PATTERNS

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About

The live file is available for free at `http://github.com/glimeuxe/papers`, where updates are made on an occasional basis. I thank all those who have made suggestions to improve the text.

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Classical Logic

2.1 Prologue

What could it mean for something to be *true*? Throughout history, various definitions for the word "truth" have been proposed.

Definition 1 (Correspondence Theory of Truth)**.**
Truth is that which corresponds to reality.

Definition 2 (Coherence Theory of Truth)**.**
Truth is that which coheres with every other truth.

Ought mathematics concern itself with the universe we inhabit? In any case, the practice of mathematics seems to be based on something like the coherence theory of truth; this presents a *seeming* trilemma.

Definition 3 (Münchhausen Trilemma)**.**
Every proof is completed by circularity, infinite regress, or assumption.

Reasoning by coherence appears to obtain truth by circularity, infinite regress, or assumption. It is no secret that mathematical theories admit assumptions. Mathematical theories are also said to build upon, and thus derive from, each other. A theory from which other theories can be derived is said to be *foundational*.

Broadly speaking, a logic can be thought of as a language for reasoning about truth — that is, a system which prescribes symbols, and ways of interchanging those symbols. This text studies propositional and predicate logics, in particular.

With any human language, definitions may be said to originate in *pre-linguistic* thought. Writers sometimes choose their starting points based on what feels the most intuitive to them. For this text, I have decided to use words and phrases like "not", "and", "if. . . then", "either. . . or", "otherwise", "every", "same", in addition to a few other mathematical symbols, in my appeal to the intuition of the reader.

2.2 Language

2.2.1 Propositional Logic

Definition 4 (Propositional Formula)**.**

Let $p_\circ, \ldots, p_\bullet$ be propositional variables.

1. If p is a propositional variable, then p is a propositional formula.
2. If φ is a propositional formula, then $(\neg\varphi)$ is a propositional formula.
   If φ and φ' are propositional formulas, then $(\varphi \wedge \varphi')$ is a propositional formula.

Definition 5 (Truth Value of Propositional Formula)**.**

1. Every propositional variable is either true or false.
2. If φ is true, then $(\neg\varphi)$ is false. Otherwise, $(\neg\varphi)$ is true.
   If φ and φ' are true, then $(\varphi \wedge \varphi')$ is true. Otherwise, $(\varphi \wedge \varphi')$ is false.

2.2.2 Predicate Logic

Definition 6 (Formula)**.**

Let $x_\circ, \ldots, x_\bullet$ be variables.

1. If x is a variable, and x' is a variable, then $(x = x')$ is a formula.
   If x is a variable, and x' is a variable, then $(x \in x')$ is a formula.
2. If φ is a formula, then $(\neg\varphi)$ is a formula.
   If φ is a formula, and φ' is a formula, then $(\varphi \wedge \varphi')$ is a formula.
   If x is a variable, and φ is a formula, then $\forall x(\varphi)$ is a formula.

Definition 7 (Free Variable)**.**

1. If $(x = x')$ is a formula, then x and x' are free variables in the formula.
   If $(x \in x')$ is a formula, then x and x' are free variables in the formula.
2. If x is a free variable in φ, then x is a free variable in $(\neg\varphi)$.
   If x is a free variable in φ and φ', then x is a free variable in $(\varphi \wedge \varphi')$.
   If x is a free variable in φ, and $(x \neq x')$, then x is a free variable in $\forall x'(\varphi)$.

Notation 1 (Naïve Variable Substitution)**.**

$(x = x)\{^{\tau}_{x} \eqcirc (\tau = \tau)$
$(x = x')\{^{\tau}_{x} \eqcirc (\tau = x')$
$(x' = x)\{^{\tau}_{x} \eqcirc (x' = \tau)$
$(x' = x')\{^{\tau}_{x} \eqcirc (x' = x')$
$(x \in x)\{^{\tau}_{x} \eqcirc (\tau \in \tau)$
$(x \in x')\{^{\tau}_{x} \eqcirc (\tau \in x')$
$(x' \in x)\{^{\tau}_{x} \eqcirc (x' \in \tau)$
$(x' \in x')\{^{\tau}_{x} \eqcirc (x' \in x')$
$(\neg\varphi)\{^{\tau}_{x} \eqcirc (\neg\varphi\{^{\tau}_{x})$
$(\varphi \wedge \varphi')\{^{\tau}_{x} \eqcirc (\varphi\{^{\tau}_{x} \wedge \varphi'\{^{\tau}_{x})$
$\forall x(\varphi)\{^{\tau}_{x} \eqcirc \forall \tau(\varphi\{^{\tau}_{x})$

Definition 8 (Truth Value of Formula)**.**

1. Every variable is a set.
2. If x is the same set as x', then $(x = x')$ is true. Otherwise, $(x = x')$ is false.
   If φ is true, then $(\neg\varphi)$ is false. Otherwise, $(\neg\varphi)$ is true.
   If φ and φ' are true, then $(\varphi \wedge \varphi')$ is true. Otherwise, $(\varphi \wedge \varphi')$ is false.
   If φ is true for every possible x, then $\forall x(\varphi)$ is true. Otherwise, $\forall x(\varphi)$ is false.

2.3 Interlogue

In the interest of brevity, we shall omit pairs of delimiters (i.e. brackets) wherever we can. To do this, we shall stipulate that within every bold-faced environment (i.e. "**Definition**", "**Theorem**", etc.), and pair of delimiters within those environments, the glyphs which appear closer together ought to be read before the glyphs which appear further apart.[1]

The negation of a conjunction of formulas is equivalent to the disjunction of the negations of those formulas. Similarly, the negation of a disjunction of formulas is equivalent to the conjunction of the negations of those formulas. These equivalences are known as De Morgan laws, and are said to illustrate a "duality" between conjunction and disjunction, with respect to negation.

Notation 2 (Connective of Disjunction)**.**
$\varphi \vee \varphi' \eqcirc \neg(\neg\varphi \wedge \neg\varphi')$

Notation 3 (Connective of Implication)**.**
$\varphi \Rightarrow \varphi' \eqcirc \neg(\varphi \wedge \neg\varphi')$

Notation 4 (Connective of Equivalence)**.**
$\varphi \Leftrightarrow \varphi' \eqcirc (\varphi \Rightarrow \varphi') \wedge (\varphi' \Rightarrow \varphi)$

Notation 5 (Connective of Exclusive Disjunction)**.**
$\varphi \bigvee \varphi' \eqcirc \neg(\varphi \Leftrightarrow \varphi')$

Notation 6 (Quantifier of Existence)**.**
$\exists x(\varphi) \eqcirc \not\forall x(\neg\varphi)$

Notation 7 (Quantifier of Unique Existence)**.**
$\exists_{!} x(\varphi) \eqcirc \exists x, \forall x'(\varphi\{_x^{x'} \Leftrightarrow x = x')$

Notation 8 (Quantifier of Dichotomous Existence)**.**
$\mathsf{E} x(\varphi) \eqcirc \nexists x(\varphi) \bigvee \exists_{!} x(\varphi)$

[1] Constant symbols ought to be read before function symbols. Function symbols ought to be read before relation symbols. Relation symbols ought to be read before logical symbols.

2.4 Zermelo–Frænkel Set Theory with Choice

A set is a Many that allows itself to be thought of as a One.
GEORG CANTOR

Around the 1920s, an axiomatic set theory was proposed by Ernst Zermelo and Abraham Frænkel: this Zermelo–Frænkel set theory (ZF), when paired with the axiom of choice (AC), came to be known as Zermelo–Frænkel set theory with choice (ZFC). ZFC is commonly used as a foundational theory, and is conventionally written in a classical predicate logic.

It has been shown that if ZFC is consistent, then every formulation of ZFC must include at least one axiom schema. For brevity, we shall examine a particular formulation of ZFC which includes the axiom schema of separation, and no other axiom schema.

With the admission of ZF, AC is equivalent to various theorems, including the theorem that every vector space has a basis, and the theorem that every surjective function has a right inverse. Famously, AC in ZF also implies the Banach–Tarski paradox.

Every object in ZFC is a set. A set can be distinguished from a *proper* class: the latter is, in some sense, "larger" than the former. However, in the interest of brevity, we shall not always make this distinction.

Axiom 1 (Empty Set).
$\exists e, \nexists x \in e$

Definition 9 (Empty Set).
$e = \varnothing \Leftrightarrow \nexists x \in e$

Axiom 2 (Extensionality).
$\forall A, B, \forall x(x \in A \Leftrightarrow x \in B)(A = B)$

Definition 10 (Subset).
$A \subseteq B \Leftrightarrow \forall x \in A(x \in B)$

Axiom 3 (Union).
$\forall N, \exists U, \forall x \in X \in N(x \in U)$

Definition 11 (Unary Union).
$U = \bigcup N \Leftrightarrow \forall x \in X \in N(x \in U)$

Definition 12 (Binary Union).
$A \cup B = \bigcup\{A, B\}$

Definition 13 (Space of Free Variables).
Let φ be a formula, and $x_\circ, \ldots, x_\bullet$ be the free variables in φ.
$\mathrm{free}(\varphi) = \bigcup\{\{x_\circ\}, \ldots, \{x_\bullet\}\}$

Axiom 4 (Power Set).
$\forall X, \exists P, \forall x \subseteq X(x \in P)$

Definition 14 (Power Set).
$x \in \mathscr{P}(X) \Leftrightarrow x \subseteq X$

Axiom Schema 1 (Separation).
Let φ be a formula, and $\text{free}(\varphi) \subseteq \{d, D\}$.
$\forall D, \exists I, \forall d \in D(d \in I \Leftrightarrow \varphi)$

Definition 15.
Let φ be a formula, and $\text{free}(\varphi) \subseteq \{d, D\}$.
$I = \{d \in D \mid \varphi\} \Leftrightarrow \forall d \in D(d \in I \Leftrightarrow \varphi)$

Definition 16 (Unary Intersection).
$\bigcap N = \{i \mid \forall X \in N(i \in X)\}$

Definition 17 (Binary Intersection).
$A \cap B = \bigcap\{A, B\}$

Definition 18 (Successor).
$\mathscr{S}(n) = n \cup \{n\}$

Axiom 5 (Infinity).
$\exists R \supseteq \{\varnothing\}, \forall r \in R(\mathscr{S}(r) \in R)$

Axiom 6 (Regularity).
$\forall O \neq \varnothing, \exists o \in O(o \cap O = \varnothing)$

Definition 19 (Set of Pairwise Disjoint Sets).
$\not\cap(X) \Leftrightarrow \forall A, B \in X(A \neq B \Rightarrow A \cap B = \varnothing)$

Definition 20 (Set of Non-Empty Sets).
$\not\supseteq_{\{\varnothing\}}(X) \Leftrightarrow X \not\supseteq \{\varnothing\}$

Axiom 7 (Choice).
$\forall B(\not\cap(B) \land \not\supseteq_{\{\varnothing\}}(B) \Rightarrow \exists B', \forall S \in B, \exists_! s \in S(s \in B'))$

2.5 Interlogue

A tuple can be seen as a totally-ordered finite set. A string can be seen as a tuple.

Definition 21 (Binary Difference).
$X \setminus Y = \{x \in X \mid x \notin Y\}$

Definition 22 (Reflexive Relation).
Let R be a relation symbol.
$\mathsf{Reflexive}\,(X, \mathsf{R}) \Leftrightarrow \forall a \in X(a \mathrel{\mathsf{R}} a)$

Definition 23 (Antisymmetric Relation).
Let R be a relation symbol.
$\mathsf{Antisymmetric}\,(X, \mathsf{R}) \Leftrightarrow \forall a, b \in X(a \mathrel{\mathsf{R}} b \wedge b \mathrel{\mathsf{R}} a \Rightarrow a = b)$

Definition 24 (Transitive Relation).
Let R be a relation symbol.
$\mathsf{Transitive}\,(X, \mathsf{R}) \Leftrightarrow \forall a, b, c \in X(a \mathrel{\mathsf{R}} b \wedge b \mathrel{\mathsf{R}} c \Rightarrow a \mathrel{\mathsf{R}} c)$

Definition 25 (Partial-Ordering Relation).
Let R be a relation symbol.
$\mathsf{PartialOrdering}\,(X, \mathsf{R}) \Leftrightarrow \mathsf{Reflexive}\,(X, \mathsf{R}) \wedge \mathsf{Antisymmetric}\,(X, \mathsf{R}) \wedge \mathsf{Transitive}\,(X, \mathsf{R})$

Definition 26 (Interval).
Let $\mathsf{PartialOrdering}\,(X, \preceq)$.
$X_{[a,b]} = \{x \in X \mid a \preceq x \preceq b\}$
$X_{[a,b)} = \{x \in X \mid a \preceq x \prec b\}$
$X_{(a,b]} = \{x \in X \mid a \prec x \preceq b\}$
$X_{(a,b)} = \{x \in X \mid a \prec x \prec b\}$
$X_{[a,\infty)} = X_{\succeq a} = \{x \in X \mid a \preceq x\}$
$X_{(a,\infty)} = X_{\succ a} = \{x \in X \mid a \prec x\}$
$X_{(-\infty,b]} = X_{\preceq b} = \{x \in X \mid x \preceq b\}$
$X_{(-\infty,b)} = X_{\prec b} = \{x \in X \mid x \prec b\}$

Definition 27 (Space of Natural Numbers).
$\mathbb{N} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, \dots\} = \bigcap\{R \supseteq \{\varnothing\} \mid \forall r \in R(\mathcal{S}(r) \in R)\}$

Definition 28 (Order of Natural Numbers).
Let $a, b \in \mathbb{N}$.
$a \leq b \Leftrightarrow a \in b \vee a = b$

Definition 29 (Tuple).
Let $n \in \mathbb{N}_{\geq 1}$.
$\langle\rangle = \varnothing$
$\langle x_1 \rangle = x_1$
$\langle x_1, \dots, x_n, x_{\mathcal{S}(n)} \rangle = \{\{\langle x_1, \dots, x_n \rangle\}, \{\langle x_1, \dots, x_n \rangle, x_{\mathcal{S}(n)}\}\}$

Definition 30 (Detuple)**.**
Let $n \in \mathbb{N}_{\geq 1}$.
$\Diamond(\langle\rangle) = \varnothing$
$\Diamond(\langle x_1 \rangle) = \{x_1\}$
$\Diamond(\langle x_1, \ldots, x_n, x_{\mathcal{S}(n)} \rangle) = \Diamond(\langle x_1, \ldots, x_n \rangle) \cup \{x_{\mathcal{S}(n)}\}$

Definition 31 (Projector)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $k \in \mathbb{N}_{[1,n]}$.
$\langle x_1, \ldots, x_n \rangle_{[k]} = x_k$

