Alexander S. Kechris and Benjamin D. Miller

Topics in Orbit Equivalence Theory

Springer Internal Project Number

June 16, 2004

Springer

Berlin Heidelberg New York
Hong Kong London
Milan Paris Tokyo

Preface

(A) These notes provide an introduction to some topics in orbit equivalence theory, a branch of ergodic theory. One of the main concerns of ergodic theory is the structure and classification of measure preserving (or more generally measure-class preserving) actions of groups. By contrast, in orbit equivalence theory one focuses on the equivalence relation induced by such an action, i.e., the equivalence relation whose classes are the orbits of the action. This point of view originated in the pioneering work of Dye in the late 1950's, in connection with the theory of operator algebras. Since that time orbit equivalence theory has been a very active area of research in which a number of remarkable results have been obtained.

Roughly speaking, two main and opposing phenomena have been discovered, which we will refer to as *elasticity* (not a standard terminology) and *rigidity*. To explain them, we will need to introduce first the basic concepts of orbit equivalence theory.

In these notes we will only consider countable, discrete groups Γ. If such a group Γ acts in a Borel way on a standard Borel space X, we denote by E_Γ^X the corresponding equivalence relation on X:

$$xE_\Gamma^X y \Leftrightarrow \exists\gamma \in \Gamma(\gamma \cdot x = y).$$

If μ is a probability (Borel) measure on X, the action *preserves* μ if $\mu(\gamma \cdot A) = \mu(A)$, for any Borel set $A \subseteq X$ and $\gamma \in \Gamma$. The action (or the measure) is *ergodic* if every Γ-invariant Borel set is null or conull.

Suppose now Γ acts in a Borel way on X with invariant probability measure μ and Δ acts in a Borel way on Y with invariant probability measure ν. Then these actions are *orbit equivalent* if there are conull invariant Borel sets $A \subseteq X$, $B \subseteq Y$ and a Borel isomorphism $\pi : A \to B$ which sends μ to ν (i.e., $\pi_*\mu = \nu$) and for $x, y \in A$:

$$xE_\Gamma^X y \Leftrightarrow \pi(x)E_\Delta^Y\pi(y).$$

We can now describe these two competing phenomena:

(I) Elasticity: For amenable groups there is exactly one orbit equivalence type of non-atomic probability measure preserving ergodic actions. More precisely, if Γ, Δ are amenable groups acting in a Borel way on X, Y with non-atomic, invariant, ergodic probability measures μ, ν, respectively, then these two actions are orbit equivalent. This follows from a combination of Dye's work with subsequent work of Ornstein-Weiss in the 1980's. Thus the equivalence relation induced by such an action of an amenable group does not "encode" or "remember" anything about the group (beyond the fact that it is amenable). For example, any two free, measure preserving ergodic actions of the free abelian groups $\mathbb{Z}^m, \mathbb{Z}^n$ $(m \neq n)$ are orbit equivalent.

(II) Rigidity: As originally discovered by Zimmer in the 1980's, for many non-amenable groups Γ we have the opposite situation: The equivalence relation induced by a probability measure preserving action of Γ "encodes" or "remembers" a lot about the group (and the inducing action). For example, a recent result of Furman, strengthening an earlier theorem of Zimmer, asserts that if the canonical action of $\mathrm{SL}_n(\mathbb{Z})$ on $\mathbb{T}^n$ $(n \geq 3)$ is orbit equivalent to a free, non-atomic probability measure preserving, ergodic action of a countable group Γ, then Γ is isomorphic to $\mathrm{SL}_n(\mathbb{Z})$ and under this isomorphism the actions are also Borel isomorphic (modulo null sets). Another recent result, due to Gaboriau, states that if the free groups F_m, F_n $(1 \leq m, n \leq \aleph_0)$ have orbit equivalent free probability measure preserving Borel actions, then $m = n$. (This should be contrasted with the result mentioned in (I) above about $\mathbb{Z}^m, \mathbb{Z}^n$.)

(B) These notes are divided into three chapters. The first, very short, chapter contains a quick introduction to some basic concepts of ergodic theory.

The second chapter is primarily an exposition of the "elasticity" phenomenon described above. Some topics included here are: amenability of groups, the concept of hyperfiniteness for equivalence relations, Dye's Theorem to the effect that hyperfinite equivalence relations with non-atomic, invariant ergodic probability measures are Borel isomorphic (modulo null sets), quasi-invariant measures and amenable equivalence relations, and the Connes-Feldman-Weiss Theorem that amenable equivalence relations are hyperfinite a.e. We also include topics concerning amenability and hyperfiniteness in the Borel and generic (Baire category) contexts, like the result that finitely generated groups of polynomial growth always give rise to hyperfinite equivalence relations (without neglecting null sets), that generically (i.e., on a comeager set) every countable Borel equivalence relation is hyperfinite, and finally that, also generically, a countable Borel equivalence relation admits no invariant Borel probability measure, and therefore all generically aperiodic, non-smooth countable Borel equivalence relations are Borel isomorphic modulo meager sets.

The third chapter contains an exposition of the theory of costs for equivalence relations and groups, originated by Levitt, and mainly developed by

Gaboriau, who used this theory to prove the rigidity results about free groups mentioned in (II) above.

(C) In order to make it easier for readers who are familiar with the material in Chapter II but would like to study the theory of costs, we have made Chapter III largely independent of Chapter II. This explains why several basic definitions and facts, introduced in Chapter II (or even in Chapter I), are again repeated in Chapter III. We apologize for this redundancy to the reader who starts from the beginning.

Chapter II grew out of a set of rough notes prepared by the first author in connection with teaching a course on orbit equivalence at Caltech in the Fall of 2001. It underwent substantial modification and improvement under the input of the second author, who prepared the current final version. Chapter III is based on a set of lecture notes written-up by the first author for a series of lectures at the joint Caltech-UCLA Logic Seminar during the Fall and Winter terms of the academic year 2000-2001. Various revised forms of these notes have been available on the web since that time.

(D) As the title of these notes indicates, this is by no means a comprehensive treatment of orbit equivalence theory. Our choice of topics was primarily dictated by the desire to keep these notes as elementary and self-contained as possible. In fact, the prerequisites for reading these notes are rather minimal: a basic understanding of measure theory, functional analysis, and classical descriptive set theory. Also helpful, but not necessary, would be some familiarity with the theory of countable Borel equivalence relations (see, e.g., Feldman-Moore [FM], Dougherty-Jackson-Kechris [DJK], and Jackson-Kechris-Louveau [JKL]).

Since this is basically a set of informal lecture notes, we have not attempted to present a detailed picture of the historical development of the subject nor a comprehensive list of references to the literature.

Acknowledgments. The authors would like to thank D. Gaboriau, G. Hjorth, A. Louveau, J. Pavelich, O. Raptis, S. Solecki, B. Velickovic, and the anonymous referees, for many valuable comments and suggestions or for allowing us to include their unpublished results. Work on this book was partially supported by NSF Grant DMS-9987437 and by a Guggenheim Fellowship.

Los Angeles
June 2004

Alexander S. Kechris
Benjamin D. Miller

Contents

I

Orbit Equivalence

1 Group Actions and Equivalence Relations

Suppose X is a standard Borel space and Γ is a countable (discrete) group. A *Borel action* of Γ on X is a Borel map $(\gamma, x) \mapsto \gamma \cdot x$ such that

1. $\forall x \in X \, (1 \cdot x = x)$, and
2. $\forall x \in X \forall \gamma_1, \gamma_2 \in \Gamma \, (\gamma_1 \cdot (\gamma_2 \cdot x) = (\gamma_1 \gamma_2) \cdot x)$.

We also say that X is a *Borel Γ-space*. Given $x \in X$, the *orbit* of x is

$$\Gamma \cdot x = \{\gamma \cdot x : \gamma \in \Gamma\},$$

and the *stabilizer* of x is given by

$$\Gamma_x = \{\gamma \in \Gamma : \gamma \cdot x = x\}.$$

We say that the action of Γ on X is *free* if

$$\forall x \in X \, (\Gamma_x = \{1\}) \, ,$$

that is, if $\gamma \cdot x \neq x$ whenever $\gamma \neq 1$. The *equivalence relation induced by the action of Γ* is given by

$$x E_\Gamma^X y \Leftrightarrow \exists \gamma \in \Gamma \, (\gamma \cdot x = y) \, ,$$

and the *quotient space* associated with the action of Γ is

$$X/\Gamma = X/E_\Gamma^X = \{\Gamma \cdot x : x \in X\}.$$

Example 1.1. The (left) *shift action* of Γ on X^Γ is given by

$$\gamma \cdot p(\delta) = p(\gamma^{-1}\delta).$$

For example, when $\Gamma = \mathbb{Z}$ and $X = 2 = \{0, 1\}$, we have

$$m \cdot p(n) = p(n-m),$$

so that

$$pE_{\mathbb{Z}}^{2^{\mathbb{Z}}} q \Leftrightarrow \exists m \in \mathbb{Z} \forall k \in \mathbb{Z}\, (p(k) = q(m+k))\,.$$

Similarly, the *right shift* of Γ on X^Γ is given by

$$\gamma \cdot p(\delta) = p(\delta\gamma),$$

and the *conjugation action* of Γ on X^Γ is given by

$$\gamma \cdot p(\delta) = p(\gamma^{-1}\delta\gamma).$$

Example 1.2. If $X = G$ is a standard Borel group and $\Gamma \subseteq G$, then Γ acts on G by left translation, that is,

$$\gamma \cdot g = \gamma g.$$

Here the orbit of $g \in G$ is the right coset Γg, so that

$$g E_\Gamma^G h \Leftrightarrow \Gamma g = \Gamma h$$

and the quotient space G/Γ coincides with the space of right cosets of Γ. For example, take $G = \mathrm{SO}(3)$, the compact metrizable group of 3-dimensional rotations, and Γ the copy of F_2 (the free group on 2 generators) sitting inside of $\mathrm{SO}(3)$, whose two generators are given by rotation by $\cos^{-1}(1/3)$ around $\mathbf{k}$ and rotation by $\cos^{-1}(1/3)$ around $\mathbf{i}$. Then the left translation action gives a Borel action of F_2 on $\mathrm{SO}(3)$. Also, F_2 acts on S^2, the unit sphere in $\mathbb{R}^3$ (this action is related to the geometrical paradoxes of Hausdorff-Banach-Tarski; see [W]).

An equivalence relation E on X is (*finite*) *countable* if its equivalence classes are (finite) countable. We use $[E]$ to denote the group of Borel automorphisms of X whose graphs are contained in E. We use $[[E]]$ to denote the set of partial Borel automorphisms of X whose graphs are contained in E. (A *partial Borel automorphism* is a Borel bijection $\phi : A \to B$, where $A = \mathrm{dom}(\phi), B = \mathrm{rng}(\phi)$ are Borel subsets of X. As usual the *graph* of a function is the set $\mathrm{graph}(f) = \{(x,y) : f(x) = y\}$.)

Theorem 1.3 (Feldman-Moore [FM]). *Let E be a countable Borel equivalence relation on X. Then there is a countable group Γ and a Borel action of Γ on X such that $E = E_\Gamma^X$. Moreover, Γ and the action can be chosen so that*

$$xEy \Leftrightarrow \exists g \in \Gamma (g^2 = 1 \;\&\; g \cdot x = y).$$

Proof. As $E \subseteq X^2$ has countable sections, it follows from Theorem 18.10 of [K] that

$$E = \bigcup_{n \in \mathbb{N}} F_n,$$

for some sequence $\{F_n\}_{n\in\mathbb{N}}$ of Borel graphs. We can assume that $F_n \cap F_m = \emptyset$ if $n \neq m$. Let $F_{n,m} = F_n \cap F_m^{-1}$, where for $F \subseteq X^2$,

$$F^{-1} = \{(y,x) : (x,y) \in F\}.$$

Since X is Borel isomorphic to a subset of $\mathbb{R}$, it follows that $X^2 \setminus \Delta_X$, where $\Delta_X = \{(x,x) : x \in X\}$, is of the form

$$X^2 \setminus \Delta_X = \bigcup_{p\in\mathbb{N}} (A_p \times B_p),$$

where A_p, B_p are disjoint Borel subsets of X. It follows that $F_{n,m,p} = F_{n,m} \cap (A_p \times B_p)$ is of the form

$$F_{n,m,p} = \mathrm{graph}(f_{n,m,p}),$$

for some Borel bijection $f_{n,m,p} : D_{n,m,p} \to R_{n,m,p}$, where

$$D_{n,m,p} \cap R_{n,m,p} = \emptyset.$$

Now define a sequence $\{g_{n,m,p}\}$ of Borel automorphisms of X by

$$g_{n,m,p}(x) = \begin{cases} f_{n,m,p}(x) & \text{if } x \in D_{n,m,p}, \\ f^{-1}_{n,m,p}(x) & \text{if } x \in R_{n,m,p}, \\ x & \text{otherwise.} \end{cases}$$

Note that $g_{n,m,p}$ is an involution and

$$E = \bigcup \mathrm{graph}(g_{n,m,p}),$$

thus the induced action of $\Gamma = \langle g_{n,m,p} \rangle$ on X is as desired. $\dashv$

2 Invariant Measures

By a *measure* on a standard Borel space X we mean a non-zero σ-finite Borel measure on X. If $\mu(X) < \infty$ we call μ *finite*, and if $\mu(X) = 1$ we call μ a *probability measure*. Given a countable Borel equivalence relation E on X, we say that μ is *E-invariant* if

$$\forall f \in [E]\, (f_*\mu = \mu)\,,$$

where $f_*\mu(A) = \mu(f^{-1}(A))$.

Proposition 2.1. *The following are equivalent:*

(a) μ *is* E*-invariant,*
(b) μ *is* Γ*-invariant, whenever* Γ *is a countable group acting in a Borel fashion on* X *such that* $E = E^X_\Gamma$*,*
(c) μ *is* Γ*-invariant for some countable group* Γ *acting in a Borel fashion on* X *such that* $E = E^X_\Gamma$*, and*
(d) $\forall \phi \in [[E]]\, (\mu(\mathrm{dom}(\phi)) = \mu(\mathrm{rng}(\phi)))$.

Proof. The proof of $(d) \Rightarrow (a) \Rightarrow (b) \Rightarrow (c)$ is straightforward. To see $(c) \Rightarrow (d)$, simply observe that, if $\mathrm{dom}(\phi) = A, \mathrm{rng}(\phi) = B$, there is a Borel partition $A = \bigcup A_n$ and a sequence $\{\gamma_n\} \subseteq \Gamma$ such that for $x \in A_n$, $\phi(x) = \gamma_n \cdot x$, so that $B = \bigcup \gamma_n \cdot A_n$. ⊣

3 Ergodicity

A measure μ is *E-ergodic* if every E-invariant Borel set is null or conull. A measure μ is *ergodic with respect to an action* of a group Γ if it is ergodic for the induced equivalence relation.

Example 3.1. Let μ be the product measure on $X = 2^\Gamma$, where Γ is a countably infinite group, and 2 has the $(1/2, 1/2)$-measure. For each finite $S \subseteq \Gamma$ and $s : S \to 2$, put

$$\mathcal{N}_s = \{f : f|\mathrm{dom}(s) = s\},$$

noting that the $\mathcal{N}_s$'s form a clopen basis for 2^Γ and $\mu(\mathcal{N}_s) = 2^{-|\mathrm{dom}(s)|}$, where $|A| = \mathrm{card}(A)$. As

$$\gamma \cdot \mathcal{N}_s = \mathcal{N}_{\gamma \cdot s},$$

it follows that μ is shift-invariant. To see that μ is ergodic with respect to the shift, we will actually show the stronger fact that it is *mixing*, that is,

$$\lim_{\gamma \to \infty} \mu(\gamma \cdot A \cap B) = \mu(A)\mu(B),$$

for $A, B \subseteq X$ Borel. (That mixing implies ergodicity follows from the fact that, when $A = B$ is invariant, mixing implies $\mu(A) = \mu(A)^2$, thus $\mu(A) \in \{0, 1\}$.) Given $A, B \subseteq X$ and $\epsilon > 0$, we can find A', B', each a finite union of basic clopen sets, such that

$$\mu(A \Delta A'), \mu(B \Delta B') < \epsilon,$$

where Δ denotes *symmetric difference*. Thus, it suffices to show the mixing condition when A, B are finite unions of basic clopen sets. Say A is supported by S and B by T, where $S, T \subseteq \Gamma$ are finite. Then off a finite set of γ's, $\gamma \cdot S \cap T = \emptyset$, thus $\gamma \cdot A$, B are independent and

$$\begin{aligned} \mu(\gamma \cdot A \cap B) &= \mu(\gamma \cdot A)\mu(B) \\ &= \mu(A)\mu(B). \end{aligned}$$

Example 3.2. Suppose G is an infinite compact metrizable group, $\Delta \leq G$ is a dense subgroup, and μ is Haar measure on G. For example, we could take $G = \mathbb{Z}_2^{\mathbb{N}}$ and $\Delta = \mathbb{Z}_2^{<\mathbb{N}}$. Then Δ acts by left-translation and μ is clearly invariant. To see that μ is ergodic, consider $L^1(G)$ and its dual $L^\infty(G)$. G acts on $L^1(G)$ via left shift, i.e.,

$$g \cdot p(h) = p(g^{-1}h).$$

This is a continuous action, i.e., if $g_n \to g$ in G and $p_n \to p$ in $L^1(G)$, then $g_n \cdot p_n \to g \cdot p$ in $L^1(G)$. So, by duality, it induces a continuous action of G on the unit ball of $L^\infty(G)$, with the weak*-topology, given by

$$g \cdot \Lambda(p) = \Lambda(g^{-1} \cdot p).$$

Now, if $A \subseteq G$ is Δ-invariant, let $\chi_A = \Lambda \in L^\infty(G)$, so that

$$\forall \delta \in \Delta \,(\delta \cdot \Lambda = \Lambda)\,.$$

But since Δ is dense in G and the action is continuous, this means that

$$\forall g \in G \,(g \cdot \Lambda = \Lambda)\,,$$

or equivalently, that

$$\forall g \forall^*_\mu h \left(\chi_A(g^{-1}h) = \chi_A(h)\right),$$

where "$\forall^*_\mu$" means "for μ-almost all." It follows from Fubini's Theorem that

$$\forall^*_\mu h \forall^*_\mu g \left(g^{-1}h \in A \Leftrightarrow h \in A\right),$$

so if $\mu(A) > 0$ we can find $h \in A$ such that

$$\forall^*_\mu g \left(g^{-1}h \in A \Leftrightarrow h \in A\right),$$

thus $\mu(A) = 1$.

It should be noted, however, that translation is not mixing! To see this, fix $\delta_n \in \Delta$ with $\delta_n \to g \neq 1$, and let N be a sufficiently small compact neighborhood of 1 such that $N \cap gN = \emptyset$. Then

$$\mu(N \cap \delta_n \cdot N) \to 0,$$

while $\mu(N)\mu(\delta_n \cdot N) = \mu(N)^2 > 0$.

For each countable Borel equivalence relation E on X, let $\mathcal{I}_E$ be the set of invariant probability measures for E, and let $\mathcal{EI}_E$ be the set of ergodic, invariant probability measures for E. Then we have:

Theorem 3.3. (Ergodic Decomposition – Farrell [F], Varadarajan [V]) *Let E be a countable Borel equivalence relation on X. Then $\mathcal{I}_E, \mathcal{EI}_E$ are Borel sets in the standard Borel space $P(X)$ of probability measures on X.*

Now suppose $\mathcal{I}_E \neq \emptyset$. Then $\mathcal{EI}_E \neq \emptyset$, and there is a Borel surjection $\pi : X \to \mathcal{EI}_E$ such that

1. *π is E-invariant,*
2. *if $X_e = \{x : \pi(x) = e\}$, then $e(X_e) = 1$ and $E|X_e$ has a unique invariant measure, namely e, and*
3. *if $\mu \in \mathcal{I}_E$, then $\mu = \int \pi(x)\ d\mu(x)$.*

Moreover, π is uniquely determined in the sense that, if π' is another such map, then $\{x : \pi(x) \neq \pi'(x)\}$ is null with respect to all measures in $\mathcal{I}_E$.

4 Isomorphism and Orbit Equivalence

We say that (X, E), (Y, F) are (Borel) *isomorphic* ($E \cong_B F$) if there is a Borel bijection $\pi : X \to Y$ such that

$$\forall x, y \in X \,(xEy \Leftrightarrow \pi(x) F \pi(y))\,.$$

We say that (X, E, μ), (Y, F, ν) are (Borel almost everywhere) *isomorphic* if there are conull Borel invariant sets $A \subseteq X$, $B \subseteq Y$ and a Borel isomorphism π of $(A, E|A)$ with $(B, F|B)$ such that $\pi_* \mu = \nu$. Note that ergodicity is preserved under isomorphism. If Γ acts in a Borel fashion on X, Y, then the actions are (Borel) *isomorphic* if there is a Borel bijection $\pi : X \to Y$ such that

$$\forall x \in X \forall \gamma \in \Gamma \,(\pi(\gamma \cdot x) = \gamma \cdot \pi(x))\,.$$

Finally, if Γ acts in a measure preserving fashion on (X, μ), (Y, ν), then the actions are (Borel almost everywhere) *isomorphic* or *conjugate* if there are conull Borel invariant sets $A \subseteq X$, $B \subseteq Y$ and a Borel isomorphism π of the actions of Γ on A, B such that $\pi_* \mu = \nu$. It is a classical problem of ergodic theory to classify (e.g., when $\Gamma = \mathbb{Z}$) the probability measure preserving, ergodic actions up to conjugacy. (Recall here the results of Ornstein and others.) Actions of Γ, Δ on X, Y, respectively, are *orbit equivalent* if $E_\Gamma^X \cong_B E_\Delta^Y$. Similarly measure preserving actions of Γ, Δ on $(X, \mu), (Y, \nu)$, respectively, are *orbit equivalent* if there are conull Borel invariant sets $A \subseteq X$, $B \subseteq Y$ and a Borel isomorphism π of E_Γ^A, E_Δ^B with $\pi_* \mu = \nu$.

II

Amenability and Hyperfiniteness

5 Amenable Groups

Suppose X is a set. A *finitely additive probability measure* (f.a.p.m.) on X is a map

$$\mu : \mathrm{Power}(X) \to [0,1],$$

where $\mathrm{Power}(X) = \{A : A \subseteq X\}$, such that $\mu(X) = 1$ and $\mu(A \cup B) = \mu(A) + \mu(B)$, if $A \cap B = \emptyset$. A f.a.p.m. μ on a countable group Γ is *left-invariant* if

$$\forall \gamma \in \Gamma \forall A \subseteq \Gamma \left(\mu(\gamma A) = \mu(A)\right),$$

and a countable group Γ is *amenable* if it admits a left-invariant f.a.p.m.

Example 5.1. Suppose Γ is finite. Then

$$\mu(A) = |A|/|\Gamma|$$

defines a left-invariant f.a.p.m. on Γ, thus Γ is amenable.

Example 5.2. The group $\Gamma = \mathbb{Z}$ is amenable. Let $\mathcal{U}$ be a non-atomic ultrafilter on $\mathbb{N}$. Then

$$\mu(A) = \lim_{n \to \mathcal{U}} \frac{|A \cap \{-n, \ldots, n\}|}{2n+1}$$

defines a left-invariant f.a.p.m. on Γ. Here $\lim_{n \to \mathcal{U}} a_n$, where $\{a_n\}$ is a bounded sequence of reals, denotes the unique real a such that for each neighborhood N of a the set $\{n : a_n \in N\}$ is in $\mathcal{U}$.

A *mean* on Γ is a linear functional $m : l^\infty(\Gamma) \to \mathbb{C}$ such that

(i) m is positive, i.e., $f \geq 0 \Rightarrow m(f) \geq 0$, and
(ii) $m(1) = 1$, where 1 denotes the constant function with value 1.

Proposition 5.3. *If m is a mean, then $||m|| = 1$.*

Proof. For $f \in l^\infty(\Gamma)$, find $\alpha \in \mathbb{T}$ such that $\alpha m(f) = |m(f)|$. Then

$$\begin{aligned} |m(f)| &= \alpha m(f) \\ &= m(\alpha f) \\ &= \mathrm{Re}\; m(\alpha f) \\ &= m(\mathrm{Re}\; \alpha f) \\ &\leq ||\mathrm{Re}\; \alpha f||_\infty \\ &\leq ||\alpha f||_\infty \\ &\leq ||f||_\infty, \end{aligned}$$

and the claim follows. ⊣

There is a canonical correspondence between means and f.a.p.m.'s, given by associating to each mean m the f.a.p.m. defined by

$$\mu(A) = m(1_A),$$

where 1_A = the *characteristic function* of A, and by associating to each f.a.p.m. μ the mean defined by

$$m(f) = \int f \; d\mu.$$

A mean m is *left-invariant* if

$$\forall \gamma \in \Gamma(m(\gamma \cdot f) = m(f)),$$

where $\gamma \cdot f(\delta) = f(\gamma^{-1}\delta)$. Thus

Proposition 5.4. *Γ is amenable $\Leftrightarrow$ Γ admits a left-invariant mean.*

By considering $\gamma \mapsto \gamma^{-1}$, it is easy to see that left-invariance can be replaced with right-invariance in the definition of amenability. In fact,

Proposition 5.5. *Γ is amenable $\Leftrightarrow$ Γ admits a 2-sided invariant f.a.p.m.*

Proof. Let μ_l be a left-invariant f.a.p.m. on Γ, define $\mu_r(A) = \mu_l(A^{-1})$, and observe that

$$\mu(A) = \int \mu_l(A\gamma^{-1}) \; d\mu_r(\gamma)$$

gives the desired 2-sided invariant f.a.p.m. ⊣

Next we turn to closure properties of amenability.

Proposition 5.6. *Suppose Γ is a countable group.*

(i) *If Γ is amenable and $\Delta \leq \Gamma$, then Δ is amenable.*
(ii) *If $N \trianglelefteq \Gamma$, then Γ is amenable $\Leftrightarrow$ $N, \Gamma/N$ are amenable. In particular, it follows that the amenable groups are closed under epimorphic images and finite products.*
(iii) *Γ is amenable $\Leftrightarrow$ every finitely generated subgroup of Γ is amenable.*

Proof. To see (i), let μ be a left-invariant f.a.p.m. on Γ, let $T \subseteq \Gamma$ consist of one point from every right coset $\Delta\gamma$, $\gamma \in \Gamma$, and note that

$$\nu(A) = \mu(AT)$$

is a left-invariant f.a.p.m. on Δ.

To see ($\Rightarrow$) of (ii), it remains to check that if Γ is amenable, then Γ/N is amenable. But if μ is a left-invariant f.a.p.m. for Γ, then

$$\nu(A) = \mu\left(\bigcup A\right)$$

defines a left-invariant f.a.p.m. on Γ/N.

To see ($\Leftarrow$) of (ii), let μ be a left-invariant f.a.p.m. on N. Then μ induces a f.a.p.m. μ_C on each $C \in \Gamma/N$, given by

$$\mu_C(A) = \mu(\gamma^{-1}A),$$

where $\gamma \in C$. As μ is left-invariant, μ_C is independent of the choice of γ. Now let ν be a left-invariant f.a.p.m. on Γ/N, and observe that

$$\lambda(A) = \int_{\Gamma/N} \mu_C(A \cap C)\ d\nu(C)$$

defines a left-invariant f.a.p.m. on Γ.

To see (iii), suppose that $\{\Gamma_n\}_{n\in\mathbb{N}}$ is an increasing, *exhaustive* (i.e., whose union is Γ) sequence of amenable subgroups of Γ, let μ_n be a left-invariant f.a.p.m. on Γ_n, fix a non-atomic ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and observe that

$$\mu(A) = \lim_{n\to\mathcal{U}} \mu_n(A \cap \Gamma_n)$$

defines a left-invariant f.a.p.m. on Γ. ⊣

Corollary 5.7. *Solvable groups are amenable.*

Proof. First, we note that abelian groups are amenable. By Proposition 5.6, it suffices to show that finitely generated abelian groups are amenable. But all such groups are direct products of finite groups with copies of $\mathbb{Z}$, and are therefore amenable by Proposition 5.6 and Examples 5.1 and 5.2.

Now suppose G is solvable. Then we can find

$$\{1\} = H_0 \trianglelefteq H_1 \trianglelefteq \cdots \trianglelefteq H_n = G$$

such that H_{i+1}/H_i is abelian, for all $i < n$, and the corollary now follows from repeated applications of Proposition 5.6. ⊣

Given two sets $X, Y \subseteq \Gamma$, we write $X \sim Y$ if there are partitions $X_1, \ldots, X_n$ of X and $Y_1, \ldots, Y_n$ of Y and group elements $\gamma_1, \ldots, \gamma_n \in \Gamma$ such that $\gamma_i X_i = Y_i$, for all $1 \leq i \leq n$. A group Γ is *paradoxical* if there are disjoint sets $A, B \subseteq \Gamma$ such that $A \sim B \sim \Gamma$.

Proposition 5.8. *If Γ is paradoxical, then Γ is not amenable.*

Proof. Suppose that μ is a left-invariant mean on Γ, fix disjoint sets $A, B \subseteq \Gamma$ such that $A \sim B \sim \Gamma$, and note that

$$\begin{aligned}\mu(\Gamma) &\geq \mu(A \cup B)\\ &= \mu(A) + \mu(B)\\ &= 2\mu(\Gamma),\end{aligned}$$

contradicting the fact that $\mu(\Gamma) = 1$. ⊣

In fact, we have (see, e.g., [W]):

Theorem 5.9 (Tarski). *Γ is not amenable $\Leftrightarrow$ Γ is paradoxical.*

In particular, to see that F_2 is not amenable it suffices to observe

Proposition 5.10. *F_2 is paradoxical.*

Proof. Let a, b be the generators of F_2. For each $x \in \{a^{\pm 1}, b^{\pm 1}\}$, let $S(x)$ denote the set of reduced words which begin with x. Clearly F_2 is the disjoint union of these sets with $\{1\}$. Noting that

$$a \cdot S(a^{-1}) = F_2 \setminus S(a),$$

it follows that $S(a) \cup S(a^{-1}) \sim F_2$, and similarly, $S(b) \cup S(b^{-1}) \sim F_2$. ⊣

It follows that no group which contains an isomorphic copy of F_2 can be amenable. Day conjectured that this is the only way that a group can fail to be amenable (this became known as the *Von Neumann Conjecture*). Although Tits showed that this conjecture is true for linear groups (subgroups of $\mathrm{GL}_n(F)$, where F is a field), Olshanskii showed that the conjecture itself is false (see [W]).

A countable group Γ satisfies the *Reiter condition* if

$$\forall \epsilon > 0 \forall \gamma_1, \ldots, \gamma_n \in \Gamma \exists f \in l^1(\Gamma)\ (f \geq 0, ||f||_1 = 1, \text{ and } \\ \forall 1 \leq i \leq n\ (||f - \gamma_i \cdot f||_1 < \epsilon),$$

or equivalently, if there is a sequence $\{f_n\}_{n \in \mathbb{N}}$ of elements of $l^1(\Gamma)$ such that $\forall n \in \mathbb{N} (f_n \geq 0$ and $||f_n||_1 = 1)$, and

$$\forall \gamma \in \Gamma (||f_n - \gamma \cdot f_n||_1 \to 0).$$

A countable group Γ satisfies the *Følner condition* if

$$\forall \epsilon > 0 \forall \gamma_1, \ldots, \gamma_n \exists F \subseteq \Gamma \text{ finite} \forall 1 \leq i \leq n \left(\frac{|\gamma_i F \Delta F|}{|F|} < \epsilon \right),$$

or equivalently, if there is a sequence $\{F_n\}_{n \in \mathbb{N}}$ of finite subsets of Γ such that

$$\forall \gamma \in \Gamma \left(\frac{|\gamma F_n \Delta F_n|}{|F_n|} \to 0 \right).$$

Such a sequence $\{F_n\}_{n\in\mathbb{N}}$ is called a *Følner sequence.*

Remark 5.11. We could similarly say $\{F_n\}_{n\in\mathbb{N}}$ is a *right Følner sequence* if

$$\forall \gamma \in \Gamma \left(\frac{|F_n \gamma \Delta F_n|}{|F_n|} \to 0 \right).$$

There is a canonical correspondence between Følner sequences and right Følner sequences, given by $\{F_n\}_{n\in\mathbb{N}} \mapsto \{F_n^{-1}\}_{n\in\mathbb{N}}$. In particular, a group Γ admits a Følner sequence if and only if it admits a right Følner sequence.

Remark 5.12. If $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$ and

$$\forall \epsilon > 0 \exists F \subseteq \Gamma \text{ finite} \forall 1 \le i \le n \forall \delta \in \{\pm 1\} \left(\frac{|\gamma_i^\delta F \Delta F|}{|F|} < \epsilon \right),$$

then Γ satisfies the Følner condition. For if $\gamma = \gamma_{k_1}^{\delta_1} \cdots \gamma_{k_m}^{\delta_m}$ and

$$\forall 1 \le i \le n \forall \delta \in \{\pm 1\} \left(\frac{|\gamma_i^\delta F \Delta F|}{|F|} < \epsilon/m \right),$$

then

$$\begin{aligned}
\frac{|\gamma F \Delta F|}{|F|} &= \frac{|\gamma_{k_1}^{\delta_1} \cdots \gamma_{k_m}^{\delta_m} F \Delta F|}{|F|} \\
&\le \frac{\sum_{i<m} |\gamma_{k_1}^{\delta_1} \cdots \gamma_{k_{i+1}}^{\delta_{i+1}} F \Delta \gamma_{k_1}^{\delta_1} \cdots \gamma_{k_i}^{\delta_i} F|}{|F|} \\
&= \frac{\sum_{i<m} |\gamma_{k_{i+1}}^{\delta_{i+1}} F \Delta F|}{|F|} \\
&< \epsilon.
\end{aligned}$$

Example 5.13. A Følner sequence for $G = \mathbb{Z}^k$ is given by $F_n = [-n, n]^k$.

Theorem 5.14. *Suppose Γ is a countable group. Then the following are equivalent:*

(i) *Γ is amenable.*
(ii) **(Day)** *Γ satisfies the Reiter condition.*
(iii) **(Følner)** *Γ satisfies the Følner condition.*

Proof. To see (iii) $\Rightarrow$ (i), let $\{F_n\}_{n\in\mathbb{N}}$ be a Følner sequence, let $\mathcal{U}$ be a non-atomic ultrafilter on $\mathbb{N}$, and observe that

$$\mu(A) = \lim_{n \to \mathcal{U}} \frac{|A \cap F_n|}{|F_n|}$$

defines a f.a.p.m. on Γ. Noting that

$$\begin{aligned} |\mu(\gamma A) - \mu(A)| &= \lim_{n \to \mathcal{U}} \frac{\big||\gamma A \cap F_n| - |A \cap F_n|\big|}{|F_n|} \\ &= \lim_{n \to \mathcal{U}} \frac{\big||A \cap \gamma^{-1} F_n| - |A \cap F_n|\big|}{|F_n|} \\ &\leq \lim_{n \to \mathcal{U}} \frac{|\gamma^{-1} F_n \Delta F_n|}{|F_n|} \\ &= 0, \end{aligned}$$

it follows that μ is left-invariant.

To see (i) $\Rightarrow$ (ii), we will use

Theorem 5.15 (Mazur). *In a Banach space, the weak closure of a convex set is the same as its norm closure.*

Proof. If A is convex and $x \notin \overline{A}$, then it follows from the Hahn-Banach Theorem that we can find x^* such that

$$\text{Re } x^*(x) < \inf_{a \in A} \text{Re } x^*(a),$$

which implies that $x \notin \overline{A}^w$. ⊣

Set $K = \{f \in l^1(\Gamma) : ||f||_1 = 1, f \geq 0\}$, and fix a mean $m \in l^\infty(\Gamma)^*$. Clearly m is in the unit ball of $l^\infty(\Gamma)^*$.

Claim 5.16. *m is in the weak*-closure of K.*

Proof. Suppose not. As $\overline{K}^{w^*}$ is compact convex, it follows from the Hahn-Banach Theorem that there exists $\phi_0 \in l^\infty(\Gamma)$ separating m from K, i.e., such that

$$\forall f \in K \left(\text{Re } \langle \phi_0, f \rangle \leq \text{Re } \langle \phi_0, m \rangle - \epsilon\right),$$

for some $\epsilon > 0$. By replacing ϕ_0 with Re ϕ_0, we may assume that ϕ_0 is real-valued. Let $a_0 = \sup(\phi_0)$. Then $a_0 \geq \phi_0(\gamma_0) \geq a_0 - \epsilon/2$, for some $\gamma_0 \in \Gamma$. If $f = 1_{\{\gamma_0\}}$, then

$$\begin{aligned} a_0 &\geq m(\phi_0) \\ &= \langle \phi_0, m \rangle \\ &\geq \langle \phi_0, f \rangle + \epsilon \\ &= \phi_0(\gamma_0) + \epsilon \geq a_0 - \epsilon/2 + \epsilon \\ &= a_0 + \epsilon/2, \end{aligned}$$

a contradiction. ⊣

It follows that there is a net $f_i \to^{w^*} m$ (i.e., $\langle f_i, \phi\rangle \to m(\phi)$ for all $\phi \in l^\infty(\Gamma)$), with $f_i \in K$.

Next we observe that $\gamma \cdot f_i \to^{w^*} \gamma \cdot m = m$, where $\gamma \cdot m(\phi) = m(\gamma^{-1} \cdot \phi)$. To see this, note that

$$\begin{aligned}\sum_{\delta\in\Gamma} \gamma \cdot f_i(\delta)\phi(\delta) &= \sum_{\delta\in\Gamma} f_i(\gamma^{-1}\delta)\phi(\delta) \\ &= \sum_{\delta\in\Gamma} f_i(\delta)\phi(\gamma\delta) \\ &= \sum_{\delta\in\Gamma} f_i(\delta)\gamma^{-1} \cdot \phi(\delta) \\ &= \left\langle f_i, \gamma^{-1} \cdot \phi\right\rangle \\ &\to m(\gamma^{-1} \cdot \phi) \\ &= \gamma \cdot m(\phi).\end{aligned}$$

So $f_i - \gamma \cdot f_i \to^{w^*} 0$, hence $f_i - \gamma \cdot f_i \to^{w} 0$, as the subspace topology on $l^1(\Gamma)$ (when viewed as a subspace of $(l^\infty)^*$ with the weak-$*$ topology) is the same as the weak topology.

Fix now $\epsilon > 0$, $\gamma_1, \dots, \gamma_n \in \Gamma$. Note that

$$\{\{f - \gamma_i \cdot f\}_{1\leq i\leq n} : f \in K\}$$

is convex in $(l^1(\Gamma))^n$. (Note that when X is a Banach space, the weak topology on X^n in the n-fold product of the weak topology on X.) Then $0 \in (l^1(\Gamma))^n$ is in the weak closure of

$$\{\{f - \gamma_i \cdot f\}_{1\leq i\leq n} : f \in K\}.$$

So by Mazur's Theorem it is in the strong closure, so there is an $f \in K$ such that

$$\sum_{1\leq i\leq n} ||f - \gamma_i \cdot f||_1 < \epsilon,$$

which completes the proof of (i) $\Rightarrow$ (ii).

To see (ii) $\Rightarrow$ (iii), fix $\epsilon > 0$ and $\gamma_1, \dots, \gamma_n \in \Gamma$. We will find a finite set $F \subseteq \Gamma$ such that

$$\forall 1 \leq i \leq n \left(\frac{|F \Delta \gamma_i F|}{|F|} < \epsilon\right).$$

Fix $f \in l^1(\Gamma)$ such that $f \geq 0$, $||f||_1 = 1$, and

$$\forall 1 \leq i \leq n \left(||f - \gamma_i \cdot f||_1 < (\epsilon/n)||f||_1 = \epsilon/n\right).$$

Consider the truncation function

$$E_a(t) = \begin{cases} 1 \text{ if } t \geq a, \\ 0 \text{ if } t < a. \end{cases}$$

Note that $\int_0^\infty E_a(t)\ da = t$, for $t \geq 0$, and also

$$\forall t, t' \geq 0 \left(|t - t'| = \int_0^\infty |E_a(t) - E_a(t')|\ da \right).$$

Setting

$$\begin{aligned} f_a(\gamma) &= E_a \circ f(\gamma) \\ &= \begin{cases} 1 \text{ if } f(\gamma) \geq a, \\ 0 \text{ if } f(\gamma) < a \end{cases} \\ &= 1_{\{\gamma \in \Gamma : f(\gamma) \geq a\}}, \end{aligned}$$

it follows that

$$\begin{aligned} \int_0^\infty \sum_{\gamma \in \Gamma} |f_a(\gamma) - \gamma_i \cdot f_a(\gamma)|\ da &= \sum_{\gamma \in \Gamma} \int_0^\infty |f_a(\gamma) - \gamma_i \cdot f_a(\gamma)|\ da \\ &= \sum_{\gamma \in \Gamma} |f(\gamma) - \gamma_i \cdot f(\gamma)| \\ &= ||f - \gamma_i \cdot f||_1 \\ &< (\epsilon/n)||f||_1 \\ &= (\epsilon/n) \sum_{\gamma \in \Gamma} f(\gamma) \\ &= (\epsilon/n) \sum_{\gamma \in \Gamma} \int_0^\infty f_a(\gamma)\ da \\ &= (\epsilon/n) \int_0^\infty \sum_{\gamma \in \Gamma} f_a(\gamma)\ da \\ &= (\epsilon/n) \int_0^\infty ||f_a||_1\ da, \end{aligned}$$

thus for some $a > 0$ we must have

$$\forall 1 \leq i \leq n (||f_a - \gamma_i \cdot f_a||_1 < \epsilon ||f_a||_1).$$

Setting $F = \{\gamma : f_a(\gamma) = 1\} = \{\gamma : f(\gamma) \geq a\}$, it follows that F is a non-empty, finite subset of Γ, and

$$|F \Delta \gamma_i F| = ||f_a - \gamma_i \cdot f_a||_1 < \epsilon ||f_a||_1 = \epsilon |F|,$$

which completes the proof of Theorem 5.14. $\dashv$

Proposition 5.17. *Every countable amenable group Γ admits an increasing, exhaustive Følner sequence.*

Proof. Clearly we may assume that Γ is infinite. Let $\{F'_n\}_{n\in\mathbb{N}}$ be a Følner sequence for Γ. We begin by noting that $|F'_n| \to \infty$. To see this, fix $k \in \mathbb{N}$, let $\gamma_1, \dots, \gamma_k$ be distinct elements of Γ, and choose $N \in \mathbb{N}$ sufficiently large that

$$\forall n \geq N \forall 1 \leq i \leq k \left(|\gamma_i F'_n \Delta F'_n|/|F'_n| \leq 1/k\right).$$

Given $n \geq N$, it follows that if any of the above ratios is non-zero, then $|F'_n| \geq k$. But if all of the ratios are zero, then $\gamma_1\gamma, \dots, \gamma_k\gamma$ are distinct elements of F'_n, for any $\gamma \in F'_n$, so again $|F'_n| \geq k$.

Thus, by passing to a subsequence, we may assume that for all $n > 0$,

$$|F'_n| > 2n\left(n + 1 + \sum_{i<n} |F'_i|\right).$$

Let $\{\gamma_n\}_{n\in\mathbb{N}}$ be an enumeration of Γ, and define

$$F_n = \{\gamma_0, \dots, \gamma_n\} \cup (F'_0 \cup \dots \cup F'_n).$$

Noting that for all $\gamma \in \Gamma$,

$$\begin{aligned}
\frac{|\gamma F_n \Delta F_n|}{|F_n|} &\leq \frac{2(n+1) + \sum_{i\leq n} |\gamma F'_i \Delta F'_i|}{|F'_n|} \\
&\leq \frac{2\left(n+1+\sum_{i<n} |F'_i|\right)}{|F'_n|} + \frac{|\gamma F'_n \Delta F'_n|}{|F'_n|} \\
&< \frac{1}{n} + \frac{|\gamma F'_n \Delta F'_n|}{|F'_n|} \\
&\to 0,
\end{aligned}$$

it follows that $\{F_n\}_{n\in\mathbb{N}}$ is an increasing, exhaustive Følner sequence. ⊣

A *graph* G with vertex set X is a set $G \subseteq X^2$ such that $(x,x) \notin G$ and $(x,y) \in G \Leftrightarrow (y,x) \in G$, for all $x, y \in X$. The *boundary* of $F \subseteq X$ with respect to G is

$$\partial_G F = \partial F = \{x \in F : \exists y \notin F((x,y) \in G)\}.$$

The graph G is *amenable* if for all $\epsilon > 0$, there is a non-empty finite set $F \subseteq X$ such that

$$|\partial F|/|F| < \epsilon,$$

or equivalently, if there is a sequence $\{F_n\}_{n\in\mathbb{N}}$ of non-empty, finite subsets of X such that $|\partial F_n|/|F_n| \to 0$. Such a sequence is called a *Følner sequence* for the graph.

Proposition 5.18. *Every countable amenable graph with no finite connected components admits an increasing, exhaustive Følner sequence.*

Proof. Suppose G is such a graph, let $\{F'_n\}_{n\in\mathbb{N}}$ be a Følner sequence for G, and note that $|F'_n| \to \infty$, since G has no finite connected components. Thus, by passing to a subsequence, we may assume that for all $n > 0$,

$$|F'_n| > n \left(n + 1 + \sum_{i<n} |\partial F'_i| \right),$$

for all $n \in \mathbb{N}$. Let $\{x_n\}_{n\in\mathbb{N}}$ be an enumeration of X, and define

$$F_n = \{x_0, \ldots, x_n\} \cup (F'_0 \cup \cdots \cup F'_n).$$

Noting that

$$\begin{aligned}
\frac{|\partial F_n|}{|F_n|} &\leq \frac{n + 1 + \sum_{i\leq n} |\partial F'_i|}{|F'_n|} \\
&\leq \frac{n + 1 + \sum_{i<n} |\partial F'_i|}{|F'_n|} + \frac{|\partial F'_n|}{|F'_n|} \\
&\leq \frac{1}{n} + \frac{|\partial F'_n|}{|F'_n|} \\
&\to 0,
\end{aligned}$$

it follows that $\{F_n\}_{n\in\mathbb{N}}$ is an increasing, exhaustive Følner sequence. ⊣

Given a countable group Γ and $\gamma_1, \ldots, \gamma_n \in \Gamma$, the *Cayley graph* induced by $\gamma_1, \ldots, \gamma_n$ is the graph, with vertex set $\langle \gamma_1, \ldots, \gamma_n \rangle$, given by

$$(\gamma, \delta) \in C_{\gamma_1,\ldots,\gamma_n} \Leftrightarrow \exists 1 \leq i \leq n(\gamma\gamma_i^{\pm 1} = \delta).$$

Proposition 5.19. *The following are equivalent:*

1. *Γ is amenable.*
2. *For all $\gamma_1, \ldots, \gamma_n \in \Gamma$, $C_{\gamma_1,\ldots,\gamma_n}$ is amenable.*

Proof. By Proposition 5.6, it suffices to show that if $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$, then

$$\Gamma \text{ is amenable } \Leftrightarrow C_{\gamma_1,\ldots,\gamma_n} \text{ is amenable.}$$

The main observation here is that

$$|\partial F| \leq \left| \bigcup_{1\leq i\leq n, \delta\in\{\pm 1\}} F\gamma_i^\delta \Delta F \right| \leq (2n + 1)|\partial F|.$$

To see this, simply observe that

$$\partial F = \bigcup_{1\leq i\leq n, \delta\in\{\pm 1\}} F \setminus F\gamma_i^\delta$$

by the definition of boundary, and

$$\left| \bigcup_{1 \leq i \leq n, \delta \in \{\pm 1\}} F\gamma_i^\delta \setminus F \right| \leq 2n \left| \bigcup_{1 \leq i \leq n, \delta \in \{\pm 1\}} F \setminus F\gamma_i^\delta \right|,$$

as every vertex of $C_{\gamma_1,\ldots,\gamma_n}$ is of degree $\leq 2n$.

To see that if Γ is amenable, then $C_{\gamma_1,\ldots,\gamma_n}$ is amenable, fix $\epsilon > 0$, find a finite set $F \subseteq \Gamma$ such that

$$\forall 1 \leq i \leq n \forall \delta \in \{\pm 1\} \left(|F\gamma_i^\delta \Delta F|/|F| < \epsilon/2n \right),$$

and observe that

$$|\partial F|/|F| \leq \sum_{1 \leq i \leq n, \delta \in \{\pm 1\}} |F\gamma_i^\delta \Delta F|/|F| < \epsilon.$$

To see that if $C_{\gamma_1,\ldots,\gamma_n}$ is amenable, then Γ is amenable, fix $\epsilon > 0$, find a finite set $F \subseteq \Gamma$ such that

$$|\partial F|/|F| < \epsilon/(2n+1),$$

and observe that

$$|F\gamma_i^\delta \Delta F|/|F| \leq (2n+1)|\partial F|/|F| < \epsilon,$$

for all $1 \leq i \leq n$ and $\delta \in \{\pm 1\}$. Thus Γ is amenable by Remark 5.12. $\dashv$

6 Hyperfiniteness

Our goal is to show that any two non-atomic probability measure preserving ergodic actions of amenable groups are orbit equivalent. This will be done in two steps:

1. (Dye) Any two non-atomic probability measure preserving ergodic actions of $\mathbb{Z}$ are orbit equivalent.
2. (Ornstein-Weiss) Any probability measure preserving action of an amenable group is orbit equivalent to a probability measure preserving action of $\mathbb{Z}$.

