PATTERNS

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About

The live file is available for free at `http://github.com/glimeuxe/papers`, where updates are made on an occasional basis. I thank all those who have made suggestions to improve the text.

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Theories

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Classical Logic

2.1 Prologue

What could it mean for something to be *true*? Throughout history, various definitions for the word "truth" have been proposed.

Definition 1 (Correspondence Theory of Truth)**.**
Truth is that which corresponds to reality.

Definition 2 (Coherence Theory of Truth)**.**
Truth is that which coheres with every other truth.

Ought mathematics concern itself with the universe we inhabit? In any case, the practice of mathematics seems to be based on something like the coherence theory of truth; this presents a *seeming* trilemma.

Definition 3 (Münchhausen Trilemma)**.**
Every proof is completed by circularity, infinite regress, or assumption.

Reasoning by coherence appears to obtain truth by circularity, infinite regress, or assumption. It is no secret that mathematical theories admit assumptions. Mathematical theories are also said to build upon, and thus derive from, each other. A theory from which other theories can be derived is said to be *foundational.*

Broadly speaking, a logic can be thought of as a language for reasoning about truth — that is, a system which prescribes symbols, and ways of interchanging those symbols. This text studies propositional and predicate logics, in particular.

With any human language, definitions may be said to originate in *pre-linguistic* thought. Writers sometimes choose their starting points based on what feels the most intuitive to them. For this text, I have decided to use words and phrases like "not", "and", "if. . . then", "either. . . or", "otherwise", "every", "same", in addition to a few other mathematical symbols, in my appeal to the intuition of the reader.

2.2 Language

2.2.1 Propositional Logic

Definition 4 (Propositional Formula)**.**

Let $p_\circ, \dots, p_\bullet$ be propositional variables.

1. If p is a propositional variable, then p is a propositional formula.
2. If φ is a propositional formula, then $(\neg\varphi)$ is a propositional formula.
   If φ and φ' are propositional formulas, then $(\varphi \wedge \varphi')$ is a propositional formula.

Definition 5 (Truth Value of Propositional Formula)**.**

1. Every propositional variable is either true or false.
2. If φ is true, then $(\neg\varphi)$ is false. Otherwise, $(\neg\varphi)$ is true.
   If φ and φ' are true, then $(\varphi \wedge \varphi')$ is true. Otherwise, $(\varphi \wedge \varphi')$ is false.

2.2.2 Predicate Logic

Definition 6 (Formula)**.**

Let $x_\circ, \dots, x_\bullet$ be variables.

1. If x is a variable, and x' is a variable, then $(x = x')$ is a formula.
   If x is a variable, and x' is a variable, then $(x \in x')$ is a formula.
2. If φ is a formula, then $(\neg\varphi)$ is a formula.
   If φ is a formula, and φ' is a formula, then $(\varphi \wedge \varphi')$ is a formula.
   If x is a variable, and φ is a formula, then $\forall x(\varphi)$ is a formula.

Definition 7 (Free Variable)**.**

1. If $(x = x')$ is a formula, then x and x' are free variables in the formula.
   If $(x \in x')$ is a formula, then x and x' are free variables in the formula.
2. If x is a free variable in φ, then x is a free variable in $(\neg\varphi)$.
   If x is a free variable in φ and φ', then x is a free variable in $(\varphi \wedge \varphi')$.
   If x is a free variable in φ, and $(x \neq x')$, then x is a free variable in $\forall x'(\varphi)$.

Notation 1 (Naïve Variable Substitution)**.**

$(x = x)\{^{\tau}_{x} \eqcirc (\tau = \tau)$

$(x = x')\{^{\tau}_{x} \eqcirc (\tau = x')$

$(x' = x)\{^{\tau}_{x} \eqcirc (x' = \tau)$

$(x' = x')\{^{\tau}_{x} \eqcirc (x' = x')$

$(x \in x)\{^{\tau}_{x} \eqcirc (\tau \in \tau)$

$(x \in x')\{^{\tau}_{x} \eqcirc (\tau \in x')$

$(x' \in x)\{^{\tau}_{x} \eqcirc (x' \in \tau)$

$(x' \in x')\{^{\tau}_{x} \eqcirc (x' \in x')$

$(\neg\varphi)\{^{\tau}_{x} \eqcirc (\neg\varphi\{^{\tau}_{x})$

$(\varphi \wedge \varphi')\{^{\tau}_{x} \eqcirc (\varphi\{^{\tau}_{x} \wedge \varphi'\{^{\tau}_{x})$

$\forall x(\varphi)\{^{\tau}_{x} \eqcirc \forall\tau(\varphi\{^{\tau}_{x})$

Definition 8 (Truth Value of Formula)**.**

1. Every variable is a set.
2. If x is the same set as x', then $(x = x')$ is true. Otherwise, $(x = x')$ is false.
   If φ is true, then $(\neg\varphi)$ is false. Otherwise, $(\neg\varphi)$ is true.
   If φ and φ' are true, then $(\varphi \wedge \varphi')$ is true. Otherwise, $(\varphi \wedge \varphi')$ is false.
   If φ is true for every possible x, then $\forall x(\varphi)$ is true. Otherwise, $\forall x(\varphi)$ is false.

2.3 Interlogue

In the interest of brevity, we shall omit pairs of delimiters (i.e. brackets) wherever we can. To do this, we shall stipulate that within every bold-faced environment (i.e. "**Definition**", "**Theorem**", etc.), and pair of delimiters within those environments, the glyphs which appear closer together ought to be read before the glyphs which appear further apart.[1]

The negation of a conjunction of formulas is equivalent to the disjunction of the negations of those formulas. Similarly, the negation of a disjunction of formulas is equivalent to the conjunction of the negations of those formulas. These equivalences are known as De Morgan laws, and are said to illustrate a "duality" between conjunction and disjunction, with respect to negation.

Notation 2 (Connective of Disjunction).
$\varphi \vee \varphi' \eqcirc \neg(\neg\varphi \wedge \neg\varphi')$

Notation 3 (Connective of Implication).
$\varphi \Rightarrow \varphi' \eqcirc \neg(\varphi \wedge \neg\varphi')$

Notation 4 (Connective of Equivalence).
$\varphi \Leftrightarrow \varphi' \eqcirc (\varphi \Rightarrow \varphi') \wedge (\varphi' \Rightarrow \varphi)$

Notation 5 (Connective of Exclusive Disjunction).
$\varphi \boldsymbol{\vee} \varphi' \eqcirc \neg(\varphi \Leftrightarrow \varphi')$

Notation 6 (Quantifier of Existence).
$\exists x(\varphi) \eqcirc \not\forall x(\neg\varphi)$

Notation 7 (Quantifier of Unique Existence).
$\exists_! x(\varphi) \eqcirc \exists x, \forall x'(\varphi\{^{x'}_{x} \Leftrightarrow x = x')$

Notation 8 (Quantifier of Dichotomous Existence).
$\mathsf{E}x(\varphi) \eqcirc \nexists x(\varphi) \vee \exists_! x(\varphi)$

[1]Constant symbols ought to be read before function symbols. Function symbols ought to be read before relation symbols. Relation symbols ought to be read before logical symbols.

2.4 Zermelo–Frænkel Set Theory with Choice

Around the 1920s, an axiomatic set theory was proposed by Ernst Zermelo and Abraham Frænkel: this Zermelo–Frænkel set theory (ZF), when paired with the axiom of choice (AC), came to be known as Zermelo–Frænkel set theory with choice (ZFC). ZFC is commonly used as a foundational theory, and is conventionally written in a classical predicate logic.

It has been shown that if ZFC is consistent, then every formulation of ZFC must include at least one axiom schema. For brevity, we shall examine a particular formulation of ZFC which includes the axiom schema of separation, and no other axiom schema.

With the admission of ZF, AC is equivalent to various theorems, including the theorem that every vector space has a basis, and the theorem that every surjective function has a right inverse. Famously, AC in ZF also implies the Banach–Tarski paradox.

Every object in ZFC is a set. A set can be distinguished from a *proper* class: the latter is, in some sense, a "larger collection" than the former. However, in the interest of brevity, we shall not always make this distinction.

Axiom 1 (Empty Set).
$\exists N, \forall x (x \notin N)$

Definition 9 (Empty Set).
$N = \varnothing \Leftrightarrow \forall x (x \notin N)$

Axiom 2 (Extensionality).
$\forall X, Y (\forall m (m \in X \Leftrightarrow m \in Y) \Rightarrow X = Y)$

Definition 10 (Subset).
$X \subseteq Y \Leftrightarrow \forall m (m \in X \Rightarrow m \in Y)$

Axiom 3 (Pairing).
$\forall c, c', \exists C (c \in C \wedge c' \in C)$

Definition 11 (Couple Set).
$C = \{c, c'\} \Leftrightarrow c \in C \wedge c' \in C$

Definition 12 (Singleton Set).
$\{c\} = \{c, c\}$

Axiom 4 (Union).
$\forall W, \exists U, \forall w \in W, \forall u \in w (u \in U)$

Definition 13 (Unary Union Function).
$U = \bigcup W \Leftrightarrow \forall w \in W, \forall u \in w (u \in U)$

Definition 14 (Binary Union Function).
$x \cup x' = \bigcup \{x, x'\}$

Definition 15 (Space of Free Variables).
Let $x_\circ, \ldots, x_\bullet$ be the free variables in φ.
$\mathrm{free}(\varphi) = \bigcup\{\{x_\circ\}, \ldots, \{x_\bullet\}\}$

Axiom 5 (Power Set).
$\forall X, \exists P, \forall x(x \subseteq X \Rightarrow x \in P)$

Definition 16 (Power Set).
$x \in \mathscr{P}(X) \Leftrightarrow x \subseteq X$

Axiom Schema 1 (Separation).
Let φ be a formula, and $\mathrm{free}(\varphi) \subseteq \{d, D, f\}$.
$\forall D, \exists F, \forall f(f \in D \land \varphi \Rightarrow f \in F)$

Definition 17.
Let φ be a formula, and $\mathrm{free}(\varphi) \subseteq \{d, D, f\}$.
$F = \{d \in D \mid \varphi\} \Leftrightarrow \forall f(f \in D \land \varphi \Rightarrow f \in F)$

Definition 18.
Let φ be a formula, and $\mathrm{free}(\varphi) \subseteq \{d, D, f\}$.
$F = \{d \mid \varphi\} \Leftrightarrow \forall f(\varphi \Rightarrow f \in F)$

Definition 19 (Unary Intersection Function).
$\bigcap W = \{u \in \bigcup W \mid \forall w \in W(u \in w)\}$

Definition 20 (Binary Intersection Function).
$x \cap x' = \bigcap\{x, x'\}$

Definition 21 (Successor Function).
$\mathscr{S}(n) = n \cup \{n\}$

Axiom 6 (Infinity).
$\exists R(\varnothing \in R \land \forall r \in R(\mathscr{S}(r) \in R))$

Axiom 7 (Regularity).
$\forall O \neq \varnothing, \exists o \in O(o \cap O = \varnothing)$

Definition 22 (Set of Pairwise Disjoint Sets).
$\not\cap(X) \Leftrightarrow \forall x, x' \in X(x \neq x' \Rightarrow x \cap x' = \varnothing)$

Axiom 8 (Choice).
$\forall B(\not\cap(B) \land \forall S \in B(S \neq \varnothing) \Rightarrow \exists B', \forall S \in B, \exists_! s \in S(s \in B'))$

2.5 Interlogue

Definition 23 (Binary Difference Function).
$X \setminus Y = \{x \in X \mid x \notin Y\}$

Definition 24 (Reflexive Relation).
Let R be a relation symbol.
$\mathsf{Reflexive}(X, \mathsf{R}) \Leftrightarrow \forall a \in X(a \mathrel{\mathsf{R}} a)$

Definition 25 (Antisymmetric Relation).
Let R be a relation symbol.
$\mathsf{Antisymmetric}(X, \mathsf{R}) \Leftrightarrow \forall a, b \in X(a \mathrel{\mathsf{R}} b \land b \mathrel{\mathsf{R}} a \Rightarrow a = b)$

Definition 26 (Transitive Relation).
Let R be a relation symbol.
$\mathsf{Transitive}(X, \mathsf{R}) \Leftrightarrow \forall a, b, c \in X(a \mathrel{\mathsf{R}} b \land b \mathrel{\mathsf{R}} c \Rightarrow a \mathrel{\mathsf{R}} c)$

Definition 27 (Partial-Ordering Relation).
Let R be a relation symbol.
$\mathsf{PartialOrdering}(X, \mathsf{R}) \Leftrightarrow \mathsf{Reflexive}(X, \mathsf{R}) \land \mathsf{Antisymmetric}(X, \mathsf{R}) \land \mathsf{Transitive}(X, \mathsf{R})$

Definition 28 (Interval).
Let $\mathsf{PartialOrdering}(X, \preceq)$.
$X_{[a,b]} = \{x \in X \mid a \preceq x \preceq b\}$
$X_{[a,b)} = \{x \in X \mid a \preceq x \prec b\}$
$X_{(a,b]} = \{x \in X \mid a \prec x \preceq b\}$
$X_{(a,b)} = \{x \in X \mid a \prec x \prec b\}$
$X_{[a,\infty)} = X_{\succeq a} = \{x \in X \mid a \preceq x\}$
$X_{(a,\infty)} = X_{\succ a} = \{x \in X \mid a \prec x\}$
$X_{(-\infty,b]} = X_{\preceq b} = \{x \in X \mid x \preceq b\}$
$X_{(-\infty,b)} = X_{\prec b} = \{x \in X \mid x \prec b\}$

Definition 29 (Space of Natural Numbers).
$\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \dots\} = \bigcap\{R \mid \varnothing \in R \land \forall r \in R(\mathcal{S}(r) \in R)\}$

Definition 30 (Order of Natural Numbers).
Let $a, b \in \mathbb{N}$.
$a \leq b \Leftrightarrow a \in b \lor a = b$

Definition 31 (Tuple).
Let $n \in \mathbb{N}_{\geq 1}$.
$\langle\rangle = \varnothing$
$\langle x_1 \rangle = \{\{\langle\rangle\}, \{\langle\rangle, x_1\}\}$
$\langle x_1, \dots, x_n, x_{\mathcal{S}(n)} \rangle = \{\{\langle x_1, \dots, x_n \rangle\}, \{\langle x_1, \dots, x_n \rangle, x_{\mathcal{S}(n)}\}\}$

Definition 32 (Detuple Function)**.**

Let $n \in \mathbb{N}_{\geq 1}$.