Definition 32 (Finite Cartesian Product)**.**
Let $n \in \mathbb{N}_{\geq 1}$.
$X_1 \times \cdots \times X_n = \{\langle x_1, \ldots, x_n \rangle \mid x_1 \in X_1 \wedge \ldots \wedge x_n \in X_n\}$

Definition 33 (Space of Functions)**.**
$\{X \to Y\} = \{f \subseteq X \times Y \mid \forall x \in X, \exists! y \in Y(\langle x, y \rangle \in f)\}$

Definition 34 (Function)**.**
$f : X \to Y \Leftrightarrow f \in \{X \to Y\}$
$f(x) = y \Leftrightarrow \langle x, y \rangle \in f$

Definition 35 (Identity)**.**
$\mathrm{id}(x) = x$

Definition 36 (Space of Strings)**.**
$A^* = \bigcup\{A^l \mid l \in \mathbb{N}\}$

Definition 37 (String)**.**
Let $n \in \mathbb{N}_{\geq 1}$.
$\texttt{""} = \langle\rangle$
$\texttt{"}x_1 \ldots x_n\texttt{"} = \langle \texttt{"}x_1\texttt{"}, \ldots, \texttt{"}x_n\texttt{"} \rangle$

Definition 38 (Mutually Exclusive Set of Formulas)**.**
$\not\wedge(\Phi) \Leftrightarrow \forall \texttt{"}\varphi\texttt{"}, \texttt{"}\varphi'\texttt{"} \in \Phi(\texttt{"}\varphi\texttt{"} \neq \texttt{"}\varphi'\texttt{"} \Rightarrow \neg(\varphi \wedge \varphi'))$

Definition 39 (Collectively Exhaustive Set of Formulas)**.**
Let $n \in \mathbb{N}_{\geq 1}$.
$\vee(\{\texttt{"}\varphi_1\texttt{"}, \ldots, \texttt{"}\varphi_n\texttt{"}\}) \Leftrightarrow \varphi_1 \vee \ldots \vee \varphi_n$

Definition 40 (Logical Partition)**.**
$\veebar(\Phi) \Leftrightarrow \Phi \neq \varnothing \wedge \not\wedge(\Phi) \wedge \vee(\Phi)$

Definition 41 (Piecewise Function)**.**
Let $f : X \to Y$, and $n \in \mathbb{N}_{\geq 1}$, and $\veebar(\{\texttt{"}\varphi_1\texttt{"}, \ldots, \texttt{"}\varphi_{\mathcal{S}(n)}\texttt{"}\})$.

$$f(x) = \begin{cases} y_1, & \varphi_1 \\ \vdots & \vdots \\ y_n, & \varphi_n \\ y_{\mathcal{S}(n)}, & \varphi_{\mathcal{S}(n)} \end{cases} \Leftrightarrow \forall x \in X, \forall i \in [1, \mathcal{S}(n)](\varphi_i \Rightarrow \langle x, y_i \rangle \in f)$$

Notation 9 (Finite Arity Quantification)**.**

Let $k, n \in \mathbb{N}_{\geq 1}$, and φ be a formula.

$\forall x_{i,j} \underset{j \in \langle 1,\ldots,k \rangle}{\overset{i \in \langle 1,\ldots,n \rangle}{\in}} X_i(\varphi) \coloneqq \forall x_{1,1}, \ldots, x_{1,k} \in X_1, \ldots, \forall x_{n,1}, \ldots, x_{n,k} \in X_n(\varphi)$

Definition 42 (Inductive Closure)**.**

Let $k, n \in \mathbb{N}_{\geq 1}$.

$\mathscr{S}_k(X_1, \ldots, X_n, F) \Leftrightarrow \forall x_{i,j} \underset{j \in \langle 1,\ldots,k \rangle}{\overset{i \in \langle 1,\ldots,n \rangle}{\in}} X_i, \forall f \in F(f(x_{1,1}, \ldots, x_{n,k}) \in X_n)$

Definition 43 (Inductive Tuple Closure)**.**

Let $k, n \in \mathbb{N}_{\geq 1}$.

$\widehat{\mathscr{S}}_k(X_1, \ldots, X_n, F) \Leftrightarrow \forall x_{i,j} \underset{j \in \langle 1,\ldots,k \rangle}{\overset{i \in \langle 1,\ldots,n \rangle}{\in}} X_i, \forall f \in F(\langle x_{1,1}, \ldots, x_{n,k}, f(x_{1,1}, \ldots, x_{n,k}) \rangle \in X_n)$

2.6 Metalanguage

Previously, we defined propositional and predicate logics. Now, we shall define propositional and predicate metalogics that mimic, and thus help to more formally define, their logical counterparts. To avoid circularity, a metalogic ought to be distinguished from its logic. However, in the interest of brevity, we shall not always make such a distinction.

2.6.1 Propositional Logic

Definition 44 (Space of Propositional Variables).
$\mathbb{X}_0 = \bigcup\{\{"p_1", \ldots, "p_n"\} \mid n \in \mathbb{N}\}$

Definition 45 (Propositional Alphabet).
$\wp_0 = \mathbb{X}_0 \cup \{"\neg", "\wedge", "(", ")"\}$

Definition 46 (Concatenation Function of Negation).
$\mathrm{neg}("\varphi") = "(\neg\varphi)"$

Definition 47 (Concatenation Function of Conjunction).
$\mathrm{conj}("\varphi", "\varphi'") = "(\varphi \wedge \varphi')"$

Definition 48 (Space of Propositional Formulas).
$\mathbb{W}_0 = \bigcap\{\Phi \subseteq \wp_0^* \mid \Phi \supseteq \mathbb{X}_0 \wedge c\!\wp_1(\Phi, \{\mathrm{neg}\}) \wedge c\!\wp_2(\Phi, \{\mathrm{conj}\})\}$

Definition 49 (Truth Function of Negation).
$\mathrm{not} : 2 \to 2$
$\mathrm{not}(1) = 0$
$\mathrm{not}(0) = 1$

Definition 50 (Truth Function of Conjunction).
$\mathrm{and} : 2^2 \to 2$
$\mathrm{and}(1, 1) = 1$
$\mathrm{and}(1, 0) = 0$
$\mathrm{and}(0, 1) = 0$
$\mathrm{and}(0, 0) = 0$

Definition 51 (Space of Truth Assignments).
$\mathbb{T}_0 = \{\mathbb{X}_0 \to 2\}$

Definition 52 (Valuation of Propositional Formula).
Let $t \in \mathbb{T}_0$.
$v_0^t : \mathbb{W}_0 \to 2$

$$v_0^t(\varphi) = \begin{cases} t(p), & \exists p \in \mathbb{X}_0(\varphi = p) \\ \mathrm{not}(v_0^t(\psi)), & \exists \psi \in \mathbb{W}_0(\varphi = \mathrm{neg}(\psi)) \\ \mathrm{and}(v_0^t(\psi), v_0^t(\psi')), & \exists \psi, \psi' \in \mathbb{W}_0(\varphi = \mathrm{conj}(\psi, \psi')) \end{cases}$$

2.6.2 Predicate Logic

$\mathbb{W}_{1,1}$ is a space of atomic (i.e. quantifier-less) formulas.

Definition 53 (Space of Variables).
$\mathbb{X}_1 = \bigcup\{\{"x_1", \ldots, "x_n"\} \mid n \in \mathbb{N}\}$

Definition 54 (Predicate Alphabet).
$\wp_1 = \mathbb{X}_1 \cup \{"=", "\in", "\neg", "\wedge", "(", ")"\}$

Definition 55 (Concatenation Function of Equality).
$\mathrm{eq}("x", "x'") = "(x = x')"$

Definition 56 (Concatenation Function of Membership).
$\mathrm{in}("x", "x'") = "(x \in x')"$

Definition 57 (Concatenation Function of Universal Quantification).
$\mathrm{fa}("x", "\varphi") = "\forall x(\varphi)"$

Definition 58 (Space of Formulas).
$\mathbb{W}_{1,1} = \bigcap\{\Phi \subseteq \wp_1^* \mid \Phi \supseteq \mathbb{X}_1 \wedge \infty_2(\Phi, \{\mathrm{eq}, \mathrm{in}\})\}$
$\mathbb{W}_1 = \bigcap\{\Phi \subseteq \wp_1^* \mid \Phi \supseteq \mathbb{W}_{1,1} \wedge \infty_1(\Phi, \{\mathrm{neg}\}) \wedge \infty_2(\Phi, \{\mathrm{conj}\}) \wedge \infty_1(\mathbb{X}_1, \Phi, \{\mathrm{fa}\})\}$

Definition 59 (Space of Free Variables).
$\mathrm{free} : \mathbb{W}_1 \to \mathscr{P}(\mathbb{X}_1)$

$$\mathrm{free}(\varphi) = \begin{cases} \{x, x'\}, & \exists x, x' \in \mathbb{X}_1(\varphi = \mathrm{eq}(x, x')) \\ \{x, x'\}, & \exists x, x' \in \mathbb{X}_1(\varphi = \mathrm{in}(x, x')) \\ \mathrm{free}(\psi), & \exists \psi \in \mathbb{W}_1(\varphi = \mathrm{neg}(\psi)) \\ \mathrm{free}(\psi) \cup \mathrm{free}(\psi'), & \exists \psi, \psi' \in \mathbb{W}_1(\varphi = \mathrm{conj}(\psi, \psi')) \\ \mathrm{free}(\psi) \setminus \{x\}, & \exists x \in \mathbb{X}_1, \exists \psi \in \mathbb{W}_1(\varphi = \mathrm{fa}(x, \psi)) \end{cases}$$

Definition 60 (Space of Sentences).
$\mathbb{W}_1^{\bowtie} = \{\varphi \in \mathbb{W}_1 \mid \mathrm{free}(\varphi) = \varnothing\}$

Definition 61.
Let $n \in \mathbb{N}_{\geq 1}$, and $\varphi_1, \ldots, \varphi_n \in \mathbb{W}_1$.
🆕$\{\varphi_1, \ldots, \varphi_n\} = \mathbb{X}_1 \setminus (\Diamond(\varphi_1) \cup \cdots \cup \Diamond(\varphi_n))$

Definition 62 (Variable Substitution)**.**
Let $s, s' \in \mathbb{X}_1$.
$.|_s^{s'} : \mathbb{W}_1 \to \mathbb{W}_1$

$$\varphi|_s^{s'} = \begin{cases} \mathrm{eq}(x,x')|_s^{s'} = \begin{cases} \mathrm{eq}(s',s'), & x,x' = s \\ \mathrm{eq}(s',x'), & x = s \wedge x' \neq s \\ \mathrm{eq}(x,s'), & x \neq s \wedge x' = s \\ \mathrm{eq}(x,x'), & \text{otherwise} \end{cases}, & \exists x,x' \in \mathbb{X}_1(\varphi = \mathrm{eq}(x,x')) \\ \mathrm{in}(x,x')|_s^{s'} = \begin{cases} \mathrm{in}(s',s'), & x,x' = s \\ \mathrm{in}(s',x'), & x = s \wedge x' \neq s \\ \mathrm{in}(x,s'), & x \neq s \wedge x' = s \\ \mathrm{in}(x,x'), & \text{otherwise} \end{cases}, & \exists x,x' \in \mathbb{X}_1(\varphi = \mathrm{in}(x,x')) \\ \mathrm{neg}(\psi)|_s^{s'} = \mathrm{neg}(\psi|_s^{s'}), & \exists \psi \in \mathbb{W}_1(\varphi = \mathrm{neg}(\psi)) \\ \mathrm{conj}(\psi,\psi')|_s^{s'} = \mathrm{conj}(\psi|_s^{s'}, \psi'|_s^{s'}), & \exists \psi,\psi' \in \mathbb{W}_1(\varphi = \mathrm{conj}(\psi,\psi')) \\ \mathrm{fa}(x,\psi)|_s^{s'} = \begin{cases} \mathrm{fa}(x,\psi), & x = s \\ \mathrm{fa}(x,\psi|_s^{s'}), & x \neq s \wedge x \notin \Diamond(s'), \\ \mathrm{fa}(\text{🆕}\{s',\psi\}, \psi|_x^{\text{🆕}\{s',\psi\}}|_s^{s'}), & \text{otherwise} \end{cases} & \exists x \in \mathbb{X}_1, \exists \psi \in \mathbb{W}_1(\varphi = \mathrm{fa}(x,\psi)) \end{cases}$$

Definition 63 (Space of Structures)**.**
$\mathrm{ars}(F,R) = \{F \cup R \to \mathbb{N}\}$
$\mathrm{ins}(U,F,R,a) = \{F \cup R \to \bigcup\{\{U^{a(f)} \to U\} \mid f \in F\} \cup \bigcup\{\mathscr{P}(U^{a(r)}) \mid r \in R\}\}$
$\mathcal{M} = \{\langle U, \langle F,R,a\rangle, i\rangle \mid U \neq \varnothing \wedge F \cap R = \varnothing \wedge a \in \mathrm{ars}(F,R) \wedge i \in \mathrm{ins}(U,F,R,a)\}$

Definition 64 (Space of Variable Assignments)**.**
Let $m \in \mathcal{M}$.
$\mathbb{T}_1^m = \bigcup\{\{\mathrm{free}(\varphi) \to m_{[1]}\} \mid \varphi \in \mathbb{W}_1\} \cup \{\mathrm{id}\}$

Definition 65 (Valuation of Formula)**.**
Let $m \in \mathcal{M}$, and $t \in \mathbb{T}_1^m$.
$v_1^{m,t} : \mathbb{W}_1 \to 2$

$$v_1^{m,t}(\varphi) = \begin{cases} v_1^{m,t}(\mathrm{eq}(x,x')) = \begin{cases} 1, & t(x) = t(x') \\ 0, & \text{otherwise} \end{cases}, & \exists x,x' \in \mathbb{X}_1(\varphi = \mathrm{eq}(x,x')) \\ v_1^{m,t}(\mathrm{in}(x,x')) = \begin{cases} 1, & t(x) \in t(x') \\ 0, & \text{otherwise} \end{cases}, & \exists x,x' \in \mathbb{X}_1(\varphi = \mathrm{in}(x,x')) \\ \mathrm{not}(v_1^{m,t}(\psi)), & \exists \psi \in \mathbb{W}_1(\varphi = \mathrm{neg}(\psi)) \\ \mathrm{and}(v_1^{m,t}(\psi), v_1^{m,t}(\psi')), & \exists \psi,\psi' \in \mathbb{W}_1(\varphi = \mathrm{conj}(\psi,\psi')) \\ v_1^{m,t}(\mathrm{fa}(x,\psi)) = \begin{cases} 1, & \forall s \in m_{[1]}(v_1^{m,t}(\psi|_s^x) = 1) \\ 0, & \text{otherwise} \end{cases}, & \exists x \in \mathbb{X}_1, \psi \in \mathbb{W}_1(\varphi = \mathrm{fa}(x,\psi)) \end{cases}$$

2.7 Satisfiability and Definability

A logical formula is said to be *tautological* only if it is "always true", *satisfiable* only if it is "sometimes true", and *contradictory* only if it is "never true". The property of being tautological can be seen as opposite to the property of being contradictory, while the property of being satisfiable can be seen as complementary to the property of being contradictory.

