INVARIANT UNIFORMIZATIONS AND QUASI-TRANSVERSALS

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Abstract. We establish a dichotomy characterizing the class of $(E \times \Delta(Y))$-invariant Borel sets $R \subseteq X \times Y$, whose vertical sections are countable, that admit $(E \times \Delta(Y))$-invariant Borel uniformizations, where X and Y are Polish spaces and E is a Borel equivalence relation on X. We achieve this by establishing a dichotomy characterizing the class of Borel equivalence relations $F \subseteq E$, where F has countable index below E and satisfies an additional technical definability condition, for which there is a Borel set intersecting each E-class in a non-empty finite union of F-classes.

Introduction

Endow $\mathbb{N}$ with the discrete topology, and $\mathbb{N}^{\mathbb{N}}$ with the corresponding product topology. A topological space is *analytic* if it is a continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$, and *Polish* if it is separable and admits a compatible complete metric. A subset of a topological space is *Borel* if it is in the smallest σ-algebra containing the open sets, and *co-analytic* if its complement is analytic. Every Polish space is analytic (see, for example, [Kec95, Theorem 7.9]), and Souslin's theorem ensures that a subset of an analytic Hausdorff space is Borel if and only if it is analytic and co-analytic (see, for example, [Kec95, 14.11][1]).

A *homomorphism* from a binary relation R on a set X to a binary relation S on a set Y is a function $\phi\colon X \to Y$ for which $(\phi \times \phi)(R) \subseteq S$, a *reduction* of R to S is a homomorphism from R to S that is also a homomorphism from $\sim R$ to $\sim S$, and an *embedding* of R into S is an injective reduction of R to S. More generally, an *embedding* of a sequence $(R_i)_{i \in I}$ of binary relations on a set X into a sequence $(S_i)_{i \in I}$ of binary relations on a set Y is a function $\phi\colon X \to Y$ that is an embedding of R_i into S_i for all $i \in I$.

2010 *Mathematics Subject Classification.* Primary 03E15, 28A05.

Key words and phrases. Glimm-Effros, Lusin-Novikov, quotient, transversal, uniformization.

The author was partially supported by FWF grant P29999.

[1]While the results in [Kec95] are stated for Polish spaces, the proofs of those to which we refer go through just as easily in the generality discussed here.

The *diagonal* on X is given by $\Delta(X) = \{(x, y) \in X \times X \mid x = y\}$. Define $I(X) = X \times X$, and let $\mathbb{E}_0$ denote the equivalence relation on $2^{\mathbb{N}}$ given by $c \mathbin{\mathbb{E}_0} d \iff \exists n \in \mathbb{N} \forall m \geq n \; c(m) = d(m)$.

The *product* of binary relations R on X and S on Y is the binary relation given by $(x, y) \; (R \times S) \; (x', y') \iff (x \; R \; x'$ and $y \; S \; y')$. The *vertical sections* of a set $R \subseteq X \times Y$ are the sets of the form $R_x = \{y \in Y \mid (x, y) \in R\}$, where $x \in X$. A *partial uniformization* of a set $R \subseteq X \times Y$ over an equivalence relation F on Y is a set $U \subseteq R$ such that $F \restriction U_x = I(U_x)$ for all $x \in X$.

Given an equivalence relation E on a set X, the *E-saturation* of a set $Y \subseteq X$ is given by $[Y]_E = \{x \in X \mid \exists y \in Y \; x \; E \; y\}$, and a set $Y \subseteq X$ is *E-complete* if $X = [Y]_E$. A *quasi-transversal* of E over a subequivalence relation F is an E-complete set $Y \subseteq X$ for which there exists $k \in \mathbb{N}$ such that every $(E \restriction Y)$-class is contained in a union of at most k F-classes. The following fact is a generalization of the Glimm–Effros dichotomy for countable Borel equivalence relations:

Theorem 1. *Suppose that X is an analytic Hausdorff space, E is a Borel equivalence relation on X, F is a countable-index Borel subequivalence relation of E, and the projection onto the left coordinate of every $(\Delta(X) \times F)$-invariant Borel partial uniformization of E over F is Borel. Then exactly one of the following holds:*

1. *There is a partition $(B_n)_{n \in \mathbb{N}}$ of X into E-invariant Borel sets with the property that there is an F-invariant Borel quasi-transversal of $E \restriction B_n$ over $F \restriction B_n$ for all $n \in \mathbb{N}$.*
2. *There is a continuous embedding $\pi \colon 2^{\mathbb{N}} \times \mathbb{N} \hookrightarrow X$ of $(\mathbb{E}_0 \times I(\mathbb{N}), \Delta(2^{\mathbb{N}}) \times \Delta(\mathbb{N}))$ into (E, F) for which $[\pi(2^{\mathbb{N}} \times \mathbb{N})]_F$ is E-invariant.*

Following the usual abuse of language, we say that a Borel equivalence relation is *countable* if all of its equivalence classes are countable. The special case of Theorem 1 where E is countable originally arose in a conversation with Marks, and was used to eliminate the need for determinacy in an argument due to Thomas.

A *uniformization* of a set $R \subseteq X \times Y$ is a set $U \subseteq R$ such that $|U_x| = 1$ for all $x \in \mathrm{proj}_X(R)$. A Borel equivalence relation E on an analytic Hausdorff space X is *smooth* if there is a Borel reduction $\pi \colon X \to 2^{\mathbb{N}}$ of E to equality. Kechris has shown that the smooth Borel equivalence relations are precisely those with the property that every $(E \times \Delta(Y))$-invariant Borel set $R \subseteq X \times Y$ with countable vertical sections has an $(E \times \Delta(Y))$-invariant Borel uniformization (see [Kec20, Theorem 1.5]). He also asked the finer question as to the circumstances under which a given $(E \times \Delta(Y))$-invariant Borel set $R \subseteq X \times Y$ admits such a uniformization. The following fact refines Kechris's result and answers his question:

Theorem 2. *Suppose that X and Y are Polish spaces, E is a Borel equivalence relation on X, and $R \subseteq X \times Y$ is an $(E \times \Delta(Y))$-invariant Borel set whose vertical sections are countable. Then exactly one of the following holds:*

1. *There is an $(E \times \Delta(Y))$-invariant Borel uniformization of R.*
2. *There are a continuous embedding $\pi_X \colon 2^{\mathbb{N}} \times \mathbb{N} \hookrightarrow X$ of $\mathbb{E}_0 \times I(\mathbb{N})$ into E and a continuous injection $\pi_Y \colon 2^{\mathbb{N}} \times \mathbb{N} \hookrightarrow Y$ such that $R \cap (\pi_X(2^{\mathbb{N}} \times \mathbb{N}) \times Y) = (\pi_X \times \pi_Y)(\mathbb{E}_0 \times I(\mathbb{N}))$.*

In §1, we establish a generalization of Theorem 1 in which F need not be contained in E, while simultaneously strengthening it so as to ensure that, in condition (2), distinct points map to points that are inequivalent with respect to a given smooth countable Borel subequivalence relation of E satisfying an additional technical property.

