GALOIS PARSIMONY

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Abstract. In philosophy, Occam's razor is a principle stating that "simpler" explanations ought to be preferred over "less simple" ones. In mathematics, a sentential Galois connection is a Galois connection between a space of first-order sentences and a space of models. In this paper, we connect Occam's razor to sentential Galois connections, establishing an association between the older philosophical principle and newer mathematical idea.

1. Introduction

Historically, Occam's razor has been stated as, "entia non sunt multiplicanda praeter necessitatem," which roughly translates to "entities must not be multiplied beyond necessity."[1] The razor is occasionally described as a principle of parsimony. It has been used to guide human reasoning in many areas, serving as a general "heuristic" or "rule" for selecting between competing explanations. In physics, for instance, one often has to deal with competing hypotheses. Assuming the hypotheses *can* be tested experimentally, Occam's razor could motivate prioritising the "simplest" one. One might ask, however, why simplicity or parsimony should be a reliable guide to truth. We propose that an examination of first-order mathematical models can offer insight.

Fixing a language $\mathcal{L}$, let $\mathbb{S}$ be the space of sentences in $\mathcal{L}$, and $\mathbb{M}$ be the space of models in $\mathcal{L}$.

Definition 1 (Subject of Set of Sentences)**.** Let $\Phi \subseteq \mathbb{S}$.

$$\operatorname{subj}(\Phi) = \{m \in \mathbb{M} : \forall \varphi \in \Phi (m \vDash \varphi)\}$$

Definition 2 (Theory of Set of Models)**.** Let $M \subseteq \mathbb{M}$.

$$\operatorname{th}(M) = \{\varphi \in \mathbb{S} : \forall m \in M (m \vDash \varphi)\}$$

For a set of sentences Φ, the subject of Φ is the set of models satisfying every sentence in Φ. Similarly, for a set of models M, the theory of M is the set of sentences satisfied by every model in M.

Theorem 1 (Sentential Galois Connection)**.** *Let* $\Phi \subseteq \mathbb{S}$*, and* $M \subseteq \mathbb{M}$*.*

$$\Phi \subseteq \operatorname{th}(M) \iff M \subseteq \operatorname{subj}(\Phi)$$

Proof. Let $\Phi \subseteq \mathbb{S}$, and $M \subseteq \mathbb{M}$.

In the forward direction, if Φ is a subset of $\operatorname{th}(M)$, then Φ is a set of sentences satisfied by every model in M. This means $\operatorname{subj}(\Phi)$ is the set containing *every*

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model for which every sentence in Φ is satisfied. Thus, if M is a *particular* set of models for which every sentence in Φ is satisfied, then M contains no more model than $\mathrm{subj}(\Phi)$, i.e. M is a subset of $\mathrm{subj}(\Phi)$.

Similarly, in the backward direction, if M is a subset of $\mathrm{subj}(\Phi)$, then M is a set of models satisfying every sentence in Φ. This means $\mathrm{th}(M)$ is the set containing *every* sentence satisfied by every model in M. Thus, if Φ is a *particular* set of sentences for which every model in M is satisfied, then Φ contains no more sentence than $\mathrm{th}(M)$, i.e. Φ is a subset of $\mathrm{th}(M)$. □

To make intuitive sense of Theorem 1, one can consider how subsets of the space of sentences relate to subsets of the space of models. For any set of sentences $\Phi \subseteq \mathbb{S}$, one can consider its subject $\mathrm{subj}(\Phi) \subseteq \mathbb{M}$. Increasing the number of sentences in Φ will either maintain or decrease the number of models in $\mathrm{subj}(\Phi)$, since a conjunction with more sentences will either be satisfied by the same set of models, or a smaller set of models. Intuitively, adding to one's set of requirements (on models) can only ever keep or shrink the set of models satisfying what one requires. Similarly, for any set of models $M \subseteq \mathbb{M}$, one can consider its theory $\mathrm{th}(M) \subseteq \mathbb{S}$. Increasing the number of models in M will either maintain or decrease the number of models in $\mathrm{th}(M)$, since a larger set of models will either satisfy the same set of sentences, or a smaller set of sentences. Intuitively, adding to one's set of models can only ever keep or shrink one's set of requirements (on models).

2. The Association

A hypothesis can be defined as a "testable" set of assumptions. Under this definition, every hypothesis is a set of assumptions, but not every set of assumptions is a hypothesis. Broadly, what we defined as *subjects* and *theories* could correspond to *hypotheses* and *universes*, respectively, in a different (e.g. the "usual" physical) setting. In particular, a set of sentences can be seen as a set of assumptions, and a set of models can be seen as a set of universes. Thus, if H is said to be a hypothesis, then subj(H) is said to be the set of universes satisfying H. Suppose one has a finite hypothesis H whose subject is a finite set of universes U. One may (already) possess the intuition that increasing the number of assumptions in H could only ever decrease U, if changing H *had* to either increase or decrease the size of U. This intuition would (directly) correspond to a particular case of the backward direction of Theorem 1, which extends this intuition to infinite hypotheses and sets of universes. In this way, "simplicity" in the hypothesis of one's assumptions could correspond to "generality" in the explanatory ability of the hypothesis.

References

[1] Jonathan Schaffer, *What not to multiply without necessity*, Australasian Journal of Philosophy **93** (2015), no. 4, 644–664, DOI 10.1080/00048402.2014.992447. `https://doi.org/10.1080/00048402.2014.992447`.