A set is said to be *definable* only if there exists a logical formula whose truth is equivalent to existence of the set. The cardinality of the space of truth assignments is larger than that of the space of propositional formulas. Similarly, the cardinality of the space of structures is larger than that of the space of sentences.

2.7.1 Propositional Logic

Definition 66 (Tautological Set of Propositional Formulas).
Let $\Phi \subseteq \mathbb{W}_0$.
$\mathsf{Tautological}\,(\Phi) \Leftrightarrow \forall t \in \mathbb{T}_0, \forall \varphi \in \Phi(v_0^t(\varphi) = 1)$

Definition 67 (Contradictory Set of Propositional Formulas).
Let $\Phi \subseteq \mathbb{W}_0$.
$\mathsf{Contradictory}\,(\Phi) \Leftrightarrow \forall t \in \mathbb{T}_0, \forall \varphi \in \Phi(v_0^t(\varphi) = 0)$

Definition 68 (Satisfiable Set of Propositional Formulas).
Let $\Phi \subseteq \mathbb{W}_0$.
$\mathsf{Satisfiable}\,(\Phi) \Leftrightarrow \exists t \in \mathbb{T}_0, \forall \varphi \in \Phi(v_0^t(\varphi) = 1)$

Definition 69 (Definable Set of Truth Assignments).
Let $T \subseteq \mathbb{T}_0$.
$\mathsf{Definable}\,(T) \Leftrightarrow \exists \varphi \in \mathbb{W}_0, \forall t \in T(v_0^t(\varphi) = 1)$

Definition 70 (Subject of Set of Propositional Formulas).
Let $\Phi \subseteq \mathbb{W}_0$.
$\mathrm{subj}(\Phi) = \{t \in \mathbb{T}_0 \mid \forall \varphi \in \Phi(v_0^t(\varphi) = 1)\}$

Definition 71 (Theory of Set of Truth Assignments).
Let $T \subseteq \mathbb{T}_0$.
$\mathrm{th}(T) = \{\varphi \in \mathbb{W}_0 \mid \forall t \in T(v_0^t(\varphi) = 1)\}$

Theorem 1 (Propositional Galois Connection).
Let $\Phi \subseteq \mathbb{W}_0$, and $T \subseteq \mathbb{T}_0$.
$\Phi \subseteq \mathrm{th}(T) \Leftrightarrow T \subseteq \mathrm{subj}(\Phi)$

Proposition 1 (Existence of Unsatisfiable Set of Propositional Formulas).
$\exists \Phi \subseteq \mathbb{W}_0(\neg\,\mathsf{Satisfiable}\,(\Phi))$

Proposition 2 (Existence of Undefinable Set of Truth Assignments).
$\exists T \subseteq \mathbb{T}_0(\neg\,\mathsf{Definable}\,(T))$

2.7.2 Predicate Logic

Definition 72 (Tautological Set of Sentences).
Let $\Phi \subseteq \mathbb{W}_1^{\bowtie}$.
$\mathsf{Tautological}\,(\Phi) \Leftrightarrow \forall m \in \mathcal{M}, \forall \varphi \in \Phi(v_1^{m,\mathrm{id}}(\varphi) = 1)$

Definition 73 (Contradictory Set of Sentences).
Let $\Phi \subseteq \mathbb{W}_1^{\bowtie}$.
$\mathsf{Contradictory}\,(\Phi) \Leftrightarrow \forall m \in \mathcal{M}, \forall \varphi \in \Phi(v_1^{m,\mathrm{id}}(\varphi) = 0)$

Definition 74 (Satisfiable Set of Sentences).
Let $\Phi \subseteq \mathbb{W}_1^{\bowtie}$.
$\mathsf{Satisfiable}\,(\Phi) \Leftrightarrow \exists m \in \mathcal{M}, \forall \varphi \in \Phi(v_1^{m,\mathrm{id}}(\varphi) = 1)$

Definition 75 (Definable Set of Structures).
Let $M \subseteq \mathcal{M}$.
$\mathsf{Definable}\,(M) \Leftrightarrow \exists \varphi \in \mathbb{W}_1^{\bowtie}, \forall m \in M(v_1^{m,\mathrm{id}}(\varphi) = 1)$

Definition 76 (Subject of Set of Sentences).
Let $\Phi \subseteq \mathbb{W}_1^{\bowtie}$.
$\mathrm{subj}(\Phi) = \{m \in \mathcal{M} \mid \forall \varphi \in \Phi(v_1^{m,\mathrm{id}}(\varphi) = 1)\}$

Definition 77 (Theory of Set of Structures).
Let $M \subseteq \mathcal{M}$.
$\mathrm{th}(M) = \{\varphi \in \mathbb{W}_1^{\bowtie} \mid \forall m \in M(v_1^{m,\mathrm{id}}(\varphi) = 1)\}$

Theorem 2 (Predicate Galois Connection).
Let $\Phi \subseteq \mathbb{W}_1^{\bowtie}$, and $T \subseteq \mathbb{T}_1$.
$\Phi \subseteq \mathrm{th}(T) \Leftrightarrow T \subseteq \mathrm{subj}(\Phi)$

Proposition 3 (Existence of Unsatisfiable Set of Sentences).
$\exists \Phi \subseteq \mathbb{W}_1^{\bowtie}(\neg\,\mathsf{Satisfiable}\,(\Phi))$

Proposition 4 (Existence of Undefinable Set of Structures).
$\exists M \subseteq \mathcal{M}(\neg\,\mathsf{Definable}\,(M))$

2.8 Soundness and Completeness

For a logic, soundness can be seen as the property that “every proof has truth”, and completeness can be seen as the property that “every truth has proof”. Taken together, soundness and completeness establish a “correspondence” between syntactic notions of proof and semantic notions of truth.

Definition 78 (Modus Ponens Function)**.**
$\mathrm{pon}(\texttt{"}\varphi\texttt{"}, \texttt{"}(\varphi \Rightarrow \varphi')\texttt{"}) = \texttt{"}\varphi'\texttt{"}$

2.8.1 Propositional Logic

In 1879, Gottlob Frege proposed a set of six axiom schemata for propositional logic. Years later, Jan Łukasiewicz showed that modifications to Frege’s schema could reduce the number of schemata to three. Alonzo Church helped popularise this new schema, referring to it as “P2”. $\mathscr{A}_0$ encodes this schema.

Definition 79 (Space of Propositional Axioms)**.**
$\mathscr{A}_{0,1} = \{\texttt{"}\varphi \Rightarrow (\psi \Rightarrow \varphi)\texttt{"} \mid \varphi, \psi \in \mathbb{W}_0\}$
$\mathscr{A}_{0,2} = \{\texttt{"}(\varphi \Rightarrow (\psi \Rightarrow \chi)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \chi))\texttt{"} \mid \varphi, \psi, \chi \in \mathbb{W}_0\}$
$\mathscr{A}_{0,3} = \{\texttt{"}(\neg\varphi \Rightarrow \neg\psi) \Rightarrow (\psi \Rightarrow \varphi)\texttt{"} \mid \varphi, \psi \in \mathbb{W}_0\}$
$\mathscr{A}_0 = \bigcup\{\mathscr{A}_{0,1}, \mathscr{A}_{0,2}, \mathscr{A}_{0,3}\}$

Definition 80 (Propositional Proof System)**.**
Let $\Gamma \subseteq \mathbb{W}_0$.
$\mathscr{D}_0(\Gamma) = \bigcap\{\Phi \subseteq \wp_0^* \mid \Phi \supseteq \mathscr{A}_0 \cup \Gamma \wedge \mathscr{S}_1(\Phi, \{\mathrm{pon}\})\}$

Definition 81 (Space of Propositional Proof Strings)**.**
Let $\Gamma \subseteq \mathbb{W}_0$.
$\widehat{\mathscr{D}}_0(\Gamma) = \bigcap\{\Phi \subseteq \wp_0^* \mid \Phi \supseteq \mathscr{A}_0 \cup \Gamma \wedge \widehat{\mathscr{S}}_1(\Phi, \{\mathrm{pon}\})\}$

Definition 82 (Propositional Syntactic Entailment)**.**
Let $\Gamma \subseteq \mathbb{W}_0$, and $\varphi \in \mathbb{W}_0$.
$\Gamma \vdash \varphi \Leftrightarrow \varphi \in \mathscr{D}_0(\Gamma)$

Definition 83 (Propositional Semantic Entailment)**.**
Let $\Gamma \subseteq \mathbb{W}_0$, and $\varphi \in \mathbb{W}_0$.
$\Gamma \vDash \varphi \Leftrightarrow \forall t \in \mathbb{T}_0(\forall \gamma \in \Gamma(v_0^t(\gamma) = 1) \Rightarrow v_0^t(\varphi) = 1)$

Proposition 5 (Finitaryness of Propositional Proof System)**.**
Let $\Gamma \subseteq \mathbb{W}_0$, and $\varphi \in \mathscr{D}_0(\Gamma)$.
$\exists n \in \mathbb{N}_{\geq 1}, \exists\langle\psi_1, \ldots, \psi_n\rangle \in \widehat{\mathscr{D}}_0(\Gamma)(\psi_n = \varphi)$

Theorem 3 (Soundness of Propositional Proof System)**.**
Let $\Gamma \subseteq \mathbb{W}_0$, and $\varphi \in \mathbb{W}_0$.
$\Gamma \vdash \varphi \Rightarrow \Gamma \vDash \varphi$

Theorem 4 (Completeness of Propositional Proof System)**.**
Let $\Gamma \subseteq \mathbb{W}_0$, and $\varphi \in \mathbb{W}_0$.
$\Gamma \vDash \varphi \Rightarrow \Gamma \vdash \varphi$

Definition 84 (Consistent Set of Propositional Formulas)**.**
Let $\Gamma \subseteq \mathbb{W}_0$.
$\mathsf{Consistent}\,(\Gamma) \Leftrightarrow \forall \varphi \in \mathbb{W}_0(\neg(\Gamma \vdash \varphi \wedge \Gamma \nvdash \varphi))$

Proposition 6 (Satisfiable Set of Propositional Formulas)**.**
Let $\Gamma \subseteq \mathbb{W}_0$.
$\mathsf{Satisfiable}\,(\Gamma) \Leftrightarrow \forall \varphi \in \mathbb{W}_0(\neg(\Gamma \vDash \varphi \wedge \Gamma \nvDash \varphi))$

Theorem 5.
Let $\Gamma \subseteq \mathbb{W}_0$.
$\mathsf{Consistent}\,(\Gamma) \Leftrightarrow \mathsf{Satisfiable}\,(\Gamma)$

Proposition 7 (Propositional Explosion)**.**
Let $\alpha, \varphi \in \mathbb{W}_0$.
$\{\alpha, \neg\alpha\} \vdash \varphi$

2.8.2 Predicate Logic

$\mathscr{A}_{1,1}$ encodes a "uniform replacement", within every propositional formula, of propositional variables for formulas. $\mathscr{A}_{1,2}$ encodes universal instantiation. $\mathscr{A}_{1,3}$ encodes universal generalisation.

Definition 85 (Space of Predicate Axioms)**.**
$\mathscr{A}_{1,1} = \{\tau|_p^\varphi \mid \tau \in \mathbb{W}_0 \wedge p \in \mathbb{X}_0 \wedge \varphi \in \mathbb{W}_1 \wedge \varnothing \vDash \tau\}$
$\mathscr{A}_{1,2} = \{\text{"}\forall x(\varphi) \Rightarrow \varphi|_x^{x'}\text{"} \mid x, x' \in \mathbb{X}_1 \wedge \varphi \in \mathbb{W}_1\}$
$\mathscr{A}_{1,3} = \{\text{"}\varphi \Rightarrow \forall x(\varphi|_x^{x'})\text{"} \mid x, x' \in \mathbb{X}_1 \wedge \varphi \in \mathbb{W}_1\}$
$\mathscr{A}_1 = \bigcup\{\mathscr{A}_{1,1}, \mathscr{A}_{1,2}, \mathscr{A}_{1,3}\}$

Definition 86 (Predicate Proof System)**.**
Let $\Gamma \subseteq \mathbb{W}_1$.
$\mathscr{D}_1(\Gamma) = \bigcap\{\Phi \subseteq \mathfrak{o}_1^* \mid \Phi \supseteq \mathscr{A}_1 \cup \Gamma \wedge \mathscr{co}_1(\Phi, \{\text{pon}\})\}$

Definition 87 (Space of Predicate Proof Strings)**.**
Let $\Gamma \subseteq \mathbb{W}_1$.
$\widehat{\mathscr{D}}_1(\Gamma) = \bigcap\{\Phi \subseteq \mathfrak{o}_1^* \mid \Phi \supseteq \mathscr{A}_1 \cup \Gamma \wedge \widehat{\mathscr{co}}_1(\Phi, \{\text{pon}\})\}$

Definition 88 (Predicate Syntactic Entailment)**.**
Let $\Gamma \subseteq \mathbb{W}_1$, and $\varphi \in \mathbb{W}_1$.
$\Gamma \vdash \varphi \Leftrightarrow \varphi \in \mathscr{D}_1(\Gamma)$

Definition 89 (Predicate Semantic Entailment)**.**
Let $\Gamma \subseteq \mathbb{W}_1$, and $\varphi \in \mathbb{W}_1$, and $M \subseteq \mathcal{M}$.
$\langle M, \Gamma\rangle \vDash \varphi \Leftrightarrow \forall m \in M, \forall t \in \mathbb{T}_1^m(\forall \gamma \in \Gamma(v_1^{m,t}(\gamma) = 1) \Rightarrow v_1^{m,t}(\varphi) = 1)$

Proposition 8 (Finitaryness of Predicate Proof System)**.**
Let $\Gamma \subseteq \mathbb{W}_1$, and $\varphi \in \mathscr{D}_1(\Gamma)$.
$\exists n \in \mathbb{N}_{\geq 1}, \exists\langle\psi_1, \ldots, \psi_n\rangle \in \widehat{\mathscr{D}}_1(\Gamma)(\psi_n = \varphi)$

Theorem 6 (Soundness of Predicate Proof System)**.**

Let $\Gamma \subseteq \mathbb{W}_1$, and $\varphi \in \mathbb{W}_1$.