In §2, we establish a strengthening of Theorem 2 characterizing the circumstances under which $\mathrm{proj}_X(R)$ is a countable union of E-invariant Borel sets on which R admits an $((E \times F) \upharpoonright R)$-invariant Borel quasi-uniformization over a given countable Borel equivalence relation F. Here, a *quasi-uniformization* of a set $R \subseteq X \times Y$ over an equivalence relation F on Y is a set $U \subseteq R$ for which there exists $k \in \mathbb{Z}^+$ such that U_x is contained in a non-empty union of at most k F-classes for all $x \in \mathrm{proj}_X(R)$.

1. Quasi-transversals

While the following two facts are consequences of their well-known analogs for $\mathbb{E}_0$, we provide proofs for the reader's convenience:

Proposition 1.1. *Suppose that $B \subseteq 2^{\mathbb{N}} \times \mathbb{N}$ is a non-meager set with the Baire property. Then there exists $(c, m) \in 2^{\mathbb{N}} \times \mathbb{N}$ with the property that $B \cap ([c]_{\mathbb{E}_0} \times \{m\})$ is infinite.*

Proof. Fix $n \in \mathbb{N}$ and $s \in 2^{<\mathbb{N}}$ for which B is comeager in $\mathcal{N}_s \times \{n\}$ (see, for example, [Kec95, Proposition 8.26]). It is sufficient to show that for all $k \in \mathbb{N}$, there are comeagerly-many $c \in \mathcal{N}_s$ with the property that $B \cap ([c]_{\mathbb{E}_0} \times \mathbb{N}) \cap (\mathcal{N}_s \times \{n\})$ has at least element k elements.

For each permutation σ of 2^k, let ϕ_σ be the corresponding homeomorphism of $\mathcal{N}_s \times \{n\}$, given by $\phi_\sigma(s \frown t \frown c)(0) = s \frown \sigma(t) \frown c$ for all $c \in 2^{\mathbb{N}}$ and $t \in 2^k$. Then there are comeagerly-many $c \in \mathcal{N}_s$ with the property that $\phi_\sigma(c, n) \in B$ for all permutations σ of 2^k (see, for example, [Kec95, Exercise 8.45]), and clearly $B \cap ([c]_{\mathbb{E}_0} \times \mathbb{N}) \cap (\mathcal{N}_s \times \{n\})$ has at least 2^k elements for every such c. ⊠

Proposition 1.2. *Suppose that E and F are equivalence relations on $2^{\mathbb{N}} \times \mathbb{N}$ with the Baire property, every E-class is a countable union of $(E \cap F)$-classes, and $F \cap (\mathbb{E}_0 \times \Delta(\mathbb{N})) = \Delta(2^{\mathbb{N}}) \times \Delta(\mathbb{N})$. Then E and F are meager.*

Proof. Suppose, towards a contradiction, that F is not meager. As F has the Baire property, the Kuratowski-Ulam theorem (see, for example, [Kec95, Theorem 8.41]) yields an F-class C with the Baire property that is not meager. But $(\mathbb{E}_0 \times \Delta(\mathbb{N})) \upharpoonright C \nsubseteq \Delta(2^{\mathbb{N}}) \times \Delta(\mathbb{N})$ by Proposition 1.1, the desired contradiction. It follows that F is meager.

The Kuratowski-Ulam theorem now ensures that every F-class is meager, in which case every $(E \cap F)$-class is meager, so every E-class is meager, thus E is meager. ⊠

An *invariant embedding* of an equivalence relation E on X into an equivalence relation F on Y is an embedding $\phi \colon X \hookrightarrow Y$ of E into F for which $\phi(X)$ is F-invariant.

Proposition 1.3. *Suppose that $U \subseteq 2^{\mathbb{N}} \times \mathbb{N}$ is a non-empty open set. Then there is a continuous invariant embedding $\pi \colon 2^{\mathbb{N}} \times \mathbb{N} \hookrightarrow U$ of $\mathbb{E}_0 \times I(\mathbb{N})$ into $(\mathbb{E}_0 \times I(\mathbb{N})) \upharpoonright U$.*

Proof. Fix $S \subseteq (\bigcup_{n \in \mathbb{N}} 2^{2n}) \times \mathbb{N}$ such that $\{\mathcal{N}_s \times \{n\} \mid (s, n) \in S\}$ partitions U, as well as an injective enumeration $((s_k, n_k), t_k)_{k \in \mathbb{N}}$ of $S \times \{c \in 2^{\mathbb{N}} \mid \exists n \in \mathbb{N} \forall m \geq n \ c(m) = 0\}$, and define $\pi \colon 2^{\mathbb{N}} \times \mathbb{N} \hookrightarrow U$ by

$$\pi(c,k)(0)(i) = \begin{cases} s_k(i) & \text{if } i < |s_k|, \\ c((i-1)/2) & \text{if } i \geq |s_k| \text{ is odd}, \\ t_k((i - 2|s_k|)/2) & \text{if } i \geq 2|s_k| \text{ is even, and} \\ c((i - |s_k|)/2) & \text{otherwise}, \end{cases}$$

and $\pi(c,k)(1) = n_k$. ⊠

A *homomorphism* from a sequence $(R_i)_{i \in I}$ of binary relations on a set X to a sequence $(S_i)_{i \in I}$ of binary relations on a set Y is a function $\phi \colon X \to Y$ that is a homomorphism from R_i to S_i for all $i \in I$.

Proposition 1.4. *Suppose that R is a meager binary relation on $2^{\mathbb{N}} \times \mathbb{N}$. Then there is a continuous injective homomorphism $\phi \colon 2^{\mathbb{N}} \times \mathbb{N} \hookrightarrow 2^{\mathbb{N}} \times \mathbb{N}$ from $(\mathbb{E}_0 \times I(\mathbb{N}), \sim(\mathbb{E}_0 \times I(\mathbb{N})))$ to $(\mathbb{E}_0 \times I(\mathbb{N}), \sim R)$ such that $\forall c \in 2^{\mathbb{N}} \ \phi([c]_{\mathbb{E}_0} \times \mathbb{N})$ is an $(\mathbb{E}_0 \times I(\mathbb{N}))$-class.*

Proof. Set $d_0 = r_0 = 1$ and $\ell_0 = 0$, and fix a decreasing sequence $(U_n)_{n \in \mathbb{N}}$ of dense open symmetric subsets of $(2^{\mathbb{N}} \times \mathbb{N}) \times (2^{\mathbb{N}} \times \mathbb{N})$ whose intersection is disjoint from R, as well as $\phi_0 \colon 2^0 \times d_0 \leftrightarrow 2^{\ell_0} \times r_0$.