$\lozenge(\langle\rangle) = \varnothing$

$\lozenge(\langle x_1 \rangle) = \{x_1\}$

$\lozenge(\langle x_1, \ldots, x_n, x_{\mathcal{S}(n)} \rangle) = \lozenge(\langle x_1, \ldots, x_n \rangle) \cup \{x_{\mathcal{S}(n)}\}$

Definition 33 (Projection Function)**.**

Let $n \in \mathbb{N}_{\geq 1}$, and $k \in \mathbb{N}_{[1,n]}$.

$\langle x_1, \ldots, x_n \rangle_{[k]} = x_k$

Definition 34 (n-ary Cartesian Product)**.**

Let $n \in \mathbb{N}_{\geq 1}$.

$X_1 \times \cdots \times X_n = \{\langle x_1, \ldots, x_n \rangle \mid x_1 \in X_1 \wedge \ldots \wedge x_n \in X_n\}$

Definition 35 (Space of Functions)**.**

$\{X \to Y\} = \{f \subseteq X \times Y \mid \forall x \in X, \exists! y \in Y(\langle x, y \rangle \in f)\}$

Definition 36 (Function)**.**

$f : X \to Y \Leftrightarrow f \in \{X \to Y\}$

$f(x) = y \Leftrightarrow \langle x, y \rangle \in f$

Definition 37 (Identity Function)**.**

$\mathrm{id}(x) = x$

Definition 38 (Space of Strings)**.**

$A^* = \bigcup\{A^i \mid i \in \mathbb{N}\}$

Definition 39 (String)**.**

Let $n \in \mathbb{N}_{\geq 1}$.

$\texttt{""} = \langle\rangle$

$\texttt{"}x_1 \ldots x_n\texttt{"} = \langle \texttt{"}x_1\texttt{"}, \ldots, \texttt{"}x_n\texttt{"} \rangle$

Definition 40 (Mutually Exclusive Set of Formulas)**.**

$\not\wedge(\Phi) \Leftrightarrow \forall \texttt{"}\varphi\texttt{"}, \texttt{"}\varphi'\texttt{"} \in \Phi(\texttt{"}\varphi\texttt{"} \neq \texttt{"}\varphi'\texttt{"} \Rightarrow \neg(\varphi \wedge \varphi'))$

Definition 41 (Collectively Exhaustive Set of Formulas)**.**

$\vee(\{\texttt{"}\varphi_1\texttt{"}, \ldots, \texttt{"}\varphi_n\texttt{"}\}) \Leftrightarrow \varphi_1 \vee \ldots \vee \varphi_n$

Definition 42 (Logical Partition)**.**

$\perp\!\!\!\perp(\Phi) \Leftrightarrow \not\varnothing(\Phi) \wedge \not\wedge(\Phi) \wedge \vee(\Phi)$

Definition 43 (Piecewise Function)**.**

Let $f : X \to Y$, and $n \in \mathbb{N}_{\geq 1}$, and $\perp\!\!\!\perp(\{\texttt{"}\varphi_1\texttt{"}, \ldots, \texttt{"}\varphi_{\mathcal{S}(n)}\texttt{"}\})$.

$$f(x) = \begin{cases} y_1, & \varphi_1 \\ \vdots & \vdots \\ y_n, & \varphi_n \\ y_{\mathcal{S}(n)}, & \varphi_{\mathcal{S}(n)} \end{cases} \Leftrightarrow \forall x \in X, \forall i \in \mathbb{N}_{[1,\mathcal{S}(n)]}(\varphi_i \Rightarrow \langle x, y_i \rangle \in f)$$

$(\text{otherwise} \Leftrightarrow \varphi_{\mathcal{S}(n)}) \Leftrightarrow (\varphi_{\mathcal{S}(n)} \Leftrightarrow \neg(\varphi_1 \vee \ldots \vee \varphi_n))$

Definition 44.
Let $n, s \in \mathbb{N}_{\geq 1}$.
$\forall x_{i,j} \overset{i\in\langle 1,\dots,n\rangle}{\underset{j\in\langle 1,\dots,s\rangle}{\in}} X_i(\varphi) \Leftrightarrow \forall x_{1,1},\dots,x_{1,s} \in X_1,\dots,\forall x_{n,1},\dots,x_{n,s} \in X_n(\varphi)$

Definition 45.
Let $n, s \in \mathbb{N}_{\geq 1}$.
$\wp_k(X_1,\dots,X_n,F) \Leftrightarrow \forall x_{i,j} \overset{i\in\langle 1,\dots,n\rangle}{\underset{j\in\langle 1,\dots,s\rangle}{\in}} X_i, \forall f \in F(f(x_{1,1},\dots,x_{n,s}) \in X_n)$

Definition 46.
Let $k, n \in \mathbb{N}_{\geq 1}$.
$\widehat{\wp}_k(X_1,\dots,X_n,F) \Leftrightarrow \forall x_{i,j} \overset{i\in\langle 1,\dots,n\rangle}{\underset{j\in\langle 1,\dots,s\rangle}{\in}} X_i, \forall f \in F(\langle x_{1,1},\dots,x_{n,s}, f(x_{1,1},\dots,x_{n,s})\rangle \in X_n)$

2.6 Metalanguage

Previously, we defined propositional and predicate logics. Now, we shall define propositional and predicate metalogics that mimic, and thus help to more formally define, their logical counterparts. To avoid circularity, a metalogic ought to be distinguished from its logic. However, in the interest of brevity, we shall not always make such a distinction.

2.6.1 Propositional Logic

Definition 47 (Space of Propositional Variables).
$\mathcal{X}_0 = \bigcup\{\{"p_1", \ldots, "p_n"\} \mid n \in \mathbb{N}\}$

Definition 48 (Propositional Alphabet).
$\mathscr{A}_0 = \mathcal{X}_0 \cup \{"\neg", "\wedge", "(", ")"\}$

Definition 49 (Concatenation Function of Negation).
$\mathrm{neg}("\varphi") = "(\neg\varphi)"$

Definition 50 (Concatenation Function of Conjunction).
$\mathrm{conj}("\varphi", "\varphi'") = "(\varphi \wedge \varphi')"$

Definition 51 (Space of Propositional Formulas).
$\mathcal{F}_0 = \bigcap\{\Phi \subseteq \mathscr{A}_0^* \mid \mathcal{X}_0 \subseteq \Phi \wedge \mathscr{cl}_1(\Phi, \{\mathrm{neg}\}) \wedge \mathscr{cl}_2(\Phi, \{\mathrm{conj}\})\}$

Definition 52 (Truth Function of Negation).
$\mathrm{not} : 2 \to 2$
$\mathrm{not}(1) = 0$
$\mathrm{not}(0) = 1$

Definition 53 (Truth Function of Conjunction).
$\mathrm{and} : 2^2 \to 2$
$\mathrm{and}(1, 1) = 1$
$\mathrm{and}(1, 0) = 0$
$\mathrm{and}(0, 1) = 0$
$\mathrm{and}(0, 0) = 0$

Definition 54 (Space of Truth Assignments).
$\mathcal{T}_0 = \{\mathcal{X}_0 \to 2\}$

Definition 55 (Valuation of Propositional Formula).
Let $t \in \mathcal{T}_0$.
$\mathscr{v}_0^t : \mathcal{F}_0 \to 2$

$$\mathscr{v}_0^t(\varphi) = \begin{cases} t(p), & \exists p \in \mathcal{X}_0(\varphi = p) \\ \mathrm{not}(\mathscr{v}_0^t(\psi)), & \exists \psi \in \mathcal{F}_0(\varphi = \mathrm{neg}(\psi)) \\ \mathrm{and}(\mathscr{v}_0^t(\psi), \mathscr{v}_0^t(\psi')), & \text{otherwise} \end{cases}$$

2.6.2 Predicate Logic

Definition 56 (Space of Variables).
$\mathcal{X}_1 = \bigcup\{\{"x_1", \ldots, "x_n"\} \mid n \in \mathbb{N}\}$

Definition 57 (Predicate Alphabet).
$\wp_1 = \mathcal{X}_1 \cup \{"=", "\in", "\neg", "\wedge", "(", ")"\}$

Definition 58 (Concatenation Function of Equality).
$\mathrm{eq}("x", "x'") = "(x = x')"$

Definition 59 (Concatenation Function of Membership).
$\mathrm{in}("x", "x'") = "(x \in x')"$

Definition 60 (Concatenation Function of Universal Quantification).
$\mathrm{fa}("x", "\varphi") = "\forall x(\varphi)"$

Definition 61 (Space of Formulas).
$\mathcal{F}_{1,1} = \bigcap\{\Phi \subseteq \wp_1^* \mid \mathcal{X}_1 \subseteq \Phi \wedge \text{ꝏ}_2(\Phi, \{\mathrm{eq}, \mathrm{in}\})\}$
$\mathcal{F}_1 = \bigcap\{\Phi \subseteq \wp_1^* \mid \mathcal{F}_{1,1} \subseteq \Phi \wedge \text{ꝏ}_1(\Phi, \{\mathrm{neg}\}) \wedge \text{ꝏ}_2(\Phi, \{\mathrm{conj}\}) \wedge \text{ꝏ}_1(\mathcal{X}_1, \Phi, \{\mathrm{fa}\})\}$

Definition 62 (Space of Free Variables).
$\mathrm{free} : \mathcal{F}_1 \to \mathscr{P}(\mathcal{X}_1)$

$$\mathrm{free}(\varphi) = \begin{cases} \{x, x'\}, & \exists x, x' \in \mathcal{X}_1(\varphi = \mathrm{eq}(x, x')) \\ \{x, x'\}, & \exists x, x' \in \mathcal{X}_1(\varphi = \mathrm{in}(x, x')) \\ \mathrm{free}(\psi), & \exists \psi \in \mathcal{F}_1(\varphi = \mathrm{neg}(\psi)) \\ \mathrm{free}(\psi) \cup \mathrm{free}(\psi'), & \exists \psi, \psi' \in \mathcal{F}_1(\varphi = \mathrm{conj}(\psi, \psi')) \\ \mathrm{free}(\psi) \setminus \{x\}, & \text{otherwise} \end{cases}$$

Definition 63 (Space of Sentences).
$\mathcal{F}_1^{\bowtie} = \{\varphi \in \mathcal{F}_1 \mid \mathrm{free}(\varphi) = \varnothing\}$

Definition 64.
Let $n \in \mathbb{N}_{\geq 1}$.
$\star(S_1, \ldots, S_n) = \mathcal{X}_1 \setminus (\Diamond(S_1) \cup \cdots \cup \Diamond(S_n))$