We will start with step 1, by studying, in some detail, the equivalence relations induced by $\mathbb{Z}$-actions.

A Borel equivalence relation E on X is *smooth* if it admits a *Borel separating family*, i.e., a sequence $\{B_n\}_{n\in\mathbb{N}}$ of Borel subsets of X such that

$$\forall x, y \in X \left(xEy \Leftrightarrow \forall n \in \mathbb{N}(x \in B_n \Leftrightarrow y \in B_n) \right).$$

By considering the map $x \mapsto \{1_{B_n}(x)\}_{n\in\mathbb{N}}$, it is easy to see that E is smooth if and only if there is a Borel function $\phi : X \to Y$, with Y standard Borel, such that

$$\forall x, y \in X \, (xEy \Leftrightarrow \phi(x) = \phi(y)) \, .$$

An invariant Borel set $A \subseteq X$ is *smooth* (for E) if $E|A$ is smooth.

Example 6.1. Every finite Borel equivalence relation E is smooth. To see this, let $<$ be a Borel linear ordering of X, and define $\phi : X \to X$ by

$$\phi(x) = \text{the } <\text{-least element of } [x]_E.$$

Remark 6.2. Suppose E is countable and smooth, as witnessed by $\phi : X \to Y$. As ϕ is countable-to-1, it follows from Exercise 18.14 of [K] that $\mathrm{rng}(\phi)$ is Borel. Thus, we may assume that ϕ is surjective.

Given any Borel equivalence relation E on X, we use X/E to denote the set of equivalence classes of E, equipped with the *quotient Borel structure* (i.e., $A \subseteq X/E$ is in the quotient Borel structure iff $\bigcup A$ is Borel in X).

Proposition 6.3. *A countable Borel equivalence relation E is smooth if and only if its quotient X/E is standard Borel.*

Proof. To see ($\Rightarrow$), suppose that $\phi : X \to Y$ is a surjection verifying the smoothness of E and let $\tilde{\phi}$ be the induced bijection from X/E to Y. It suffices to check that $\tilde{\phi}$ induces an isomorphism of the quotient Borel structure of X/E with the standard Borel structure of Y. Noting that ϕ is countable-to-1, it follows from Exercise 18.14 of [K] that ϕ sends Borel sets to Borel sets. Thus, an E-invariant set B is Borel if and only if $\tilde{\phi}(B) = \phi(B)$ is Borel.

To see ($\Leftarrow$), note that if the quotient Borel structure on X/E is standard Borel, then in particular, we can find a countable family $\{B_n\}_{n\in\mathbb{N}}$ of sets in the quotient Borel structure on X/E which separates points. But any such family clearly induces a Borel separating family for E. ⊣

A *complete section* for E is a set $B \subseteq X$ which intersects every class of E. A *transversal* for E is a set $B \subseteq X$ which intersects each class of E in exactly one point. A *selector* for E is a function $f : X \to X$, whose graph is contained in E, and whose image is a transversal. Clearly a Borel equivalence relation admits a Borel transversal iff it admits a Borel selector, and if it admits a Borel transversal, then it is smooth (but not conversely, see [K], 18.D).

Proposition 6.4. *If E is countable Borel and smooth, then E admits a Borel selector. Moreover, if all of the classes of E are of size $n \in \mathbb{N} \cup \{\infty\}$, then there is a sequence $\{f_i\}_{i<n}$ of Borel selectors for E, whose images partition X.*

Proof. Let $\phi : X \to Y$ be a surjection witnessing that E is smooth, and define $R \subseteq Y \times X$ by

$$(y, x) \in R \Leftrightarrow \phi(x) = y.$$

As R is Borel and has countable sections, it follows from Theorem 18.10 of [K] that there is a Borel uniformization $f : Y \to X$ of R. It follows that $f \circ \phi$ is the desired Borel selector.

Now suppose all of the classes of E are of size $n \in \mathbb{N} \cup \{\infty\}$. Then it follows from 18.15 of [K] that there is a sequence $\{f_i\}_{i<n}$ of Borel uniformizations of R with disjoint ranges whose union is X, thus $\{f_i \circ \phi\}_{i<n}$ is a sequence of Borel selectors whose images partition X. ⊣

A countable Borel equivalence relation E on X is *hyperfinite* if there is an increasing, exhaustive sequence $\{F_n\}_{n\in\mathbb{N}}$ of finite Borel subequivalence relations of E, i.e., if there exists a sequence of finite Borel equivalence relations $F_0 \subseteq F_1 \subseteq \cdots$ with $\bigcup_n F_n = E$.

Example 6.5. The equivalence relation E_0, on $2^{\mathbb{N}}$, given by

$$xE_0y \Leftrightarrow \exists n \in \mathbb{N} \forall m \geq n(x_m = y_m),$$

is hyperfinite, as witnessed by the equivalence relations

$$xF_ny \Leftrightarrow \forall m \geq n(x_m = y_m).$$

But E_0 is not smooth. To see this, notice that if $\Delta = \mathbb{Z}_2^{<\mathbb{N}}, G = \mathbb{Z}_2^{\mathbb{N}}$ are as in Example 3.2 and Δ acts by translation on G, then the corresponding equivalence relation is E_0. It follows that μ (as in 3.2 again) is E_0-ergodic, which easily implies that there can be no Borel transversal T for E_0, since then the probability measure ν on T defined by $\nu(A) = \mu([A]_{E_0})$ would be non-atomic and 2-valued, which is clearly impossible. (Here we use $[A]_E$ to denote $\{x \in X : \exists y \in A(xEy)\}$, the *E-saturation* of A.)

Theorem 6.6 (Slaman-Steel [S^2], Weiss [We]). *Suppose E is a Borel equivalence relation. The following are equivalent:*

1. *E is hyperfinite.*
2. *E is induced by a Borel $\mathbb{Z}$-action.*

Proof. To see (1) $\Rightarrow$ (2), we begin with the case that E is smooth. By breaking E into countably many invariant Borel pieces, we may assume that every E-class is of cardinality $n \in \mathbb{N} \cup \{\infty\}$. Suppose $n \in \mathbb{N}$. Fix a Borel linear ordering $<$ of X. Letting x^+ denote the successor of x in the restriction of $<$ to $[x]_E$, it follows that

$$T(x) = \begin{cases} x^+ & \text{if } \exists y \in [x]_E(x < y), \\ \min_<[x]_E & \text{otherwise,} \end{cases}$$

is as desired. If $n = \infty$, then we can partition X into Borel transversals $\{B_n\}_{n\in\mathbb{Z}}$ of E. Put $T(x) = y$, whenever xEy, $x \in B_n$, and $y \in B_{n+1}$. Then T is a Borel automorphism that induces E.

Now suppose $\{F_n\}_{n\in\mathbb{N}}$ is an increasing, exhaustive sequence of finite Borel subequivalence relations of E, with $F_0 = \varDelta_X =$ equality on X. As X/F_n is standard, we can find a Borel linear ordering $<_n$ of X/F_n. Given distinct xEy, let $n(x, y)$ denote the largest $n \in \mathbb{N}$ such that $(x, y) \notin F_n$, and define

$$x < y \Leftrightarrow x \neq y \text{ and } xEy \text{ and } [x]_{F_{n(x,y)}} <_{n(x,y)} [y]_{F_{n(x,y)}}.$$

It is clear that $<$ discretely orders the classes of E. The set of E-classes on which $<$ has a least or largest element clearly forms a Borel invariant set on which E is smooth. So we may assume that $<$ is a $\mathbb{Z}$-ordering on each of the classes of E. It follows that the successor function for $<$ is the desired Borel automorphism that induces E.

To prove (2) $\Rightarrow$ (1), we first need some preliminaries. Given any Borel equivalence relation E, a *vanishing sequence of markers* is a decreasing, vanishing (i.e., having empty intersection) sequence of Borel complete sections for E. A countable Borel equivalence relation is *aperiodic* if its classes are infinite. We now have:

Lemma 6.7 (Marker Lemma). *Every aperiodic countable Borel equivalence relation admits a vanishing sequence of markers.*

Proof. (Slaman-Steel) We can assume our equivalence relation E lives on $X = 2^{\mathbb{N}}$. Let $s_n(x)$ denote the lexicographically least $s \in 2^n$ such that $|[x]_E \cap \mathcal{N}_s| = \infty$, and define

$$x \in A_n \Leftrightarrow x|n = s_n(x).$$

It is clear that $\{A_n\}_{n\in\mathbb{N}}$ is a decreasing sequence of Borel sets, each of which intersects every class of E in infinitely many points. Noting that

$$A = \bigcap_{n\in\mathbb{N}} A_n$$

intersects each class of E in at most one point, it follows that $B_n = A_n \setminus A$ is as desired. $\dashv$

Now suppose, without loss of generality, that E is induced by an aperiodic Borel automorphism T. Note that on any smooth for E invariant Borel set it is easy to find an increasing, exhaustive sequence of finite Borel subequivalence relations. In each E-class C we define the $\mathbb{Z}$-ordering

$$x <_C y \Leftrightarrow \exists n > 0 (T^n(x) = y).$$

Then we can assume that there is a vanishing sequence of markers $\{B_n\}_{n\in\mathbb{N}}$ such that for each such C, $B_n \cap C$ is unbounded in both directions in $<_C$. This is because, starting with a vanishing sequence of markers $\{A_n\}_{n\in\mathbb{N}}$ for E, the union of the set of all classes C in which $A_n \cap C$ has a least or largest element in $<_C$, for some $n \in \mathbb{N}$, is an invariant Borel set on which E is smooth.

Let F_n be the finite subequivalence relation of E defined by

$$xF_ny \Leftrightarrow x = y \text{ or } \left(xEy \text{ and if } C = [x]_E = [y]_E, \text{ then } [x,y] \text{ avoids } B_n\right),$$

where $[x,y]$ denotes the closed interval between x, y in $<_C$. It is clear that $\{F_n\}_{n\in\mathbb{N}}$ is an increasing, exhaustive sequence of finite Borel subequivalence relations of E. ⊣

Remark 6.8. It follows that an aperiodic countable Borel equivalence relation is hyperfinite exactly when there is a Borel assignment of $\mathbb{Z}$-orderings to its classes. In fact, suppose that $\mathcal{L} \subseteq E$ is a Borel forest of lines, such that any two E-related points x, y are connected via a path through $\mathcal{L}$ (so $\mathcal{L}$ arranges the classes of E like $\mathbb{Z}$-orderings, but there is no distinguished direction.) Then the above proof shows that E is hyperfinite.

It should be noted, however, that not every such $\mathcal{L}$ is induced by a Borel automorphism. (An example of this sort was first pointed out by S. Adams.) That is, one can find $\mathcal{L} \subseteq E$ as above, such that for every Borel automorphism T of the underlying space, there is some $(x,y) \in \mathcal{L}$ such that $y \notin \{T^{\pm 1}(x)\}$. To see this, let σ be the *odometer* on $2^{\mathbb{N}}$, that is, the automorphism given by

$$\sigma(x) = \begin{cases} 0^n 1y \text{ if } x = 1^n 0y, \\ 00\ldots \text{ if } x = 11\ldots. \end{cases}$$

Intuitively, σ is "addition by $100\ldots$ with right carry". Note that, off of the eventually constant sequences, the equivalence relation induced by σ is E_0. Let $i : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the map which flips each digit of its input, and let F be the equivalence relation induced by i. As i is an involution, F is smooth. Set $E = (E_\sigma \vee F)/F$ (here $E_\sigma \vee F$ denotes the smallest equivalence relation containing E_σ, F), and define $\mathcal{L} \subseteq E$ by

$$([x]_F, [y]_F) \in \mathcal{L} \Leftrightarrow \exists m \in \{0,1\}, n \in \{\pm 1\}(x = i^m \circ \sigma^n(y)).$$

It follows easily from the fact that $i \circ \sigma = \sigma^{-1} \circ i$ that $\mathcal{L}$ is a Borel forest of lines, and that any two E-related points are connected via a path through $\mathcal{L}$. Now suppose that $T : 2^{\mathbb{N}}/F \to 2^{\mathbb{N}}/F$ is a Borel automorphism, and let A_1 be the E-invariant set on which T induces $\mathcal{L}$, i.e.,

$$u \in A_1 \Leftrightarrow \forall (y,z) \in \mathcal{L} \cap ([u]_E \times [u]_E)(y \in \{T^{\pm 1}(z)\}),$$

and let $A \subseteq 2^{\mathbb{N}}$ be defined by

$$x \in A \Leftrightarrow [x]_F \in A_1,$$

so that A is Borel and $(E_\sigma \vee F)$-invariant. It is enough to show that if μ is the usual product measure on $2^{\mathbb{N}}$, then A is μ-null. Otherwise, since μ is E_σ-ergodic, so $(E_\sigma \vee F)$-ergodic, $\mu(A) = 1$. Define $B \subseteq A$ by

$$x \in B \Leftrightarrow \sigma(x) \in T([x]_F).$$

It is easily verified that B is E_σ-invariant and consists of exactly one E_σ-class of every $(E_\sigma \vee F)|A$-class, thus $B \cap i(B) = \emptyset$ and $B \cup i(B) = A$, so $\mu(B) = 1/2$. Since B is E_σ-invariant, this is a contradiction. See [M] for more on Borel forests of lines.

Proposition 6.9. *Below E and F are countable Borel equivalence relations and B is a Borel set.*

(i) If $E \subseteq F$ and F is hyperfinite, then E is hyperfinite.
(ii) If E is hyperfinite, then $E|B$ is hyperfinite. If B is a complete section for E and $E|B$ is hyperfinite, then E is hyperfinite.
(iii) If E and F are hyperfinite, then $E \times F$ is hyperfinite.

Proof. The proofs of (i), (iii), and the first part of (ii) are straightforward. To see the second part of (ii), let $\{F_n\}_{n\in\mathbb{N}}$ be an increasing, exhaustive sequence of finite Borel subequivalence relations of $E|B$, let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of Borel involutions whose graphs cover E, let $n(x)$ be the least $n \in \mathbb{N}$ such that $f_n(x) \in B$, and set $f(x) = f_{n(x)}(x)$. Defining $F'_n \subseteq E$ by

$$xF'_n y \Leftrightarrow x = y \text{ or } (f(x)F_n f(y) \text{ and } n(x), n(y) \leq n),$$

it follows that $\{F'_n\}_{n\in\mathbb{N}}$ is an increasing, exhaustive sequence of finite Borel subequivalence relations of E, thus E is hyperfinite. $\dashv$

Remark 6.10. If E is hyperfinite, then we can find an increasing, exhaustive sequence $\{F_n\}_{n\in\mathbb{N}}$ of finite subequivalence relations of E such that $|[x]_{E_n}| \leq n$, for all $n \in \mathbb{N}$ and $x \in X$. To see this assume, without loss of generality, that E is aperiodic, let $\{F'_n\}_{n\in\mathbb{N}}$ be an increasing, exhaustive sequence of finite Borel subequivalence relations of E with $F'_0 = \Delta_X$, and for each $x \in X$ and $n \in \mathbb{N}$, define

$$k_n(x) = \max\{k \in \mathbb{N} : |[x]_{F'_k}| \leq n\}.$$

Define $F_n \subseteq E$ by

$$xF_n y \Leftrightarrow xF'_{k_n(x)}y.$$

Noting that $xF_n y \Rightarrow k_n(x) = k_n(y)$, it follows that $\{F_n\}_{n\in\mathbb{N}}$ is as desired.

Given (X, E, μ), we call E *hyperfinite μ-a.e.* if there is an invariant conull Borel set $A \subseteq X$ such that $E|A$ is hyperfinite.

Theorem 6.11 (Dye [D], Krieger [Kr]). *Suppose E is the union of an increasing sequence of hyperfinite Borel equivalence relations and $\mu \in P(X)$. Then E is hyperfinite μ-a.e.*

Proof. The proof hinges on the following lemma.

Lemma 6.12. *Suppose E is the union of an increasing sequence of hyperfinite Borel equivalence relations, $R \subseteq E$ is a Borel subrelation of E with finite* *sections (not necessarily an equivalence relation), $\mu \in P(X)$, and $\epsilon > 0$. Then there is a finite Borel subequivalence relation $F \subseteq E$ such that*

$$\mu(\{x \in X : R_x \subseteq [x]_F\}) > 1 - \epsilon,$$

where $R_x = \{y \in X : (x, y) \in R\}$.

Proof. Let $\{E_n\}_{n \in \mathbb{N}}$ be an increasing, exhaustive sequence of hyperfinite subequivalence relations of E, and fix $n \in \mathbb{N}$ sufficiently large such that

$$\mu(\{x \in X : R_x \subseteq [x]_{E_n}\}) > 1 - \epsilon.$$

Now let $\{F_m\}_{m \in \mathbb{N}}$ be an increasing, exhaustive sequence of finite Borel subequivalence relations of E_n, choose $m \in \mathbb{N}$ sufficiently large that

$$\mu(\{x \in X : R_x \subseteq [x]_{F_m}\}) > 1 - \epsilon,$$

and set $F = F_m$. $\dashv$

By 1.3, we can find an increasing, exhaustive sequence $\{R_n\}_{n \in \mathbb{N}}$ of Borel subrelations of E with finite sections. Set $F_0 = \Delta_X$, and define $\{F_n\}_{n \in \mathbb{N}}$ recursively, as follows. Given a finite Borel subequivalence relation $F_n \subseteq E$, it follows from Proposition 6.9 that E/F_n is the union of an increasing sequence of hyperfinite subequivalence relations. By Lemma 6.12, we can find another finite Borel subequivalence relation F of E/F_n, such that

$$\mu_{F_n}(\{[x]_{F_n} \in X/F_n : ([R_n]_{F_n \times F_n})_{[x]_{F_n}} \subseteq [[x]_{F_n}]_F\}) > 1 - 1/n,$$

where μ_{F_n} is the measure, on X/F_n, induced by μ. Letting F_{n+1} be the equivalence relation on X induced by F, it follows that

$$\left\{x \in X : [x]_E \not\subseteq \bigcup_{n \in \mathbb{N}} [x]_{F_n}\right\} = \bigcup_{n \in \mathbb{N}} \bigcap_{m > n} \{x \in X : (R_m)_x \not\subseteq [x]_{F_{m+1}}\},$$

and this latter set is clearly null. As $\{F_n\}_{n \in \mathbb{N}}$ is clearly increasing, it follows that E is hyperfinite a.e. $\dashv$

Problem 6.13. Is the union of an increasing sequence of hyperfinite Borel equivalence relations always hyperfinite?

7 Dye's Theorem

In this section, we show

Theorem 7.1 (Dye [D]). *All non-atomic probability measure preserving ergodic actions of $\mathbb{Z}$ are orbit equivalent.*

Remark 7.2. In view of 6.6, this can be equivalently formulated as follows. Let (X, E, μ), (Y, F, ν) be almost everywhere aperiodic hyperfinite Borel equivalence relations with invariant, ergodic probability measures. Then they are isomorphic.

Note that in the presence of the other conditions, almost everywhere aperiodicity is equivalent to saying that μ, ν are non-atomic.

Before we present the proof of Theorem 7.1, we develop some preliminaries which are also important in their own right. An *fsr (finite partial subequivalence relation)* of a countable Borel equivalence relation E is a finite Borel equivalence relation F, defined on a Borel set $\mathrm{dom}(F) \subseteq X$, such that $F \subseteq E$. Let $[X]^{<\infty}$ denote the standard Borel space of finite subsets of X, and let $[E]^{<\infty}$ denote the Borel subset of $[X]^{<\infty}$ of pairwise E-related finite subsets of X. Given a set $\Phi \subseteq [E]^{<\infty}$, we say that an fsr $F \subseteq E$ is *Φ-maximal* if

1. $\forall x \in \mathrm{dom}(F)([x]_F \in \Phi)$ and
2. $\forall S \in [X \setminus \mathrm{dom}(F)]^{<\infty}(S \notin \Phi)$.

Lemma 7.3. *Suppose E is a countable Borel equivalence relation and $\Phi \subseteq [E]^{<\infty}$ is Borel. Then E admits a Φ-maximal fsr.*

Proof. Define a graph G on $[E]^{<\infty}$ by

$$(S, T) \in G \Leftrightarrow S \neq T \text{ and } S \cap T \neq \emptyset.$$

We begin by noting that G admits a Borel $\aleph_0$-*coloring*, i.e., a Borel map $c : [E]^{<\infty} \to I$, with I a countable (discrete) set, such that

$$(S, T) \in G \Rightarrow c(S) \neq c(T).$$

Let $\{g_n\}_{n \in \mathbb{N}}$ be a sequence of Borel involutions whose graphs cover E, and let $<$ be a Borel linear ordering of X. Given $S \in [E]^{<\infty}$, let $\{x_i\}_{i<n}$ be the $<$-increasing enumeration of S, and let $c(S)$ be the lexicographically least sequence $\{k_{ij}\}_{i,j<|S|}$ of natural numbers such that

$$\forall i, j < |S| \left(g_{k_{ij}} \cdot x_i = x_j\right).$$

Now suppose, towards a contradiction, that c is not a coloring. Then we can find $(S, T) \in G$ such that $c(S) = c(T)$. Clearly $|S| = |T|$, so let $\{x_i\}_{i<n}$, $\{y_i\}_{i<n}$ be the $<$-increasing enumerations of S, T, and fix $i, j < n$ such that $x_i = y_j$. Then

$$\begin{aligned} i < j &\Leftrightarrow x_i < x_j \\ &\Leftrightarrow x_i < g_{k_{ij}}(x_i) \\ &\Leftrightarrow y_j < g_{k_{ij}}(y_j) \\ &\Leftrightarrow y_j < y_i \\ &\Leftrightarrow j < i, \end{aligned}$$

thus $i = j$, and $x_i = y_i$. It follows that for all $l < n$,

$$\begin{aligned} x_l &= g_{k_{il}}(x_i) \\ &= g_{k_{il}}(y_i) \\ &= y_l, \end{aligned}$$

thus $S = T$, contradicting our assumption that $(S, T) \in G$.

We can of course assume that $c : [E]^{<\infty} \to \mathbb{N}$. Now recursively define a sequence $\{F_n\}_{n \in \mathbb{N}}$ of fsr's by putting $x F_n y$ if

$$\exists S \in \Phi \left(c(S) = n \text{ and } x, y \in S \text{ and } S \cap \bigcup_{m<n} \operatorname{dom}(F_m) = \emptyset \right).$$

Noting that $\operatorname{dom}(F_m) \cap \operatorname{dom}(F_n) = \emptyset$ for $m \neq n$, it follows that

$$F = \bigcup_{n \in \mathbb{N}} F_n$$

is an fsr of E. It is also clear that $[x]_F \in \Phi$, for all $x \in \operatorname{dom}(F)$. Now suppose that $S \in [E]^{<\infty}$ and $S \in \Phi$. Then $c(S) = n$ for some $n \in \mathbb{N}$, and it follows that S intersects $\operatorname{dom}(F_m)$, for some $m \leq n$, thus F is Φ-maximal. ⊣

One useful corollary of this lemma is the following.

Proposition 7.4. *Suppose $n > 0$ and E is an aperiodic countable Borel equivalence relation. Then E contains a finite Borel subequivalence relation, all of whose classes are of cardinality n.*

Proof. Say E lives on X. By Proposition 7.3, we can find a maximal fsr F whose classes are of cardinality n. Let

$$A = \{x \in X : [x]_E \text{ is the union of } F\text{-classes}\}.$$

It is clear that $C \setminus \operatorname{dom}(F)$ has at most $n-1$ elements, for each E-class C. So $X \setminus A$ is a smooth set and we can easily define a finite Borel subequivalence relation $E' \subseteq E|(X \setminus A)$, whose classes are of cardinality n. Then $(F|A) \cup E'|(X \setminus A)$ is as desired. ⊣

Theorem 7.5 (Rokhlin's Lemma). *Suppose T is an aperiodic Borel automorphism, $\mu \in P(X)$, $n \geq 1$, and $\epsilon > 0$. Then there is a Borel complete section $A \subseteq X$ such that*

(i) $T^i(A) \cap T^j(A) = \emptyset$, *if* $0 \leq i < j < n$,
(ii) $\mu(X \setminus \bigcup_{i<n} T^i(A)) < \epsilon$, *and*
(iii) $\mu(A) \leq 1/n$.

Proof. Let $\{A_m\}_{m \in \mathbb{N}}$ be a vanishing sequence of markers. As it is clear how to proceed when E_T, the equivalence relation induced by T, is smooth, we may assume that each A_m is *doubly recurrent* for T, i.e.,

$$\forall x \in A_m \exists i < 0 < j (T^i(x), T^j(x) \in A_m).$$

Given a doubly recurrent Borel complete section $B \subseteq X$, define

$$d_l(x, B) = \min\{k \in \mathbb{N} : T^{-k}(x) \in B\}, d_r(x, B) = \min\{k \in \mathbb{N} : T^k(x) \in B\}.$$

Now define

$$x \in B_m \Leftrightarrow d_l(x, A_m) < n \text{ or } d_r(x, A_m) < n.$$

As the A_m's are vanishing, it follows that the B_m's are vanishing as well. In particular, we can find $m \in \mathbb{N}$ such that $\mu(B_m) < \epsilon$. For each $k < n$, put

$$x \in C_k \Leftrightarrow d_l(x, A_m) \cong k (\text{mod } n) \text{ and } d_r(x, A_m) \geq n.$$

As these sets are pairwise disjoint, it follows that $\mu(C_k) \leq 1/n$, for some $k < n$. It is clear that $A = C_k$ satisfies (i), and (ii) follows from the fact that $X \setminus B_m \subseteq \bigcup_{i<n} T^i(A)$. ⊣

For Borel automorphisms S, T on X, $\mu \in P(X)$, let

$$d_\mu(S, T) = \mu(\{x \in X : S(x) \neq T(x)\}).$$

Corollary 7.6. *Suppose T is an aperiodic Borel automorphism on X, μ is a non-atomic probability measure on X, $n \geq 1$, and $\epsilon > 0$. Then there is a Borel automorphism S in $[E_T]$, of period exactly n, such that $d_\mu(S, T) \leq 1/n + \epsilon$.*

Proof. Let $A' \subseteq X$ be the set obtained by applying Theorem 7.5 to T^{-1}, set $A = T^{-(n-1)}(A')$, and observe that

(i) $T^i(A) \cap T^j(A) = \emptyset$, if $0 \leq i < j < n$,
(ii) $\mu(X \setminus B) < \epsilon$, where $B = \bigcup_{i<n} T^i(A)$, and
(iii) $\mu(T^{n-1}(A)) \leq 1/n$.

Define $T_1 : B \to B$ by

$$T_1(x) = \begin{cases} T(x) & \text{if } x \notin T^{n-1}(A), \\ T^{-(n-1)}(x) & \text{otherwise.} \end{cases}$$

As it is clear how to proceed when E_T is smooth, we may assume that $X \setminus B$ intersects every class of E_T in infinitely many points. It follows from Proposition 7.4 that we can find a Borel equivalence relation $F \subseteq E|(X \setminus B)$, all of whose classes are of cardinality n. We can clearly find a Borel automorphism T_2 of period n which generates F. It follows that

$$T(x) = \begin{cases} T_1(x) & \text{if } x \in B, \\ T_2(x) & \text{otherwise,} \end{cases}$$

is as desired. ⊣

By using a little ergodic theory, we can strengthen Theorem 7.5. A Borel set $B \subseteq X$ is *(ϵ, n)-Rokhlin* if $\{T^i(B)\}_{i<n}$ partitions a set of measure $> 1 - \epsilon$.

Proposition 7.7. *Suppose T is an aperiodic Borel automorphism on X, μ is a T-invariant probability measure, $n \geq 1$, $\epsilon > 0$, and $\mu(A) > 1 - \epsilon$. Then A contains an (ϵ, n)-Rokhlin set.*

Proof. By the Birkhoff Ergodic Theorem,

$$m_A(x) = \lim_{k \to \infty} \frac{|A \cap \{T^i(x)\}_{i<k-n}|}{k}$$

exists μ-a.e., is T-invariant, and

$$\mu(A) = \int m_A \, d\mu.$$

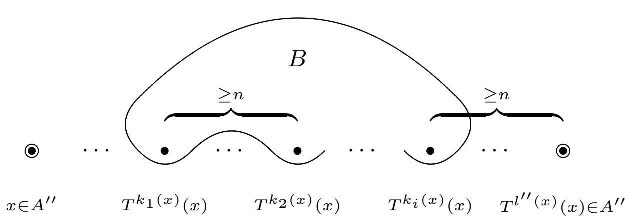
Fix $0 < \delta < (\mu(A) - (1 - \epsilon))/2$ and for almost all $x \in X$, put

$$l(x) = \min \left\{ l > n : \forall k \geq l \ \left| m_A(x) - \frac{|A \cap \{T^i(x)\}_{i<k-n}|}{k} \right| < \delta \right\}.$$

Choose $l \in \mathbb{N}$ sufficiently large that

$$A' = \{x \in X : l(x) \leq l\}$$

is of measure $> 1 - \delta$. By throwing away a null set, we can assume that A' is doubly recurrent for T.



Let $\{A'_k\}_{k \in \mathbb{N}}$ be a vanishing sequence of markers for $E_T|A'$. Again, by throwing away a null set we may assume that each A'_k is doubly recurrent for T. For each $k \in \mathbb{N}$ and $x \in A'_k$, let $l'_k(x)$ be the least positive natural number such that $T^{l'_k(x)}(x) \in A'_k$, choose $k \in \mathbb{N}$ sufficiently large that the E_T-saturation of

$$A'' = \{x \in A'_k : l'_k(x) \geq l\}$$

is of measure $> 1 - \delta$, and let $l''(x)$ be the least positive natural number such that $T^{l''(x)}(x) \in A''$. Then $l''(x) \geq l'_k(x) \geq l$, for $x \in A''$. Once more, we may assume that A'' is doubly recurrent, by throwing away a null set. For each $x \in A''$, put $k_0(x) = -n$ and let $k_{i+1}(x)$ be the least natural number such that $T^{k_{i+1}(x)}(x) \in A$ and $k_i(x) + n \leq k_{i+1}(x) \leq l''(x) - n$, if such a number exists.

Set $B = \{T^{k_i(x)}(x) : i > 0 \text{ and } k_i(x) \text{ is defined}\}$, and note that $\{T^i(B)\}_{i<n}$ is a pairwise disjoint family whose union contains

$$C = \{T^i(x) \in A : x \in A'' \text{ and } i < l''(x) - n\}.$$

For simplicity, put $[A'']_T = [A'']_{E_T}$. For $x \in [A'']_T$, let $\phi(x) = T^{-i}(x)$, where $i \in \mathbb{N}$ is the least natural number such that $T^{-i}(x) \in A''$. Then

$$\begin{aligned}
\mu(C) &= \int_{[A'']_T} 1_C(x)\ d\mu(x) \\
&= \int_{A''} |C \cap \{T^i(x)\}_{i<l''(x)}|\ d\mu(x) \\
&= \int_{[A'']_T} \frac{|C \cap \{T^i(\phi(x))\}_{i<l''(\phi(x))}|}{l''(\phi(x))}\ d\mu(x) \\
&= \int_{[A'']_T} \frac{|A \cap \{T^i(\phi(x))\}_{i<l''(\phi(x))-n}|}{l''(\phi(x))}\ d\mu(x) \\
&\geq \int_{[A'']_T} m_A(\phi(x))\ d\mu(x) - \delta \\
&= \int_{[A'']_T} m_A(x)\ d\mu(x) - \delta \\
&> \int m_A(x)\ d\mu(x) - 2\delta \\
&= \mu(A) - 2\delta \\
&> 1 - \epsilon,
\end{aligned}$$

thus B is an (ϵ, n)-Rokhlin set. ⊣

Remark 7.8. In the statement of Proposition 7.7, the requirement that μ is T-invariant can be dropped. First, it can be weakened to T-quasi-invariance (see Section 8) by using the Hurewicz Ergodic Theorem in place of the Birkhoff Ergodic Theorem. The general case then follows from the same trick as used in the proof of Corollary 10.2.

Remark 7.9. As in Theorem 7.5, we can modify the above proof so as to ensure that the (ϵ, n)-Rokhlin subset of A is of measure $\leq 1/n$.

We need one more basic fact before we start the proof of Dye's Theorem.

Lemma 7.10. *Suppose E is a countable Borel equivalence relation on X and $\mu \in P(X)$ is E-invariant and ergodic. The following are equivalent:*

(i) $\mu(A) = \mu(B)$.
(ii) $\exists \phi \in [[E]]$ $(\mathrm{dom}(\phi) \subseteq A, \mathrm{rng}(\phi) \subseteq B,$ *and*
$$\mu(\mathrm{dom}(\phi) \setminus A) = \mu(\mathrm{rng}(\phi) \setminus B) = 0).$$

Proof. It suffices to show (i) $\Rightarrow$ (ii). Let $\Gamma = \{\gamma_n\}$ be a countable sequence of Borel involutions whose graphs cover E, and define pairwise disjoint Borel sets $A_n \subseteq A$ recursively by

$$A_n = \left(A \setminus \bigcup_{m<n} A_m \right) \cap \gamma_n^{-1} \left(B \setminus \bigcup_{m<n} \gamma_m(A_m) \right).$$

Set $A' = \bigcup A_n$, let $\phi : A' \to B$ be the Borel injection which sends $x \in A_n$ to $\gamma_n(x)$, and put $B' = \mathrm{rng}(\phi)$.

The main observation here is that for any x, either $A \cap [x]_E \subseteq A'$ or $B \cap [x]_E \subseteq B'$. To see this, suppose that $y \in [x]_E \cap A \setminus A'$, $z \in [x]_E \cap B \setminus B'$ and find a natural number n such that $\gamma_n(y) = z$. Then $y \in A_n$, a contradiction.

It follows from ergodicity that, by neglecting an invariant null set and reversing the roles of A and B if necessary, we may assume that $A = A'$. But it then follows from invariance that $\mu(B \setminus B') = 0$. $\dashv$

One could show the preceding lemma using a simpler exhaustion argument. The advantage of the argument presented above is that it produces a ϕ which simultaneously works for every invariant, ergodic measure ν with the property that $\nu(A) = \nu(B)$.

We now describe the setup for the proof of Dye's Theorem and prove some basic technical lemmas.

Given a countable Borel equivalence relation E on a space X with a non-atomic probability measure μ, which is E-invariant and ergodic, an *array* (see [Kr] or [Su]) is a system

$$\mathcal{A} = \langle A_0, \ldots, A_{n-1}, \phi_0, \ldots, \phi_{n-1} \rangle$$

of Borel sets and $\phi_i \in [[E]]$ such that

(i) $\{A_i\}_{i<n}$ is a partition of a conull subset of X, and
(ii) $\phi_i : A_0 \to A_i$, $\phi_0 = \mathrm{identity}|A_0$.

Given an array $\mathcal{A}$, we use $\bigcup \mathcal{A}$ to denote $A_0 \cup \cdots \cup A_{n-1}$, and $\phi_{i,j}$ to denote $\phi_j \circ \phi_i^{-1} : A_i \to A_j$. It is useful to think of $\phi_{i,j}$ as a *link* between A_i, A_j. The *rank* of $\mathcal{A}$ is given by $\mathrm{rank}(\mathcal{A}) = n$. A *refinement* of $\mathcal{A}$ of rank k is a system $\mathcal{B} = \langle B_0, \ldots, B_{k-1}, \psi_0, \ldots, \psi_{k-1} \rangle$, where $\{B_i\}$ is a partition of a conull subset of A_0. This gives rise to a *subarray* $\mathcal{AB}$ of $\mathcal{A}$ defined by

$$\mathcal{AB} = \left\langle B_{\langle i,j \rangle}, \phi_{\langle i,j \rangle} \right\rangle_{\langle i,j \rangle < nk},$$

where $\langle i, j\rangle = ik + j$, $\phi_{\langle i,j\rangle} = \phi_i \circ \psi_j$, $B_{\langle 0,0\rangle} = B_0$, and $B_{\langle i,j\rangle} = \phi_{\langle i,j\rangle}(B_0)$, for $i < n, j < k$. Note that

$$\phi_{\langle i,j\rangle,\langle i',j\rangle} = \phi_{i,i'}|B_{\langle i,j\rangle},$$

i.e., the link of $\langle i, j\rangle$, $\langle i', j\rangle$ is the same as the link of i, i' (restricted to its domain). For almost all $x \in X$, put

$$\mathrm{Orbit}_{\mathcal{A}}(x) = \{\phi_{i,j}(x) : x \in A_i \text{ and } 0 \le j \le n-1\}.$$

If $\{C_0, \ldots, C_{k-1}\}$ are pairwise disjoint, $\delta > 0$, then $A \in_\delta \{C_0, \ldots, C_{k-1}\}$ means that there exists $I \subseteq \{0, \ldots, k-1\}$ such that

$$\mu\left(A \varDelta \bigcup_{i \in I} C_i\right) < \delta.$$

We use $A \in_\delta \mathcal{A}$ to denote $A \in_\delta \{A_0, \ldots, A_{n-1}\}$.

Lemma 7.11. *Suppose $\mathcal{A}$ is an array, $A \subseteq X$ is Borel, and $\delta > 0$. Then for all sufficiently large k, there is a refinement $\mathcal{B}$ of rank k, such that $A \in_\delta \mathcal{AB}$.*

Proof. Set $r = \mathrm{rank}(\mathcal{A})$, let $\{S_0, \ldots, S_{n-1}\}$ be the atoms of the Boolean algebra generated by $\{\phi_i^{-1}(A \cap A_i) : i < r\}$, and suppose $k > n/\delta$. Choose natural numbers $p_0, \ldots, p_{n-1}$ such that

$$\forall i < n(p_i/kr \le \mu(S_i) < (p_i + 1)/kr).$$

For each $i < n$, let $\{S_{ij} : j < p_i\}$ be a pairwise disjoint collection of subsets of S_i, each of measure $1/kr$. It follows from Lemma 7.10 that we can find a refinement $\mathcal{B}$ of $\mathcal{A}$ of rank k which includes all of the S_{ij}'s. Now set

$$A' = \bigcup\{\phi_m(S_{ij}) : m < r, S_i \subseteq \phi_m^{-1}(A \cap A_m), j < p_i\}.$$

Clearly $A' \subseteq A$ and

$$\begin{aligned}\mu(A \setminus A') &\le \mu\left(\bigcup\{\phi_m(S_i \setminus \bigcup_{j<p_i} S_{ij}) : m < r, i < n\}\right)\\ &\le r\sum_{i<n} 1/kr\\ &< \delta,\end{aligned}$$

thus $A \in_\delta \mathcal{AB}$. $\dashv$

Now suppose $T : X \to X$ is a Borel automorphism which induces E.

Lemma 7.12. *Suppose $\mathcal{A}$ is an array and $\delta > 0$. Then for all sufficiently large k, there is a refinement $\mathcal{B}$ of rank 2^k, such that*

$$\mu(\{x : T(x) \notin \mathrm{Orbit}_{\mathcal{AB}}(x)\}) < \delta.$$

Proof. Let $T' : A_0 \to A_0$ be a Borel automorphism which induces $E|A_0$. We can assume that T' is aperiodic by throwing away a null set. Apply Corollary 7.6 to obtain a sequence $\{T_n\}_{n\in\mathbb{N}}$ of Borel automorphisms of A_0, such that T_n is of period exactly 2^n and $d_{\mu_0}(T', T_n) < 1/2^{n-1}$, where μ_0 is the normalized restriction of μ to A_0. Then

$$\bigcap_{n\in\mathbb{N}} \bigcup_{m>n} \{x \in A_0 : T_m(x) \neq T'(x)\}$$

is clearly null, so by neglecting an invariant null Borel set, we may assume that for all $x \in A_0$ there exists k such that

$$\forall m \geq k(T_m(x) = T'(x)).$$

Let ϕ be the map which sends $x \in A_i$ to $\phi_i^{-1}(x) \in A_0$. Then for all $x \in \mathrm{dom}(\phi) = \bigcup_i A_i$, there exists k such that

$$\forall m \geq k(\phi(x) E_{T_m} \phi(T(x))).$$

Thus, by taking k sufficiently large, we can ensure that

$$\mu(\{x \in X : \phi(x) E_{T_k} \phi(T(x))\}) > 1 - \delta.$$

Letting B_0 be a transversal of E_{T_k}, it follows that

$$\mathcal{B} = \left\langle B_0, T_k(B_0), \ldots, T_k^{2^k-1}(B_0), T_k^0, T_k^1, \ldots, T_k^{2^k-1} \right\rangle.$$

is the desired refinement. ⊣

We are now ready to prove Dye's Theorem.

Theorem 7.13 (Dye [D]). *All non-atomic probability measure preserving ergodic actions of $\mathbb{Z}$ are orbit equivalent.*

Proof. For $n_i \geq 2$, $n_i \in \mathbb{N}$, let $n_i = \{0, \ldots, n_i - 1\}$ have the uniform measure and $\prod n_i$ have the product measure. Define $E_{0,(n_i)}$ on $\prod n_i$ by

$$x E_{0,(n_i)} y \Leftrightarrow \exists n \forall m \geq n(x_m = y_m),$$

so that $E_0 = E_{0,2}$, where 2 denotes the constant 2 sequence. Note that

$$(2^{\mathbb{N}}, E_0, \mu_2), \left(\prod 2^{m_i}, E_{0,(2^{m_i})}, \mu_{(2^{m_i})}\right) \text{ are isomorphic,}$$

for any $m_i \geq 1$. So it is enough to show that if T is an aperiodic Borel automorphism on X, $\mu \in P(X)$ is T-invariant and ergodic, and T induces the equivalence relation E, then there exist $m_i \geq 1$ such that

$$(X, E, \mu), \left(\prod 2^{m_i}, E_{0,(2^{m_i})}, \mu_{(2^{m_i})}\right) \text{ are isomorphic.}$$

Fix a sequence $\{X_i\}_{i\in\mathbb{N}}$ of Borel subsets of X which separate points, and in which every X_i appears infinitely often. Find, by Lemmas 7.11 and 7.12, indices $m_i \geq 1$ ($i \geq 0$), and arrays $\mathcal{A}_i$ ($i \geq -1$) such that

(1) $\mathcal{A}_{-1} = (X, \text{identity})$,
(2) $\mathcal{A}_{i+1} = \mathcal{A}_i \mathcal{B}_{i+1}$, where $\mathcal{B}_{i+1}$ is a refinement of rank $2^{m_{i+1}}$ ($i \geq -1$),
(3) $X_i \in_{2^{-i}} \mathcal{A}_i$, ($i \geq 0$), and
(4) $\mu(\{x : T(x) \in \text{Orbit}_{\mathcal{A}_i}(x)\}) > 1 - 2^{-i-1}$ ($i \geq 0$).

Write the elements of $\mathcal{A}_i$ as $A_{\langle a_0,\ldots,a_i\rangle}$, where $a_j < 2^{m_j}$, and denote by $\phi_{\langle a_0,\ldots,a_i\rangle,\langle a'_0,\ldots,a'_i\rangle}$ the corresponding links. Then for any finite sequence s, the link of $\langle a_0,\ldots,a_i\rangle^\frown s$, $\langle a'_0,\ldots,a'_i\rangle^\frown s$ is the same as that of $\langle a_0,\ldots,a_i\rangle$, $\langle a'_0,\ldots,a'_i\rangle$ (restricted to its domain). Let

$$X_1 = \bigcap_{i\in\mathbb{N}} \bigcup \mathcal{A}_i,$$

noting that $\mu(X_1) = 1$. For $x \in X_1$, let $\phi(x) \in \prod 2^{m_i}$ be given by $\phi(x) = \langle a_0, a_1, \ldots\rangle$, where $x \in \bigcap_i A_{\langle a_0,\ldots,a_i\rangle}$.

Claim 7.14. *ϕ is injective a.e.*

Proof. For each $i \in \mathbb{N}$, fix X'_i in the Boolean algebra generated by the sets of $\mathcal{A}_i$, such that $\mu(X_i \Delta X'_i) < 2^{-i}$. Now set

$$N = \bigcap_{i\in\mathbb{N}} \bigcup_{j>i} (X_j \Delta X'_j),$$

and observe that N is a null set off of which ϕ is injective. ⊣

Also note that, because of (4),

$$\mu(\{x : \exists i(T(x) \in \text{Orbit}_{\mathcal{A}_i}(x))\}) = 1.$$

So fix a conull invariant $X_0 \subseteq X_1$, such that ϕ is 1-1 on X_0 and

$$\forall x \in X_0 \exists i(T(x) \in \text{Orbit}_{\mathcal{A}_i}(x)).$$

Put $Y_0 = \phi(X_0) \subseteq \prod 2^{m_i}$, $\nu = \phi_*\mu$. Then for any basic open set $\mathcal{N}_{\langle a_0,\ldots,a_{k-1}\rangle} \subseteq \prod 2^{m_i}$,

$$\begin{aligned}\nu(\mathcal{N}_{\langle a_0,\ldots,a_{k-1}\rangle}) &= \mu(\phi^{-1}(A_{\langle a_0,\ldots,a_{k-1}\rangle} \cap X_0)) \\ &= \mu(\phi^{-1}(A_{\langle a_0,\ldots,a_{k-1}\rangle})) \\ &= 2^{-m_0-\cdots-m_{k-1}},\end{aligned}$$

thus $\nu = \mu_{0,(2^{m_i})}$. Also $\mu_{0,(2^{m_i})}(Y_0) = 1$.

Claim 7.15. Y_0 *is* $E_{0,(2^{m_i})}$*-invariant.*

Proof. Fix $\langle a_0, a_1, \ldots \rangle \in Y_0$ and $x \in X_0$ such that $\phi(x) = \langle a_0, a_1, \ldots \rangle$. If $\langle b_0, b_1, \ldots \rangle \, E_{0,(2^{m_i})} \, \langle a_0, a_1, \ldots \rangle$, then $\forall i \geq k(a_i = b_i)$, for some $k \in \mathbb{N}$. Put

$$\phi_{\langle a_0,\ldots,a_{k-1}\rangle,\langle b_0,\ldots,b_{k-1}\rangle}(x) = y,$$

so $y \in A_{\langle b_0,\ldots,b_{i-1}\rangle}$, for all $i \geq k$. Since yEx, we have that $y \in X_0$, and since $\phi(y) = \langle b_0, b_1, \ldots \rangle$, it follows that $\langle b_0, b_1, \ldots \rangle \in Y_0$. ⊣

It remains to show that for all $x, y \in X_0$,

$$xEy \Leftrightarrow \phi(x) E_{0,(2^{m_i})} \phi(y).$$

To see ($\Leftarrow$), say $\phi(x) = \langle a_0, a_1, \ldots \rangle, \phi(y) = \langle b_0, b_1, \ldots \rangle$, and $a_i = b_i$ for all $i \geq k$. Setting

$$y' = \phi_{\langle a_0,\ldots,a_{k-1}\rangle,\langle b_0,\ldots,b_{k-1}\rangle}(x),$$

it follows that $y' \in X_0$ and $\phi(y') = \langle b_0, b_1, \ldots \rangle = \phi(y)$, thus $y = y'Ex$.

To see ($\Rightarrow$), it suffices to show that $\phi(x) E_{0,(2^{m_i})} \phi \circ T(x)$. Since $x \in X_0$, let i be such that $T(x) \in \mathrm{Orbit}_{\mathcal{A}_i}(x)$. If $\phi(x) = \langle a_0, a_1, \ldots \rangle$, $\phi \circ T(x) = \langle b_0, b_1, \ldots \rangle$, then

$$\phi_{\langle a_0,\ldots,a_i,a_{i+1},\ldots,a_k\rangle,\langle b_0,\ldots,b_i,a_{i+1},\ldots,a_k\rangle}(x) = T(x),$$

for all $k \geq i$, thus

$$\phi \circ T(x) = \langle b_0, \ldots, b_i, a_{i+1}, \ldots \rangle,$$

so $\forall k \geq i+1(a_k = b_k)$, and $\phi(x) E_{0,(2^{m_i})} \phi \circ T(x)$. ⊣

8 Quasi-invariant Measures

We will now embark on proving the second result mentioned in the beginning of Section 6, i.e., that measure preserving actions of amenable groups are orbit equivalent to $\mathbb{Z}$-actions. We will actually consider a more general version of this result in which invariance of the measure is not assumed and which also applies to a wider class of equivalence relations.

Consider (X, E, μ), where E is a countable Borel equivalence relation and μ is a probability measure. We say that μ is E*-quasi-invariant* if $[A]_E$ is null whenever A is null. For measures μ, ν on X, $\mu \ll \nu$ (μ is *absolutely continuous* with respect to ν) means that every ν-null set is μ-null. Also, μ, ν are *equivalent*, $\mu \sim \nu$, if $\mu \ll \nu$ and $\nu \ll \mu$. The next fact is easily proven as in Proposition 2.1.

