m \in \mathcal{M}$, and $T \subseteq \mathcal{T}_1^m$.

$\exists \varphi \in \mathcal{F}_1(\forall t \in T(\mathfrak{v}_1^{m,t}(\varphi) = \mathbf{T}))$

**DEFINITION LXXXVI** — SUBJECT OF SET OF SENTENCES

Let $\Phi \subseteq \mathcal{S}$.

$\mathrm{subject}_1(\Phi) = \{m \in \mathcal{M} \mid \forall \varphi \in \Phi(\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = \mathbf{T})\}$

**DEFINITION LXXXVII** — SUBJECT OF SET OF FORMULAS

Let $m \in \mathcal{M}$, and $\Phi \subseteq \mathcal{F}_1$.

$\mathrm{subject}_1^m(\Phi) = \{t \in \mathcal{T}_1^m \mid \forall \varphi \in \Phi(\mathfrak{v}_1^{m,t}(\varphi) = \mathbf{T})\}$

**DEFINITION LXXXVIII** — THEORY OF SET OF STRUCTURES

Let $M \subseteq \mathcal{M}$.

$\mathrm{theory}_1(M) = \{\varphi \in \mathcal{S} \mid \forall m \in M(\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = \mathbf{T})\}$

**DEFINITION LXXXIX** — THEORY OF SET OF VARIABLE ASSIGNMENTS

Let $m \in \mathcal{M}$, and $T \subseteq \mathcal{T}_1^m$.

$\mathrm{theory}_1^m(T) = \{\varphi \in \mathcal{F}_1 \mid \forall t \in T(\mathfrak{v}_1^{m,t}(\varphi) = \mathbf{T})\}$

**THEOREM V** — EXISTENCE OF UNSATISFIABLE SET OF SENTENCES

$\exists \Phi \subseteq \mathcal{S}(\mathrm{subject}_1(\Phi) = \varnothing)$

**THEOREM VI** — EXISTENCE OF UNDEFINABLE SET OF STRUCTURES

$\exists M \subseteq \mathcal{M}(\mathrm{theory}_1(M) = \varnothing)$

**THEOREM VII** — EXISTENCE OF UNSATISFIABLE SET OF FORMULAS

$\forall m \in \mathcal{M} (\exists \Phi \subseteq \mathcal{F}_1 ((\text{subject}_1^m(\Phi) = \varnothing)))$

**THEOREM VIII** — EXISTENCE OF UNDEFINABLE SET OF VARIABLE ASSIGNMENTS

$\forall m \in \mathcal{M} (\exists T \subseteq \mathcal{T}_1^m ((\text{theory}_1^m(T) = \varnothing)))$

**THEOREM IX**

$\forall \Phi \subseteq \mathcal{S} (\forall M \subseteq \mathcal{M} ((\Phi \subseteq \text{theory}_1(M)) \Leftrightarrow (M \subseteq \text{subject}_1(\Phi))))$

**THEOREM X**

$\forall m \in \mathcal{M} (\forall \Phi \subseteq \mathcal{F}_1 (\forall T \subseteq \mathcal{T}_1^m ((\Phi \subseteq \text{theory}_1^m(T)) \Leftrightarrow (T \subseteq \text{subject}_1^m(\Phi)))))$

# 9

# Soundness and Completeness

For a logic, soundness and completeness concern the truth of every proof, and the proof of every truth, respectively. Soundness can be seen as the property that every proof has truth, while completeness can be seen as the property that every truth has proof. Taken together, soundness and completeness establish a correspondence between notions of proof, which are *syntactic*, and notions of truth, which are *semantic*.

**DEFINITION XC** — MODUS PONENS INFERENCE FUNCTION

$\text{ponens} : \Phi^2 \to \Phi$

$\text{ponens}(\text{"}\varphi\text{"}, \text{"}(\varphi \Rightarrow \varphi')\text{"}) = \text{"}\varphi'\text{"}$

## 9.1 Propositional Logic

**DEFINITION XCI** — SPACE OF AXIOMS FOR PROPOSITIONAL LOGIC

$\mathcal{A}_0' = \{\text{"}(\varphi \Rightarrow (\psi \Rightarrow \varphi))\text{"} \mid (\varphi, \psi \in \mathcal{F}_0)\}$

$\mathcal{A}_0'' = \{\text{"}((\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi)))\text{"} \mid (\varphi, \psi, \chi \in \mathcal{F}_0)\}$

$\mathcal{A}_0''' = \{\text{"}(((\neg\varphi) \Rightarrow (\neg\psi)) \Rightarrow (\psi \Rightarrow \varphi))\text{"} \mid (\varphi, \psi \in \mathcal{F}_0)\}$

$\mathcal{A}_0 = \bigcup\{\mathcal{A}_0', \mathcal{A}_0'', \mathcal{A}_0'''\}$

**DEFINITION XCII** — PROOF SYSTEM FOR PROPOSITIONAL LOGIC

Let $\Gamma \subseteq \mathcal{F}_0$.