Definition 65 (Variable Substitution)**.**
Let $s, s' \in \mathcal{X}_1$.
$.|_s^{s'} : \mathcal{F}_1 \to \mathcal{F}_1$

$$\varphi|_s^{s'} = \begin{cases} \mathrm{eq}(x,x')|_s^{s'} = \begin{cases} \mathrm{eq}(s',s'), & x, x' = s \\ \mathrm{eq}(s',x'), & x = s \wedge x' \neq s \\ \mathrm{eq}(x,s'), & x \neq s \wedge x' = s \\ \mathrm{eq}(x,x'), & \text{otherwise} \end{cases}, & \exists x, x' \in \mathcal{X}_1(\varphi = \mathrm{eq}(x,x')) \\ \mathrm{in}(x,x')|_s^{s'} = \begin{cases} \mathrm{in}(s',s'), & x, x' = s \\ \mathrm{in}(s',x'), & x = s \wedge x' \neq s \\ \mathrm{in}(x,s'), & x \neq s \wedge x' = s \\ \mathrm{in}(x,x'), & \text{otherwise} \end{cases}, & \exists x, x' \in \mathcal{X}_1(\varphi = \mathrm{in}(x,x')) \\ \mathrm{neg}(\psi)|_s^{s'} = \mathrm{neg}(\psi|_s^{s'}), & \exists \psi \in \mathcal{F}_1(\varphi = \mathrm{neg}(\psi)) \\ \mathrm{conj}(\psi,\psi')|_s^{s'} = \mathrm{conj}(\psi|_s^{s'}, \psi'|_s^{s'}), & \exists \psi, \psi' \in \mathcal{F}_1(\varphi = \mathrm{conj}(\psi,\psi')) \\ \mathrm{fa}(x,\psi)|_s^{s'} = \begin{cases} \mathrm{fa}(x,\psi), & x = s \\ \mathrm{fa}(x,\psi|_s^{s'}), & x \neq s \wedge x \notin \Diamond(s'), \text{ otherwise} \\ \mathrm{fa}(\star(s',\psi), \psi|_x^{\star(s',\psi)}|_s^{s'}), & \text{otherwise} \end{cases} \end{cases}$$

Definition 66 (Space of Structures)**.**
$\mathscr{a}_{F,R} = \{F \cup R \to \mathbb{N}\}$
$\mathscr{i}_{U,F,R,a} = \{F \cup R \to \bigcup\{\{U^{a(f)} \to U\} \mid f \in F\} \cup \bigcup\{\mathscr{P}(U^{a(r)}) \mid r \in R\}\}$
$\mathscr{m} = \{\langle U, \langle F, R, a \rangle, i \rangle \mid U \neq \varnothing \wedge F \cap R = \varnothing \wedge a \in \mathscr{a}_{F,R} \wedge i \in \mathscr{i}_{U,F,R,a}\}$

Definition 67 (Structure of Predicate Logic)**.**
$\mathbb{s} = \langle U_{\mathbb{s}}, \langle \varnothing, \{\text{"}\in\text{"}\}, a_{\mathbb{s}} \rangle, i_{\mathbb{s}} \rangle$
$\mathbb{s} \in \mathscr{m}$
$a_{\mathbb{s}}(\text{"}\in\text{"}) = 2$
$i_{\mathbb{s}}(\text{"}\in\text{"}) = \{\langle x, X \rangle \mid x \in X\}$

Definition 68 (Space of Variable Assignments)**.**
Let $m \in \mathscr{m}$.
$\mathcal{T}_1^m = \bigcup\{\{\mathrm{free}(\varphi) \to m_{[1]}\} \mid \varphi \in \mathcal{F}_1\} \cup \{\mathrm{id}\}$

Definition 69 (Valuation of Formula)**.**
Let $m \in \mathscr{m}$, and $t \in \mathcal{T}_1^m$.
$\mathscr{v}_1^{m,t} : \mathcal{F}_1 \to 2$

$$\mathscr{v}_1^{m,t}(\varphi) = \begin{cases} \mathscr{v}_1^{m,t}(\mathrm{eq}(x,x')) = \begin{cases} 1, & t(x) = t(x') \\ 0, & \text{otherwise} \end{cases}, & \exists x, x' \in \mathcal{X}_1(\varphi = \mathrm{eq}(x,x')) \\ \mathscr{v}_1^{m,t}(\mathrm{in}(x,x')) = \begin{cases} 1, & t(x) \in t(x') \\ 0, & \text{otherwise} \end{cases}, & \exists x, x' \in \mathcal{X}_1(\varphi = \mathrm{in}(x,x')) \\ \mathrm{not}(\mathscr{v}_1^{m,t}(\psi)), & \exists \psi \in \mathcal{F}_1(\varphi = \mathrm{neg}(\psi)) \\ \mathrm{and}(\mathscr{v}_1^{m,t}(\psi), \mathscr{v}_1^{m,t}(\psi')), & \exists \psi, \psi' \in \mathcal{F}_1(\varphi = \mathrm{conj}(\psi,\psi')) \\ \mathscr{v}_1^{m,t}(\mathrm{fa}(x,\psi)) = \begin{cases} 1, & \forall s \in m_{[1]}(\mathscr{v}_1^{m,t}(\psi|_s^x) = 1) \\ 0, & \text{otherwise} \end{cases}, & \text{otherwise} \end{cases}$$

2.7 Satisfiability and Definability

A logical formula is said to be *tautological* only if it is "always true", *satisfiable* only if it is "sometimes true", and *contradictory* only if it is "never true". The property of being tautological can be seen as opposite to the property of being contradictory, while the property of being satisfiable can be seen as complementary to the property of being contradictory.

A set is said to be *definable* only if there exists a logical formula whose truth is equivalent to existence of the set. The cardinality of the space of truth assignments is larger than that of the space of propositional formulas. Similarly, the cardinality of the space of structures is larger than that of the space of sentences.

2.7.1 Propositional Logic

Definition 70 (Tautological Set of Propositional Formulas).
Let $\Phi \subseteq \mathcal{F}_0$.
$\mathsf{Tautological}\,(\Phi) \Leftrightarrow \forall t \in \mathcal{T}_0, \forall \varphi \in \Phi(\mathcal{v}_0^t(\varphi) = 1)$

Definition 71 (Contradictory Set of Propositional Formulas).
Let $\Phi \subseteq \mathcal{F}_0$.
$\mathsf{Contradictory}\,(\Phi) \Leftrightarrow \forall t \in \mathcal{T}_0, \forall \varphi \in \Phi(\mathcal{v}_0^t(\varphi) = 0)$

Definition 72 (Satisfiable Set of Propositional Formulas).
Let $\Phi \subseteq \mathcal{F}_0$.
$\mathsf{Satisfiable}\,(\Phi) \Leftrightarrow \exists t \in \mathcal{T}_0, \forall \varphi \in \Phi(\mathcal{v}_0^t(\varphi) = 1)$

Definition 73 (Definable Set of Truth Assignments).
Let $T \subseteq \mathcal{T}_0$.
$\mathsf{Definable}\,(T) \Leftrightarrow \exists \varphi \in \mathcal{F}_0, \forall t \in T(\mathcal{v}_0^t(\varphi) = 1)$

Definition 74 (Subject of Set of Propositional Formulas).
Let $\Phi \subseteq \mathcal{F}_0$.
$\mathrm{subj}(\Phi) = \{t \in \mathcal{T}_0 \mid \forall \varphi \in \Phi(\mathcal{v}_0^t(\varphi) = 1)\}$

Definition 75 (Theory of Set of Truth Assignments).
Let $T \subseteq \mathcal{T}_0$.
$\mathrm{th}(T) = \{\varphi \in \mathcal{F}_0 \mid \forall t \in T(\mathcal{v}_0^t(\varphi) = 1)\}$

Theorem 1 (Propositional Subject–Theory Galois Connection).
Let $\Phi \subseteq \mathcal{F}_0$, and $T \subseteq \mathcal{T}_0$.
$\Phi \subseteq \mathrm{th}(T) \Leftrightarrow T \subseteq \mathrm{subj}(\Phi)$

Proposition 1 (Existence of Unsatisfiable Set of Propositional Formulas).
$\exists \Phi \subseteq \mathcal{F}_0(\neg\,\mathsf{Satisfiable}\,(\Phi))$

Proposition 2 (Existence of Undefinable Set of Truth Assignments).
$\exists T \subseteq \mathcal{T}_0(\neg\,\mathsf{Definable}\,(T))$

2.7.2 Predicate Logic

Definition 76 (Tautological Set of Sentences)**.**
Let $\Phi \subseteq \mathcal{F}_1^{\bowtie}$.
$\mathsf{Tautological}\,(\Phi) \Leftrightarrow \forall m \in \mathcal{M}, \forall \varphi \in \Phi(\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = 1)$

Definition 77 (Contradictory Set of Sentences)**.**
Let $\Phi \subseteq \mathcal{F}_1^{\bowtie}$.
$\mathsf{Contradictory}\,(\Phi) \Leftrightarrow \forall m \in \mathcal{M}, \forall \varphi \in \Phi(\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = 0)$

Definition 78 (Satisfiable Set of Sentences)**.**
Let $\Phi \subseteq \mathcal{F}_1^{\bowtie}$.
$\mathsf{Satisfiable}\,(\Phi) \Leftrightarrow \exists m \in \mathcal{M}, \forall \varphi \in \Phi(\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = 1)$

Definition 79 (Definable Set of Structures)**.**
Let $M \subseteq \mathcal{M}$.
$\mathsf{Definable}\,(M) \Leftrightarrow \exists \varphi \in \mathcal{F}_1^{\bowtie}, \forall m \in M(\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = 1)$

Definition 80 (Subject of Set of Sentences)**.**
Let $\Phi \subseteq \mathcal{F}_1^{\bowtie}$.
$\mathrm{subj}(\Phi) = \{m \in \mathcal{M} \mid \forall \varphi \in \Phi(\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = 1)\}$

Definition 81 (Theory of Set of Structures)**.**
Let $M \subseteq \mathcal{M}$.
$\mathrm{th}(M) = \{\varphi \in \mathcal{F}_1^{\bowtie} \mid \forall m \in M(\mathfrak{v}_1^{m,\mathrm{id}}(\varphi) = 1)\}$

Theorem 2 (Predicate Subject–Theory Galois Connection)**.**
Let $\Phi \subseteq \mathcal{F}_1^{\bowtie}$, and $T \subseteq \mathcal{T}_1$.
$\Phi \subseteq \mathrm{th}(T) \Leftrightarrow T \subseteq \mathrm{subj}(\Phi)$

Proposition 3 (Existence of Unsatisfiable Set of Sentences)**.**
$\exists \Phi \subseteq \mathcal{F}_1^{\bowtie}(\neg\,\mathsf{Satisfiable}\,(\Phi))$

Proposition 4 (Existence of Undefinable Set of Structures)**.**
$\exists M \subseteq \mathcal{M}(\neg\,\mathsf{Definable}\,(M))$

2.8 Soundness and Completeness

For a logic, soundness can be seen as the property that "every proof has truth", and completeness can be seen as the property that "every truth has proof". Taken together, soundness and completeness establish a "correspondence" between syntactic notions of proof and semantic notions of truth.

Definition 82 (Modus Ponens Function).
$\mathrm{pon}(\texttt{"}\varphi\texttt{"}, \texttt{"}(\varphi \Rightarrow \varphi')\texttt{"}) = \texttt{"}\varphi'\texttt{"}$

2.8.1 Propositional Logic

Definition 83 (Space of Propositional Axioms).
$\mathscr{A}_{0,1} = \{\texttt{"}\varphi \Rightarrow (\psi \Rightarrow \varphi)\texttt{"} \mid \varphi, \psi \in \mathscr{F}_0\}$
$\mathscr{A}_{0,2} = \{\texttt{"}(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi))\texttt{"} \mid \varphi, \psi, \chi \in \mathscr{F}_0\}$
$\mathscr{A}_{0,3} = \{\texttt{"}(\neg\varphi \Rightarrow \neg\psi) \Rightarrow (\psi \Rightarrow \varphi)\texttt{"} \mid \varphi, \psi \in \mathscr{F}_0\}$
$\mathscr{A}_0 = \bigcup\{\mathscr{A}_{0,1}, \mathscr{A}_{0,2}, \mathscr{A}_{0,3}\}$

Definition 84 (Propositional Proof System).
Let $\Gamma \subseteq \mathscr{F}_0$.
$\mathscr{D}_0(\Gamma) = \bigcap\{\Phi \subseteq \mathscr{s}_0^* \mid \mathscr{A}_0 \cup \Gamma \subseteq \Phi \wedge \mathscr{cs}_1(\Phi, \{\mathrm{pon}\})\}$

Definition 85 (Propositional Syntactic Entailment).
Let $\Gamma \subseteq \mathscr{F}_0$, and $\varphi \in \mathscr{F}_0$.
$\Gamma \vdash \varphi \Leftrightarrow \varphi \in \mathscr{D}_0(\Gamma)$

Definition 86 (Propositional Semantic Entailment).
Let $\Gamma \subseteq \mathscr{F}_0$, and $\varphi \in \mathscr{F}_0$.
$\Gamma \vDash \varphi \Leftrightarrow \forall t \in \mathscr{T}_0(\forall\gamma \in \Gamma(\mathscr{v}_0^t(\gamma) = 1) \Rightarrow \mathscr{v}_0^t(\varphi) = 1)$

