

PERIODIC PERMUTATIONS AND THE SUCCESSOR

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ABSTRACT. We investigate pairs of conjugacy classes of periodic permutations of \mathbb{Z} whose product contains the successor function.

INTRODUCTION

Let $\text{Sym}(X)$ denote the *symmetric group* of all permutations of X . The *orbit* of a point $x \in X$ under a permutation τ of X is given by $[x]_\tau = \{\tau^i(x) \mid i \in \mathbb{Z}\}$. We say that τ is *almost trivial* if $\tau(x) = x$ for cofinitely many $x \in X$, an *almost involution* if τ^2 is almost trivial, and (σ) -*periodic* if every orbit is finite. Define $C(\tau) = \sum_{x \in X} 1 - 2/|[x]_\tau|$ and $\text{Cl}(\tau) = \{\sigma \circ \tau \circ \sigma^{-1} \mid \sigma \in \text{Sym}(X)\}$. The *successor function* on \mathbb{Z} is given by $S^\mathbb{Z}(i) = i + 1$ for all $i \in \mathbb{Z}$. Here we prove the following:

Theorem A. *Suppose that ρ and σ are almost involutions of \mathbb{Z} for which $S^\mathbb{Z} \in \text{Cl}(\rho)\text{Cl}(\sigma)$. Then $C(\rho) + C(\sigma) \geq -1$.*

Theorem B. *Suppose that ρ and σ are non-almost-trivial permutations of \mathbb{Z} that are not both almost involutions. Then $S^\mathbb{Z} \in \text{Cl}(\rho)\text{Cl}(\sigma)$.*

The special case of Theorem B where neither ρ nor σ is an almost involution follows from [Mor89, Theorem A]. As far as I am aware, however, the special case of Theorem B where ρ or σ is an almost involution was not previously known. Regardless, the real purpose of this paper is to introduce ideas and language—in the simplest possible context—that can be used to investigate the finite-order elements R and S of the full group of an aperiodic Borel automorphism T for which $T \in \text{Cl}(R)\text{Cl}(S)$. This topic will be explored in a future paper.

In §1, we prove Theorem A. In §2, we note a symmetry that eliminates the need to repeat several arguments in the paper. In §3, we establish a technical fact concerning the removal of fixed points. In §4, we describe the simplest finite approximations to pairs (ρ, σ) for which $S^\mathbb{Z} \in \text{Cl}(\rho)\text{Cl}(\sigma)$. In §5, we use these as building blocks to construct extensions of more general finite approximations. And in §6, we prove the special case of Theorem B where ρ or σ has finite order.

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1. THE CASE OF TWO ALMOST INVOLUTIONS

For all $R \subseteq X^2$, define $\text{graph}_R(\tau) = \text{graph}(\tau) \cap R$.

Proposition 1.1. *Suppose that \leq is a linear ordering of a finite set F and $\tau \in \text{Sym}(F)$. Then $|\text{graph}_{\geq}(\tau)| \geq 1$ and $|\text{graph}_{<}(\tau)| \leq |F| - 1$.*

Proof. Let x be the \leq -maximal element of F . Then $x \geq \tau(x)$, so $|\text{graph}_{\geq}(\tau)| \geq 1$. But $|\text{graph}(\tau)| = |F|$, thus $|\text{graph}_{<}(\tau)| \leq |F| - 1$. \square

Define $\mathcal{O}(\tau) = \{[x]_\tau \mid x \in X\}$, $\text{Per}_K(\tau) = \{x \in X \mid |[x]_\tau| \in K\}$, and $\mathcal{O}_K(\tau) = \mathcal{O}(\tau \upharpoonright \text{Per}_K(\tau))$ for all sets K of cardinals.

Define $\text{graph}'_R(\tau) = \text{graph}(\tau \upharpoonright \sim \text{Per}_2(\tau)) \cap R$.

Proposition 1.2. *Suppose that τ is an almost involution of a set X and \preceq is a binary relation on X whose restriction to each orbit of τ is a linear order. Then $C(\tau) \geq |\text{graph}'_{<}(\tau)| - |\text{graph}'_{\preceq}(\tau)|$.*

Proof. As $|\text{graph}'_{<}(\tau)| \leq \sum_{O \in \mathcal{O}_{\geq 3}(\tau)} (|O| - 1)$ and $|\text{graph}'_{\preceq}(\tau)| \geq |\mathcal{O}_{\geq 3}(\tau)|$ by Proposition 1.1, the desired result follows from the fact that $C(\tau) = \sum_{O \in \mathcal{O}(\tau)} (|O| - 2) = \sum_{O \in \mathcal{O}_{\geq 3}(\tau)} (|O| - 1) - |\mathcal{O}_{\geq 3}(\tau)| - |\mathcal{O}_1(\tau)|$. \square

The disjoint union of $\tau_0, \tau_1 \in \text{Sym}(\mathbb{Z})$ is given by $(\tau_0 \amalg \tau_1)(i, k) = (\tau_k(i), k)$ for all $i \in \mathbb{Z}$ and $k < 2$. Let \preceq denote any binary relation on $\mathbb{Z} \times 2$ such that $(i, k) \preceq (j, k) \iff i \leq j$ for all $i, j \in \mathbb{Z}$ and $k < 2$. Theorem A follows from Proposition 1.2 and:

Proposition 1.3. *Suppose that $\tau_0, \tau_1 \in \text{Sym}(\mathbb{Z})$ and $S^{\mathbb{Z}} = \tau_0 \circ \tau_1$. Then $|\text{graph}'_{<}(\tau_0 \amalg \tau_1)| - |\text{graph}'_{\preceq}(\tau_0 \amalg \tau_1)| \geq -1$.*

Proof. Define $I, J: \text{graph}(\tau_0 \amalg \tau_1) \rightarrow (\mathbb{Z} \times 2)^2$ by

$$I((i, k), (j, k)) = \begin{cases} ((j, k), (i, k)) & \text{if } i, j \in \text{Per}_2(\tau_k) \text{ and} \\ ((i, k), (j, k)) & \text{otherwise} \end{cases}$$

and

$$J((i, k), (j, k)) = ((j - (1 - k), 1 - k), (i + k, 1 - k))$$

for all $i, j \in \mathbb{Z}$ and $k < 2$.

Lemma 1.4. $J(\text{graph}(\tau_0 \amalg \tau_1)) \subseteq \text{graph}(\tau_0 \amalg \tau_1)$.

Proof. Suppose that $((i, k), (j, k)) \in \text{graph}(\tau_0 \amalg \tau_1)$.