$\Gamma \vdash \varphi \Rightarrow \langle \mathcal{M}, \Gamma \rangle \vDash \varphi$

Theorem 7 (Completeness of Predicate Proof System)**.**

Let $\Gamma \subseteq \mathbb{W}_1$, and $\varphi \in \mathbb{W}_1$.

$\langle \mathcal{M}, \Gamma \rangle \vDash \varphi \Rightarrow \Gamma \vdash \varphi$

Definition 90 (Consistent Set of Formulas)**.**

Let $\Gamma \subseteq \mathbb{W}_1$.

$\mathsf{Consistent}\,(\Gamma) \Leftrightarrow \forall \varphi \in \mathbb{W}_1 (\neg(\Gamma \vdash \varphi \land \Gamma \nvdash \varphi))$

Proposition 9 (Satisfiable Set of Formulas)**.**

Let $\Gamma \subseteq \mathbb{W}_1$.

$\mathsf{Satisfiable}\,(\Gamma) \Leftrightarrow \forall \varphi \in \mathbb{W}_1 (\neg(\langle \mathcal{M}, \Gamma \rangle \vDash \varphi \land \langle \mathcal{M}, \Gamma \rangle \nvDash \varphi))$

Theorem 8.

Let $\Gamma \subseteq \mathbb{W}_1$.

$\mathsf{Consistent}\,(\Gamma) \Leftrightarrow \mathsf{Satisfiable}\,(\Gamma)$

Proposition 10 (Predicate Explosion)**.**

Let $\alpha, \varphi \in \mathbb{W}_1$.

$\{\alpha, \neg\alpha\} \vdash \varphi$

2.9 Interlogue

An injective function is called an injection, a surjective function is called a *s*urjection, and a bijective function is called a bijection. A pair of sets is said to be *equinumerous* (i.e. have equal cardinality) only if there exists a bijection between them. A set is said to be *finite* (i.e. have finite cardinality) only if there exists a bijection from the set to $\mathbb{N}_{[1,n]}$ for some natural number n. A set is said to be *infinite* (i.e. have infinite cardinality) only if it is not finite.

Definition 91 (Injective Function)**.**
Let $f : X \to Y$.
$\mathsf{Injective}\,(f) \Leftrightarrow \forall y \in Y, \reflectbox{$\exists$} x \in X(f(x) = y)$

Definition 92 (Surjective Function)**.**
Let $f : X \to Y$.
$\mathsf{Surjective}\,(f) \Leftrightarrow \forall y \in Y, \exists x \in X(f(x) = y)$

Definition 93 (Bijective Function)**.**
Let $f : X \to Y$.
$\mathsf{Bijective}\,(f) \Leftrightarrow \mathsf{Injective}\,(f) \wedge \mathsf{Surjective}\,(f)$

Definition 94.
$f : X \leftrightarrow Y \Leftrightarrow f : X \to Y \wedge \mathsf{Bijective}\,(f)$

Definition 95 (Injectionality of Sets)**.**
$X \mathrel{\mathsf{inj}} Y \Leftrightarrow \exists f : X \to Y(\mathsf{Injective}\,(f))$

Definition 96 (Surjectionality of Sets)**.**
$X \mathrel{\mathsf{surj}} Y \Leftrightarrow \exists f : X \to Y(\mathsf{Surjective}\,(f))$

Definition 97 (Bijectionality of Sets)**.**
$X \mathrel{\mathsf{bij}} Y \Leftrightarrow \exists f : X \leftrightarrow Y$

Definition 98 (Order of Cardinals)**.**
$|X| \leq |Y| \Leftrightarrow X \mathrel{\mathsf{inj}} Y$

Theorem 9 (Schröder–Bernstein)**.**
$X \mathrel{\mathsf{inj}} Y \wedge Y \mathrel{\mathsf{inj}} X \Rightarrow X \mathrel{\mathsf{bij}} Y$

Proposition 11.
$X \mathrel{\mathsf{inj}} Y \Leftrightarrow Y \mathrel{\mathsf{surj}} X$

Proposition 12.
Let $|C| \leq |\mathbb{N}| \leq |I|$.
$|I \cup C| = |I|$

Theorem 10 (Cantor)**.**
$|X| < |\mathscr{P}(X)|$

2.10 Compactness and Maximality

Previously, we established equivalence between consistency and satisfiability. In the interest of brevity, we shall occasionally omit mention of consistency in favor of satisfiability, keeping in mind that every property which applies to the latter, also applies to the former, in equivalent fashion.

2.10.1 Propositional Logic

Theorem 11 (Propositional Compactness).
Let $\Gamma \subseteq \mathbb{W}_0$.
$\mathsf{Satisfiable}(\Gamma) \Leftrightarrow \forall \Gamma' \subseteq \Gamma(|\Gamma'| < |\mathbb{N}| \Rightarrow \mathsf{Satisfiable}(\Gamma'))$

Definition 99 (Maximal Set of Propositional Formulas).
Let $\Gamma \subseteq \mathbb{W}_0$.
$\mathsf{Maximal}(\Gamma) \Leftrightarrow \forall \varphi \in \mathbb{W}_0(\Gamma \vDash \varphi \vee \Gamma \nvDash \varphi)$

Theorem 12 (Propositional Lindenbaum).
Let $\Gamma \subseteq \mathbb{W}_0$.
$\mathsf{Satisfiable}(\Gamma) \Leftrightarrow \exists \Gamma' \supseteq \Gamma(\mathsf{Satisfiable}(\Gamma') \wedge \mathsf{Maximal}(\Gamma'))$

2.10.2 Predicate Logic

Theorem 13 (Predicate Compactness).
Let $\Gamma \subseteq \mathbb{W}_1$.
$\mathsf{Satisfiable}(\Gamma) \Leftrightarrow \forall \Gamma' \subseteq \Gamma(|\Gamma'| < |\mathbb{N}| \Rightarrow \mathsf{Satisfiable}(\Gamma'))$

Definition 100 (Maximal Set of Formulas).
Let $\Gamma \subseteq \mathbb{W}_1$.
$\mathsf{Maximal}(\Gamma) \Leftrightarrow \forall \varphi \in \mathbb{W}_1(\langle \mathcal{M}, \Gamma \rangle \vDash \varphi \vee \langle \mathcal{M}, \Gamma \rangle \nvDash \varphi)$

Theorem 14 (Predicate Lindenbaum).
Let $\Gamma \subseteq \mathbb{W}_1$.
$\mathsf{Satisfiable}(\Gamma) \Leftrightarrow \exists \Gamma' \supseteq \Gamma(\mathsf{Satisfiable}(\Gamma') \wedge \mathsf{Maximal}(\Gamma'))$

2.11 Gödel Incompleteness

A space of sentences which are true for a structure of natural numbers is known as a theory of natural arithmetic. A theory is said to be “sufficiently strong” only if it contains “sufficiently many” sentences.

Previously, we defined a proof system which is finitary, sound, and complete with respect to a theory of predicate logic. In the 1920s, it was wondered whether one could develop a proof system which is verifiable, sound, and complete with respect to every theory of mathematics. Around the 1930s, Kurt Gödel demonstrated that, if “verifiable” is taken to mean “recursively axiomatisable”, then no such proof system exists for “sufficiently strong” theories of natural arithmetic. Gödel also showed that no consistent proof system in which such theories of natural arithmetic are derivable can prove its *own* consistency. These theorems have come to be known as the Gödel incompleteness theorems. For the sake of concreteness, we shall state these theorems for a particular theory of natural arithmetic.

Famously, Gödel’s incompleteness theorems have been shown to apply to an axiomatic theory of natural arithmetic known as Peano arithmetic (PA). The strengthened finite Ramsey theorem, which can be seen as a sentence about the natural numbers, is provable in ZFC, but not in PA. Additionally, ZFC can prove PA consistent, but PA cannot prove *itself* consistent.

Definition 101.
$\mathbb{n} = \langle \mathbb{N}, \langle \{\texttt{"0"}, \texttt{"}\mathcal{S}\texttt{"}\}, \varnothing, a_{\mathbb{n}} \rangle, i_{\mathbb{n}} \rangle$
$\mathbb{n} \in \mathcal{M}$
$a_{\mathbb{n}}(\texttt{"0"}) = 0$
$a_{\mathbb{n}}(\texttt{"}\mathcal{S}\texttt{"}) = 1$
$i_{\mathbb{n}}(\texttt{"0"}) = 0$
$i_{\mathbb{n}}(\texttt{"}\mathcal{S}\texttt{"}) = \mathcal{S}$

Proposition 13.
Let $M \subseteq \mathcal{M}$.
$\mathrm{th}(\mathcal{M}) \subseteq \mathrm{th}(M)$

Theorem 15 (First Gödel Incompleteness)**.**
$\nexists \Gamma \subseteq \mathbb{W}_1, \forall \varphi \in \mathrm{th}(\{\mathbb{n}\})(\langle \{\mathbb{n}\}, \Gamma \rangle \vDash \varphi \Rightarrow \Gamma \vdash \varphi)$

Theorem 16 (Second Gödel Incompleteness)**.**
Let $\Gamma \subseteq \mathbb{W}_1$, and $\mathrm{th}(\{\mathbb{n}\}) \subseteq \mathcal{D}_1(\Gamma)$.
$\mathsf{Consistent}\,(\mathcal{D}_1(\Gamma)) \Leftrightarrow \Gamma \nvdash \texttt{"}\mathsf{Consistent}\,(\mathcal{D}_1(\Gamma))\texttt{"}$

2.12 Epilogue

In the interest of brevity, we shall occasionally overload our notation for sets. But to avoid ambiguity, we shall make three stipulations, in order:

1. If a denotation x can be read as either set a or set a', and the cardinality of the class of a is less than that of the class of a', then x ought to be read as a.
2. If a denotation x can be read as either denotation d or denotation d', and d is not defined for every set, and d' is defined for every set, then x ought to be read as d.
3. If a denotation x can be read as either denotation d or denotation d', and d is defined after d', then x ought to be read as d.

The image of a function is a subset of its codomain. Notions of "inversion" for a function are defined between its domain and image. A function is *surjective* only if its codomain and image are equal.

Definition 102 (Inverse of Function)**.**
$f^{-1} : Y \leftrightarrow X \Leftrightarrow f : X \leftrightarrow Y$

Proposition 14.
Let $f : X \leftrightarrow Y$, and $x \in X$, and $y \in Y$.
$f^{-1}(f(x)) = f(f^{-1}(x)) = x$
$f(f^{-1}(y)) = f^{-1}(f(y)) = y$

Definition 103 (Domain of Function)**.**
Let $f : X \to Y$.
$\mathrm{dom}(f) = X$

Definition 104 (Codomain of Function)**.**
Let $f : X \to Y$.
$\mathrm{cod}(f) = Y$

Definition 105 (Image of Function)**.**
Let $f : X \to Y$, and $X' \subseteq X$.
$\mathrm{im}_{X'}(f) = \{f(x') \mid x' \in X'\}$
$\mathrm{im}(f) = \mathrm{im}_X(f)$

Definition 106 (Inverse Image of Function)**.**
Let $f : X \to Y$, and $Y' \subseteq Y$.
$\mathrm{im}^{-1}_{Y'}(f) = \{x' \mid f(x') \in Y'\}$
$\mathrm{im}^{-1}(f) = \mathrm{im}^{-1}_Y(f)$

Definition 107 (Minimum)**.**
Let $\mathsf{PartialOrdering}\,(X, \preceq)$.
$\min(X) = x_* \Leftrightarrow \exists x_* \in X, \forall x \in X(x_* \preceq x)$

Definition 108 (Maximum)**.**
Let $\mathsf{PartialOrdering}\,(X, \preceq)$.
$\max(X) = x_* \Leftrightarrow \exists x_* \in X, \forall x \in X(x \preceq x_*)$

3

Abstract Algebra

Definition 109 (Pot)**.**
Let $n \in \mathbb{N}_{\geq 1}$.
$\langle a, b \rangle = a \Leftrightarrow a = b$
$\langle a, b \rangle = \langle a, b \rangle \Leftrightarrow a \neq b$

Definition 110.
Let $\circ$ be a function symbol.
$\mathsf{Closed}\,(X, \circ) \Leftrightarrow \forall a, b \in X (a \circ b \in X)$

Definition 111.
Let $\circ$ be a function symbol.
$\mathsf{Commutative}\,(X, \circ) \Leftrightarrow \forall a, b \in X (a \circ b = b \circ a)$

Definition 112.
Let ①, ② be function symbols.
$\mathsf{Associative}\,(\langle X, Y \rangle, \langle ①, ② \rangle) \Leftrightarrow \forall a \in X, \forall b, c \in Y (a \mathbin{①} (b \mathbin{②} c) = (a \mathbin{①} b) \mathbin{②} c)$

Definition 113.
Let ①, ② be function symbols.
$\mathsf{L\text{-}Distributive}\,(\langle X, Y \rangle, \langle ①, ② \rangle) \Leftrightarrow \forall a \in X, \forall b, c \in Y (a \mathbin{①} (b \mathbin{②} c) = (a \mathbin{①} b) \mathbin{②} (a \mathbin{①} c))$
$\mathsf{R\text{-}Distributive}\,(\langle X, Y \rangle, \langle ①, ② \rangle) \Leftrightarrow \forall a \in X, \forall b, c \in Y ((b \mathbin{②} c) \mathbin{①} a = (b \mathbin{①} a) \mathbin{②} (c \mathbin{①} a))$
$\mathsf{Distributive}\,(\langle X, Y \rangle, \langle ①, ② \rangle) \Leftrightarrow \mathsf{L\text{-}Distributive}\,(\langle X, Y \rangle, \langle ①, ② \rangle) \wedge \mathsf{R\text{-}Distributive}\,(\langle X, Y \rangle, \langle ①, ② \rangle)$

Definition 114.
Let $\circ$ be a function symbol.
$\mathsf{L\text{-}Identive}\,(\langle S, X \rangle, \circ) \Leftrightarrow \exists e \in S, \forall a \in X (e \circ a = a)$
$\mathsf{R\text{-}Identive}\,(\langle S, X \rangle, \circ) \Leftrightarrow \exists e \in S, \forall a \in X (e \circ a = a)$
$\mathsf{Identive}\,(\langle S, X \rangle, \circ) \Leftrightarrow \mathsf{L\text{-}Identive}\,(\langle S, X \rangle, \circ) \wedge \mathsf{R\text{-}Identive}\,(\langle S, X \rangle, \circ)$

Definition 115 (Identity Element)**.**
Let $\circ$ be a function symbol.
$\ell(X, \circ) = e \Leftrightarrow \forall x \in X (x \circ e = e \circ x = x)$