Proposition 5 (Finitaryness of Propositional Proof System).
Let $\Gamma \subseteq \mathscr{F}_0$, and $\varphi \in \mathscr{D}_0(\Gamma)$.
$\exists n \in \mathbb{N}_{\geq 1}, \exists\langle\psi_1, \ldots, \psi_n\rangle \in \bigcap\{\Phi \subseteq \mathscr{s}_0^* \mid \mathscr{A}_0 \cup \Gamma \subseteq \Phi \wedge \widehat{\mathscr{cs}}_1(\Phi, \{\mathrm{pon}\})\}(\psi_n = \varphi)$

Theorem 3 (Soundness of Propositional Proof System).
Let $\Gamma \subseteq \mathscr{F}_0$, and $\varphi \in \mathscr{F}_0$.
$\Gamma \vdash \varphi \Rightarrow \Gamma \vDash \varphi$

Theorem 4 (Completeness of Propositional Proof System).
Let $\Gamma \subseteq \mathscr{F}_0$, and $\varphi \in \mathscr{F}_0$.
$\Gamma \vDash \varphi \Rightarrow \Gamma \vdash \varphi$

Definition 87 (Consistent Set of Propositional Formulas).
Let $\Gamma \subseteq \mathscr{F}_0$.
$\mathsf{Consistent}\,(\Gamma) \Leftrightarrow \forall\varphi \in \mathscr{F}_0(\neg(\Gamma \vdash \varphi \wedge \Gamma \nvdash \varphi))$

Definition 88 (Satisfiable Set of Propositional Formulas).
Let $\Gamma \subseteq \mathscr{F}_0$.
$\mathsf{Satisfiable}\,(\Gamma) \Leftrightarrow \forall\varphi \in \mathscr{F}_0(\neg(\Gamma \vDash \varphi \wedge \Gamma \nvDash \varphi))$

Theorem 5 (Propositional Consistency–Satisfiability Equivalence).
Let $\Gamma \subseteq \mathscr{F}_0$.
$\mathsf{Consistent}\,(\Gamma) \Leftrightarrow \mathsf{Satisfiable}\,(\Gamma)$

Proposition 6 (Consistency of Propositional Proof System).
Let $\Gamma \subseteq \mathscr{F}_0$.
$\mathsf{Consistent}\,(\mathscr{D}_0(\Gamma)) \Leftrightarrow \mathsf{Consistent}\,(\Gamma)$

Proposition 7 (Propositional Explosion).
Let $\Gamma \subseteq \mathscr{F}_0$.
$\mathsf{Contradictory}\,(\Gamma) \Leftrightarrow \forall \varphi \in \mathscr{F}_0(\Gamma \vdash \varphi)$

Theorem 6 (Propositional Proof-by-Contradiction).
Let $\alpha \in \mathscr{F}_0$, and $\Gamma \subseteq \mathscr{F}_0$.
$\Gamma \cup \{\alpha\} \vdash \varphi \wedge \Gamma \cup \{\alpha\} \vdash \neg\varphi \Rightarrow \Gamma \vdash \neg\alpha$

2.8.2 Predicate Logic

Definition 89 (Space of Predicate Axioms).
$\mathscr{A}_{1,1} = \{\tau|_p^\varphi \mid \tau \in \mathscr{F}_0 \wedge p \in \mathscr{X}_0 \wedge \varphi \in \mathscr{F}_1 \wedge \varnothing \vDash \tau\}$
$\mathscr{A}_{1,2} = \{\text{"}\forall x(\varphi) \Rightarrow \varphi|_x^{x'}\text{"} \mid x, x' \in \mathscr{X}_1 \wedge \varphi \in \mathscr{F}_1\}$
$\mathscr{A}_{1,3} = \{\text{"}\varphi \Rightarrow \forall x(\varphi|_x^{x'})\text{"} \mid x, x' \in \mathscr{X}_1 \wedge \varphi \in \mathscr{F}_1\}$
$\mathscr{A}_1 = \bigcup\{\mathscr{A}_{1,1}, \mathscr{A}_{1,2}, \mathscr{A}_{1,3}\}$

Definition 90 (Predicate Proof System).
Let $\Gamma \subseteq \mathscr{F}_1$.
$\mathscr{D}_1(\Gamma) = \bigcap\{\Phi \subseteq \wp_1^* \mid \mathscr{A}_1 \cup \Gamma \subseteq \Phi \wedge \mathscr{w}_1(\Phi, \{\text{pon}\})\}$

Definition 91 (Predicate Syntactic Entailment).
Let $\Gamma \subseteq \mathscr{F}_1$, and $\varphi \in \mathscr{F}_1$.
$\Gamma \vdash \varphi \Leftrightarrow \varphi \in \mathscr{D}_1(\Gamma)$

Definition 92 (Predicate Semantic Entailment).
Let $\Gamma \subseteq \mathscr{F}_1$, and $\varphi \in \mathscr{F}_1$, and $M \subseteq \mathscr{M}$.
$\langle M, \Gamma\rangle \vDash \varphi \Leftrightarrow \forall m \in M, \forall t \in \mathscr{T}_1^m(\forall \gamma \in \Gamma(\mathscr{v}_1^{m,t}(\gamma) = 1) \Rightarrow \mathscr{v}_1^{m,t}(\varphi) = 1)$

Proposition 8 (Finitaryness of Proof System).
Let $\Gamma \subseteq \mathscr{F}_1$, and $\varphi \in \mathscr{D}_1(\Gamma)$.
$\exists n \in \mathbb{N}_{\geq 1}, \exists\langle\psi_1, \ldots, \psi_n\rangle \in \bigcap\{\Phi \subseteq \wp_1^* \mid \mathscr{A}_1 \cup \Gamma \subseteq \Phi \wedge \widehat{\mathscr{w}}_1(\Phi, \{\text{pon}\})\}(\psi_n = \varphi)$

Theorem 7 (Soundness of Proof System).
Let $\Gamma \subseteq \mathscr{F}_1$, and $\varphi \in \mathscr{F}_1$.
$\Gamma \vdash \varphi \Rightarrow \langle\mathscr{M}, \Gamma\rangle \vDash \varphi$

Theorem 8 (Completeness of Proof System).
Let $\Gamma \subseteq \mathscr{F}_1$, and $\varphi \in \mathscr{F}_1$.
$\langle\mathscr{M}, \Gamma\rangle \vDash \varphi \Rightarrow \Gamma \vdash \varphi$

Definition 93 (Consistent Set of Formulas)**.**
Let $\Gamma \subseteq \mathscr{F}_1$.
$\mathsf{Consistent}\,(\Gamma) \Leftrightarrow \forall \varphi \in \mathscr{F}_1(\neg(\Gamma \vdash \varphi \wedge \Gamma \nvdash \varphi))$

Definition 94 (Satisfiable Set of Formulas)**.**
Let $\Gamma \subseteq \mathscr{F}_1$.
$\mathsf{Satisfiable}\,(\Gamma) \Leftrightarrow \forall \varphi \in \mathscr{F}_1(\neg(\langle \mathcal{M}, \Gamma\rangle \vDash \varphi \wedge \langle \mathcal{M}, \Gamma\rangle \nvDash \varphi))$

Theorem 9 (Predicate Consistency–Satisfiability Equivalence)**.**
Let $\Gamma \subseteq \mathscr{F}_1$.
$\mathsf{Consistent}\,(\Gamma) \Leftrightarrow \mathsf{Satisfiable}\,(\Gamma)$

Proposition 9 (Consistency of Predicate Proof System)**.**
Let $\Gamma \subseteq \mathscr{F}_1$.
$\mathsf{Consistent}\,(\mathscr{D}_1(\Gamma)) \Leftrightarrow \mathsf{Consistent}\,(\Gamma)$

Proposition 10 (Predicate Explosion)**.**
Let $\Gamma \subseteq \mathscr{F}_1$.
$\mathsf{Contradictory}\,(\Gamma) \Leftrightarrow \forall \varphi \in \mathscr{F}_1(\langle \mathcal{M}, \Gamma\rangle \vdash \varphi)$

Theorem 10 (Predicate Proof-by-Contradiction)**.**
Let $\alpha \in \mathscr{F}_1$, and $\Gamma \subseteq \mathscr{F}_1$.
$\langle \mathcal{M}, \Gamma \cup \{\alpha\}\rangle \vdash \varphi \wedge \langle \mathcal{M}, \Gamma \cup \{\alpha\}\rangle \vdash \neg\varphi \Rightarrow \langle \mathcal{M}, \Gamma\rangle \vdash \neg\alpha$

2.9 Interlogue

An injective function is called an injection, a surjective function is called a *s*urjection, and a bijective function is called a bijection. A pair of sets is said to be *equinumerous* (i.e. have equal cardinality) only if there exists a bijection between them. A set is said to be *finite* (i.e. have finite cardinality) only if there exists a bijection from the set to $\mathbb{N}_{[1,n]}$ for some natural number n. A set is said to be *infinite* (i.e. have infinite cardinality) only if it is not finite.

Definition 95 (Injective Function)**.**
Let $f : X \to Y$.
$\mathsf{Injective}\,(f) \Leftrightarrow \forall y \in Y, \mathsf{E} x \in X(f(x) = y)$

Definition 96 (Surjective Function)**.**
Let $f : X \to Y$.
$\mathsf{Surjective}\,(f) \Leftrightarrow \forall y \in Y, \exists x \in X(f(x) = y)$

Definition 97 (Bijective Function)**.**
Let $f : X \to Y$.
$\mathsf{Bijective}\,(f) \Leftrightarrow \mathsf{Injective}\,(f) \wedge \mathsf{Surjective}\,(f)$

Definition 98 (Functionality of Sets)**.**
$X \;\mathsf{fun}\; Y \Leftrightarrow \exists f : X \to Y$

Definition 99 (Injectionality of Sets)**.**
$X \;\mathsf{inj}\; Y \Leftrightarrow \exists f : X \to Y(\mathsf{Injective}\,(f))$

Definition 100 (Surjectionality of Sets)**.**
$X \;\mathsf{surj}\; Y \Leftrightarrow \exists f : X \to Y(\mathsf{Surjective}\,(f))$

Definition 101 (Order of Cardinals)**.**
$|X| \leq |Y| \Leftrightarrow X \;\mathsf{inj}\; Y$

Theorem 11 (Schröder–Bernstein)**.**
$|X| \leq |Y| \wedge |Y| \leq |X| \Rightarrow |X| = |Y|$

Proposition 11.
$X \;\mathsf{inj}\; Y \Leftrightarrow Y \;\mathsf{surj}\; X$

Definition 102 (Finite Set)**.**
$\mathsf{Finite}\,(X) \Leftrightarrow |X| < |\mathbb{N}|$

Definition 103 (Countably Infinite Set)**.**
$\mathsf{CountablyInfinite}\,(X) \Leftrightarrow |X| = |\mathbb{N}|$

Definition 104 (Countable Set)**.**
$\mathsf{Countable}\,(X) \Leftrightarrow \mathsf{Finite}\,(X) \vee \mathsf{CountablyInfinite}\,(X)$

Proposition 12.
Let $\mathsf{Infinite}\,(X)$, and $\mathsf{Countable}\,(Y)$.
$|X \cup Y| = |X|$

Theorem 12 (Cantor)**.**
$|X| < |\mathscr{P}(X)|$

2.10 Compactness and Maximality

Previously, we established equivalence between consistency and satisfiability. In the interest of brevity, we shall occasionally omit mention of consistency in favor of satisfiability, keeping in mind that every property which applies to the latter, also applies to the former, in equivalent fashion.