If $k = 0$, then $\tau_0(i) = j = S^{\mathbb{Z}}(j - 1) = (\tau_0 \circ \tau_1)(j - 1)$, so $i = \tau_1(j - 1)$, thus $J((i, 0), (j, 0)) = ((j - 1, 1), (i, 1)) \in \text{graph}(\tau_0 \amalg \tau_1)$.

If $k = 1$, then $\tau_1(i) = j$, so $\tau_0(j) = (\tau_0 \circ \tau_1)(i) = S^{\mathbb{Z}}(i) = i + 1$, thus $J((i, 1), (j, 1)) = ((j, 0), (i + 1, 0)) \in \text{graph}(\tau_0 \amalg \tau_1)$. \square

Lemma 1.5. $J(\text{graph}_{<}(\tau_0 \amalg \tau_1)) = \text{graph}_{\preceq}(\tau_0 \amalg \tau_1)$.

Proof. Note that $((i, k), (j, k)) \in \text{graph}_{\prec}(\tau_0 \amalg \tau_1) \iff i < j \iff j - 1 \geq i \iff j \geq i + 1 \iff J((i, k), (j, k)) \in \text{graph}_{\succeq}(\tau_0 \amalg \tau_1)$. \square

The *length* of $((i, k), (j, k)) \in \text{graph}(\tau_0 \amalg \tau_1)$ is $|((i, k), (j, k))| = |i - j|$.

Lemma 1.6. *Suppose that $((i, k), (j, k)) \in \text{graph}(\tau_0 \amalg \tau_1)$.*

- (1) *If $i < j$, then $|J((i, k), (j, k))| = |((i, k), (j, k))| - 1$.*
- (2) *If $i \geq j$, then $|J((i, k), (j, k))| = |((i, k), (j, k))| + 1$.*

Proof. If $i < j$, then $i + 1 \leq j$, so

$$|(j - 1) - i| = |j - (i + 1)| = j - (i + 1) = (j - i) - 1 = |i - j| - 1,$$

thus $|J((i, k), (j, k))| = |((i, k), (\pi, j))| - 1$. If $i \geq j$, then $i + 1 > j$, so

$$|(j - 1) - i| = |j - (i + 1)| = (i + 1) - j = (i - j) + 1 = |i - j| + 1,$$

thus $|J((i, k), (j, k))| = |((i, k), (\pi, j))| + 1$. \square

Let G be the group generated by I and J . The *orbit* of $((i, k), (j, k))$ under G is given by $[((i, k), (j, k))]_G = \{g \cdot ((i, k), (j, k)) \mid g \in G\}$. Set $\mathcal{O}(G) = \{[((i, k), (j, k))]_G \mid ((i, k), (j, k)) \in \text{graph}(\tau_0 \amalg \tau_1)\}$.

Lemma 1.7. *Suppose that $O \in \mathcal{O}(G)$. Then $\text{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O \neq \emptyset$.*

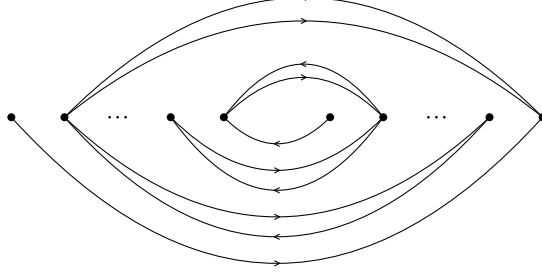
Proof. Fix $((i, k), (j, k)) \in O$. We can assume that $((i, k), (j, k)) \notin \text{graph}'_{\succeq}(\tau_0 \amalg \tau_1)$. By replacing $((i, k), (j, k))$ with $I((i, k), (j, k))$ if necessary, we can therefore assume that $((i, k), (j, k)) \in \text{graph}_{\prec}(\tau_0 \amalg \tau_1)$. For all $n \in \mathbb{N}$, define $((i_n, k), (j_n, k)) = (I \circ J)^n((i, k), (j, k))$. Note that if $((i_n, k), (j_n, k)) \in \text{graph}_{\prec}(\tau_0 \amalg \tau_1)$, then $J((i_n, k), (j_n, k)) \in \text{graph}_{\succeq}(\tau_0 \amalg \tau_1)$ and $|J((i_n, k), (j_n, k))| = |((i_n, k), (j_n, k))| - 1$ by Lemmas 1.5 and 1.6. Observe further that if $J((i_n, k), (j_n, k)) \notin \text{graph}'_{\succeq}(\tau_0 \amalg \tau_1)$, then $((i_{n+1}, k), (j_{n+1}, k)) \in \text{graph}_{\prec}(\tau_0 \amalg \tau_1)$. Setting $n = |i - j| - 1$, it follows that if $J((i_m, k), (j_m, k)) \notin \text{graph}'_{\succeq}(\tau_0 \amalg \tau_1)$ for all $m < n$, then $J((i_n, k), (j_n, k)) = 0$, thus $J((i_n, k), (j_n, k)) \in \text{graph}'_{\succeq}(\tau_0 \amalg \tau_1)$. \square

For all $i, j \in \mathbb{Z}$, we slightly abuse the usual notation by using (i, j) , $[i, j)$, $(i, j]$, and $[i, j]$ to denote the corresponding intervals of integers.

Lemma 1.8. *Suppose that $O \in \mathcal{O}(G)$.*

- (1) *If $|O| < \aleph_0$, then $|\text{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O| = |\text{graph}'_{\prec}(\tau_0 \amalg \tau_1) \cap O| = 1$.*
- (2) *If $|O| = \aleph_0$, then O is a cofinite subset of $\text{graph}(\tau_0 \amalg \tau_1)$, $|\text{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O| = 1$, and $\text{graph}'_{\prec}(\tau_0 \amalg \tau_1) \cap O = \emptyset$.*

Proof. By Lemma 1.7, there exists $((i, k), (j, k)) \in \text{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O$. For all $n \in \mathbb{N}$, define $((i_n, k), (j_n, k)) = (I \circ J)^n((i, k), (j, k))$. If $n \in \mathbb{N}$ and $((i_n, k), (j_n, k)) \in \text{graph}_{\succeq}(\tau_0 \amalg \tau_1)$, then Lemma 1.5 ensures that $J((i_n, k), (j_n, k)) \in \text{graph}_{\prec}(\tau_0 \amalg \tau_1)$. If $J((i_n, k), (j_n, k)) \notin \text{graph}'_{\prec}(\tau_0 \amalg \tau_1)$, then $((i_{n+1}, k), (j_{n+1}, k)) \in \text{graph}_{\succeq}(\tau_0 \amalg \tau_1)$.