Lemma 1.5. *Suppose that $n \in \mathbb{N}$, $d_n, \ell_n, r_n \in \mathbb{N}$, and $\phi_n \colon 2^n \times d_n \leftrightarrow 2^{\ell_n} \times r_n$ is a bijection. Then there exist $d_{n+1} > d_n$, $\ell_{n+1} > \ell_n$, $r_{n+1} > r_n$, and a bijection $\phi_{n+1} \colon 2^{n+1} \times d_{n+1} \leftrightarrow 2^{\ell_{n+1}} \times r_{n+1}$ such that:*

1. $\forall i < 2 \forall (t, m) \in 2^n \times d_n$ $(\phi_n(t, m)(0) \sqsubseteq \phi_{n+1}(t \frown (i), m)(0)$ *and* $\phi_n(t, m)(1) = \phi_{n+1}(t \frown (i), m)(1))$.
2. $\forall i, j < 2 \forall (t, m) \in (2^n \times 2^n) \times (d_n \times d_n)$
   $(i = j \iff \forall \ell \in [\ell_n, \ell_{n+1})$
   $\phi_{n+1}(t(0) \frown (i), m(0))(0)(\ell) = \phi_{n+1}(t(1) \frown (j), m(1))(0)(\ell))$.
3. $\forall (t, m) \in (2^n \times 2^n) \times (d_n \times d_n)$
   $\prod_{i<2} \mathcal{N}_{\phi_{n+1}(t(i) \frown (i), m(i))(0)} \times \{\phi_{n+1}(t(i) \frown (i), m(i))(1)\} \subseteq U_n$.

Proof. Fix an enumeration $(t_k, m_k)_{k < 4^n d_n^2}$ of $(2^n \times 2^n) \times (d_n \times d_n)$, as well as any pair $u_0 \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ such that $\forall i < 2$ $u_0(i) \not\sqsubseteq u_0(1 - i)$. Given $k < 4^n d_n^2$ and $u_k \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$, fix $u_{k+1} \in 2^{<\mathbb{N}} \times 2^{<\mathbb{N}}$ such that:

- $\forall i < 2$ $u_k(i) \sqsubseteq u_{k+1}(i)$.
- $\prod_{i<2} \mathcal{N}_{\phi_n(t_k(i), m_k(i))(0) \frown u_{k+1}(i)} \times \{\phi_n(t_k(i), m_k(i))(1)\} \subseteq U_n$.

Fix $\ell_{n+1} > \ell_n$ and $u \in 2^{\ell_{n+1} - \ell_n} \times 2^{\ell_{n+1} - \ell_n}$ such that $u_{4^n d_n^2}(i) \sqsubseteq u(i)$ for all $i < 2$. Set $d_{n+1} = 2^{\ell_{n+1} - \ell_n} d_n$ and $r_{n+1} = 2 r_n$. Then $2^{n+1} d_{n+1} = 2^{\ell_{n+1} - \ell_n + 1} 2^n d_n = 2^{\ell_{n+1} - \ell_n + 1} 2^{\ell_n} r_n = 2^{\ell_{n+1}} r_{n+1}$, in which case there is a bijection $\phi_{n+1} \colon 2^{n+1} \times d_{n+1} \leftrightarrow 2^{\ell_{n+1}} \times r_{n+1}$ with the property that $\phi_{n+1}(t \frown (i), m)(0) = \phi_n(t, m)(0) \frown u(i)$ and $\phi_{n+1}(t \frown (i), m)(1) = \phi_n(t, m)(1)$ for all $(t, m) \in 2^n \times d_n$. ⊠

As $\phi_n(t, m) \sqsubset \phi_{n+1}(t \frown (i), m)$ for all $i < 2$, $n \in \mathbb{N}$, and $(t, m) \in 2^n \times d_n$, we obtain a continuous function $\phi \colon 2^{\mathbb{N}} \times \mathbb{N} \to 2^{\mathbb{N}} \times \mathbb{N}$ by setting $\phi(c, m) = \bigcup_{n > m} \phi_n(c \restriction n, m)$ for all $c \in 2^{\mathbb{N}}$ and $m \in \mathbb{N}$.

To see that ϕ is a homomorphism from $\mathbb{E}_0 \times I(\mathbb{N})$ to $\mathbb{E}_0 \times I(\mathbb{N})$, observe that if $c \in \mathbb{E}_0 \times I(\mathbb{N})$, then there exists $n \geq \max_{i<2} c(i)(1)$ with the property that $\forall m \geq n$ $c(0)(0)(m) = c(1)(0)(m)$, in which case $\forall m \geq \ell_n$ $\phi(c(0))(0)(m) = \phi(c(1))(0)(m)$.

To see that ϕ is a homomorphism from $\sim(\mathbb{E}_0 \times I(\mathbb{N}))$ to $\sim R$, note that if $c \in \sim(\mathbb{E}_0 \times I(\mathbb{N}))$, then there are infinitely many $n \geq \max_{i<2} c(i)(1)$ with the property that $(\phi(c(i)))_{i<2} \in \prod_{i<2} \mathcal{N}_{\phi_{n+1}(c(i)(0) \restriction (n+1), c(i)(1))(0)} \times \{\phi_{n+1}(c(i)(0) \restriction (n+1), c(i)(1))(1)\} \subseteq U_n$, so $(\phi(c(i)))_{i<2} \in \sim R$.

It remains to note that if $(c, m) \in 2^{\mathbb{N}} \times \mathbb{N}$, then $\phi([(c, m)]_{\mathbb{E}_0 \times I(\mathbb{N})}) = \bigcup_{n > m} \phi([c]_{F_n} \times d_n) = \bigcup_{n > m} [\phi(c, m)]_{F_{\ell_n} \times I(r_n)} = [\phi(c, m)]_{\mathbb{E}_0 \times I(\mathbb{N})}$, where $(F_n)_{n \in \mathbb{N}}$ is the increasing sequence of subequivalence relations of $\mathbb{E}_0$ given by $c \mathrel{F_n} d \iff \forall m \geq n$ $c(m) = d(m)$ for all $n \in \mathbb{N}$. ⊠

Given $n \in \mathbb{N}$ and an equivalence relation F on $2^n \times (n+1)$, let F^* denote the corresponding equivalence relation on $2^{\mathbb{N}} \times (n+1)$ given by $(c, \ell) \mathrel{F^*} (d, m) \iff ((c \restriction n, \ell) \mathrel{F} (d \restriction n, m)$ and $\forall k \geq n$ $c(k) = d(k))$. A *one-step extension* of F is an equivalence relation F' on $2^{n+1} \times (n+2)$

such that $(s, \ell) \; F \; (t, m) \iff (s \frown (i), \ell) \; F' \; (t \frown (i), m)$ for all $i < 2$ and $(s, \ell), (t, m) \in 2^n \times (n+1)$, and such an extension is *splitting* if it has the further property that $\neg(s \frown (i), \ell) \; F' \; (t \frown (1-i), m)$ for all $i < 2$ and $(s, \ell), (t, m) \in 2^n \times (n+1)$. A sequence $(F_n)_{n \in \mathbb{N}}$ is *suitable* if F_0 is the unique equivalence relation on $2^0 \times 1$, and F_{n+1} is a splitting one-step extension of F_n for all $n \in \mathbb{N}$.