2.10.1 Propositional Logic

Theorem 13 (Propositional Compactness)**.**
Let $\Gamma \subseteq \mathcal{F}_0$.
$\mathsf{Satisfiable}\,(\Gamma) \Leftrightarrow \forall \Gamma' \subseteq \Gamma(\mathsf{Finite}\,(\Gamma') \Rightarrow \mathsf{Satisfiable}\,(\Gamma'))$

Definition 105 (Maximal Set of Propositional Formulas)**.**
Let $\Gamma \subseteq \mathcal{F}_0$.
$\mathsf{Maximal}\,(\Gamma) \Leftrightarrow \forall \varphi \in \mathcal{F}_0(\Gamma \vDash \varphi \vee \Gamma \nvDash \varphi)$

Theorem 14 (Propositional Lindenbaum)**.**
Let $\Gamma \subseteq \mathcal{F}_0$.
$\mathsf{Satisfiable}\,(\Gamma) \Leftrightarrow \exists \Gamma' \supseteq \Gamma(\mathsf{Satisfiable}\,(\Gamma') \wedge \mathsf{Maximal}\,(\Gamma'))$

2.10.2 Predicate Logic

Theorem 15 (Predicate Compactness)**.**
Let $\Gamma \subseteq \mathcal{F}_1$.
$\mathsf{Satisfiable}\,(\Gamma) \Leftrightarrow \forall \Gamma' \subseteq \Gamma(\mathsf{Finite}\,(\Gamma') \Rightarrow \mathsf{Satisfiable}\,(\Gamma'))$

Definition 106 (Maximal Set of Formulas)**.**
Let $\Gamma \subseteq \mathcal{F}_1$.
$\mathsf{Maximal}\,(\Gamma) \Leftrightarrow \forall \varphi \in \mathcal{F}_1(\langle \mathcal{M}, \Gamma \rangle \vDash \varphi \vee \langle \mathcal{M}, \Gamma \rangle \nvDash \varphi)$

Theorem 16 (Predicate Lindenbaum)**.**
Let $\Gamma \subseteq \mathcal{F}_1$.
$\mathsf{Satisfiable}\,(\Gamma) \Leftrightarrow \exists \Gamma' \supseteq \Gamma(\mathsf{Satisfiable}\,(\Gamma') \wedge \mathsf{Maximal}\,(\Gamma'))$

2.11 Gödel Incompleteness

A space of sentences which are true for a structure of natural numbers is known as a theory of natural arithmetic. A theory is said to be “sufficiently strong” only if it contains “sufficiently many” sentences.

Previously, we defined a proof system which is finitary, sound, and complete with respect to a theory of predicate logic. In the 1920s, it was wondered whether one could develop a proof system which is verifiable, sound, and complete with respect to every theory of mathematics. Around the 1930s, Kurt Gödel demonstrated that, if “verifiable” is taken to mean “recursively axiomatisable”, then no such proof system exists for “sufficiently strong” theories of natural arithmetic. Gödel also showed that no consistent proof system in which such theories of natural arithmetic are derivable can prove its *own* consistency. These theorems have come to be known as the Gödel incompleteness theorems. For the sake of concreteness, we shall state these theorems for a particular theory of natural arithmetic.

Famously, Gödel's incompleteness theorems have been shown to apply to an axiomatic theory of natural arithmetic known as Peano arithmetic (PA). The strengthened finite Ramsey theorem, which can be seen as a sentence about the natural numbers, is provable in ZFC, but not in PA. Additionally, ZFC can prove PA consistent, but PA cannot prove *itself* consistent.

Definition 107.
$\mathbb{n} = \langle \mathbb{N}, \langle \{\texttt{"0"}, \texttt{"}\mathscr{S}\texttt{"}\}, \varnothing, a_{\mathbb{n}} \rangle, i_{\mathbb{n}} \rangle$
$\mathbb{n} \in \mathcal{M}$
$a_{\mathbb{n}}(\texttt{"0"}) = 0$
$a_{\mathbb{n}}(\texttt{"}\mathscr{S}\texttt{"}) = 1$
$i_{\mathbb{n}}(\texttt{"0"}) = 0$
$i_{\mathbb{n}}(\texttt{"}\mathscr{S}\texttt{"}) = \mathscr{S}$

Proposition 13.
Let $M \subseteq \mathcal{M}$.
$\mathrm{th}(\mathcal{M}) \subseteq \mathrm{th}(M)$

Theorem 17 (First Gödel Incompleteness)**.**
$\nexists \Gamma \subseteq \mathscr{F}_1, \forall \varphi \in \mathrm{th}(\{\mathbb{n}\})(\langle \{\mathbb{n}\}, \Gamma \rangle \vDash \varphi \Rightarrow \Gamma \vdash \varphi)$

Theorem 18 (Second Gödel Incompleteness)**.**
Let $\Gamma \subseteq \mathscr{F}_1$, and $\mathrm{th}(\{\mathbb{n}\}) \subseteq \mathscr{D}_1(\Gamma)$.
$\mathsf{Consistent}\,(\mathscr{D}_1(\Gamma)) \Leftrightarrow \Gamma \nvdash \texttt{"}\mathsf{Consistent}\,(\mathscr{D}_1(\Gamma))\texttt{"}$

2.12 Epilogue

For brevity, we shall occasionally overload our notation for sets. To avoid ambiguity, we shall make three stipulations. Firstly, we shall stipulate that if a denotation x can be read as either set a or set a', and $|\{a\}| < |\{a'\}|$, then x ought to be read as a. Secondly, we shall stipulate that if a denotation x can be read as either denotation d or denotation d', and d is not defined for every set, and d' is defined for every set, then x ought to be read as d. Thirdly, we shall stipulate that if a denotation x can be read as either denotation d or denotation d', and d is defined after d', then x ought to be read as d.

Definition 108 (Vector)**.**
Let $n \in \mathbb{N}_{\geq 1}$.
$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \langle x_1, \ldots, x_n \rangle$

Definition 109 (Element of Vector)**.**
Let $k, n \in \mathbb{N}_{\geq 1}$, and $k \in \mathbb{N}_{[1,n]}$, and $\boldsymbol{v} \in V^n$.
$v_k = \boldsymbol{v}_{\texttt{[}k\texttt{]}}$

Definition 110 (Prototype of Vector)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{v} \in V^n$.
$[x_i]_{1..i..n} = \boldsymbol{v} \Leftrightarrow \forall i \in \mathbb{N}_{[1,n]}(x_i = v_i)$

Definition 111 (Matrix)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$.
$\begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{bmatrix} = \langle \langle x_{1,1}, \ldots, x_{1,n} \rangle, \ldots, \langle x_{m,1}, \ldots, x_{m,n} \rangle \rangle$

Definition 112 (Element of Matrix)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $i \in \mathbb{N}_{[1,m]}$, and $j \in \mathbb{N}_{[1,n]}$, and $X^{m \times n}$.
$M_{i,j} = M_{\texttt{[}i\texttt{]}\,\texttt{[}j\texttt{]}}$

Definition 113 (Prototype of Matrix)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $X^{m \times n}$.
$[x_{i,j}]_{\substack{1..i..m, \\ 1..j..n}} = M \Leftrightarrow \forall i \in \mathbb{N}_{[1,m]}, \forall j \in \mathbb{N}_{[1,n]}(x_{i,j} = M_{i,j})$

Definition 114 (Domain of Function)**.**
Let $f : X \to Y$.
$\mathrm{dom}(f) = X$

Definition 115 (Codomain of Function)**.**
Let $f : X \to Y$.
$\mathrm{cod}(f) = Y$

Definition 116 (Image of Function)**.**
Let $f : X \to Y$, and $A \subseteq \mathrm{dom}(f)$.
$f[A] = \{f(a) \in \mathrm{cod}(f) \mid a \in A\}$

Definition 117 (Preimage of Function)**.**
Let $f : X \to Y$, and $B \subseteq \operatorname{cod}(f)$.
$f^{-1}[B] = \{a \in \operatorname{dom}(f) \mid f(a) \in B\}$

Definition 118 (Minimum)**.**
Let $\mathsf{PartialOrdering}\,(X, \preceq)$.
$\min(X) = x_* \Leftrightarrow \exists x_* \in X, \forall x \in X(x_* \preceq x)$

Definition 119 (Maximum)**.**
Let $\mathsf{PartialOrdering}\,(X, \preceq)$.
$\max(X) = x_* \Leftrightarrow \exists x_* \in X, \forall x \in X(x \preceq x_*)$

Definition 120.
$f : X_{\subseteq} \to Y_{\subseteq} \Leftrightarrow \exists A \subseteq X, \exists B \subseteq Y(f : A \to B)$

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Definition 121 (Pot)**.**
Let $n \in \mathbb{N}_{\geq 1}$.
$\langle x_1, \ldots, x_n \rangle = x_1 \Leftrightarrow x_1 = \cdots = x_n$
$\langle x_1, \ldots, x_n \rangle = \langle x_1, \ldots, x_n \rangle \Leftrightarrow x_1 \neq \ldots \neq x_n$

Definition 122.
Let $\circ$ be a function symbol.
$\mathsf{Closed}\,(X, \circ) \Leftrightarrow \forall a, b \in X (a \circ b \in X)$

Definition 123.
Let $\circ$ be a function symbol.
$\mathsf{Commutative}\,(X, \circ) \Leftrightarrow \forall a, b \in X (a \circ b = b \circ a)$

Definition 124.
Let ①, ② be function symbols.
$\mathsf{Associative}\,(\langle X, Y \rangle, \langle ①, ② \rangle) \Leftrightarrow \forall a \in X, \forall b, c \in Y (a \,①\, (b \,②\, c) = (a \,①\, b) \,②\, c)$

Definition 125.
Let ①, ② be function symbols.
$\mathsf{L\text{-}Distributive}\,(\langle X, Y \rangle, \langle ①, ② \rangle) \Leftrightarrow \forall a \in X, \forall b, c \in Y (a \,①\, (b \,②\, c) = (a \,①\, b) \,②\, (a \,①\, c))$
$\mathsf{R\text{-}Distributive}\,(\langle X, Y \rangle, \langle ①, ② \rangle) \Leftrightarrow \forall a \in X, \forall b, c \in Y ((b \,②\, c) \,①\, a = (b \,①\, a) \,②\, (c \,①\, a))$
$\mathsf{Distributive}\,(\langle X, Y \rangle, \langle ①, ② \rangle) \Leftrightarrow \mathsf{L\text{-}Distributive}\,(\langle X, Y \rangle, \langle ①, ② \rangle) \wedge \mathsf{R\text{-}Distributive}\,(\langle X, Y \rangle, \langle ①, ② \rangle)$

Definition 126.
Let $\circ$ be a function symbol.
$\mathsf{L\text{-}Identive}\,(\langle S, X \rangle, \circ) \Leftrightarrow \exists e \in S, \forall a \in X (e \circ a = a)$
$\mathsf{R\text{-}Identive}\,(\langle S, X \rangle, \circ) \Leftrightarrow \exists e \in S, \forall a \in X (e \circ a = a)$
$\mathsf{Identive}\,(\langle S, X \rangle, \circ) \Leftrightarrow \mathsf{L\text{-}Identive}\,(\langle S, X \rangle, \circ) \wedge \mathsf{R\text{-}Identive}\,(\langle S, X \rangle, \circ)$

Definition 127 (Identity Element)**.**
Let $\circ$ be a function symbol.
$\ell(X, \circ) = e \Leftrightarrow \forall x \in X (x \circ e = e \circ x = x)$

Definition 128.
Let $\circ$ be a function symbol.
$\mathsf{L\text{-}Invertive}\,(\langle S, X \rangle, \circ) \Leftrightarrow \forall a \in X, \exists a' \in S (a' \circ a = \ell(X, \circ))$
$\mathsf{R\text{-}Invertive}\,(\langle S, X \rangle, \circ) \Leftrightarrow \forall a \in X, \exists a' \in S (a \circ a' = \ell(X, \circ))$
$\mathsf{Invertive}\,(\langle S, X \rangle, \circ) \Leftrightarrow \mathsf{L\text{-}Invertive}\,(\langle S, X \rangle, \circ) \wedge \mathsf{R\text{-}Invertive}\,(\langle S, X \rangle, \circ)$

Definition 129 (Group)**.**
Let $\circ$ be a function symbol.
$\mathsf{Group}\,(X,\circ) \;\Leftrightarrow\; \mathsf{Closed}\,(X,\circ) \wedge \mathsf{Associative}\,(X,\circ) \wedge \mathsf{Identive}\,(X,\circ) \wedge \mathsf{Invertive}\,(X,\circ)$

Definition 130 (Field)**.**
Let ①, ② be function symbols.
$\mathsf{Field}\,(X,\langle ①,②\rangle) \;\Leftrightarrow\; \mathsf{Group}\,(X,①) \wedge \mathsf{Group}\,(X\setminus\{\ell(X,①)\},②) \wedge \mathsf{Distributive}\,(X,\langle ①,②\rangle)$

Definition 131 (Abelian Group)**.**
Let $\circ$ be a function symbol.
$\mathsf{Abel}\,(X,\circ) \;\Leftrightarrow\; \mathsf{Group}\,(X,\circ) \wedge \mathsf{Commutative}\,(X,\circ)$

Definition 132 (Vector Space)**.**
Let $\mathsf{Field}\,(F,\langle +,\cdot\rangle)$.
$V \geqslant F \;\Leftrightarrow\; \mathsf{Abel}\,(V,+) \wedge \mathsf{Group}\,(\langle F,V\rangle,\cdot) \wedge \mathsf{Distributive}\,(\langle F,V\rangle,\langle +,\cdot\rangle)$

4

Elementary Number Theory

4.1 Prologue

Definition 133 (Composition of Functions)**.**
Let $f : X \to Y$, and $g : Y \to Z$.
$g \circ f = \{\langle x, z\rangle \mid \langle x, y\rangle \in f \wedge \langle y, z\rangle \in g\}$

Definition 134 (Function-Induced Equivalence Relation)**.**
Let $f : X \to Y$.
$x \sim_f y \Leftrightarrow f(x) = f(y)$

4.2 Natural, Integer, and Rational Number Systems

The space of natural numbers is said to be embeddable in the space of integers, and the space of integers is said to be embeddable in the space of rational numbers. A number from a set is said to have an (embedding) identity for every other set in which the number is embedded. Formally, a number's identity in a set ought to be distinguished from its identity in another set. However, for the sake of brevity, we shall not always make such a distinction.