Proposition 8.1. *The following are equivalent:*

(a) μ *is* E*-quasi-invariant,*
(b) μ *is* Γ*-quasi-invariant (i.e.,* $\gamma_*\mu \sim \mu$ *for all* $\gamma \in \Gamma$*), whenever* Γ *is a group acting in a Borel fashion on* X *such that* $E = E_\Gamma^X$*,*
(c) μ *is* Γ*-quasi-invariant for some countable group* Γ *acting in a Borel fashion on* X *such that* $E = E_\Gamma^X$*, and*
(d) $\forall \phi \in [[E]]\, (\mu(\mathrm{dom}(\phi)) = 0 \Rightarrow \mu(\mathrm{rng}(\phi)) = 0)$.

Assume μ is E-quasi-invariant. We define two measures M_r,M_l on E as follows:

$$M_l(A) = \int |A_x|\ d\mu(x),$$

where $A \subseteq E$ is Borel, $A_x = \{y \in X : (x,y) \in A\}$, and $|B| = \mathrm{card}(B)$. Equivalently, for any nonnegative Borel f,

$$\int f(x,y)\ dM_l(x,y) = \int \sum_{y \in [x]_E} f(x,y)\ d\mu(x).$$

Let also

$$M_r(A) = \int |A^y|\ d\mu(y),$$

where $A^y = \{x \in X : (x,y) \in A\}$. Note that for $\phi \in [[E]]$,

$$M_l(\mathrm{graph}(\phi)) = \mu(\mathrm{dom}(\phi)), M_r(\mathrm{graph}(\phi)) = \mu(\mathrm{rng}(\phi)),$$

thus M_l, M_r are σ-finite.

Proposition 8.2. $M_l \sim M_r$.

Proof. Let $A \subseteq E$ be a Borel set such that $M_l(A) = 0$. By Theorem 18.10 of [K], we can find a sequence $\{\phi_i\}$ of Borel functions in $[[E]]$ whose graphs partition A. So

$$M_l(A) = \sum_i M_l(\mathrm{graph}(\phi_i)) = 0,$$

therefore $M_l(\mathrm{graph}(\phi_i)) = \mu(\mathrm{dom}(\phi_i)) = 0$, thus $\mu(\mathrm{rng}(\phi_i)) = 0$ and it follows that

$$\begin{aligned} M_r(A) &= \sum_i M_r(\mathrm{graph}(\phi_i)) \\ &= \sum_i \mu(\mathrm{rng}(\phi_i)) \\ &= 0. \end{aligned}$$

Thus $M_r \ll M_l$, and $M_l \ll M_r$ by the same argument. ⊣

Similarly one can see that if μ is actually E-invariant, then $M_r = M_l$.

Consider the Radon-Nikodym derivative

$$D_\mu(x,y) = D(x,y) = (dM_l/dM_r)(x,y),$$

for $(x,y) \in E$. D is a Borel map from E to $\mathbb{R}^+$, and for any Borel $f : E \to \mathbb{R}^+$,

$$\int f(x,y)\ dM_l(x,y) = \int f(x,y)D(x,y)\ dM_r(x,y),$$

so for Borel $A \subseteq E$:

$$M_l(A) = \int_A D(x,y)\ dM_r(x,y),$$

and D is uniquely characterized M_r-a.e. by this last property. Also $D^{-1} = dM_r/dM_l$, M_l-a.e. (equivalently, M_r-a.e.)

Using D we can define the measure $|\cdot|_y$ on each $[y]_E$ by

$$|x|_y = |\{x\}|_y = D(x,y).$$

Then for Borel $A \subseteq E$:

$$\begin{aligned} M_l(A) &= \int_A D(x,y)\ dM_r(x,y) \\ &= \int \sum_{x\in[y]_E} 1_A(x,y)D(x,y)\ d\mu(y) \\ &= \int |A^y|_y\ d\mu(y), \end{aligned}$$

so $M_l(A) = \int |A_x|\ d\mu(x) = \int |A^y|_y\ d\mu(y)$.

In particular, for any $\phi \in [[E]]$ and Borel $B \subseteq \mathrm{rng}(\phi)$, we have

$$\begin{aligned} \mu(\phi^{-1}(B)) &= M_l(\mathrm{graph}(\phi|\phi^{-1}(B))) \\ &= \int_B D(\phi^{-1}(y),y)\ d\mu(y). \end{aligned}$$

So, if $f \in [E]$, then $(df_*\mu/d\mu)(y) = D(f^{-1}(y),y)$, μ-a.e.

A map $\alpha : E \to G$, where G is a group, is called a *cocycle* if

$$\alpha(x,z) = \alpha(y,z)\alpha(x,y),$$

for $xEyEz$. If this only happens on an E-invariant set A with $\mu(A) = 1$, we say that α is a *cocycle a.e.*

Proposition 8.3. *$D : E \to \mathbb{R}^+$ is a cocycle a.e.*

Proof. We begin with a lemma.

Lemma 8.4. *If ρ, σ are probability measures on X, g is a Borel automorphism of X and $\rho \ll \sigma$, then $g_*\rho \ll g_*\sigma$ and $d(g_*\rho)/d(g_*\sigma) = (d\rho/d\sigma) \circ g^{-1}$, $g_*\sigma$-a.e.*

Proof. The first assertion is obvious. To prove the second, note that for any Borel nonnegative h,

$$\begin{aligned}\int h \; d(g_*\rho) &= \int h \circ g \; d\rho \\ &= \int (h \circ g)(d\rho/d\sigma) \; d\sigma \\ &= \int h(g(x))(d\rho/d\sigma)(x) \; d\sigma(x) \\ &= \int h(g(x))(d\rho/d\sigma) \circ g^{-1} \circ g(x) \; d\sigma(x) \\ &= \int h(x)(d\rho/d\sigma) \circ g^{-1}(x) \; d(g_*\sigma(x)),\end{aligned}$$

so $d(g_*\rho)/d(g_*\sigma) = (d\rho/d\sigma) \circ g^{-1}$, $g_*\sigma$-a.e. ⊣

It follows that if $f, g \in [E]$:

$$\begin{aligned}d((fg)_*\mu)/d(f_*\mu) &= d(f_*(g_*\mu))/d(f_*\mu) \\ &= (d(g_*\mu)/d\mu) \circ f^{-1}, \mu - \text{a.e.},\end{aligned}$$

so by the Chain Rule:

$$d((fg)_*\mu)/d\mu = \left((d(g_*\mu)/d\mu) \circ f^{-1}\right)(d(f_*\mu)/d\mu), \mu - \text{a.e.},$$

or

$$D(g^{-1}f^{-1}(y), y) = D(g^{-1}f^{-1}(y), f^{-1}(y))D(f^{-1}(y), y), \mu - \text{a.e.}$$

It follows that there is an E-invariant Borel set A with $\mu(A) = 1$, so that for $xEyEz$ in A we have $D(x, z) = D(y, z)D(x, y)$. ⊣

9 Amenable Equivalence Relations

We will now define a notion of amenability for equivalence relations, in the presence of a measure, originally due to Zimmer. There are several equivalent definitions of which we will consider one used in Kaimanovich [Ka], which is the analog of the Reiter condition. See also [JKL] where an appropriate purely Borel version was (independently) developed.

Let E be a countable Borel equivalence relation on X and $\mu \in P(X)$. We say that (X, E, μ) is *amenable*, or E is *μ-amenable*, if there is a sequence $\lambda^n : E \to \mathbb{R}$ of non-negative Borel functions such that

(i) $\lambda^n_x \in l^1([x]_E)$, where $\lambda^n_x(y) = \lambda^n(x,y)$, for xEy,
(ii) $||\lambda^n_x||_1 = 1$, and
(iii) there is a Borel E-invariant set $A \subseteq X$ with $\mu(A) = 1$, such that $||\lambda^n_x - \lambda^n_y||_1 \to 0$ for all xEy with $x, y \in A$.

Remark 9.1. If μ is E-quasi-invariant, then (iii) is equivalent to $||\lambda^n_x - \lambda^n_y||_1 \to 0$, M_l-a.e.

Proposition 9.2. *Let Γ be a countable amenable group acting in a Borel way on X and $\mu \in P(X)$. Then E^X_Γ is μ-amenable.*

Proof. By the (right) Reiter condition for Γ, we can find a sequence $\{f_n\}_{n\in\mathbb{N}}$ with $f_n \in l^1(\Gamma)$, $f_n \geq 0$, $||f_n||_1 = 1$ and $||f_n - \gamma * f_n||_1 \to 0$, $\forall\gamma \in \Gamma$, where $\gamma * f(\delta) = f(\delta\gamma)$. It follows that

$$\lambda^n(x,y) = \sum_{\gamma\in\Gamma, \gamma\cdot x = y} f_n(\gamma)$$

witnesses the amenability of E^X_Γ. ⊣

Proposition 9.3. *If (X, E, μ) is amenable and $Y \subseteq X$ is Borel with $\mu(Y) > 0$, then so is $(Y, E|Y, \mu_Y)$, where $\mu_Y = (\mu|Y)/\mu(Y)$.*

Proof. Let $\{\lambda^n\}$ witness the amenability of E. We can assume that Y is a complete section, so let $F : X \to Y$ be Borel with $F(x)Ex$, and note that

$$\sigma^n(x,y) = \sum_{z\in F^{-1}(\{y\})} \lambda^n(x,z)$$

witnesses the amenability of $(Y, E|Y, \mu_Y)$. ⊣

Example 9.4. Suppose that $\Gamma = F_2$ acts on $X = 2^{F_2}$ via the shift and μ is the usual product measure. Then E^X_Γ is not μ-amenable. To see this, suppose that $\{\lambda^n\}$ witnesses that E^X_Γ is μ-amenable, and define

$$f_n(\gamma) = \int \lambda^n_x(\gamma\cdot x)\ d\mu(x).$$

Then

$$\begin{aligned} ||f_n||_1 &= \sum_{\gamma\in\Gamma} \int \lambda^n_x(\gamma\cdot x)\ d\mu(x) \\ &= \int \sum_{\gamma\in\Gamma} \lambda^n_x(\gamma\cdot x)\ d\mu(x) \\ &= 1, \end{aligned}$$

and, recalling that $\gamma * f(\delta) = f(\delta\gamma)$,

$$\begin{aligned}
||f_n - \gamma * f_n||_1 &= \sum_{\delta\in\Gamma} \left| \int \lambda^n_x(\delta\cdot x)\ d\mu(x) - \int \lambda^n_x(\delta\gamma\cdot x)\ d\mu(x) \right| \\
&= \sum_{\delta\in\Gamma} \left| \int \lambda^n_x(\delta\cdot x)\ d\mu(x) - \int \lambda^n_{\gamma^{-1}\cdot x}(\delta\cdot x)\ d\mu(x) \right| \\
&\le \sum_{\delta\in\Gamma} \int \left| \lambda^n_x(\delta\cdot x) - \lambda^n_{\gamma^{-1}\cdot x}(\delta\cdot x) \right|\ d\mu(x) \\
&= \int ||\lambda^n_x - \lambda^n_{\gamma^{-1}\cdot x}||_1\ d\mu(x) \\
&\to 0,
\end{aligned}$$

contradicting the fact that F_2 is not amenable.

We will consider below Borel graphs $\mathcal{G} \subseteq E$, where E is a countable Borel equivalence relation on X and μ is an E-quasi-invariant probability measure on X. We will denote by $\mathcal{G}_0$ the set $\{x \in X : \mathcal{G}_x \neq \emptyset\}$ (here $\mathcal{G}_x = \{y : (x,y) \in \mathcal{G}\}$ is the set of all $\mathcal{G}$-neighbors of x). We call $\mathcal{G}$ *bounded* if there is a constant $M > 0$ such that

$$1/M \le D(x,y) \le M,$$

for all $(x,y) \in \mathcal{G}$, and the degree of each vertex in $\mathcal{G}$ is bounded by M. (Here D is the Radon-Nikodym derivative as defined in Section 8.)

Recall that the boundary of $A \subseteq X$ is defined by

$$\partial_{\mathcal{G}} A = \{x \in A : \exists y((x,y) \in \mathcal{G} \text{ and } y \notin A)\}.$$

We say that $\mathcal{G}$ satisfies the *Følner condition* if, for every E-class $[x]_E$ and any $\epsilon > 0$, there is a finite, non-empty set $A \subseteq [x]_E$ such that

$$|\partial_{\mathcal{G}} A|_x / |A|_x < \epsilon.$$

Here $|\cdot|_x$ is the measure on $[x]_E$ defined in Section 8. Note that, neglecting null sets, if we choose another point $x' \in [x]_E$, then as $D(y,x') = D(y,x)D(x,x')$, it follows that

$$|\partial_{\mathcal{G}} A|_x / |A|_x = |\partial_{\mathcal{G}} A|_{x'} / |A|_{x'}.$$

Following [Ka], we say that (X, E, μ) satisfies the *Følner condition* if for every bounded Borel graph $\mathcal{G} \subseteq E$, there is an invariant Borel set Y with $\mu(Y) = 1$, such that $\mathcal{G}|Y$ satisfies the Følner condition. For the next lemma, see [CFW], Lemma 8 and [Ka], p. 1003.

Lemma 9.5. *If (X, E, μ) is amenable, where μ is an E-quasi-invariant probability measure, then (X, E, μ) satisfies the Følner condition.*

Proof. Assume (X, E, μ) is amenable. Fix a bounded graph $\mathcal{G} \subseteq E$ and $\epsilon > 0$. By a simple exhaustion argument, it is enough to show that for a positive set of z's, there is a non-empty finite set $A \subseteq [z]_E$ with $|\partial_{\mathcal{G}} A|_z / |A|_z \leq 2\epsilon$. We can also assume that $\mu(\mathcal{G}_0) > 0$. Fix $\{\lambda^n\}_{n \in \mathbb{N}}$ witnessing the amenability of E.

Claim 9.6. *There exists* $n \in \mathbb{N}$ *such that*

$$\int_{\mathcal{G}} ||\lambda^n_x - \lambda^n_y||_1 \; dM_l(x, y) < \epsilon \int_{\mathcal{G}_0} ||\lambda^n_x||_1 \; d\mu(x).$$

Proof. Note that $M_l(\mathcal{G}) < \infty$ and $||\lambda^n_x - \lambda^n_y||_1 \to 0$, M_l-a.e., so by the Lebesgue Dominated Convergence Theorem,

$$\int_{\mathcal{G}} ||\lambda^n_x - \lambda^n_y||_1 \; dM_l(x, y) \to 0,$$

while

$$\int_{\mathcal{G}_0} ||\lambda^n_x||_1 \; d\mu(x) = \int_{\mathcal{G}_0} d\mu = \mu(\mathcal{G}_0) > 0,$$

so if n is large enough,

$$\int_{\mathcal{G}} ||\lambda^n_x - \lambda^n_y||_1 \; dM_l(x, y) < \epsilon \int_{\mathcal{G}_0} ||\lambda^n_x||_1 \; d\mu(x),$$

which completes the proof of the claim. $\dashv$

Put $\Lambda(x, y) = \lambda^n(x, y)$, with n as in the claim. Recall that

$$E_a(t) = \begin{cases} 1 \text{ if } t \geq a, \\ 0 \text{ if } t < a. \end{cases}$$

For $a \in [0, \infty]$ and $(x, y) \in E$, put

$$F_a(x, y) = E_a(\Lambda(x, y)) = \begin{cases} 1 \text{ if } \Lambda_x(y) \geq a, \\ 0 \text{ if } \Lambda_x(y) < a. \end{cases}$$

Noting that

$$\int_0^\infty E_a(t) \; da = t, \int_0^\infty |E_a(t) - E_a(t')| \; da = |t - t'|$$

for $t, t' \geq 0$, it follows that $\int_0^\infty F_a(x, y) \; da = \Lambda_x(y)$ and

$$\int_0^\infty |F_a(x, z) - F_a(y, z)| \; da = |\Lambda_x(z) - \Lambda_y(z)|.$$

If $\Pi_a = \{(x, y) \in E : F_a(x, y) = 1\} = \{(x, y) \in E : \Lambda_x(y) \geq a\}$, then notice that $|(\Pi_a)_x| < \infty$, for each $a > 0$, and

$$\int_{\mathcal{G}} \left(\int_0^\infty |(\Pi_a)_x \Delta (\Pi_a)_y| \ da \right) \ dM_l(x,y) < \epsilon \int_{\mathcal{G}_0} \left(\int_0^\infty |(\Pi_a)_x| \ da \right) \ d\mu(x).$$

To see this, notice that for $(x,y) \in E$:

$$\begin{aligned}
\int_0^\infty |(\Pi_a)_x \Delta (\Pi_a)_y| \ da &= \int_0^\infty \sum_{z \in [x]_E = [y]_E} |F_a(x,z) - F_a(y,z)| \ da \\
&= \sum_{z \in [x]_E = [y]_E} \int_0^\infty |F_a(x,z) - F_a(y,z)| \ da \\
&= \sum_{z \in [x]_E = [y]_E} |\Lambda_x(z) - \Lambda_y(z)| \\
&= ||\Lambda_x - \Lambda_y||_1,
\end{aligned}$$

and similarly for any x,

$$\begin{aligned}
\int_0^\infty |(\Pi_a)_x| \ da &= \int_0^\infty \sum_{z \in [x]_E} F_a(x,z) \ da \\
&= \sum_{z \in [x]_E} \int_0^\infty F_a(x,z) \ da \\
&= \sum_{z \in [x]_E} \Lambda_x(z) \\
&= ||\Lambda_x||_1.
\end{aligned}$$

So, by Fubini,

$$\int_0^\infty \left[\epsilon \int_{\mathcal{G}_0} |(\Pi_a)_x| \ d\mu(x) - \int_{\mathcal{G}} |(\Pi_a)_x \Delta (\Pi_a)_y| \ dM_l(x,y) \right] \ da > 0,$$

thus there exists $a > 0$ such that, putting $\Pi = \Pi_a$, we get

$$\int_{\mathcal{G}} |\Pi_x \Delta \Pi_y| \ dM_l(x,y) < \epsilon \int_{\mathcal{G}_0} |\Pi_x| \ d\mu(x) < \infty.$$

Now note that

$$\begin{aligned}
\int_{\mathcal{G}} |\Pi_x \Delta \Pi_y| \ dM_l(x,y) &= \int_{\mathcal{G}_0} \sum_{y \in \mathcal{G}_x} |\Pi_x \Delta \Pi_y| \ d\mu(x) \\
&= \int_{\mathcal{G}_0} \sum_{y \in \mathcal{G}_x} \sum_{z \in [x]_E} |1_\Pi(x,z) - 1_\Pi(y,z)| \ d\mu(x) \\
&= \int_{\mathcal{G}_0} \sum_{z \in [x]_E} \sum_{y \in \mathcal{G}_x} |1_\Pi(x,z) - 1_\Pi(y,z)| \ d\mu(x) \\
&= \int 1_{\mathcal{G}_0}(x) \sum_{y \in \mathcal{G}_x} |1_\Pi(x,z) - 1_\Pi(y,z)| \ dM_l(x,z) \\
&\geq \int_\Pi |\mathcal{G}_x \setminus \Pi^z| \ dM_l(x,z),
\end{aligned}$$

where the latter inequality holds since, if $(x,z) \in \Pi$ and $y \in \mathcal{G}_x \setminus \Pi^z$, then $|1_\Pi(x,z) - 1_\Pi(y,z)| = 1$ and $y \in \mathcal{G}_x$, so

$$1_\Pi(x,z)\,|G_x \setminus \Pi^z| \le 1_{\mathcal{G}_0}(x) \sum_{y \in \mathcal{G}_x} |1_\Pi(x,z) - 1_\Pi(y,z)|\,.$$

If we let

$$(x,z) \in \partial_{\mathcal{G}}\Pi \Leftrightarrow (x,z) \in \Pi \text{ and } \exists y((x,y) \in \mathcal{G} \text{ and } y \notin \Pi^z),$$

so that $(\partial_{\mathcal{G}}\Pi)^z = \partial_{\mathcal{G}}\Pi^z$, we have

$$1_{\partial_{\mathcal{G}}\Pi}(x,z) \le 1_\Pi(x,z)|\mathcal{G}_x \setminus \Pi^z|,$$

since if $(x,z) \in \partial_{\mathcal{G}}\Pi$, then $(x,z) \in \Pi$ and $|G_x \setminus \Pi^z| \neq \emptyset$. So

$$\begin{aligned}
\int_\Pi |\mathcal{G}_x \setminus \Pi^z|\ dM_l(x,z) &= \int 1_\Pi(x,z)|\mathcal{G}_x \setminus \Pi^z|\ dM_l(x,z) \\
&\ge \int 1_{\partial_{\mathcal{G}}\Pi}(x,z)\ dM_l(x,z) \\
&= M_l(\partial_{\mathcal{G}}\Pi) \\
&= \int |(\partial_{\mathcal{G}}\Pi)^z|_z\ d\mu(z).
\end{aligned}$$

To summarize,

$$\int_{\mathcal{G}} |\Pi_x \Delta \Pi_y|\ \ dM_l(x,y) \ge \int |\partial_{\mathcal{G}}\Pi^z|_z\ \ d\mu(z).$$

Also

$$\begin{aligned}
\int_{\mathcal{G}_0} |\Pi_x|\ d\mu(x) &= M_l(\Pi \cap (\mathcal{G}_0 \times X)) \\
&= \int |(\Pi \cap (\mathcal{G}_0 \times X))^z|_z\ \ d\mu(z) \\
&= \int |\Pi^z \cap \mathcal{G}_0|_z\ \ d\mu(z),
\end{aligned}$$

so

$$\int |\partial_{\mathcal{G}}\Pi^z|_z\ d\mu(z) < \epsilon \int |\Pi^z \cap \mathcal{G}_0|_z\ d\mu(z) < \infty.$$

It follows that for a set of positive measure of z's, we have

(i) $0 < |\Pi^z \cap \mathcal{G}_0|_z < \infty$ and
(ii) $\epsilon|\Pi^z \cap \mathcal{G}_0|_z > |\partial_{\mathcal{G}}\Pi^z|_z = |\partial_{\mathcal{G}}(\Pi^z \cap \mathcal{G}_0)|_z$.

Let $B = \Pi^z \cap \mathcal{G}_0 \subseteq [z]_E$ and choose a finite $A \subseteq B$ with $|B|_z \le 1.5|A|_z$ and $|B \setminus A|_z \le (0.5\epsilon|A|_z)/M^2$, where $1/M \le D(u,v) \le M$ for $(u,v) \in \mathcal{G}$

and the $\mathcal{G}$-degree of any x is $\leq M$. Then, if $u \in \partial_{\mathcal{G}} A$, either $u \in \partial_{\mathcal{G}} B$ or all $\mathcal{G}$-neighbors of u are in B and at least one is not in A, so there is a $\mathcal{G}$-neighbor of u in $B \setminus A$, and, as every element has at most M neighbors, and $|u|_z = D(u,v)|v|_z \leq M|v|_z$ (since we can assume, throwing away a null set, that D is a cocycle), when $(u,v) \in \mathcal{G}, (u,z) \in E$, we have:

$$\begin{aligned} |\partial_{\mathcal{G}} A|_z &\leq |\partial_{\mathcal{G}} B|_z + M^2 |B \setminus A|_z \\ &\leq \epsilon |B|_z + (M^2 \cdot 0.5 \cdot \epsilon |A|_z)/M^2 \\ &\leq 1.5\epsilon |A|_z + 0.5\epsilon |A|_z \\ &= 2\epsilon |A|_z, \end{aligned}$$

which completes the proof of Lemma 9.5. ⊣

A graph $\mathcal{G} \subseteq E$ *generates* E if the connected components of $\mathcal{G}$ are the E-classes, i.e., if any two E-related elements of X are connected by a path through $\mathcal{G}$. We note below that any countable Borel equivalence relation can be generated by a graph with the Følner condition.

Proposition 9.7. *Suppose E is a countable Borel equivalence relation on X and μ is an E-quasi-invariant measure. Then, neglecting null sets, there is a Borel graph $\mathcal{G}$, satisfying the Følner condition, which generates E.*

Proof. Neglecting null sets, we can assume that E is aperiodic and $D = D_\mu$ is a cocycle. Let $\{A_n\}_{n \in \mathbb{N}}$ be a vanishing sequence of markers with $A_0 = X$, and set

$$B_n = A_n \setminus A_{n+1}.$$

By neglecting an invariant Borel set on which E is smooth, we may assume that each $E|B_n$ is aperiodic. By Proposition 7.4, we can find finite Borel equivalence relations $F_n \subseteq E|B_n$, such that each F_n-class is of cardinality n. Let T be a Borel transversal of $F = \bigcup_{n \in \mathbb{N}} F_n$ with the property that

$$\forall x \in T \forall y \in [x]_F (D(y,x) \geq 1),$$

let $\mathcal{G}'$ be a Borel graph that generates $E|T$, let $\mathcal{G}'' = F \setminus \Delta_X$, and put $\mathcal{G} = \mathcal{G}' \cup \mathcal{G}''$. It is clear that $\mathcal{G}$ generates E. Noting that for each E-class C there are infinitely many $n \in \mathbb{N}$ such that $\mathrm{dom}(F_n) \cap C \neq \emptyset$, it follows that $\mathcal{G}$ satisfies the Følner condition. ⊣

Remark 9.8. By the remarks at the end of Section 3.4 of [JKL], every countable Borel equivalence relation is generated by a *locally finite* Borel graph, i.e., a graph in which every point has only finitely many neighbors. By using such a graph for $\mathcal{G}'$ above, we obtain a locally finite graph which satisfies the Følner condition and generates E. An argument similar to that given in Section 3.4 of [JKL] shows that if E is generated by a bounded Borel graph, then so is the restriction of E to any Borel set. It follows from this and the proof of Proposition 9.7 that, in the context of 9.7, if E is generated by a bounded

Borel graph, then it is generated by a bounded Borel graph which satisfies the Følner condition, neglecting null sets. As every countable Borel equivalence relation is the union of an increasing sequence of bounded Borel graphs (by Theorem 1.3), it similarly follows that, in the context of 9.7 again, we can find an increasing, exhaustive sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ of bounded Borel subgraphs of E which satisfy the Følner condition.

10 Amenability vs. Hyperfiniteness

We will next prove the result of Connes-Feldman-Weiss [CFW], which identifies amenability and hyperfiniteness in the measure theoretic context. It was historically preceded by the result of Ornstein-Weiss [OW] that proves the same conclusion when E is induced by a Borel action of an amenable group.

Theorem 10.1 (Connes-Feldman-Weiss [CFW]). *Let E be a countable Borel equivalence relation on X and μ an E-quasi-invariant probability measure. If E is μ-amenable, then E is hyperfinite μ-a.e.*

Corollary 10.2. *Let E be a countable Borel equivalence relation on X and $\mu \in P(X)$. If E is μ-amenable, then E is hyperfinite μ-a.e.*

Proof. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of Borel automorphisms whose graphs cover E, and set

$$\mu' = \sum_{n\in\mathbb{N}} (g_n)_*\mu/2^{n+1},$$

so that $\mu \ll \mu'$, μ' is E-quasi-invariant, and $\mu(A) = \mu'(A)$ for any invariant Borel set A. In particular, E is μ'-amenable. By Theorem 10.1, E is hyperfinite μ'-a.e., and it follows that E is hyperfinite μ-a.e. $\dashv$

Remark 10.3. Corollary 10.2 can be also proved by noticing that if $\mu \in P(X)$, then there is a Borel complete section $A \subseteq X$ with $\mu(A) = 1$ such that $\mu|A$ is $E|A$-quasi-invariant (and using 9.3 and 6.9). This fact can be shown as follows (by adapting an argument of Woodin from the Baire Category context): We may assume, without loss of generality, that $X = [0,1]$ and μ is Lebesgue measure. Let $\mathcal{U}_n$ enumerate the rational intervals. Also, fix a sequence of Borel automorphisms $f_n : X \to X$ such that

$$E = \bigcup_{n\in\mathbb{N}} \mathrm{graph}(f_n).$$

For each pair of natural numbers (m,n) for which it is possible, fix a Borel set $B_{mn} \subseteq \mathcal{U}_m$ such that $\mu(B_{mn}) \geq \mu(\mathcal{U}_m)/2$ and $\mu(f_n^{-1}(B_{mn})) = 0$. Define

$$B = X \setminus \bigcup_{m,n} f_n^{-1}(B_{mn}),$$

and suppose, towards a contradiction, that there is a Borel null set $B' \subseteq B$ such that $[B']_{E|B}$ is non-null. Set $B_n = B \cap f_n^{-1}(B)$, and note that

$$[B']_{E|B} = \bigcup_{n \in \mathbb{N}} f_n(B' \cap B_n).$$

In particular, it follows that there exists $n \in \mathbb{N}$ such that

$$\mu(f_n(B' \cap B_n)) > 0.$$

By the Lebesgue density theorem, there exists $m \in \mathbb{N}$ such that

$$\mu(f_n(B' \cap B_n) \cap \mathcal{U}_m) > \mu(\mathcal{U}_m)/2.$$

It follows that B_{mn} exists, and since $\mu(B_{mn}) \geq \mu(\mathcal{U}_m)/2$, we have that

$$f_n(B' \cap B_n) \cap B_{mn} \neq \emptyset.$$

It then follows that $B \cap f_n^{-1}(B_{mn}) \neq \emptyset$, a contradiction. Thus, the set

$$A = B \cup (X \setminus [B]_E)$$

is the desired complete section.

Let E be a countable Borel equivalence relation on X, suppose μ is an E-quasi-invariant probability measure and D is the Radon-Nikodym derivative as defined in Section 8. The proof of 10.1 is immediate from the following two lemmas.

Lemma 10.4. *The following are equivalent:*

1. *For any bounded Borel graph $\mathcal{G} \subseteq E$ and any $\epsilon > 0$, there is finite Borel equivalence relation $F \subseteq E$ such that $\mu(\{x \in X : \mathcal{G}_x \not\subseteq [x]_F\}) < \epsilon$.*
2. *E is hyperfinite μ-a.e.*

Proof. The proof of (2) $\Rightarrow$ (1) is straightforward. To see (1) $\Rightarrow$ (2), let $\{g_n\}_{n \in \mathbb{N}}$ be a sequence of Borel involutions whose graphs cover E, with $g_0 =$ identity. For each $m > 0$ and $n \in \mathbb{N}$, put

$$X_{m,n} = \{x \in X : 1/m \leq D(x, g_n(x)) \leq m\}.$$

Let $\{k_n\}_{n \in \mathbb{N}}$ be a sequence of natural numbers in which every natural number appears infinitely often, and define $\mathcal{G}_n \subseteq E$ by

$$(x, y) \in \mathcal{G}_n \Leftrightarrow \exists m \leq n (x, y \in X_{m,k_m} \text{ and } y = g_{k_m}(x)).$$

It follows that $\{\mathcal{G}_n\}_{n \in \mathbb{N}}$ is an increasing, exhaustive sequence of bounded Borel subgraphs of E. Now let $\{F_n\}_{n \in \mathbb{N}}$ be a sequence of finite Borel subequivalence relations of E such that, for all $n \in \mathbb{N}$,

$$\mu(\{x \in X : (\mathcal{G}_n)_x \not\subseteq [x]_{F_n}\}) < 1/2^n.$$

Setting $E_n = \bigcap_{m>n} F_m$, so that $E_0 \subseteq E_1 \subseteq \cdots$ are finite Borel subequivalence relations of F, it is clear that, for all $n \in \mathbb{N}$,

$$\mu(\{x \in X : (\mathcal{G}_n)_x \not\subseteq [x]_{E_n}\}) < 1/2^n.$$

Thus

$$\left\{x \in X : [x]_E \not\subseteq \bigcup_{n\in\mathbb{N}} [x]_{E_n}\right\} \subseteq \bigcup_{n\in\mathbb{N}} \bigcap_{m>n} \{x \in X : (\mathcal{G}_m)_x \not\subseteq [x]_{E_m}\},$$

and this latter set is clearly null. It follows that E is hyperfinite μ-a.e. $\dashv$

Lemma 10.5. *Suppose E is μ-amenable, $\mathcal{G}$ is a bounded Borel subgraph of E, and $\epsilon > 0$. Then there is a finite Borel subequivalence relation $F \subseteq E$ such that*

$$\mu(\{x \in X : \mathcal{G}_x \not\subseteq [x]_F\}) \leq \epsilon.$$

Proof. Fix a natural number $M - 1$ witnessing that $\mathcal{G}$ is bounded, and recursively define a sequence $\{F_n\}_{n\in\mathbb{N}}$ of fsr's of E by putting

$$X_n = X \setminus \bigcup_{m<n} \mathrm{dom}(F_m),$$

$\mathcal{G}_n = \mathcal{G} \cap (X_n \times X_n)$, and letting F_n be a maximal fsr of $E|X_n$ such that

$$\forall x \in \mathrm{dom}(F_n)\ |\partial_{\mathcal{G}_n}[x]_{F_n}|_x/|[x]_{F_n}|_x < \epsilon/M^3.$$

Noting that $\{\mathrm{dom}(F_n)\}_{n\in\mathbb{N}}$ is pairwise disjoint, it follows that

$$F_\infty = \bigcup_{n\in\mathbb{N}} F_n$$

is an fsr of E. Put $X_\infty = X \setminus \mathrm{dom}(F_\infty) = \bigcap_n X_n$ and suppose, towards a contradiction, that $\mu(X_\infty) > 0$. Setting $\mathcal{G}_\infty = \mathcal{G} \cap (X_\infty \times X_\infty)$, it follows from 9.3 and 9.5 that we can find $x \in X_\infty$ and a finite set $S \subseteq [x]_E \cap X_\infty$ such that

$$|\partial_{\mathcal{G}_\infty} S|_x/|S|_x < \epsilon/M^3.$$

As S is finite and $\mathcal{G}$ is locally finite, we can find n such that $\partial_{\mathcal{G}_n} S = \partial_{\mathcal{G}_\infty} S$, thus

$$|\partial_{\mathcal{G}_n} S|_x/|S|_x < \epsilon/M^3,$$

contradicting the maximality of F_n.

Thus, throwing away a null invariant Borel set, which is harmless as we can define F to be equality in that set, we can assume that $X = \mathrm{dom}(F_\infty)$, i.e., $F = F_\infty$ is a finite Borel subequivalence relation of E. We will show that this F works.

First note that if $\mathcal{G}_x \not\subseteq [x]_F = [x]_{F_n}$, where $x \in \mathrm{dom}(F_n)$, then there is $y \in \mathcal{G}_x$ with $y \notin [x]_{F_n}$. Say $y \in \mathrm{dom}(F_m)$. If $m \geq n$, clearly $y \in X_n$, so $x \in \partial_{\mathcal{G}_n}[x]_{F_n}$. If $m < n$, then $y \in \partial_{\mathcal{G}_m}[y]_{F_m}$. Thus

$$\{x \in X : \mathcal{G}_x \not\subseteq [x]_F\} \subseteq \bigcup_{n\in\mathbb{N}} \{x \in X : \exists y \in \{x\} \cup \mathcal{G}_x (y \in \partial_{\mathcal{G}_n}[y]_{F_n})\}.$$

We will now use the following.

Sublemma 10.6. *(i) If $B \subseteq X$ is Borel, then $\mu\left(\bigcup_{x\in B}\{x\} \cup \mathcal{G}_x\right) \leq M^3\mu(B)$. (ii) If Ω is a Borel transversal for a Borel fsr R of E, then for any $A \subseteq$* $\mathrm{dom}(R)$, $\mu(A) = \int_\Omega |[x]_R \cap A|_x \ d\mu(x)$.

Granting this, it follows that if

$$B_n = \{y \in \mathrm{dom}(F_n) : y \in \partial_{\mathcal{G}_n}[y]_{F_n}\},$$

then, using 10.6 (ii) for $R = F_n$, $A = B_n$, and $A = \mathrm{dom}(F_n)$, we have

$$\begin{aligned} \mu(B_n) &= \int_\Omega |[x]_{F_n} \cap B_n|_x \ d\mu(x) \\ &= \int_\Omega |\partial_{\mathcal{G}_n}[x]_{F_n}|_x \ d\mu(x) \\ &\leq \frac{\epsilon}{M^3}\int_\Omega |[x]_{F_n}|_x \ d\mu(x) \\ &= \frac{\epsilon}{M^3}\mu(\mathrm{dom}(F_n)). \end{aligned}$$

Next, using 10.6 (i), we have that

$$\begin{aligned} \mu(\{x \in X : \exists y \in \{x\} \cup \mathcal{G}_x (y \in \partial_{\mathcal{G}_n}[y]_{F_n})\} &= \mu\left(\bigcup_{x\in B_n} \{x\} \cup \mathcal{G}_x\right) \\ &\leq M^3\mu(B_n) \leq \epsilon\mu(\mathrm{dom}(F_n)). \end{aligned}$$

Therefore

$$\mu(\{x \in X : \mathcal{G}_x \not\subseteq [x]_F\}) \leq \sum_n \epsilon\mu(\mathrm{dom}(F_n)) = \epsilon$$

and the proof is complete, modulo the

Proof of Sublemma 10.6. (i) Applying 18.15 of [K], we can find a sequence $\{f_m\}_{m<M}$ of Borel functions with domains Borel subsets of B such that for $x \in B$,

$$\{x\} \cup \mathcal{G}_x = \{f_m(x) : m < M \text{ and } f_m(x) \text{ is defined}\}.$$

Clearly each f_m is at most M-to-1, so again by using 18.15 of [K], we can find $g_{m,n} \in [[E]]$ $(m < M, n < N)$ such that $\mathrm{graph}(f_m)$ is the disjoint union of graph($g_{m,n}^{-1}$), $n < M$. Thus $\mathrm{dom}(g_{m,n}) \subseteq \bigcup_{x \in B} \{x\} \cup \mathcal{G}_x$, $\mathrm{rng}(g_{m,n}) \subseteq B$, and $\bigcup_{x \in B} \{x\} \cup \mathcal{G}_x = \bigcup_{m,n} \mathrm{dom}(g_{m,n})$. Now, by Section 8,

$$\mu(\mathrm{dom}(g_{m,n})) = \int_{\mathrm{rng}(g_{m,n})} D(g_{m,n}^{-1}(y), y) \ d\mu(y) \leq M\mu(B),$$

so $\mu\left(\bigcup_{x \in B} \{x\} \cup \mathcal{G}_x\right) \leq M^2 \cdot M\mu(B) = M^3 \mu(B)$.

(ii) Write Ω as a disjoint union of Borel sets $\Omega = \bigcup_n \Omega_n$ such that for $x \in \Omega_n$, $|[x]_R \cap A| = n$. Let then $\phi_1^n, \ldots, \phi_n^n \in [[E]]$ be such that $\mathrm{dom}(\phi_i^n) = \Omega_n$ and $[x]_R \cap A = \{\phi_1^n(x), \ldots, \phi_n^n(x)\}, \forall x \in \Omega_n$. Then, by Section 8,

$$\mu(\phi_i^n(C)) = \int_C D(\phi_i^n(y), y) \ d\mu(y),$$

for any Borel $C \subseteq \mathrm{dom}(\phi_i^n) = \Omega_n$, and A is the disjoint union of $\phi_i^n(\Omega_n)$, so

$$\begin{aligned}
\mu(A) &= \sum_n \sum_{i=1}^n \int_{\Omega_n} D(\phi_i^n(x), x) \ d\mu(x) \\
&= \sum_n \int_{\Omega_n} \sum_{i=1}^n D(\phi_i^n(x), x) \ d\mu(x) \\
&= \sum_n \int_{\Omega_n} |[x]_R \cap A|_x \ d\mu(x) \\
&= \int_{\Omega} |[x]_R \cap A|_x \ d\mu(x),
\end{aligned}$$

which completes the proof. ⊣

By putting together the results of Sections 7 and 10, we now have

Theorem 10.7 (Dye [D], Ornstein-Weiss [OW]). *Any two non-atomic, probability measure preserving ergodic actions of amenable groups are orbit equivalent.*

Hjorth [H1] has recently proved the converse of this result, so that we have that a (countable) group is amenable iff it has exactly one, up to orbit equivalence, non-atomic probability measure preserving ergodic action.

11 Groups of Polynomial Growth

Weiss [We] raised the question of whether a Borel action of an amenable group gives rise to a hyperfinite equivalence relation. (This is in the pure Borel context; no measures are present.) Weiss (unpublished) proved this for the group $\mathbb{Z}^n$ and this was later extended to all finitely generated groups of polynomial growth.

Theorem 11.1 (Jackson-Kechris-Louveau [JKL]). *Let Γ be a finitely generated group of polynomial growth. Then for any Borel action of Γ on a standard Borel space X, E_Γ^X is hyperfinite.*

Proof. We begin with a lemma.

Lemma 11.2. *Let E, F be countable Borel equivalence relations on X with $E \subseteq F$ and $[F : E] < \infty$, i.e., every F-class contains only finitely many E-classes. If E is hyperfinite, then so is F.*

Proof. We can assume that for some $k \geq 1$, each F-class contains exactly k E-classes. Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of Borel involutions whose graphs cover F. Recursively define $f_1, \ldots, f_k : X \to X$ by $f_1(x) = x$ and $f_{i+1}(x) = g_N(x)$, where is N is least such that $(g_N(x), f_j(x)) \notin E$, for all $j \leq i$. Then $f_1(x), \ldots, f_k(x)$ belong to the different E-classes contained in $[x]_F$. Let $\{E_n\}_{n\in\mathbb{N}}$ be an increasing, exhaustive sequence of finite Borel subequivalence relations of E. Put

$$xF_ny \Leftrightarrow \forall i \leq k(f_{\pi(i)}(x)E_nf_i(y)),$$

where π is the unique permutation of $\{1, \ldots, k\}$ such that $f_{\pi(i)}(x)Ef_i(y)$, $\forall i \leq k$. It follows that $\{F_n\}_{n\in\mathbb{N}}$ is an increasing, exhaustive sequence of finite Borel subequivalence relations of F. $\dashv$

Let E be a countable Borel equivalence relation on X. A *cascade* is a sequence of Borel complete sections $X = S_0 \supseteq S_1 \supseteq \cdots$ and Borel *retractions* $f_n : S_n \to S_{n+1}$ (i.e., $f_n|S_{n+1} =$ identity).

A cascade $\{S_n, f_n\}$ defines a sequence of equivalence relations E_n given by

$$xE_ny \Leftrightarrow f_nf_{n-1}\cdots f_0(x) = f_nf_{n-1}\cdots f_0(y).$$

Clearly $\{E_n\}_{n\in\mathbb{N}}$ is an increasing sequence of smooth Borel subequivalence relations of E, and if each f_n is finite-to-1, then each E_n is finite. Thus $E_\infty = \bigcup_{n\in\mathbb{N}} E_n$ is hyperfinite.

Suppose now a locally finite Borel graph $\mathcal{G} \subseteq E$ generates E. A *kernel* for $\mathcal{G}$ is a set $B \subseteq X$ which is maximal with the property that no two points of B are $\mathcal{G}$-neighbors.

Claim 11.3. *$\mathcal{G}$ admits a Borel kernel.*

Proof. First note that $\mathcal{G}$ admits an $\aleph_0$-coloring: Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of Borel sets such that for any distinct $x, x_1, \ldots, x_k \in X$ there is n with $x \in X_n, x_1, \ldots, x_k \notin X_n$, and define $c : X \to \mathbb{N}$ by $c(x) =$ least n with $x \in X_n$ and $\mathcal{G}_x \cap X_n = \emptyset$. Clearly c is a Borel coloring. Put $C_n = c^{-1}(\{n\})$, and define inductively $B_0 = C_0$,

$$B_{n+1} = B_n \cup (C_{n+1} \setminus \{x : \exists y((x, y) \in \mathcal{G} \text{ and } y \in B_n)\}).$$

Then $B = \bigcup_n B_n$ is a Borel kernel. ⊣

Let $\mathcal{G}^n = \{(x,y) :$ the distance from x to y in $\mathcal{G}$ is $\leq n\}$, and note that for any Borel $Y \subseteq X$, there is a Borel kernel for $\mathcal{G}^n|Y$. For any fixed sequence $0 \leq k_0 < k_1 < \cdots$, we can therefore define a cascade $\{S_n, f_n\}$, with S_{n+1} a Borel kernel for $\mathcal{G}^{k_n}|S_n$, and $(x, f_n(x)) \in \mathcal{G}^{k_n}$, so that any two elements of S_{n+1} have distance $> k_n$ in $\mathcal{G}$ and $f_n(x)$ has distance $\leq k_n$ from x. For any $x \in X$, put

$$(x)_n = f_n f_{n-1} \cdots f_0(x) \in S_{n+1}.$$

We clearly have that the distance from x to $(x)_n$ is $\leq k_0 + k_1 + \cdots k_n$.

Suppose we could choose $\{k_n\}$ so that there is a constant c, such that for infinitely many n and any $x \in X$, if we consider the ball of radius $k_0 + k_1 + \cdots k_{n-1} + 2k_n$ around x in $\mathcal{G}$, then there are no more than c elements in it, any two of which have distance $> k_n$ from each other. Then it is easy to see that, if $\{E_n\}$ are the equivalence relations associated with $\{S_n, f_n\}$, then there are at most c elements in each E-class which are pairwise E_∞-inequivalent. Thus $[E : E_\infty] < \infty$ and E is hyperfinite. Indeed, if $x_0, x_1, \ldots, x_c$ are E-equivalent but pairwise E_∞-inequivalent elements, and the distance from x_0 to each x_i is $\leq M$, choose n large enough so that the above condition holds and $n \geq M$. Then $(x_0)_n, \ldots, (x_c)_n$ are distinct elements, any two of which are at distance $> k_n$ apart, and they are in the ball of radius

$$M + k_0 + k_1 + \cdots + k_n \leq k_0 + k_1 + \cdots + k_{n-1} + 2k_n$$

around x_0, a contradiction.

Return now to (Γ, X, E_Γ^X). Fix a finite symmetric set $A = \{a_1, \ldots, a_m\}$ generating Γ such that

$$N_A(n) = |\{\gamma \in \Gamma : \exists \gamma_1, \ldots, \gamma_n \in A(\gamma_1 \cdots \gamma_n = \gamma)\}|$$

grows polynomially, i.e., $N_A(n) \leq kn^d$, for some $d, k \in \mathbb{N}$ and all $n \in \mathbb{N}$. We will use $B_A(n)$ to denote

$$\{\gamma \in \Gamma : \exists \gamma_1, \ldots, \gamma_n \in A(\gamma_1 \cdots \gamma_n = \gamma)\}.$$

Let $\mathcal{G}$ be the graph induced by A, i.e.,

$$(x,y) \in \mathcal{G} \Leftrightarrow \exists a \in A(a \cdot x = y).$$

Clearly $\mathcal{G}$ is locally finite and generates $E = E_\Gamma^X$. Choose S_n, f_n as above, associated with this $\mathcal{G}$ and $k_n = 2^n$, so that

$$k_0 + k_1 + \cdots + k_{n-1} + 2k_n \leq 2^{n+2}.$$

We claim that there is a c such that, for infinitely many n, there are no more than c elements in any ball of radius 2^{n+2} in $\mathcal{G}$, any two of which are of distance $> 2^n$ from each other. Indeed, if we have a sequence $x_1, \ldots, x_c$ of distinct such elements in the ball of radius 2^{n+2} around x, find $\gamma_1, \dots, \gamma_c$ in $B_A(2^{n+2})$ with $\gamma_i \cdot x = x_i$. Then clearly

$$B_A(2^{n-1})\gamma_1, \dots, B_A(2^{n-1})\gamma_c$$

are pairwise disjoint and contained in $B_A(2^{n+3})$, so $N_A(2^{n+3}) \geq cN_A(2^{n-1})$. Thus, if the claim fails, then for any c there is n_0 such that $N_A(2^{n+3}) \geq cN_A(2^{n-1})$ for $n \geq n_0 + 1$, so for all $k \geq 1$,

$$N_A(2^{4k+n_0}) \geq c^k N_A(2^{n_0}),$$

thus

$$(16^d/c)^k \geq N_A(2^{n_0})/(k \cdot 2^{n_0 d}),$$

a contradiction if $c > 16^d$. $\dashv$

12 Generic Hyperfiniteness

In this and the next section we will see that countable Borel equivalence relations behave dramatically different with respect to Baire category compared with measure. Generically, i.e., on a comeager invariant Borel set, they are hyperfinite and admit no invariant probability measure.

Theorem 12.1 (Hjorth-Kechris [HK], Sullivan-Weiss-Wright [SW^2], Woodin). *Let E be a countable Borel equivalence relation on a Polish space. Then there is a comeager invariant Borel set C such that $E|C$ is hyperfinite.*

Proof. The proof below is a simplification (in particular, avoiding forcing by using the Kuratowski-Ulam theorem) of an argument of Miri Segal.

Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence of Borel involutions whose graphs cover E, and fix a Borel linear ordering $<$ of X. For each Borel set $S \subseteq X$ and each n, define the equivalence relation on S:

$$xF_n^S y \Leftrightarrow x = y \text{ or } g_n \cdot x = y.$$

Each F_n^S-class has at most 2 elements, so let $\Phi_n(S)$ be the subset of S consisting of the smallest element of each class, and let $f_n^S : S \to \Phi_n(S)$ assign to each $x \in S$ the smallest element of its F_n^S-class. So f_n^S is a Borel retraction. Notice that if S is a complete section, so is $\Phi_n(S)$.

Associate to each $\alpha \in \mathbb{N}^\mathbb{N}$ the cascade $\{S_n^\alpha, f_n^\alpha\}$ defined by $S_0^\alpha = X$, $S_{n+1}^\alpha = \Phi_{\alpha(n)}(S_n^\alpha)$, and $f_n^\alpha = f_{\alpha_n}^{S_n^\alpha}$. Let $\{E_n^\alpha\}_{n\in\mathbb{N}}$ be the associated equivalence relations, and put $E_\infty^\alpha = \bigcup_{n\in\mathbb{N}} E_n^\alpha$. We will show that there is an α and a comeager invariant Borel set C with $E_\infty^\alpha|C = E|C$. This follows immediately from the following claim, where "$\forall^*$" means "for comeager many."

Claim 12.2. $\forall^* \alpha \in \mathbb{N}^{\mathbb{N}} \forall^* x \in X([x]_E = [x]_{E^\alpha_\infty})$.

Indeed, by the claim, we can find $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $C = \{x \in X : [x]_E = [x]_{E^\alpha_\infty}\}$ is comeager. Clearly C is a comeager E-invariant Borel set with $E^\alpha_\infty|C = E|C$.

To prove the claim, note that, by the Kuratowski-Ulam theorem (the analog of Fubini for category), it is enough to show that for all $x \in X$,

$$\forall^* \alpha \in \mathbb{N}^{\mathbb{N}}([x]_E = [x]_{E^\alpha_\infty}).$$

As the intersection of countably many comeager sets is comeager, it is enough to show that for any fixed $y \in [x]_E$,

$$\forall^* \alpha \in \mathbb{N}^{\mathbb{N}}(y \in [x]_{E^\alpha_\infty}).$$

Clearly $A = \{\alpha \in \mathbb{N}^{\mathbb{N}} : y \in [x]_{E^\alpha_\infty} = \bigcup_n [x]_{E^\alpha_n}\}$ is open, so it is enough to show it is dense. So fix a basic neighborhood $\mathcal{N}_s = \{\alpha \in \mathbb{N}^{\mathbb{N}} : s \subseteq \alpha\}$ of $\mathbb{N}^{\mathbb{N}}$, where $s \in \mathbb{N}^{n+1}$. Consider the finite cascade $S_0, f_0, S_1, \ldots, S_n, f_n, S_{n+1}$ associated (in the obvious way) to s. Then $x' = f_n f_{n-1} \cdots f_0(x) E f_n f_{n-1} \cdots f_0(y) = y'$, so find k with $g_k(x') = y'$. Fix $\alpha_0 \supseteq s$ with $\alpha_0(n+1) = k$. Then clearly $\alpha_0 \in A$ (as $(x, y) \in E^{\alpha_0}_{n+1}$), so $A \cap \mathcal{N}_s \neq \emptyset$. $\dashv$

13 Generic Compressibility

Let E be a countable Borel equivalence relation on X, and let $D : E \to \mathbb{R}^+$ be a Borel cocycle. E is *D-aperiodic* if $|[x]_E|_x$ is infinite, for all $x \in X$, where $|\cdot|_x$ is defined as in Section 8. The following extends a result of Wright [Wr].

Theorem 13.1 (Kechris-Miller). *Let E be a countable Borel equivalence relation on a Polish space X and let $D : E \to \mathbb{R}^+$ be a Borel cocycle. Suppose E is D-aperiodic. Then there is an invariant comeager Borel set C such that for any E-quasi-invariant probability measure μ whose induced cocycle is D μ-a.e., we have $\mu(C) = 0$.*

Proof. We first note that if F is a smooth D-aperiodic Borel equivalence relation on a Borel set $A \subseteq X$, $F \subseteq E$, and μ is an E-quasi-invariant probability measure whose induced cocycle is D μ-a.e., then $\mu(A) = 0$. To see this, let $A_0 \subseteq A$ be a Borel transversal for F and let $\{\phi_n\} \subseteq [[F]]$ be such that $\mathrm{dom}(\phi_n) = A_0$, $\{\mathrm{rng}(\phi_n)\}$ is a partition of A, and $\phi_0 = $ identity on A_0. Then, by Section 8, $\mu(\mathrm{rng}(\phi_n)) = \int_{A_0} D(\phi_n(y), y)\, d\mu(y)$, so

$$\mu(A) = \sum_n \mu(\mathrm{dom}(\phi_n)) = \int_{A_0} |[y]_F|_y \, d\mu(y).$$

It follows that $\mu(A_0) = 0$, and thus $\mu(A) = 0$.

Next fix a Borel $\aleph_0$-coloring $c : [E]^{<\infty} \to \mathbb{N}$ for the graph

$$(S,T) \in \mathcal{G} \Leftrightarrow S \neq T \text{ and } S \cap T \neq \emptyset$$

(see the proof of 7.3). For each $n \in \mathbb{N}$ and $S \in [E]^{<\infty}, S \neq \emptyset$, note that there is at most one $T \in [E]^{<\infty}$, $T \supseteq S$ with $c(T) = n$; denote it by $\Phi_n(S)$, if it exists.

Let now $\{A_n\}_{n\in\mathbb{N}}$ be a vanishing sequence of markers for E. Set $k_0(x) = 0$ and

$$k_{n+1}(x) = \min\{k \in \mathbb{N} : A_{k_n(x)} \cap [x]_E \not\subseteq A_k\},$$

and define $B_n \subseteq X$ by

$$x \in B_n \Leftrightarrow x \in A_{k_n(x)} \setminus A_{k_{n+1}(x)}.$$

Then $\{B_n\}_{n\in\mathbb{N}}$ is a partition of X into Borel complete sections for E. If $|[x]_E \cap B_n|_x < \infty$, then $\{y \in [x]_E \cap B_n : D(y,x) \geq D(y',x), \forall y' \in [x]_E \cap B_n\}$ is finite, non-empty and independent of x in its E-class (since D is a cocycle). It follows that there is a smooth invariant Borel set $X_0 \subseteq X$ such that for $x \in X_1 = X \setminus X_0$ and each n, $|B_n \cap [x]_E|_x = \infty$.

For each $\alpha \in \mathbb{N}^{\mathbb{N}}$ we will define next an increasing sequence of fsr's $\{F^\alpha_n\}_{n\in\mathbb{N}}$ of $E|X_1$, so that $B_0 \cap X_1$ is a transversal for each F^α_n and for each $b \in B_0 \cap X_1$, $[b]_{F^\alpha_n} \subseteq B_0 \cup \cdots \cup B_n$. We start with

$$F^\alpha_0 = \text{ equality on } B_0,$$

Assume F^α_n is given, in order to construct F^α_{n+1}. For each $b \in B_0$, we define

$$[b]_{F^\alpha_{n+1}} = \Phi_{\alpha(n)}([b]_{F^\alpha_n}),$$

if $\Phi_{\alpha(n)}([b]_{F^\alpha_n})$ is defined and is of the form $[b]_{F^\alpha_n} \cup S$, with $S \subseteq B_{n+1} \cap [b]_E$. Otherwise, $[b]_{F^\alpha_{n+1}} = [b]_{F^\alpha_n}$. It is easy to check that this makes sense, i.e., if $b_1 \neq b_2 \in B_0$, then $[b_1]_{F^\alpha_{n+1}} \cap [b_2]_{F^\alpha_{n+1}} = \emptyset$.

For $\alpha \in \mathbb{N}^{\mathbb{N}}$, put $F^\alpha_\infty = \bigcup_n F^\alpha_n$.

Claim 13.2. $\forall^*\alpha\forall^*x[x \in X_0$ or $(x \in X_1$ and $\forall b \in B_0 \cap [x]_E(|[b]_{F^\alpha_\infty}|_b = \infty))]$.

Granting this claim, it follows that we can find $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that

$$C = X_0 \cup \{x \in X_1 : \forall b \in B_0 \cap [x]_E(|[b]_{F^\alpha_\infty}|_b = \infty)\}$$

is an invariant Borel comeager set. If now μ is an E-quasi-invariant probability measure on X whose induced cocycle is D μ-a.e., then $\mu(C) = 0$, by applying the remarks at the beginning of this proof to $A = C$, $F = E|X_0 \cup F^\alpha_\infty$.

It remains to prove the claim. By Kuratowski-Ulam, it is enough to show that for any $x \in X_1$,

$$\forall^*\alpha\forall b \in B_0 \cap [x]_E(|[b]_{F^\alpha_\infty}|_b = \infty).$$

Since the intersection of countably many comeager sets is comeager, it is enough to show that for any $x \in X_1, b \in B_0 \cap [x]_E$,

$$\forall^* \alpha(|[b]_{F^\alpha_\infty}|_b = \infty).$$

Now $\{\alpha : |[b]_{F^\alpha_\infty}|_b = \infty\} = \bigcap_n \{\alpha : |[b]_{F^\alpha_\infty}|_b > n\}$ and the latter sets are clearly open, so it is enough to show that each

$$\{\alpha : |[b]_{F^\alpha_\infty}|_b > n\}$$

is dense. Fix then a basic open subset $\mathcal{N}_s$ of $\mathbb{N}^{\mathbb{N}}$, with $s \in \mathbb{N}^n$. Define $F^s_0, \ldots, F^s_n$ by using the same recipe as before. Since $|B_{n+1} \cap [b]_E|_b = \infty$, we can find a finite $S \subseteq B_{n+1} \cap [b]_E$ with $|S|_b > n$. Let then $\alpha \supseteq s$ be such that $\alpha(n) = c([b]_{F^s_n} \cup S)$. Then clearly $[b]_{F^\alpha_{n+1}} = [b]_{F^s_n} \cup S$, so $|[b]_{F^\alpha_{n+1}}|_b > n$ and thus $|[b]_{F^\alpha_\infty}|_b > n$, therefore $\alpha \in \mathcal{N}_s \cap \{\alpha : |[b]_{F^\alpha_\infty}|_b > n\}$. ⊣

For $D \equiv 1$, the above result says that there is an invariant comeager Borel set C such that $E|C$ admits no invariant probability measure. Nadkarni [N] showed that an aperiodic countable Borel equivalence relation E admits no invariant probability measure if and only if E is *compressible*, i.e., if there is a map $\phi \in [[E]]$ such that $\mathrm{dom}(\phi) = X$ and $X \setminus \mathrm{rng}(\phi)$ is a complete section. We therefore have

Corollary 13.3. *Suppose E is an aperiodic countable Borel equivalence relation on a Polish space X. Then there is an invariant comeager Borel set C such that $E|C$ is compressible.*

It should be noted that Nadkarni's Theorem is unnecessary to derive this as a corollary, as any countable Borel equivalence relation which contains an aperiodic smooth equivalence relation defined on a complete Borel section is clearly compressible.

Corollary 13.4. *Suppose E, F are aperiodic countable Borel equivalence relations on Polish spaces X, Y, respectively, such that for any invariant comeager Borel sets $A \subseteq X, B \subseteq Y$, $E|A, F|B$ are not smooth. Then there exist invariant comeager Borel sets $C \subseteq X, D \subseteq Y$ with $E|C \cong_B F|D$.*

Proof. By 12.1, 13.3, and Theorem 9.1 of [DJK]. ⊣

III

Costs of Equivalence Relations and Groups

14 Preliminaries

We review here some standard terminology and notation.

A *relation* R on a set X is a set of ordered pairs from $X, R \subseteq X^2$. If R is a relation, we write interchangeably

$$xRy \Leftrightarrow (x, y) \in R.$$

We also let $R_x = \{y : (x, y) \in R\}, R^y = \{x : (x, y) \in R\}$. If R is a relation, its *inverse*, R^{-1}, is defined by $R^{-1} = \{(y, x) : (x, y) \in R\}$. Finally, if $Y \subseteq X$, the restriction of R to Y, $R|Y$, is defined by $R|Y = R \cap Y^2$.

If f is a function, we denote by $\mathrm{dom}(f)$ and $\mathrm{rng}(f)$ its domain and range respectively, and by $\mathrm{graph}(f) = \{(x, y) : f(x) = y\}$ its graph.

A *graph* G with *vertex set* X is a *non-reflexive* (i.e. $(x, x) \notin G, \ \forall x \in X$), *symmetric* (i.e., $G = G^{-1}$) relation on X. The *neighbors* of $x \in X$ in the graph G are the $y \in X$ such that $(x, y) \in G$. The cardinality of the set of neighbors of X is called the *degree* of x, in symbols $d_G(x)$. Denoting by $|A| = \mathrm{card}(A)$ the cardinality of A, we thus have

$$d_G(x) = |G_x|.$$

A graph G in which $d_G(x) < \infty$, for every x, is called *locally finite*. A *G-path* from x to y is a finite sequence of vertices $x = x_0, x_1, \ldots, x_n = y$ such that $(x_i, x_{i+1}) \in G$, $\forall i < n$, and $x_i \neq x_j$ if $i \neq j$, except possibly for $i = 0, \ j = n$. Consider the equivalence relation on X given by

$$xEy \Leftrightarrow \exists \text{ a } G\text{-path from } x \text{ to } y.$$

Its equivalence classes are the *connected components* of G. If there is only one connected component, we call G *connected*.

A *G-cycle* is a G-path $x = x_0, x_1, \ldots, x_n = x_0$, $n \geq 3$, starting and ending at the same point. A graph G is *acyclic* if it contains no G-cycles. An acyclic connected graph is called a *tree*. Thus G is a tree if for any $x \neq y$ there is a unique G-path $x = x_0, x_1, \dots, x_n = y$. A *rooted tree* is a tree with a distinguished vertex x_0, called its *root*. For any vertex x, let $x_0, x_1, \dots, x_{n-1}, x_n = x$ be the unique path from x_0 to x. The neighbors of x different from x_{n-1} are called the *children* of x.

15 Countable Borel Equivalence Relations

Let X be a standard Borel space. A Borel equivalence relation E on X is *countable* if every equivalence class $[x]_E,\ x \in X$, is countable.

If Γ is a countable group and $(g, x) \mapsto g \cdot x$ is a Borel action of Γ on X, then the orbit equivalence relation

$$xE_\Gamma^X y \Leftrightarrow \exists g \in \Gamma\ (g \cdot x = y)$$

is countable. Conversely we have (see 1.3):

Theorem 15.1 (Feldman-Moore [FM]). *If E is a countable Borel equivalence relation on X, there is a countable group Γ and a Borel action of Γ on X with $E_\Gamma^X = E$. Moreover, Γ and the action can be chosen so that*

$$xEy \Leftrightarrow \exists g \in \Gamma(g^2 = 1\ \&\ g \cdot x = y).$$

We call a countable equivalence relation E on X *aperiodic* if every equivalence class $[x]_E$ is infinite. The following is a most useful fact (see 6.7).

Proposition 15.2 (Marker Lemma). *Let E be an aperiodic countable Borel equivalence relation on X. Then there is a sequence $\{S_n\}$ of Borel sets $S_n \subseteq X$ such that*

(i) $S_0 \supseteq S_1 \supseteq S_2 \supseteq \dots,$
(ii) $\bigcap_n S_n = \emptyset,$
(iii) Each S_n is a complete section for E, i.e., it meets every equivalence class.

We refer to such a sequence $\{S_n\}$ as a *vanishing sequence of markers*.

Finally, given Borel equivalence relations E, F on X, Y respectively, we say that E is *Borel reducible* to F, in symbols

$$E \leq_B F,$$

if there is a Borel map $f : X \to Y$ such that

$$xEy \Leftrightarrow f(x)Ff(y).$$

16 More on Invariant Measures

A *measure* on a standard Borel space X is a non-zero σ-finite Borel measure on X. If μ is a measure on X, then μ is *finite* if $\mu(X) < \infty$ and a *probability measure* if $\mu(X) = 1$. If μ is a measure on X and $A \subseteq X$ is a Borel set, then $\mu|A$ is the measure on A defined by $(\mu|A)(B) = \mu(B)$, for any a Borel set $B \subseteq A$.

Let now E be a countable Borel equivalence relation and μ a measure on X. The following is easy to check.

Proposition 16.1. *The following are equivalent:*

(i) There is a countable group Γ and a Borel action of Γ on X with $E_\Gamma^X = E$ such that μ is Γ-invariant.
(ii) For all countable groups Γ and Borel actions of Γ on X with $E_\Gamma^X = E$, μ is Γ-invariant.
(iii) For all Borel bijections $f : A \to B$, with A, B Borel subsets of X, such that $f(x)Ex,\ \forall x \in A$, we have that $\mu(A) = \mu(B)$.
(iv) For all Borel maps $f : A \to X$, A a Borel subset of X, such that $f(x)Ex,\ \forall x \in A$, we have $\mu(f(A)) \leq \mu(A)$.

If these equivalent conditions are satisfied, we say that μ is *E-invariant.*

We denote by $[E]$ the set of all Borel automorphisms f of X with $f(x)Ex$, for all x, and by $[[E]]$ the set of all partial Borel automorphisms $f : A \to B$, A, B Borel subsets of X, with $f(x)Ex,\ \forall x \in A$. Thus μ is E-invariant iff μ is f-invariant for all $f \in [E]$ iff μ is f-invariant for all $f \in [[E]]$.

Suppose now μ is E-invariant. We define a measure M on E as follows:

$$M(A) = \int |A_x|\ d\mu(x),$$

for $A \subseteq E$ Borel. Here $|S| = \mathrm{card}(S)$ (which is ∞ if S is infinite), and $A_x = \{y : (x, y) \in A\}$. Of course we could also define $M'(A) = \int |A^y|\ d\mu(y)$, where $A^y = \{x : (x, y) \in A\}$, but the invariance of μ implies that $M = M'$. To see this, recall from 15.1 that $E = \bigcup_{i\in\mathbb{N}} \mathrm{graph}(g_i)$, with $g_i \in [E]$. So if $A \subseteq E$ is Borel, $A = \bigcup_{i\in\mathbb{N}}[\mathrm{graph}(g_i) \cap A]$ and $\mathrm{graph}(g_i) \cap A = \mathrm{graph}(f_i)$, with $f_i \in [[E]]$, so we can write A as a countable disjoint union of graphs of $f \in [[E]]$. Thus it is enough to show that $M(\mathrm{graph}(f)) = M'(\mathrm{graph}(f))$, for $f \in [[E]]$. If $\mathrm{dom}(f) = C$, $\mathrm{rng}(f) = D$, then $M(\mathrm{graph}(f)) = \mu(C), M'(\mathrm{graph}(f)) = \mu(D)$, so we are done.

Now define an equivalence relation $\tilde{E}$ on E as follows.

$$(x, y)\tilde{E}(z, w) \Leftrightarrow xEz\ (\Leftrightarrow xEyEzEw).$$

This is clearly a countable Borel equivalence relation on E.

Proposition 16.2. *M is $\tilde{E}$-invariant.*

Proof. Let Γ be a countable group acting in a Borel way on X so that $E_\Gamma^X = E$. Then if Γ^2 acts on E by $(g,h)\cdot(x,y) = (g\cdot x, h\cdot y)$, clearly $E_{\Gamma^2}^E = \tilde{E}$. So it is enough to show that this action preserves M. Now for $A \subseteq E$, A Borel, we have

$$\begin{aligned} M((g,h)\cdot A) &= M(\{(g\cdot x, h\cdot y) : (x,y)\in A\}) \\ &= \int |B_x| \; d\mu(x), \end{aligned}$$

where $B = \{(g\cdot x, h\cdot y) : (x,y)\in A\}$. But

$$\begin{aligned} y \in B_x &\Leftrightarrow (x,y)\in B \Leftrightarrow (g^{-1}\cdot x, h^{-1}\cdot y)\in A \\ &\Leftrightarrow h^{-1}\cdot y \in A_{g^{-1}\cdot x}, \end{aligned}$$

so $B_x = h\cdot A_{g^{-1}\cdot x}$, thus

$$\begin{aligned} M((g,h)\cdot A) &= \int |A_{g^{-1}\cdot x}| \; d\mu(x) \\ &= \int |A_x| \; d\mu(x) = M(A), \end{aligned}$$

by the Γ-invariance of μ. ⊣

Finally, when μ is also *E-ergodic*, i.e., every E-invariant Borel set is either null or conull, we also have a converse to 16.1(iii), see 7.10.

Proposition 16.3. *Let E be a countable Borel equivalence relation on X and suppose a measure μ is E-invariant ergodic. If $A, B \subseteq X$ are Borel sets with $\mu(A) = \mu(B)$, then there is $f \in [[E]]$ with* $\mathrm{dom}(f) = A'$, $\mathrm{rng}(f) = B'$, *where $A' \subseteq A$, $B' \subseteq B$ and $\mu(A \setminus A') = \mu(B \setminus B') = 0$.*

17 Graphings of Equivalence Relations

A (locally countable Borel) *graph* on a standard Borel space X is a graph $\mathcal{G}$ on X, such that $\mathcal{G} \subseteq X^2$ is Borel, and every $x \in X$ has at most countably many neighbors. Let E be a countable Borel equivalence relation. A (Borel) *graphing of E* is a graph $\mathcal{G}$ such that the connected components of $\mathcal{G}$ are exactly the E-equivalence classes. We then say that $\mathcal{G}$ *generates* E.

It will be also convenient to consider another concept of graph, which for distinction we will call an L-*graph* (L stands for Levitt). This is simply a countable family $\Phi = \{\varphi_i\}_{i\in I}$ of partial Borel isomorphisms, $\varphi_i : A_i \to B_i$, where A_i, B_i are Borel subsets of X. We call Φ *finite* if I is finite. Φ is an L-*graphing of E* if Φ *generates* E, i.e., $xEy \Leftrightarrow x = y$ or there is a sequence

$i_1, \dots, i_k \in I$ and $\epsilon_1, \dots, \epsilon_k \in \{\pm 1\}$ such that $x = \varphi_{i_1}^{\epsilon_1} \dots \varphi_{i_k}^{\epsilon_k}(y)$ (in particular each φ_i is in $[[E]]$, which we can abbreviate as $\Phi \subseteq [[E]]$).

Note that for every L-graph $\Phi = \{\varphi_i\}$ one has an associated graph $\mathcal{G}_\Phi$, which generates the same equivalence relation, defined as follows: $x\mathcal{G}_\Phi y$ iff $x \neq y$ & $\exists i[\varphi_i(x) = y$ or $\varphi_i(y) = x]$. Conversely, for every graph $\mathcal{G}$ we can find an L-graph $\Phi_\mathcal{G}$ such that $\mathcal{G} = \mathcal{G}_{\Phi_\mathcal{G}}$. To see this, fix a Borel ordering $<$ on X and find a countable family of partial Borel functions $\{f_i\}_{i \in \mathbb{N}}$ with disjoint graphs such that $\forall x \in X$:

$$\{(x, y) : x\mathcal{G}y \ \& \ x < y\} = \bigcup_{i \in \mathbb{N}} \mathrm{graph}(f_i).$$

Now f_i is countable-to-one, therefore let $\{g_{i,j}\}_{j \in K_i}$ be partial Borel automorphisms with disjoint graphs such that $\mathrm{graph}(f_i)^{-1} = \bigcup_j \mathrm{graph}(g_{i,j})$. Let $\Phi_\mathcal{G} = \{g_{i,j}\}$.

If μ is E-invariant and $\mathcal{G}$ is a graphing of $E|A$ for a conull E-invariant Borel set A, then we call $\mathcal{G}$ a *graphing of E a.e.* Similarly we define an L-*graphing a.e.*

We use the following notation concerning L-graphs. Let $\Phi = \{\varphi_i\}_{i \in I}$, $\Psi = \{\psi_j\}_{j \in J}$ be L-graphs. By $\Phi \sqcup \Psi$ we denote the disjoint union of Φ, Ψ, defined by replacing J by a copy J', via a bijection $j \mapsto j'$, so that $I \cap J' = \emptyset$, and letting $\Phi \sqcup \Psi = \{\varphi_i, \psi_{j'}\}_{i \in I, j' \in J'}$, where $\psi_{j'} = \psi_j$. Similarly we define $\Phi_0 \sqcup \Phi_1 \sqcup \Phi_2 \sqcup \dots$ We write $\Phi \subseteq \Psi$ if $I \subseteq J$ and $\varphi_i = \psi_i$, for $i \in I$. In this case we also let $\Psi \setminus \Phi = \{\psi_i\}_{i \in J \setminus I}$. If $\Phi_0 \subseteq \Phi_1 \subseteq \dots, \Phi_i = \{\varphi_j\}_{j \in I_i}$, where $I_0 \subseteq I_1 \subseteq \dots$, then we let $\bigcup_n \Phi_n = \{\varphi_j\}_{j \in \bigcup_n I_n}$. Finally, if $A \subseteq X$, we put $\Phi|A = \{\varphi_i|A\}_{i \in I}$.

18 Cost of an Equivalence Relation

We will now start the systematic development of the theory of costs originated in Levitt [L] and mainly developed in Gaboriau [G1], [G2], [G3]. The results of Levitt and Gaboriau discussed below are contained in these papers.

Let E be a countable Borel equivalence relation on X and μ an E-invariant measure. If $\mathcal{G} \subseteq E$, we define its (μ-)*cost* by

$$C_\mu(\mathcal{G}) = \frac{1}{2} M(\mathcal{G})$$

(the factor $\frac{1}{2}$ is a normalizing constant). Thus $0 \leq C_\mu(\mathcal{G}) \leq \infty$, and $0 < C_\mu(\mathcal{G})$, if $\mathcal{G}$ is a graphing of E, provided that it is not the case that almost all E-classes are singletons. (To see that $C_\mu(\mathcal{G}) > 0$, notice that $\mathcal{G}$ is a complete section for $\tilde{E}$, so $M(\mathcal{G}) > 0$.) Also $C_\mu(\mathcal{G})$ is $\frac{1}{2}$ of the integral of the degree $d_\mathcal{G}(x)$ of a vertex in the graph $\mathcal{G}$, i.e., $C_\mu(\mathcal{G}) = \frac{1}{2} \int d_\mathcal{G}(x) \ d\mu(x)$.

If now $\Phi = \{\varphi_i\}_{i \in I} \subseteq [[E]]$ is an L-graph, define its (μ-)*cost* by

$$\begin{aligned} C_\mu(\Phi) &= \sum_{i\in I} \mu(\mathrm{dom}(\varphi_i)) \\ &= \sum_{i\in I} \mu(\mathrm{rng}(\varphi_i)). \end{aligned}$$

Notice that

$$C_\mu(\Phi) = \tfrac{1}{2} \int \sum_{i\in I} (1_{A_i}(x) + 1_{B_i}(x))\ d\mu(x),$$

where $A_i = \mathrm{dom}(\varphi_i), B_i = \mathrm{rng}(\varphi_i)$, and $1_A =$ the characteristic function of A. Thus if $\mathcal{G}_\Phi$ is the graph associated to Φ, then $|(\mathcal{G}_\Phi)_x| \leq \sum_{i\in I}(1_{A_i}(x)+1_{B_i}(x))$, so $C_\mu(\mathcal{G}_\Phi) = \frac{1}{2}M(\mathcal{G}_\Phi) \leq C_\mu(\Phi)$.

Conversely, let $\mathcal{G} \subseteq E$ be a graph and let $\Phi_\mathcal{G}$ be the associated L-graph, as in Section 17. Then in the notation used there and noticing that $(x,y) \mapsto (y,x)$ is in $[\tilde{E}]$, we have

$$\begin{aligned} C_\mu(\mathcal{G}) &= \frac{1}{2} M(\mathcal{G}) = M(\{(x,y) : x\mathcal{G}y \ \&\ x < y\}) \\ &= M\left(\bigcup_{i\in\mathbb{N}} \mathrm{graph}(f_i)\right) = \sum_{i\in\mathbb{N}} M(\mathrm{graph}(f_i)) \\ &= \sum_{i\in\mathbb{N}} M(\mathrm{graph}(f_i)^{-1}) = \sum_{i\in\mathbb{N}} \sum_{j\in K_i} M(\mathrm{graph}(g_{i,j})) = C_\mu(\Phi_\mathcal{G}). \end{aligned}$$

Thus we have

$$\inf\{C_\mu(\mathcal{G}) : \mathcal{G} \text{ is a graphing of } E\} = \inf\{C_\mu(\Phi) : \Phi \text{ is an L-graphing of } E\}.$$

This common quantity is called the (μ-)*cost* of E, and denoted by:

$$C_\mu(E).$$

Thus $0 \leq C_\mu(E) \leq \infty$. Clearly $C_\mu(\Delta_X) = 0$, where Δ_X is the equality on X.

Note the following easy fact: If $X = \bigcup_n A_n$ is a countable partition of X into E-invariant Borel sets of positive measure, then

$$C_\mu(E) = \sum_n C_{\mu|A_n}(E|A_n).$$

We now compute an estimate for the descriptive complexity of the function $\mu \mapsto C_\mu(E)$.

Proposition 18.1. *Let E be a countable Borel equivalence relation on X, $\mathcal{I}_E$ the standard Borel space of E-invariant probability measures on X (a Borel subset of the standard Borel space of probability measures on X). Then for each $r \in \mathbb{R}$, the set*

$$\{\mu \in \mathcal{I}_E : C_\mu(E) < r\}$$

is analytic.

Proof. Fix a countable sequence of Borel automorphisms $\{g_k\}$ in $[E]$, whose graphs cover E, fix a countable Boolean algebra of Borel sets $\{A_\ell\}$ which generates the Borel sets of X, let $\{\theta_n\}$ be an enumeration of $\{g_k|A_\ell\}$, and for each $S \subseteq \mathbb{N}$, put $\Theta_S = \{\theta_n\}_{n\in S}$. We will show that

$$C_\mu(E) < r \Leftrightarrow \exists S \subseteq \mathbb{N}\,(\Theta_S \text{ is an L-graphing of } E \ \mu\text{-a.e. and } C_\mu(\Theta_S) < r)\,.$$

Of course, ($\Leftarrow$) is clear. To see ($\Rightarrow$), suppose that Ψ is an L-graphing of E with $C_\mu(\Psi) < r$, and fix $\epsilon > 0$ such that $C_\mu(\Psi) < r - 2\epsilon$. Note that, by splitting up the domains of the elements of Ψ, we may assume that Ψ is of the form $\{g_{i_k}|B_k\}$, where each B_k is Borel. Let $\{k_n\}$ be an enumeration of the natural numbers in which every natural number appears infinitely often, and recursively choose ℓ_n such that

$$\mu\left(A_{\ell_n} \Delta \left(B_{k_n} \setminus \bigcup_{m<n, k_m=k_n} A_{\ell_m}\right)\right) < \epsilon/2^n.$$

This can be done because $\{A_\ell\}$ is dense in the measure algebra of μ, by 17.43 of [K]. Find $S \subseteq \mathbb{N}$ such that $\Theta_S = \{g_{k_n}|A_{\ell_n}\}$, and note that for all k,

$$\begin{aligned}
\sum_{k_n=k} \mu(A_{\ell_n}) = & \sum_{k_n=k} \mu\left(A_{\ell_n} \cap \left(B_k \setminus \bigcup_{m<n, k_m=k_n} A_{\ell_m}\right)\right) + \\
& \sum_{k_n=k} \mu\left(A_{\ell_n} \setminus \left(B_k \setminus \bigcup_{m<n, k_m=k_n} A_{\ell_m}\right)\right) \\
< & \sum_{k_n=k} \mu\left(B_k \cap \left(A_{\ell_n} \setminus \bigcup_{m<n, k_m=k_n} A_{\ell_m}\right)\right) + \sum_{k_n=k} \epsilon/2^n \\
\leq & \ \mu(B_k) + \sum_{k_n=k} \epsilon/2^n,
\end{aligned}$$

thus

$$\begin{aligned}
C_\mu(\Theta_S) &= \sum_k \sum_{k=k_n} \mu(A_{\ell_n}) \\
&< \sum_k \left(\mu(B_k) + \sum_{k_n=k} \epsilon/2^n\right) \\
&= C_\mu(\Psi) + \sum_n \epsilon/2^n \\
&= C_\mu(\Psi) + 2\epsilon \\
&< r.
\end{aligned}$$

It remains to check that Θ_S is an L-graphing of E μ-a.e., and this follows from the observation that for all k, $\mu\big(B_k \setminus \bigcup_{k_n=k} A_{\ell_n}\big) = 0$. $\dashv$

Problem 18.2. Is the function $\mu \mapsto C_\mu(E)$ Borel?

In connection to this problem, Hjorth proved the following result, which shows that the question of whether $\mu \mapsto C_\mu(E)$ is Borel is equivalent to the question of whether $\{\mu \in \mathcal{I}_E : C_\mu(E) < \infty\}$ is Borel.

Theorem 18.3 (Hjorth). *Let E be a countable Borel equivalence relation on $X, \mathcal{I}_E$ the standard Borel space of E-invariant probability measures on X. Consider the analytic set*

$$M_f = \{\mu \in \mathcal{I}_E : C_\mu(E) < \infty\}.$$

Then $\mu \mapsto C_\mu(E)$ is Borel on M_f.

Proof. It is enough to show that for each $r \in \mathbb{R}$,

$$\{\mu \in M_f : C_\mu(E) < r\}$$

is co-analytic. To see this, it is enough to show, for $\mu \in M_f$ and using the notation of 18.1, that $C_\mu(E) < r$ exactly when

$$\forall S \subseteq \mathbb{N}\big(\Theta_S \text{ is an L-graphing of } E\ \mu\text{-a.e. of finite } \mu\text{-cost } \Rightarrow \exists S' \subseteq \mathbb{N} \text{ finite}$$
$$\exists N \in \mathbb{N}\big(C_\mu\big(\Theta_{S'} \sqcup \Theta_{S\setminus\{0,\ldots,N\}} \sqcup \{\theta_i | D(\theta_i, \Theta_{S'})\}_{i \in S\cap\{0,\ldots,N\}}\big) < r\big)\big),$$

where, for $\theta \in [[E]]$ and Θ an L-graph,

$$D(\theta, \Theta) = \{x \in \mathrm{dom}(\theta) : (x, \theta(x)) \notin R_\Theta\},$$

and R_Θ is the equivalence relation generated by Θ.

To see ($\Leftarrow$), simply take $S \subseteq \mathbb{N}$ such that Θ_S an L-graphing of E μ-a.e. of finite cost and note that $\Theta_{S'} \sqcup \Theta_{S\setminus\{0,\ldots,N\}} \sqcup \{\theta_i | D(\theta_i, \Theta_{S'})\}_{i\in S\cap\{0,\ldots,N\}}$ is an L-graphing of E μ-a.e. To see ($\Rightarrow$), suppose $\Theta_{S''}$ is an L-graphing of E with $C_\mu(\Theta_{S''}) < r - 2\epsilon$, take $S \subseteq \mathbb{N}$ with Θ_S an L-graphing of E μ-a.e. of finite cost, and choose N sufficiently large that $C_\mu\big(\Theta_{S\setminus\{0,\ldots,N\}}\big) < \epsilon$. Noting, for all $i \in S$ and all $x \in \mathrm{dom}(\theta_i)$, that $(x, \theta_i(x)) \in R_{\Theta_{S''\cap\{0,\ldots,n\}}}$ for n sufficiently large, it follows that for all i,

$$\lim_{n\to\infty} \mu\big(D\big(\theta_i, \Theta_{S''\cap\{0,\ldots,n\}}\big)\big) = 0.$$

Thus, for n sufficiently large, $\mu(D(\theta_i, \Theta_{S''\cap\{0,\ldots,n\}})) < \epsilon/(N+1)$, for all $i \in S \cap \{0, \ldots, N\}$, so $S' = S'' \cap \{0, \ldots, n\}$ is as desired. ⊣

We now use 18.1 to determine the behavior of cost under measure disintegration.

Let X, Y be standard Borel spaces and $f : X \to Y$ a Borel map. Let μ be a probability measure on X and put $\nu = f_*\mu$ (i.e., $\nu(A) = \mu(f^{-1}(A))$). Then there is a Borel map $y \mapsto \mu_y$ from Y into the standard Borel space $P(X)$ of probability measures on X such that (i) $\forall^*_\nu y(\mu_y(f^{-1}(\{y\})) = 1)$ and (ii)

$\mu = \int \mu_y \, d\nu(y)$, i.e., $\mu(A) = \int \mu_y(A) \, d\nu(y)$, for any Borel set $A \subseteq X$. This $y \mapsto \mu_y$ is unique, ν-a.e., satisfying (i), (ii) (see 17.35 of [K]).

Now note that if E is a countable Borel equivalence relation on X and f is E*-invariant*, i.e., $f(x) = f(y)$ whenever xEy, so that each fiber $f^{-1}(\{y\})$ is also E-invariant, then μ_y is E-invariant, ν-a.e. Because if Γ is a countable group of Borel automorphisms of X so that $xEy \Leftrightarrow \exists g \in \Gamma(g(x) = y)$, and $g_*\mu_y = \nu_y$, then $\nu_y(f^{-1}(\{y\})) = 1$, ν-a.e., as $f^{-1}(\{y\})$ is invariant under g. Also for any Borel set $A \subseteq X$, $\mu(A) = \mu(g^{-1}(A)) = \int \mu_y(g^{-1}(A)) \, d\nu(y) = \int \nu_y(A) \, d\nu(y)$. So $\nu_y = \mu_y$ ν-a.e., i.e., μ_y is Γ-invariant ν-a.e.

We now have the following formula connecting $C_\mu(E)$ and $C_{\mu_y}(E)$.

Proposition 18.4. *Let E be a countable Borel equivalence relation on X and μ an E-invariant probability measure. Let $f : X \to Y$ be an E-invariant Borel map, $\nu = f_*\mu$, and $\{\mu_y\}_{y \in Y}$ the disintegration of μ given by f. Then*

$$C_\mu(E) = \int C_{\mu_y}(E) \, d\nu(y).$$

Proof. It is easy to check that if Φ is an L-graphing of E μ-a.e., then Φ is an L-graphing of E μ_y-a.e., for ν-a.e. y, and

$$C_\mu(\Phi) = \int C_{\mu_y}(\Phi | f^{-1}(\{y\})) \, d\nu(y) = \int C_{\mu_y}(\Phi) \, d\nu(y).$$

Now, given $\epsilon > 0$, find an L-graphing Φ of E μ-a.e. such that $\epsilon + C_\mu(E) \geq C_\mu(\Phi)$. Then

$$\begin{aligned} \epsilon + C_\mu(E) &\geq C_\mu(\Phi) \\ &= \int C_{\mu_y}(\Phi) \, d\nu(y) \\ &\geq \int C_{\mu_y}(E) \, d\nu(y), \end{aligned}$$

so $C_\mu(E) \geq \int C_{\mu_y}(E) \, d\nu(y)$.

Now since $y \in Y \mapsto C_{\mu_y}(E)$ is ν-measurable (by 18.1), let $F : Y \to \mathbb{R}$ be Borel with $F(y) = C_{\mu_y}(E)$ ν-a.e. Let $A \subseteq Y$ be Borel with $\nu(A) = 1$ such that $F(y) = C_{\mu_y}(E)$, $\mu_y(f^{-1}(\{y\})) = 1$, and μ_y is E-invariant, $\forall y \in A$.

Fix $\epsilon > 0$. Then, for all $y \in A$, there is an L-graphing Φ of E μ_y-a.e. with $C_{\mu_y}(\Phi) \leq C_{\mu_y}(E) + \epsilon$. Using the conventions and notations of 18.1, it follows that for all $y \in A$,

$$\exists S \subseteq \mathbb{N}(\Theta_S \text{ is an L-graphing of } E \ \mu_y\text{-a.e. and } C_{\mu_y}\big(\Theta_S\big) \leq F(y) + \epsilon).$$

Since the conditions in parenthesis are Borel, it follows, by the Jankov-von Neumann Uniformization Theorem (see 18.1 of [K]), that there is a ν-measurable function $S : A \to \text{Power}(\mathbb{N})$ which, for $y \in A$, gives $S(y)$ satisfying these conditions. By shrinking A a bit, we can assume that S is Borel.

Put $\Phi_y = \Theta_{S(y)}|f^{-1}(\{y\})$, so that $C_{\mu_y}(\Phi_y) \leq C_{\mu_y}(E) + \epsilon,\ \forall y \in A$. Then

$$\Phi = \left\{\theta_n | f^{-1}(\{y : n \in S(y)\})\right\}_{n \in \mathbb{N}}$$

is an L-graphing of E μ-a.e., so

$$\begin{aligned} C_\mu(E) &\leq C_\mu(\Phi) \\ &= \int C_{\mu_y}(\Phi)\ d\nu(y) \\ &= \int C_{\mu_y}(\Phi_y)\ d\nu(y) \\ &\leq \int (C_{\mu_y}(E) + \epsilon)\ d\nu(y) \\ &= \int C_{\mu_y}(E)\ d\nu(y) + \epsilon, \end{aligned}$$

thus $C_\mu(E) \leq \int C_{\mu_y}(E)\ d\nu(y)$ and the proof is complete. ⊣

An important case of measure disintegration is the ergodic decomposition of an invariant measure.

For a countable Borel equivalence relation E, we denote by $\mathcal{EI}_E$ the standard Borel space of E-ergodic invariant probability measures (again a Borel subset of the standard Borel space $P(X)$ of probability measures on X, see [DJK], Section 4). Recall that a measure μ is E-ergodic if every Borel E-invariant set is either null or conull.

We now state the ergodic decomposition theorem of Farrell, Varadarajan (see [F], [V]).

Theorem 18.5. (Ergodic Decomposition – Farrell [F], Varadarajan [V]) *Let E be a countable Borel equivalence relation on a standard Borel space X and assume $\mathcal{I}_E \neq \emptyset$. Then $\mathcal{EI}_E \neq \emptyset$ and there is a Borel surjection $\pi : X \to \mathcal{EI}_E$ such that:*

(i) π is E-invariant.
(ii) If $X_e = \{x : \pi(x) = e\}$, for $e \in \mathcal{EI}_E$, then $e(X_e) = 1$ (and in fact e is the unique E-ergodic invariant measure on $E|X_e$).
(iii) For any $\mu \in \mathcal{I}_E$, $\mu = \int \pi(x)\ d\mu(x) (= \int e\ d\nu(e)$, where $\nu = \pi_*\mu)$.

We then have:

Corollary 18.6. *In the notation of 18.4,18.5, for any $\mu \in \mathcal{I}_E$,*

$$C_\mu(E) = \int C_{\pi(x)}(E)\ d\mu(x) = \int C_e(E)\ d\nu(e).$$

19 Treeings of an Equivalence Relation

A graphing $\mathcal{T}$ of an equivalence relation E is called a *treeing* if $\mathcal{T}$ is acyclic. An L-graphing $\Phi = \{\varphi_i\}$ of E is called a *treeing* if for each non-empty (formal) reduced word $w = \varphi_{i_1}^{\epsilon_1} \dots \varphi_{i_n}^{\epsilon_n} (\epsilon_i \in \{\pm 1\})$, in the symbols $\{\varphi_i\}$, the set $\{x : x \in \mathrm{dom}(w) \ \& \ w(x) = x\}$ is empty.

If μ is an E-invariant measure, then we say that $\mathcal{T}$ is a *treeing* μ*-a.e.* if $\mu(\{x : \mathcal{T}|[x]_E \text{ is not acyclic}\}) = 0$. Similarly Φ is called a *treeing* μ*-a.e.* if $\mu(\{x : x \in \mathrm{dom}(w) \ \& \ w(x) = x\}) = 0$, for all non-empty reduced words w.

Note that if Φ is an L-treeing (resp., a.e.), then $\mathcal{G}_\Phi$ is also a treeing (resp., a.e.) and $C_\mu(\mathcal{G}_\Phi) = C_\mu(\Phi)$ as $|(\mathcal{G}_\Phi)_x| = \sum_{i \in I} (x_{A_i}(x) + \chi_{B_i}(x))$, (resp., a.e.), if $\Phi = \{\varphi_i\}_{i \in I}$, $\mathrm{dom}(\varphi_i) = A_i$, $\mathrm{rng}(\varphi_i) = B_i$. Also if $\mathcal{T}$ is a treeing (resp., a.e.), then $\Phi_\mathcal{T}$ is a treeing (resp., a.e.).

The relevance of treeings to costs is illustrated by the following fact.

Proposition 19.1 (Gaboriau). *Let E be a countable Borel equivalence relation and let μ be E-invariant with $C_\mu(E) < \infty$. If $\mathcal{G}$ is a graphing of E which attains the cost of E, i.e., $C_\mu(\mathcal{G}) = C_\mu(E)$, then $\mathcal{G}$ is a treeing of E a.e. Similarly for* L*-graphings.*

Proof. By the preceding remarks it is enough to prove this for L-graphings. Indeed if $C_\mu(\mathcal{G}) = C_\mu(E)$, then $C_\mu(\Phi_\mathcal{G}) = C_\mu(\mathcal{G})$, so $C_\mu(\Phi_\mathcal{G}) = C_\mu(E)$ and then $\Phi_\mathcal{G}$ is a treeing a.e. and as $\mathcal{G} = \mathcal{G}_{\Phi_\mathcal{G}}$, $\mathcal{G}$ is a treeing a.e.

Let $\Phi = \{\varphi_i\}$ have $C_\mu(\Phi) = C_\mu(E) < \infty$, and let $w = \varphi_{i_n}^{\epsilon_n} \dots \varphi_{i_1}^{\epsilon_1}$ be a reduced non-empty word of least length with $\mu(\{x \in \mathrm{dom}(w) : w(x) = x\}) > 0$, towards a contradiction. Then if $w_j = \varphi_{i_j}^{\epsilon_j} \dots \varphi_{i_1}^{\epsilon_1}$, $1 \leq j \leq n$,

$$\{x \in \mathrm{dom}(w) : w(x) = x \text{ and two of } x, w_1(x), \dots, w_{n-1}(x) \text{ are equal}\}$$

has measure 0 (by the minimality of n). So $A = \{x : x \in \mathrm{dom}(w) \ \& \ w(x) = x \ \& \ x, w_1(x), \dots, w_{n-1}(x) \text{ are distinct}\}$ has positive finite measure. By changing the Polish topology of X but not its Borel structure, we can assume that A and $\mathrm{dom}(\varphi_i)$, $\mathrm{rng}(\varphi_i)$ are clopen and φ_i is a homeomorphism, for each i. It follows that we can find a Borel set $B \subseteq A$ with $\infty > \mu(B) > 0$ and $B, w_1(B), \dots, w_{n-1}(B)$ pairwise disjoint. Then assuming $\epsilon_1 = 1$, without loss of generality, $\Phi' = \{\varphi_i\}_{i \in I, i \neq i_1} \cup \{\varphi_{i_1} | (A_{i_1} \setminus B)\}$, where $A_{i_1} = \mathrm{dom}(\varphi_{i_1}) \supseteq A$, is still an L-graphing of E and $C_\mu(\Phi') < C_\mu(\Phi)$, a contradiction. $\dashv$

One of the main results of Gaboriau's theory is the converse of this proposition. We will prove later the following (see 20.1, 21.3 and 27.10).

Theorem 19.2 (Gaboriau). *Let E be a countable Borel equivalence relation and μ an E-invariant measure. If $\mathcal{T}$ is a treeing of E a.e., then $C_\mu(\mathcal{T}) = C_\mu(E)$. Similarly for* L*-treeings.*

Notice that this also implies a positive answer to 18.2, if E is *treeable*, i.e., admits a treeing.

20 The Cost of a Smooth Equivalence Relation

Recall that a countable Borel equivalence relation E is *smooth* if it has a *Borel transversal*, i.e., a Borel set meeting every equivalence class in exactly one point. If μ is a measure and there is a conull Borel set $A \subseteq X$ such that $E|A$ is smooth we say that E is *smooth μ-a.e.* Note that if μ is finite and E-invariant, then E is smooth μ-a.e. iff E is *periodic μ-a.e.* (i.e., for μ-a.e. x, $[x]_E$ is finite). This is because if an equivalence relation R on a space Y is smooth and aperiodic and $Z \subseteq Y$ is a Borel transversal, then there is an infinite sequence $f_n : Z \to Y$ of Borel injections such that $z \in Z \;\&\; n \neq m \Rightarrow f_n(z) \neq f_m(z)$ and $f_n(z)Ez$, for all n and $z \in Z$.

We can easily calculate the cost of a smooth equivalence relation as follows:

Proposition 20.1 (Levitt). *Let E be a countable Borel equivalence relation on X, μ an E-invariant measure. If E is smooth and $T \subseteq X$ is a Borel transversal, then $C_\mu(E) = \mu(X \setminus T)$ and any treeing (resp.* L*-treeing of E) realizes $C_\mu(E)$.*

Proof. Fix a graphing $\mathcal{G}$ of E. Since E is smooth, we can easily find a treeing $\mathcal{T}$ of E such that $\mathcal{T} \subseteq \mathcal{G}$. Thus $C_\mu(\mathcal{G}) \geq C_\mu(\mathcal{T})$. So it is enough to show that if $\mathcal{T}$ is a treeing of E, then $C_\mu(\mathcal{T}) = \mu(X \setminus T)$. Since the map $s(x, y) = (y, x)$ is in $[\tilde{E}]$, it is enough to show that there is $\mathcal{T}' \subseteq \mathcal{T}$ such that $s(\mathcal{T}') \cap \mathcal{T}' = \emptyset$, $s(\mathcal{T}') \cup \mathcal{T}' = \mathcal{T}$ and $M(\mathcal{T}') = \mu(X \setminus T)$.