Proposition 1.6. *Suppose that $(F_n)_{n \in \mathbb{N}}$ is a suitable sequence. Then there is a clopen transversal U of the equivalence relation $F^* = \bigcup_{n \in \mathbb{N}} F_n^*$.*

Proof. Fix the unique transversal S_0 of F_0, and given a transversal S_n of F_n, fix a transveral $S_{n+1} \supseteq \{(t \frown (i), m) \mid i < 2 \text{ and } (t, m) \in S_n\}$ of F_{n+1}. Set $S^* = \{(t \frown c, m) \mid c \in 2^{\mathbb{N}} \text{ and } (t, m) \in S\}$ for all $n \in \mathbb{N}$ and $S \subseteq 2^n \times (n+1)$, and define $U = \bigcup_{n \in \mathbb{N}} S_n^*$. ⊠

We can now establish our primary technical result.

Theorem 1.7. *Suppose that X is an analytic Hausdorff space, E is a Borel equivalence relation on X, F is a countable-index Borel subequivalence relation of E for which the projection onto the left coordinate of every $(\Delta(X) \times F)$-invariant Borel partial uniformization of E over F is Borel, and $F_\perp$ is a Borel subequivalence relation of E for which the E-saturation of every $F_\perp$-invariant Borel partial quasi-transversal of E over $F_\perp$ is Borel. Then at least one of the following holds:*

1. *There is a partition $(B_n)_{n \in \mathbb{N}}$ of X into E-invariant Borel sets such that at least one of the following holds for all $n \in \mathbb{N}$:*
   (a) *There is an F-invariant $(E \upharpoonright B_n)$-complete Borel partial quasi-transversal $A_n \subseteq B_n$ of F over $F \cap F_\perp$.*
   (b) *There is an F_*-invariant Borel quasi-transversal $A_n \subseteq B_n$ of $E \upharpoonright B_n$ over $F_* \upharpoonright B_n$, for some $F_* \in \{F, F_\perp\}$.*
2. *There exist a suitable sequence $(F_n)_{n \in \mathbb{N}}$ and a continuous homomorphism $\pi \colon 2^{\mathbb{N}} \times \mathbb{N} \to X$ from $(F^* \setminus (\Delta(2^{\mathbb{N}}) \times \Delta(\mathbb{N})), (\mathbb{E}_0 \times I(\mathbb{N})) \setminus F^*)$ to $(F \setminus F_\perp, E \setminus (F \cup F_\perp))$ with the property that $\forall c \in 2^{\mathbb{N}} \; [\pi([c]_{\mathbb{E}_0} \times \mathbb{N})]_F$ is an E-class, where $F^* = \bigcup_{n \in \mathbb{N}} F_n^*$.*

Proof. By [dRM20, Remark 2.14], there are $(\Delta(X) \times F)$-invariant Borel partial uniformizations R_n of E over F for which $E = \bigcup_{n \in \mathbb{N}} R_n$.

Lemma 1.8. *Every $(\Delta(X) \times F)$-invariant Borel partial uniformization R of E over F is contained in a $(\Delta(X) \times F)$-invariant Borel uniformization S of E over F.*

Proof. Set $S_0 = R$, recursively define $S_{n+1} = (R_n \setminus (\mathrm{proj}_0(S_n) \times Y)) \cup S_n$ for all $n \in \mathbb{N}$, and observe that the set $S = \bigcup_{n \in \mathbb{N}} S_n$ is as desired. ⊠

We can clearly assume that $R_0 = F$, and by Lemma 1.8, we can assume that each R_n is a $(\Delta(X) \times F)$-invariant Borel uniformization of E over F.

We can also assume that $F \setminus F_\perp \neq \emptyset$, since otherwise X is a transversal of F over $F \cap F_\perp$.

Finally, we can assume that $E \setminus (F \cup F_\perp) \neq \emptyset$. To see this, suppose otherwise, and define $A = \{x \in X \mid [x]_E \nsubseteq [x]_F\}$. Note that if $x \in A$, then there exists $y \in [x]_E \setminus [x]_F$, in which case $[y]_F \subseteq [x]_E \setminus [x]_F \subseteq [x]_{F_\perp}$ and $[y]_{F_\perp} = [x]_{F_\perp}$, so $[x]_E = [y]_E = [y]_F \cup [y]_{F_\perp} = [x]_{F_\perp}$, thus A is a partial transversal of E over $F_\perp$. By [dRM20, Proposition 2.1], there is an $F_\perp$-invariant Borel partial transversal $B \subseteq X$ of E over $F_\perp$ containing A. Then $\sim[B]_E$ is an E-invariant Borel partial transversal of E over F.

It now follows that there are continuous surjections $\phi_X \colon \mathbb{N}^\mathbb{N} \twoheadrightarrow X$, $\phi_{F\setminus F_\perp} \colon \mathbb{N}^\mathbb{N} \twoheadrightarrow F\setminus F_\perp$, $\phi_{E\setminus(F\cup F_\perp)} \colon \mathbb{N}^\mathbb{N} \twoheadrightarrow E\setminus(F\cup F_\perp)$, and $\phi_{R_n} \colon \mathbb{N}^\mathbb{N} \twoheadrightarrow R_n$ for all $n \in \mathbb{N}$. Define $\phi_{E\setminus F_\perp} \colon \mathbb{N}^\mathbb{N} \times 2 \twoheadrightarrow E \setminus F_\perp$ by

$$\phi_{E\setminus F_\perp}(b, i) = \begin{cases} \phi_{F\setminus F_\perp}(b) & \text{if } i = 1, \text{ and} \\ \phi_{E\setminus(F\cup F_\perp)}(b) & \text{otherwise.} \end{cases}$$

We will recursively define a decreasing sequence $(B^\alpha)_{\alpha<\omega_1}$ of E-invariant Borel subsets of X, off of which condition (1) holds. We begin by setting $B^0 = X$. For all limit ordinals $\lambda < \omega_1$, we set $B^\lambda = \bigcap_{\alpha<\lambda} B^\alpha$. To describe the construction at successor ordinals, we require several preliminaries.

An *approximation* is a sextuple $a = (n^a, D^a, F^a, \psi^a_X, \psi^a_R, \psi^a_{E\setminus F_\perp})$ with the property that $n^a \in \mathbb{N}$, D^a is a lexicographically downward-closed subset of $(n^a+1)\times 2^{n^a}$ containing $n^a \times 2^{n^a}$, F^a is an equivalence relation on D^a, $\psi^a_* \colon D^a \to \mathbb{N}^{n^a}$ for all $* \in \{X, R\}$, and $\psi^a_{E\setminus F_\perp} \colon \sim\Delta(D^a) \to \mathbb{N}^{n^a}$.

If a is an approximation for which $D^a \neq (n^a + 1) \times 2^{n^a}$, then a *one-step extension* of a is an approximation b such that:

- $n^a = n^b$.
- $D^a = D^b \setminus \{\max_{\mathrm{lex}} D^b\}$.
- $F^a = F^b \restriction D^a$.
- $\forall * \in \{X, R\}\ \psi^a_* = \psi^b_* \restriction D^a$.
- $\psi^a_{E\setminus F_\perp} = \psi^b_{E\setminus F_\perp} \restriction \sim\Delta(D^a)$.