Definition 116.
Let $\circ$ be a function symbol.
$\mathsf{L\text{-}Invertive}\,(\langle S, X \rangle, \circ) \Leftrightarrow \forall a \in X, \exists a' \in S (a' \circ a = \ell(X, \circ))$
$\mathsf{R\text{-}Invertive}\,(\langle S, X \rangle, \circ) \Leftrightarrow \forall a \in X, \exists a' \in S (a \circ a' = \ell(X, \circ))$
$\mathsf{Invertive}\,(\langle S, X \rangle, \circ) \Leftrightarrow \mathsf{L\text{-}Invertive}\,(\langle S, X \rangle, \circ) \wedge \mathsf{R\text{-}Invertive}\,(\langle S, X \rangle, \circ)$

Definition 117 (Group)**.**
Let $\circ$ be a function symbol.
$\mathsf{Group}\,(X, \circ) \Leftrightarrow \mathsf{Closed}\,(X, \circ) \wedge \mathsf{Associative}\,(X, \circ) \wedge \mathsf{Identive}\,(X, \circ) \wedge \mathsf{Invertive}\,(X, \circ)$

Definition 118 (Field)**.**
Let ①, ② be function symbols.
$\mathsf{Field}\,(X, \langle ①, ② \rangle) \Leftrightarrow \mathsf{Group}\,(X, ①) \wedge \mathsf{Group}\,(X \setminus \{\ell(X, ①)\}, ②) \wedge \mathsf{Distributive}\,(X, \langle ①, ② \rangle)$

Definition 119 (Abelian Group)**.**
Let $\circ$ be a function symbol.
$\mathsf{Abel}\,(X, \circ) \Leftrightarrow \mathsf{Group}\,(X, \circ) \wedge \mathsf{Commutative}\,(X, \circ)$

Definition 120 (Vector Space)**.**
Let $\mathsf{Field}\,(F, \langle +, * \rangle)$.
$V \geqslant F \Leftrightarrow \mathsf{Abel}\,(V, +) \wedge \mathsf{Group}\,(\langle F, V \rangle, *) \wedge \mathsf{Distributive}\,(\langle F, V \rangle, \langle +, * \rangle)$

4

Elementary Number Theory

4.1 Prologue

Definition 121 (Composition of Functions)**.**
Let $f : X \to Y$, and $g : Y \to Z$.
$g \circ f = \{\langle x, z\rangle \mid \langle x, y\rangle \in f \wedge \langle y, z\rangle \in g\}$

Definition 122 (Function-Induced Equivalence Relation)**.**
Let $f : X \to Y$.
$x \sim_f y \Leftrightarrow f(x) = f(y)$

4.2 Number Systems

The space of natural numbers is said to be embeddable in the space of integers, and the space of integers is said to be embeddable in the space of rational numbers. A number from a set is said to have an (embedding) identity for every other set in which the number is embedded. Formally, a number's identity in a set ought to be distinguished from its identity in another set. However, for the sake of brevity, we shall not always make such a distinction.

Definition 123 (Natural Successor Function).
$\mathrm{suc} : \mathbb{N} \to \mathbb{N}$
$\mathrm{suc}(n) = n \cup \{n\}$

Definition 124 (Addition of Natural Numbers).
Let $a, b \in \mathbb{N}$.
$a + b = \mathrm{suc}^{\circ(b)}(a)$

Definition 125 (Multiplication of Natural Numbers).
Let $a, b \in \mathbb{N}$.
$a * b = \mathrm{suc}^{\circ(b)^{\circ(b)}}(a)$

Definition 126 (Prototype of Integer).
$[\![\langle a, b\rangle]\!]_{\mathbb{Z}} = \{\langle c, d\rangle \in \mathbb{N}^2 \mid \exists a, b \in \mathbb{N}(a + c = b + d)\}$

Definition 127 (Space of Integers).
$\mathbb{Z} = \{[\![\langle a, b\rangle]\!]_{\mathbb{Z}} \mid a, b \in \mathbb{N}\}$

Definition 128 (Integer Identity of Natural Number).
Let $n \in \mathbb{N}$.
$n_{\mathbb{Z}} = [\![\langle n, 0\rangle]\!]_{\mathbb{Z}}$

Definition 129 (Order of Integers).
Let $a, b, c, d \in \mathbb{N}$, and $[\![\langle a, b\rangle]\!]_{\mathbb{Z}}, [\![\langle c, d\rangle]\!]_{\mathbb{Z}} \in \mathbb{Z}$.
$[\![\langle a, b\rangle]\!]_{\mathbb{Z}} \leq [\![\langle c, d\rangle]\!]_{\mathbb{Z}} \Leftrightarrow a + d \leq b + c$

Definition 130 (Addition of Integers).
Let $a, b, c, d \in \mathbb{N}$, and $[\![\langle a, b\rangle]\!]_{\mathbb{Z}}, [\![\langle c, d\rangle]\!]_{\mathbb{Z}} \in \mathbb{Z}$.
$[\![\langle a, b\rangle]\!]_{\mathbb{Z}} + [\![\langle c, d\rangle]\!]_{\mathbb{Z}} = [\![\langle a + c, b + d\rangle]\!]_{\mathbb{Z}}$

Definition 131 (Multiplication of Integers).
Let $a, b, c, d \in \mathbb{N}$, and $[\![\langle a, b\rangle]\!]_{\mathbb{Z}}, [\![\langle c, d\rangle]\!]_{\mathbb{Z}} \in \mathbb{Z}$.
$[\![\langle a, b\rangle]\!]_{\mathbb{Z}} * [\![\langle c, d\rangle]\!]_{\mathbb{Z}} = [\![\langle ac + bd, ad + bc\rangle]\!]_{\mathbb{Z}}$

Definition 132 (Subtraction of Integers).
Let $a, b, c \in \mathbb{Z}$.
$a - b = c \Leftrightarrow a = b + c$

Definition 133 (Negation of Integer).
Let $n \in \mathbb{Z}$.
$-n = 0 - n$

Definition 134 (Prototype of Rational Number)**.**
Let $a, b \in \mathbb{Z}$.
$\llbracket\langle a, b\rangle\rrbracket_{\mathbb{Q}} = \{\langle c, d\rangle \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0_{\mathbb{Z}}\}) \mid \exists a, b \in \mathbb{Z}(ad = bc)\}$

Definition 135 (Space of Rational Numbers)**.**
$\mathbb{Q} = \{\llbracket\langle a, b\rangle\rrbracket_{\mathbb{Q}} \mid a \in \mathbb{Z} \wedge b \in \mathbb{Z} \setminus \{0_{\mathbb{Z}}\}\}$

Definition 136 (Rational Identity of Integer)**.**
Let $n \in \mathbb{Z}$.
$n_{\mathbb{Q}} = \llbracket\langle n, 1_{\mathbb{Z}}\rangle\rrbracket_{\mathbb{Q}}$

Definition 137 (Order of Rational Numbers)**.**
Let $a, b, c, d \in \mathbb{Z}$, and $\llbracket\langle a, b\rangle\rrbracket_{\mathbb{Q}}, \llbracket\langle c, d\rangle\rrbracket_{\mathbb{Q}} \in \mathbb{Q}$.
$\llbracket\langle a, b\rangle\rrbracket_{\mathbb{Q}} \leq \llbracket\langle c, d\rangle\rrbracket_{\mathbb{Q}} \Leftrightarrow ad \leq bc$

Definition 138 (Addition of Rational Numbers)**.**
Let $a, b, c, d \in \mathbb{Z}$, and $\llbracket\langle a, b\rangle\rrbracket_{\mathbb{Q}}, \llbracket\langle c, d\rangle\rrbracket_{\mathbb{Q}} \in \mathbb{Q}$.
$\llbracket\langle a, b\rangle\rrbracket_{\mathbb{Q}} + \llbracket\langle c, d\rangle\rrbracket_{\mathbb{Q}} = \llbracket\langle ad + bc, bd\rangle\rrbracket_{\mathbb{Q}}$

Definition 139 (Multiplication of Rational Numbers)**.**
Let $a, b, c, d \in \mathbb{Z}$, and $\llbracket\langle a, b\rangle\rrbracket_{\mathbb{Q}}, \llbracket\langle c, d\rangle\rrbracket_{\mathbb{Q}} \in \mathbb{Q}$.
$\llbracket\langle a, b\rangle\rrbracket_{\mathbb{Q}} * \llbracket\langle c, d\rangle\rrbracket_{\mathbb{Q}} = \llbracket\langle ac, bd\rangle\rrbracket_{\mathbb{Q}}$

Definition 140 (Subtraction of Rational Numbers)**.**
Let $a, b, c \in \mathbb{Q}$.
$a - b = c \Leftrightarrow a = b + c$

Definition 141 (Negation of Rational Number)**.**
Let $q \in \mathbb{Q}$.
$-q = 0 - q$

Definition 142 (Division of Rational Numbers)**.**
Let $a, b \in \mathbb{Q}$, and $b \neq 0$.
$\frac{a}{b} = c \Leftrightarrow a = bc$

Definition 143 (Partition)**.**
$X \mathbin{\dot\vee} U \Leftrightarrow \varnothing \notin X \wedge \not\cap(X) \wedge \bigcup X = U$

Definition 144 (Bipartition)**.**
$X \mathbin{\dot\vee}_2 U \Leftrightarrow \{X, X^{\complement}\} \mathbin{\dot\vee} U$

Definition 145 (Riterval)**.**
$X_{()} = \{S \subseteq X \mid \forall s \in S, \exists s' \in S(s \prec s') \wedge \forall s \in S, \exists s' \in S(s' \prec s)\}$
$X_{(-\infty,)} = \{S \subseteq X \mid \forall s \in S, \forall x \in X(x \prec s \Rightarrow x \in S) \wedge \forall s \in S, \exists s' \in S(s \prec s')\}$
$X_{(,\infty)} = \{S \subseteq X \mid \forall s \in S, \forall x \in X(s \prec x \Rightarrow x \in S) \wedge \forall s \in S, \exists s' \in S(s' \prec s)\}$

Definition 146 (Space of Real Numbers)**.**
$\mathbb{R} = \{Q \in \mathbb{Q}_{(-\infty,)} \mid Q \mathbin{\dot\vee}_2 \mathbb{Q}\}$

Definition 147 (Real Identity of Rational Number)**.**
Let $q \in \mathbb{Q}$.
$q_{\mathbb{R}} = \{p \in \mathbb{Q} \mid p < q\}$

Definition 148 (Order of Real Numbers)**.**
Let $x, y \in \mathbb{R}$.
$x \leq y \Leftrightarrow x \subseteq y$

Definition 149 (Addition of Real Numbers)**.**
Let $A, B \in \mathbb{R}$.
$X + Y = \{q \in \mathbb{Q} \mid \exists x \in X, \exists y \in Y(q \leq x + y)\}$

Definition 150 (Subtraction of Real Numbers)**.**
Let $x, y \in \mathbb{R}$.
$x - y = z \Leftrightarrow x = y + z$

Definition 151 (Negation of Real Number)**.**
Let $x \in \mathbb{R}$.
$-x = 0 - x$

Definition 152 (Multiplication of Real Numbers)**.**
Let $X, Y \in \mathbb{R}_{>0}$.
$X * Y = \{q \in \mathbb{Q} \mid \exists x \in X, \exists y \in Y(q \leq xy)\}$
$XY = -X * -Y = -(X * -Y) = -(-XY)$

Definition 153 (Division of Real Numbers)**.**
Let $x \in \mathbb{R}$, and $y \in \mathbb{R} \setminus \{0\}$.
$\frac{x}{y} = z \Leftrightarrow x = yz$
$\frac{x}{y} = \frac{-x}{-y} = -\frac{x}{-y} = -\frac{-x}{y}$

Definition 154 (Real Exponentiation Function)**.**
Let $x \in \mathbb{R}$.
$\mathsf{e}^x = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{n} x}{n!}$

Definition 155 (Real Logarithmic Function)**.**
Let $x \in \mathbb{R}_{>0}$.
$\ln(x) = y \Leftrightarrow x = \exp(y)$

Definition 156 (Exponentiation of Real Numbers)**.**
Let $n \in \mathbb{Z}$, and $x \in \mathbb{R}_{>0}$, and $y \in \mathbb{R}$.
$x^y = \exp(y \ln(x))$
$0^x = 0$
$(-x)^n = \prod_{i=1}^{n}(-x)$

Definition 157 (Root of Real Numbers)**.**
Let $n \in \mathbb{Z}_{\geq 1}$, and $x \in \mathbb{R}$.
$\sqrt[n]{x} = x^{\frac{1}{n}}$

Definition 158 (Logarithm of Real Numbers)**.**
Let $x, y \in \mathbb{R}_{>0}$.
$\log_x(y) = \dfrac{\ln(y)}{\ln(x)}$

4.3 Interlogue

Definition 159.

Let $n \in \mathbb{Z}_{\geq 0}$.

$! : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 1}$

$0! = 1$

$n! = n * \ldots * 1$

Definition 160.

Let $n \in \mathbb{N}_{\geq 1}$.
$\sum\{x\} = x$
$\sum\{x_1, \ldots, x_n, x_{n+1}\} = \sum\{x_1, \ldots, x_n\} + x_{n+1}$

Definition 161.

Let $n \in \mathbb{N}_{\geq 1}$.
$\prod\{x\} = x$
$\prod\{x_1, \ldots, x_n, x_{n+1}\} = \prod\{x_1, \ldots, x_n\} + x_{n+1}$

Definition 162 (Pi)**.**

$\pi = 4 \sum_{i=0}^{\infty} \frac{(-1)^i}{2i+1}$

4.4 Divisibility and Primality

An integer is said to divide another integer only if the former is a factor of the latter, or the latter is a multiple of the former. For an integer, the word “factor” is synonymous with the word “divisor”, while the word “multiple” is synonymous with the word “product”.