FIGURE 1. A finite orbit of G .

Suppose now that n is the least natural number with the property that $J((i_n, k), (j_n, k)) \in \text{graph}'_{\prec}(\tau_0 \amalg \tau_1)$. Then $((i, k), (j, k))$ is in $\text{graph}'_{\prec}(\tau_0 \amalg \tau_1) \cap O$, $J((i_n, k), (j_n, k))$ is in $\text{graph}'_{\prec}(\tau_0 \amalg \tau_1) \cap O$, and the pairs $J((i_m, k), (j_m, k)), ((i_{m+1}, k), (j_{m+1}, k)) \in \text{Per}_2(\tau_0 \amalg \tau_1)^2$, for $m < n$, make up the rest of O , so condition (1) holds.

Finally, suppose that there is no $n \in \mathbb{N}$ for which $J((i_n, k), (j_n, k)) \in \text{graph}'_{\prec}(\tau_0 \amalg \tau_1)$. Then $((i, k), (j, k))$ is in $\text{graph}'_{\prec}(\tau_0 \amalg \tau_1) \cap O$ and the pairs $J((i_n, k), (j_n, k)), ((i_{n+1}, k), (j_{n+1}, k)) \in \text{Per}_2(\tau_0 \amalg \tau_1)^2$, for $n \in \mathbb{N}$, make up the rest of O , in which case $\text{graph}'_{\prec}(\tau_0 \amalg \tau_1) \cap O = \emptyset$. Moreover, a straightforward induction shows that, for all $n \in \mathbb{N}$, the following hold:

- $i_{2n} = i + n$ and $j_{2n} = j - n$.
- $\pi = \tau_0 \implies i_{2n+1} = i + n$ and $j_{2n+1} = j - (n + 1)$.
- $\pi = \tau_1 \implies i_{2n+1} = i + (n + 1)$ and $j_{2n+1} = j - n$.

Then $\text{graph}(\tau_0 \amalg \tau_1) \setminus O \subseteq ([i, j] \times 2)^2$, so condition (2) holds. \square

As at most one orbit of G can be cofinite, Lemma 1.8 ensures that $|\text{graph}'_{\prec}(\tau_0 \amalg \tau_1)| - |\text{graph}'_{\prec}(\tau_0 \amalg \tau_1)| \geq (|\mathcal{O}(G)| - 1) - |\mathcal{O}(G)| = -1$. \square

2. DUALS

We use $f: X \hookrightarrow Y$ to denote a partial injection of X into Y . For all $\sigma: \mathbb{Z} \hookrightarrow \mathbb{Z}$, define $\bar{\sigma}: \mathbb{Z} \hookrightarrow \mathbb{Z}$ by $\bar{\sigma}(i) = -\sigma^{-1}(-i)$ for all $i \in \mathbb{Z}$.

Proposition 2.1. *Suppose that $\sigma: \mathbb{Z} \hookrightarrow \mathbb{Z}$. Then $\sigma = \bar{\bar{\sigma}}$.*

Proof. If $i \in \mathbb{Z}$, then $\bar{\bar{\sigma}}(i) = -(\bar{\sigma})^{-1}(-i)$, so $\bar{\sigma}(-\bar{\bar{\sigma}}(i)) = -i$. But $\bar{\sigma}(-\bar{\bar{\sigma}}(i)) = -\sigma^{-1}(\bar{\bar{\sigma}}(i))$, so $i = \sigma^{-1}(\bar{\bar{\sigma}}(i))$, thus $\sigma(i) = \bar{\bar{\sigma}}(i)$. \square

Proposition 2.2. *Suppose that $\rho, \sigma: \mathbb{Z} \hookrightarrow \mathbb{Z}$. Then $\overline{\rho \circ \sigma} = \bar{\sigma} \circ \bar{\rho}$.*

Proof. Observe that

$$\begin{aligned} (\bar{\sigma} \circ \bar{\rho})(i) &= -\sigma^{-1}(-(-\rho^{-1}(-i))) \\ &= -(\sigma^{-1} \circ \rho^{-1})(-i) \\ &= -(\rho \circ \sigma)^{-1}(-i) \\ &= \bar{\rho} \circ \bar{\sigma}(i) \end{aligned}$$

for all $i \in \mathbb{Z}$. \(\square\)

Define $\mathcal{F} = \{(\rho: \mathbb{Z} \hookrightarrow \mathbb{Z}, \sigma: \mathbb{Z} \hookrightarrow \mathbb{Z}) \mid \rho \circ \sigma = S^{\mathbb{Z}} \upharpoonright \text{dom}(\rho \circ \sigma)\}$.

Proposition 2.3. $(\rho, \sigma) \in \mathcal{F} \iff (\bar{\sigma}, \bar{\rho}) \in \mathcal{F}$.

Proof. Note that if $i \in \mathbb{Z}$ and $\rho, \sigma: \mathbb{Z} \hookrightarrow \mathbb{Z}$, then $(\rho \circ \sigma)(i) = i + 1 \iff (\rho \circ \sigma)^{-1}(i + 1) = i \iff \bar{\rho} \circ \bar{\sigma}(-i - 1) = -i$, so the desired result follows from Proposition 2.2. \(\square\)

Let $(i_0 \ i_1 \ \dots \ i_n)$ denote the permutation of $\{i_m \mid m \leq n\}$ sending i_m to i_{m+1} for all $m < n$.

Proposition 2.4. Suppose that $n \geq 1$, $(i_m)_{m \leq n}$ is strictly increasing, $\rho = (i_0 \ i_1 \ \dots \ i_n)$, and $\sigma = (-i_n \ -i_{n-1} \ \dots \ -i_0)$. Then $\rho = \bar{\sigma}$.