If a is an approximation for which $D^a = (n^a+1)\times 2^{n^a}$, then a *one-step extension* of a is an approximation b such that:

- $n^b = n^a + 1$.
- $D^b = n^b \times 2^{n^b}$.

- $\forall i < 2 \forall (m, s), (n, t) \in D^a$
  $((m, s) \ F^a \ (n, t) \iff (m, s \frown (i)) \ F^b \ (n, t \frown (i))$ and
  $\neg (m, s \frown (i)) \ F^b \ (n, t \frown (1 - i)))$.
- $\forall * \in \{X, R\} \forall i < 2 \forall (n, t) \in D^a \ \psi^a_*(n, t) \sqsubseteq \psi^b_*(n, t \frown (i))$.
- $\forall i < 2 \forall ((m, s), (n, t)) \in {\sim}\Delta(D^a)$
  $\psi^a_{E \setminus F_\perp}((m, s), (n, t)) \sqsubseteq \psi^b_{E \setminus F_\perp}((m, s \frown (i)), (n, t \frown (i)))$.

A *configuration* is a sextuple $\gamma = (n^\gamma, D^\gamma, F^\gamma, \psi^\gamma_X, \psi^\gamma_R, \psi^\gamma_{E \setminus F_\perp})$ with the property that $n^\gamma \in \mathbb{N}$, D^γ is a lexicographically downward-closed subset of $(n^\gamma + 1) \times 2^{n^\gamma}$ containing $n^\gamma \times 2^{n^\gamma}$, F^γ is an equivalence relation on D^γ, $\psi^\gamma_* \colon D^\gamma \to \mathbb{N}^\mathbb{N}$ for all $* \in \{X, R\}$, $\psi^\gamma_{E \setminus F_\perp} \colon {\sim}\Delta(D^\gamma) \to \mathbb{N}^\mathbb{N}$, $(\phi_{R_n} \circ \psi^\gamma_R)(n, t) = ((\phi_X \circ \psi^\gamma_X)(0, t), (\phi_X \circ \psi^\gamma_X)(n, t))$ for all $(n, t) \in D^\gamma$, and $(\phi_{E \setminus F_\perp} \circ (\psi^\gamma_{E \setminus F_\perp} \times \mathbf{1}_{F^\delta}))((m, s), (n, t)) = ((\phi_X \circ \psi^\gamma_X)(m, s), (\phi_X \circ \psi^\gamma_X)(n, t))$ for all distinct $(m, s), (n, t) \in D^\gamma$. We say that γ is *compatible* with an E-invariant set $X' \subseteq X$ if $(\phi_X \circ \psi^\gamma_X)(D^\gamma) \subseteq X'$, and *compatible* with an approximation a if:

- $(n^a, D^a, F^a) = (n^\gamma, D^\gamma, F^\gamma)$.
- $\forall * \in \{X, R\} \forall (n, t) \in D^a \ \psi^a_*(n, t) \sqsubseteq \psi^\gamma_*(n, t)$.
- $\forall ((m, s), (n, t)) \in {\sim}\Delta(D^a)$
  $\psi^a_{E \setminus F_\perp}((m, s), (n, t)) \sqsubseteq \psi^\gamma_{E \setminus F_\perp}((m, s), (n, t))$.

We say that an approximation a is *X'-terminal* if no configuration is compatible with both X' and a one-step extension of a.

For each configuration γ such that $D^\gamma \neq (n^\gamma + 1) \times 2^{n^\gamma}$, let t^γ be the lexicographically minimal element of 2^{n^γ} for which $(n^\gamma, t^\gamma) \notin D^\gamma$ and set $C^\gamma = (R_{n^\gamma})_{(\phi_X \circ \psi^\gamma_X)(0, t^\gamma)}$. For each approximation a with the property that $D^a \neq (n^a + 1) \times 2^{n^a}$ and each set $X' \subseteq X$, define $A'(a, X') = \bigcup \{C^\gamma \mid \gamma$ is compatible with a and $X'\}$.

Lemma 1.9. *Suppose that $X' \subseteq X$ is E-invariant and a is an X'-terminal approximation for which $D^a \neq (n^a + 1) \times 2^{n^a}$. Then $A'(a, X')$ is a partial quasi-transversal of F over $F \cap F_\perp$.*

Proof. Suppose, towards a contradiction, that there is a configuration γ, compatible with a and X', with the property that C^γ contains strictly more than $|D^\gamma|$ $(F \cap F_\perp)$-classes, in which case there exists $y \in C^\gamma \setminus [(\phi_X \circ \psi^\gamma_X)(D^\gamma)]_{F \cap F_\perp}$. Define $n^\delta = n^a$, as well as $D^\delta = D^a \cup \{(n^a, t^a)\}$, and fix an extension ψ^δ_X of ψ^γ_X to D^δ for which $(\phi_X \circ \psi^\delta_X)(n^a, t^a) = y$. Let F^δ be the equivalence relation on D^δ given by $F^\delta \restriction D^\gamma = F^\gamma \restriction D^\gamma$ and $(n, t) \ F^\delta \ (n^a, t^a) \iff (\phi_X \circ \psi^\delta_X)(n, t) \ F \ (\phi_X \circ \psi^\delta_X)(n^a, t^a)$ for all $(n, t) \in D^\delta$, fix an extension ψ^δ_R of ψ^γ_R to D^δ for which $(\phi_R \circ \psi^\delta_R)(n^a, t^a) = y$, and fix an extension $\psi^\delta_{E \setminus F_\perp}$ of $\psi^\gamma_{E \setminus F_\perp}$ to ${\sim}\Delta(D^\delta)$ such that $(\phi_{E \setminus F_\perp} \circ (\psi^\delta_{E \setminus F_\perp} \times \mathbf{1}_{F^\delta}))((m, s), (n, t)) = ((\phi_X \circ \psi^\delta_X)(m, s), (\phi_X \circ \psi^\delta_X)(n, t))$ for all distinct $(m, s), (n, t) \in D^\delta$

such that $(n^a, t^a) \in \{(m, s), (n, t)\}$. Then δ is compatible with a one-step extension of a, contradicting the fact that a is X'-terminal. ⊠

Set $\overline{X} = X \times \{F, F_\perp\}$ and $\overline{E} = E \times I(\{F, F_\perp\})$, and define $\overline{F}$ on $\overline{X}$ by $(x, F_*) \ \overline{F} \ (x', F'_*) \iff (F_* = F'_* \text{ and } x \ F_* \ x')$. For each configuration γ, set $A^\gamma = (\phi_X \circ \psi_X^\gamma)(D^\gamma)$, and for each approximation a with the property that $D^a = (n^a + 1) \times 2^{n^a}$ and each E-invariant set $X' \subseteq X$, define $\mathscr{A}(a, X') = \{A^\gamma \mid \gamma \text{ is compatible with } a \text{ and } X'\}$ and $\overline{\mathscr{A}}(a, X') = \{A \times \{F, F_\perp\} \mid A \in \mathscr{A}(a, X')\}$. We say that a family $\overline{\mathscr{A}}$ of subsets of $\overline{X}$ is $\overline{F}$*-intersecting* if the $\overline{F}$-saturations of any two sets in the family have a point in common, and $\overline{E}$*-locally* $\overline{F}$*-intersecting* if, for every $\overline{E}$-class C, the family $\overline{\mathscr{A}} \restriction C = \{A \in \overline{\mathscr{A}} \mid A \subseteq C\}$ is $\overline{F}$-intersecting.