Theorem 17 (Euclidean Division)**.**
Let $a \in \mathbb{Z}$, and $b \in \mathbb{Z}_{\geq 1}$.
$\exists_! q \in \mathbb{Z}, \exists_! r \in \mathbb{Z}_{[0,b-1]}(a = bq + r)$

Definition 163 (Quotient)**.**
Let $a \in \mathbb{Z}$, and $b \in \mathbb{Z}_{\geq 1}$.
$a \,\texttt{div}\, b = q \Leftrightarrow \exists r \in \mathbb{Z}_{[0,b-1]}(a = bq + r)$

Definition 164 (Remainder)**.**
Let $a \in \mathbb{Z}$, and $b \in \mathbb{Z}_{\geq 1}$.
$a \,\texttt{mod}\, b = r \Leftrightarrow \exists q \in \mathbb{Z}(a = bq + r)$

Definition 165 (Divisibility)**.**
Let $n, N \in \mathbb{Z}$.
$n \mid N \Leftrightarrow \exists n' \in \mathbb{Z}(N = nn')$

Definition 166 (Proper Divisibility)**.**
Let $n, N \in \mathbb{Z}$.
$n \parallel N \Leftrightarrow n \mid N \wedge 0 < n < N$

Proposition 15 (Transitivity of Divisibility)**.**
Let $a, b, c \in \mathbb{Z}$.
$a \mid b \wedge b \mid c \Rightarrow a \mid c$

Proposition 16 (Additivity of Divisibility)**.**
Let $a, b, c \in \mathbb{Z}$.
$a \mid b \wedge a \mid c \Rightarrow a \mid b + c$

Proposition 17 (Multiplicativity of Divisibility)**.**
Let $a, b, c \in \mathbb{Z}$.
$a \mid b \wedge a \mid c \Rightarrow a \mid bc$

Definition 167 (Space of Prime Numbers)**.**
$\divideontimes = \{p \in \mathbb{Z}_{\geq 2} \mid \forall n \in \mathbb{Z}_{\geq 2}(n \mid p \Leftrightarrow n = p)\}$

Theorem 18 (Euclid)**.**
$\mathsf{Infinite}\,(\divideontimes)$

Theorem 19 (Wilson)**.**
$p \in \divideontimes \Leftrightarrow p + 1 \mid p! + 1$

Theorem 20 (Bertrand–Chebyshev)**.**
Let $n \in \mathbb{Z}_{\geq 2}$.
$\exists p \in \divideontimes(n < p < 2n)$

5

Linear Algebra

5.1 Prologue

Definition 168 (Initial Segment)**.**
Let $\mathsf{PartialOrdering}\,(X, \preceq)$.
$A \subseteq_{\downarrow} X \Leftrightarrow A \subseteq X \wedge \forall a, a' \in X(a' \preceq a \Rightarrow a' \in A)$

5.2 Vectors and Matrices

A vector can be defined as a tuple, and a matrix can be defined as a vector. We shall stipulate that wherever a bold-faced letter denotes a vector, its regular-faced counterpart denotes an element of the vector. Similarly, we shall stipulate that wherever a bold-faced capital letter denotes a matrix, its lowercase counterpart denotes an element of the matrix.

Definition 169 (Vector)**.**
Let $n \in \mathbb{N}_{\geq 1}$.

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \langle x_1, \ldots, x_n \rangle$$

Definition 170 (Element of Vector)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{a} \in X^n$, and $i \in \mathbb{N}_{[1,n]}$.
$a_i = \boldsymbol{a}_{[i]}$

Definition 171 (Template of Vector)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{a} \in X^n$, and $I \subseteq_{\downarrow} [1, n]$.
$[a_i : i \in I] = \boldsymbol{b} \Leftrightarrow \exists \boldsymbol{b} \in X^{\max(I)}, \forall i \in I(a_i = b_i)$

Definition 172 (Matrix)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$.

$$\begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{bmatrix} = \langle \langle x_{1,1}, \ldots, x_{1,n} \rangle, \ldots, \langle x_{m,1}, \ldots, x_{m,n} \rangle \rangle$$

Definition 173 (Element of Matrix)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in X^{m \times n}$, and $i \in [1, m]$, and $j \in [1, n]$.
$a_{i,j} = \boldsymbol{A}_{[i]}{}_{[j]}$

Definition 174 (Template of Matrix)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in X^{m \times n}$, and $I \subseteq_{\downarrow} [1, m]$, and $J \subseteq_{\downarrow} [1, n]$.
$[a_{i,j} : i \in I, j \in J] = \boldsymbol{B} \Leftrightarrow \exists \boldsymbol{B} \in X^{\max(I) \times \max(J)}, \forall i \in I, \forall j \in J(a_{i,j} = b_{i,j})$

Definition 175 (Row Vector of Matrix)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in X^{m \times n}$, and $i \in [1, m]$.
$\boldsymbol{a}_i = \boldsymbol{a}_{i,*} = \boldsymbol{A}_{[i]}$

Definition 176 (Column Vector of Matrix)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in X^{m \times n}$, and $i \in [1, m]$.
$\boldsymbol{a}_{*,j} = [a_{i,k} : i \in [1, m]]$

5.3 The Real Vector System

Definition 177 (Addition of Real Vectors)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$.
$\boldsymbol{u} + \boldsymbol{v} = [u_i + v_i : i \in [1, n]]$

Definition 178 (Subtraction of Real Vectors)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$.
$\boldsymbol{u} - \boldsymbol{v} = \boldsymbol{w} \Leftrightarrow \boldsymbol{u} = \boldsymbol{v} + \boldsymbol{w}$

Definition 179 (Negation of Real Vector)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{v} \in \mathbb{R}^n$.
$-\boldsymbol{v} = 0^n - \boldsymbol{v}$

Definition 180 (Scalar Multiplication of Real Vectors)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $c \in \mathbb{R}$, and $\boldsymbol{v} \in \mathbb{R}^n$.
$c * \boldsymbol{v} = [cv_i : i \in [1, n]]$

Definition 181 (Dot Product of Real Vectors)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$.
$\boldsymbol{u} \cdot \boldsymbol{v} = \displaystyle\sum_{i=1}^{n} u_i v_i$

Definition 182 (Euclidean Norm of Real Vector)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{v} \in \mathbb{R}^n$.
$\|\boldsymbol{v}\| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$

Definition 183 (Cross Product of Real Vectors)**.**
Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$.
$\boldsymbol{u} \times \boldsymbol{v} = \begin{bmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{bmatrix}$

Definition 184 (Unitary Real Vector)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{v} \in \mathbb{R}^n \setminus \{0^n\}$.
$\hat{\boldsymbol{v}} = \dfrac{\boldsymbol{v}}{\|\boldsymbol{v}\|}$

5.4 Trigonometric Functions

Definition 185 (Sine)**.**

$\sin : \mathbb{R} \to \mathbb{R}_{[-1,1]}$

$\sin(x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)!}$

Definition 186 (Cosine)**.**

$\cos : \mathbb{R} \to \mathbb{R}_{[-1,1]}$

$\cos(x) = \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i}}{(2i)!}$

Definition 187 (Tangent)**.**

$\tan : \mathbb{R} \setminus \left\{ \frac{\pi}{2} + n\pi \mid n \in \mathbb{Z} \right\} \to \mathbb{R}$

$\tan(x) = \frac{\sin(x)}{\cos(x)}$

Definition 188 (Principal Inverse of Sine)**.**

$\arcsin : \mathbb{R}_{[-1,1]} \to \mathbb{R}_{[\frac{-\pi}{2}, \frac{\pi}{2}]}$

$\arcsin(x) = y \Leftrightarrow \sin(y) = x$

Definition 189 (Principal Inverse of Cosine)**.**

$\arccos : \mathbb{R}_{[-1,1]} \to \mathbb{R}_{[0,\pi]}$

$\arccos(x) = y \Leftrightarrow \cos(y) = x$

Definition 190 (Principal Inverse of Tangent)**.**

$\arctan : \mathbb{R} \to \mathbb{R}_{(-\frac{\pi}{2}, \frac{\pi}{2})}$

$\arctan(x) = y \Leftrightarrow \tan(y) = x$

5.5 Interlogue

Definition 191.

Let $k, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$.

$\bullet\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\} = \displaystyle\sum_{i=1}^{n} (\boldsymbol{v}_1)_{[i]} \ldots (\boldsymbol{v}_k)_{[i]}$

Proposition 18.

Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$.

$\boldsymbol{u}^{\intercal}\boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v}$

Proposition 19 (Antisymmetry of Cross Product of Real Vectors)**.**

Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$.

$\boldsymbol{u} \times \boldsymbol{v} = -(\boldsymbol{v} \times \boldsymbol{u})$

Proposition 20 (Orthogonality of Cross Product of Real Vectors)**.**

Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$.

$(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{u} = (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{v} = 0$

Theorem 21.

Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$.

$\|\boldsymbol{u} \times \boldsymbol{v}\| = \sqrt{\|\boldsymbol{u}\|^2 \|\boldsymbol{v}\|^2 - (\boldsymbol{u} \cdot \boldsymbol{v})^2}$

Definition 192 (Angle between Real Vectors)**.**

Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, and $\boldsymbol{u}, \boldsymbol{v} \neq 0^n$.

$\angle(\boldsymbol{u}, \boldsymbol{v}) = \arccos\left(\dfrac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\right)$

Theorem 22.

Let $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3$.

$\angle(\boldsymbol{u}, \boldsymbol{v}) = \sin^{-1}\left(\dfrac{\|\boldsymbol{u} \times \boldsymbol{v}\|}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|}\right)$

Theorem 23.

Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$, and $\boldsymbol{u}, \boldsymbol{v} \neq 0^n$.

$0 \leq \angle(\boldsymbol{u}, \boldsymbol{v}) \leq \dfrac{\pi}{2} \Leftrightarrow \boldsymbol{u} \cdot \boldsymbol{v} \geq 0$

5.6 The Real Matrix System

Informally, the product obtained from the multiplication of real matrices can be visually described by a procedure involving "projections".

Definition 193 (Addition of Real Matrices)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{m \times n}$.
$\boldsymbol{A} + \boldsymbol{B} = [a_{i,j} + b_{i,j} : i \in [1, m], j \in [1, n]]$

Definition 194 (Subtraction of Real Matrices)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}^{m \times n}$.
$\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{C} \Leftrightarrow \boldsymbol{A} = \boldsymbol{B} + \boldsymbol{C}$

Definition 195 (Negation of Real Matrix)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$.
$-\boldsymbol{A} = 0^{m \times n} - \boldsymbol{A}$

Definition 196 (Scalar Multiplication of Real Matrices)**.**
Let $c \in \mathbb{R}$, and $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$.
$c * \boldsymbol{A} = [c a_{i,j} : i \in [1, m], j \in [1, n]]$

Definition 197 (Multiplication of Real Matrices)**.**
Let $m, n, p \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, and $\boldsymbol{B} \in \mathbb{R}^{n \times p}$.
$\boldsymbol{A} * \boldsymbol{B} = [\boldsymbol{a}_i \cdot \boldsymbol{b}_{*,j} : i \in [1, m], j \in [1, p]]$

Definition 198 (Frobenius Norm of Real Matrix)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$.
$$\|A\| = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} (a_{i,j})^2}$$

Definition 199 (Transpose of Real Matrix)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$.
$\boldsymbol{A}^{\intercal} = [a_{i,j} : i \in [1, n], j \in [1, m]]$

Definition 200 (Trace of Real Square Matrix)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{n \times n}$.
$$\operatorname{tr}(\boldsymbol{A}) = \sum_{i=1}^{n} a_{i,i}$$

Definition 201 (Symmetric Extension of Real Square Matrix)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{n \times n}$.
$\operatorname{syx}(\boldsymbol{A}) = \boldsymbol{B} \Leftrightarrow \forall i, j \in \mathbb{N}_{[1,n]}(a_{i,j} = b_{i,j}) \wedge \forall i \in \mathbb{N}_{[1,n]}, \forall k \in \mathbb{N}_{[1,n-1]}(a_{i,i-k} = b_{i,i+k})$

Definition 202 (Positive Diagon of Real Square Matrix)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{n \times n}$.
$\searrow^{+}(A) = \{[a_{i,i+k} : i \in [1, n]] \mid k \in [0, n-1]\}$

Definition 203 (Negative Diagon of Real Square Matrix)**.**

Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{n \times n}$.

$\nearrow^{-}(A) = \{\left[a_{i+(n-1),i+k} : i \in [1,n]\right] \mid k \in [0, n-1]\}$

Definition 204 (Determinant of Real Square Matrix)**.**

Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{n \times n}$.

$\det(A) = \bullet(\searrow^{+}(\mathrm{syx}(\boldsymbol{A}))) - \bullet(\nearrow^{-}(\mathrm{syx}(\boldsymbol{A})))$

5.7 Interlogue

In a real vector space, a hyperplane is an affine space of codimension 1. The vector normal to a hyperplane in a real vector space is unique up to scalar multiplication.

Definition 205 (Implicit Hyperplane)**.**
Let $k \in \mathbb{N}_{\geq 1}$, and $r \in \mathbb{R}$, and $\boldsymbol{n} \in \mathbb{R}^{k+1} \setminus \{0^{k+1}\}$.
$\text{hyp}_k(r, \boldsymbol{n}) = \{\boldsymbol{x} \in \mathbb{R}^{k+1} \mid \boldsymbol{n} \cdot \boldsymbol{x} = r\}$

Definition 206 (Parametric Hyperplane)**.**
Let $k \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{r} \in \mathbb{R}^{k+1}$, and $\boldsymbol{n} \in \mathbb{R}^{k+1} \setminus \{0^{k+1}\}$.
$\text{hyp}_k(\boldsymbol{r}, \boldsymbol{n}) = \{\boldsymbol{x} \in \mathbb{R}^{k+1} \mid \boldsymbol{n} \cdot (\boldsymbol{x} - \boldsymbol{r}) = 0^{k+1}\}$

Theorem 24.
Let $k \in \mathbb{N}_{\geq 1}$, and $r \in \mathbb{R}$, and $\boldsymbol{n} \in \mathbb{R}^{k+1} \setminus \{0^{k+1}\}$, and $\boldsymbol{r} \in \mathbb{R}^{k+1}$.
$\text{hyp}_k(r, \boldsymbol{n}) = \text{hyp}_k(\boldsymbol{r}, \boldsymbol{n}) \Leftrightarrow \boldsymbol{n} \cdot \boldsymbol{r} = r$

5.8 Linear Functionality

"Linear function" is synonymous with "linear transformation".