Proof. If $m < n$, then $\bar{\sigma}(i_m) = -\sigma^{-1}(-i_m) = -(-i_{m+1}) = i_{m+1}$. \(\square\)

3. ELIMINATING FIXED POINTS

For all $k \in \mathbb{N}$, let $\text{par}(k)$ denote the remainder when k is divided by two. For all $\rho, \sigma \in \text{Sym}(X)$, set $\delta(\rho, \sigma) = \{x \in X \mid \rho(x) \neq \sigma(x)\}$ and

$$\text{Mal}(\rho, \sigma) = \{x \in \text{Per}_{\mathbb{N}+3}(\sigma) \mid |[x]_{\sigma} \setminus \text{Per}_1(\rho)| = 1\}.$$

Proposition 3.1. Suppose that $m \geq 1$, ρ and σ are permutations of a set X , and $\forall n \geq 3 \ \neg 0 < |\text{Mal}(\rho, \sigma) \cap \text{Per}_{2\mathbb{N}+n}(\sigma)| < \aleph_0$. Then there are permutations ρ' and σ' of X such that:

- (1) $\rho \circ \sigma = \rho' \circ \sigma'$,
- (2) $\delta(\rho, \rho') = \delta(\sigma^{-1}, (\sigma')^{-1}) = \text{Mal}(\rho, \sigma) \cap \text{Per}_1(\rho)$,
- (3) $\text{Mal}(\rho, \sigma) \cap \text{Per}_1(\rho) \subseteq \text{Per}_m(\rho')$, and
- (4) $\forall n \geq 3 \ \text{Mal}(\rho, \sigma) \cap \text{Per}_n(\sigma) \subseteq \text{Per}_n(\sigma')$.

Proof. Define $Y = \text{Mal}(\rho, \sigma)$ and $Z = Y \setminus \text{Per}_1(\rho)$. For all $n \geq 3$, set $Y_n = \text{Per}_{2\mathbb{N}+n}(\sigma) \cap Y$ and $Z_n = \text{Per}_{2\mathbb{N}+n}(\sigma) \cap Z$. Fix an equivalence relation F_4 on Z_4 whose classes all have cardinality m^2 , as well as $\pi_{0,1}, \pi_{0,2} \in \text{Sym}(Z_4)$, whose graphs are contained in F_4 , such that the orbits of $\pi_{0,1}$, $\pi_{0,2}$, and $\pi_{0,3} = (\pi_{0,1} \circ \pi_{0,2})^{-1}$ all have cardinality m . For all $n \in (\mathbb{N}+3) \setminus \{4\}$, fix an equivalence relation F_n on Z_n whose classes all have cardinality m , fix $\pi_{\text{par}(n), n-2} \in \text{Sym}(Z_n)$ whose orbits coincide

with the equivalence classes of F_n , and set $\pi_{\text{par}(n),n-1} = \pi_{\text{par}(n),n-2}^{-1}$. Then the support of $\pi = \text{id}_{X \setminus (Y \setminus Z)} \cup \bigcup_{p < 2, n \geq 1} \sigma^n \circ \pi_{p,n} \circ \sigma^{-n}$ is $Y \setminus Z$, so $\rho' = \rho \circ \pi$ and $\sigma' = \pi^{-1} \circ \sigma$ satisfy conditions (1)–(3).

Lemma 3.2. *Suppose that $\ell \leq n - 1$. Then*

$$(\sigma')^\ell \upharpoonright Z_n = (\sigma^\ell \circ \pi_{\text{par}(n),\ell}^{-1} \circ \cdots \circ \pi_{\text{par}(n),1}^{-1}) \upharpoonright Z_n. \quad (*)$$

Proof. The case $\ell = 0$ is trivial. If $\ell > 0$ and $(*)$ holds at $\ell - 1$, then

$$\begin{aligned} (\sigma')^\ell \upharpoonright Z_n &= (\sigma' \circ (\sigma')^{\ell-1}) \upharpoonright Z_n \\ &= (\sigma' \circ \sigma^{\ell-1} \circ \pi_{\text{par}(n),\ell-1}^{-1} \circ \cdots \circ \pi_{\text{par}(n),1}^{-1}) \upharpoonright Z_n \\ &= (\pi^{-1} \circ \sigma^\ell \circ \pi_{\text{par}(n),\ell-1}^{-1} \circ \cdots \circ \pi_{\text{par}(n),1}^{-1}) \upharpoonright Z_n \\ &= (\sigma^\ell \circ \pi_{\text{par}(n),\ell}^{-1} \circ \sigma^{-\ell} \circ \sigma^\ell \circ \pi_{\text{par}(n),\ell-1}^{-1} \circ \cdots \circ \pi_{\text{par}(n),1}^{-1}) \upharpoonright Z_n \\ &= (\sigma^\ell \circ \pi_{\text{par}(n),\ell}^{-1} \circ \cdots \circ \pi_{\text{par}(n),1}^{-1}) \upharpoonright Z_n, \end{aligned}$$

so $(*)$ also holds at ℓ . \square

For all $n \geq 3$, set $Y'_n = \text{Per}_n(\sigma) \cap Y$ and $Z'_n = \text{Per}_n(\sigma) \cap Z$. Lemma 3.2 ensures that $Y'_n = \bigcup_{\ell \leq n-1} \sigma^\ell(Z'_n) = \bigcup_{\ell \leq n-1} (\sigma')^\ell(Z'_n)$ and

$$\begin{aligned} (\sigma')^n \upharpoonright Z'_n &= (\sigma' \circ (\sigma')^{n-1}) \upharpoonright Z'_n \\ &= (\sigma' \circ \sigma^{n-1} \circ \pi_{\text{par}(n),n-1}^{-1} \circ \cdots \circ \pi_{\text{par}(n),1}^{-1}) \upharpoonright Z'_n \\ &= (\sigma' \circ \sigma^{n-1}) \upharpoonright Z'_n \\ &= (\sigma' \circ \sigma^{-1}) \upharpoonright Z'_n \\ &= \text{id}_{Z'_n}, \end{aligned}$$

so condition (4) also holds. \square

We write $\rho \cong \sigma$ to indicate that ρ and σ are isomorphic.

Proposition 3.3. *Suppose that $m \geq 1$, ρ and σ are permutations of a set X , $\forall n \geq 3 \neg 0 < |\text{Mal}(\rho, \sigma) \cap \text{Per}_{2\mathbb{N}+n}(\sigma)| < \aleph_0$, and $\text{Per}_m(\rho)$ is infinite. Then there are permutations $\rho' \cong \rho \upharpoonright \sim(\text{Mal}(\rho, \sigma) \cap \text{Per}_1(\rho))$ and $\sigma' \cong \sigma$ of X for which $\rho \circ \sigma = \rho' \circ \sigma'$.*

Proof. Proposition 3.1 yields $\rho', \sigma' \in \text{Sym}(X)$ such that $\rho \circ \sigma = \rho' \circ \sigma'$ and $|\mathcal{O}_\kappa(\rho \upharpoonright \sim(\text{Mal}(\rho, \sigma) \cap \text{Per}_1(\rho)))| = |\mathcal{O}_\kappa(\rho')|$ and $|\mathcal{O}_\kappa(\sigma)| = |\mathcal{O}_\kappa(\sigma')|$ for all cardinals κ . \square

4. BUILDING BLOCKS

Set $\mathcal{F}(i, j] = \{(\rho, \sigma) \in \mathcal{F} \mid \rho: (i, j] \hookrightarrow (i, j] \text{ and } \sigma: (i, j) \hookrightarrow (i, j)\}$, noting that $\forall (\rho, \sigma) \in \mathcal{F}(i, j] \text{ dom}(\rho \circ \sigma) = (i, j - 1]$.