For $x \in X \setminus T$, let $\sigma(x) =$ the unique $t \in T$ such that xEt. Define

$$\begin{aligned} x\mathcal{T}'y \Leftrightarrow \quad & x \in X \setminus T \;\&\; \text{if } x_0 = x, x_1, \dots, \\ & x_n = \sigma(x) \text{ is the unique path in } \mathcal{T} \\ & \text{from } x \text{ to } \sigma(x), \text{ then } y = x_1. \end{aligned}$$

Clearly $s(\mathcal{T}') \cap \mathcal{T}' = \emptyset$ and $s(\mathcal{T}') \cup \mathcal{T}' = \mathcal{T}$, so it is enough to check that $M(\mathcal{T}') = \mu(X \setminus T)$. But note that $x\mathcal{T}'y \Rightarrow x \in X \setminus T$ and $\forall x \in (X \setminus T)\exists! y(x, y) \in \mathcal{T}'$, so this is obvious. ⊣

In particular, if for all $x \in X$, $|[x]_E| = n$, then every transversal T of X has the property that $n \cdot \mu(T) = \mu(X)$, thus if $\mu(X) < \infty$, $C_\mu(E) = \mu(X) - \mu(X)/n = (1 - \frac{1}{n})\mu(X)$. If on the other hand $\mu(X) = \infty$, and $n > 1$, $\mu(X \setminus T) = \infty$, so $C_\mu(E) = \infty$. In any case,

$$C_\mu(E) = \left(1 - \frac{1}{n}\right)\mu(X),$$

using the convention $0 \cdot \infty = 0$.

In general, for an arbitrary, not necessarily smooth E, let $X_n = \{x \in X : |[x]_E| = n\}$, $n = 1, 2, \dots, \infty$. Then

$$\begin{aligned} C_\mu(E) &= C_{\mu|X_\infty}(E|X_\infty) + \sum_{n=1}^{\infty} C_{\mu|X_n}(E|X_n) \\ &= C_{\mu|X_\infty}(E|X_\infty) + \sum_{n=2}^{\infty} \left(1 - \frac{1}{n}\right) \mu(X_n). \end{aligned}$$

Remark 20.2. If G is a finite connected graph with vertex set V, then it is an elementary fact that

$$\frac{1}{2}|G| = \frac{1}{2} \sum_{x \in V} d_G(x) \geq |V| - 1, \tag{$*$}$$

and equality holds exactly when G is a tree.

Proposition 20.1 can be viewed as a generalization of this fact. Indeed, if we take $X = V$, $E = V \times V$, $T = \{v_0\}$ for some fixed $v_0 \in V$, μ the counting measure on X, then $C_\mu(E) = \inf\{\frac{1}{2}|G| : G$ is a connected graph on $V\} = |V| - 1 = \mu(X \setminus T)$, and any tree with vertex set V realizes $C_\mu(E)$.

We can also use $(*)$ to prove 20.1, when μ is a finite measure, in which case $|[x]_E| < \infty$, μ-a.e. (x). Indeed, first notice the following simple fact:

If $f : X \to [0, \infty]$ is Borel, then

$$\int f(x)\ d\mu(x) = \int_{x \in T} \left(\sum_{yEx} f(y) \right)\ d\mu(x).$$

To see this, notice that, by the usual approximation arguments, it is enough to prove this for characteristic functions, i.e., to show that if $A \subseteq X$ is Borel, then

$$\mu(A) = \int_{x \in T} |A \cap [x]_E|\ d\mu(x).$$

This is easy to prove, writing $T = \bigcup_n T_n$, where $T_n = \{x \in T : |A \cap [x]_E| = n\}$, $n = 1, 2, \ldots, \infty$, and using the E-invariance of μ.

So if $\mathcal{G}$ is any graphing of E, then

$$\begin{aligned} C_\mu(\mathcal{G}) &= \frac{1}{2} \int d_{\mathcal{G}}(x)\ d\mu(x) \\ &= \int_{x \in T} \left(\frac{1}{2} \sum_{yEx} d_{\mathcal{G}}(y) \right)\ d\mu(x) \\ &\geq \int_{x \in T} (|[x]_E| - 1)\ d\mu(x) \\ &= \mu(X \setminus T), \end{aligned}$$

and equality holds if $\mathcal{G}$ is a treeing.

21 The Cost of a Complete Section

The following result is crucial for many computations.

Theorem 21.1 (Gaboriau). *Let E be a countable Borel equivalence relation on X, $S \subseteq X$ a Borel complete section for E and μ an E-invariant measure. Then*

$$C_\mu(E) = C_{\mu|S}(E|S) + \mu(X \setminus S).$$

We will derive this from the following, very useful in its own right, lemma whose idea of the proof is due independently to Gaboriau, and Jackson-Kechris-Louveau (who used it to prove the last statement of this lemma).

Lemma 21.2. *In the notation of 21.1, for every graphing $\mathcal{G}$ of E, there is a Borel subequivalence relation V of E in which S is a transversal, a treeing $\mathcal{T}_V$ of V with $\mathcal{T}_V \subseteq \mathcal{G}$, so that $\mathcal{T}_V \cap S^2 = \emptyset$, and a graphing $\mathcal{G}_S$ of $E|S$ such that $C_\mu(\mathcal{T}_V) = \mu(X \setminus S)$ and $C_\mu(\mathcal{G}) \geq \mu(X \setminus S) + C_\mu(\mathcal{G}_S)$. Moreover, for each graphing $\mathcal{G}'_S$ of $E|S$, $\mathcal{G}'_S \cup \mathcal{T}_V$ is a graphing of E. Finally, if $\mathcal{G}$ is a treeing of E, $\mathcal{G}_S$ is a treeing of $E|S$.*

To see that 21.2 $\Rightarrow$ 21.1, we note that $C_\mu(\mathcal{G}) \geq \mu(X \setminus S) + C_{\mu|S}(\mathcal{G}_S) \Rightarrow C_\mu(\mathcal{G}) \geq \mu(X \setminus S) + C_{\mu|S}(E|S)$, so as $\mathcal{G}$ was an arbitrary graphing of E, $C_\mu(E) \geq \mu(X \setminus S) + C_{\mu|S}(E|S)$. Now fix a graphing $\mathcal{G}'_S$ of $E|S$. Then $\mathcal{G}'_S \cap \mathcal{T}_V = \emptyset$, and $\mathcal{G}'_S \cup \mathcal{T}_V$ is a graphing of E, so $C_\mu(\mathcal{G}'_S \cup \mathcal{T}_V) = C_\mu(\mathcal{G}'_S) + C_\mu(\mathcal{T}_V)$, and $C_\mu(E) \leq C_\mu(\mathcal{G}'_S \cup \mathcal{T}_V) = C_{\mu|S}(\mathcal{G}'_S) + \mu(X \setminus S)$, so, taking the infimum over $\mathcal{G}'_S$, we get $C_\mu(E) \leq C_{\mu|S}(E|S) + \mu(X \setminus S)$.

Proof of Lemma 21.2. Fix a sequence of Borel functions $\{g_n\}$ of X with $xEy \Leftrightarrow \exists n(g_n(x) = y)$. Fix $x \in X \setminus S$. Let n be the least length of a $\mathcal{G}$-path $x'_0 = x,\ x'_1, \dots, x'_n = z \in S$ from x to S. Among all such paths, choose the "lexicographically least one" defined as follows: For uEv let $\ell(u, v)$ be the least ℓ with $g_\ell(u) = v$. Then $x_0 = x, x_1, \dots, x_n = y$ is the lexicographically least path if $(\ell(x_0, x_1), \ell(x_1, x_2), \dots, \ell(x_{n-1}, x_n))$ is lexicographically least among all $(\ell(x'_0, x'_1), \dots, \ell(x'_{n-1}, x'_n))$ as above.

We call this $x_0 = x, x_1, \dots x_n = y$ the *canonical $\mathcal{G}$-path* from x to S. Put also $\pi(x) = y$ and define $\pi(y) = y$ if $y \in S$.

Notice that if $x_0 = x, x_1, \dots, x_n = y$ is the canonical $\mathcal{G}$-path from x to S, then $x_1, x_2, \dots x_n = y$ is the canonical $\mathcal{G}$-path from x_1 to S (provided $n \geq 2$).

Let now V be defined by

$$xVy \Leftrightarrow \pi(x) = \pi(y).$$

Clearly V is a subequivalence relation of E and S is a transversal for V.

Define now $\mathcal{T}_V$ as follows: Let $\mathcal{T}_1$ consist of all (x, x') such that $x \in X \setminus S$ and $x = x_0, x_1 = x', \dots, x_n = y$ is the canonical $\mathcal{G}$-path from x to S. Let $\mathcal{T}_2 = (\mathcal{T}_1)^{-1} = \{(x', x) : (x, x') \in \mathcal{T}_1\}$ and put $\mathcal{T}_V = \mathcal{T}_1 \cup \mathcal{T}_2$. Clearly $\mathcal{T}_V \subseteq \mathcal{G}$.

By the above observation, there is a $\mathcal{T}_V$-path from any $x \in X \setminus S$ to $\pi(x)$, thus $\mathcal{T}_V$ is a graphing of V.

If ρ is the function with domain $X \setminus S$ such that $\rho(x) = x'$ (in the notation above), then $\mathcal{T}_1 = \text{graph}(\rho)$, so clearly $\mathcal{T}$ is a tree. Moreover, $M(\mathcal{T}_1) = M(\text{graph}(\rho)) = \mu(X \setminus S)$ and thus $M(\mathcal{T}_V) = 2M(\mathcal{T}_1) = 2\mu(X \setminus S)$, so $C_\mu(\mathcal{T}_V) = \mu(X \setminus S)$.

Now let $\mathcal{G}^* = \{(x, y) : x\mathcal{G}y \text{ and } \pi(x) \neq \pi(y)\}$. For $(x, y) \in \mathcal{G}^*$, let $\varphi(x, y) = (\pi(x), \pi(y))$. Then put $\mathcal{G}_S = \varphi(\mathcal{G}^*)$. Clearly $\mathcal{G}_S \subseteq E|S$ and as $\varphi(x, y)\tilde{E}(x, y)$, it follows that $M(\mathcal{G}_S) = M(\varphi(\mathcal{G}^*)) \leq M(\mathcal{G}^*)$. Also $\mathcal{G}^* \cap \mathcal{T}_V = \emptyset$, $\mathcal{G}^* \cup \mathcal{T}_V \subseteq \mathcal{G}$, so $M(\mathcal{G}) \geq M(\mathcal{G}^*) + M(\mathcal{T}_V) \geq M(\mathcal{G}_S) + 2\mu(X \setminus S)$, thus $C_\mu(\mathcal{G}) \geq C_{\mu|S}(\mathcal{G}_S) + \mu(X \setminus S)$.

We now check that $\mathcal{G}_S$ is a graphing of $E|S$. Let $x, y \in S$ with xEy. Let $x_0 = x, x_1, \dots, x_n = y$ be a $\mathcal{G}$-path. Then clearly $\pi(x_0) = \pi(x) = x$, $\pi(x_1), \dots, \pi(x_n) = \pi(y) = y$ is such that for each i, either $\pi(x_i) = \pi(x_{i+1})$ or else $x_i\mathcal{G}_S x_{i+1}$. So there is a $\mathcal{G}_S$-path from x to y.

Finally, assume $\mathcal{G}$ is a treeing, in order to verify that $\mathcal{G}_S$ is also a treeing. If $x_0 = x, x_1, \dots, x_n = x$ is a $\mathcal{G}_S$-cycle, then by definition and the fact that $\mathcal{G}$ is a treeing there is a unique sequence $y_0, y_1', y_1, y_2', y_2, y_3' \dots, y_{n-1}, y_n'$ with $\pi(y_i) = \pi(y_i') = x_i$, for $i \leq n$, and $y_i\mathcal{G}y_{i+1}'$, for $i \leq n-1$. As $\pi(y_i) = \pi(y_i')$ there is a $\mathcal{G}$-path from y_i to y_i' contained in $\pi^{-1}(\{x_i\})$, for $1 \leq i \leq n-1$, and a $\mathcal{G}$-path from y_n' to y_0 contained in $\pi^{-1}(\{x_0\})$. This gives a $\mathcal{G}$-cycle, which is a contradiction. ⊣

Corollary 21.3. *If E is an aperiodic countable Borel equivalence relation on X and μ is E-invariant, then $C_\mu(E) \geq \mu(X)$.*

Proof. Let, by 15.2, $\{S_n\}$ be a vanishing sequence of markers. If $\mathcal{G}$ is a graphing of E, then, by 21.2, $C_\mu(\mathcal{G}) \geq \mu(X \setminus S_n)$. But $X = X \setminus \bigcap_n S_n = \bigcup_n (X \setminus S_n)$ and $\{X \setminus S_n\}$ is an increasing sequence, so $\mu(X) = \lim_{n\to\infty} \mu(X \setminus S_n)$, thus $C_\mu(\mathcal{G}) \geq \mu(X)$. ⊣

Corollary 21.4. *If E is a countable Borel equivalence relation on X, μ an E-invariant measure, and $X_n = \{x \in X : |[x]_E| = n\}$, $n = 1, 2, \dots, \infty$, then $C_\mu(E) \geq \sum_{n=2}^\infty (1 - \frac{1}{n})\mu(X_n) + \mu(X_\infty) \geq \frac{1}{2}\mu(X \setminus X_1)$.*

In particular, $C_\mu(E) = 0$ iff $\mu(X \setminus X_1) = 0$ iff $E = \Delta_X$ a.e.

As a consequence, $\mu(X \setminus X_1) = \infty \Rightarrow C_\mu(E) = \infty$. *In view of this, we will mostly restrict ourselves from now on to finite measures μ.*

22 Cost and Hyperfiniteness

A countable Borel equivalence relation E on X is *hyperfinite* if $E = E_{\mathbb{Z}}^X$ for some Borel action of $\mathbb{Z}$ on X or equivalently $E = \bigcup_n E_n$, $E_0 \subseteq E_1 \subseteq \dots$, with each E_n a finite (i.e., having finite equivalence classes) Borel equivalence relation (see 6.6). Note that every hyperfinite equivalence relation is treeable.

Proposition 22.1 (Levitt). *Let E be aperiodic and hyperfinite, and μ be an E-invariant measure. If $\mathcal{T}$ is a treeing of E, then $C_\mu(\mathcal{T}) = C_\mu(E) = \mu(X)$. So any aperiodic hyperfinite E has cost $C_\mu(E) = \mu(X)$.*

In particular, if E is hyperfinite but not necessarily aperiodic, and $X_n = \{x \in X : |[x]_E| = n\}$, $n = 1, 2, \ldots, \infty$, then $C_\mu(E) = \sum_{n=2}^{\infty}(1 - \frac{1}{n})\mu(X_n) + \mu(X_\infty)$, and if $\mathcal{T}$ is a treeing of E, then $C_\mu(\mathcal{T}) = C_\mu(E)$.

Similarly for L*-treeings.*

Proof. Write $E = \bigcup_n E_n$, $E_n \subseteq E_{n+1}$, E_n finite. Let $<$ be a Borel partial order of X such that $< |[x]_E$ is isomorphic to the usual order on $\mathbb{Z}$, $\forall x \in X$. Let

$$x \in T_n \Leftrightarrow x \text{ is the } <\text{-least element of } [x]_{E_n}.$$

Then T_n is a Borel transversal for E_n, $T_0 \supseteq T_1 \supseteq T_2 \supseteq \ldots$ and $\bigcap_n T_n = \emptyset$.

Let $\mathcal{T}_n = \mathcal{T} \cap E_n$ and let F_n be the subequivalence relation of E_n whose classes are the connected components of $\mathcal{T}_n$. Thus $\mathcal{T}_n$ is a treeing of F_n. Find a Borel transversal S_n for F_n with $S_n \supseteq T_n$. Then

$$\begin{aligned} C_\mu(\mathcal{T}_n) &= C_\mu(F_n) \\ &= \mu(X \setminus S_n) \quad \text{(by 20.1)}. \end{aligned}$$

Since $X \setminus S_n \subseteq X \setminus T_n$, $C_\mu(\mathcal{T}_n) \leq \mu(X \setminus T_n)$. Now $\mathcal{T} = \bigcup_n \mathcal{T}_n$, and $\mathcal{T}_n \subseteq \mathcal{T}_{n+1}$, so $C_\mu(\mathcal{T}) = \lim_n C_\mu(\mathcal{T}_n) \leq \lim_n \mu(X \setminus T_n) = \mu(X)$. So $C_\mu(\mathcal{T}) \leq \mu(X)$, and, by 7.3, $\mu(X) \geq C_\mu(\mathcal{T}) \geq C_\mu(E) \geq \mu(X)$. $\dashv$

The next result is the converse of 22.1.

Theorem 22.2 (Levitt). *Let E be an aperiodic countable Borel equivalence relation on X and let μ be finite and E-invariant. Then the following are equivalent:*

(i) E is hyperfinite a.e. (i.e., $E|A$ is hyperfinite for a conull Borel set A).
(ii) $C_\mu(E) = \mu(X)$ and $C_\mu(E)$ is attained.

Proof. (i) $\Rightarrow$ (ii): follows from 22.1.

(ii) $\Rightarrow$ (i): (A version of the proof given in Gaboriau [G2], Proof of III.3(2), p. 59.) Let $\{S_n\}$ be a vanishing sequence of markers for E. By throwing away a null set we can assume that $E|S_n$ is aperiodic for each n.

Fix a graphing $\mathcal{G}$ of E with $C_\mu(\mathcal{G}) = C_\mu(E) = \mu(X)$. Apply 21.2 to X, S_0 to get a subequivalence relation $V_0 \subseteq E$, with transversal S_0, a treeing $\mathcal{T}_{V_0} = \mathcal{T}_0 \subseteq \mathcal{G}$ of V_0 and a corresponding graphing $\mathcal{G}_{S_0} = \mathcal{G}_0$ of $E|S_0$ such that $C_\mu(\mathcal{T}_0) = \mu(X \setminus S_0)$. As $\mu(X) < \infty$, $[x]_{V_0}$ is finite a.e., so throwing away a null set we assume that V_0 is finite. Now apply the same procedure to $(S_0,\ E_0|S_0,\ \mu|S_0,\ \mathcal{G}_0)$ and S_1, to get a finite subequivalence relation V_1 of $E|S_0$ with transversal S_1, treeing $\mathcal{T}_1 \subseteq \mathcal{G}_0$ and corresponding graphing $\mathcal{G}_1$ of $E|S_1$, so that $C_\mu(\mathcal{T}_1) = \mu(S_0) - \mu(S_1)$, etc. Let R_n be the equivalence relation generated by $\mathcal{T}_0 \cup \mathcal{T}_1 \cup \cdots \cup \mathcal{T}_n$. Inductively we see that S_n is a transversal for R_n, and R_n

is finite. Also clearly $R_0 \subseteq R_1 \subseteq R_2 \subseteq \dots$. From the construction in 21.2 we also see that there is Borel $\pi_i : S_{i-1} \to S_i$ (where $S_{-1} = X$) with $\pi_i(x) E x$, and $\mathcal{G}'_i \subseteq \mathcal{G}_{i-1} \setminus \mathcal{T}_i$ (where $\mathcal{G}_{-1} = \mathcal{G}$) such that if $\tilde{\pi}_i(x,y) = (\pi_i(x), \pi_i(y))$, $\tilde{\pi}_i(\mathcal{G}'_i) = \mathcal{G}_i$. Let $\rho_i : \mathcal{G}_i \to \mathcal{G}'_i$ be a Borel inverse of $\tilde{\pi}_i$. Then $\mathcal{T}_0$, $\rho_0(\mathcal{T}_1), \rho_0\rho_1(\mathcal{T}_2), \dots$ are pairwise disjoint subsets of $\mathcal{G}$ and $\mathcal{T}_0 \cup \rho_0(\mathcal{T}_1) \cup \dots \cup \rho_0\rho_1 \dots \rho_{n-1}(\mathcal{T}_n)$ is a graphing of R_n, so $\mathcal{T}_0 \cup \rho_0(\mathcal{T}_1) \cup \dots \cup \rho_0\rho_1 \dots \rho_{n-1}(\mathcal{T}_n) \cup \dots$ is a graphing of $\bigcup_n R_n$. But

$$
\begin{aligned}
&C_\mu(\mathcal{T}_0 \cup \rho_0(\mathcal{T}_1) \cup \dots \cup \rho_0\rho_1 \dots \rho_{n-1}(\mathcal{T}_n) \cup \dots) \\
&= C_\mu(\mathcal{T}_0) + C_\mu(\rho_0(\mathcal{T}_1)) + C_\mu(\rho_0\rho_1(\mathcal{T}_2)) + \dots \\
&= C_\mu(\mathcal{T}_0) + C_\mu(\mathcal{T}_1) + C_\mu(\mathcal{T}_2) + \dots \\
&= (\mu(X) - \mu(S_0)) + (\mu(S_0) - \mu(S_1)) + (\mu(S_1) - \mu(S_2)) + \dots \\
&= \mu(X) \\
&= C_\mu(\mathcal{G}).
\end{aligned}
$$

Since $\mathcal{T}_0 \cup \rho_0(\mathcal{T}_1) \cup \dots \subseteq \mathcal{G}$, this implies that $M(\mathcal{G} \setminus (\mathcal{T}_0 \cup \rho_0(\mathcal{T}_1) \cup \dots)) = 0$, i.e., $\mathcal{G} = \mathcal{T}_0 \cup \rho_0(\mathcal{T}_1) \cup \dots$ a.e., which means that there is a Borel E-invariant set $A \subseteq X$ such that $\mu(A) = \mu(X)$ and $\mathcal{G}|A = (\mathcal{T}_0 \cup \rho_0(\mathcal{T}_1) \cup \dots)|A$ and thus, since $\mathcal{G}$ is a graphing of E and $\mathcal{T}_0 \cup \rho_0(\mathcal{T}_1) \cup \dots$ of $\bigcup_n R_n$, it follows that $E|A = \bigcup_n R_n|A$, i.e., E is hyperfinite a.e.

Alternative proof of (ii)⇒(i) in 22.2: (A version of the proof in Levitt [L].) By 19.1, fix a treeing $\mathcal{T}$ of E with $C_\mu(\mathcal{T}) = C_\mu(E) = \mu(X)$. Since $C_\mu(\mathcal{T})$ is finite, $d_\mathcal{T}(x) < \infty$ a.e., so we can assume that $\mathcal{T}$ is locally finite.

First consider the special case where the degree $d_\mathcal{T}(x)$ of a.e. vertex x is ≥ 2. Then, as

$$\mu(X) = C_\mu(\mathcal{T}) = \frac{1}{2} \int d_\mathcal{T}(x)\, d\mu(x),$$

we have $d_\mathcal{T}(x) = 2$ μ-a.e. So in this case, we may assume, without loss of generality, that $d_\mathcal{T}(x) = 2$ for all x. Then, by the usual argument which shows that every equivalence relation induced by a Borel $\mathbb{Z}$-action is hyperfinite, we can show that E is hyperfinite:

Fix a countable group $\Gamma = \{g_n\}$ and a Borel action of Γ on X inducing E. Every $\mathcal{T}$-neighbor of x is of the form $g_n \cdot x$ for some n. We call the $\mathcal{T}$-neighbor of x which is of the form $g_n \cdot x$ for the smallest possible n the *right neighbor* of x and the other one the *left neighbor* of x. We say that $y \in [x]_E$ is to the *right* of x if the unique $\mathcal{T}$-path from x to y passes through the right neighbor of x. Similarly we define what it means to be to the *left* of x.

Now let $\{S_n\}$ be a vanishing sequence of markers for E. Since $\bigcap_n S_n = \emptyset$, $S_n \cap C$ is infinite for each E-class C. If, for some E-class C and some n, there is $x \in S_n \cap C$ such that all other elements of $S_n \cap C$ are to the right of x, such an x is unique. So in the union of these classes, Y, we can define a Borel selector, thus $E|Y$ is smooth, so hyperfinite. Similarly with right replaced by left. So we can assume that for every n and every $x \in S_n$ there are elements of S_n to the right as well as to the left of x.

Finally, for any xEy, let $[x, y] = \{x_0, x_1, \ldots, x_n\}$, where the sequence of points $x_0 = x, x_1, \ldots, x_n = y$ is the unique $\mathcal{T}$-path from x to y, and define

$$xE_ny \Leftrightarrow x = y \text{ or } [x, y] \cap S_n = \emptyset.$$

Clearly E_n are finite Borel equivalence relations, $E_n \subseteq E_{n+1}$ (as $S_n \supseteq S_{n+1}$) and $\bigcup_n E_n = E$ (as $\bigcap_n S_n = \emptyset$), so E is hyperfinite.

We now consider the general case, where $d_{\mathcal{T}}(x)$ may be < 2 on a set of positive measure.

Define a sequence $X_0 = X \supseteq X_1 \supseteq \cdots \supseteq X_n \supseteq \ldots$ of Borel sets as follows:

$$x \in X_{n+1} \Leftrightarrow x \in X_n \text{ and } d_{\mathcal{T}|X_n}(x) \geq 2.$$

Let also

$$X_\omega = \bigcap_n X_n.$$

It is easy to check that $\mathcal{T}|X_n$ is a treeing of $E|X_n$. This is because, assuming $\mathcal{T}|X_n$ is a treeing of $E|X_n, X_{n+1}$ is a convex subset of X_n, i.e., if $x, y \in X_{n+1}, xEy$, then all the vertices in the unique $\mathcal{T}|X_n$-path from x to y are also in X_{n+1}. Similarly for $\mathcal{T}|X_\omega$ and $E|X_\omega$.

As $\mathcal{T}$ is locally finite, it follows that

$$d_{\mathcal{T}|X_\omega}(x) = \lim_{n\to\infty} d_{\mathcal{T}|X_n}(x) \geq 2,$$

for all $x \in X_\omega$.

Now notice that

$$C_\mu(\mathcal{T}|X_{n+1}) = C_\mu(\mathcal{T}|X_n) - \mu(X_n \setminus X_{n+1}),$$

so, for all $n \geq 1$,

$$C_\mu(\mathcal{T}|X_n) = C_\mu(\mathcal{T}) - \mu(X \setminus X_n),$$

and thus

$$C_\mu(\mathcal{T}|X_\omega) = C_\mu(\mathcal{T}) - \mu(X \setminus X_\omega).$$

This means that if $\mu(X_\omega) > 0$, then

$$C_{\mu|X_\omega}(\mathcal{T}|X_\omega) = C_{\mu|X_\omega}(E|X_\omega) = \mu(X_\omega).$$

Thus we can apply the special case above to $E|X_\omega$ to conclude that $E|X_\omega$ is hyperfinite a.e., and thus so is $E|[X_\omega]_E$. So it is enough to show that $E|(X \setminus [X_\omega]_E)$ is hyperfinite.

Recall that an infinite path through a tree T is a sequence $\mathbf{x} = \{x_i\}_{i\in\mathbb{N}}$ such that $(x_i, x_{i+1}) \in T$, for all i, and $x_i \neq x_j$ if $i \neq j$. We call two such paths equivalent if $\exists n \exists m \forall i (x_{n+i} = y_{m+i})$. An equivalence class of paths is called an *end* of T. For each end e and vertex x of T there is a unique infinite path $\mathbf{x} = \{x_i\}_{i\in\mathbb{N}}$ with $x_0 = x$ and $\mathbf{x} \in e$, called the *geodesic* from x to e and denoted by $[x, e]$. Finally, a *line* of T is a sequence $\{x_i\}_{i \in \mathbb{Z}}$ with $(x_i, x_{i+1}) \in T$, for all i, and $x_i \neq x_j$, if $i \neq j$.

Now if $x \notin [X_\omega]_E$, clearly $\mathcal{T}|[x]_E$ has at most one end (since any two distinct ends determine a line, which would then have to be contained in X_ω), and by König's Lemma, which asserts that every infinite locally finite tree has an infinite path, it has exactly one end. For each such x, let $f(x) \in X^{\mathbb{N}}$ be the geodesic from x to the unique end of $\mathcal{T}|[x]_E$. Clearly for $x, y \notin [X_\omega]_E$,

$$xEy \Leftrightarrow f(x)E_t(X^{\mathbb{N}})f(y),$$

where $E_t(X^{\mathbb{N}})$ is the *tail equivalence relation* on $X^{\mathbb{N}}$:

$$\{x_k\}E_t(X^{\mathbb{N}})\{y_\ell\} \Leftrightarrow \exists k \exists \ell \forall i (x_{k+i} = y_{k+i}).$$

So $E|(X \setminus [X_\omega]_E) \leq_B E_t(X^{\mathbb{N}})$ via f, thus, as shown in [DJK], 8.1 and 5.1, $E|(X \setminus [X_\omega]_E)$ is hyperfinite. $\dashv$

The following is a corollary of the alternative proof of 22.2, (ii)$\Rightarrow$(i).

Theorem 22.3. (Adams [A1]) *Let E be an aperiodic countable Borel equivalence relation on X and let μ be finite and E-invariant. If E is hyperfinite and $\mathcal{T}$ is a treeing of E, then $\mathcal{T}|[x]_E$ has at most two ends for μ-a.e. (x).*

23 Joins

We start with some useful lemmas.

Lemma 23.1 (Gaboriau). *Let $E_0 \subseteq E$ be countable Borel equivalence relations on X with E_0 aperiodic, and let μ be an E-invariant finite measure. Let $\epsilon > 0$ and let $\mathcal{G}_0$ be a graphing of E_0 with $C_\mu(\mathcal{G}_0) \leq C_\mu(E_0) + \epsilon$. Then there is a graphing $\mathcal{G} \supseteq \mathcal{G}_0$ of E with $C_\mu(\mathcal{G}) \leq C_\mu(E) + (C_\mu(E_0) - \mu(X)) + 3\epsilon$.*

Moreover, if E_0 is hyperfinite but not necessarily aperiodic and $\mathcal{G}_0$ is a treeing of E_0, then for every $\epsilon > 0$ there is a graphing $\mathcal{G} \supseteq \mathcal{G}_0$ of E with $C_\mu(\mathcal{G}) \leq C_\mu(E) + \epsilon$.

Proof. Let S be a Borel complete section for E_0 with $\mu(S) \leq \epsilon$. Find a graphing $\mathcal{G}_S$ of $E|S$ with $C_\mu(\mathcal{G}_S) \leq C_{\mu|S}(E|S) + \epsilon$. Using 21.1, we get $C_\mu(\mathcal{G}_S) \leq C_\mu(E) - (\mu(X) - \mu(S)) + \epsilon \leq C_\mu(E) - \mu(X) + 2\epsilon$. Put $\mathcal{G}_0 \cup \mathcal{G}_S = \mathcal{G}$. This is a graphing of E and

$$\begin{aligned} C_\mu(\mathcal{G}) &\leq C_\mu(\mathcal{G}_0) + C_\mu(\mathcal{G}_S) \\ &\leq C_\mu(E_0) + (C_\mu(E) - \mu(X)) + 3\epsilon. \end{aligned}$$

To prove the second assertion, let $X_n = \{x \in X : |[x]_{E_0}| = n\}$, $n = 1, 2, \ldots, \infty$, let, for $n < \infty$, S_n be a Borel transversal for $E_0|X_n$, let S_∞ be a complete section for $E_0|X_\infty$ with $\mu(S_\infty) \leq \frac{\epsilon}{2}$, and put $S = \bigcup_{n=1}^{\infty} S_n \cup S_\infty$.

Fix a graphing $\mathcal{G}_S$ of $E|S$ with $C_\mu(\mathcal{G}_S) \leq C_\mu(E) - \mu(X) + \mu(S) + \frac{\epsilon}{2}$ as before. Then if $\mathcal{G} = \mathcal{G}_0 \cup \mathcal{G}_S$,

$$\begin{aligned}
C_\mu(\mathcal{G}) &\leq C_\mu(\mathcal{G}_0) + C_\mu(\mathcal{G}_S) \\
&= C_\mu(E_0|X_\infty) + \sum_{n=2}^{\infty}\left(1 - \frac{1}{n}\right)\mu(X_n) + C_\mu(\mathcal{G}_S) \\
&\leq \mu(X_\infty) + \sum_{n=2}^{\infty}\left(1 - \frac{1}{n}\right)\mu(X_n) + C_\mu(E) + \frac{\epsilon}{2} - \mu(X) + \mu(S) \\
&\leq \mu(X_\infty) + \sum_{n=2}^{\infty}\left(1 - \frac{1}{n}\right)\mu(X_n) + C_\mu(E) + \\
&\qquad \frac{\epsilon}{2} - \mu(X) + \frac{\epsilon}{2} + \sum_{n=1}^{\infty}\frac{1}{n}\mu(X_n) \\
&= C_\mu(E) + \epsilon.
\end{aligned}$$

⊣

Lemma 23.2 (Jackson-Kechris-Louveau [JKL]). *Let E be an aperiodic countable Borel equivalence relation. Then there is an aperiodic hyperfinite $E_0 \subseteq E$.*

Proof. (Miller) Let $\{S_n\}$ be a vanishing sequence of markers. Let also $E = E_\Gamma^X$ for some countable group $\Gamma = \{\gamma_n\}$ and Borel action of Γ on X. Given $x \in X$, let $n(x)$ be least such that $x \notin S_{n(x)}$, let $m(x)$ be least such that $\gamma_{m(x)} \cdot x \in S_{n(x)}$, and put $g(x) = \gamma_{m(x)} \cdot x$. Suppose $i < j$, set $S = S_{n(g^i(x))}$, and note that $g^i(x) \notin S$ but $g^j(x) \in S$. In particular, $g^i(x) \neq g^j(x)$. Now define

$$xE_0y \Leftrightarrow \exists k \exists \ell (g^k(x) = g^\ell(y)).$$

Clearly E_0 is an aperiodic subequivalence relation of E and $E_0 \leq_B E_t(X^{\mathbb{N}})$, via the map $x \mapsto (g^k(x))$, so, as in the last paragraph of the proof of 22.2, E_0 is hyperfinite. ⊣

Remark 23.3. Note that the first proof of (ii)⇒(i) in 22.2 actually shows that for any graphing $\mathcal{G}$ of a countable aperiodic Borel equivalence relation E, there is an acyclic subgraph $\mathcal{G}_0 \subseteq \mathcal{G}$ which generates an aperiodic hyperfinite subequivalence relation $E_0 \subseteq E$. Indeed, in the notation of that proof, each R_n is smooth with transversal S_n, and $\mathcal{T}_0 \cup \rho_0(\mathcal{T}_1) \cup \rho_0\rho_1(\mathcal{T}_2) \cup \cdots \cup \rho_0\rho_1 \dots \rho_{n-1}(\mathcal{T}_n)$ in a treeing of R_n. Put $E_0 = \bigcup_n R_n, \mathcal{G}_0 = \mathcal{T}_0 \cup \rho_0(\mathcal{T}_1) \cup \rho_0\rho_1(\mathcal{T}_2) \cup \dots$ Then $\mathcal{G}_0 \subseteq \mathcal{G}$, $\mathcal{G}_0$ is a treeing of E_0, E_0 is hyperfinite by [DJK] 5.1, being the increasing union of a sequence of smooth countable Borel equivalence relations, and finally E_0 is aperiodic, since every E_0-class meets every S_n and $\bigcap_n S_n = \emptyset$.

If $\{R_i\}_{i \in I}$ is a family of relations on a set X, we denote by $\bigvee_{i \in I} R_i$ the smallest equivalence relation containing all the R_i's, and call it the *join* of $\{R_i\}$. If $I = \{1, 2\}$ we simply write $R_1 \vee R_2$.

If μ is $E_1 \vee E_2$-invariant, where E_1, E_2 are equivalence relations on X, then $C_\mu(E_1 \vee E_2) \leq C_\mu(E_1) + C_\mu(E_2)$, and there are examples to show this is best possible in general (see 27.2). However, we have

Proposition 23.4 (Gaboriau). *Let E_1, E_2 be countable Borel equivalence relations with $E_1 \cap E_2$ aperiodic and μ a finite $(E_1 \vee E_2)$-invariant measure. Then*

$$C_\mu(E_1 \vee E_2) \leq C_\mu(E_1) + C_\mu(E_2) - \mu(X).$$

In particular, if $C_\mu(E_1) = C_\mu(E_2) = \mu(X)$, then $C_\mu(E_1 \vee E_2) = \mu(X)$.

Proof. Find aperiodic hyperfinite $E_0 \subseteq E_1 \cap E_2$ by 23.2. Then let $\mathcal{G}_0$ be a treeing of E_0 of cost $C_\mu(\mathcal{G}_0) = C_\mu(E_0) = \mu(X)$. Find then for each $\epsilon > 0$, $\mathcal{G}_i \supseteq \mathcal{G}_0$, a graphing of E_i with $C_\mu(\mathcal{G}_i) \leq C_\mu(E_i) + \epsilon,\ i = 1, 2$. Then $\mathcal{G}_1 \cup (\mathcal{G}_2 \setminus \mathcal{G}_0)$ is a graphing of $E_1 \vee E_2$ of cost $\leq C_\mu(\mathcal{G}_1) + C_\mu(\mathcal{G}_2) - C_\mu(\mathcal{G}_0) \leq C_\mu(E_1) + C_\mu(E_2) - \mu(X) + 2\epsilon$. So $C_\mu(E_1 \vee E_2) \leq C_\mu(E_1) + C_\mu(E_2) - \mu(X)$. ⊣

There are examples where $C_\mu(E_1) = C_\mu(E_2) = \mu(X)$, $E_1 \cap E_2 = \Delta_X$ and $C_\mu(E_1 \vee E_2) > \mu(X)$ (see again 27.2).

We also have the following generalization.

Proposition 23.5 (Gaboriau). *Let $E, \{E_n\}_{n \geq 1}$ be countable Borel equivalence relations on X with $E = \bigvee_n E_n$ and $\bigcap_n E_n$ aperiodic, and let μ be a finite E-invariant measure. Then*

$$C_\mu(E) - \mu(X) \leq \sum_n (C_\mu(E_n) - \mu(X)).$$

In particular, if $C_\mu(E_n) = \mu(X)$, for all μ, then $C_\mu(E) = \mu(X)$. Also if $E_n \subseteq E_{n+1}$ for each n, so that $E = \bigcup_n E_n$, and $C_\mu(E_n) \to \mu(X)$, then $C_\mu(E) = \mu(X)$.

Proof. Let $E_0 \subseteq \bigcap_n E_n$ be aperiodic hyperfinite. Fix a treeing $\mathcal{G}_0$ of E_0 of cost $C_\mu(\mathcal{G}_0) = \mu(X)$ and, for each $\epsilon > 0, n \geq 1$, let $\mathcal{G}'_n \supseteq \mathcal{G}_0$ be a graphing of E_n with cost $C_\mu(\mathcal{G}'_n) \leq C_\mu(E_n) + \frac{\epsilon}{2^n}$. If $\mathcal{G}_n = \mathcal{G}'_n \setminus \mathcal{G}_0$, then $C_\mu(\mathcal{G}'_n) = C_\mu(\mathcal{G}_0) + C_\mu(\mathcal{G}'_n \setminus \mathcal{G}_0) \leq C_\mu(E_n) + \frac{\epsilon}{2^n}$, so $\mu(X) + C_\mu(\mathcal{G}_n) \leq C_\mu(E_n) + \frac{\epsilon}{2^n}$ or $C_\mu(\mathcal{G}_n) \leq C_\mu(E_n) - \mu(X) + \frac{\epsilon}{2^n}$, thus, since $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \dots$ is a graphing of E,

$$\begin{aligned} C_\mu(E) &\leq C_\mu(\mathcal{G}_0 \cup \mathcal{G}_1 \cup \dots) \\ &\leq C_\mu(\mathcal{G}_0) + \sum_{n \geq 1} C_\mu(\mathcal{G}_n) \\ &\leq \mu(X) + \sum_{n \geq 1} (C_\mu(E_n) - \mu(X)) + \epsilon, \end{aligned}$$

so $C_\mu(E) \leq \mu(X) + \sum_n (C_\mu(E_n) - \mu(X))$.

If the E_n are increasing and $C_\mu(E_n) \to \mu(X)$, then for each $\epsilon > 0$ we can find a subsequence $\{k_n\}$ with $C_\mu(E_{k_n}) - \mu(X) < \frac{\epsilon}{2^n}$ so $C_\mu(E) \leq \mu(X) + \epsilon$ (using the above estimate for $\{E_{k_n}\}$) and thus $C_\mu(E) = \mu(X)$. ⊣

Note that the proof of 23.5 also shows the following.

Proposition 23.6 (Gaboriau). *Let E, $\{E_n\}_{n=1}^{\infty}$ be countable Borel equivalence relations on X with $E = \bigvee_n E_n$, and let μ be a finite E-invariant measure. If $E_0 \subseteq \bigcap_{n=1}^{\infty} E_n$ is hyperfinite, then*

$$C_\mu(E) - C_\mu(E_0) \leq \sum_{n=1}^{\infty} (C_\mu(E_n) - C_\mu(E_0)).$$

24 Commuting Equivalence Relations

Let R, S be relations on a set X. We say that R, S *commute*, in symbols $R \Box S$, if $R \circ S = S \circ R$ (where $(x, y) \in R \circ S \Leftrightarrow \exists z[(x, z) \in R \ \& \ (z, y) \in S]$). The following is easy to check:

Proposition 24.1. *The following are equivalent for all equivalence relations R, S:*

(i) $R \Box S$,
(ii) $R \circ S = R \vee S$, and
(iii) within each $R \vee S$-class, every R-class meets every S-class.

Let now E_1, E_2 be countable Borel equivalence relations on X, let $E = E_1 \vee E_2$, and let μ be E-invariant. Then the trivial estimate $C_\mu(E_1 \vee E_2) \leq C_\mu(E_1) + C_\mu(E_2)$, which is in general best possible, even if E_1, E_2 are aperiodic, can be improved if $E_1 \Box E_2$.

The next result was originally proved by Pavelich, in a weaker form ($C_\mu(E) \leq C_\mu(E_1) + 2C_\mu(E_2) - 2\mu(X)$), and improved in the current version by Solecki.

Theorem 24.2 (Pavelich [P], Solecki). *Let E_1, E_2 be commuting aperiodic countable Borel equivalence relations on X, let $E = E_1 \vee E_2$, and let μ be a finite E-invariant measure. Then*

$$C_\mu(E) \leq C_\mu(E_1) + C_\mu(E_2) - \mu(X).$$

So, if $C_\mu(E_1) = C_\mu(E_2) = \mu(X)$, then $C_\mu(E) = \mu(X)$.

Proof. Let

$$A = \{x \in X : [x]_{E_1 \cap E_2} \text{ is finite}\}.$$

If $A = \emptyset$, we are done by 23.4. So we can assume that $A \neq \emptyset$. Note that

$$\{x \in X : [x]_{E_1} \cap A \text{ is non-empty and finite}\}$$

has μ-measure 0 (since E_1 is aperiodic), so we can assume that it is empty, i.e., we can assume that $E_1 | A$ is aperiodic. Let then, by 15.2, $\{S_n\}$ be a vanishing sequence of markers for $E_1 | A$ (in particular $S_n \subseteq A$).

By 23.2, let $F \subseteq E_2$ be aperiodic hyperfinite. For each n, consider the set

$$\{x \in X : [x]_F \cap S_n \text{ is non-empty and finite}\}.$$

As before it has μ-measure 0, and so we can assume that it is empty, i.e., we can assume that for each n, $F|S_n$ is aperiodic.

Claim 24.3. *For each n, $(E_1 \vee F|S_n) \cap E_2$ is aperiodic.*

Granting this, we have, from 23.4,

$$\begin{aligned} C_\mu(E) &\le C_\mu(E_1 \vee F|S_n) + C_\mu(E_2) - \mu(X) \\ &\le C_\mu(E_1) + C_\mu(F|S_n) + C_\mu(E_2) - \mu(X). \end{aligned}$$

Now $\mu(S_n) \to 0$ as $n \to \infty$, and since $C_\mu(F|S_n) = \mu(S_n)$, it follows that

$$C_\mu(E) \le C_\mu(E_1) + C_\mu(E_2) - \mu(X).$$

Proof of 24.3. We have to show that for each $x \in X, [x]_{(E_1 \vee F|S_n)\cap E_2}$ is infinite.

Case 1. $x \notin A$. Since

$$[x]_{(E_1 \vee F|S_n)\cap E_2} \supseteq [x]_{E_1 \cap E_2},$$

we are obviously done in this case.

Case 2: $x \in A$. Since S_n is a complete section for $E_1|A$, find $y \in S_n$ such that xE_1y. Now, since $F|S_n$ is aperiodic, the set $[y]_F \cap S_n$ is infinite. If $u, v \in [y]_F \cap S_n$, then uFv, so also uE_2v, and $u, v \in A$. Thus $E_1|([y]_F \cap S_n)$ has finite classes, so we can find an infinite subset $\{u_1, u_2, \dots\} \subseteq [y]_F \cap S_n$, so that $(u_i, u_j) \notin E_1$, if $i \neq j$. Now $xE_1yE_2u_i$, so, by commutativity, we can find v_i such that $xE_2v_iE_1u_i$. Since $(u_i, u_j) \notin E_1$, if $i \neq j$, it follows that $v_i \neq v_j$, if $i \neq j$. Also $v_i \in [x]_{E_2} \cap [x]_{(E_1 \vee F|S_n)} = [x]_{(E_1 \vee F|S_n)\cap E_2}$, so this set is infinite, and we are done. $\dashv$

Thus, by 23.4 and 24.2, we have the same estimate for $E_1 \vee E_2$ if either $E_1 \cap E_2$ is aperiodic or E_1, E_2 are aperiodic and commute.

There are examples of E_1, E_2 with $E_1 \square E_2$, and $C_\mu(E_1) = 1, C_\mu(E_2) = \infty, C_\mu(E_1 \vee E_2) > 1$ for some $(E_1 \vee E_2)$-invariant measure μ (see the paragraph after 33.1). There are also examples of E_1, E_2 with $E_1 \square E_2$, such that every $(E_1 \vee E_2)$-class contains only finitely many E_1-classes and $C_\mu(E_1) > 1, C_\mu(E_2) = 1$ and $C_\mu(E_1 \vee E_2) > 1$, for some $(E_1 \vee E_2)$-invariant measure μ. To see this, consider the free group $F_2 = \langle a, b\rangle$ with two generators, and the shift action of F_2 on 2^{F_2}

$$g \cdot x(h) = x(g^{-1}h),$$

restricted to its free part

$$X = \{x \in 2^{F_2} : \forall g \in F_2(g \neq 1 \Rightarrow g \cdot x \neq x)\}.$$

Let E_1 be the equivalence relation induced by this action. For $x \in 2^{F_2}$, let $\bar{x}(g) = 1 - x(g)$. Let also $A = \langle a \rangle$ be the subgroup of F_2 generated by a, and let E_2 be the relation on X defined by

$$xE_2y \Leftrightarrow \exists g \in A(g \cdot x = y \text{ or } g \cdot x = \bar{y}).$$

Noting that for any $g \in F_2$, $x \in 2^{F_2}$,

$$\overline{g \cdot x} = g \cdot \bar{x},$$

it is easily seen that this is an equivalence relation, $E_1 \square E_2$, and if $E = E_1 \vee E_2$, then

$$xEy \Leftrightarrow \exists g \in F_2(g \cdot x = y \text{ or } g \cdot x = \bar{y}).$$

Thus, if μ is the usual product measure on 2^{F_2} (with $2 = \{0, 1\}$ having the $(1/2, 1/2)$-measure), then μ-almost every E-class contains exactly 2 E_1-classes. Now by 27.10, $C_\mu(E_1) = 2$ and so, by 25.4, $C_\mu(E) > 1$, while by 22.2 $C_\mu(E_2) = 1$, since E_2 is hyperfinite.

However the following is open:

Problem 24.4. Let μ be a probability measure on X, E_1, E_2 commuting aperiodic countable Borel equivalence relations on X, let $E = E_1 \vee E_2$ and assume μ is E-invariant. If $C_\mu(E_1), C_\mu(E_2) < \infty$ and each E-class contains infinitely many E_1-classes and infinitely many E_2-classes, is it true that $C_\mu(E) \leq \min\{C_\mu(E_1), C_\mu(E_2)\}$?

We do not even know a counterexample to the following stronger version:

Problem 24.5. Under the hypothesis of 24.4, is it actually true that $C_\mu(E) = 1$?