Lemma 1.10. *Suppose that $X' \subseteq X$ and a is an X'-terminal approximation for which $D^a = (n^a + 1) \times 2^{n^a}$. Then $\overline{\mathscr{A}}(a, X')$ is $\overline{E}$-locally $\overline{F}$-intersecting.*

Proof. Suppose, towards a contradiction, that there are configurations γ_0 and γ_1, both compatible with a and X', such that A^{γ_0} and A^{γ_1} are contained in the same E-class, but have disjoint F-saturations and disjoint $F_\perp$-saturations. Set $n^\delta = n^a + 1$ and $D^\delta = n^\delta \times 2^{n^\delta}$, define functions $\psi_*^\delta \colon D^\delta \to \mathbb{N}^\mathbb{N}$ by $\psi_*^\delta(n, t \frown (i)) = \psi_*^{\gamma_i}(n, t)$ for all $* \in \{X, R\}$, $i < 2$, and $(n, t) \in D^\delta$, let F^δ be the equivalence relation on D^δ given by $(m, s) \ F^\delta \ (n, t) \iff (\phi_X \circ \psi_X^\delta)(m, s) \ F \ (\phi_X \circ \psi_X^\delta)(n, t)$ for all $(m, s), (n, t) \in D^\delta$, and fix $\psi_{E \setminus F_\perp}^\delta \colon {\sim}\Delta(D^\delta) \to \mathbb{N}^\mathbb{N}$ such that $\psi_{E \setminus F_\perp}^\delta((m, s \frown (i)), (n, t \frown (i))) = \psi_{E \setminus F_\perp}^{\gamma_i}((m, s), (n, t))$ for all $i < 2$ and distinct $(m, s), (n, t) \in D^a$ and

$$\begin{aligned}(\phi_{E \setminus F_\perp} \circ (\psi_{E \setminus F_\perp}^\delta \times \mathbf{1}_{F^\delta}))((m, s \frown (i)), (n, t \frown (1 - i))) \\ = ((\phi_X \circ \psi_X^\delta)(m, s \frown (i)), (\phi_X \circ \psi_X^\delta)(n, t \frown (1 - i)))\end{aligned}$$

for all $i < 2$ and $(m, s), (n, t) \in D^a$. Then δ is compatible with a one-step extension of a, contradicting the fact that a is X'-terminal. ⊠

Suppose that a is B^α-terminal. If $D^a \neq (n^a + 1) \times 2^{n^a}$, then Lemma 1.9 and [dRM20, Proposition 2.1] yield an F-invariant Borel partial quasi-transversal $A(a, B^\alpha)$ of F over $F \cap F_\perp$ containing $A'(a, B^\alpha)$, in which case we define $B(a, B^\alpha) = [A(a, B^\alpha)]_E$. A set $Y \subseteq X$ *punctures* a family $\mathscr{A}$ of subsets of X if $A \cap Y \neq \emptyset$ for all $A \in \mathscr{A}$. If $D^a = (n^a + 1) \times 2^{n^a}$, then Lemma 1.10 and [dRM20, Proposition 4.1] yield an $\overline{F}$-invariant Borel partial quasi-transversal $\overline{A}(a, B^\alpha)$ of $\overline{E}$ over $\overline{F}$ puncturing $\overline{\mathscr{A}}(a, B^\alpha)$, and it follows that the set $A_{F_*}(a, B^\alpha) = \{x \in X \mid (x, F_*) \in \overline{A}(a, B^\alpha)\}$ is an F_*-invariant Borel partial quasi-transversal of E over F_* for all $F_* \in \{F, F_\perp\}$, and $\bigcup_{F_* \in \{F, F_\perp\}} A_{F_*}(a, B^\alpha)$ punctures $\mathscr{A}(a, B^\alpha)$, in which case we define $B(a, B^\alpha) = \bigcup_{F_* \in \{F, F_\perp\}} [A_{F_*}(a, B^\alpha)]_E$.

Let $B^{\alpha+1}$ be the set obtained from B^α by subtracting the union of the sets of the form $B(a, B^\alpha)$, where a varies over all B^α-terminal approximations.

Lemma 1.11. *Suppose that $\alpha < \omega_1$ and a is a non-$B^{\alpha+1}$-terminal approximation. Then a has a non-B^α-terminal one-step extension.*

Proof. Fix a one-step extension b of a for which there is a configuration γ compatible with b and $B^{\alpha+1}$. Then $(\phi_X \circ \phi^\gamma_X)(D^\gamma) \subseteq B^{\alpha+1}$, so b is not B^α-terminal. ⊠

Fix $\alpha < \omega_1$ such that the families of B^α- and $B^{\alpha+1}$-terminal approximations coincide, and let a_0 be the approximation given by $n^{a_0} = 0$ and $D^{a_0} = 1 \times 2^0$. As $\overline{\mathscr{A}}(a_0, X') = \{\{(x, F_*) \mid F_* \in \{F, F_\perp\}\} \mid x \in X'\}$ for all E-invariant sets $X' \subseteq X$, we can assume that a_0 is not B^α-terminal, since otherwise $B^{\alpha+1} = \emptyset$, so condition (1) holds.

By recursively applying Lemma 1.11, we obtain non-B^α-terminal one-step extensions a'_{n+1} of a'_n for all $n \in \mathbb{N}$. Let $(a_n)_{n\in\mathbb{N}}$ be the unique subsequence such that $D^{a_n} = (n+1) \times 2^n$ for all $n \in \mathbb{N}$. Define $F_n = F^{a_n}_n$ for all $n \in \mathbb{N}$, $\psi_*\colon 2^\mathbb{N} \times \mathbb{N} \to \mathbb{N}^\mathbb{N}$ by $\psi_*(c, m) = \bigcup_{n \geq m} \psi^{a_n}_*(m, c(0) \restriction n)$ for all $* \in \{X, R\}$, and $\psi_{E \setminus F_\perp} \colon (\mathbb{E}_0 \times I(\mathbb{N})) \setminus (\Delta(2^\mathbb{N}) \times \Delta(\mathbb{N})) \to \mathbb{N}^\mathbb{N}$ by $\psi_{E\setminus F_\perp}((b, \ell), (c, m)) = \bigcup_{n \geq n((b,\ell),(c,m))} \psi^{a_n}_{E \setminus F_\perp}((\ell, b \restriction n), (m, c \restriction n))$, where $n((b, \ell), (c, m))$ is the least natural number $n \geq \max\{\ell, m\}$ such that $\forall k \geq n \; b(k) = c(k)$. We will show that the function $\pi = \phi_X \circ \psi_X$ is as desired.