Proposition 4.1. *If $i < j$ and $(\rho, \sigma) \in \mathcal{F}(i, j]$, then $\rho(j) = i + 1$.*

Proof. Observe that $\rho((i, j - 1]) = (\rho \circ \sigma)((i, j - 1]) = (i + 1, j]$. \square

Set $\mathcal{F}[i, j] = \{(\rho, \sigma) \in \mathcal{F} \mid \rho: (i, j) \hookrightarrow (i, j) \text{ and } \sigma: [i, j] \hookrightarrow [i, j]\}$, this time noting that $\forall(\rho, \sigma) \in \mathcal{F}(i, j] \ S^{\mathbb{Z}}(j - 1) \notin \text{rng}(\rho)$, and therefore $\forall(\rho, \sigma) \in \mathcal{F}(i, j] \ \text{dom}(\rho \circ \sigma) = [i, j - 1]$.

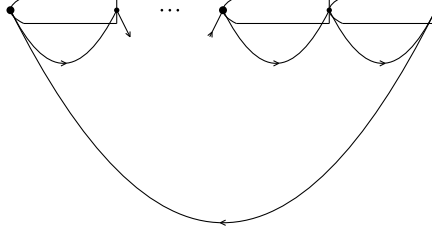


FIGURE 2. The extension provided by Proposition 4.2.

Proposition 4.2. Suppose that $n \geq 1$, $(i_m)_{m \leq n}$ is strictly increasing, $\forall m < n \ (\rho_m, \sigma_m) \in \mathcal{F}(i_m, i_{m+1}]$, $\rho = \bigcup_{m < n} \rho_m$, and $\sigma = (i_0 \ i_1 \ \dots \ i_n) \cup \bigcup_{m < n} \sigma_m$. Then $(\rho, \sigma) \in \mathcal{F}[i_0, i_n + 1)$.

Proof. As $[i_0, i_n) = \{i_m \mid m < n\} \cup \bigcup_{m < n} (i_m, i_{m+1} - 1]$, it follows that

$$\begin{aligned} (\rho, \sigma) \in \mathcal{F}[i_0, i_n + 1) &\iff \forall k \in [i_0, i_n) \ (\rho \circ \sigma)(k) = k + 1 \\ &\iff \forall m < n \ (\rho \circ \sigma)(i_m) = i_m + 1 \\ &\iff \forall m < n \ \rho(i_{m+1}) = i_m + 1 \\ &\iff \forall m < n \ \rho_m(i_{m+1}) = i_m + 1, \end{aligned}$$

so Proposition 4.1 yields the desired result. \square

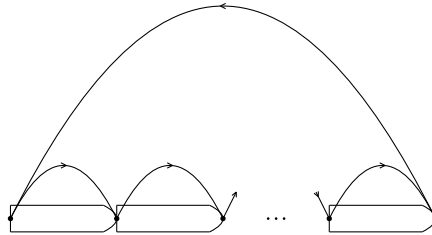


FIGURE 3. The extension provided by Proposition 4.3.

Proposition 4.3. Suppose that $n \geq 1$, $(i_m)_{m \leq n}$ is strictly increasing, $\forall m < n \ (\rho_m, \sigma_m) \in \mathcal{F}[i_m, i_{m+1})$, $\rho = (i_0 \ i_1 \ \dots \ i_n) \cup \bigcup_{m < n} \rho_m$, and $\sigma = \bigcup_{m < n} \sigma_m$. Then $(\rho, \sigma) \in \mathcal{F}(i_0 - 1, i_n]$.

Proof. By Propositions 2.1, 2.3, 2.4, and 4.2. \square



FIGURE 4. Building blocks from Propositions 4.5 and 4.6.

Proposition 4.4. *Suppose that $i \in \mathbb{Z}$. Then $(\emptyset, \text{id}_{\{i\}}) \in \mathcal{F}[i, i+1)$ and $(\text{id}_{\{i\}}, \emptyset) \in \mathcal{F}(i-1, i]$.*

Proof. As $[i, i] = (i-1, i-1] = \emptyset$, the definitions of $\mathcal{F}[i, i+1)$ and $\mathcal{F}(i-1, i]$ yield that $(\rho, \sigma) \in \mathcal{F}[i, i+1) \iff (\rho = \emptyset \text{ and } \text{dom}(\sigma) = \{i\})$ and $(\rho, \sigma) \in \mathcal{F}(i-1, i] \iff (\text{dom}(\rho) = \{i\} \text{ and } \sigma = \emptyset)$. \square

Proposition 4.5. *Suppose that $i < j$ are integers. Then the pair $(\text{id}_{(i,j)}, (i \ i+1 \ \dots \ j-1))$ is in $\mathcal{F}[i, j)$.*

Proof. If $i+1 = j$, then this follows from Proposition 4.4. Otherwise, Proposition 4.4 ensures that $(\text{id}_{\{k\}}, \emptyset) \in \mathcal{F}(k-1, k]$ for all $k \in (i, j)$, so Proposition 4.2 yields the desired result. \square

Proposition 4.6. *Suppose that $m \geq 1$ and $(i_k)_{k < m}$ is a strictly increasing sequence of integers. Then the pair*

$$((i_0 \ i_1 \ \dots \ i_{m-1}) \cup \bigcup_{k < m-1} \text{id}_{(i_k, i_{k+1})}, \bigcup_{k < m-1} (i_k \ i_k+1 \ \dots \ i_{k+1}-1))$$

is in $\mathcal{F}(i_0-1, i_{m-1}]$.

Proof. If $m = 1$, then this follows from Proposition 4.4. Otherwise, $(\text{id}_{(i_k, i_{k+1})}, (i_k \ i_k+1 \ \dots \ i_{k+1}-1)) \in \mathcal{F}[i_k, i_{k+1})$ for all $k < m-1$ by Proposition 4.5, so Proposition 4.3 yields the desired result. \square

5. EXTENSION

Given $n \geq 3$ and $\rho, \sigma: X \hookrightarrow X$, we say that a fixed point x of ρ is *n-malleable* if $x \in \text{Per}_n(\sigma)$, $[x]_\sigma \subseteq \text{dom}(\rho)$, and $|[x]_\sigma \setminus \text{Per}_1(\rho)| = 1$.