As an application of 24.2, we have:

Corollary 24.6. *Suppose E is a countable Borel equivalence relation on X, μ is a finite measure on X, and E is μ-invariant. If $E = \bigvee_{i \in \mathbb{N}} E_i$, where each E_i is an aperiodic Borel equivalence relation with $C_\mu(E_i) = \mu(X)$, and $E_i \square E_{i+1}$, $\forall i \in \mathbb{N}$, then $C_\mu(E) = \mu(X)$.*

Proof. By 24.2, $C_\mu(E_i \vee E_{i+1}) = \mu(X)$. Let $F_n = \bigvee_{i=1}^n E_i$. We prove, inductively on n, that $C_\mu(F_n) = \mu(X)$. Indeed, for $n = 0$ this is true by assumption. Assume $C_\mu(F_n) = \mu(X)$. Now $F_{n+1} = F_n \vee E_{n+1} = F_n \vee (E_n \vee E_{n+1})$, and $F_n \cap (E_n \vee E_{n+1}) \supseteq E_n$ is aperiodic, so $C_\mu(F_{n+1}) \leq C_\mu(F_n) + C_\mu(E_n \vee E_{n+1}) - \mu(X) = C_\mu(F_n) = \mu(X)$, so $C_\mu(F_{n+1}) = \mu(X)$.

Since $F_n \subseteq F_{n+1}$ and $E = \bigcup_n F_n$ it follows from 23.5 that $C_\mu(E) = \mu(X)$. ⊣

We now prove the following lemma, due to Kechris.

Lemma 24.7. *Let E_1, E_2 be countable Borel equivalence relations on X and let μ be $(E_1 \vee E_2)$-invariant. Let $\mathcal{G}_1$ be a graphing of E_1 with $C_\mu(\mathcal{G}_1) < \infty$, and assume that for every $(a, b) \in \mathcal{G}_1 \setminus E_2$ there are infinitely many (a', b') such that $aE_2a'\mathcal{G}_1b'E_2b$. Then for every $\epsilon > 0$, there is $\mathcal{G}_1' \subseteq \mathcal{G}_1$ with $C_\mu(\mathcal{G}_1') < \epsilon$ such that for any graphing $\mathcal{G}_2'$ of an equivalence relation $E_2' \supseteq E_2$, $\mathcal{G}_1' \cup \mathcal{G}_2'$ is a graphing of an equivalence relation containing $E_1 \vee E_2'$. In particular, this is true if $\mathcal{G}_1 \circ E_2 = E_2 \circ \mathcal{G}_1$ and E_2 is aperiodic.*

Proof. If $\mathcal{G}_1 \subseteq E_2$ there is nothing to prove. Otherwise, on $X_1 = \mathcal{G}_1 \setminus E_2$ define the equivalence relation

$$(a, b)F(c, d) \Leftrightarrow \{[a]_{E_2}, [b]_{E_2}\} = \{[c]_{E_2}, [d]_{E_2}\}.$$

We claim that this is aperiodic. Indeed, if $(a, b) \in \mathcal{G}_1 \setminus E_2$, then there are infinitely many (a', b') such that $[a]_{E_2} = [a']_{E_2}$, $[b]_{E_2} = [b']_{E_2}$ and $(a', b') \in \mathcal{G}_1$, thus $(a, b)R(a', b')$.

Then using a vanishing sequence of markers for X_1 and using the fact that $M(X_1) < \infty$, we see that there is $\mathcal{G}_1' \subseteq \mathcal{G}_1 \setminus E_2$ with $M(\mathcal{G}_1') < 2\epsilon$, thus $C_\mu(\mathcal{G}_1') < \epsilon$, and $\mathcal{G}_1'$ a complete section for F. This clearly works. ⊣

Corollary 24.8. *Let E_1, E_2 be countable Borel equivalence relations on X and let μ be finite and $(E_1 \vee E_2)$-invariant. Let $\mathcal{G}_i$ be a graphing of E_i, with $C_\mu(\mathcal{G}_i) < \infty$, and let $E_i' \subseteq E_i$ $(i = 1, 2)$ be aperiodic hyperfinite, with $\mathcal{G}_1 \circ E_2' = E_2' \circ \mathcal{G}_1$, $\mathcal{G}_2 \circ E_1' = E_1' \circ \mathcal{G}_2$. Then $C_\mu(E_1 \vee E_2) = \mu(X)$.*

Proof. Fix graphings $\mathcal{G}_1', \mathcal{G}_2'$ of E_1', E_2' resp. of cost $\mu(X)$. For $\epsilon > 0$ find, by 24.7, $\mathcal{G}_1'' \subseteq \mathcal{G}_1$ of cost $< \epsilon$ such that $\mathcal{G}_1'' \cup \mathcal{G}_2'$ graphs $E_1 \vee E_2'$ and $\mathcal{G}_2'' \subseteq \mathcal{G}_2$ of cost $< \epsilon$ such that $\mathcal{G}_1' \cup \mathcal{G}_2''$ graphs $E_1' \vee E_2$. Then $\mathcal{G}_1'' \cup \mathcal{G}_2' \cup \mathcal{G}_2''$ graphs $E_1 \vee E_2$ and has cost $< \mu(X) + 2\epsilon$. ⊣

If R is a countable Borel equivalence relation on X and μ is R-invariant and S is a countable Borel equivalence relation on Y and ν is S-invariant, then if $R \times S = \{((x, y), (x', y')) \in (X \times Y)^2 : xRx', yRy'\}$, $\mu \times \nu$ is $R \times S$-invariant. Notice that if $E_1 = R \times \Delta_Y$, $E_2 = \Delta_X \times S$, then $E_1 \vee E_2 = R \times S$ and $E_1 \Box E_2$. Actually we have a much stronger commutativity here: If $R' \subseteq R$, $S' \subseteq S$, then $(R' \times \Delta_Y) \circ (\Delta_X \times S') = (\Delta_X \times S') \circ (R' \times \Delta_Y) = R' \times S'$.

We can now prove the following (somewhat stronger form of a) result of Gaboriau.

Theorem 24.9 (Gaboriau). *Let R, S be countable aperiodic Borel equivalence relations on X, Y resp. and let μ be an $(R \times S)$-invariant finite measure on $X \times Y$. Then $C_\mu(R \times S) = \mu(X \times Y)$.*

Proof. We can assume of course that μ is a probability measure. Let π_X, π_Y be the two projections of $X \times Y$ and let $\mu_X = (\pi_X)_*\mu$, $\mu_Y = (\pi_Y)_*\mu$. Then it is easy to check that μ_X is R-invariant, and μ_Y is S-invariant. Clearly (using 23.2) R is an increasing union $R = \bigcup_n R_n$ of aperiodic equivalence relations of finite cost and similarly we can write $S = \bigcup_n S_n$. As $R \times S$ is the increasing union of $\{R_n \times S_n\}$, we can assume (using 23.5) that R, S have finite cost. So fix a graphing $\mathcal{G}$ of R of finite cost, a graphing $\mathcal{H}$ of S of finite cost, and aperiodic hyperfinite subequivalence relations $R' \subseteq R,\ S' \subseteq S$. Let $\mathcal{G}_1 = \mathcal{G} \times \Delta_Y$, $\mathcal{G}_2 = \Delta_X \times \mathcal{H}$, $E'_1 = R' \times \Delta_Y$, $E'_2 = \Delta_X \times S'$, $E_1 = R \times \Delta_X$, $E_2 = \Delta_X \times S$. Then $C_\mu(\mathcal{G}_1) = C_{\mu_X}(\mathcal{G})$, $C_\mu(\mathcal{G}_2) = C_{\mu_Y}(\mathcal{H})$, so $\mathcal{G}_1, \mathcal{G}_2$ have finite cost, thus all the hypotheses of 24.8 are satisfied, and so $C_\mu(E_1 \vee E_2) = C_\mu(R \times S) = 1$. ⊣

We next provide some generalizations of this result.

Let $E \subseteq F$ be two countable Borel equivalence relations on the standard Borel space X. We say that E is *normal* in F, in symbols

$$E \trianglelefteq F,$$

if there is a countable group of Borel automorphisms $\{g_i\}_{i \in \mathbb{N}}$ of X which generates F, i.e.,

$$xFy \Leftrightarrow \exists i (g_i(x) = y),$$

and each g_i preserves E, i.e.,

$$xEy \Rightarrow g_i(x) E g_i(y).$$

For example, if Γ is a countable group acting in a Borel way on X, and $N \trianglelefteq F$ is a normal subgroup, then clearly $E_N^X \trianglelefteq E_\Gamma^X$. Also, in the notation preceding 24.9, we clearly have $R \times \Delta_Y \trianglelefteq R \times S$ and $\Delta_X \times S \trianglelefteq R \times S$.

Proposition 24.10. *Let E, F be aperiodic countable Borel equivalence relations on X and let μ be an F-invariant finite measure on X. If $E \trianglelefteq F$, then for any Borel equivalence relation E' with $E \subseteq E' \subseteq F$, we have $C_\mu(E') \geq C_\mu(F)$.*

Proof. Let F be generated by the Borel automorphisms $\{g_i\}_{i \in \mathbb{N}}$ which preserve E. Consider the graphing $\mathcal{G}_n = \bigcup_{i=0}^n [\mathrm{graph}(g_i) \cup \mathrm{graph}(g_i^{-1})]$. Then $C_\mu(\mathcal{G}_n) < \infty$ and $\mathcal{G}_n \circ E = E \circ \mathcal{G}_n$, so, by Lemma 24.7, if $\epsilon > 0$ there is $\mathcal{G}'_n \subseteq \mathcal{G}_n$ with $C_\mu(\mathcal{G}'_n) < \frac{\epsilon}{2^n}$, such that for any graphing $\mathcal{G}'$ of E' we have that $\mathcal{G}'_n \cup \mathcal{G}'$ graphs $E_n \vee E'$, where E_n is the equivalence relation generated by $\mathcal{G}_n$. Take now $\mathcal{G}'$ with $C_\mu(\mathcal{G}') \leq C_\mu(E') + \epsilon$. Then $\mathcal{G} = \mathcal{G}' \cup \bigcup_n \mathcal{G}'_n$ graphs F with cost $C_\mu(\mathcal{G}) \leq C_\mu(E') + \epsilon + \sum_n \frac{\epsilon}{2^n} = C_\mu(E') + 3\epsilon$. So $C_\mu(F) \leq C_\mu(E')$. ⊣

Corollary 24.11. *Let E, F be as in 24.10 and assume that there is an aperiodic Borel equivalence relation $E_1 \subseteq F$ with cost $C_\mu(E_1) = \mu(X)$ such that $E_1 \trianglelefteq E \vee E_1$. Then $C_\mu(F) = \mu(X)$.*

Proof. By 24.10, if $E' = E \vee E_1$, then $E_1 \trianglelefteq E'$, so $C_\mu(E') = \mu(X)$ and also $C_\mu(E') \geq C_\mu(F)$, so $C_\mu(F) = \mu(X)$. ⊣

Notice that 24.11 implies 24.9 by taking $E = R \times \Delta_Y, F = R \times S, E_1 = \Delta_X \times E_0$, where $E_0 \subseteq S$ is aperiodic hyperfinite.

25 Subequivalence Relations of Finite Index

If $E \subseteq F$ are equivalence relations and every F-class contains exactly n E-classes, we say that *index* of E in F is n, in symbols

$$[F : E] = n.$$

We say that E has *finite index* in F, in symbols

$$[F : E] < \infty,$$

if every F-class contains only finitely many E-classes.

Here we will discuss the relation of $C_\mu(E),\ C_\mu(F)$ if $[F : E] < \infty$. The following facts have been noted, independently, by Gaboriau, Kechris, and Miller, and partially generalize analogous results of Gaboriau [G2] for groups (see Section 34).

Proposition 25.1. *Let $E \subseteq F$ be aperiodic countable Borel equivalence relations on X and let μ be finite and F-invariant. If $[F : E] < \infty$, then $C_\mu(F) \leq C_\mu(E)$.*

Proof. It is enough to consider the case when $[F : E] = n$, for some fixed $n > 1$.

Fix $\epsilon > 0$ and let $A \subseteq X$ be a Borel complete section for E with $\mu(A) \leq \frac{\epsilon}{2(n-1)}$. Define Borel $f_i : A \to A,\ i = 1, \dots, n$, so that $f_1(x) = x, f_2(x), \dots, f_n(x)$ belong to different E-classes. Fix a graphing $\mathcal{G}$ of E of cost $C_\mu(\mathcal{G}) \leq C_\mu(E) + \epsilon/2$. Then $\mathcal{G} \cup \bigcup_{i=2}^n [\mathrm{graph}(f_i) \cup \mathrm{graph}(f_i)^{-1}]$ is a graphing of F and has cost $\leq C_\mu(\mathcal{G}) + \sum_{i=2}^n \mu(A) \leq C_\mu(\mathcal{G}) + (n-1)\frac{\epsilon}{2(n-1)} \leq C_\mu(E) + \epsilon$. So $C_\mu(F) \leq C_\mu(E)$. ⊣

Corollary 25.2. *If $E \subseteq F$ are aperiodic countable Borel equivalence relations on X and μ is a finite F-invariant measure, then if $[F : E] < \infty$,*

$$C_\mu(E) = \mu(X) \Rightarrow C_\mu(F) = \mu(X).$$

Proposition 25.3. *Let $E \subseteq F$ be aperiodic countable Borel equivalence relations on X and let μ be a finite F-invariant measure. If $[F : E] = n$, then*

$$C_\mu(E) - \mu(X) \leq n(C_\mu(F) - \mu(X)).$$

Proof. We claim first that it is enough to show that $C_\mu(E) \leq nC_\mu(F)$.

Indeed, granting this, let $A \subseteq X$ be a Borel complete section for E with $\mu(A) < \epsilon$. Apply this inequality to $E|A,\ F|A$ (which are aperiodic after throwing away a null set) to get

$$C_{\mu|A}(E|A) \leq nC_{\mu|A}(F|A)$$

or

$$C_\mu(E) - (\mu(X) - \mu(A)) \le n(C_\mu(F) - (\mu(X) - \mu(A)))$$

or

$$\begin{aligned} C_\mu(E) - \mu(X) &\le n(C_\mu(F) - \mu(X)) + (n-1)\mu(A) \\ &\le n(C_\mu(F) - \mu(X)) + (n-1)\epsilon. \end{aligned}$$

So letting $\epsilon \to 0$ we are done.

To prove $C_\mu(E) \le nC_\mu(F)$, fix a graphing $\mathcal{G}$ of F. For each E-class C, let D be the F-class it is contained in. Using the method of 21.2 assign to each $x \in D$, $\pi_C(x) \in C$, so that if for $(x, y) \in \mathcal{G}$ with $\pi_C(x) \neq \pi_C(y)$, we let $\varphi_C(x, y) = (\pi_C(x),\ \pi_C(y))$, then $\{\varphi_C(x, y) : x, y \in D,\ \pi_C(x) \neq \pi_C(y),\ (x, y) \in \mathcal{G}\}$ is a connected graph on C. Thus $\mathcal{G}' = \{\varphi_C(x, y) : (x, y) \in \mathcal{G},\ C$ is an E-class contained in $[x]_F \pi_C(x) \neq \pi_C(y)\}$ is a graphing of E, so it is enough to show that $C_\mu(\mathcal{G}') \le nC_\mu(\mathcal{G})$.

Let

$$\begin{aligned} (x, y)R(u, v) \Leftrightarrow (x, y) \in \mathcal{G}, (u, v) \in \mathcal{G}',\ &\text{and} \\ \text{if } C = [u]_E,\ \text{then } \varphi_C(x, y) = (u, v)&. \end{aligned}$$

Then clearly for each $(x, y) \in \mathcal{G}$,

$$\begin{aligned} R_{(x,y)} &= \{(u, v) : (x, y)R(u, v)\} \\ &= \{\varphi_C(x, y) : C \text{ is an } E\text{-class contained in } [x]_F\}, \end{aligned}$$

so $|R_{(x,y)}| \le n$. Thus there are n Borel functions $F_1, \dots, F_n,\ F_i : \mathcal{G}_i \to \mathcal{G}'$, where $\mathcal{G}_i \subseteq \mathcal{G}$ is Borel, such that $R_{(x,y)} = \{F_1(x, y), \dots, F_n(x, y)\}$ and so $\mathcal{G}' = F_1(\mathcal{G}_1) \cup \cdots \cup F_n(\mathcal{G}_n)$. Since $F_i(x, y)\tilde{E}(x, y)$, we have that $M(F_i(\mathcal{G}_i)) \le M(\mathcal{G}_i) \le M(\mathcal{G})$, so $C_\mu(\mathcal{G}') \le nC_\mu(\mathcal{G})$. ⊣

Corollary 25.4. *Let $E \subseteq F$ be aperiodic countable Borel equivalence relations on X and let μ be a finite F-invariant measure. If $[F : E] < \infty$, then*

$$C_\mu(E) = \mu(X) \Leftrightarrow C_\mu(F) = \mu(X).$$

Proof. Again it is enough to consider the case where $[F : E] = n$. Then the result follows from 25.2, 25.3. ⊣

The following is open:

Problem 25.5. In the notation of 25.3, do we actually have

$$C_\mu(E) - \mu(X) = n(C_\mu(F) - \mu(X))?$$

As we will see later on (in 34.1), Gaboriau has shown that the analog of this for groups is valid.

There are some special cases under which one can establish a positive answer to 25.5. We call an equivalence relation E *treeable* if it admits a treeing. If μ is a measure, we similarly define what it means for E to be *treeable* μ*−a.e.*

Proposition 25.6. *Let $E \subseteq F$ be aperiodic countable Borel equivalence relations on X and let μ be a finite F-invariant measure. If $[F : E] = n$ and F is treeable, then*

$$C_\mu(E) - \mu(X) = n(C_\mu(F) - \mu(X)).$$

Proof. As in the proof of 25.3, it is enough to show that $C_\mu(E) \geq nC_\mu(F) - n^2\mu(X)$ (we are using here the fact that treeability is preserved under restriction to any Borel subset of X, which follows from 21.2).

Fix a treeing $\mathcal{T}$ of F. Then, by 19.2, $C_\mu(\mathcal{T}) = C_\mu(E)$. As in the proof of 25.3, using the method of 21.2, assign to each E-class C and to each $x \in D = [C]_F$, $\pi_C(x) \in C$, so that if for $(x, y) \in \mathcal{T}$ with $\pi_C(x) \neq \pi_C(y)$, we put $\varphi_C(x, y) = (\pi_C(x), \pi_C(y))$, then $\{\varphi_C(x, y) : x, y \in D,\ (x, y) \in \mathcal{G}\}$ is a tree (connected acyclic graph) on C. Then if $\mathcal{T}'$ consists of all $\varphi_C(x, y)$ as C varies over the E-classes $C \subseteq D$, clearly $\mathcal{T}'$ is a treeing of E, and thus, by 19.2 again, it is enough to show that $C_\mu(\mathcal{T}') \geq nC_\mu(\mathcal{T}) - n^2\mu(X)$.

Let $\mathcal{T}_0 = \{(x, y) \in \mathcal{T} : \pi_C(x) \neq \pi_C(y)$, for all E-classes C contained in $[x]_F\}$, $\mathcal{T}_1 = \mathcal{T} \setminus \mathcal{T}_0$.

We first check that $C_\mu(\mathcal{T}_1) \leq (n-1)\mu(X)$. For that it is enough to check that there are $n - 1$ functions $\theta_i : X \to X$, $i = 1, \ldots, n-1$ such that $\mathcal{T}_1 \subseteq \bigcup_{i=1}^{n-1}[\text{graph}(\theta_i) \cup \text{graph}(\theta_i)^{-1}]$. First notice that there are $n-1$ functions $f_1(x), \ldots, f_{n-1}(x)$, such that $\{\pi_C(x) : C$ an E-class contained in $[x]_F$ and $x \notin C\} = \{f_1(x), \ldots, f_{n-1}(x)\}$. Next, by the construction in 21.2, there is a (unique) $\mathcal{T}$-path $x_0^i = x, x_1^i, \ldots, x_{\ell_i}^i = f_i(x)$, from x to $f_i(x)$. Put $\theta_i(x) = x_1^i$.

Now assume that $(x, y) \in \mathcal{T}_1$. Then for some C, $\pi_C(x) = \pi_C(y)$ and thus, by definition of $\pi_C(x)$, at least one of x, y is not in C (else $\pi_C(x) = x$, $\pi_C(y) = y$). Say $x \notin C$. If $y \in C$, then $\pi_C(x) = \pi_C(y) = y$, so clearly $y = \theta_i(x)$, where $f_i(x) = \pi_C(x)$. So assume also that $y \notin C$. Since there is a unique $\mathcal{T}$-path from x to $z = \pi_C(x) = f_i(x)$ (for some i) and a unique $\mathcal{T}$-path from y to $z = \pi_C(y) = f_j(x)$ (for some j), either $y = \theta_i(x)$ or $x = \theta_j(y)$, i.e., $(x, y) \in \text{graph}(\theta_i)$ or $(x, y) \in \text{graph}(\theta_j)^{-1}$, so we are done.

Now consider $\mathcal{T}_0$. There are n functions $\rho_i : \mathcal{T}_0 \to \mathcal{T}'$, $i = 1, \ldots, n$, such that $\{\rho_i(x, y) : i = 1, \ldots, n\} = \{\varphi_C(x, y) : C$ an E-class contained in $[x]_F\}$. Let $\bigcup_{i=1}^n \rho_i(\mathcal{T}_0) = \mathcal{T}''$. Since $\mathcal{T}$ is a treeing, recalling again the construction of 21.2, for each $(x', y') \in \mathcal{T}'$, if $C = [x']_E$, then there is a unique $(x, y) \in \mathcal{T}$ with $\varphi_C(x, y) = (x', y')$. It follows that there is a unique i and unique $(x, y) \in \mathcal{T}$ with $\rho_i(x, y) = (x', y')$. Thus each $\rho_i : \mathcal{T}_0 \to \mathcal{T}'$ is 1-1 and $\rho_i(\mathcal{T}_0) \cap \rho_j(\mathcal{T}_0) = \emptyset$, if $i \neq j$. So $C_\mu(\mathcal{T}'') = \sum_{i=1}^n C_\mu(\rho_i(\mathcal{T}_0)) = \sum_{i=1}^n C_\mu(\mathcal{T}_0) = n \cdot C_\mu(\mathcal{T}_0)$ (since $\rho_i(x, y)\tilde{F}(x, y)$). Thus we have

$$\begin{aligned}
C_\mu(\mathcal{T}') &\geq C_\mu(\mathcal{T}'') \\
&= nC_\mu(\mathcal{T}_0) \\
&= n(C_\mu(\mathcal{T}) - C_\mu(\mathcal{T}_1)) \\
&\geq nC_\mu(\mathcal{T}) - n^2\mu(X).
\end{aligned}$$

⊣

Finally, another special case under which a positive answer to 25.5 can be established is the following.

Proposition 25.7. *Let F be an aperiodic countable Borel equivalence relation on X and let μ be a finite F-invariant measure on X. Let $X = X_1 \cup \cdots \cup X_n$ be a partition of X into complete Borel sections. Let $E \subseteq F$ be the equivalence relation determined by this partition, i.e.,*

$$xEy \Leftrightarrow x \in F \ \& \ \exists i \leq n(x, y \in X_i).$$

(Thus $[F : E] = n$.) We have

$$C_\mu(E) - \mu(X) = n(C_\mu(F) - \mu(X)).$$

Proof. Since each X_i is a complete Borel section for F, and $F|X_i = E|X_i$, we have, by 21.1, $C_{\mu|X_i}(E|X_i) = C_{\mu|X_i}(F|X_i) = C_\mu(F) - \mu(X) + \mu(X_i)$. Since each X_i is E-invariant, we have (see Section 18) that $C_\mu(E) = \sum_{i=1}^n C_{\mu|X_i}(E|X_i) = nC_\mu(F) - n\mu(X) + \mu(X)$ and the proof is complete. ⊣

26 Cheap Equivalence Relations

Let E be an aperiodic countable Borel equivalence relation on X and μ an E-invariant finite measure on X. Then the smallest possible cost of E is $\mu(X)$. In that case we will call E *cheap*. Otherwise we call E *expensive*. We have not yet seen any examples of expensive E's. This will be done in Section 27. Here we will summarize the basic properties of cheap equivalence relations that we proved in earlier sections.

Proposition 26.1. *i) Every hyperfinite aperiodic Borel equivalence relation is cheap and attains its cost. Conversely, every cheap Borel equivalence relation which attains its cost is hyperfinite a.e.*

ii) A countable Borel equivalence relation is cheap iff its restriction to some complete Borel section is cheap iff its restriction to any complete Borel section is cheap.

iii) The join of a sequence of cheap equivalence relations is cheap if their intersection is aperiodic.

iv) The join of a sequence of cheap equivalence relations, which has the property that each member of the sequence commutes with the next one, is cheap.

v) The product of any two aperiodic countable Borel equivalence relations is cheap.

vi) Any aperiodic countable Borel equivalence relation that has finite index over or under a cheap equivalence relation is cheap.

Proof. i) is 22.2. ii) follows immediately from 21.1. iii) is a consequence of 23.5. iv) is 24.6. v) is 24.9, and vi) is 25.4. ⊣

27 Free and Amalgamated Joins

Let E_1, E_2 be countable Borel equivalence relations on X, $E = E_1 \vee E_2$. We say that E_1, E_2 are *independent*, in symbols $E_1 \perp E_2$, if for any sequence of points $x_0, x_1, x_2, \dots, x_{2n} = x_0, n > 0$, with $x_{2i} E_1 x_{2i+1} E_2 x_{2i+2}$ $(i = 0, 1, \dots, n-1)$, there is some $j < 2n$ with $x_j = x_{j+1}$. Clearly then $E_1 \cap E_2 = \Delta_X$. Notice also that this condition is equivalent to saying that there is no sequence $x_0, x_1, \dots, x_n = x_0$ with $n > 1$, $x_i \neq x_j$, if $0 \leq i < j < n$, and $x_0 E_1 x_1 E_2 x_2 E_1 x_3 \dots$ Thus this condition is also equivalent to saying that $E_1 \cap E_2 = \Delta_X$ and if $(x, y) \in E$, $x \neq y$, then there is a unique sequence $x_0 = x, x_1, \dots, x_n = y$, with x_i distinct, $(x_j, x_{j+1}) \in E_1 \cup E_2$ if $j \leq n-1$, and if $n > 1$, $x_j E_i x_{j+1} E_{i'} x_{j+2}$, $0 \leq j \leq n-2$, $i \neq i' \in \{1, 2\}$.

A typical example of independent equivalence relations is produced as follows. First recall that an action of a group Γ on a set X is *free* if $g \cdot x \neq x$ for $g \neq 1$. Let now $\Gamma = \Gamma_1 * \Gamma_2$ be the free product of two countable groups, let Γ act in a Borel way on X and suppose the action is free. Thus so are the actions of $\Gamma_1, \Gamma_2 \subseteq \Gamma$. Let $E_1 = E^X_{\Gamma_1}$, $E_2 = E^X_{\Gamma_2}$, $E = E^X_{\Gamma}$. Then $E = E_1 \vee E_2$. Moreover E_1, E_2 are independent. Indeed, if $x_0, x_1, x_2, \dots, x_{2n} = x_0$ violates independence, let $g_0, g_1, \dots$ be unique with $g_i \cdot x_i = x_{i+1}$ $(i = 0, \dots, 2n-1)$, $g_i \in \Gamma_1 \setminus \{1\}$ if i is even, $g_i \in \Gamma_2 \setminus \{1\}$, if i is odd. Then $g_0 g_1 \dots g_{2n-1} \cdot x_0 = x_0$ so, by freeness again, $g_0 g_1 \dots g_{2n-1} = 1$, violating the fact that $\Gamma = \Gamma_1 * \Gamma_2$.

Another example of independent equivalence relations is the following: Let $\mathcal{T}$ be a treeing of E and let $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ be a partition of $\mathcal{T}$. Then if E_1, E_2 are the equivalence relations generated by $\mathcal{T}_1, \mathcal{T}_2$, then $E = E_1 \vee E_2$ and $E_1 \perp E_2$.

Notice that, in some sense, independence is orthogonal to commutativity. If E_1, E_2 are independent and commute, then there is a partition of X into Borel sets $X = X_1 \cup X_2$, which are $(E_1 \vee E_2)$-invariant, and $E_1 | X_1 = \Delta_{X_1}$, $E_2 | X_2 = \Delta_{X_2}$.

If E_1, E_2 are independent, then we call $E = E_1 \vee E_2$ the *free join* of E_1, E_2 and denote it by

$$E = E_1 * E_2.$$

If E_1, E_2 are independent (μ-)a.e., i.e., independent when restricted to a conull E-invariant set, then we say that E is the free join of E_1, E_2 a.e.

This concept can be generalized as follows: Let E_1, E_2 be as before and let $E_3 \subseteq E_1 \cap E_2$. We say that E_1, E_2 are *independent over* E_3, in symbols $E_1 \perp_{E_3} E_2$, if for any sequence $x_0, x_1, \dots, x_{2n} = x_0$ $(n > 0)$ as in the definition of independence, we must now have some $j < 2n$ with $x_j E_3 x_{j+1}$. Clearly then $E_3 = E_1 \cap E_2$ (as otherwise any $(x, y) \in (E_1 \cap E_2) \setminus E_3$ would produce the counterexample: x, y, x). Again this condition is equivalent to saying that there is no sequence $x_0, x_1, \dots, x_n = x_0$ with $n > 1$, $(x_i, x_j) \notin E_3$, if $0 \leq i < j < n$, and $x_0 E_1 x_1 E_2 x_2 E_1 x_3 \dots$

Remark 27.1. For further reference, notice that if $E_1 \perp_{E_3} E_2$, then for any $x_0, x_1, \dots, x_{2n} = x_0$, with $n > 0$ and $x_{2i} E_1 x_{2i+1} E_2 x_{2i+2}$ $(i = 0, 1, \dots, n-1)$, there is $j < 2n-1$ with $(x_j, x_{j+1}) \in E_3$. This is easily proved by induction on n.

Again a typical example of independent equivalence relations over their intersection is produced by free actions of amalgamated products of groups. If $\Gamma = \Gamma_1 *_{\Gamma_2} \Gamma_3$ and Γ acts freely in a Borel way on X, then if $E^X_{\Gamma_i} = E_i$, we have $E_1 \perp_{E_3} E_2$.

Another example is as follows: If $\mathcal{T}$ is a treeing of E and $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ with $\mathcal{T}_1 \cap \mathcal{T}_2 = \mathcal{T}_3$, then if E_i is the equivalence relation generated by $\mathcal{T}_i$, we have $E_1 \perp_{E_3} E_2$.

If E_1, E_2 are independent over $E_3 = E_1 \cap E_2$, we call $E = E_1 \vee E_2$ the *amalgamated free join* of E_1, E_2 over E_3, in symbols

$$E = E_1 *_{E_3} E_2.$$

We similarly define the a.e. notions.

The main result of Gaboriau's theory is now the following.

Theorem 27.2 (Gaboriau). *Let E_1, E_2 be countable Borel equivalence relations on X, let $E = E_1 \vee E_2,\ E_3 = E_1 \cap E_2$ and assume that $E_1 \perp_{E_3} E_2$. Let μ be a finite measure which is E-invariant. Then, if E_3 is hyperfinite, and $C_\mu(E_i) < \infty,\ i = 1, 2$, we have*

$$C_\mu(E_1 *_{E_3} E_2) = C_\mu(E_1) + C_\mu(E_2) - C_\mu(E_3).$$

In particular, if $E_1 \perp E_2$,

$$C_\mu(E_1 * E_2) = C_\mu(E_1) + C_\mu(E_2).$$

The condition that E_3 is hyperfinite is necessary. Let F_n $(n = 1, 2, \dots)$ be the free group with n generators and let F_∞ be the free group with $\aleph_0$ generators. Take $\Gamma = F_2 *_{F_6} F_3$ and consider a free action of Γ on X with invariant probability measure μ (for example, consider the action of Γ on 2^Γ by shift, restricted to its free part, i.e., the set $\{x \in 2^\Gamma : \forall g \neq 1 (g \cdot x \neq x)\}$, and $\mu =$ the product measure on 2^Γ, with $2 = \{0, 1\}$ having the $(\frac{1}{2}, \frac{1}{2})$-measure). Then if $E_1 = E^X_{F_2},\ E_2 = E^X_{F_3},\ E_3 = E^X_{F_6}$, and if the formula above was correct, we would have $C_\mu(E_1 *_{E_3} E_2) = C_\mu(E_1) + C_\mu(E_2) - C_\mu(E_3)$. But it is easy to see, as an application of 27.10, that the cost of the equivalence relation induced by a free action of F_n with respect to an invariant probability measure is n, so we would get $C_\mu(E_1 *_{E_3} E_2) = 2 + 3 - 6 < 0$, which is absurd.

Problem 27.3. Does 27.2 hold even without the restriction that $C_\mu(E_i) < \infty,\ i = 1, 2$?

Proof of 27.2. First we note that

$$C_\mu(E_1 *_{E_3} E_2) \leq C_\mu(E_1) + C_\mu(E_2) - C_\mu(E_3).$$

This follows immediately from 23.6. So the main problem is to show that

$$C_\mu(E_1 *_{E_3} E_2) \geq C_\mu(E_1) + C_\mu(E_2) - C_\mu(E_3).$$

Suppose R_1, R_2 are countable Borel equivalence relations on some space Y and let Φ be an L-graphing of $R = R_1 \vee R_2$. We call Φ *decomposable* if $\Phi = \Phi_1 \sqcup \Phi_2$ with $\Phi_1 \subseteq [[R_1]],\ \Phi_2 \subseteq [[R_2]]$.

Lemma 27.4. *Let R_1, R_2 be countable Borel equivalence relations on Y, $R = R_1 \vee R_2$, and let ν be a finite measure which is R-invariant. Let $R_3 = R_1 \cap R_2$ and assume that $R_1 \perp_{R_3} R_2$ (so that $R = R_1 *_{R_3} R_2$) and R_3 is hyperfinite. Then if Φ is a decomposable* L*-graphing of R, we have that $C_\nu(\Phi) \geq C_\nu(R_1) + C_\nu(R_2) - C_\nu(R_3)$.*

Proof. We will derive this from the following sublemma.

Sublemma 27.5. *In the notation of 27.4, and letting $\Phi = \Phi_1 \sqcup \Phi_2$, where $\Phi_i \subseteq [[R_i]],\ i = 1, 2$, we can find* L*-graphs Ψ_1, Ψ_2, with $\Psi_1, \Psi_2 \subseteq [[R_3]]$ such that $\Psi_1 \sqcup \Psi_2$ is an* L*-treeing of a subequivalence relation of R_3 and $\Phi_i \sqcup \Psi_i$ is an* L*-graphing of $R_i, i = 1, 2$.*

To see that 27.5 implies 27.4, notice that $C_\nu(\Phi) = C_\nu(\Phi_1) + C_\nu(\Phi_2) = C_\nu(\Phi_1 \sqcup \Psi_1) + C_\nu(\Phi_2 \sqcup \Psi_2) - C_\nu(\Psi_1 \sqcup \Psi_2) \geq C_\nu(R_1) + C_\nu(R_2) - C_\nu(R_3)$, since if R_3' is the equivalence relation generated by $\Psi_1 \sqcup \Psi_2$, then $R_3' \subseteq R_3$, so R_3' is hyperfinite, and so, by 22.1 and 23.1, $C_\nu(\Psi_1 \sqcup \Psi_2) = C_\nu(R_3') \leq C_\nu(R_3)$.

Proof of 27.5. We will find Ψ_1, Ψ_2 such that $\Psi_1 \sqcup \Psi_2$ is an L-treeing of a subequivalence relation of R_3, and if we let R_i' be the equivalence relation generated by $\Phi_i \sqcup \Psi_i,\ i = 1, 2$, then we have

$$R_3 \cap R_1' = R_3 \cap R_2' \tag{$*$}$$

To see that this suffices assume, say, that $R_1 \neq R_1'$, towards a contradiction. Then there is some sequence $x_0 = x, x_1, \ldots, x_m, x_{m+1} = y$, so that $(x, y) \in R_1 \setminus R_1'$ and $(x_i, x_{i+1}) \in R_1' \cup R_2'$ (as $\Phi_1 \sqcup \Phi_2$, and so $(\Phi_1 \sqcup \Psi_1) \cup (\Phi_2 \sqcup \Psi_2)$, L-graphs $R_1 \vee R_2$). Choose such a sequence with the least possible m. Then clearly $(x_i, x_{i+1}), (x_{i+1}, x_{i+2})$ belong to different R_i' and $(x, x_1), (x_m, y) \in R_2'$, as $(x, y) \in R_1$. If $m = 0$, then $(x, y) \in R_2' \subseteq R_2$, so $(x, y) \in R_3$, thus, by (*) $(x, y) \in R_1'$, a contradiction. So $m > 0$ and m is even, say $m = 2(n-1)$, with $n > 1$, and the sequence above is $x_0 = x, x_1, \ldots, x_{2n-2}, x_{2n-1} = y$. Then since $R_1 \perp_{R_3} R_2$, there is some $i \leq 2n - 2$ (see Remark 27.1) with $(x_i, x_{i+1}) \in R_3$. If (x_i, x_{i+1}) is, say, in R_1', then it must also be in R_2', and thus we can reduce the length of this sequence by at least 2, a contradiction.

So it only remains to construct Ψ_1, Ψ_2.

Let $R_3 = \bigcup_{n=1}^\infty R_3^n$ with $R_3^0 = \Delta_Y$, and $R_3^n \subseteq R_3^{n+1}$ finite Borel equivalence relations. We will inductively define L-graphs Ψ_1^n, Ψ_2^n such that $\Psi_1^n \subseteq \Psi_1^{n+1}$ and $\Psi_2^n \subseteq \Psi_2^{n+1}$, $\Psi_1^n \sqcup \Psi_2^n$ is an L-treeing of a subequivalence relation contained in R_3^n, and such that, if we denote by R_Θ the equivalence relation generated by an L-graph Θ, we have

$$\begin{aligned}(x,y)\in R_3^n \Rightarrow &(1)\ (x,y)\in R_{\Psi_1^n\sqcup\Psi_2^n} \Rightarrow (x,y)\in R_{\Phi_1\sqcup\Psi_1^n}\cap R_{\Phi_2\sqcup\Psi_2^n}.\\ &(2)\ \text{If } n \text{ is even, } (x,y)\in R_{\Phi_1\sqcup\Psi_1^n} \Rightarrow (x,y)\in R_{\Phi_2\sqcup\Psi_2^n}.\\ &(3)\ \text{If } n \text{ is odd, } (x,y)\in R_{\Phi_2\sqcup\Psi_2^n} \Rightarrow (x,y)\in R_{\Phi_1\sqcup\Psi_1^n}\end{aligned}$$

Granting that these have been constructed, it is clear that $\Psi_1 = \bigcup_n \Psi_1^n,\ \Psi_2 = \bigcup_n \Psi_2^n$ works.

We start with $\Psi_1^0 = \Psi_2^0 = \emptyset$. We now assume we have constructed Ψ_1^n, Ψ_2^n, and we proceed to construct $\Psi_1^{n+1}, \Psi_2^{n+1}$. The two cases being similar, we assume that $n+1$ is even (when it is odd we interchange 1 and 2). Let D be a Borel selector for the finite equivalence relation $R_3^{n+1}\cap R_{\Phi_1\sqcup\Psi_1^n}\cap R_{\Phi_2\sqcup\Psi_2^n}$. Fix a Borel linear order $<$ on X. We will define $\Psi_2' = \{\psi\}$, so that ψ is a partial Borel automorphism with graph contained in D^2. Consider a $R_3^{n+1}\cap R_{\Phi_1\sqcup\Psi_1^n}$-class C and the $R_3^{n+1}\cap R_{\Phi_1\sqcup\Psi_1^n}\cap R_{\Phi_2\sqcup\Psi_2^n}$-classes $C_1,\dots,C_k$ contained in it. Let $D\cap C_i = \{x_i\},\ i=1,\dots,k$, and suppose $C_1,\dots,C_k$ have been numbered so that $x_1<\dots<x_k$. Then put $\psi(x_i) = x_{i+1}$, for $1\le i\le k-1$.

This defines Ψ_2'. Finally let $\Psi_1^{n+1} = \Psi_1^n,\ \Psi_2^{n+1} = \Psi_2^n\sqcup\Psi_2'$. It is straightforward to see that this works. (The verification that $\Psi_1^{n+1}\sqcup\Psi_2^{n+1}$ is an L-treeing uses (1) above.) $\dashv$

Return now to E_1, E_2. If we could find decomposable graphings of $E = E_1\vee E_2$ that approach the cost of E, then we would be done by 27.4. This may not be possible directly for E but an unfolding trick can still be used to apply 27.4.

Fix L-graphings Ω_1,Ω_2 of E_1,E_2 resp., of finite cost. Let $\Omega = \Omega_1\sqcup\Omega_2$, a graphing of E. Let now $\epsilon>0$ and Λ be an L-graphing of E with $C_\mu(\Lambda)\le C_\mu(E)+\epsilon/3$. Then by decomposing, if necessary, the domain of each $\lambda\in\Lambda$ into countably many disjoint pieces, which does not affect $C_\mu(\Lambda)$, we can assume that each $\lambda\in\Lambda$ is equal (on its domain) to a composition of members of Ω and their inverses (as both Λ,Ω are L-graphings of E). Now choose N large enough so that $\sum_{i>N} C_\mu(\{\omega_i\})\le\frac{\epsilon}{3}$, where $\Omega=\{\omega_i\}_{i\in I}$, with $I\subseteq\mathbb{N}$.

Call a formal product $\omega_{i_1}^{\pm1}\dots\omega_{i_k}^{\pm1}$, where $\omega_{i_j}\in\Omega$, an *Ω-word*, and similarly define Λ-words. Enumerate the Λ-words in a sequence $v_1,v_2,\dots$ and let for $i\le N,\ i\in I$:

$$W_n^i = \{x\in X : \omega_i(x)\ne v_j(x),\ \forall j=1,\dots,n\}.$$

Then $W_n^i\supseteq W_{n+1}^i$ and $\bigcap_n W_n^i=\emptyset$, so $\mu(W_n^i)\to 0$. Choose N_0 with $\mu(W_{N_0}^i)\le\frac{\epsilon}{3(N+1)}$. Let Λ_0 be the finite subset of Λ used in $v_1,\dots,v_{N_0}$ and let

$$\Theta = \Lambda_0\sqcup\{\omega_i|W_{N_0}^i\}_{i\in I,\ i\le N}\sqcup\{\omega_i\}_{i\in I,\ i>N}.$$

Then $C_\mu(\Theta)\le C_\mu(E)+\frac{\epsilon}{3}+(N+1)\cdot\frac{\epsilon}{3(N+1)}+\frac{\epsilon}{3} = C_\mu(E)+\epsilon$, and Θ is an L-graphing of E.

Also Θ can be written as $\Theta = \Theta_0 \sqcup \Theta_1$, where $\Theta_0 = \Lambda_0$ is finite, every $\theta \in \Theta_0$ can be written on its domain as an Ω-word, and every $\theta \in \Theta_1$ is equal on its domain to an element of Ω. If Θ_0 was empty, then clearly Θ would be decomposable, so $C_\mu(\Theta) \geq C_\mu(E_1) + C_\mu(E_2) - C_\mu(E_3)$ by 27.4 and we would be done. We will show that this inequality is still true, even if $\Theta_0 \neq \emptyset$.

For each $\theta \in \Theta_0$, say $\theta : A_\theta \to B_\theta$, we can find an Ω-word equal to Θ on its domain A_θ and thus we can fix $k(\theta) \geq 1$, Borel sets $A_\theta^0 = A_\theta,\ A_\theta^1, \dots, A_\theta^{k(\theta)-1},\ A_\theta^{k(\theta)} = B_\theta$, and $\varphi_\theta^i : A_\theta^i \to A_\theta^{i+1}$ $(i < k(\theta))$ partial Borel isomorphisms, so that $\varphi_\theta^i = \omega_{l(i,\theta)}^{\pm 1} | A_\theta^i$, for some $\omega_{l(i,\theta)} \in \Omega$, and $\theta = \varphi_\theta^{k(\theta)-1} \varphi_\theta^{k(\theta)-2} \dots \varphi_\theta^0$. Clearly $\{\varphi_\theta^i : \theta \in \Theta_0, i < k(\theta)\} \sqcup \Theta_1$ is decomposable, but it has too large of a cost, since each $\theta \in \Theta_0$ has been now replaced by $k(\theta) - 1$ maps of the same cost. The trick here is to "disjointify" these φ_θ^i.

Fix a copy $\bar{A}_\theta^i$ of A_θ^i, $0 < i < k(\theta)$, so that all $\bar{A}_\theta^i$ are disjoint from each other and from X. Let $\bar{X} = X \sqcup \bigsqcup_{\theta \in \Theta_0, 0 < i < k(\theta)} \bar{A}_\theta^i$. Putting a copy of $\mu | A_\theta^i$ on each $\bar{A}_\theta^i$, this defines a finite measure $\bar{\mu}$ on $\bar{X}$ (with $\bar{\mu} | \bar{A}_\theta^i$ a copy of $\mu | A_\theta^i$ and $\bar{\mu} | X$ a copy of μ). Let also $\bar{\varphi}_\theta^i : \bar{A}_\theta^i \to \bar{A}_\theta^{i+1}$ $(i < k(\theta))$ be a copy of φ_θ^i.

Define a projection $\pi : \bar{X} \to X$ as follows: If $x \in X$, $\pi(x) = x$. If $\bar{x} \in \bar{A}_\theta^i$ and $\bar{x}$ is the copy of $x \in A_\theta^i$, then $\pi(\bar{x}) = x$.

Let $R_i = \pi^{-1}(E_i)$, $i = 1, 2, 3$, $R = \pi^{-1}(E)$, where $E = E_1 \vee E_2$ $(= E_1 *_{E_3} E_2)$. Clearly then $R = R_1 \vee R_2$, $R_3 = R_1 \cap R_2$ is hyperfinite (this is obvious, since π is finite-to-1), and it is trivial to check that $R_1 \perp_{R_3} R_2$, so $R = R_1 *_{R_3} R_2$. Note also that $R_i | X = E_i$, $R | X = E$. Moreover, X is a complete section for R_i, R and so $C_{\bar{\mu}}(R_i) - \bar{\mu}(\bar{X}) = C_\mu(E_i) - \mu(X)$, and similarly for R, E.

Consider the L-graph in $\bar{X}$ given by $\bar{\Theta} = \{\bar{\varphi}_\theta^i\}_{\theta \in \Theta_0,\ i < k(\theta)} \sqcup \Theta_1$. It is clear that it is an L-graphing of R and this in particular shows that $\bar{\mu}$ is R-invariant. It is also clear that $\bar{\Theta}$ is decomposable. So by 27.4 (for $Y = \bar{X}$, $\nu = \bar{\mu}$), $C_{\bar{\mu}}(\bar{\Theta}) \geq C_{\bar{\mu}}(R_1) + C_{\bar{\mu}}(R_2) - C_{\bar{\mu}}(R_3)$. Thus

$$C_{\bar{\mu}}(\bar{\Theta}) - \bar{\mu}(\bar{X}) \geq (C_{\bar{\mu}}(R_1) - \bar{\mu}(\bar{X})) + (C_{\bar{\mu}}(R_2) - \bar{\mu}(\bar{X})) - (C_{\bar{\mu}}(R_3) - \bar{\mu}(\bar{X})).$$

But clearly $C_{\bar{\mu}}(\bar{\Theta}) - \bar{\mu}(\bar{X}) = C_\mu(\Theta) - \mu(X)$, so

$$C_\mu(\Theta) - \mu(X) \geq (C_\mu(E_1) + C_\mu(E_2) - C_\mu(E_3)) - \mu(X),$$

or $C_\mu(\Theta) \geq C_\mu(E_1) + C_\mu(E_2) - C_\mu(E_3)$, and since $C_\mu(\Theta) \leq C_\mu(E) + \epsilon$, we have $C_\mu(E) + \epsilon \geq C_\mu(E_1) + C_\mu(E_2) - C_\mu(E_3)$. Since $\epsilon > 0$ is arbitrary the proof is complete. $\dashv$

One can generalize the preceding to infinite joins. Let $\{E_i\}_{i \in I}$ be a countable family of countable Borel equivalence relations on X and F a countable Borel equivalence relation, $F \subseteq \bigcap_i E_i$. We say that $\{E_i\}$ is independent over F, in symbols $\perp_F^i E_i$, if for any sequence $x_0, x_1, \dots, x_n = x_0$, with $n > 1$, if $x_0 E_{i_0} x_1 E_{i_1} x_2 \dots x_{n-1} E_{i_{n-1}} x_0$, where $i_k \neq i_{k+1}$, if $k < n - 2$, and $i_{n-1} \neq i_0$, there is $j < n$, with $x_j F x_{j+1}$. (Notice then that $F = E_i \cap E_j$ $(= \bigcap_{i \in I} E_i)$ for any $i \neq j$.) If this happens, we write

$$\bigvee_i E_i = *_F^i E_i,$$

and call $*_F^i E_i$ the *amalgamated free join of* $\{E_i\}$ *over* F. An example of this is an equivalence relation induced by a free Borel action of an amalgamated product $*_H^i G_i$ of countable groups G_i over H.