$\mathcal{P}_0(\Gamma) = \bigcap\{(\Phi \subseteq \mathcal{C}_0^*) \mid (((\mathcal{A}_0 \cup \Gamma) \subseteq \Phi) \wedge \mathscr{C}_1^2(\Phi, \{\text{ponens}\}))\}$

**DEFINITION XCIII**

Let $\Gamma, \Phi \subseteq \mathcal{F}_0$.

$(\Gamma \vdash_0 \Phi) \Leftrightarrow \forall\varphi \in \Phi((\varphi \in \mathcal{P}_0(\Gamma)))$

**DEFINITION XCIV**

Let $\Gamma, \Phi \subseteq \mathcal{F}_0$.

$(\Gamma \vDash_0 \Phi) \Leftrightarrow \forall t \in \mathcal{T}_0(\forall\gamma \in \Gamma(\forall\varphi \in \Phi(((\mathfrak{v}_0^t(\gamma) = \mathbf{T}) \Rightarrow (\mathfrak{v}_0^t(\varphi) = \mathbf{T})))))$

**THEOREM XI** — FINITARYNESS OF PROOF SYSTEM FOR PROPOSITIONAL LOGIC

$\forall\Gamma \subseteq \mathcal{F}_0(\forall\varphi \in \mathcal{P}_0(\Gamma)(\exists\Psi \subseteq \mathcal{P}_0(\Gamma)(((|\Psi| < \infty) \wedge \forall\psi \in \Psi(((\psi = \varphi) \vee (\psi \in (\mathcal{A}_0 \cup \Gamma))))))))$

**THEOREM XII** — SOUNDNESS OF PROOF SYSTEM FOR PROPOSITIONAL LOGIC

$\forall\Gamma, \Phi \subseteq \mathcal{F}_0((\Gamma \vdash_0 \Phi) \Rightarrow (\Gamma \vDash_0 \Phi))$

**THEOREM XIII** — COMPLETENESS OF PROOF SYSTEM FOR PROPOSITIONAL LOGIC

$\forall\Gamma, \Phi \subseteq \mathcal{F}_0((\Gamma \vDash_0 \Phi) \Rightarrow (\Gamma \vdash_0 \Phi))$

**DEFINITION XCV** — CONSISTENT SET OF PROPOSITIONAL FORMULAS

Let $\Gamma \subseteq \mathcal{F}_0$.

$\text{consistent}_0(\Gamma) \Leftrightarrow \forall\varphi \in \mathcal{F}_0(\neg(((\Gamma \vdash_0 \varphi) \wedge (\Gamma \nvdash_0 \varphi))))$

**DEFINITION XCVI** — SATISFIABLE SET OF PROPOSITIONAL FORMULAS

Let $\Gamma \subseteq \mathcal{F}_0$.

$\text{satisfiable}_0(\Gamma) \Leftrightarrow \forall\varphi \in \mathcal{F}_0(\neg(((\Gamma \vDash_0 \varphi) \wedge (\Gamma \nvDash_0 \varphi))))$

**THEOREM XIV**

$\forall\Gamma \subseteq \mathcal{F}_0((\text{consistent}_0(\Gamma) \Leftrightarrow \text{satisfiable}_0(\Gamma)))$

## 9.2 Predicate Logic

**DEFINITION XCVII** — SPACE OF AXIOMS FOR PREDICATE LOGIC

$\mathcal{A}_1' = \{(\tau)_{p\mapsto\varphi} \mid (((\varnothing \vDash_0 \tau) \wedge (p \in \mathcal{X}_0)) \wedge (\varphi \in \mathcal{F}_1))\}$

$\mathcal{A}_1'' = \{\text{"}(\forall x(\varphi) \Rightarrow (\varphi)_{x\mapsto x'})\text{"} \mid ((x, x' \in \mathcal{X}_1) \wedge \varphi \in \mathcal{F}_1)\}$

$\mathcal{A}_1''' = \{\text{"}(\varphi \Rightarrow \forall x((\varphi)_{x\mapsto x'}))\text{"} \mid ((x, x' \in \mathcal{X}_1) \wedge \varphi \in \mathcal{F}_1)\}$

$\mathcal{A}_1 = \bigcup\{\mathcal{A}_1', \mathcal{A}_1'', \mathcal{A}_1'''\}$

**DEFINITION XCVIII** — PROOF SYSTEM FOR PREDICATE LOGIC

Let $\Gamma \subseteq \mathcal{F}_1$.