Definition 135 (Natural Successor Function).
$\mathrm{suc} : \mathbb{N} \to \mathbb{N}$
$\mathrm{suc}(n) = n \cup \{n\}$

Definition 136 (Addition of Natural Numbers).
Let $a, b \in \mathbb{N}$.
$a + b = \mathrm{suc}^{\circ(b)}(a)$

Definition 137 (Multiplication of Natural Numbers).
Let $a, b \in \mathbb{N}$.
$a \cdot b = \mathrm{suc}^{\circ(b)^{\circ(b)}}(a)$

Definition 138 (Prototype of Integer).
$[\![\langle a, b\rangle]\!]_{\mathbb{Z}} = \{\langle c, d\rangle \in \mathbb{N}^2 \mid \exists a, b \in \mathbb{N}(a + c = b + d)\}$

Definition 139 (Space of Integers).
$\mathbb{Z} = \{[\![\langle a, b\rangle]\!]_{\mathbb{Z}} \mid a, b \in \mathbb{N}\}$

Definition 140 (Integer Identity of Natural Number).
Let $n \in \mathbb{N}$.
$n_{\mathbb{Z}} = [\![\langle n, 0\rangle]\!]_{\mathbb{Z}}$

Definition 141 (Order of Integers).
Let $a, b, c, d \in \mathbb{N}$, and $[\![\langle a, b\rangle]\!]_{\mathbb{Z}}, [\![\langle c, d\rangle]\!]_{\mathbb{Z}} \in \mathbb{Z}$.
$[\![\langle a, b\rangle]\!]_{\mathbb{Z}} \leq [\![\langle c, d\rangle]\!]_{\mathbb{Z}} \Leftrightarrow a + d \leq b + c$

Definition 142 (Addition of Integers).
Let $a, b, c, d \in \mathbb{N}$, and $[\![\langle a, b\rangle]\!]_{\mathbb{Z}}, [\![\langle c, d\rangle]\!]_{\mathbb{Z}} \in \mathbb{Z}$.
$[\![\langle a, b\rangle]\!]_{\mathbb{Z}} + [\![\langle c, d\rangle]\!]_{\mathbb{Z}} = [\![\langle a + c, b + d\rangle]\!]_{\mathbb{Z}}$

Definition 143 (Multiplication of Integers).
Let $a, b, c, d \in \mathbb{N}$, and $[\![\langle a, b\rangle]\!]_{\mathbb{Z}}, [\![\langle c, d\rangle]\!]_{\mathbb{Z}} \in \mathbb{Z}$.
$[\![\langle a, b\rangle]\!]_{\mathbb{Z}} \cdot [\![\langle c, d\rangle]\!]_{\mathbb{Z}} = [\![\langle a \cdot c + b \cdot d, a \cdot d + b \cdot c\rangle]\!]_{\mathbb{Z}}$

Definition 144 (Subtraction of Integers).
Let $a, b, c \in \mathbb{Z}$.
$a - b = c \Leftrightarrow a = b + c$

Definition 145 (Negation of Integer).
Let $n \in \mathbb{Z}$.
$-n = 0 - n$

Definition 146 (Prototype of Rational Number)**.**
Let $a, b \in \mathbb{Z}$.
$[\![\langle a, b\rangle]\!]_{\mathbb{Q}} = \{\langle c, d\rangle \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0_{\mathbb{Z}}\}) \mid \exists a, b \in \mathbb{Z}(a \cdot d = b \cdot c)\}$

Definition 147 (Space of Rational Numbers)**.**
$\mathbb{Q} = \{[\![\langle a, b\rangle]\!]_{\mathbb{Q}} \mid a \in \mathbb{Z} \land b \in \mathbb{Z} \setminus \{0_{\mathbb{Z}}\}\}$

Definition 148 (Rational Identity of Integer)**.**
Let $n \in \mathbb{Z}$.
$n_{\mathbb{Q}} = [\![\langle n, 1_{\mathbb{Z}}\rangle]\!]_{\mathbb{Q}}$

Definition 149 (Order of Rational Numbers)**.**
Let $a, b, c, d \in \mathbb{Z}$, and $[\![\langle a, b\rangle]\!]_{\mathbb{Q}}, [\![\langle c, d\rangle]\!]_{\mathbb{Q}} \in \mathbb{Q}$.
$[\![\langle a, b\rangle]\!]_{\mathbb{Q}} \leq [\![\langle c, d\rangle]\!]_{\mathbb{Q}} \Leftrightarrow a \cdot d \leq b \cdot c$

Definition 150 (Addition of Rational Numbers)**.**
Let $a, b, c, d \in \mathbb{Z}$, and $[\![\langle a, b\rangle]\!]_{\mathbb{Q}}, [\![\langle c, d\rangle]\!]_{\mathbb{Q}} \in \mathbb{Q}$.
$[\![\langle a, b\rangle]\!]_{\mathbb{Q}} + [\![\langle c, d\rangle]\!]_{\mathbb{Q}} = [\![\langle a \cdot d + b \cdot c, b \cdot d\rangle]\!]_{\mathbb{Q}}$

Definition 151 (Multiplication of Rational Numbers)**.**
Let $a, b, c, d \in \mathbb{Z}$, and $[\![\langle a, b\rangle]\!]_{\mathbb{Q}}, [\![\langle c, d\rangle]\!]_{\mathbb{Q}} \in \mathbb{Q}$.
$[\![\langle a, b\rangle]\!]_{\mathbb{Q}} \cdot [\![\langle c, d\rangle]\!]_{\mathbb{Q}} = [\![\langle a \cdot c, b \cdot d\rangle]\!]_{\mathbb{Q}}$

Definition 152 (Subtraction of Rational Numbers)**.**
Let $a, b, c \in \mathbb{Q}$.
$a - b = c \Leftrightarrow a = b + c$

Definition 153 (Negation of Rational Number)**.**
Let $q \in \mathbb{Q}$.
$-q = 0 - q$

Definition 154 (Division of Rational Numbers)**.**
Let $a, b \in \mathbb{Q}$, and $b \neq 0$.
$\dfrac{a}{b} = c \Leftrightarrow a = b \cdot c$

4.3 Interlogue

Definition 155.

Let $n \in \mathbb{Z}_{\geq 0}$.

$! : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 1}$

$0! = 1$

$n! = n \cdot \ldots \cdot 1$

Definition 156.

Let $n \in \mathbb{N}_{\geq 1}$.

$\sum\{x\} = x$

$\sum\{x_1, \ldots, x_n, x_{n+1}\} = \sum\{x_1, \ldots, x_n\} + x_{n+1}$

Definition 157.

Let $n \in \mathbb{N}_{\geq 1}$.

$\prod\{x\} = x$

$\prod\{x_1, \ldots, x_n, x_{n+1}\} = \prod\{x_1, \ldots, x_n\} + x_{n+1}$

4.4 Divisibility and Primality

An integer is said to divide another integer only if the former is a factor of the latter, or the latter is a multiple of the former. For an integer, the word “factor” is synonymous with the word “divisor”, while the word “multiple” is synonymous with the word “product”.

Theorem 19 (Euclidean Division)**.**
Let $a \in \mathbb{Z}$, and $b \in \mathbb{Z}_{\geq 1}$.
$\exists_! q \in \mathbb{Z}, \exists_! r \in \mathbb{Z}_{[0,b-1]}(a = bq + r)$

Definition 158 (Quotient)**.**
Let $a \in \mathbb{Z}$, and $b \in \mathbb{Z}_{\geq 1}$.
$a \texttt{ div } b = q \Leftrightarrow \exists r \in \mathbb{Z}_{[0,b-1]}(a = bq + r)$

Definition 159 (Remainder)**.**
Let $a \in \mathbb{Z}$, and $b \in \mathbb{Z}_{\geq 1}$.
$a \texttt{ mod } b = r \Leftrightarrow \exists q \in \mathbb{Z}(a = bq + r)$

Definition 160 (Divisibility)**.**
Let $n, N \in \mathbb{Z}$.
$n \mid N \Leftrightarrow \exists n' \in \mathbb{Z}(N = nn')$

Definition 161 (Proper Divisibility)**.**
Let $n, N \in \mathbb{Z}$.
$n \parallel N \Leftrightarrow n \mid N \wedge 0 < n < N$

Proposition 14 (Transitivity of Divisibility)**.**
Let $a, b, c \in \mathbb{Z}$.
$a \mid b \wedge b \mid c \Rightarrow a \mid c$

Proposition 15 (Additivity of Divisibility)**.**
Let $a, b, c \in \mathbb{Z}$.
$a \mid b \wedge a \mid c \Rightarrow a \mid b + c$

Proposition 16 (Multiplicativity of Divisibility)**.**
Let $a, b, c \in \mathbb{Z}$.
$a \mid b \wedge a \mid c \Rightarrow a \mid bc$

Definition 162 (Space of Prime Numbers)**.**
$⋇ = \{p \in \mathbb{Z}_{\geq 2} \mid \forall n \in \mathbb{Z}_{\geq 2}(n \mid p \Leftrightarrow n = p)\}$

Theorem 20 (Euclid)**.**
$\mathsf{Infinite}\,(⋇)$

Theorem 21 (Wilson)**.**
$p \in ⋇ \Leftrightarrow p + 1 \mid p! + 1$

Theorem 22 (Bertrand–Chebyshev)**.**
Let $n \in \mathbb{Z}_{\geq 2}$.
$\exists p \in ⋇(n < p < 2 \cdot n)$

5

Real Analysis

5.1 Prologue

Definition 163 (Partition).
$X \veebar U \Leftrightarrow \varnothing \notin X \wedge \not\cap(X) \wedge \bigcup X = U$

Definition 164 (Bipartition).
$X \veebar_2 U \Leftrightarrow \{X, X^{\complement}\} \veebar U$

Definition 165 (Riterval).
$X_{()} = \{S \subseteq X \mid \forall s \in S, \exists s' \in S(s \prec s') \wedge \forall s \in S, \exists s' \in S(s' \prec s)\}$
$X_{(-\infty,)} = \{S \subseteq X \mid \forall s \in S, \forall x \in X(x \prec s \Rightarrow x \in S) \wedge \forall s \in S, \exists s' \in S(s \prec s')\}$
$X_{(,\infty)} = \{S \subseteq X \mid \forall s \in S, \forall x \in X(s \prec x \Rightarrow x \in S) \wedge \forall s \in S, \exists s' \in S(s' \prec s)\}$

5.2 The Real Number System

Definition 166 (Space of Real Numbers)**.**

$\mathbb{R} = \{Q \in \mathbb{Q}_{(-\infty,)} \mid Q \forall_2 \mathbb{Q}\}$

Definition 167 (Real Identity of Rational Number)**.**

Let $q \in \mathbb{Q}$.

$q_{\mathbb{R}} = \{p \in \mathbb{Q} \mid p < q\}$

Definition 168 (Order of Real Numbers)**.**

Let $x, y \in \mathbb{R}$.

$x \leq y \Leftrightarrow x \subseteq y$

Definition 169 (Addition of Real Numbers)**.**

Let $A, B \in \mathbb{R}$.

$X + Y = \{q \in \mathbb{Q} \mid \exists x \in X, \exists y \in Y(q \leq x + y)\}$

Definition 170 (Subtraction of Real Numbers)**.**

Let $x, y \in \mathbb{R}$.

$x - y = z \Leftrightarrow x = y + z$

Definition 171 (Negation of Real Number)**.**

Let $x \in \mathbb{R}$.

$-x = 0 - x$

Definition 172 (Multiplication of Real Numbers)**.**

Let $X, Y \in \mathbb{R}_{>0}$.

$X \cdot Y = \{q \in \mathbb{Q} \mid \exists x \in X, \exists y \in Y(q \leq xy)\}$

$XY = -X \cdot -Y = -(X \cdot -Y) = -(-XY)$

Definition 173 (Division of Real Numbers)**.**

Let $x \in \mathbb{R}$, and $y \in \mathbb{R} \setminus \{0\}$.

$\frac{x}{y} = z \Leftrightarrow x = yz$

$\frac{x}{y} = \frac{-x}{-y} = -\frac{x}{-y} = -\frac{-x}{y}$

Definition 174 (Real Exponentiation Function)**.**

Let $x \in \mathbb{R}$.