To see that $\forall c \in 2^\mathbb{N} \; [\pi([c]_{\mathbb{E}_0} \times \mathbb{N})]_F$ is an E-class, we will show that if $c \in 2^\mathbb{N}$ and $m \in \mathbb{N}$, then $(\phi_{R_m} \circ \psi_R)(c, m) = (\pi(c, 0), \pi(c, m))$. As $X \times X$ is a Hausdorff space, it is sufficient to show that if U is an open neighborhood of $(\pi(c, 0), \pi(c, m))$ and V is an open neighborhood of $(\phi_{R_m} \circ \psi_R)(c, m)$, then $U \cap V \neq \emptyset$. Towards this end, fix $n \geq m$ such that $\phi_X(\mathcal{N}_{\psi^{a_n}_X(0, c\restriction n)}) \times \phi_X(\mathcal{N}_{\psi^{a_n}_X(m, c\restriction n)}) \subseteq U$ and $\phi_{R_m}(\mathcal{N}_{\psi^{a_n}_R(m, c\restriction n)}) \subseteq V$. As a_n is not B^α-terminal, there is a configuration γ compatible with a_n, in which case $((\phi_X \circ \psi^\gamma_X)(0, c \restriction n), (\phi_X \circ \psi^\gamma_X)(m, c \restriction n)) \in U$ and $(\phi_{R_m} \circ \phi^\gamma_R)(m, c \restriction n) \in V$, thus $U \cap V \neq \emptyset$.

It now only remains to establish that π is a homomorphism from $(F^* \setminus (\Delta(2^\mathbb{N}) \times \Delta(\mathbb{N})), (\mathbb{E}_0 \times I(\mathbb{N})) \setminus F^*)$ to $(F \setminus F_\perp, (E \setminus (F \cup F_\perp))$. We will show the stronger fact that if (b, ℓ) and (c, m) are distinct but $(\mathbb{E}_0 \times I(\mathbb{N}))$-equivalent, then $(\phi_{E \setminus F_\perp} \circ (\psi_{E\setminus F_\perp} \times \mathbf{1}_{F^*}))((b, \ell), (c, m)) = (\pi(b, \ell), \pi(c, m))$. As $X \times X$ is a Hausdorff space, it is sufficient to show that if U is an open neighborhood of $(\pi(b, \ell), \pi(c, m))$ and V is an open neighborhood of $(\phi_{E \setminus F_\perp} \circ (\psi_{E\setminus F_\perp} \times \mathbf{1}_{F^*}))((b, \ell), (c, m))$, then

$U \cap V \neq \emptyset$. Towards this end, set $n = n((b,\ell),(c,m))$, and note that $\phi_X(\mathcal{N}_{\psi_X^{a_n}(\ell,b \upharpoonright n)}) \times \phi_X(\mathcal{N}_{\psi_X^{a_n}(m,c \upharpoonright n)}) \subseteq U$ and $\phi_{E \setminus F_\perp}(\mathcal{N}_{\psi_{E \setminus F_\perp}^{a_n}((\ell,b \upharpoonright n),(m,c \upharpoonright n))} \times \{\mathbf{1}_{F^*}((b,\ell),(c,m))\}) \subseteq V$. As a_n is not B^α-terminal, there exists a configuration γ compatible with a_n, so $((\phi_X \circ \psi_X^\gamma)(\ell, b \upharpoonright n), (\phi_X \circ \psi_X^\gamma)(m, c \upharpoonright n)) \in U$ and $\phi_E(\psi_{E \setminus F_\perp}^\gamma((\ell, b \upharpoonright n), (m, c \upharpoonright n)), \mathbf{1}_{F^*}((b,\ell),(c,m))) \in V$, and it follows that $U \cap V \neq \emptyset$. ⊠

Remark 1.12. The apparent use of choice beyond $\mathtt{DC}$ in the above argument can be eliminated by first running the analog of the argument without [dRM20, Proposition 2.1] and replacing the use of [dRM20, Propositions 4.1] with the use of its weakening without any definability constraints on the partial quasi-transversal puncturing the family (which can be proven in the same manner, but without using [dRM20, Proposition 2.1]), in order to obtain an upper bound $\alpha' < \omega_1$ on the least ordinal $\alpha < \omega_1$ for which the sets of B^α- and $B^{\alpha+1}$-terminal approximations coincide.

The *composition* of sets $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ is given by $R \circ S = \{(x,z) \in X \times Z \mid \exists y \in Y \ x \mathrel{R} y \mathrel{S} z\}$.

Theorem 1.13. *Suppose that X is an analytic Hausdorff space, E is a Borel equivalence relation on X, F is a Borel equivalence relation on X for which every E-class is a countable union of $(E \cap F)$-classes and the projection onto the left coordinate of every $(\Delta(X) \times (E \cap F))$-invariant Borel partial uniformization of E over $E \cap F$ is Borel, and $F_\perp$ is a smooth countable Borel subequivalence relation of E for which $E = (E \cap F) \circ F_\perp$. Then exactly one of the following holds:*

1. *There is a partition $(B_n)_{n \in \mathbb{N}}$ of X into E-invariant Borel sets with the property that there is an $(E \cap F)$-invariant Borel quasi-transversal $A_n \subseteq B_n$ of $E \upharpoonright B_n$ over $(E \cap F) \upharpoonright B_n$ for all $n \in \mathbb{N}$.*
2. *There is a continuous embedding $\pi \colon 2^{\mathbb{N}} \times \mathbb{N} \hookrightarrow X$ of $(\mathbb{E}_0 \times I(\mathbb{N}), \Delta(2^{\mathbb{N}}) \times \Delta(\mathbb{N}))$ into $(E, F \cup F_\perp)$ for which $[\pi(2^{\mathbb{N}} \times \mathbb{N})]_{E \cap F}$ is E-invariant.*

Proof. To see that conditions (1) and (2) are mutually exclusive, note that if both hold, then there exists $n \in \mathbb{N}$ for which $\pi^{-1}(B_n)$ is not meager, thus $\pi^{-1}(A_n)$ is a non-meager Borel partial quasi-transversal of $\mathbb{E}_0 \times I(\mathbb{N})$, contradicting Proposition 1.1.

Note that if $A \subseteq X$ is an E-invariant Borel set for which there is an $F_\perp$-invariant Borel quasi-transversal of $E \upharpoonright A$ over $F_\perp \upharpoonright A$, then the smoothness of $F_\perp$ and [HKL90, Theorem 1.1] ensure that $E \upharpoonright A$ is smooth. Moreover, if $B \subseteq X$ is an E-invariant Borel set for which there is an $(E \upharpoonright B)$-complete $(E \cap F)$-invariant Borel partial quasi-transversal of $E \cap F$ over $E \cap F \cap F_\perp$, then the fact that $E = (E \cap F) \circ F_\perp$ ensures that B is a partial quasi-transversal of E over $F_\perp$, so $E \restriction B$ is smooth.