$\mathcal{P}_1(\Gamma) = \bigcap\{(\Phi \subseteq \mathcal{C}_1^*) \mid (((\mathcal{A}_1 \cup \Gamma) \subseteq \Phi) \wedge \mathscr{C}_1^2(\Phi, \{\text{ponens}\}))\}$

**DEFINITION XCIX**

Let $\Gamma, \Phi \subseteq \mathcal{F}_1$.

$(\Gamma \vdash_1 \Phi) \Leftrightarrow \forall\varphi \in \Phi((\varphi \in \mathcal{P}_1(\Gamma)))$

**DEFINITION C**

Let $\Gamma, \Phi \subseteq \mathcal{F}_1$.

$(\Gamma \vDash_1 \Phi) \Leftrightarrow \forall m \in \mathcal{M}(\forall v \in \mathcal{I}_1^m(\forall\gamma \in \Gamma(\forall\varphi \in \Phi((\mathfrak{v}_1^{m,v}(\gamma) = \mathbf{T}) \Rightarrow (\mathfrak{v}_1^{m,v}(\varphi) = \mathbf{T})))))$

**THEOREM XV** — FINITARYNESS OF PROOF SYSTEM FOR PREDICATE LOGIC

$\forall\Gamma \subseteq \mathcal{F}_1(\forall\varphi \in \mathcal{P}_1(\Gamma)(\exists\Psi \subseteq \mathcal{P}_1(\Gamma)(((|\Psi| < \infty) \wedge \forall\psi \in \Psi(((\psi = \varphi) \vee (\psi \in (\mathcal{A}_1 \cup \Gamma))))))))$

**THEOREM XVI** — SOUNDNESS OF PROOF SYSTEM FOR PREDICATE LOGIC

$\forall\Gamma, \Phi \subseteq \mathcal{F}_1((\Gamma \vdash_1 \Phi) \Rightarrow (\Gamma \vDash_1 \Phi))$

**THEOREM XVII** — COMPLETENESS OF PROOF SYSTEM FOR PREDICATE LOGIC

$\forall\Gamma, \Phi \subseteq \mathcal{F}_1((\Gamma \vDash_1 \Phi) \Rightarrow (\Gamma \vdash_1 \Phi))$

**DEFINITION CI** — CONSISTENT SET OF FORMULAS

Let $\Gamma \subseteq \mathcal{F}_1$.

$\text{consistent}_1(\Gamma) \Leftrightarrow \forall\varphi \in \mathcal{F}_1(\neg(((\Gamma \vdash_1 \varphi) \wedge (\Gamma \nvdash_1 \varphi))))$

**DEFINITION CII** — SATISFIABLE SET OF FORMULAS

Let $\Gamma \subseteq \mathcal{F}_1$.

$\text{satisfiable}_1(\Gamma) \Leftrightarrow \forall\varphi \in \mathcal{F}_1(\neg(((\Gamma \vDash_1 \varphi) \wedge (\Gamma \nvDash_1 \varphi))))$

**THEOREM XVIII**

$\forall\Gamma \subseteq \mathcal{F}_1((\text{consistent}_1(\Gamma) \Leftrightarrow \text{satisfiable}_1(\Gamma)))$

# 10

# Compactness and Maximisability

Previously, we established equivalence between consistency and satisfiability. In the interest of brevity, we shall, henceforth, occasionally omit discussion of consistency in favor of satisfiability, keeping in mind that every property which applies to the latter, also applies to the former, in equivalent fashion.

## 10.1 Propositional Logic

**THEOREM XIX** — PROPOSITIONAL COMPACTNESS

Let $\Gamma \subseteq \mathscr{F}_0$.

$\mathrm{satisfiable}_0(\Gamma) \Leftrightarrow \forall \Gamma' \subseteq \Gamma(((|\Gamma'| < \infty) \wedge \mathrm{satisfiable}_0(\Gamma')))$

**DEFINITION CIII** — MAXIMAL SET OF PROPOSITIONAL FORMULAS

Let $\Gamma \subseteq \mathscr{F}_0$.