Definition 207 (Linear Real Function)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}^m$.
$\mathsf{Linear}\,(f) \Leftrightarrow \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n (f(\boldsymbol{u} + \boldsymbol{v}) = f(\boldsymbol{u}) + f(\boldsymbol{v})) \wedge \forall \lambda \in \mathbb{R}, \forall \boldsymbol{x} \in \mathbb{R}^n (f(\lambda \boldsymbol{x}) = \lambda f(\boldsymbol{x}))$

Definition 208.
$f_{\mathsf{L}} : X \to Y \Leftrightarrow f : X \to Y \wedge \mathsf{Linear}\,(f)$

Proposition 21.
Let $m, n \in \mathbb{N}_{\geq 1}$, and $f_{\mathsf{L}} : \mathbb{R}^n \to \mathbb{R}^m$.
$f(0^n) = 0^m$

Theorem 25.
$\{f : \mathbb{R}^n \to \mathbb{R}^m \mid \mathsf{Linear}\,(f) \wedge m, n \in \mathbb{N}_{\geq 1}\}$ bij $\{\boldsymbol{A} \in \mathbb{R}^{m \times n} \mid m, n \in \mathbb{N}_{\geq 1}\}$

Theorem 26.
Let $m, n \in \mathbb{N}_{\geq 1}$, and $f_{\mathsf{L}} : \mathbb{R}^n \to \mathbb{R}^m$, and $\boldsymbol{v} \in \mathbb{R}^n$, and $\boldsymbol{E} \in \widehat{\mathscr{B}}(\mathbb{R}^n)$.
$f(\boldsymbol{v}) = [f(\boldsymbol{e}_i) : i \in [1, n]]^{\mathsf{T}}\, \boldsymbol{v}$

5.9 Invertibility, Symmetry, and Diagonality

A (diagonally) symmetric matrix can be seen as a matrix whose "diagonally opposed" entries are equal. A diagonal matrix can be seen as a matrix whose "non-diagonal" entries are zero. For a matrix, diagonality implies symmetry. For a real matrix, diagonality implies a determinant of 1.

Definition 209 (Invertible Real Square Matrix)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{n \times n}$.
$\mathsf{Invertible}\,(\boldsymbol{A}) \Leftrightarrow \exists \boldsymbol{A}^{-1} \in \mathbb{R}^{n \times n}(\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = 1_{\mathbb{R}^{n \times n}})$

Definition 210 (Symmetric Real Square Matrix)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{n \times n}$.
$\mathsf{Symmetric}\,(\boldsymbol{A}) \Leftrightarrow \boldsymbol{A}^{\intercal} = \boldsymbol{A}$

Definition 211 (Antisymmetric Real Square Matrix)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{n \times n}$.
$\mathsf{Antisymmetric}\,(\boldsymbol{A}) \Leftrightarrow \boldsymbol{A}^{\intercal} = -\boldsymbol{A}$

Definition 212 (Diagonal Real Square Matrix)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{n \times n}$.
$\mathsf{Diagonal}\,(\boldsymbol{A}) \Leftrightarrow \forall i, j \in [1, n](i \neq j \Rightarrow a_{i,j} = 0)$

Proposition 22.
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{n \times n}$.
$\mathsf{Diagonal}\,(\boldsymbol{A}) \Rightarrow \mathsf{Symmetric}\,(\boldsymbol{A})$

Theorem 27.
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{n \times n}$.
$\mathsf{Diagonal}\,(\boldsymbol{A}) \Rightarrow \det(\boldsymbol{A}) = 1$

5.10 Column and Row Pictures

Theorem 28 (Real Column Picture)**.**

Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, and $\boldsymbol{x} \in \mathbb{R}^n$.

$$\boldsymbol{Ax} = \sum_{i=1}^{n} x_i \boldsymbol{a}_{*,i}$$

Theorem 29 (Real Row Picture)**.**

Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, and $\boldsymbol{x} \in \mathbb{R}^n$.

$\boldsymbol{Ax} = [\boldsymbol{a}_i \cdot \boldsymbol{x} : i \in [1, m]]$

5.11 Independence and Spanning

Definition 213 (Span of Real Vectors)**.**
Let $k, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in \mathbb{R}^n$.

$$\operatorname{span}(\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\}) = \left\{ \boldsymbol{x} \in \mathbb{R}^n \mid \exists \lambda_1, \ldots, \lambda_k \in \mathbb{R} \left(\sum_{i=1}^{k} \lambda_i \boldsymbol{v}_i = \boldsymbol{x} \right) \right\}$$

Definition 214 (Independent Set of Real Vectors)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $V \subseteq \mathbb{R}^n$.
$\mathsf{Independent}\,(V) \Leftrightarrow |V| \leq n \wedge \nexists \boldsymbol{v} \in V(\boldsymbol{v} \in \operatorname{span}(V \setminus \{\boldsymbol{v}\}))$

Definition 215 (Spanning Set of Real Vectors)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $V \subseteq \mathbb{R}^n$.
$\mathsf{Spanning}\,(V) \Leftrightarrow |V| \geq n \wedge \forall \boldsymbol{x} \in \mathbb{R}^n(\boldsymbol{x} \in \operatorname{span}(V))$

Definition 216 (Basis of Real Vector Space)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $V \subseteq \mathbb{R}^n$.
$\mathsf{Basis}\,(V) \Leftrightarrow \mathsf{Independent}\,(V) \wedge \mathsf{Spanning}\,(V)$

Definition 217 (Space of Bases of Real Vector Space)**.**
Let $n \in \mathbb{N}_{\geq 1}$.
$\mathscr{B}(\mathbb{R}^n) = \{V \subseteq \mathbb{R}^n \mid \mathsf{Basis}\,(V)\}$

Definition 218 (Space of Ordered Bases of Real Vector Space)**.**
Let $n \in \mathbb{N}_{\geq 1}$.
$\widehat{\mathscr{B}}(\mathbb{R}^n) = \{\langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \rangle \in (\mathbb{R}^n)^n \mid \mathsf{Basis}\,(\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\})\}$

Definition 219 (Dimension of Real Vector Space)**.**
Let $n \in \mathbb{N}_{\geq 1}$.
$\dim(\mathbb{R}^n) = |B| \Leftrightarrow B \in \mathscr{B}(\mathbb{R}^n)$

5.12 Rank and Nullity

The row space of a matrix is the span of the row vectors of the matrix. The column space of a matrix is the span of the column vectors of the matrix. The null space of a matrix is the kernel of the linear function of the matrix.

Definition 220 (Space of Row Vectors of Real Matrix)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$.
$\text{rows}(\boldsymbol{A}) = \{\boldsymbol{a}_{i,*} \mid i \in [1, m]\}$

Definition 221 (Row Space of Real Matrix)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$.
$\mathscr{R}(\boldsymbol{A}) = \text{span}(\text{rows}(\boldsymbol{A}))$

Definition 222 (Space of Column Vectors of Real Matrix)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$.
$\text{cols}(\boldsymbol{A}) = \{\boldsymbol{a}_{*,j} \mid j \in [1, n]\}$

Definition 223 (Column Space of Real Matrix)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$.
$\mathscr{C}(\boldsymbol{A}) = \text{span}(\text{cols}(\boldsymbol{A}))$

Theorem 30.
Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$.
$\dim(\mathscr{R}(\boldsymbol{A})) = \dim(\mathscr{C}(\boldsymbol{A}))$

Definition 224 (Rank of Real Matrix)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$.
$\text{rank}(\boldsymbol{A}) = \dim(\mathscr{R}(\boldsymbol{A})) = \dim(\mathscr{C}(\boldsymbol{A}))$

Definition 225 (Null Space of Matrix)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$.
$\mathscr{N}(\boldsymbol{A}) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{A}\boldsymbol{x} = 0^m\}$

Definition 226 (Nullity of Matrix)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$.
$\text{nullity}(\boldsymbol{A}) = \dim(\mathscr{N}(\boldsymbol{A}))$

Theorem 31 (Rank–Nullity)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$.
$\text{rank}(\boldsymbol{A}) + \text{nullity}(\boldsymbol{A}) = n$

5.13 Eigenvalues and Eigenvectors

Definition 227 (Eigenpair of Real Matrix)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{A} \in \mathbb{R}^{m \times n}$.
$\langle \lambda, \boldsymbol{x} \rangle \ \mathsf{eig}\ \boldsymbol{A} \Leftrightarrow \exists \lambda \in \mathbb{R}, \exists \boldsymbol{x} \in \mathbb{R}^n \setminus \{0^n\}(\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x})$

6

Real Analysis

6.1 Prologue

Definition 228 (Space of Extended Real Numbers)**.**
$\underline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$

Definition 229 (Space of Partial Functions)**.**
$\{X \rightharpoonup Y\} = \{f \subseteq X \times Y \mid \forall x \in X, \exists y \in Y(\langle x, y\rangle \in f)\}$

Definition 230 (Partial Function)**.**
$f : X \rightharpoonup Y \Leftrightarrow f \in \{X \rightharpoonup Y\}$
$f(x) = y \Leftrightarrow \langle x, y\rangle \in f$

6.2 Limits and Derivatives

To find $f(x, y)$'s gradient whole,
break f for ev'ry input role;
down input tree, each path you trace,
combine path rates, then sum with grace.

Definition 231 (Distance Function of $\mathbb{R}^n$)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$.
$\operatorname{dist}(a, b) = \|\boldsymbol{a} - \boldsymbol{b}\|$

Definition 232 (Limit of Real Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $\boldsymbol{x}_* \in \operatorname{dom}(f)$.
$\lim_{\boldsymbol{x}_*} f : \operatorname{dom}(f) \rightharpoonup \underline{\mathbb{R}}$

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}_*} f(\boldsymbol{x}) = \begin{cases} y_*, & \forall \varepsilon \in \mathbb{R}_{>0}, \exists \delta \in \mathbb{R}_{>0}(\operatorname{dist}(\boldsymbol{x}, \boldsymbol{x}_*) < \delta \Rightarrow \operatorname{dist}(f(\boldsymbol{x}), y_*) < \varepsilon) \\ -\infty, & \forall \varepsilon \in \mathbb{R}_{>0}, \exists \delta \in \mathbb{R}_{>0}(\operatorname{dist}(\boldsymbol{x}, \boldsymbol{x}_*) < \delta \Rightarrow f(\boldsymbol{x}) < \varepsilon) \\ \infty, & \forall \varepsilon \in \mathbb{R}_{>0}, \exists \delta \in \mathbb{R}_{>0}(\operatorname{dist}(\boldsymbol{x}, \boldsymbol{x}_*) < \delta \Rightarrow f(\boldsymbol{x}) > \varepsilon) \end{cases}$$

Definition 233 (Partial Derivative of Real Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $k \in \mathbb{N}_{[1,n]}$.
$\partial_k f : \operatorname{dom}(f) \rightharpoonup \underline{\mathbb{R}}$

$$\partial_k f(\boldsymbol{x}) = \lim_{\varepsilon \to 0} \left(\frac{f(\boldsymbol{x}|_{x_k}^{x_k + \varepsilon}) - f(\boldsymbol{x})}{\varepsilon} \right)$$

Definition 234 (Gradient of Real Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$.
$\boldsymbol{\nabla} f : \operatorname{dom}(f) \rightharpoonup \underline{\mathbb{R}}^n$
$\boldsymbol{\nabla} f(\boldsymbol{x}) = [\partial_i f(\boldsymbol{x}) : [1, n]]$

Definition 235 (Directional Derivative of Real Function on Real Unit Vector)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $\hat{\boldsymbol{u}} \in \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x}\| = 1\}$.
$\boldsymbol{\nabla}_{\boldsymbol{u}} f : \mathbb{R}^n \rightharpoonup \mathbb{R}$
$\boldsymbol{\nabla}_{\boldsymbol{u}} f(\boldsymbol{x}) = \boldsymbol{u} \cdot \boldsymbol{\nabla} f(\boldsymbol{x})$

6.3 Continuity and Differentiability

Definition 236 (Continuous Real Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $D \subseteq \mathrm{dom}(f)$.
$\mathsf{Continuous}\,(f/D) \Leftrightarrow \forall \boldsymbol{d} \in D \left(\lim\limits_{\boldsymbol{x} \to \boldsymbol{d}} f(\boldsymbol{x}) = f(\boldsymbol{d})\right)$

Definition 237 (Differentiable Real Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $D \subseteq \mathrm{dom}(f)$.
$\mathsf{Differentiable}\,(f/D) \Leftrightarrow \forall \boldsymbol{d} \in D, \exists \boldsymbol{\nabla} f(\boldsymbol{d})$

Proposition 23.
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $D \subseteq \mathrm{dom}(f)$.
$\mathsf{Differentiable}\,(f/D) \Rightarrow \mathsf{Continuous}\,(f/D)$

Definition 238.
$f : X \overset{\mathsf{c}}{\to} Y \Leftrightarrow f : X \to Y \wedge \mathsf{Continuous}\,(f/X)$

6.4 Interlogue

For a symmetric matrix, the property of indefiniteness can be seen as the complement of the conjunction of the properties of positive-definiteness and negative-definiteness.

Definition 239 (Hessian)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$.
$\operatorname{hes} f : \mathbb{R}^n \rightharpoonup \underline{\mathbb{R}}^n$
$\operatorname{hes} f(\boldsymbol{x}) = [\partial_{i,j} f(\boldsymbol{x}) : i \in [1,n], j \in [1,n]]$

Definition 240 (Positive-Definite Real Matrix)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $M \in \mathbb{R}^{n \times n}$, and $\mathsf{Symmetric}\,(M)$.
$\mathsf{PositiveDefinite}\,(M) \Leftrightarrow \forall \boldsymbol{x} \in \mathbb{R}^n \setminus \{0^n\}(\boldsymbol{x}^\intercal M \boldsymbol{x} > 0)$

Definition 241 (Negative-Definite Real Matrix)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $M \in \mathbb{R}^{n \times n}$, and $\mathsf{Symmetric}\,(M)$.
$\mathsf{NegativeDefinite}\,(M) \Leftrightarrow \forall \boldsymbol{x} \in \mathbb{R}^n \setminus \{0^n\}(\boldsymbol{x}^\intercal M \boldsymbol{x} < 0)$

Definition 242 (Indefinite Real Matrix)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $M \in \mathbb{R}^{n \times n}$, and $\mathsf{Symmetric}\,(M)$.
$\mathsf{Indefinite}\,(M) \Leftrightarrow \exists \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n \setminus \{0^n\}(\boldsymbol{x}^\intercal M \boldsymbol{x} > 0 \wedge \boldsymbol{y}^\intercal M \boldsymbol{y} < 0)$

Definition 243 (Jacobian)**.**
Let $m, n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}^m$.
$\operatorname{jac} f : \mathbb{R}^n \rightharpoonup \underline{\mathbb{R}}^{m \times n}$
$\operatorname{jac} f(\boldsymbol{x}) = [\partial_j f_i(\boldsymbol{x}) : i \in [1,m], j \in [1,n]]$

6.5 Interlogue

Definition 244 (Level Set)**.**
Let $f : X \to Y$, and $h \in \mathrm{im}(f)$.
$\mathrm{lev}_f(h) = \{x \in \mathrm{dom}(f) \mid f(x) = h\}$

Definition 245 (Interior of Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : X \to Y$.
$\mathrm{int}(f) = \{i \in \mathrm{dom}(f) \mid \exists \varepsilon \in \mathbb{R}_{>0}, \forall i' \in \mathrm{dom}(f)(\mathrm{dist}(i, i') < \varepsilon \Rightarrow i' \in \mathrm{dom}(f))\}$