$\mathsf{e}^x = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{n} x}{n!}$

Definition 175 (Real Logarithmic Function)**.**

Let $x \in \mathbb{R}_{>0}$.

$\ln(x) = y \Leftrightarrow x = \exp(y)$

Definition 176 (Exponentiation of Real Numbers)**.**
Let $n \in \mathbb{Z}$, and $x \in \mathbb{R}_{>0}$, and $y \in \mathbb{R}$.
$x^y = \exp(y \ln(x))$
$0^x = 0$
$(-x)^n = \prod_{i=1}^{n}(-x)$

Definition 177 (Root of Real Numbers)**.**
Let $n \in \mathbb{Z}_{\geq 1}$, and $x \in \mathbb{R}$.
$\sqrt[n]{x} = x^{\frac{1}{n}}$

Definition 178 (Logarithm of Real Numbers)**.**
Let $x, y \in \mathbb{R}_{>0}$.
$\log_x(y) = \dfrac{\ln(y)}{\ln(x)}$

Definition 179 (Space of Extended Real Numbers)**.**
$\underline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$

5.3 Interlogue

Definition 180 (Pi).

$$\pi = 4\sum_{i=0}^{\infty}\frac{(-1)^i}{2i+1}$$

Definition 181 (Sine).

$$\sin(x) = \sum_{i=0}^{\infty}\frac{(-1)^i x^{2i+1}}{(2i+1)!}$$

Definition 182 (Cosine).

$$\cos(x) = \sum_{i=0}^{\infty}\frac{(-1)^i x^{2i}}{(2i)!}$$

5.4 The Real Vector System

Definition 183 (Addition of Real Vectors)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$.
$\boldsymbol{u} + \boldsymbol{v} = [u_i + v_i]_{1..i..n}$

Definition 184 (Subtraction of Real Vectors)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$.
$\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{w} \Leftrightarrow \boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$

Definition 185 (Scalar Multiplication of Real Vectors)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $c \in \mathbb{R}$, and $\boldsymbol{v} \in \mathbb{R}^n$.
$c \cdot \boldsymbol{v} = [cv_i]_{1..i..n}$

Proposition 17.
Let $n \in \mathbb{N}_{\geq 1}$.
$\mathbb{R}^n \geqslant \mathbb{R}$

Definition 186 (Dot Product of Real Vectors)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$.
$\boldsymbol{u} \cdot \boldsymbol{v} = \displaystyle\sum_{i=1}^{n} u_i v_i$

Definition 187 (Euclidean Norm of Real Vectors)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{v} \in \mathbb{R}^n$.
$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$

Proposition 18.
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, and $\boldsymbol{u}, \boldsymbol{v} \neq 0^n$.
$\exists_! \theta \in \mathbb{R}_{[0,\pi]} \left(\cos(\theta) = \dfrac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} \right)$

Definition 188 (Angle between Real Vectors)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, and $\boldsymbol{u}, \boldsymbol{v} \neq 0^n$.
$\angle(\boldsymbol{u}, \boldsymbol{v}) = \cos^{-1} \left(\dfrac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|} \right)$

Theorem 23.
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$.
$0 \leq \angle(\boldsymbol{u}, \boldsymbol{v}) < \dfrac{\pi}{2} \Leftrightarrow \boldsymbol{u} \cdot \boldsymbol{v} > 0$
$\angle(\boldsymbol{u}, \boldsymbol{v}) = \dfrac{\pi}{2} \Leftrightarrow \boldsymbol{u} \cdot \boldsymbol{v} = 0$
$\dfrac{\pi}{2} < \angle(\boldsymbol{u}, \boldsymbol{v}) \leq \pi \Leftrightarrow \boldsymbol{u} \cdot \boldsymbol{v} < 0$

Definition 189 (Cross Product of Real Vectors)**.**
Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$.
$$\boldsymbol{u} \times \boldsymbol{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$$

Proposition 19.

Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$.

$\boldsymbol{u} \times \boldsymbol{v} = -(\boldsymbol{v} \times \boldsymbol{u})$

Theorem 24.

Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$.

$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{u} = (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{v} = 0$

Theorem 25.

Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$.

$\|\boldsymbol{u} \times \boldsymbol{v}\| = \sqrt{\|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2 - (\boldsymbol{u} \cdot \boldsymbol{v})^2}$

Theorem 26.

Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$.

$\angle(\boldsymbol{u}, \boldsymbol{v}) = \sin^{-1}\left(\frac{\|\boldsymbol{u} \times \boldsymbol{v}\|}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\right)$

5.5 Interlogue

Definition 190 (Space of Partial Functions).
$\{X \rightharpoonup Y\} = \{f \subseteq X \times Y \mid \forall x \in X, \exists y \in Y(\langle x, y\rangle \in f)\}$

Definition 191 (Partial Function).
$f : X \rightharpoonup Y \Leftrightarrow f \in \{X \rightharpoonup Y\}$
$f(x) = y \Leftrightarrow \langle x, y\rangle \in f$

Definition 192 (Circle).
Let $\boldsymbol{c} \in \mathbb{R}^2$, and $r \in \mathbb{R}_{\geq 0}$.
$\text{circle}(\boldsymbol{c}, r) = \{\boldsymbol{x} \in \mathbb{R}^2 \mid (x - c_1)^2 + (y - c_2)^2 = r^2\}$

Definition 193 (Plane).
Let $\boldsymbol{n}, \boldsymbol{p} \in \mathbb{R}^3$.
$\text{plane}(\boldsymbol{n}, \boldsymbol{p}) = \{\boldsymbol{x} \in \mathbb{R}^3 \mid \boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{p}) = 0\}$

5.6 Limits and Derivatives

To find $f(x, y)$'s gradient whole,
break f for ev'ry input role;
down input tree, each path you trace,
combine path rates, then sum with grace.

Definition 194 (Distance Function of $\mathbb{R}^n$)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$.
$\operatorname{dist}(a, b) = \|\boldsymbol{a} - \boldsymbol{b}\|$

Definition 195 (Limit of Real Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $\boldsymbol{x}_* \in \operatorname{dom}(f)$.
$\lim\limits_{\boldsymbol{x}_*} f : \operatorname{dom}(f) \rightharpoonup \underline{\mathbb{R}}$

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}_*} f(\boldsymbol{x}) = \begin{cases} y_*, & \forall \varepsilon \in \mathbb{R}_{>0}, \exists \delta \in \mathbb{R}_{>0} (\operatorname{dist}(\boldsymbol{x}, \boldsymbol{x}_*) < \delta \Rightarrow \operatorname{dist}(f(\boldsymbol{x}), y_*) < \varepsilon) \\ -\infty, & \forall \varepsilon \in \mathbb{R}_{>0}, \exists \delta \in \mathbb{R}_{>0} (\operatorname{dist}(\boldsymbol{x}, \boldsymbol{x}_*) < \delta \Rightarrow f(\boldsymbol{x}) < \varepsilon) \\ \infty, & \forall \varepsilon \in \mathbb{R}_{>0}, \exists \delta \in \mathbb{R}_{>0} (\operatorname{dist}(\boldsymbol{x}, \boldsymbol{x}_*) < \delta \Rightarrow f(\boldsymbol{x}) > \varepsilon) \end{cases}$$

Definition 196 (Partial Derivative of Real Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $k \in \mathbb{N}_{[1,n]}$.
$\partial_k f : \operatorname{dom}(f) \rightharpoonup \underline{\mathbb{R}}$
$$\partial_k f(\boldsymbol{x}) = \lim_{\varepsilon \to 0} \left(\frac{f(\boldsymbol{x}|_{x_k}^{x_k + \varepsilon}) - f(\boldsymbol{x})}{\varepsilon} \right)$$

Definition 197 (Gradient of Real Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$.
$\boldsymbol{\nabla} f : \operatorname{dom}(f) \rightharpoonup \underline{\mathbb{R}}^n$
$\boldsymbol{\nabla} f(\boldsymbol{x}) = [\partial_i f(\boldsymbol{x})]_{1..i..n}$

Definition 198 (Directional Derivative of Real Function on Real Unit Vector)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $\hat{\boldsymbol{u}} \in \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\| = 1\}$.
$\boldsymbol{\nabla}_{\boldsymbol{u}} f : \mathbb{R}^n \rightharpoonup \mathbb{R}$
$\boldsymbol{\nabla}_{\boldsymbol{u}} f(\boldsymbol{x}) = \boldsymbol{u} \cdot \boldsymbol{\nabla} f(\boldsymbol{x})$

5.7 Continuity and Differentiability

Definition 199 (Continuous Real Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $D \subseteq \mathrm{dom}(f)$.
$\mathsf{Continuous}\,(f/D) \Leftrightarrow \forall \boldsymbol{d} \in D \left(\lim_{\boldsymbol{x} \to \boldsymbol{d}} f(\boldsymbol{x}) = f(\boldsymbol{d}) \right)$

Definition 200 (Differentiable Real Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $D \subseteq \mathrm{dom}(f)$.
$\mathsf{Differentiable}\,(f/D) \Leftrightarrow \forall \boldsymbol{d} \in D, \exists \boldsymbol{\nabla} f(\boldsymbol{d})$

Proposition 20.
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $D \subseteq \mathrm{dom}(f)$.
$\mathsf{Differentiable}\,(f/D) \Rightarrow \mathsf{Continuous}\,(f/D)$

Definition 201.
$f : X \overset{\mathsf{c}}{\to} Y \Leftrightarrow f : X \to Y \wedge \mathsf{Continuous}\,(f/X)$

5.8 Interlogue

For a symmetric matrix, the property of indefiniteness can be seen as the complement of the conjunction of the properties of positive-definiteness and negative-definiteness.

Definition 202 (Hessian)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$.
$\operatorname{hes} f : \mathbb{R}^n \rightharpoonup \underline{\mathbb{R}}^n$
$\operatorname{hes} f(\boldsymbol{x}) = [\partial_{i,j} f(\boldsymbol{x})]_{\substack{1..i..n,\\ 1..j..n}}$

Definition 203 (Symmetric Matrix)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $M \in X^{n \times n}$.
$\mathsf{Symmetric}\,(M) \Leftrightarrow \forall i, j \in \mathbb{N}_{[1,n]}(M_{i,j} = M_{j,i})$

Definition 204 (Positive-Definite Real Matrix)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $M \in \mathbb{R}^{n \times n}$, and $\mathsf{Symmetric}\,(M)$.
$\mathsf{PositiveDefinite}\,(M) \Leftrightarrow \forall \boldsymbol{x} \in \mathbb{R}^n \setminus \{0^n\}(\boldsymbol{x}^\intercal M \boldsymbol{x} > 0)$

Definition 205 (Negative-Definite Real Matrix)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $M \in \mathbb{R}^{n \times n}$, and $\mathsf{Symmetric}\,(M)$.
$\mathsf{NegativeDefinite}\,(M) \Leftrightarrow \forall \boldsymbol{x} \in \mathbb{R}^n \setminus \{0^n\}(\boldsymbol{x}^\intercal M \boldsymbol{x} < 0)$

Definition 206 (Indefinite Real Matrix)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $M \in \mathbb{R}^{n \times n}$, and $\mathsf{Symmetric}\,(M)$.
$\mathsf{Indefinite}\,(M) \Leftrightarrow \exists \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n \setminus \{0^n\}(\boldsymbol{x}^\intercal M \boldsymbol{x} > 0 \wedge \boldsymbol{y}^\intercal M \boldsymbol{y} < 0)$

Definition 207 (Jacobian)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}^m$.
$\operatorname{jac} f : \mathbb{R}^n \rightharpoonup \underline{\mathbb{R}}^{m \times n}$
$\operatorname{jac} f(\boldsymbol{x}) = [\partial_j f_i(\boldsymbol{x})]_{\substack{1..i..m,\\ 1..j..n}}$

5.9 Optimisation

Definition 208 (Level Set)**.**
Let $f : X \to Y$, and $h \in \text{img}(f)$.
$\text{lev}_f(h) = \{x \in \text{dom}(f) \mid f(x) = h\}$

Definition 209 (Local Minimum of Real Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $\boldsymbol{x}_* \in \text{dom}(f)$.
$\boldsymbol{x}_* \ \mathsf{locmin}\ f \Leftrightarrow \exists \varepsilon \in \mathbb{R}_{>0}, \forall \boldsymbol{x} \in \text{dom}(f)(\text{dist}(\boldsymbol{x}_*, \boldsymbol{x}) \leq \varepsilon \Rightarrow f(\boldsymbol{x}) \geq f(\boldsymbol{x}_*))$