$\mathrm{maximal}_0(\Gamma) \Leftrightarrow \forall \varphi \in \mathscr{F}_0(((\Gamma \vDash_0 \varphi) \vee (\Gamma \nvDash_0 \varphi)))$

**THEOREM XX** — PROPOSITIONAL LINDENBAUM

$\forall \Gamma \subseteq \mathscr{F}_0((\mathrm{satisfiable}_0(\Gamma) \Leftrightarrow \exists \Gamma' \supseteq \Gamma((\mathrm{satisfiable}_0(\Gamma') \wedge \mathrm{maximal}_0(\Gamma')))))$

## 10.2 Predicate Logic

**THEOREM XXI** — PREDICATE COMPACTNESS

$\forall \Gamma \subseteq \mathscr{F}_1(\,(\text{satisfiable}_1(\Gamma) \Leftrightarrow \forall \Gamma' \subseteq \Gamma(\,(\,(|\Gamma'| < \infty) \wedge \text{satisfiable}_1(\Gamma')\,)\,)\,)\,)$

**DEFINITION CIV** — MAXIMAL SET OF FORMULAS

Let $\Gamma \subseteq \mathscr{F}_1$.

$\text{maximal}_1(\Gamma) \Leftrightarrow \forall \varphi \in \mathscr{F}_1(\,(\,(\Gamma \vDash_1 \varphi) \vee (\Gamma \nvDash_1 \varphi)\,)\,)$

**THEOREM XXII** — PREDICATE LINDENBAUM

$\forall \Gamma \subseteq \mathscr{F}_1(\,(\text{satisfiable}_1(\Gamma) \Leftrightarrow \exists \Gamma' \supseteq \Gamma(\,(\text{maximal}_1(\Gamma) \wedge \text{satisfiable}_1(\Gamma)\,)\,)\,)\,)$

# 11

# Gödel Incompleteness

*Everything fits in the number.*

— IAMBLICHUS

The natural numbers have, historically, been seen as a "staple" of mathematics. Indeed, various historical accounts suggest that the ancient Pythagoreans viewed the natural numbers as "fundamental to reality". Famously, Carl Friedrich Gauss dubbed number theory the "queen of mathematics". A space of sentences which are true for a structure of natural numbers is known as a *theory of arithmetic*.

Previously, we defined a proof system which is finitary, sound, and complete with respect to the space of tautological formulas. In the 1920s, there was an interest in founding mathematics upon formal methods of proof. In particular, it was wondered whether one could develop a proof system which is verifiable, sound, and complete with respect to *theories of mathematics*. Around the 1930s, Kurt Gödel demonstrated that, if "verifiable" is taken to mean "recursively axiomatisable", then no such proof system exists for "sufficiently strong" theories of arithmetic. Gödel also showed that no consistent proof system in which such theories of arithmetic are derivable can prove its *own* consistency. These theorems have come to be known as *Gödel's incompleteness theorems*. A theory is said to be "sufficiently strong" only if it makes "sufficiently many" claims about (the objects in) its universe (i.e. contains "sufficiently many" sentences). For the sake of concreteness, we shall state the incompleteness theorems for a particular theory of arithmetic known as Robinson arithmetic.

**DEFINITION CV** — STRUCTURE OF NATURAL NUMBERS

$(\mathbb{N} = \langle \mathbb{N}, \langle (\{"0", "1", "i", "+", "\times"\} \cup \varnothing), a_{\mathbb{N}} \rangle, i_{\mathbb{N}} \rangle) \in \mathcal{m}$

$a_{\mathbb{N}}("0") = 0$

$a_{\mathbb{N}}("1") = 0$

$a_{\mathbb{N}}("i") = 1$

$a_{\mathbb{N}}("+") = 2$

$a_{\mathbb{N}}("\times") = 2$

$i_{\mathbb{N}}("0") = 0$

$i_{\mathbb{N}}("1") = 1$

$i_{\mathbb{N}}("i") = i$

$i_{\mathbb{N}}("+") = \{\langle x, x', i^{x'}(x) \rangle \mid (x, x', y \in \mathbb{N})\}$

$i_{\mathbb{N}}("\times") = \{\langle x, x', (i^{x'})^{x'}(x) \rangle \mid (x, x', y \in \mathbb{N})\}$

**THEOREM XXIII** — FIRST GÖDEL INCOMPLETENESS

$\nexists \Gamma \subseteq \mathscr{F}_1 (\forall \Phi \subseteq \mathrm{theory}_1(\{\mathbb{N}\})(((\Gamma \vDash_1 \Phi) \Rightarrow (\Gamma \vdash_1 \Phi))))$

**THEOREM XXIV** — SECOND GÖDEL INCOMPLETENESS

$\forall \Gamma \subseteq \mathscr{F}_1(((\mathrm{theory}_1(\{\mathbb{N}\}) \subseteq \mathscr{P}_1(\Gamma)) \Rightarrow (\mathrm{consistent}_1(\mathscr{P}_1(\Gamma)) \Leftrightarrow (\text{"}\,\mathrm{consistent}_1(\mathscr{P}_1(\Gamma))\text{"} \notin \mathscr{P}_1(\Gamma)))))$