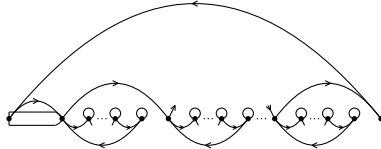


FIGURE 5. The extension provided by Proposition 5.1.

Proposition 5.1. *Suppose that $i < j$, $m \geq 2$, $n_k \geq 3$ for all $k < m-2$, and $(\rho, \sigma) \in \mathcal{F}[i, j]$. Then there exists $(\rho', \sigma') \in \mathcal{F}(i-1, j + \sum_{k < m-2} n_k]$ such that:*

- ρ' is obtained from ρ by adding a single cycle of length m and $n_k - 1$ n_k -malleable fixed points for all $k < m-2$.
- σ' is obtained from σ by adding a cycle of length n_k for all $k < m-2$.

Proof. Recursively define $i_0 = i$, $i_1 = j$, and $i_k = i_{k-1} + n_{k-2}$ for all $2 \leq k \leq m-1$. Set $(\rho_0, \sigma_0) = (\rho, \sigma)$. For all $1 \leq k \leq m-2$, Proposition 4.5 ensures that $(\rho_k, \sigma_k) = (\text{id}_{(i_k, i_{k+1})}, (i_k \ i_k + 1 \ \cdots \ i_{k+1} - 1))$ is in $\mathcal{F}[i_k, i_{k+1}]$. So $(\rho', \sigma') = ((i_0 \ i_1 \ \cdots \ i_{m-1}) \cup \bigcup_{k \leq m-2} \rho_k, \bigcup_{k \leq m-2} \sigma_k)$ is in $\mathcal{F}(i_0 - 1, i_{m-1}]$ by Proposition 4.3. But $i_0 - 1 = i - 1$ and $i_{m-1} = j + \sum_{k < m-2} n_k$. \square

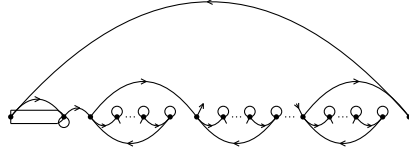


FIGURE 6. The extension provided by Proposition 5.2.

Proposition 5.2. *Suppose that $i < j$, $m \geq 3$, $n_k \geq 3$ for all $k < m-3$, and $(\rho, \sigma) \in \mathcal{F}[i, j]$. Then there exists $(\rho', \sigma') \in \mathcal{F}(i-1, j+1 + \sum_{k < m-3} n_k]$ such that:*

- ρ' is obtained from ρ by adding a single cycle of length m and $n_k - 1$ n_k -malleable fixed points for all $k < m-3$.
- σ' is obtained from σ by adding a fixed point and a cycle of length n_k for all $k < m-3$.

Proof. Recursively define $i_0 = i$, $i_1 = j$, $i_2 = j+1$, and $i_k = i_{k-1} + n_{k-3}$ for all $3 \leq k \leq m-1$. Set $(\rho_0, \sigma_0) = (\rho, \sigma)$. For all $1 \leq k \leq m-2$, Proposition 4.5 ensures that $(\rho_k, \sigma_k) = (\text{id}_{(i_k, i_{k+1})}, (i_k \ i_k + 1 \ \cdots \ i_{k+1} - 1))$ is in $\mathcal{F}[i_k, i_{k+1}]$. So $(\rho', \sigma') = ((i_0 \ i_1 \ \cdots \ i_{m-1}) \cup \bigcup_{k \leq m-2} \rho_k, \bigcup_{k \leq m-2} \sigma_k)$ is in $\mathcal{F}(i_0 - 1, i_{m-1}]$ by Proposition 4.3. But $i_0 - 1 = i - 1$ and $i_{m-1} = j + 1 + \sum_{k < m-3} n_k$. \square

Proposition 5.3. *Suppose that $i < j$, $n \geq 3$, $n_k \geq 3$ for all $k < n-3$, and $(\rho, \sigma) \in \mathcal{F}[i, j]$. Then there exists $(\rho', \sigma') \in \mathcal{F}[i-2 - \sum_{k < n-3} (n_k + 1), j+1]$ such that:*

- ρ' is obtained from ρ by adding $n-2$ cycles of length two and $n_k - 1$ n_k -malleable fixed points for all $k < n-3$.

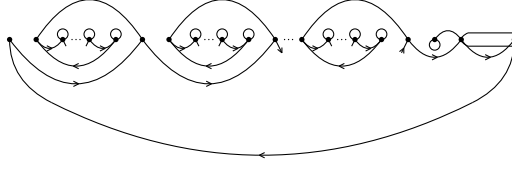


FIGURE 7. The extension provided by Proposition 5.3.

- σ' is obtained from σ by adding a single fixed point, a cycle of length n , and a cycle of length of n_k for all $k < n - 3$.

Proof. Recursively define $i_{n-1} = j$, $i_{n-2} = i$, $i_{n-3} = i - 2$, and $i_k = i_{k+1} - (n_k + 1)$ for all $k \leq n - 4$. Set $(\rho_{n-2}, \sigma_{n-2}) = (\rho, \sigma)$. For all $k \leq n - 3$, Proposition 4.6 implies that

$$(\rho_k, \sigma_k) = (\text{id}_{(i_k+1, i_{k+1})} \cup (i_k + 1 \ i_{k+1}), (i_k + 1 \ i_k + 2 \ \cdots \ i_{k+1} - 1))$$

is in $\mathcal{F}(i_k, i_{k+1}]$. So $(\rho', \sigma') = (\bigcup_{k \leq n-2} \rho_k, (i_0 \ i_1 \ \cdots \ i_{n-1}) \cup \bigcup_{k \leq n-2} \sigma_k)$ is in $\mathcal{F}[i_0, i_{n-1} + 1)$ by Proposition 4.2. But $i - 2 - \sum_{k < n-3} (n_k + 1) = i_0$ and $j + 1 = i_{n-1} + 1$. \square

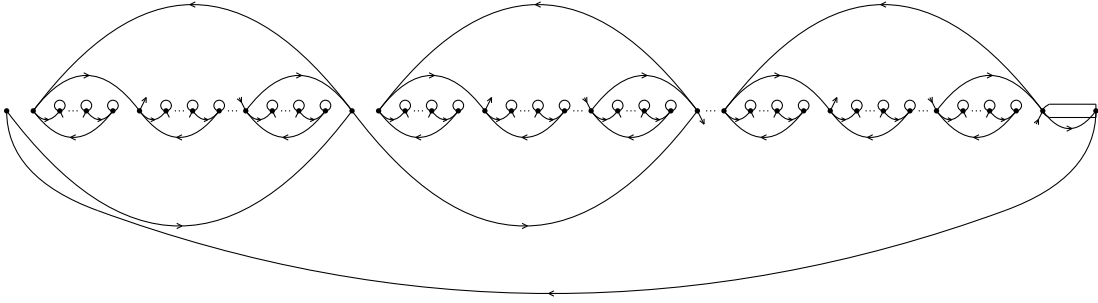


FIGURE 8. The extension provided by Proposition 5.4.