By [dRM20, Theorem 2.6] and Theorem 1.7, we can therefore assume that there is a suitable sequence $(F_n)_{n \in \mathbb{N}}$ and a continuous homomorphism $\phi \colon 2^{\mathbb{N}} \times \mathbb{N} \to X$ from $(F^* \setminus (\Delta(2^{\mathbb{N}}) \times \Delta(\mathbb{N})), (\mathbb{E}_0 \times I(\mathbb{N})) \setminus F^*)$ to $((E \cap F) \setminus F_\perp, E \setminus (F \cup F_\perp))$ such that $\forall c \in 2^{\mathbb{N}}\ [\phi([c]_{\mathbb{E}_0} \times \mathbb{N})]_{E \cap F}$ is an E-class, where $F^* = \bigcup_{n \in \mathbb{N}} F_n^*$. As Proposition 1.6 yields a clopen transversal $U \subseteq 2^{\mathbb{N}} \times \mathbb{N}$ of F^*, Proposition 1.3 gives rise to a continuous invariant embedding $\chi \colon 2^{\mathbb{N}} \times \mathbb{N} \hookrightarrow U$ of $\mathbb{E}_0 \times I(\mathbb{N})$ into $(\mathbb{E}_0 \times I(\mathbb{N})) \restriction U$, in which case $\phi \circ \chi$ is a continuous homomorphism from $(\mathbb{E}_0 \times I(\mathbb{N})) \setminus (\Delta(2^{\mathbb{N}}) \times \Delta(\mathbb{N}))$ to $E \setminus (F \cup F_\perp)$ with the property that $\forall c \in 2^{\mathbb{N}}\ [(\phi \circ \chi)([c]_{\mathbb{E}_0} \times \mathbb{N})]_{E \cap F}$ is an E-class. As Proposition 1.1 ensures that the preimages E' and F' of E and F under $(\phi \circ \chi) \times (\phi \circ \chi)$ are meager, Proposition 1.4 yields a continuous injective homomorphism $\psi \colon 2^{\mathbb{N}} \times \mathbb{N} \hookrightarrow 2^{\mathbb{N}} \times \mathbb{N}$ from $(\mathbb{E}_0 \times I(\mathbb{N}), \sim(\mathbb{E}_0 \times I(\mathbb{N})))$ to $(\mathbb{E}_0 \times I(\mathbb{N}), \sim(E' \cup F'))$ with the property that $\forall c \in 2^{\mathbb{N}}\ \psi([c]_{\mathbb{E}_0} \times \mathbb{N})$ is an $(\mathbb{E}_0 \times I(\mathbb{N}))$-class. Define $\pi = \phi \circ \chi \circ \psi$. ⊠

2. Uniformizations

As a corollary of Theorem 1.13, we obtain the following:

Theorem 2.1. *Suppose that X and Y are Polish spaces, E is a Borel equivalence relation on X, F is a countable Borel equivalence relation on Y, and $R \subseteq X \times Y$ is an $(E \times \Delta(Y))$-invariant Borel set whose vertical sections are contained in countable unions of F-classes. Then exactly one of the following holds:*

1. *There is a partition $(B_n)_{n \in \mathbb{N}}$ of $\mathrm{proj}_X(R)$ into E-invariant Borel sets with the property that there is an $((E \times F) \restriction R)$-invariant Borel quasi-uniformization of $R \cap (B_n \times Y)$ for all $n \in \mathbb{N}$.*
2. *There are continuous embeddings $\pi_X \colon 2^{\mathbb{N}} \times \mathbb{N} \hookrightarrow X$ of $\mathbb{E}_0 \times I(\mathbb{N})$ into E and $\pi_Y \colon 2^{\mathbb{N}} \times \mathbb{N} \hookrightarrow Y$ of $\Delta(2^{\mathbb{N}}) \times \Delta(\mathbb{N})$ into F such that $R \cap (\pi_X(2^{\mathbb{N}} \times \mathbb{N}) \times Y) = [(\pi_X \times \pi_Y)(\mathbb{E}_0 \times I(\mathbb{N}))]_{(\Delta(X) \times F) \restriction R}$.*

Proof. To see that conditions (1) and (2) are mutually exclusive, note that if both hold, then there exists $n \in \mathbb{N}$ for which $\pi_X^{-1}(B_n)$ is not meager, in which case the pullback of the corresponding $((E \times F) \restriction R)$-invariant Borel quasi-uniformization of $R \cap (B_n \times Y)$ through $\pi_X \times \pi_Y$ is a non-meager Borel quasi-transversal of $\mathbb{E}_0 \times I(\mathbb{N})$, contradicting Proposition 1.1.

Suppose now that condition (1) fails. Then Theorem 1.13 yields a continuous embedding $\pi \colon 2^{\mathbb{N}} \times \mathbb{N} \hookrightarrow R$ of $(\mathbb{E}_0 \times I(\mathbb{N}), \Delta(2^{\mathbb{N}}) \times \Delta(\mathbb{N}))$ into

$(E \times I(Y), (I(X) \times F) \cup (\Delta(X) \times I(Y)))$ for which $[\pi(2^{\mathbb{N}} \times \mathbb{N})]_{(E \times F) \restriction R}$ is $((E \times I(Y)) \restriction R)$-invariant. Set $\pi_X = \mathrm{proj}_X \circ \pi$ and $\pi_Y = \mathrm{proj}_Y \circ \pi$. ⊠

As a corollary, we obtain the following generalization of Theorem 2:

Theorem 2.2. *Suppose that X and Y are Polish spaces, E is a Borel equivalence relation on X, F is a smooth countable Borel equivalence relation on Y, and $R \subseteq X \times Y$ is an $(E \times \Delta(Y))$-invariant Borel set whose vertical sections are contained in countable unions of F-classes. Then exactly one of the following holds:*

(1) *There is an $((E \times F) \restriction R)$-invariant Borel uniformization of R over F.*
(2) *There are continuous embeddings $\pi_X : 2^{\mathbb{N}} \times \mathbb{N} \hookrightarrow X$ of $\mathbb{E}_0 \times I(\mathbb{N})$ into E and $\pi_Y : 2^{\mathbb{N}} \times \mathbb{N} \hookrightarrow Y$ of $\Delta(2^{\mathbb{N}}) \times \Delta(\mathbb{N})$ into F such that $R \cap (\pi_X(2^{\mathbb{N}} \times \mathbb{N}) \times Y) = [(\pi_X \times \pi_Y)(\mathbb{E}_0 \times I(\mathbb{N}))]_{(\Delta(X) \times F) \restriction R}$.*

Proof. By Theorem 2.1, it is sufficient to show that if every vertical section of R is contained in a union of finitely-many F-classes, then there is a Borel uniformization of R. But this is a straightforward consequence of the original Lusin–Novikov uniformization theorem. ⊠

Acknowledgements. I would like to thank Alexander Kechris for asking the questions that led to this work, as well as Julia Millhouse for pointing out several typos.

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