Definition 246 (Equality-Constrained Function)**.**
Let $f : A \to B$, and $\boldsymbol{g} : A \to B^k$.
$f|_{\boldsymbol{g}} = f' \Leftrightarrow f' : \{x \in \mathrm{dom}(f) \mid \boldsymbol{g}(x) = 0^k\} \to B$

6.6 Optimisation

Definition 247 (Local Minimum of Real Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $\boldsymbol{x}_* \in \mathrm{dom}(f)$.
$\boldsymbol{x}_* \;\mathsf{locmin}\; f \Leftrightarrow \exists \varepsilon \in \mathbb{R}_{>0}, \forall \boldsymbol{x} \in \mathrm{dom}(f)(\mathrm{dist}(\boldsymbol{x}_*, \boldsymbol{x}) \leq \varepsilon \Rightarrow f(\boldsymbol{x}) \geq f(\boldsymbol{x}_*))$

Definition 248 (Local Maximum of Real Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $\boldsymbol{x}_* \in \mathrm{dom}(f)$.
$\boldsymbol{x}_* \;\mathsf{locmax}\; f \Leftrightarrow \exists \varepsilon \in \mathbb{R}_{>0}, \forall \boldsymbol{x} \in \mathrm{dom}(f)(\mathrm{dist}(\boldsymbol{x}, \boldsymbol{x}_*) \leq \varepsilon \Rightarrow f(\boldsymbol{x}) \leq f(\boldsymbol{x}_*))$

Definition 249 (Local Extremum of Real Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $\boldsymbol{x}_* \in \mathrm{dom}(f)$.
$\boldsymbol{x}_* \;\mathsf{locext}\; f \Leftrightarrow \boldsymbol{x}_* \;\mathsf{locmin}\; f \vee \boldsymbol{x}_* \;\mathsf{locmax}\; f$

Definition 250 (Critical Point of Real Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $\boldsymbol{x}_* \in \mathrm{dom}(f)$.
$\boldsymbol{x}_* \;\mathsf{crit}\; f \Leftrightarrow \boldsymbol{\nabla} f(\boldsymbol{x}_*) = 0^n \vee \nexists \boldsymbol{\nabla} f(\boldsymbol{x}_*)$

Theorem 32.
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $\boldsymbol{x}_* \in \mathrm{dom}(f)$.
$\boldsymbol{x}_* \;\mathsf{crit}\; f \wedge \mathsf{NegativeDefinite}(\mathrm{hes}(f)) \Rightarrow \boldsymbol{x}_* \;\mathsf{locmax}\; f$
$\boldsymbol{x}_* \;\mathsf{crit}\; f \wedge \mathsf{PositiveDefinite}(\mathrm{hes}(f)) \Rightarrow \boldsymbol{x}_* \;\mathsf{locmin}\; f$

Theorem 33.
Let $f : \mathbb{R} \to \mathbb{R}$, and $x_* \in \mathrm{dom}(f)$.
$x_* \;\mathsf{crit}\; f \wedge \boldsymbol{\nabla}^2 f(x_*) < 0 \Rightarrow x_* \;\mathsf{locmax}\; f$
$x_* \;\mathsf{crit}\; f \wedge \boldsymbol{\nabla}^2 f(x_*) > 0 \Rightarrow x_* \;\mathsf{locmin}\; f$

Theorem 34.
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $\boldsymbol{x}_* \in \mathrm{dom}(f)$.
$\boldsymbol{x}_* \;\mathsf{locext}\; f \Rightarrow \boldsymbol{\nabla} f(\boldsymbol{x}_*) = 0^n$

Theorem 35 (Lagrange Multiplier)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $k \in \mathbb{N}_{[1,n]}$, and $\boldsymbol{g} : \mathbb{R}^n \to \mathbb{R}^k$, and $\boldsymbol{x}_* \in \mathrm{int}(f)$.
$\boldsymbol{x}_* \;\mathsf{locext}\; f|_{\boldsymbol{g}} \wedge \mathsf{Independent}(\boldsymbol{\nabla} \boldsymbol{g}(\boldsymbol{x}_*)) \Rightarrow \exists! \boldsymbol{\lambda} \in \mathbb{R}^k (\boldsymbol{\nabla} f(\boldsymbol{x}) = \boldsymbol{\lambda} \cdot \boldsymbol{\nabla} \boldsymbol{g}(\boldsymbol{x}))$

6.7 Integrals

Definition 251 (Integral of Real Function)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $R = \mathbb{R}_{[a_1,b_1]} \times \cdots \times \mathbb{R}_{[a_n,b_n]}$.

$$\int_R f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = \lim_{N \to \infty} \sum_{i_1=1}^{N} \cdots \sum_{i_n=1}^{N} \left(f\left(\Cap_{i_1}^{a_1,b_1,N}, \ldots, \Cap_{i_n}^{a_n,b_n,N} \right) \prod_{j=1}^{n} \Cap_{i_j,i_j-1}^{a_j,b_j,N} \right)$$

Definition 252 (Line Integral of Real Function on Real Scalar Field)**.**
Let $a, b \in \mathbb{R}$, and $n \in \mathbb{N}_{\geq 1}$, and $f : \mathbb{R}^n \to \mathbb{R}$, and $\boldsymbol{p} : \mathbb{R}_{[a,b]} \to \mathbb{R}^n$, and $C = \mathrm{im}(\boldsymbol{p})$.

$$\int_C f(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = \int_a^b f(\boldsymbol{p}(t)) \|\boldsymbol{\nabla} \boldsymbol{p}(t)\| \,\mathrm{d}t$$

Definition 253 (Line Integral of Real Function on Real Vector Field)**.**
Let $a, b \in \mathbb{R}$, and $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$, and $\boldsymbol{p} : \mathbb{R}_{[a,b]} \to \mathbb{R}^n$, and $C = \mathrm{im}(\boldsymbol{p})$.

$$\int_C \boldsymbol{f}(\boldsymbol{x}) \cdot \mathrm{d}\boldsymbol{x} = \int_a^b \boldsymbol{f}(\boldsymbol{p}(t)) \cdot \boldsymbol{\nabla} \boldsymbol{p}(t) \,\mathrm{d}t$$

Theorem 36 (Gradient)**.**
Let $a, b \in \mathbb{R}$, and $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{\nabla f} : \mathbb{R}^n \to \mathbb{R}^n$, and $\boldsymbol{p} : \mathbb{R}_{[a,b]} \to \mathbb{R}^n$, and $C = \mathrm{im}(\boldsymbol{p})$.

$$\int_C \boldsymbol{\nabla f}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} = \boldsymbol{f}(\boldsymbol{p}(b)) - \boldsymbol{f}(\boldsymbol{p}(a))$$

6.8 Interlogue

Theorem 37 (Fubini)**.**
Let $f : \mathbb{R}^2 \to \mathbb{R}$, and $a, b, c, d \in \mathbb{R}$, and $R = \mathbb{R}_{[a,b]} \times \mathbb{R}_{[c,d]}$.
$\mathsf{Continuous}\,(f/R) \Rightarrow \displaystyle\int_c^d \int_a^b f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \int_c^d f(x, y)\,\mathrm{d}y\,\mathrm{d}x$

Theorem 38 (Horizontal Simplicity)**.**
Let $a, b \in \mathbb{R}$, and $f : \mathbb{R}^2 \to \mathbb{R}$, and $l, u : \mathbb{R}_{[a,b]} \xrightarrow{\mathsf{c}} \mathbb{R}$, and $R = \{\langle x, y\rangle \in \mathbb{R}^2 \mid a \leq y \leq b \wedge l(y) \leq x \leq u(y)\}$.
$\displaystyle\iint_R f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \int_a^b \int_{l(y)}^{u(y)} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$

Theorem 39 (Vertical Simplicity)**.**
Let $a, b \in \mathbb{R}$, and $f : \mathbb{R}^2 \to \mathbb{R}$, and $l, u : \mathbb{R}_{[a,b]} \xrightarrow{\mathsf{c}} \mathbb{R}$, and $R = \{\langle x, y\rangle \in \mathbb{R}^2 \mid a \leq x \leq b \wedge l(x) \leq y \leq u(x)\}$.
$\displaystyle\iint_R f(x, y)\,\mathrm{d}y\,\mathrm{d}x = \int_a^b \int_{l(x)}^{u(x)} f(x, y)\,\mathrm{d}y\,\mathrm{d}x$

6.9 Divergence and Curl

Definition 254 (Divergence of Real Vector Field)**.**
Let $n \in \mathbb{N}_{\geq 1}$, and $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^n$.
$\operatorname{div} \boldsymbol{f} : \mathbb{R}^n \rightharpoonup \mathbb{R}$
$\operatorname{div} \boldsymbol{f}(\boldsymbol{x}) = \sum_{i=1}^{n} \partial_i \boldsymbol{f}(\boldsymbol{x})$

Definition 255 (Curl of Real Vector Field)**.**
Let $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^3$.
$\operatorname{curl} \boldsymbol{f} : \mathbb{R}^3 \rightharpoonup \mathbb{R}^3$
$\operatorname{curl} \boldsymbol{f}(\boldsymbol{x}) = \begin{bmatrix} \partial_2 \boldsymbol{f}_3(\boldsymbol{x}) - \partial_3 \boldsymbol{f}_2(\boldsymbol{x}) \\ \partial_3 \boldsymbol{f}_1(\boldsymbol{x}) - \partial_1 \boldsymbol{f}_3(\boldsymbol{x}) \\ \partial_1 \boldsymbol{f}_2(\boldsymbol{x}) - \partial_2 \boldsymbol{f}_1(\boldsymbol{x}) \end{bmatrix}$

III

Postamble

7

Appendix

Under ZFC, every object is a set. For this reason, we shall use the “usual” English and Greek letters to denote sets, and other notation to denote constant symbols (i.e. particular sets).

Notation 10.
$x \eqcirc (x)$

Notation 11.
Let (φ) be a formula.
$\varphi \eqcirc (\varphi)$

Notation 12.
Let φ be a formula.
$x_\circ, \ldots, x_\bullet(\varphi) \eqcirc x_\circ(\ldots(x_\bullet(\varphi)))$

Notation 13.
Let φ be a formula.
$\not\forall x(\varphi) \eqcirc \neg\forall x(\varphi)$
$\nexists x(\varphi) \eqcirc \neg\exists x(\varphi)$

Notation 14 (Bracketless Existential Quantification)**.**
Let $n \in \mathbb{N}_{\geq 1}$.
$\exists x_1, \ldots, x_n \eqcirc \exists \bullet_1, \ldots, \bullet_n(\bullet_1 = x_1 \wedge \ldots \wedge \bullet_n = x_n)$

Notation 15 (Left-Associativity of Function Symbol)**.**
Let $\circ$ be a function symbol.
$a \circ \cdots \circ y \circ z \eqcirc (a \circ \cdots \circ y) \circ z$

Notation 16 (Iterated Composition)**.**
Let $\circ$ be a function symbol, and $x_\circ = \cdots = x_\bullet$.
$x^{\circ(n)} \eqcirc (x_1 \circ \cdots \circ x_n)$

Notation 17 (Universally-Quantified Implication)**.**
Let R be a relation symbol.
$\forall x \mathrel{\mathsf{R}} x'(\varphi) \eqcirc \forall x((x \mathrel{\mathsf{R}} x') \Rightarrow \varphi)$

Notation 18 (Reflectable Relation)**.**
Let R be a relation symbol.
$(x \mathrel{\text{Я}} y) \eqcirc (y \mathrel{\mathsf{R}} x)$

Notation 19 (Cancellable Relation)**.**
Let R be a relation symbol.
$(x \not\mathsf{R}\, y) \;\doteq\; (\neg(x \mathrel{\mathsf{R}} y))$

Notation 20 (Order-Equatable Relation)**.**
Let $\underline{\mathsf{R}}$ be a relation symbol.
$(x \mathrel{\mathsf{R}} y) \;\doteq\; ((x \mathrel{\underline{\mathsf{R}}} y) \vee (x \neq y))$

Notation 21.
Let $n \in \mathbb{N}_{\geq 1}$.
$x_1, \ldots, x_n \;\doteq\; \langle x_1, \ldots, x_n \rangle$

Notation 22 (Space of Matrices)**.**
Let $m, n \in \mathbb{N}$.
$X^{m \times n} \;\doteq\; (X^n)^m$

Notation 23 (Delimited Function)**.**
Let d and d' be delimiter symbols.
$(\mathsf{d}{\bullet}\mathsf{d}' : X \to Y) \;\doteq\; (\bullet_{\mathsf{dd}'} : X \to Y)$
$(\mathsf{d}x\mathsf{d}' = y) \;\doteq\; (\bullet_{\mathsf{dd}'}(x) = y)$

Notation 24 (Sourceless Interval)**.**
Let $n \in N$.
$[1, n] \;\doteq\; N_{[1,n]}$

Notation 25 (Comprehensible Function)**.**
Let $\circ$ be a function symbol, and $\varphi_\circ, \ldots, \varphi_\bullet$ be formulas.
$\circ_{\varphi_\circ, \ldots, \varphi_\bullet} f \;\doteq\; \{f \mid \varphi_\circ \wedge \ldots \wedge \varphi_\bullet\}$

Notation 26 (Interval-Comprehensible Function)**.**
Let $\circ$ be a function symbol.
$\circ_{i=a}^{b} f \;\doteq\; \circ_{i \in [a,b]} f$
$\circ_{i=-\infty}^{b} f \;\doteq\; \circ_{i \in (-\infty,b]} f$
$\circ_{i=a}^{\infty} f \;\doteq\; \circ_{i \in [a,\infty)} f$

Notation 27.
Let $c = a * b$.
$ab \;\doteq\; c$

Notation 28 (Element of Additive Identity)**.**
$0_X \;\doteq\; \ell(X, +)$

Notation 29 (Element of Multiplicative Identity)**.**
$1_X \;\doteq\; \ell(X, *)$

Notation 30.
Let $n \in \mathbb{N}_{\geq 1}$, and $f : X \to Y$.
$\partial_{1,\ldots,n} f(\boldsymbol{x}) \;\doteq\; \partial_n(\ldots(\partial_1 f(\boldsymbol{x})))$

Notation 31.

Let $n \in \mathbb{N}_{\geq 1}$, and $f : X \to Y$.

$\boldsymbol{\nabla}^n f \eqcirc (f)^{\boldsymbol{\nabla}(n)}$

Notation 32 (Riemann Pillar)**.**

$\sqcap_i^{a,b,n} \eqcirc a + i\left(\dfrac{b-a}{n}\right)$

$\sqcap_{i,i-1}^{a,b,n} \eqcirc \sqcap_i^{a,b,n} - \sqcap_{i-1}^{a,b,n}$

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