Proposition 5.4. Suppose that $i < j$, $m \geq 2$, $n \geq 2$, $n_{k,\ell} \geq 3$ for all $k < m - 1$ and $\ell < n - 2$, and $(\rho, \sigma) \in \mathcal{F}(i, j]$. Then there exists $(\rho', \sigma') \in \mathcal{F}[i - \sum_{k < m-1, \ell < n-2} n_{k,\ell}, j + 1)$ such that:

- ρ' is obtained from ρ by adding $n - 2$ cycles of length m and $n_{k,\ell} - 1$ $n_{k,\ell}$ -malleable fixed points for all $k < m - 1$ and $\ell < n - 2$.
- σ' is obtained from σ by adding a cycle of length n and a cycle of length $n_{k,\ell}$ for all $k < m - 1$ and $\ell < n - 2$.

Proof. Recursively define $i_{n-1} = j$, $i_{n-2} = i$, $i_\ell = i_{\ell+1} - \sum_{k < m-1} n_{k,\ell}$, $i_{0,\ell} = i_\ell + 1$, and $i_{k,\ell} = i_{k-1,\ell} + n_{k-1,\ell}$ for $k \leq m - 1$ and $\ell \leq n - 3$. Set $(\rho_{n-2}, \sigma_{n-2}) = (\rho, \sigma)$. For all $\ell \leq n - 3$, Proposition 4.6 implies that the

pair (ρ_ℓ, σ_ℓ) , given by $\rho_\ell = (i_{0,\ell} \ i_{1,\ell} \ \cdots \ i_{m-1,\ell}) \cup \bigcup_{k < m-1} \text{id}_{(i_{k,\ell}, i_{k+1,\ell})}$ and $\sigma_\ell = \bigcup_{k < m-1} (i_{k,\ell} \ i_{k,\ell} + 1 \ \cdots \ i_{k+1,\ell} - 1)$, is in $\mathcal{F}(i_\ell, i_{\ell+1}]$. So Proposition 4.2 yields that $(\rho', \sigma') = (\bigcup_{\ell \leq n-2} \rho_\ell, (i_0 \ i_1 \ \cdots \ i_{n-1}) \cup \bigcup_{\ell \leq n-2} \sigma_\ell)$ is in $\mathcal{F}[i_0, i_{n-1} + 1)$. But $i - \sum_{k < m-1, \ell < n-2} n_{k,\ell} = i_0$ and $j + 1 = i_{n-1} + 1$. \square

We say that a fixed point x of ρ is *anti-malleable* if $x \in \text{Per}_2(\sigma)$, $[x]_\sigma \subseteq \text{dom}(\rho)$, and $|[x]_\sigma \setminus \text{Per}_1(\rho)| = 1$.

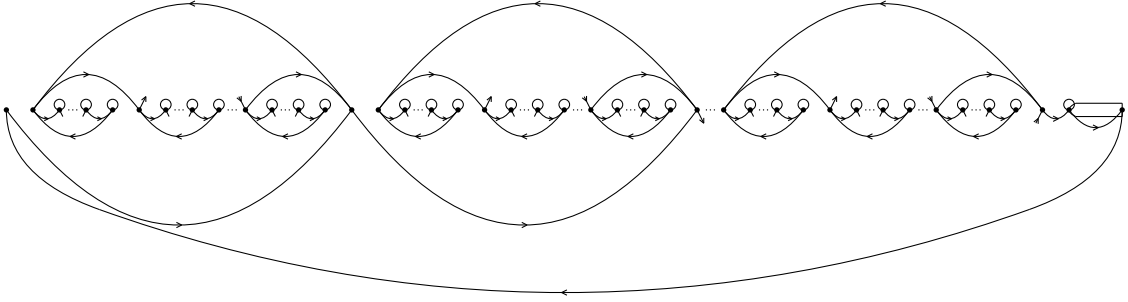


FIGURE 9. The extension provided by Proposition 5.5.

Proposition 5.5. *Suppose that $i < j$, $m \geq 2$, $n \geq 3$, $n_{k,\ell} \geq 3$ for all $k < m - 1$ and $\ell < n - 3$, and $(\rho, \sigma) \in \mathcal{F}(i, j]$. Then there exists $(\rho', \sigma') \in \mathcal{F}[i - 1 - \sum_{k < m-1, \ell < n-3} n_{k,\ell}, j + 1)$ such that:*

- ρ' is obtained from ρ by adding a single anti-malleable fixed point, $n - 3$ cycles of length m , and $n_{k,\ell} - 1$ $n_{k,\ell}$ -malleable fixed points for all $k < m - 1$ and $\ell < n - 3$.
- σ' is obtained from σ by adding a cycle of length n and a cycle of length $n_{k,\ell}$ for all $k < m - 1$ and $\ell < n - 3$.