It is clear that if $I = I_1 \cup I_2$ is a partition of I, then $(*_F^{i\in I_1} E_i) \perp_F (*_F^{i\in I_2} E_i)$, and $*_F^{i\in I} E_i = (*_F^{i\in I_1} E_i) *_F (*_F^{i\in I_2} E_i)$. Writing the formula of 27.2 as $C_\mu(E_1 *_{E_3} E_2) - C_\mu(E_3) = (C_\mu(E_1) - C_\mu(E_3)) + (C_\mu(E_2) - C_\mu(E_3))$, and defining the relative cost of S over R, when $S \supseteq R$ and R has finite cost, by

$$C_\mu^R(S) = C_\mu(S) - C_\mu(R),$$

we have $C_\mu^{E_3}(E_1 *_{E_3} E_2) = C_\mu^{E_3}(E_1) + C_\mu^{E_3}(E_2)$. By a straightforward induction then, we get that if I is finite, if F is hyperfinite and each E_i has finite cost, then

$$C_\mu^F(*_F^{i\in I} E_i) = \sum_{i\in I} C_\mu^F(E_i).$$

This also holds for infinite I, as noted by Kechris. (Earlier, Miller has pointed out that this formula follows immediately from 27.2 under the additional assumption that $\sum_{i\in I} C_\mu^F(E_i) < \infty$.)

Theorem 27.6. *Let $\{E_i\}_{i\in I}$ be a countable family of countable Borel equivalence relations on X, let $F = \bigcap_{i\in I} E_i$, and assume that $\perp_F^i E_i$. Let $E = \bigvee_{i\in I} E_i$ $(= *_F^i E_i)$ and let μ be a finite measure which is E-invariant. Then if F is hyperfinite and $C_\mu(E_i) < \infty$, $\forall i \in I$, we have*

$$C_\mu^F(*_F^i E_i) = \sum_i C_\mu^F(E_i).$$

Proof. The inequality $C_\mu^F(*_F^i E_i) \leq \sum_i C_\mu^F(E_i)$ follows from 23.6.

To prove the reverse inequality, assume, without loss of generality, that $I = \{1, 2, 3, \dots\}$. We will need the following lemmas.

Lemma 27.7 (Hjorth-Kechris). *Let E be an aperiodic countable Borel equivalence relation and let μ be an E-invariant ergodic probability measure. Then $C_\mu(E) \leq n+\epsilon$ (where $n = 1, 2, \dots,$ and $0 \leq \epsilon < 1$) iff for each $\epsilon < \delta < 1$ there is an* L*-graphing Φ of E μ-a.e. of the form $\Phi = \{\varphi_1, \dots, \varphi_n, \psi\}$, where $\varphi_i \in [E]$ and $\psi \in [[E]]$ with $\mu(\mathrm{dom}(\psi)) < \delta$.*

In particular, E has finite cost iff it can be generated a.e. by a Borel action of a finitely generated group.

Proof. The direction $\Leftarrow$ is obvious.

$\Rightarrow$: By Zimmer [Z], 9.3.2 there is an aperiodic hyperfinite $E_1 \subseteq E$, so that μ is still E_1-ergodic. Let $\varphi_1 \in [E_1]$ generate E_1. By 23.1 and Section 18 there is an L-graph Ψ such that $\{\varphi_1\} \sqcup \Psi$ is an L-graphing of E and

$n+\epsilon \le C_\mu(\{\varphi_1\} \sqcup \Psi\} = 1 + C_\mu(\Psi) < n+\delta$, thus $(n-1)+\epsilon \le C_\mu(\Psi) < (n-1)+\delta$. Say $C_\mu(\Psi) = (n-1)+\rho$, with $\epsilon \le \rho < \delta$. Since the measure μ is non-atomic, by splitting the domains of the $\psi \in \Psi$ into countably many pieces, if necessary, we can assume that $\Psi = \{\psi^i_j\}_{i=1,\dots,n-1; j\in\mathbb{N}} \sqcup \{\psi_j\}_{j\in\mathbb{N}}$, where $\sum_j C_\mu(\{\psi^i_j\}) = 1$, $\sum_j C_\mu(\{\psi_j\}) = \rho$. Fix Borel sets A^i_j, A_j such that $\mu(A^i_j) = \mu(\mathrm{dom}(\psi^i_j)) = \mu(\mathrm{rng}(\psi^i_j)), \mu(A_j) = \mu(\mathrm{dom}(\psi_j)) = \mu(\mathrm{rng}(\psi_j))$ and each of the families $\{A^i_j\}_{j\in\mathbb{N}}, \{A_j\}_{j\in\mathbb{N}}$ is pairwise disjoint. Thus $\mu(\bigcup_j A^i_j) = \sum_j \mu(A^i_j) = 1$, so by throwing away a set of measure 0 we can assume that $\bigcup_j A^i_j = X$. Since $\mu(A^i_j) = \mu(\mathrm{dom}(\psi^i_j)) = \mu(\mathrm{rng}(\psi^i_j))$, the E_1-ergodicity of μ implies, using 16.3, that, throwing away if necessary again sets of measure 0, there are bijections $\theta^i_j, \omega^i_j \in [[E_1]]$ such that $\theta^i_j : A^i_j \to \mathrm{dom}(\psi^i_j), \omega^i_j : \mathrm{rng}(\psi^i_j) \to A^i_j$, and similarly $\theta_j,\ \omega_j \in [[E_1]]$ such that $\theta_j : A_j \to \mathrm{dom}(\psi_j), \omega_j : \mathrm{rng}(\psi_j) \to A_j$. Then $\varphi_{i+1} = \bigcup_j \omega^i_j \circ \psi^i_j \circ \theta^i_j \in [E]$, $\psi = \bigcup_j \omega_j \circ \psi_j \circ \theta_j \in [[E]]$ $(i = 1,\dots,n-1)$ and $\Phi = \{\varphi_1,\dots,\varphi_n,\psi\}$ obviously works.

For the last assertion, note that if $\psi \in [[E]]$, $\psi : A \to B$ is as above, then, since $\mu(A) = \mu(B)$, we also have $\mu(X\setminus A) = \mu(X \setminus B)$, so we can find (modulo null sets) $\psi' : (X\setminus A) \to (X \setminus B)$, $\psi' \in [[E_1]]$. Then if $\varphi_{n+1} = \psi \cup \psi'$, $\{\varphi_1,\dots,\varphi_n,\varphi_{n+1}\}$ generates E. $\dashv$

Notice that ergodicity is necessary, as can be seen by using 27.10.

Lemma 27.8. *Let E be a countable Borel equivalence relation on X and μ a finite E-invariant measure. Assume $C_\mu(E) < \infty$. Let $E_0 \subseteq E$ be hyperfinite and let Φ_0 be a finite* L*-treeing of E_0. Then for each $\epsilon > 0$, there is a sequence of finite* L*-graphs $\{\Phi_n\}_{n=1}^\infty$ with $\Phi_0 \subseteq \Phi_1 \subseteq \Phi_2 \subseteq \dots$ such that $\Phi = \bigcup_n \Phi_n$ is an* L*-graphing of E, $C_\mu(\Phi \setminus \Phi_n) \le \frac{1}{n^2}$ and if R_{Φ_n} is the equivalence relation generated by Φ_n, then*

$$C_\mu(\Phi_n) \le C_\mu(R_{\Phi_n}) + \epsilon.$$

Granting this last lemma, we can complete the proof of the theorem by using a variant of an argument of Gaboriau (used in his proof of 27.10):

Fix $\epsilon > 0$ and a finite L-treeing Φ_0 of $E_0 = F$. Applying 27.8 to E_i, $i = 1,2,\dots$, find finite L-graphs $\Phi_0 \subseteq \Phi^i_1 \subseteq \Phi^i_2 \subseteq \dots$, so that $\Phi^i = \bigcup_i \Phi^i_n$ is an L-graphing of E_i, $C_\mu(\Phi^i \setminus \Phi^i_n) \le \frac{1}{n^2}$ and $C_\mu(E_i) - \frac{\epsilon}{2^i} \le C_\mu(\Phi^i_n) \le C_\mu(R_{\Phi^i_n}) + \frac{\epsilon}{2^i}$.

Let $\Phi^{\le n} = \Phi^1_n \sqcup \Phi^2_n \sqcup \dots \sqcup \Phi^n_n$, so that $\Phi^{\le n} \subseteq \Phi^{\le n+1}$, and let $\Phi = \bigcup_n \Phi^{\le n}$. Clearly each $\Phi^{\le n}$ is finite and Φ L-graphs E.

Now consider an arbitrary L-graphing $\Psi = \{\psi_1, \psi_2, \dots\}$ of E. We can assume, without affecting $C_\mu(\Psi)$, that each ψ_i is equal to a Φ-word on its domain.

Fix $m \ge 1$, and let $\Phi^{\le m} = \{\varphi_1,\dots,\varphi_{k_m}\}$, where $k_m \to \infty$. We can find large enough $N = N(m)$ and sets $D_1,\dots,D_{k_m}$ of measure $< \frac{1}{2^{k_m}}$ so that for x in $\mathrm{dom}(\varphi_i) \setminus D_i$, $\varphi_i(x)$ can be expressed as a $\Psi_N = \{\psi_1,\dots,\psi_N\}$-word (depending on x), for $i \le k_m$. Then we can find large enough $n > m$ so that each $\psi_i \in \Psi_N$ can be written as a $\Phi^{\le n}$-word on its domain. Thus

$\Psi_N \sqcup \{\varphi_i|D_i\}_{i=1}^{k_m} \sqcup (\Phi_n^1 \setminus \Phi_m^1) \sqcup \cdots \sqcup (\Phi_n^m \setminus \Phi_m^m) \sqcup (\Phi_n^{m+1} \setminus \Phi_0) \sqcup \cdots \sqcup (\Phi_n^n \setminus \Phi_0)$ is an L-graphing of $R_{\Phi^{\leq n}}$ (as $\Phi_0 \subseteq \Phi^{\leq m}$) and has cost less than or equal to

$$C_\mu(\Psi_N) + \frac{k_m}{2^{k_m}} + \sum_{i=1}^{m} C_\mu(\Phi_n^i \setminus \Phi_m^i) + \left[\sum_{i=m+1}^{n} C_\mu(\Phi_n^i)\right] - (n-m)C_\mu(\Phi_0),$$

so, as $C_\mu(\Phi_n^i \setminus \Phi_m^i) \leq C_\mu(\Phi^i \setminus \Phi_m^i) \leq \frac{1}{m^2}$ and $C_\mu(\Phi_0) = C_\mu(F)$, it has cost less than or equal to

$$C_\mu(\Psi_N) + \frac{k_m}{2^{k_m}} + \frac{1}{m} + \left[\sum_{i=m+1}^{n} C_\mu(\Phi_n^i)\right] - (n-m)C_\mu(F).$$

Now notice that we also have $\perp_F^{i\leq n} R_{\Phi_n^i}$ (since $R_{\Phi_n^i} \subseteq E_i$), so $C_\mu(R_{\Phi^{\leq n}}) = C_\mu(*_F^{i\leq n} R_{\Phi_n^i}) = [\sum_{i=1}^n C_\mu(R_{\Phi_n^i})] - (n-1)C_\mu(F)$.

Thus

$$\begin{aligned}
C_\mu(\Psi_N) + \frac{k_m}{2^{k_m}} + \frac{1}{m} + \left[\sum_{i=m+1}^{n} C_\mu(\Phi_n^i)\right] - (n-m)C_\mu(F) \\
\geq \left[\sum_{i=1}^{n} C_\mu(R_{\Phi_n^i})\right] - (n-1)C_\mu(F) \\
\geq \sum_{i=1}^{n} \left[C_\mu(\Phi_n^i) - \frac{\epsilon}{2^i}\right] - (n-1)C_\mu(F) \\
\geq \left[\sum_{i=1}^{n} C_\mu(\Phi_n^i)\right] - \epsilon - (n-1)C_\mu(F),
\end{aligned}$$

and so

$$\begin{aligned}
C_\mu(\Psi_N) &\geq -\frac{k_m}{2^{k_m}} - \frac{1}{m} - \epsilon - (m-1)C_\mu(F) + \sum_{i=1}^{m} C_\mu(\Phi_n^i) \\
&\geq -\frac{k_m}{2^{k_m}} - \frac{1}{m} - \epsilon - (m-1)C_\mu(F) + \sum_{i=1}^{m} \left[C_\mu(E_i) - \frac{\epsilon}{2^i}\right] \\
&\geq -\frac{k_m}{2^{k_m}} - \frac{1}{m} - (m-1)C_\mu(F) + \left[\sum_{i=1}^{m} C_\mu(E_i)\right] - 2\epsilon.
\end{aligned}$$

Since $C_\mu(\Psi) \geq C_\mu(\Psi_N)$, this gives

$$C_\mu(\Psi) - C_\mu(F) \geq -\frac{k_m}{2^{k_m}} - \frac{1}{m} + \sum_{i=1}^{m} [C_\mu(E_i) - C_\mu(F)] - 2\epsilon,$$

so, letting $m \to \infty$, we get

$$C_\mu(\Psi) - C_\mu(F) \geq \sum_{i=1}^{\infty} [C_\mu(E_i) - C_\mu(F)] - 2\epsilon,$$

thus

$$C_\mu(E) - C_\mu(F) \geq \sum_{i=1}^{\infty} [C_\mu(E_i) - C_\mu(F)],$$

and we are done.

Proof of Lemma 27.8. We can of course assume that $\epsilon < 1$. First fix a Borel transversal $S_{<\infty}$ for the periodic part of E_0 (i.e., the set of $x \in X$ with $[x]_{E_0}$ finite). Next notice that we can assume that E is aperiodic and μ is a probability measure. Let then $\pi : X \to \mathcal{EI}_E$ be the ergodic decomposition of E, as in 18.5, $\nu = \pi_*\mu,\ X_e = \pi^{-1}(e)$. Since, by 18.5, $C_\mu(E) = \int C_e(E)\, d\nu(e) < \infty$, we can assume, throwing away a null set, that there is a Borel partition $\mathcal{EI}_E = B_1 \cup B_2 \cup \cdots \cup B_n \cup \dots$ (finite or infinite) such that $\nu(B_n) > 0$ and for some integers $1 < M_1, M_2, \cdots < \infty$, $C_e(E) \leq M_n$, and ($e(S_{<\infty}) = 0$ or $e(S_{<\infty}) \geq \frac{1}{M_n}$), for any $e \in B_n$. Let $A_n = \pi^{-1}(B_n) = \bigcup_{e \in B_n} X_e$.

Claim 27.9. *We can find a finite* L*-graphing* Φ'_n *of* $E|A_n$ *with* $\Phi_0|A_n \subseteq \Phi'_n$ *such that* $C_\mu(\Phi'_n) \leq C_{\mu|A_n}(E|A_n) + \frac{\epsilon}{2^n}$*.*

Granting this, let

$$\tilde{\Phi}_n = \bigsqcup_{i=1}^{n} (\Phi'_i \setminus (\Phi_0|A_i)) \sqcup \Phi_0.$$

Then $\Phi_0 \subseteq \tilde{\Phi}_1 \subseteq \tilde{\Phi}_2 \subseteq \dots$, each $\tilde{\Phi}_n$ is finite, and $\Phi = \bigcup_{n=1}^{\infty} \tilde{\Phi}_n$ is an L-graphing of E. Also $R_{\tilde{\Phi}_n} = E|(A_1 \cup \cdots \cup A_n) \cup E_0|(X \setminus (A_1 \cup \cdots \cup A_n))$. So

$$\begin{aligned}
C_\mu(R_{\tilde{\Phi}_n}) &= C_\mu(E|(A_1 \cup \cdots \cup A_n)) + C_\mu(E_0|(X \setminus (A_1 \cup \cdots \cup A_n))) \\
&\geq \sum_{i=1}^{n} \left[C_\mu(\Phi'_i) - \frac{\epsilon}{2^i} \right] + C_\mu(\Phi_0|(X \setminus (A_1 \cup \cdots \cup A_n))) \\
&\geq \sum_{i=1}^{n} [C_\mu(\Phi'_i \setminus \Phi_0|A_i) + C_\mu(\Phi_0|A_i)] - \epsilon + \\
&\qquad C_\mu(\Phi_0|(X \setminus (A_1 \cup \cdots \cup A_n))) \\
&= \sum_{i=1}^{n} C_\mu(\Phi'_i \setminus \Phi_0|A_i) + C_\mu(\Phi_0) - \epsilon \\
&= C_\mu(\tilde{\Phi}_n) - \epsilon.
\end{aligned}$$

Finally,

$$
\begin{aligned}
C_\mu(\Phi \setminus \tilde{\Phi}_n) &= \sum_{i=n+1}^{\infty} C_\mu(\Phi'_i \setminus (\Phi_0|A_i)) \\
&\leq \sum_{i=n+1}^{\infty} C_\mu(\Phi'_i) \\
&\leq \sum_{i=n+1}^{\infty} \left[C_{\mu|A_i}(E|A_i) + \frac{\epsilon}{2^i} \right].
\end{aligned}
$$

Since $\sum_{i=1}^{\infty} C_{\mu|A_i}(E|A_i) = C_\mu(E) < \infty$, it follows that $C_\mu(\Phi \setminus \tilde{\Phi}_n) \downarrow 0$ as $n \to \infty$, so we can find $\ell_1 < \ell_2 < \dots$ with $C_\mu(\Phi \setminus \tilde{\Phi}_{\ell_n}) \leq \frac{1}{n^2}$. Put $\Phi_n = \tilde{\Phi}_{\ell_n}$. This clearly works.

So it only remains to prove the claim. For notational convenience put $B = B_n,\ A = A_n = \pi^{-1}(B),\ M = M_n,\ \delta = \frac{\epsilon}{2^n}$. Using a vanishing sequence of markers for the aperiodic part X_∞ of E_0 (i.e., the set of $x \in X$ with $[x]_{E_0}$ infinite), we can easily find for each $e \in B$ a Borel set $S_{\infty,e} \subseteq X_e$ which is a complete section for $E_0|(X_\infty \cap X_e)$, and either $e(X_\infty) = 0$ or else $\frac{\delta}{2} \leq e(S_{\infty,e}) \leq \delta$ (notice here that E being aperiodic, e is necessarily non-atomic). Moreover we can make sure that $S_{\infty,e}$ depends in a Borel way on e, so that $S_\infty = \bigcup_{e \in A} S_{\infty,e}$ is Borel. Now $S_e = (S_{<\infty} \cap X_e) \cup S_{\infty,e}$ is a complete section for $E_0|X_e$ and thus for $E|X_e$, for each $e \in B$. Moreover

$$
e(S_e) = e(S_{<\infty}) + e(S_{\infty,e}) \geq \min\left\{ \frac{1}{M}, \frac{\delta}{2} \right\} > \frac{1}{K},
$$

for some $K \in \mathbb{N} \setminus \{0\}$. Put $\rho = \frac{e|S_e}{e(S_e)}$. Then

$$
\begin{aligned}
C_\rho(E|S_e) &= \frac{C_{e|S_e}(E|S_e)}{e(S_e)} = \frac{C_e(E) - 1 + e(S_e)}{e(S_e)} \\
&= \frac{C_e(E) - 1}{e(S_e)} + 1 < K(M-1) + 1 = N.
\end{aligned}
$$

Since ρ is $(E|S_e)$-invariant ergodic and $E|S_e$ is aperiodic ρ-a.e., 27.7 shows that we can find an L-graphing $\Phi_{S_e} = \{\varphi_e^1, \dots, \varphi_e^N\}$ of $E|S_e$, with $C_\rho(\Phi_{S_e}) \leq C_\rho(E|S_e) + \frac{\delta}{2e(S_e)}$, so $C_e(\Phi_{S_e}) \leq C_e(E|S_e) + \frac{\delta}{2}$. Put $\Phi'_e = \Phi_0|X_e \sqcup \Phi_{S_e}$. Then, as in the proof of the second part of 23.1, Φ'_e is an L-graphing of $E|X_e$ and $C_e(\Phi'_e) \leq C_e(E|X_e) + \delta$. Since we can make sure that Φ_{S_e} depends in a Borel way on $e \in B$, if $\varphi^i = \bigcup_{e \in B} \varphi_e^i,\ i = 1, \dots, N$, and $\Phi = \{\varphi^1, \dots, \varphi^N\}$, then $\Phi' = (\Phi_0|A) \sqcup \Phi$ is an L-graphing of $E|A$, $\Phi_0|A \subseteq \Phi'$, and $C_\mu(\Phi') = \int_{e \in A} C_e(\Phi'_e)\, d\nu(e) \leq \int_{e \in A} C_e(E|X_e)\, d\nu(e) + \delta = C_{\mu|A}(E|A) + \delta$, and the proof is complete. ⊣

In particular, if $F = \Delta_X$, i.e., $\{E_i\}$ is independent, then, omitting the subscript, and assuming that $C_\mu(E_i) < \infty$ for each i, we have

$$C_\mu(*^i E_i) = \sum_i C_\mu(E_i).$$

As an immediate consequence, we have:

Corollary 27.10 (Gaboriau). *Let E be a countable Borel equivalence relation on X and μ a finite measure which is E-invariant. If $\mathcal{T}$ is a treeing of E a.e., then $C_\mu(\mathcal{T}) = C_\mu(E)$. Similarly for* L*-treeings. In particular, if $C_\mu(E) < \infty$, E is treeable a.e. iff $C_\mu(E)$ is attained.*

Proof. It is enough to prove this for L-treeings Φ. By 27.6 we can assume that $\Phi = \{\varphi\}$ and then it follows from 22.1, since the equivalence relation generated by Φ is hyperfinite. ⊣

For a shorter direct proof of this corollary, see Gaboriau [G1].

Corollary 27.11 (Gaboriau). *There is a countable Borel equivalence relation E, having an E-invariant probability measure, which does not admit a graphing $\mathcal{G}$ of bounded degree (i.e., for which there is some n so that the degree of each vertex is $\leq n$).*

Proof. Let E be the equivalence relation induced by a free action of the group F_∞ (the free group with $\aleph_0$ generators) with invariant probability measure μ. Then by 27.10, $C_\mu(E) = \infty$. If E admitted such a graphing $\mathcal{G}$ of degree $\leq n$, clearly $2C_\mu(\mathcal{G}) \leq n$, a contradiction. ⊣

Gaboriau-Jackson-Kechris-Louveau (see [JKL], 3.12 and Remark following it) have shown that every countable Borel equivalence relation admits a graphing which is locally finite (i.e., each vertex has finite degree).

Corollary 27.12 (Gaboriau). *Let E be an aperiodic countable Borel equivalence relation and μ a finite measure which is E-invariant. If $C_\mu(E) = \mu(X)$ and E is treeable, then E is hyperfinite a.e.*

Proof. This follows from 27.10 and 22.2. ⊣

Corollary 27.13 (Adams [A]). *Let R, S be aperiodic countable Borel equivalence relations on X, Y, and let μ be a finite $(R \times S)$-invariant measure on $X \times Y$. Then $R \times S$ is treeable a.e. iff R, S are hyperfinite a.e.*

Proof. By 24.9 and 27.12. ⊣

Corollary 27.14 (Jackson-Kechris-Louveau [JKL]). *Let R, S, μ be as in 27.13 and assume that $R \times S$ has a finite index subrelation which is treeable a.e. Then R, S are hyperfinite a.e.*

Proof. By 25.4, 27.12, and the fact that an equivalence relation that has finite index over a hyperfinite one is also hyperfinite. ⊣

As a further corollary, we show that $C_\mu(E)$ can obtain any value ≥ 1 (including ∞) if E is aperiodic and μ is a non-atomic probability measure, which is E-invariant and ergodic. (If E is not aperiodic, then, by 20.1, $C_\mu(E)$ can obtain only the values $1 - \frac{1}{n},\ n \geq 2$, under these assumptions on μ.)

Corollary 27.15 (Gaboriau). *For each $\alpha \in [1, \infty]$, there is an aperiodic treeable equivalence relation E and a non-atomic probability measure μ, which is E-invariant, ergodic, such that $C_\mu(E) = \alpha$.*

Proof. Write $\alpha = n + \epsilon$, where $n \in \mathbb{N} \setminus \{0\}$, and $0 \leq \epsilon < 1$, or else $n = \infty, \epsilon = 0$. Consider the free part X of the shift action of F_∞ on 2^{F_∞}, and let μ be the usual product measure. Let $F_\infty = \{g_i\}$, and let $\varphi_i(x) = g_i \cdot x$. If $n = \infty$, let E be the equivalence relation induced by this action, while if $n < \infty$, fix a Borel set A with $\mu(A) = \epsilon$ and let E be the equivalence relation generated by $\{\varphi_1, \dots, \varphi_n\} \cup \{\varphi_{n+1}|A\}$. This clearly works. $\dashv$

We next establish a partial converse to 27.6. As usual we say that $\{E_i\}$ is independent over F $(\subseteq \bigcap_i E_i)$ a.e. if the condition of independence holds for an $(\bigvee_i E_i)$-invariant conull Borel set. This means that for a.e. x, if $x_0 = x, x_1, \dots, x_n = x_0$, with $n > 1$, is such that $x_0 E_{i_0} x_1 E_{i_1} x_2 \dots x_{n-1} E_{i_{n-1}} x_0$ where $i_k \neq i_{k+1}$, if $k < n - 2$, and $i_{n-1} \neq i_0$, then for some $j < n$, we have $x_j F x_{j+1}$. If this happens, we write $\perp^i_F E_i$, a.e., and $\bigvee_i E_i = *^i_F E_i$, a.e.

Theorem 27.16 (Gaboriau). *Let $\{E_i\}_{i \in I}$ be a countable family of countable Borel equivalence relations on X and let $F \subseteq \bigcap_{i \in I} E_i$. Let $E = \bigvee_{i \in I} E_i$ and let μ be a finite measure which is E-invariant. Then if F is smooth, $\sum_i C^F_\mu(E_i) < \infty$, and*

$$C^F_\mu(E) = \sum_i C^F_\mu(E_i),$$

we have $\perp^i_F E_i$ a.e., thus $E = *^i_F E_i$, a.e.

Proof. Assume $I \subseteq \mathbb{N}$ and $\perp^i_F E_i$ a.e. fails, towards a contradiction. Consider first the case when $F = \Delta_X$. Then, as in the proof of 19.1, we can find a Borel set of positive measure $B \subseteq X$, $i_0, \dots, i_{n-1} \in I$, where $n > 1$ and $\varphi_0 \in [[E_{i_0}]], \dots, \varphi_{n-1} \in [[E_{i_{n-1}}]]$, with $i_k \neq i_{k+1}$, if $k < n - 1$, $i_{n-1} \neq i_0$, $\varphi_{n-1}\varphi_{n-2} \dots \varphi_0(x) = x$, $\forall x \in B$, and so that the sets B, $\varphi_0(B)$, $\varphi_1\varphi_0(B), \dots, \varphi_{n-2}\varphi_{n-3} \dots \varphi_0(B)$ are pairwise disjoint. Let $j_0, \dots, j_m$ $(m \leq n - 1)$ be the distinct indices among $i_0, \dots, i_{n-1}$ and let $\Phi_{j_t} = \{\varphi_k | \varphi_{k-1} \dots \varphi_0(B) : \ i_k = j_t\}$, $t = 0, \dots, m$ (where by convention $\varphi_{-1} =$ identity). Then Φ_{j_t} is an L-treeing of a finite subequivalence relation of E_{j_t} (as $i_k \neq i_{k+1}$, if $k < n - 1$, $i_{n-1} \neq i_0$, all the partial automorphisms in Φ_{j_t} have pairwise disjoint domain and ranges). Let $\mathcal{G}_{j_t}$ be the treeing associated with Φ_{j_t} as in Section 17, i.e., the union of the graphs of the partial automorphisms in Φ_{j_t} and their inverses. Fix ϵ with $\mu(B) > 2\epsilon > 0$ and, by 23.1, choose a graphing $\mathcal{G}'_{j_t} \supseteq \mathcal{G}_{j_t}$ of E_{j_t} with $C_\mu(\mathcal{G}'_{j_t}) \leq C_\mu(E_{j_t}) + \frac{\epsilon}{2^{j_t}}$, for $t = 0, \dots, m$, and let $\mathcal{G}'_j$ be a graphing of E_j, for $j \notin \{j_0, \dots, j_m\}$, with $C_\mu(\mathcal{G}'_j) \leq C_\mu(E_j) + \frac{\epsilon}{2^j}$. Then if, by renumbering, $j_0 = i_0$, clearly

$$[(\mathcal{G}'_{j_0} \cup \dots \cup \mathcal{G}'_{j_m}) \setminus (\mathrm{graph}(\varphi_0|B) \cup \mathrm{graph}(\varphi_0|B)^{-1})] \cup \bigcup_{j \notin \{j_0, \dots, j_m\}} \mathcal{G}'_j$$

is a graphing of E, so

$$\begin{aligned} C_\mu(E) &\leq [C_\mu(\mathcal{G}'_{j_0}) - \mu(B)] + C_\mu(\mathcal{G}'_{j_1}) + \dots + C_\mu(\mathcal{G}'_{j_m}) + \sum_{j \notin \{j_0, \dots, j_m\}} C_\mu(\mathcal{G}'_j) \\ &\leq \sum_i C_\mu(E_i) + \sum_{i \in I} \frac{\epsilon}{2^i} - \mu(B) \\ &\leq \sum_i C_\mu(E_i) + 2\epsilon - \mu(B) \\ &< \sum_i C_\mu(E_i), \end{aligned}$$

a contradiction.

Now consider the general case when F is smooth but not necessarily the same as Δ_X. Let T be then a Borel transversal for F. Then T is a complete section for all E_i, so

$$\begin{aligned} C_{\mu|T}(E_i|T) &= C_\mu(E_i) - (\mu(X) - \mu(T)) \\ &= C_\mu(E_i) - C_\mu(F) \\ &= C_\mu^F(E_i), \end{aligned}$$

and similarly

$$C_{\mu|T}(E|T) = C_\mu^F(E).$$

Thus $C_{\mu|T}(E|T) = \sum_i C_{\mu|T}(E_i|T)$ and $\sum_i C_{\mu|T}(E_i|T) < \infty$, so, by applying the special case to $\Delta_T,\ E_i|T,\ E|T$ (it is easy to check that $E|T = \bigvee_i E_i|T$), we conclude that $\perp^i E_i|T$, a.e., thus (as it is easy to check again) $\perp_F^i E_i$, a.e. ⊣

See the last paragraph of Section 36 for a counterexample that shows that the assumption that F is smooth cannot be replaced by the assumption that F is hyperfinite.

For further reference, note also the following fact.

Proposition 27.17. *Let $G_0, G_1, \dots, F_0, F_1, \dots$ be two sequences of countable Borel equivalence relations, such that each G_i is treeable and each F_i is smooth. Let $E_0 = G_0$, and $E_{n+1} = E_n \vee G_{n+1}$. Assume that $E_{n+1} = E_n *_{F_n} G_{n+1}$. Then $E = \bigcup_n E_n$ is treeable, and if μ is an E-invariant finite measure, then $C_\mu(E_0) \leq C_\mu(E_1) \leq \dots$ and $\lim C_\mu(E_n) = C_\mu(E)$.*

Proof. It is enough to find for each n a treeing $\mathcal{T}_n$ of E_n such that $\mathcal{T}_0 \subseteq \mathcal{T}_1 \subseteq \dots$ Then $C_\mu(E_n) = C_\mu(\mathcal{T}_n) \leq C_\mu(\mathcal{T}_{n+1}) = C_\mu(E_{n+1})$, $\mathcal{T} = \bigcup_n \mathcal{T}_n$ will be a treeing of E, and $C_\mu(E) = C_\mu(\mathcal{T}) = \lim C_\mu(\mathcal{T}_n) = \lim C_\mu(E_n)$.

We take $\mathcal{T}_0$ to be any treeing of G_0. Given $\mathcal{T}_n$, let T_n be a transversal for F_n, and, since $G_{n+1}|T_n$ is treeable (by 21.2), fix a treeing $\mathcal{T}'_{n+1}$ for it. Then take $\mathcal{T}_{n+1} = \mathcal{T}_n \cup \mathcal{T}'_{n+1}$. ⊣

28 Costs and Free Actions

Recall here the characterization of cost, in the ergodic case, given in 27.7.

Proposition 28.1. *Let E be an aperiodic countable Borel equivalence relation and let μ be an E-invariant ergodic probability measure. Then $C_\mu(E) \leq n + \epsilon$ (where $n = 1, 2, \dots,$ and $0 \leq \epsilon < 1$) iff for each $\epsilon < \delta < 1$ there is an* L-*graphing Φ of E μ-a.e. of the form $\Phi = \{\varphi_1, \dots, \varphi_n, \psi\}$, where $\varphi_i \in [E]$ and $\psi \in [[E]]$ with $\mu(\mathrm{dom}(\psi)) < \delta$.*

In particular, E has finite cost iff it can be generated a.e. by a Borel action of a finitely generated group.

It follows from 27.10 that any E induced by a free Borel action of the free group with n generators F_n with invariant probability measure μ has cost $C_\mu(E) = n$ and is of course treeable. The following gives a converse to this result, by providing a sharper version of 28.1 for treeable equivalence relations.

Theorem 28.2 (Hjorth [H]). *Let E be an aperiodic countable Borel equivalence relation and let μ be an E-invariant ergodic probability measure. If E is treeable and $C_\mu(E) = n + \epsilon$ (where $n = 1, 2, \dots,$ and $0 \leq \epsilon < 1$), then there is an* L-*treeing Φ of E μ-a.e. of the form $\Phi = \{\varphi_1, \dots, \varphi_n, \psi\}$, where $\varphi_i \in [E]$ and $\psi \in [[E]]$ with $\mu(\mathrm{dom}(\psi)) = \epsilon$.*

In particular, E is treeable with $C_\mu(E) = n$ $(n = 1, 2, \dots)$ iff E is generated by a free Borel action of F_n a.e.

This follows immediately from the following result, by a simple induction.

Theorem 28.3 (Hjorth [H]). *Let E be an aperiodic countable Borel equivalence relation on X and let μ be an E-invariant ergodic finite measure. Let $\Omega = \Phi \sqcup \Psi$ be an* L-*graphing of E with $C_\mu(\Omega) = C_\mu(E) < \infty$ (thus Ω is an* L-*treeing a.e.). If $\delta = C_\mu(\Psi) < \mu(X)$, then we can find $\varphi \in [[E]]$ with $\mu(\mathrm{dom}(\varphi)) = \delta$ such that $\Phi \sqcup \{\varphi\}$ is an* L-*graphing of E, a.e. If $\delta = C_\mu(\Psi) \geq \mu(X)$, then we can find $\varphi \in [E]$ and Ψ' such that $\Omega' = \Phi \sqcup \{\varphi\} \sqcup \Psi'$ is an* L-*graphing of E a.e. with $C_\mu(\Omega') = C_\mu(E)$, and if $\Psi = \{\psi_i\}_{i \in I}$, then $\Psi' = \{\psi_i | A'_i\}_{i \in I}$, where A'_i is Borel and $A'_i \subseteq \mathrm{dom}(\psi_i)$. Moreover, we can decompose each $\mathrm{dom}(\psi_i) \setminus A'_i$ into countably many Borel sets on each of which ψ_i can be expressed as a $\Phi \sqcup \{\varphi\}$-word.*

Proof. We will describe first a construction which will allow us to define φ when $\delta < \mu(X)$. If $\delta \geq \mu(X)$, we will define φ by an infinite repetition of this construction.

If the domain of every $\psi \in \Psi$ is null, i.e., if $C_\mu(\Psi) = 0$, we can take φ to have null domain as well. Otherwise fix a $\psi_0 = \psi_{i_0} \in \Psi$ with $\mu(\mathrm{dom}(\psi_0)) > 0$, and thus with $\mu(\{x \in \mathrm{dom}(\psi_0) : \psi_0(x) \neq x\}) > 0$, since Ω is an L-treeing a.e. Then, as in the proof of 19.1, we can find a Borel set of positive measure $A_0 \subseteq \mathrm{dom}(\psi_0)$ with $\psi_0(A_0) \cap A_0 = \emptyset$, and $\mu(A_0) < \mu(X) - \delta$, if $\delta < \mu(X)$ (note here that μ is non-atomic). Let $\varphi_1 = \psi_0|A_0$, so that $\mathrm{dom}(\varphi_1) \cap \mathrm{rng}(\varphi_1) = \emptyset$.

A *subtree* of $\mathbb{N}^{<\mathbb{N}} = \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$ is a subset $T \subseteq \mathbb{N}^{<\mathbb{N}}$ such that $\emptyset$ (= the empty sequence) $\in T$ and

$$s = (s_0, \dots, s_{n-1}) \in T \text{ and } m \leq n \Rightarrow s|m = (s_0, \dots, s_{m-1}) \in T.$$

If $s = (s_0, \dots, s_{n-1})$ we let $|s| = n$ be the *length* of s and for $j \in \mathbb{N}$ we put $(s, j) = (s_0, \dots, s_{n-1}, j) \in \mathbb{N}^{n+1}$.

We will construct, by induction on $n \geq 1$, a *finite* subtree T_n of $\mathbb{N}^{<\mathbb{N}}$, a map $s \mapsto V_s^n$, which assigns to each $s \in T_n$ a Borel subset of X of positive measure, and a map $t = (s, j) \mapsto \rho^n_{(s,j)}$ which assigns to each $t \in T_n$, of length ≥ 1, a map $\rho^n_t \in [[E]]$ with $\mathrm{dom}(\rho^n_t) = W^n_t \subseteq V^n_s$ and $\mathrm{rng}(\rho^n_t) = V^n_t$, such that the following conditions hold:

(i) $T_1 \subseteq T_2 \subseteq T_3 \subseteq \dots$,
(ii) If $t \in T_n$, $\rho^n_t = \rho^{n+1}_t$ (so also $W^n_t = W^{n+1}_t$, $V^n_t = V^{n+1}_t$),
(iii) $V^n_s \cap V^n_{s'} = \emptyset$, if $s \neq s'$, and
(iv) $W^n_{(s,j)} \cap W^n_{(s,j')} = \emptyset$, if $j \neq j'$.

Denote by $\varphi_n \in [[E]]$ the map

$$\varphi_n = \bigcup_{t \in T^n, |t| \geq 1} \rho^n_t.$$

We will also construct L-graphs $\Psi_n, n \geq 1$, such that $\Phi \sqcup \{\varphi_n\} \sqcup \Psi_n$ is an L-graphing of E of cost equal to that of E (i.e., the cost of $\Phi \sqcup \Psi$), such that, if $\Psi = \{\psi_i\}_{i \in I}$, then $\Psi_n = \{\psi_i | A^n_i\}_{i \in I}$ for some Borel sets $A^n_i \subseteq \mathrm{dom}(\psi_i)$, with $A^{n+1}_i \subseteq A^n_i$ (some A^n_i may be null). Moreover, each $\mathrm{dom}(\psi_i) \setminus A^n_i$ can be decomposed into finitely many Borel sets on each of which ψ_i can be expressed as a $\Phi \sqcup \{\varphi_n\}$-word.

Before we start the construction, we fix an enumeration $(a_n)_{n \geq 2}$ of all 5-tuples of the form (s, i, ϵ, v, w), where $s \in \mathbb{N}^{<\mathbb{N}} \setminus \{\emptyset\}, i \in I, \epsilon \in \{\pm 1\}, v, w$ are words in $\Phi \sqcup \{\hat{\varphi}\}$ ($\hat{\varphi}$ a symbol not in Φ), so that each such 5-tuple appears infinitely often in this enumeration.

We now start by taking $T_1 = \{\emptyset, (0)\}$, with $V^1_\emptyset = \mathrm{dom}(\varphi_1) = A_0, V^1_{(0)} = \mathrm{rng}(\varphi_1), \rho^1_{(0)} = \varphi_1$. (Note that this notation for φ_1 is consistent as the map corresponding to $T_1, \bigcup_{t \in T^1, |t| \geq 1} = \rho^1_{(0)}$ is indeed φ_1.) We also let $\Psi_1 = (\Psi \setminus \{\psi_0\}) \sqcup \{\psi_0 | (\mathrm{dom}(\psi_0) \setminus A_0)\}$. Thus we have transferred $\psi_0 | A_0$ from Ψ to φ_1. Clearly all the above conditions hold for $n = 1$.

Assume now $T_n, s' \mapsto V^n_{s'}, t' \mapsto \rho^n_{t'}, \Psi_n, i' \mapsto A^n_{i'}$ have been constructed satisfying the above conditions. Let $a_{n+1} = (s, i, \epsilon, v, w)$. If $\epsilon = +1$ consider the set

$$B_n = \{x \in A^n_i : w_n^{-1}(x) \in V^n_s \setminus \bigcup_{(s,j)\in T_n} W^n_{(s,j)} \ \&\ v_n\psi_i(x) \notin \bigcup_{s'\in T_n} V^n_{s'}\},$$

where it is understood that $B_n = \emptyset$, if $s \notin T_n$, and the notation v_n, w_n means that v, w are evaluated in $\Phi \sqcup \{\varphi_n\}$. If $\mu(B_n) = 0$, we do nothing, i.e., we let

$$(T_{n+1}, V^{n+1}_{s'}, \rho^{n+1}_{t'}, \Psi_{n+1}, A^n_{i'}) = (T_n, V^n_{s'}, \rho^n_{t'}, \Psi_n, A^n_{i'}).$$

If $\mu(B_n) > 0$, we define $T_{n+1} = T_n \cup \{(s,j)\}$, where j is least such that $(s,j) \notin T_n$, and define $V^{n+1}_{(s,j)} = v_n\psi_i(B_n), W^{n+1}_{(s,j)} = w_n^{-1}(B_n), \rho^{n+1}_{(s,j)} = v_n\psi_i w_n | W^{n+1}_{(s,j)}$,

$$\Psi_{n+1} = (\Psi_n \setminus \{\psi_i | A^n_i\}) \sqcup \{\psi_i | (A^n_i \setminus B_n)\},$$

so that $A^n_{i+1} = A^n_i \setminus B_n$.

Thus, in effect, we have transferred $\psi_i | B_n$ from Ψ_n to φ_{n+1}. Note also that, since $\mu(B_n) = \mu(W^{n+1}_{(s,j)}), \Phi \sqcup \{\varphi_{n+1}\} \sqcup \Psi_{n+1}$ is an L-graphing of E with cost equal to that of $\Phi \sqcup \{\varphi_n\} \sqcup \Psi_n$, i.e., that of E. So it is clear that all the conditions hold for $n+1$.

If $\epsilon = -1$, define B_n as before but replacing A^n_i by $\psi_i(A^n_i)$ and ψ_i by ψ_i^{-1}, and proceed to define T_{n+1} exactly as before and $V^{n+1}_{(s,j)} = v_n\psi_i^{-1}(B_n), W^{n+1}_{(s,j)}$ exactly as before, $\rho^n_{(s,j)} = v_n\psi_i^{-1}w_n | W^{n+1}_{(s,j)}, \Psi_{n+1} = (\Psi_n \setminus \{\psi_i | A^n_i\}) \sqcup \{\psi_i | (A^n_i \setminus \psi_i^{-1}(B_n))\}$, so that $A^n_{i+1} = A^n_i \setminus \psi_i^{-1}(B_n)$. Again all the conditions hold at $n+1$.

This completes the construction. Let $T_\infty = \bigcup_{n\geq 1} T_n, V^\infty_s = V^n_s$, if $s \in T_n, \rho^\infty_t = \rho^n_t$, if $t \in T_n, A^\infty_i = \bigcap_{n\geq 1} A^n_i, \Psi_\infty = \{\psi_i | A^\infty_i\}, \varphi_\infty = \bigcup_{n\geq 1}\varphi_n$.

We note the following fact, denoted by $(*)$:

It is impossible to find $i \in I, \epsilon \in \{\pm 1\}$, a non-null Borel set $B' \subseteq \psi_i^{\bar\epsilon}(A^\infty_i)$, where $\bar\epsilon = 0$ if $\epsilon = +1, \bar\epsilon = +1$ if $\epsilon = -1$, $s \in T_\infty, s \neq \emptyset, W' \subseteq V^\infty_s$ with $W' \cap (\bigcup_{(s,j)\in T_\infty} W_{(s,j)}) = \emptyset$, and $\rho' \in [[E]]$ with $\mathrm{dom}(\rho') = W'$, $\mathrm{rng}(\rho') = V'$, where $V' \cap \bigcup_{s'\in T_\infty} V^\infty_{s'} = \emptyset$, and words v, w in $\Phi \sqcup \{\hat\varphi\}$ such that $\rho' = v_\infty\psi' w_\infty$, where $\psi' = \psi_i^\epsilon | B'$ (thus $W' = w_\infty^{-1}(B')$), and v_∞, w_∞ means that v, w are evaluated in $\Phi \sqcup \{\varphi_\infty\}$.

Indeed, otherwise, and taking $\epsilon = +1$ for definiteness (the other can being similar), since $\varphi_\infty = \bigcup_{n\geq 1}\varphi_n, T_\infty = \bigcup_{n\geq 1} T_n$, we can assume, by shrinking B', if necessary, that for all large enough $n, s \in T_n$ and $\rho' = v_n\psi' w_n$, where v_n, w_n means that v, w are evaluated using φ_n instead of φ_∞. Choose such n so that moreover $a_{n+1} = (s, i, \epsilon, v, w)$. Clearly then $B_n \supseteq B'$ (as $V^n_s = V^\infty_s$) and so $W' \subseteq W^{n+1}_{(s,j)}$, for some j, and $V' \subseteq V^{n+1}_{(s,j)} \subseteq \bigcup_{s'\in T_\infty} V^\infty_{s'}$, a contradiction.

Let $A_n = \bigcup_{|s|=n} V^\infty_s$ for $n \in \mathbb{N}$. (Note the notation is consistent for $n = 0$, as $V^\infty_\emptyset = V^1_\emptyset = \mathrm{dom}(\varphi_i) = A_0$.) Let $A = \bigcup_n A_n$. Then clearly $A = \mathrm{dom}(\varphi_\infty) \cup \mathrm{rng}(\varphi_\infty)$, and $A_n \cap A_m = \emptyset$, if $n \neq m$. Note also that $\mathrm{rng}(\varphi_\infty) = \bigcup_{n\geq 1} A_n$, and $\mu(A \setminus \mathrm{dom}(\varphi_\infty)) = \mu(A \setminus \mathrm{rng}(\varphi_\infty)) = \mu(A_0) > 0$.

We now claim that for a.e. $x \in A$ there is (a unique) $n \geq 0$ such that $\varphi_\infty^n(x) \notin \mathrm{dom}(\varphi_\infty)$. Indeed, first note that if $x \in A_n$, then $\varphi_\infty^{-n}(x) \in A_0$, so if $C = \{x \in A : \forall n \geq 0(\varphi_\infty^n(x) \in \mathrm{dom}(\varphi_\infty))\}$, then $\varphi_\infty(C \cap A_n) = C \cap A_{n+1}$, so $\mu(C \cap A_n) = \mu(C \cap A_0)$. But the $C \cap A_n$ are pairwise disjoint, so they are null, i.e., C is null.

Now consider cases as $\delta < \mu(X)$ or $\delta \geq \mu(X)$.

(i) We first deal with the case $\delta < \mu(X)$. We argue then that $\mu(A) < \mu(X)$. Otherwise, by dropping a null set, we can assume that $A = X$. Thus $\mathrm{dom}(\varphi_\infty) \cup \mathrm{rng}(\varphi_\infty) = X$ and $X \setminus \mathrm{rng}(\varphi_\infty) = A_0$, so, as $\mu(\mathrm{dom}(\varphi_\infty)) = \mu(\mathrm{rng}(\varphi_\infty))$, we have that $\mu(X \setminus \mathrm{dom}(\varphi_\infty)) = \mu(X \setminus \mathrm{rng}(\varphi_\infty)) = \mu(A_0) < \mu(X) - \delta$, so $\mu(\mathrm{dom}(\varphi_\infty)) > \delta$, i.e., $C_\mu(\{\varphi_\infty\}) > \delta$. But then $C_\mu(\Phi \sqcup \{\varphi_\infty\} \sqcup \Psi_\infty) \geq C_\mu(\Phi) + C_\mu(\{\varphi_\infty\}) > C_\mu(\Phi) + \delta > C_\mu(E)$, contradicting the fact that $C_\mu(\Phi \sqcup \{\varphi_\infty\} \cup \Psi_\infty) = C_\mu(E)$.

Thus in this case we must have $\mu(A) < \mu(X)$.