Definition 210 (Local Maximum of Real Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $\boldsymbol{x}_* \in \text{dom}(f)$.
$\boldsymbol{x}_* \ \mathsf{locmax}\ f \Leftrightarrow \exists \varepsilon \in \mathbb{R}_{>0}, \forall \boldsymbol{x} \in \text{dom}(f)(\text{dist}(\boldsymbol{x}, \boldsymbol{x}_*) \leq \varepsilon \Rightarrow f(\boldsymbol{x}) \leq f(\boldsymbol{x}_*))$

Definition 211 (Local Extremum of Real Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $\boldsymbol{x}_* \in \text{dom}(f)$.
$\boldsymbol{x}_* \ \mathsf{locext}\ f \Leftrightarrow \boldsymbol{x}_* \ \mathsf{locmin}\ f \vee \boldsymbol{x}_* \ \mathsf{locmax}\ f$

Definition 212 (Critical Point of Real Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $\boldsymbol{x}_* \in \text{dom}(f)$.
$\boldsymbol{x}_* \ \mathsf{crit}\ f \Leftrightarrow \boldsymbol{\nabla} f(\boldsymbol{x}_*) = 0^n \vee \nexists \boldsymbol{\nabla} f(\boldsymbol{x}_*)$

Theorem 27.
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $\boldsymbol{x}_* \in \text{dom}(f)$.
$\boldsymbol{x}_* \ \mathsf{crit}\ f \wedge \mathsf{NegativeDefinite}\,(\text{hes}(f)) \Rightarrow \boldsymbol{x}_* \ \mathsf{locmax}\ f$
$\boldsymbol{x}_* \ \mathsf{crit}\ f \wedge \mathsf{PositiveDefinite}\,(\text{hes}(f)) \Rightarrow \boldsymbol{x}_* \ \mathsf{locmin}\ f$

Theorem 28.
Let $f : \mathbb{R} \to \mathbb{R}$, and $x_* \in \text{dom}(f)$.
$x_* \ \mathsf{crit}\ f \wedge \boldsymbol{\nabla}^2 f(x_*) < 0 \Rightarrow x_* \ \mathsf{locmax}\ f$
$x_* \ \mathsf{crit}\ f \wedge \boldsymbol{\nabla}^2 f(x_*) > 0 \Rightarrow x_* \ \mathsf{locmin}\ f$

Theorem 29.
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $\boldsymbol{x}_* \in \text{dom}(f)$.
$\boldsymbol{x}_* \ \mathsf{locext}\ f \Rightarrow \boldsymbol{\nabla} f(\boldsymbol{x}_*) = 0^n$

Definition 213 (Linearly Independent Real Vectors)**.**
Let $k, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$.
$$\mathsf{Independent}\,(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k) \Leftrightarrow \nexists c_1, \ldots, c_k \in \mathbb{R} \setminus \{0\} \left(\sum_{i=1}^{k} c_i \boldsymbol{v}_i = 0^n \right)$$

Definition 214 (Interior of Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : X \to Y$.
$\text{int}(f) = \{i \in \text{dom}(f) \mid \exists \varepsilon \in \mathbb{R}_{>0}, \forall i' \in \text{dom}(f)(\text{dist}(i, i') < \varepsilon \Rightarrow i' \in \text{dom}(f))\}$

Definition 215 (Equality-Constrained Function)**.**
Let $f : A \to B$, and $\boldsymbol{g} : A \to B^k$.
$f|_{\boldsymbol{g}} = f' \Leftrightarrow f' : \{x \in \text{dom}(f) \mid \forall i \in \mathbb{N}_{[1,k]}(g_i(x) = 0)\} \to B$

Theorem 30 (Lagrange Multiplier)**.**

Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $k \in \mathbb{N}_{[1,n]}$, and $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^k$, and $\boldsymbol{x}_* \in \operatorname{int}(f)$.

$\boldsymbol{x}_* \ \mathsf{locext}\ f|_{\boldsymbol{g}} \wedge \mathsf{Independent}\left(\boldsymbol{\nabla g}(\boldsymbol{x}_*)\right) \Rightarrow \exists_! \boldsymbol{\lambda} \in \mathbb{R}^k \left(\boldsymbol{\nabla} f(\boldsymbol{x}) = \boldsymbol{\lambda} \cdot \boldsymbol{\nabla g}(\boldsymbol{x})\right)$

5.10 Integrals

Definition 216 (Integral of Real Unary Function)**.**
Let $f : \mathbb{R} \to \mathbb{R}$, and $a, b \in \operatorname{dom}(f)$.

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} f(\Cap_i^{a,b,n})\ \Cap_{i,i-1}^{a,b,n}$$

Definition 217 (Double Integral of Real Binary Function)**.**
Let $f : \mathbb{R}^2 \to \mathbb{R}$, and $a, b, c, d \in \mathbb{R}$, and $R = \mathbb{R}_{[a,b]} \times \mathbb{R}_{[c,d]}$.

$$\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \lim_{n_2\to\infty} \lim_{n_1\to\infty} \sum_{j=1}^{n_2} \sum_{i=1}^{n_1} f(\Cap_i^{a,b,n_1}, \Cap_j^{c,d,n_2})\ \Cap_{i,i-1}^{a,b,n_1} \Cap_{j,j-1}^{c,d,n_2}$$

$$\int_c^d \int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

Theorem 31 (Fubini)**.**
Let $f : \mathbb{R}^2 \to \mathbb{R}$, and $a, b, c, d \in \mathbb{R}$, and $R = \mathbb{R}_{[a,b]} \times \mathbb{R}_{[c,d]}$.

$$\mathsf{Continuous}\,(f/R) \Rightarrow \int_c^d \int_a^b f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \int_c^d f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

Theorem 32 (Horizontal Simplicity)**.**
Let $a, b \in \mathbb{R}$, and $f : \mathbb{R}^2 \to \mathbb{R}$, and $l, u : \mathbb{R}_{[a,b]} \xrightarrow{\mathsf{c}} \mathbb{R}$, and $R = \{\langle x, y\rangle \in \mathbb{R}^2 \mid a \le y \le b \wedge l(y) \le x \le u(y)\}$.

$$\iint_R f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \int_{l(y)}^{u(y)} f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

Theorem 33 (Vertical Simplicity)**.**
Let $a, b \in \mathbb{R}$, and $f : \mathbb{R}^2 \to \mathbb{R}$, and $l, u : \mathbb{R}_{[a,b]} \xrightarrow{\mathsf{c}} \mathbb{R}$, and $R = \{\langle x, y\rangle \in \mathbb{R}^2 \mid a \le x \le b \wedge l(x) \le y \le u(x)\}$.

$$\iint_R f(x,y)\,\mathrm{d}y\,\mathrm{d}x = \int_a^b \int_{l(x)}^{u(x)} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

III

Postamble

6

Appendix

Under ZFC, every object is a set. For this reason, we shall use the "usual" English and Greek letters to denote sets, and other notation to denote constant symbols (i.e. particular sets).

Notation 9.
$x \ \eqqcolon\ (x)$

Notation 10.
Let (φ) be a formula.
$\varphi \ \eqqcolon\ (\varphi)$

Notation 11.
Let φ be a formula.
$x_\circ, \ldots, x_\bullet(\varphi) \ \eqqcolon\ x_\circ(\ldots(x_\bullet(\varphi)))$

Notation 12.
Let φ be a formula.
$\not\forall x(\varphi) \ \eqqcolon\ \neg\forall x(\varphi)$
$\nexists x(\varphi) \ \eqqcolon\ \neg\exists x(\varphi)$

Notation 13 (Bracketless Existential Quantification)**.**
Let $n \in \mathbb{N}_{\geq 1}$.
$\exists x_1, \ldots, x_n \ \eqqcolon\ \exists \bullet_1, \ldots, \bullet_n(\bullet_1 = x_1 \wedge \ldots \wedge \bullet_n = x_n)$

Notation 14 (Left-Associativity of Function Symbol)**.**
Let $\circ$ be a function symbol.
$a \circ \cdots \circ y \circ z \ \eqqcolon\ (a \circ \cdots \circ y) \circ z$

Notation 15 (Iterated Composition)**.**
Let $\circ$ be a function symbol, and $x_\circ = \cdots = x_\bullet$.
$x^{\circ(n)} \ \eqqcolon\ (x_1 \circ \cdots \circ x_n)$

Notation 16 (Universally-Quantified Implication)**.**
Let R be a relation symbol.
$\forall x \ \mathsf{R}\ x'(\varphi) \ \eqqcolon\ \forall x((x \ \mathsf{R}\ x') \Rightarrow \varphi)$

Notation 17 (Reflectable Relation)**.**
Let R be a relation symbol.
$(x \ \text{Я}\ y) \ \eqqcolon\ (y \ \mathsf{R}\ x)$

Notation 18 (Cancellable Relation).

Let R be a relation symbol.

$(x \not\mathsf{R}\, y) \;\coloneqq\; (\neg(x \;\mathsf{R}\; y))$

Notation 19 (Order-Equatable Relation).

Let $\underline{\mathsf{R}}$ be a relation symbol.

$(x \;\mathsf{R}\; y) \;\coloneqq\; ((x \;\underline{\mathsf{R}}\; y) \vee (x \neq y))$

Notation 20.

Let $n \in \mathbb{N}_{\geq 1}$.

$x_1, \ldots, x_n \;\coloneqq\; \langle x_1, \ldots, x_n \rangle$

Notation 21 (Space of Matrices).

Let $m, n \in \mathbb{N}$.

$X^{m \times n} \;\coloneqq\; (X^n)^m$

Notation 22 (Delimited Function).

Let d and d' be delimiter symbols.

$(\mathsf{d}\bullet\mathsf{d}' : X \to Y) \;\coloneqq\; (\bullet_{\mathsf{dd}'} : X \to Y)$

$(\mathsf{d}x\mathsf{d}' = y) \;\coloneqq\; (\bullet_{\mathsf{dd}'}(x) = y)$

Notation 23 (Comprehensible Function).

Let $\circ$ be a function symbol, and $\varphi_\circ, \ldots, \varphi_\bullet$ be formulas.

$\circ_{\varphi_\circ, \ldots, \varphi_\bullet} f \;\coloneqq\; \{f \mid \varphi_\circ \wedge \ldots \wedge \varphi_\bullet\}$

Notation 24 (Interval-Comprehensible Function).

Let $\circ$ be a function symbol.

$\circ_{i=a}^{b} f \;\coloneqq\; \circ_{i \in [a,b]} f$

$\circ_{i=-\infty}^{b} f \;\coloneqq\; \circ_{i \in (-\infty,b]} f$

$\circ_{i=a}^{\infty} f \;\coloneqq\; \circ_{i \in [a,\infty)} f$

Definition 218.

$\mathsf{Multipliable}\,(a, b) \;\Leftrightarrow\; \exists a \cdot b$

Notation 25.

Let $\mathsf{Multipliable}\,(a, b)$.

$ab \;\coloneqq\; a \cdot b$

Notation 26 (Element of Additive Identity).

$0_X \;\coloneqq\; \ell(X, +)$

Notation 27 (Element of Multiplicative Identity).

$1_X \;\coloneqq\; \ell(X, \cdot)$

Notation 28.

Let $n \in \mathbb{N}_{\geq 1}$, and $f : X \to Y$.

$\partial_{1,\ldots,n} f(\boldsymbol{x}) \;\coloneqq\; \partial_n(\ldots(\partial_1 f(\boldsymbol{x})))$

Notation 29.

Let $n \in \mathbb{N}_{\geq 1}$, and $f : X \to Y$.

$\boldsymbol{\nabla}^n f \eqcirc (f)^{\boldsymbol{\nabla}(n)}$

Notation 30 (Riemann Pillar)**.**

$\sqcap_i^{a,b,n} \eqcirc a + i\left(\dfrac{b-a}{n}\right)$

$\sqcap_{i,i-1}^{a,b,n} \eqcirc \sqcap_i^{a,b,n} - \sqcap_{i-1}^{a,b,n}$

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Bibliography

[1] Shai Ben-David. *CS245: Logic and Computation*. Fall 2015, University of Waterloo. 2015.

[2] Paul Cohen. “The Independence of the Continuum Hypothesis”. In: *Proceedings of the National Academy of Sciences* 50.6 (1963), pp. 1143–1148. DOI: `10.1073/pnas.50.6.1143`.

[3] *Dedekind Cut*. `http://ncatlab.org/nlab/show/Dedekind+cut`.

[4] Thomas Jech. *Set Theory*.

[5] Jeff Paris and Leo Harrington. “A Mathematical Incompleteness in Peano Arithmetic”. In: *Handbook of Mathematical Logic*. Ed. by Jon Barwise and H. Jerome Keisler. Amsterdam; New York: North-Holland, 1977, pp. 1133–1142. ISBN: 978-0-7204-2285-6.

[6] Terence Tao. *Math 121: Introduction to Topology*. Spring 2000, University of California, Los Angeles. 2000.