Proof. Recursively define $i_{n-1} = j$, $i_{n-2} = i$, $i_{n-3} = i - 1$, $i_\ell = i_{\ell+1} - \sum_{k < m-1} n_{k,\ell}$, $i_{0,\ell} = i_\ell + 1$, and $i_{k,\ell} = i_{k-1,\ell} + n_{k-1,\ell}$ for $k \leq m - 1$ and $\ell \leq n - 4$. Set $(\rho_{n-2}, \sigma_{n-2}) = (\rho, \sigma)$. For all $\ell \leq n - 3$, Proposition 4.6 implies that the pair (ρ_ℓ, σ_ℓ) , given by $\rho_\ell = (i_{0,\ell} \ i_{1,\ell} \ \cdots \ i_{m-1,\ell}) \cup \bigcup_{k < m-1} \text{id}_{(i_{k,\ell}, i_{k+1,\ell})}$ and $\sigma_\ell = \bigcup_{k < m-1} (i_{k,\ell} \ i_{k,\ell} + 1 \ \cdots \ i_{k+1,\ell} - 1)$, is in $\mathcal{F}(i_\ell, i_{\ell+1}]$. So $(\rho', \sigma') = (\bigcup_{\ell \leq n-2} \rho_\ell, (i_0 \ i_1 \ \cdots \ i_{n-1}) \cup \bigcup_{\ell \leq n-2} \sigma_\ell)$ is in $\mathcal{F}[i_0, i_{n-1} + 1)$ by Proposition 4.2. But $i - 1 - \sum_{k < m-1, \ell < n-3} n_{k,\ell} = i_0$ and $j + 1 = i_{n-1} + 1$. \square

6. THE MAIN RESULT

The special case of Theorem B where ρ or σ has finite order is a consequence of Propositions 2.1 and 2.3 and:

Theorem 6.1. *Suppose that $m \geq 2$, $\rho, \sigma \in \text{Sym}(\mathbb{Z})$ are periodic, and $\text{Per}_m(\rho)$ and $\text{Per}_{\geq 3}(\sigma)$ are infinite. Then $S^{\mathbb{Z}} \in \text{Cl}(\rho)\text{Cl}(\sigma)$.*

Proof. For all integers $i < j$, set $\mathcal{F}_0(i, j) = \mathcal{F}[i, j]$ and $\mathcal{F}_1(i, j) = \mathcal{F}(i, j]$. Fix an enumeration $(\pi_n, O_n)_{n \in \mathbb{N}}$ of the pairs of the form (π, O) , where $\pi \in \{\rho, \sigma\}$ and $O \in \mathcal{O}(\pi)$. Then there is an infinite set $N \subseteq \mathbb{N}$ and $p < 2$ such that $\pi_n = \sigma$, $\text{par}(|O_n|) = p$, and $3 \leq |O_n| \leq |O_{n+1}|$ for all $n \in N$. Fix $n_{-1} \in N$, set $N_0 = \mathbb{N} \setminus \{n_{-1}\}$, and apply Proposition 4.5 to find $i_0 < j_0$ and $(\rho_0, \sigma_0) \in \mathcal{F}_0(i_0, j_0)$ such that every point of $\text{dom}(\rho_0)$ is a malleable fixed point and the lone orbit of σ_0 has cardinality $|O_{n_{-1}}|$.

Suppose that k is a natural number for which we have found $i_k < j_k$, a cofinite set $N_k \subseteq \mathbb{N}$, and $(\rho_k, \sigma_k) \in \mathcal{F}_{\text{par}(k)}(i_k, j_k)$. If $k \in 2\mathbb{N}$, then let n_k be the least element of N_k for which $(\pi_{n_k} = \rho$ and $|O_{n_k}| \geq 2)$ or $(\pi_{n_k} = \sigma, |O_{n_k}| = 1, \text{ and } m \geq 3)$. If $k \in 4\mathbb{N} + 1$, then let n_k be the least element of N_k for which $(\pi_{n_k} = \sigma, |O_{n_k}| = 1, \text{ and } m = 2)$, $(\pi_{n_k} = \sigma \text{ and } |O_{n_k}| = 2)$, or $(\pi_{n_k} = \rho \text{ and } |O_{n_k}| = 1)$. And if $k \in 4\mathbb{N} + 3$, then let n_k be the least element of N_k for which $\pi_{n_k} = \sigma$ and $|O_{n_k}| \geq 3$.

Lemma 6.2. *For some finite cardinal κ_k and any subset F_k of $N \cap (N_k \setminus \{n\})$ of cardinality κ_k , there exist $i_{k+1} < i_k$, $j_{k+1} > j_k$, and $(\rho_{k+1}, \sigma_{k+1}) \in \mathcal{F}_{\text{par}(k+1)}(i_{k+1}, j_{k+1})$ such that:*

- ρ_{k+1} is obtained from ρ_k by adding a set of cycles of length k and $|O_n| - 1$ $|O_n|$ -malleable fixed points for all $n \in F_k$, as well as a cycle of length $|O_{n_k}|$ if $(\pi_{n_k} = \rho \text{ and } |O_{n_k}| \geq 2)$ and an anti-malleable fixed point if $(\pi_{n_k} = \rho \text{ and } |O_{n_k}| = 1)$.
- σ_{k+1} is obtained from σ_k by adding a cycle of length $|O_n|$ for all $n \in F_k$, as well as a cycle of length $|O_n|$ if $\pi_{n_k} = \sigma$.

Proof. If $k \in 2\mathbb{N}$, then the desired result follows from Propositions 5.1 and 5.2. Otherwise, it follows from Propositions 5.3–5.5. \square

Set $N_{k+1} = N_k \setminus (F_k \cup \{n_k\})$.

Define $\rho_\infty = \bigcup_{k \in \mathbb{N}} \rho_k$ and $\sigma_\infty = \bigcup_{k \in \mathbb{N}} \sigma_k$. As $(i_k)_{k \in \mathbb{N}}$ is strictly decreasing and $(j_k)_{k \in \mathbb{N}}$ is strictly increasing, these are permutations of \mathbb{Z} whose composition is $S^{\mathbb{Z}}$. Note that $\text{Mal}(\rho_\infty, \sigma_\infty) \cap \text{Per}_{n+2\mathbb{N}}(\sigma_\infty) = \emptyset$ for all $n \in 2\mathbb{N} + (1 - p)$. As $F_k \neq \emptyset$ for all $k \in 4\mathbb{N} + 3$, it follows that $-0 < |\text{Mal}(\rho_\infty, \sigma_\infty) \cap \text{Per}_{n+2\mathbb{N}}(\sigma_\infty)| < \aleph_0$ for all $n \in 2\mathbb{N} + p$. As the fact that $\bigcap_{k \in \mathbb{N}} N_k = \emptyset$ ensures that $\rho_\infty \upharpoonright \sim (\text{Mal}(\rho_\infty, \sigma_\infty) \cap \text{Per}_1(\rho_\infty)) \cong \rho$ and $\sigma_\infty \cong \sigma$, Proposition 3.3 yields conjugates ρ' of ρ and σ' of σ for which $\rho' \circ \sigma' = \rho_\infty \circ \sigma_\infty = S^{\mathbb{Z}}$. \square

The fact that every almost involution has finite order and [Mor89, Theorem A] therefore yield Theorem B.

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