Claim 28.4. *If F is the equivalence relation generated by $\Phi \sqcup \{\varphi_\infty\}$, then F is μ-ergodic.*

Granting this claim, we proceed as follows: If Ψ_∞ has non-0 cost, fix i with $\mu(\mathrm{dom}(\psi')) > 0$, where $\psi' = \psi_i | A_i^\infty$. Then find $B \subseteq \mathrm{dom}(\psi')$ with $0 < \mu(B) < \min\,\{\mu(A \setminus \mathrm{dom}(\varphi_\infty)), \mu(X \setminus A)\}$. By F-ergodicity and 16.3, we can find $\Phi \sqcup \{\varphi_\infty\}$-words v, w and $B' \subseteq B$ with $\mu(B') > 0$ such that w^{-1} is defined on B' and $w^{-1}(B') \subseteq A \setminus \mathrm{dom}(\varphi_\infty)$, and v is defined on $\psi'(B')$ and $v(\psi'(B)) \subseteq X \setminus A$. By shrinking B', if necessary, we can assume that $w^{-1}(B') = W' \subseteq V_s^\infty$, for some $s \in T_\infty, s \neq \emptyset$, and thus, since $w^{-1}(B') \cap \mathrm{dom}(\varphi_\infty) = \emptyset, w(B') \cap (\bigcup_{(s,j) \in T^\infty} W_{(s,j)}) = \emptyset$. Let $\rho' = v\psi' w | W'$. Then $i, \epsilon = +1, B', s, W', \rho', v, w$ violate $(*)$, a contradiction.

So we must have that Ψ_∞ has zero cost, thus, since

$$C_\mu(E) = C_\mu(\Phi) + \delta = C_\mu(\Phi) + C_\mu(\{\varphi_\infty\}) + C_\mu(\Psi_\infty),$$

we have $C_\mu(\{\varphi_\infty\}) = \mu(\mathrm{dom}(\varphi_\infty)) = \delta$. So taking $\varphi = \varphi_\infty$, we are done in this case.

It only remains to prove the claim. Assume, towards a contradiction, that μ is not F-ergodic. Let then $X = X_1 \cup X_2$ be a partition of X into F-invariant Borel sets of positive measure. Using that $0 < \mu(A) < \mu(X)$, we can assume, switching X_1, X_2 if necessary, that $\mu(A \cap X_1) > 0$ and $\mu((X \setminus A) \cap X_2) > 0$. Thus there are $A^* \subseteq A, B^* \subseteq X \setminus A$ with $0 < \mu(A^*)) \leq \mu(B^*)$ and $[A^*]_F \cap B^* = \emptyset$. By E-ergodicity and 16.3, we can assume that there is a reduced $\Phi \sqcup \{\varphi_\infty\} \sqcup \Psi_\infty$-word defined on A^* and sending A^* onto a subset of B^*. This word must contain something in Ψ_∞, since $[A^*]_F \cap B^* = \emptyset$. It follows that there is a Borel set $\tilde{A} \subseteq A$ of positive measure, $i \in I, \Phi \sqcup \{\varphi_\infty\}$-words v, w and $\epsilon \in \{\pm 1\}$, such that, letting $\psi' = \psi_i | A_i^\infty$, we have $v(\psi')^\epsilon w$ is defined on $\tilde{A}$ and $v(\psi')^\epsilon w(\tilde{A}) \cap A = \emptyset$. Take, for definiteness $\epsilon = +1$, the other case being similar. We can also assume, using the fact that for a.e.

$x \in A$ there is $n \geq 0$ with $\varphi_\infty^n(x) \notin \mathrm{dom}(\varphi_\infty)$, that for some fixed $n_0 \geq 0$ we have that $\varphi_\infty^{n_0}$ is defined on $\tilde{A}$ and $\varphi_\infty^{n_0}(\tilde{A}) \subseteq A \setminus \mathrm{dom}(\varphi_\infty)$, and in fact that moreover for some $s \in T_\infty, s \neq \emptyset, \varphi_\infty^{n_0}(\tilde{A}) = W' \subseteq V_s^\infty$. Define then ρ' with domain W' by $\rho' = v\psi' w \varphi_\infty^{-n}$. Let $B' = w(\tilde{A}) \subseteq \mathrm{dom}(\psi')$. Then $i, \epsilon = +1, B', s, W', \rho', v, w\varphi_\infty^{-n}$ violate $(*)$, a contradiction. This proves the claim and completes the proof of the theorem in the case $\delta < \mu(X)$.

(ii) We now consider the case $C_\mu(\Psi) = \delta \geq \mu(X)$. We will argue that in this case we must have (in the previous notation) $\mu(A) = \mu(X)$. Indeed assume, towards a contradiction, that $\mu(A) < \mu(X)$. Then, exactly as in the preceding paragraph, if μ is not F-ergodic (where F is the equivalence relation generated by $\Phi \sqcup \{\varphi_\infty\}$), we have a contradiction. So we can assume that μ is F-ergodic. Since $\Phi \sqcup \{\varphi_\infty\} \sqcup \Psi_\infty$ has the same cost as $\Phi \sqcup \Psi$, and clearly $\mu(\mathrm{dom}(\varphi_\infty)) < \mu(X) \leq \delta = C_\mu(\Psi)$ (since $\mathrm{dom}(\varphi_\infty) \subseteq A$), it follows that $C_\mu(\Psi_\infty) > 0$, so, exactly as in the paragraph following the statement of Claim 28.4, we have a contradiction.

So we can assume that $A = X$. Put $\tilde{\varphi} = \varphi_\infty$, $\tilde{\Psi} = \Psi_\infty$, $X^0 = A_0$. Notice that $\mu(X \setminus \mathrm{dom}(\tilde{\varphi})) = \mu(X^0) \leq \frac{1}{2}\mu(X)$. Also notice that $\tilde{\Psi} = \{\tilde{\psi}_i\}_{i \in I} = \{\psi_i | \tilde{A}_i\}_{i \in I}$, where $\tilde{A}_i \subseteq \mathrm{dom}(\psi_i)$ and $\mathrm{dom}(\psi_i) \setminus \tilde{A}_i$ can be decomposed into countably many Borel pieces, on each of which ψ_i can be written as a $\Phi \sqcup \{\tilde{\varphi}\}$-word. We now "project" everything to X^0 and repeat this construction in X^0. More precisely, note that for every $x \in X$ $(= A)$ there is a unique $N(x) = n \geq 0$ with $\tilde{\varphi}^{-n}(x) \in A_0$. Put $\pi_0(x) = \tilde{\varphi}^{-n}(x)$. Notice that $\pi_0 | (X \setminus \mathrm{dom}(\tilde{\varphi}))$ is a bijection of $X \setminus \mathrm{dom}(\tilde{\varphi})$ with X^0 μ-a.e. If $\theta : C \to D$ is in $[[E]]$, we associate to it an L-graph $\prod_0(\theta) = \{\pi_{n,m}(\theta)\}_{n,m \in \mathbb{N}}$, with $\pi_{n,m} \in [[E|X^0]]$ defined as follows: Decompose $C = \bigcup_{n,m \in \mathbb{N}} C_{n,m}$, $D = \bigcup_{n,m \in \mathbb{N}} D_{n,m}$ into pairwise disjoint Borel sets so that $\theta(C_{n,m}) = D_{n,m}$ and $N(x) = n$ for $x \in C_{n,m}$, $N(y) = m$ for $y \in D_{n,m}$. Put $\pi_{n,m}(\theta) = (\tilde{\varphi})^{-m} \theta \tilde{\varphi}^n$, with domain $(\tilde{\varphi})^{-n}(C_{n,m})$ and range $(\tilde{\varphi})^{-m}(D_{n,m})$.

Now consider X^0, $\mu|X^0$, $E|X^0$, $\Phi^0 = \prod_0(\Phi) = \bigsqcup_{k \in K} \prod_0(\varphi_k)$, where $\Phi = \{\varphi_k\}_{k \in K}$, $\Psi^0 = \prod_0(\tilde{\Psi}) = \bigsqcup_{i \in I} \prod_0(\tilde{\psi}_i)$, with $\tilde{\Psi} = \{\tilde{\psi}_i\}_{i \in I}$. Then $E|X^0$ is aperiodic (after dropping a null set) and $\mu|X^0$ is an $(E|X^0)$-invariant ergodic measure. Also $\Omega^0 = \Phi^0 \sqcup \Psi^0$ is clearly an L-graphing of $E|X^0$, since $\Phi \sqcup \{\tilde{\varphi}\} \sqcup \tilde{\Psi}$ is an L-graphing of E and $\prod_0(\tilde{\varphi})$ is an L-graph whose maps are the identity on their domain. We now claim that $C_{\mu|X^0}(\Omega^0) = C_{\mu|X^0}(E|X^0)$. Indeed, $C_{\mu|X^0}(\Omega^0) = C_\mu(\Phi) + C_\mu(\tilde{\Psi}) = C_\mu(E) - C_\mu(\{\tilde{\varphi}\}) = C_\mu(E) - \mu(\mathrm{rng}(\tilde{\varphi})) = C_\mu(E) - (\mu(X) - \mu(X^0)) = C_{\mu|X^0}(E|X^0)$, since X_0 is a complete section for E a.e. Finally, $C_{\mu|X^0}(\Psi^0) = C_\mu(\tilde{\Psi}) = C_\mu(E) - [C_\mu(\Phi) + C_\mu(\{\tilde{\varphi}\})] = C_\mu(\Psi) - C_\mu(\{\tilde{\varphi}\}) = C_\mu(\Psi) - \mu(\mathrm{rng}(\tilde{\varphi})) = C_\mu(\Psi) - (\mu(X) - \mu(X^0)) \geq \mu(X^0)$. So $E|X_0$, $\mu|X^0$, $\Omega_0 = \Phi^0 \sqcup \Psi^0$ satisfy all the hypotheses that $E, X, \Omega = \Phi \sqcup \Psi$ satisfy in the present case $(C_\mu(\Psi) \geq \mu(X))$, so we can repeat this construction in X^0 now to obtain $\tilde{\varphi}^0 \in [[E|X^0]], \tilde{\Psi}^0$ with $\Phi^0 \sqcup \{\tilde{\varphi}^0\} \sqcup \tilde{\Psi}^0$ an L-graphing of $E|X^0$ with cost $C_{\mu|X^0}(E|X^0)$, $\tilde{\Psi}^0 = \bigsqcup_{i \in I} \bigsqcup_{n,m \in \mathbb{N}} (\pi_{n,m}(\psi_i) | A_{i,n,m})$ for appropriate Borel sets $A_{i,n,m} \subseteq \mathrm{dom}(\pi_{n,m}(\tilde{\psi}_i))$, and Borel $X^1 \subseteq X^0$ of positive measure (playing the role of $X^0 = A_0$ in X) with $\mu(X^0 \setminus \mathrm{dom}(\tilde{\varphi}^0)) = \mu(X^1) \leq \frac{1}{2}\mu(X^0)$. Then we pull them back to X as follows: Define $\tilde{\Psi}_0 = \{\tilde{\psi}_i | \bigcup_{n,m\in\mathbb{N}} \tilde{\varphi}^n(A_{i,n,m})\} = \{\psi_i | A_{0,i}\}$. Let $\tilde{\varphi}_0 \supseteq \tilde{\varphi}$ extend $\tilde{\varphi}_0$ by adding to its domain all $x \in X \setminus \mathrm{dom}(\tilde{\varphi})$ with $\pi_0(x) \in \mathrm{dom}(\tilde{\varphi}^0)$ and letting $\tilde{\varphi}_0(x) = \tilde{\varphi}^0 \circ \pi_0(x)$. Then notice that $\mu(X \setminus \mathrm{dom}(\tilde{\varphi}_0)) \leq \frac{1}{4}\mu(X)$. Also $\Phi \sqcup \{\tilde{\varphi}_0\} \sqcup \tilde{\Psi}_0$ is an L-graphing of E and has cost $C_\mu(E)$, since $C_\mu(\Phi \sqcup \{\tilde{\varphi}_0\} \sqcup \tilde{\Psi}_0) = C_{\mu|X_0}(\Phi^0) + C_\mu(\{\tilde{\varphi}\}) + C_{\mu|X_0}(\{\tilde{\varphi}^0\}) + C_{\mu|X_0}(\tilde{\Psi}^0) = C_{\mu|X_0}(E|X_0) + C_\mu(\{\tilde{\varphi}\}) = C_\mu(E) - (\mu(X) - \mu(X_0)) + \mu(X) - \mu(X_0) = C_\mu(E)$. Finally, $\mathrm{dom}(\psi_i) \setminus A_{0,i}$ can be decomposed into countably many Borel sets on each of which ψ_i can be expressed as a $\Phi \sqcup \{\tilde{\varphi}_0\}$ word. Then we repeat the construction within X^1 and pull everything back to X^0 and then back to X to obtain $\tilde{\varphi}_1 \supseteq \tilde{\varphi}_0$ and $\tilde{\Psi}_1 = \{\psi_i | A_{1,i}\}_{i\in I}$, where $A_{0,i} \supseteq A_{1,i}$, etc. This way inductively we construct $\tilde{\varphi}_0 \subseteq \tilde{\varphi}_1 \subseteq \cdots \subseteq \tilde{\varphi}_n \subseteq \ldots$ in $[[E]]$ with $\mu(X \setminus \mathrm{dom}(\tilde{\varphi}_n)) \leq \frac{1}{2^{n+2}}\mu(X)$ and $\tilde{\Psi}_0, \tilde{\Psi}_1, \ldots, \tilde{\Psi}_n, \ldots$ with $\tilde{\Psi}_n = \{\tilde{\psi}_i | A_{n,i}\}_{i\in I}$, where $\mathrm{dom}(\psi_i) \supseteq A_{0,i} \supseteq A_{1,i} \supseteq \cdots \supseteq A_{n,i} \supseteq \ldots$, $\Phi \sqcup \{\tilde{\varphi}_n\} \sqcup \tilde{\Psi}_n$ L-graphs E with cost $C_\mu(E)$ and $\mathrm{dom}(\psi_i) \setminus A_{n,i}$ can be decomposed into countably many Borel pieces on each of which ψ_i can be expressed as a $\Phi \sqcup \{\tilde{\varphi}_n\}$ word. Finally put $\varphi = \bigcup_n \tilde{\varphi}_n$, so that $\mu(\mathrm{dom}(\varphi)) = \mu(\mathrm{rng}(\varphi)) = \mu(X)$, and thus after dropping a null set, we can assume that $\varphi \in [E]$, and let $\Psi' = \{\psi_i | \bigcap_n A_{n,i}\}_{i\in I}$. This clearly works, i.e., $\Omega' = \Phi \sqcup \{\varphi\} \sqcup \Psi'$ is an L-graphing of E a.e. of cost $C_\mu(E)$, $\Psi' = \{\psi_i | A'_i\}_{i\in I}$, with $A'_i = \bigcap_n A_{n,i} \subseteq \mathrm{dom}(\psi_i)$, and we can decompose $\mathrm{dom}(\psi_i) \setminus A'_i$ into countably many Borel sets on each of which ψ_i can be expressed as a $\Phi \sqcup \{\varphi\}$-word. ⊣

A careful inspection of the proof of 28.3 shows that, with some simple modifications, it can be also used to give the proof of the following version of 28.2 for infinite costs.

Theorem 28.5 (Hjorth [H]). *Let E be an aperiodic countable Borel equivalence relation and let μ be an E-invariant ergodic probability measure. If E is treeable and $C_\mu(E) = \infty$, then E is induced by a free Borel action of F_∞ a.e.*

Proof. Fix an L-treeing $\Psi = \{\psi_i\}_{i=1}^\infty$ of E with $C_\mu(\Psi) = \infty$. By decomposing the domain of each ψ_i into countably many pieces, we can assume that the domain and range of each ψ_i are disjoint. Then inductively, using the method of proof of 28.3 (for the case $\delta \geq \mu(X)$), we construct L-treeings of E a.e. $\{\Omega_i\}_{i=0}^\infty$, such that:

i) $\Omega_0 = \Psi$,
ii) $\Omega_i = \{\varphi_1, \varphi_2, \ldots, \varphi_i\} \sqcup \Psi_i$, for each $i \geq 1$, where $\varphi_i \in [E], \Psi_i = \{\psi_{i+1} | A^i_{i+1}, \psi_{i+2} | A^i_{i+2}, \ldots\}$ for some Borel sets A^i_n contained in the domain of ψ_n, for $i < n$, and such that $A^{i+1}_n \subseteq A^i_n$ for $i + 1 < n$, and the domain of each ψ_i can be decomposed into countably many pieces on each of which ψ_i can be expressed as a $\{\varphi_1, \ldots, \varphi_i\}$-word.

One observation that is needed here is that the proof of 28.3 shows that, in the notation of the statement of 28.3 in the case $\delta \geq \mu(X)$, if some ψ_i has disjoint domain and range, then we can assume that $\psi_i \subseteq \varphi$.

It is now clear that $\{\varphi_1, \varphi_2, \dots\}$ provide a free action of F_∞ that generates E a.e. $\dashv$

We conclude this section with an application of 28.3.

Recall from [DJK], 3.4 that one has the following version of the Glimm-Effros Dichotomy for countable Borel equivalence relations. Let E_0 be the following equivalence relation on $2^{\mathbb{N}}$:

$$xE_0y \Leftrightarrow \exists n \forall m \geq n(x_m = y_m).$$

Also denote by $\cong_B$ the relation of Borel isomorphism. We have then:

Theorem 28.6. (Glimm-Effros Dichotomy, see [DJK], 3.4) *For any countable Borel equivalence relation E on X, exactly one of the following holds:*

(I) There is a Borel set $A \subseteq X$ with $E_0 \cong_B E|A$:
(II) E is smooth.

It is now natural to wonder whether there is a dichotomy for the strengthening of (I), where one requires that A is E-invariant.

Problem 28.7. Prove or disprove the following dichotomy:

For any aperiodic countable Borel equivalence relation E on X exactly one of the following holds:

(I) There is a Borel E-invariant set $A \subseteq X$ with $E_0 \cong_B E|A$.
(II) There is a free Borel action of F_2 on X such that $E_{F_2}^X \subseteq E$.

Since E_0 is hyperfinite and admits an invariant probability Borel measure, it is clear that (I), (II) are incompatible.

Now it is not hard to see that alternative (I) above is equivalent to:

(I)$'$ There is an E-invariant probability measure μ on X such that E is hyperfinite μ-a.e.

Indeed, (I) $\Rightarrow$ (I)$'$ follows from the just preceding remarks. Conversely, if (I)$'$ holds, then we can use, for example, [DJK], 9.3 to conclude that there is Borel E-invariant $A \subseteq X$ with $E_0 \cong_B E|A$.

In relation to this problem, we note that one can show the following, which provides a weaker affirmative answer to 28.7 and also shows that 28.7 is true in case E is treeable.

Proposition 28.8 (Kechris-Miller). *Let E be an aperiodic countable Borel equivalence relation on X. Then one of the following holds:*

(I) There is an invariant probability measure μ such that $C_\mu(E) = 1$.
(II) There is a free Borel action of F_2 on X such that $E^X_{F_2} \subseteq E$.

Corollary 28.9. *The dichotomy 28.7 holds, when E is treeable.*

This follows immediately from 22.2.

Remark 28.10. Note that in 28.8, (I) and (II) are not incompatible, as any E induced by a free Borel action of $F_2 \times \mathbb{Z}$ with invariant probability measure satisfies both these conditions.

Proof of 28.8. We will make use of the following lemma, which was also proved independently by M. Pichot.

Lemma 28.11. *Let E be a countable Borel equivalence relation on X, $\mathcal{G}$ a graphing of E and $\mathcal{T}_0 \subseteq \mathcal{G}$ an acyclic graph. Then there is an acyclic graph $\mathcal{T}$ with $\mathcal{T}_0 \subseteq \mathcal{T} \subseteq \mathcal{G}$ and $C_\mu(\mathcal{T}) \geq C_\mu(E)$, for every E-invariant probability measure μ with $C_\mu(\mathcal{G}) < \infty$.*

Proof. Consider the standard Borel space $[X]^{<\infty}$ of finite subsets of X and its Borel subset

$$\begin{aligned} C = \{A = \{a_1, \dots, a_n\} \in [X]^{<\infty} &: \text{there is a permutation} \\ &\pi \text{ of } \{1, \dots, n\} \text{ such that } a_{\pi(1)}, \dots, a_{\pi(n)}, a_{\pi(1)} \text{ is a } \mathcal{G}-\text{cycle}). \end{aligned}$$

We call the members of C simply $\mathcal{G}$-cycles. Define a graph $\mathcal{H}$ on C by

$$(A, B) \in \mathcal{H} \Leftrightarrow A \neq B \text{ and } A \cap B \neq \emptyset.$$

We claim that there is a Borel $\aleph_0$-coloring of $\mathcal{H}$, i.e., a Borel map $c : C \to \mathbb{N}$ with $(A, B) \in \mathcal{H} \Rightarrow c(A) \neq c(B)$.

Granting this, let $C_n = c^{-1}(\{n\})$. Now inductively define $\mathcal{G}_n$ as follows: $\mathcal{G}_0 = \mathcal{G}$, and $\mathcal{G}_{n+1}$ is obtained from $\mathcal{G}_n$ by considering all $\mathcal{G}_n$-cycles A, with $A \in C_n$, and cutting off one edge in $\mathcal{G}_n \setminus \mathcal{T}_0$ connecting two vertices in A.

Then $\mathcal{G} = \mathcal{G}_0 \supseteq \mathcal{G}_1 \supseteq \cdots \supseteq \mathcal{T}_0$ and since c is a coloring, each $\mathcal{G}_n$ is a graphing of E. Thus, for any E-invariant probability measure μ with $C_\mu(\mathcal{G}) < \infty, C_\mu(\mathcal{G}_0) \geq C_\mu(\mathcal{G}_1) \geq \cdots \geq C_\mu(E)$. Put $\mathcal{T} = \bigcap_n \mathcal{G}_n$, so that $C_\mu(\mathcal{T}) = \lim_n C_\mu(\mathcal{G}_n) \geq C_\mu(E)$. It is clear that $\mathcal{T}$ is acyclic, since if A is a $\mathcal{T}$-cycle, then $A \in C_n$, for some natural number n, thus A is not a $\mathcal{G}_{n+1}$-cycle, a contradiction.

We now prove the claim: Fix a Borel ordering $<$ of C and a countable sequence of Borel involution $\{g_n\}$ on X such that $xEy \Leftrightarrow \exists n(g_n(x) = y)$ (see 1.1). For $A \in C$, write $A = \{a_1, \dots, a_n\}$ with $a_1 < \cdots < a_n$. Let $c(A) = \pi(\{g_{ij}\}_{i \leq i,j \leq n})$, where g_{ij} is least in the enumeration $\{g_n\}$ such that $g_{ij}(a_i) =$

a_j, and π is an injection of the set of finite sequences of the form $\{g_{ij}\}_{1\leq i,j\leq n}$ from $\{g_m : m \in \mathbb{N}\}$ into $\mathbb{N}$. If $A\mathcal{H}B$ and, towards a contradiction, $c(A) = c(B) = \pi(\{g_{ij}\})$, then $A = \{a_1, \ldots, a_n\}, B = \{b_1, \ldots, b_n\}$, for some n, and $a_i = b_j$ for some $1 \leq i, j \leq n$. If $i = j$, then $A = B$, a contradiction. Otherwise, say $i < j$. Then $a_j = g_{ij}(a_i) = g_{ij}(b_j) = g_{ij}^2(b_i) = b_i$, so $b_i < b_j = a_i < a_j$, a contradiction. ⊣

Now the preceding lemma implies

Lemma 28.12. *Let E be an aperiodic countable Borel equivalence relation, and μ an E-invariant probability Borel measure. If $C_\mu(E) > 1$, then there is an acyclic Borel graph $\mathcal{T} \subseteq E$ with $C_\mu(\mathcal{T}) > 1$. If F is the equivalence relation generated by $\mathcal{T}$ and μ is E-ergodic, then we can choose $\mathcal{T}$ so that μ is also F-ergodic.*

Proof. By Zimmer [Z], 9.3.2, let $E_0 \subseteq E$ be an aperiodic hyperfinite Borel equivalence relation such that if moreover μ is E-ergodic, then μ is also E_0-ergodic. Let $\mathcal{T}_0$ be a treeing of E_0. Let $\mathcal{G} \supseteq \mathcal{T}_0$ be a graphing of E. Let $\mathcal{T}_0 = \mathcal{G}_0 \subseteq \mathcal{G}_1 \subseteq \ldots$ be such that $\bigcup_n \mathcal{G}_n = \mathcal{G}$ and $C_\mu(\mathcal{G}_n) < \infty$. Let E_n be the equivalence relation generated by $\mathcal{G}_n$, so that $E_0 \subseteq E_1 \subseteq \ldots$ and $\bigcup_n E_n = E$. If $C_\mu(E_n) = 1$, for all n, then, by 23.5, $C_\mu(E) = 1$, a contradiction. So there is some n with $C_\mu(E_n) > 1$, and then by 28.11 there is an acyclic graph $\mathcal{T}_n$ with $\mathcal{T}_0 \subseteq \mathcal{T}_n \subseteq E_n$ and $C_\mu(\mathcal{T}_n) \geq C_\mu(E_n) > 1$. Take $\mathcal{T} = \mathcal{T}_n$. ⊣

We are now ready to complete the proof of 28.8.

If E admits no invariant probability measure, then, by Nadkarni's Theorem (see [DJK], 3.5), E is compressible. Then, by [DJK], 2.5, there is a Borel subequivalence relation $F \subseteq E$ which is smooth and aperiodic. Clearly then F is induced by a free Borel action of F_2. (In fact by 2.3, 3.17 of [JKL], E itself is induced by a free Borel action of F_2.) Thus alternative (II) holds.

So we can assume that E admits an invariant probability measure. Consider then the Ergodic Decomposition Theorem 18.5. If for some $e \in \mathcal{EI}_E$, we have $C_e(E) = 1$, then (I) holds. Else for all such e, $C_e(E) > 1$, and thus by Lemma 28.12, and its proof, we can find an acyclic Borel graph $\mathcal{T}_e$ generating a subequivalence relation $F_e \subseteq E|X_e$, with $c_e = C_e(F_e) = C_e(\mathcal{T}_e) > 1$. Fix n_e so that $n_e(c_e - 1) + 1 > 2$, and let $S_e \subseteq X_e$ be a Borel set with $e(S_e) = 1/n_e$ (this is possible, since e is clearly non-atomic).

Let $e' = \frac{e|S_e}{e(S_e)}$. Then

$$C_{e'}(E|S_e) = n_e c_e - (n_e - 1) = n_e(c_e - 1) + 1 > 2.$$

So by Theorem 28.2, there is a free Borel action of F_2 on an $(E|S_e)$-invariant Borel set $Y_e \subseteq S_e$ with $e'(Y_e) = 1$, such that $E_{F_2}^{Y_e} \subseteq E$. Next let $Y_1^e = Y_e, Y_2^e, \ldots, Y_{n_e}^e$ be a partition of X_e into Borel sets of e-measure $1/n_e$. By shrinking Y_e, if necessary, we can assume that there are Borel injections $\varphi_1^e =$ identity on $Y_e, \varphi_2^e : Y_e \to Y_2^e, \ldots, \varphi_{n_e}^e : Y_e \to Y_{n_e}^e$ with $\varphi_i(x)Ex$; see 16.3.

Let $Z_e = \bigcup_{i=1}^{n_e} \varphi_i^e(Y_e)$, so that $e(Z_e) = 1$. Moving the action of F_2 on Y_e to each $Y_i^e, i \geq 2$, using φ_i, we see that there is a free Borel action of F_2 on Z_e with $E_{F_2}^{Z_e} \subseteq E$. By shrinking Z_e, if necessary, we can assume that it is E-invariant. Let $Z = \bigcup_{e \in \mathcal{EI}_E} Z_e$. Then Z is an E-invariant set which is Borel, since the construction of Z_e from e is done in a Borel way. Moreover the union of the actions of F_2 on each Z_e gives a free Borel action of F_2 on Z with $E_{F_2}^Z \subseteq E$. Since $e(X \setminus Z) = 0$, for each $e \in \mathcal{EI}_E$, clearly $X \setminus Z$ is μ-null, for each E-invariant probability Borel measure μ, so, by Nadkarni's Theorem (see [DJK], 3.5) $E|(X \setminus Z)$ is compressible, so, as before, $E|(X \setminus Z)$ contains a subequivalence relation induced by a free Borel action of F_2 on $X \setminus Z$. Thus it is clear that there is a free Borel action of F_2 on X producing a subequivalence relation of E, i.e. (II) holds. ⊣

Remark 28.13. We would like to thank G. Hjorth for suggesting the use of restrictions instead of subequivalence relations, that we had originally used, in the proof of the preceding result.

From the ergodic decomposition argument, used in the preceding proof, it is clear that Problem 28.7 is closely related to the following measure theoretic version, which has been also considered earlier by Gaboriau.

Problem 28.14. Prove or disprove the following dichotomy:

Let E be a countable Borel equivalence relation and μ an E-invariant, ergodic probability measure. Then exactly one of the following holds:

(I) E is hyperfinite, μ-a.e.
(II) There is a Borel equivalence relation $F \subseteq E$ which is induced by a free Borel action of F_2, μ-a.e.

This can be considered as a "dynamic" version of the so-called "von Neumann Conjecture" (actually due to Day): Any countable group Γ is either amenable or contains F_2. (This conjecture was disproved by Olshanskii in 1980.) Note again that the argument in the proof of 28.8 shows that this is true if (I) is replaced by the weaker "$C_\mu(E) = 1$."

29 Cost of a Group

Let Γ be a countable group. We define its *cost*, $C(\Gamma)$, by

$$C(\Gamma) = \inf\{C_\mu(E)\},$$

where E ranges over those equivalence relations, with invariant probability measure μ, which are induced by a free Borel action of Γ on a standard Borel space. This definition is not vacuous since every countable group Γ admits a free Borel action on a standard Borel space with invariant probability measure.

This is clear if Γ is finite. So assume Γ is infinite. Consider then the shift action of Γ on 2^Γ, $g \cdot x(h) = x(g^{-1}h)$, and restrict it to its free part, i.e., the Borel Γ-invariant set $\{x \in 2^\Gamma : g \cdot x \neq x, \ \forall g \neq 1\}$. If μ is the usual product measure on 2^Γ, with $2 = \{0, 1\}$ having the $(\frac{1}{2}, \frac{1}{2})$-measure, then μ is Γ-invariant and concentrates on the free part.

If Γ is finite with cardinality $|\Gamma|$, then, by Section 20,

$$C(\Gamma) = 1 - \frac{1}{|\Gamma|}.$$

If Γ is infinite, then, by 21.3, $C(\Gamma) \geq 1$.

Next notice that, by 18.5, we also have

$$C(\Gamma) = \inf\{C_\mu(E)\},$$

where E ranges over those equivalence relations, with E-invariant, ergodic probability measure μ, which are induced by a free Borel action of Γ on a standard Borel space.

We now verify that $C(\Gamma)$ is attained.

Proposition 29.1 (Gaboriau). *For any countable group Γ, there is a free Borel action of Γ on a standard Borel space X with invariant probability measure μ such that $C(\Gamma) = C_\mu(E^X_\Gamma)$.*

Proof. Let E_n, X_n, μ_n be such that E_n is induced by a free Borel action of Γ on X_n with invariant probability measure μ_n, and $C_{\mu_n}(E_n) \to C(\Gamma)$. Consider the free action of Γ on $X = \prod_n X_n$ given by

$$g \cdot (x_n) = (g \cdot x_n),$$

and the product measure $\mu = \prod_n \mu_n$. Clearly it is invariant under this action. Also let $E = E^X_\Gamma$ be the induced equivalence relation. It is enough to show that

$$C_\mu(E) \leq C_{\mu_n}(E_n),$$

for each n. Take, without loss of generality $n = 0$. Let $\Phi_0 = \{\varphi_j\}_{j \in J}$, $\varphi_j : A_j \to B_j$, be an L-graphing of E_0 such that, again without loss of generality, on its domain A_j, φ_j is equal to $x \mapsto g_{i_j} \cdot x$ $(x \in A_j)$, for some $i_j \in I$. Then $\Psi_0 = \{\psi_j\}$, where $\psi_j : A_j \times \prod_{n \geq 1} X_n \to B_j \times \prod_{n \geq 1} X_n$, is given by

$$\psi_j(x_0, x_1, \dots, x_n, \dots) = (g_{i_j} \cdot x_0, g_{i_j} \cdot x_1, \dots, g_{i_j} \cdot x_n, \dots)$$

is an L-graphing of E, and $C_\mu(\Psi_0) = \sum_j \mu_0(A_j) = C_\mu(\Phi_0)$. Thus $C_\mu(E) \leq C_\mu(\Phi_0)$ and so $C_\mu(E) \leq C_{\mu_0}(E_0)$. ⊣

Remark 29.2. From 18.5 it is also clear that in 29.1 such an action can be found so that μ is also ergodic.

Denote by GROUP the standard Borel space of countable groups. We next consider the descriptive complexity of the cost function.

Proposition 29.3. *For each $r \in \mathbb{R}$, the set $\{\Gamma \in \text{GROUP} : C(\Gamma) < r\}$ is analytic.*

Proof. Consider the shift action of the free group F_∞ with $\aleph_0$ generators on $\mathbb{R}^{F_\infty}$: $g\cdot x(h) = x(g^{-1}h)$. Let E_∞ be the corresponding equivalence relation.

There is clearly a Borel function $F : \text{GROUP} \to \text{Power}(F_\infty)$ such that $F(\Gamma)$ is a normal subgroup of F_∞, $\Gamma_\infty/F(\Gamma) \cong \Gamma$. Also fix a map $\Gamma \mapsto \pi_\Gamma$ such that $\pi_\Gamma : F_\infty \to \Gamma$ is a surjective homomorphism with $\ker(\pi_\Gamma) = F(\Gamma)$.

Suppose now $\Gamma \in \text{GROUP}$ acts in a Borel way on a standard Borel space, which without loss of generality we an assume to be a Borel subset of $\mathbb{R}$. Then define $f : X \to \mathbb{R}^{F_\infty}$ by $f(x)(g) = \pi_\Gamma(g^{-1}) \cdot x$. Clearly f is a Borel bijection between X and an E_∞-invariant Borel set $Y (= f(X))$ and $xE^X_G y \Leftrightarrow f(x)E_\infty f(y)$, since $f(\pi_\Gamma(g) \cdot x) = g \cdot f(x)$, for any $x \in X$. Moreover, if the action of Γ on X is free, then for each $y \in Y$, if $\text{Stab}(y)$ is the stabilizer of y (in the shift action), then

$$\text{Stab}(y) = \ker(\pi_\Gamma) = F(\Gamma).$$

Finally, if μ is an E^X_Γ-invariant probability measure, $f_*\mu$ is an E_∞-invariant probability measure, $C_\mu(E^X_\Gamma) = C_{f_*\mu}(E_\infty)$, and by the above $f_*\mu$ concentrates on the $y \in \mathbb{R}^{F_\infty}$ with $\text{Stab}(y) = F(\Gamma)$. Conversely, if ν is a probability measure on $\mathbb{R}^{F_\infty}$ which is E_∞-invariant, and, letting $S_\Gamma = \{x \in \mathbb{R}^{F_\infty} : \text{Stab}(x) = F(\Gamma)\}$, we have $\nu(S_\Gamma) = 1$, then clearly $E_\infty|S_\Gamma$ is induced by a free Borel action of Γ, since $\Gamma \cong F_\infty/F(\Gamma)$, and of course $C_\nu(E_\infty|S_\Gamma) = C_\nu(E_\infty)$. It follows that

$$C(\Gamma) = \inf\{C_\nu(E_\infty) : \nu \in \mathcal{I}_{E_\infty} \;\&\; \nu(S_\Gamma) = 1\},$$

so

$$C(\Gamma) < r \Leftrightarrow \exists \nu \in \mathcal{I}_{E_\infty}(\nu(S_\Gamma) = 1 \;\&\; C_\nu(E_\infty) < r),$$

so, by 18.1, $\{\Gamma : C(\Gamma) < r\}$ is analytic. ⊣

As a corollary

$$\{\Gamma : C(\Gamma) = 1\}$$

is analytic. The following questions however are open.

Problem 29.4. Is the function $\Gamma \mapsto C(\Gamma)$ Borel? Is the set $\{\Gamma : C(\Gamma) = 1\}$ Borel?

We will now give an alternative definition of $C(\Gamma)$, motivated by some concepts in probability theory, which are discussed, for example, in Pemantle-Peres [P^2].

For a fixed countable group Γ, denote by GRAPH_Γ the standard Borel space of graphs on (the vertex set) Γ. Then Γ acts on GRAPH_Γ by

$$g \cdot G = G',$$

where $(x,y) \in G' \Leftrightarrow (g^{-1}x, g^{-1}y) \in G$.

We will consider Γ-invariant probability measures on GRAPH_Γ. If μ is such a measure, we can think of it as defining a *random Γ-invariant graphing of Γ* or just *random graphing of Γ*. If μ concentrates on the Borel Γ-invariant set CGRAPH_Γ of connected graphs on Γ, we call μ a *random connected graphing of Γ*.

For example, if $\{\gamma_i\}_{i \in I}$ is a set of generators for Γ and C is the corresponding Cayley graph

$$gCh \Leftrightarrow \exists i (g\gamma_i = h),$$

then C is a fixed point of the Γ-action on GRAPH_Γ, thus the Dirac measure δ_C concentrating on $\{C\}$ is a random connected graphing on Γ. So we can think of random connected graphings as generalizations of Cayley graphs.

If μ is a random graphing of Γ, we define its *expected degree*, d_μ, to be $\frac{1}{2}$ of the average degree of 1 (the identity of Γ) in a graph, i.e.,

$$d_\mu = \frac{1}{2} \int d_G(1)\ d\mu(G).$$

Note that instead of 1 we could have used any fixed element $\gamma \in \Gamma$, since

$$\begin{aligned}\int d_G(1)\ d\mu(G) &= \int d_{\gamma \cdot G}(\gamma)\ d\mu(G)\\ &= \int d_G(\gamma)\ d\mu(G),\end{aligned}$$

by the Γ-invariance of μ.

We now have

Proposition 29.5. *For any countable group Γ,*

$$C(\Gamma) = \inf\{d_\mu : \mu \textit{ is a random connected graphing of } \Gamma\}.$$

Proof. Suppose Γ acts freely and in a Borel way on the standard Borel space X and $\mathcal{G}$ is a graphing of E^X_Γ. Define the Borel map $\varphi_\mathcal{G} : X \to \mathrm{GRAPH}_\Gamma$ by

$$\varphi_\mathcal{G}(x) = \mathcal{G}(x),$$

where

$$(g,h) \in \mathcal{G}(x) \Leftrightarrow (g \cdot x, h \cdot x) \in \mathcal{G}.$$

Let $\nu = (\varphi_\mathcal{G})_* \mu$. Then ν is a random connected graphing of Γ. Moreover

$$
\begin{aligned}
C_\mu(\mathcal{G}) &= \frac{1}{2}\int |\mathcal{G}_x| \ d\mu(x) \\
&= \frac{1}{2}\int |\{y : (x,y) \in \mathcal{G}\}| \ d\mu(x) \\
&= \frac{1}{2}\int |\{g : (x, g\cdot x) \in \mathcal{G}\}| \ d\mu(x) \\
&= \frac{1}{2}\int |\{g : (1,g) \in \mathcal{G}(x)\}| \ d\mu(x) \\
&= \frac{1}{2}\int d_{\mathcal{G}(x)}(1) \ d\mu(x) \\
&= \frac{1}{2}\int d_G(1) \ d\nu(G) \\
&= d_\nu.
\end{aligned}
$$

Conversely, consider a random connected graphing ν of Γ. We would like to reverse the above steps, but unfortunately the action of Γ on GRAPH_Γ is not necessarily free ν-a.e. Instead, we use the following argument of Lyons:

Fix a free Borel action of Γ on Y with invariant probability measure ρ. Let $X = \mathrm{CGRAPH}_\Gamma \times Y$ and put on X the measure $\mu = \nu \times \rho$. Now Γ acts on CGRAPH_Γ as before, so it acts on X by $g \cdot (G,x) = (g\cdot G, g\cdot x)$. Since the action of Γ on Y is free, it follows that the action of Γ on X is free. Let $E = E_\Gamma^X$. We define a graphing $\mathcal{G}$ of E as follows: Fix $(G,x),\ (H,y) \in X$, with $(G,x)E(H,y)$. Then there is unique $g \in \Gamma$ with $g\cdot(G,x) = (H,y)$, or $g\cdot G = H,\ g\cdot x = y$. Put

$$(G,x)\mathcal{G}(H,y) \Leftrightarrow (1, g^{-1}) \in G.$$

It is easy to see that $\mathcal{G}$ is a graphing of E. We also have

$$
\begin{aligned}
C_\mu(\mathcal{G}) &= \frac{1}{2}\int |\mathcal{G}_{(G,x)}| \ d\mu(G,x) \\
&= \frac{1}{2}\int d_G(1) d\nu(G) \ d\rho(x) \\
&= d_\nu.
\end{aligned}
$$

Thus we have shown that d_ν is equal to $C_\mu(\mathcal{G})$, where $\mathcal{G}$ is a graphing of an equivalence relation E induced by a free Borel action of Γ with invariant probability measure μ, and so $d_\nu \geq C_\mu(E)$.

This completes the proof of the proposition. ⊣

Note that 29.5 gives immediately another proof of 29.3. Note also that it can be used to give another proof that $C(\Gamma) \geq 1$ for all infinite Γ:

The argument comes from the proof of Theorem 6.1 in Benjamini, Lyons, Peres, and Schramm [BLPS].

According to the so-called *Mass Transport Principle*, if $f : \Gamma \times \Gamma \to [0, \infty]$ is Γ-invariant, i.e., $f(x, y) = f(\gamma x, \gamma y),\ \forall \gamma \in \Gamma$, then $\forall \gamma \in \Gamma (\sum_{\delta \in \Gamma} f(\gamma, \delta) = \sum_{\delta \in \Gamma} f(\delta, \gamma))$. This is clear, since

$$\begin{aligned}\sum_{\delta \in \Gamma} f(\gamma, \delta) &= \sum_{\delta \in \Gamma} f(\gamma^{-1}\gamma, \gamma^{-1}\delta) \\ &= \sum_{\epsilon \in \Gamma} f(1, \epsilon) \\ &= \sum_{\delta \in \Gamma} f(1, \delta^{-1}\gamma) \\ &= \sum_{\delta \in \Gamma} f(\delta, \delta\delta^{-1}\gamma) \\ &= \sum_{\delta \in \Gamma} f(\delta, \gamma).\end{aligned}$$

It follows that if $F : \Gamma \times \Gamma \times \mathrm{GRAPH}_\Gamma \to [0, \infty]$ is Borel and Γ-invariant, i.e., $F(x, y, G) = F(\gamma x, \gamma y, \gamma \cdot G),\ \forall \gamma \in \Gamma$, then for any random graphing μ,

$$f(x, y) = \int F(x, y, G)\ d\mu(G)$$

is Γ-invariant, so $\forall \gamma \in \Gamma$,

$$\int \sum_\delta F(\gamma, \delta, G)\ d\mu(G) = \int \sum_\delta F(\delta, \gamma, G)\ d\mu(G).$$

Assume now μ is a random connected graphing of Γ. If $\mu(\{G : d_G(1) = \infty\}) > 0$, clearly $d_\mu = \infty > 1$. So we can assume that $d_G(1) < \infty$ μ-a.e.(G), and so $d_G(\gamma) < \infty$, μ-a.e.(G), in other words we can assume that μ concentrates on locally finite connected graphs. Define then $F(x, y, G)$ as follows:

$$F(x, y, G) = \begin{cases} 1, & \text{if } G \text{ is connected, } (x, y) \in G \text{, and when the edge} \\ & (x, y) \text{ is removed, the connected component of } x \\ & \text{is finite.} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly F is Γ-invariant.

Notice that $F(x, y, G) = 1$ for at most one y. Also there are at most $d_G(x) - 1$ many y such that $F(y, x, G) = 1$. Consider then the difference $\Delta(x, G) = \Delta = \sum_y F(x, y, G) - \sum_y F(y, x, G)$. (These sums are clearly finite.) If $d_G(x) = 1,\ \Delta = 1 = 2 - d_G(x)$. If $d_G(x) \geq 2$ and there is y with $F(x, y, G) = 1$, so that $\sum_y F(x, y, G) = 1$, we have, as $\sum_y F(y, x, G) \leq d_G(x) - 1$, that $\Delta \geq 1 - (d_G(x) - 1) = 2 - d_G(x)$. Finally, if $d_G(x) \geq 2$ and $F(x, y, G) = 0$, for all y, then $\sum_y F(x, y, G) \leq d_G(x) - 2$, so again $\Delta \geq 2 - d_G(x)$. Thus in any case

$$\sum_y F(x, y, G) - \sum_y F(y, x, G) \geq 2 - d_G(x),$$

thus, integrating over μ (for $x = 1$), we get

$$0 \geq 2 - 2d_\mu.$$

or

$$d_\mu \geq 1.$$

Finally, we say that a countable group Γ has *fixed price* if $C(\Gamma) = C_\mu(E)$ for every E induced by a free action of Γ with invariant probability measure μ. The following is open:

Problem 29.6. Are there countable groups that do not have fixed price?

30 Treeable Groups

For a countable group Γ, denote by TREE_Γ, the Borel Γ-invariant subset of GRAPH_Γ consisting of all trees on (the vertex set) Γ. We say that μ is a *random treeing of Γ* if μ is a random graphing of Γ and $\mu(\mathrm{TREE}_\Gamma) = 1$. We say that Γ is *treeable* if it admits a random treeing (this terminology comes from Pemantle-Peres [P^2]). The proof of 29.5 now shows the following.

Proposition 30.1. *Let Γ be a countable group. Then the following are equivalent:*

(i) Γ is treeable.
(ii) There is a free Borel action of Γ on a standard Borel space X with invariant probability measure μ such that E^X_Γ is treeable μ-a.e.

We call Γ *strongly treeable* if for *every* free Borel action of Γ on a standard Borel space X with invariant probability measure μ, the equivalence relation E^X_Γ is treeable μ-a.e.

Remark 30.2. Gaboriau [G2] uses the terminology *treeable* for what we call here *strongly treeable* and *anti-treeable* for what we call here *not treeable.*

The following is an open problem.

Problem 30.3. Is every treeable group strongly treeable?

We next consider the following analog of 19.1.

Proposition 30.4. *Let Γ be a countable group. If ν is a random connected graphing of Γ such that $d_\nu = C(\Gamma) < \infty$, then ν is a random treeing.*

Proof. Consider $X = \mathrm{CGRAPH}_\Gamma \times Y$, $\mu = \nu \times \rho$, $E, \mathcal{G}$ as in the second part of the proof of 29.5. Then $C_\mu(\mathcal{G}) = d_\nu = C(\Gamma) \leq C_\mu(E)$, so $C_\mu(\mathcal{G}) = C_\mu(E)$ and $\mathcal{G}$ is a treeing μ-a.e. So, by Fubini, let $A \subseteq Y$ be a Γ-invariant subset of CGRAPH_Γ such that $\nu(A) = 1$ and for every $G \in A$ there is some $x \in Y$ with $\mathcal{G}|[(G,x)]_E$ a tree. Now it is easy to check that $\mathcal{G}|[(G,x)]_E \cong G$, so G is a tree for $G \in A$, thus $\nu(\mathrm{TREE}_\Gamma) = 1$, and ν is a random treeing. ⊣

We also have analogs of 19.2.

Proposition 30.5 (Gaboriau). *Let Γ be a countable group and E a countable Borel equivalence relation on a standard Borel space X induced by a free action of Γ with invariant probability measure μ. If $\mathcal{T}$ is a treeing of E, then $C_\mu(\mathcal{T}) = C(\Gamma)$. In particular, every strongly treeable group has fixed price.*

Proof. Let $\Phi = \{\varphi_i\}$ be an L-treeing of E such that each $\varphi_i : A_i \to B_i$ is of the form $\varphi_i(x) = g_i \cdot x$, $\forall x \in A_i$, for some $g_i \in \Gamma$, and $C_\mu(\Phi) = C_\mu(\mathcal{T}) = C_\mu(E)$. Let also F be a countable Borel equivalence relation on a standard Borel space Y induced by a free action of Γ with invariant probability measure ν. We have to show that $C_\mu(\Phi) \leq C_\nu(F)$.

Consider the product action $g \cdot (x,y) = (g \cdot x, g \cdot y)$ of Γ on $X \times Y$. It is free and has invariant measure $\mu \times \nu$. Moreover if $\Psi = \{\psi_i\}_{i \in I}$, where $\psi_i : A_i \times Y \to B_i \times Y$ is given by $\psi_i(x,y) = (g_i \cdot x, g_i \cdot y)$, then ψ is an L-treeing of $E_\Gamma^{X \times Y}$. So

$$C_\mu(\Phi) = C_{\mu \times \nu}(\Psi) = C_{\mu \times \nu}(E_\Gamma^{X \times Y}).$$

But, as in the proof of 29.1, $C_{\mu \times \nu}(E_\Gamma^{X \times Y}) \leq C_\nu(F)$, and we are done. ⊣

Proposition 30.6. *Let Γ be a countable group and ν a random treeing of Γ. Then $C(\Gamma) = d_\nu$.*

Proof. As in the proofs of 29.5 and 30.4, from ν we get a free Borel action of Γ on a standard Borel space X with invariant measure μ and a treeing $\mathcal{T}$ of E_Γ^X such that $C_\mu(\mathcal{T}) = d_\nu$. By 30.5, $d_\nu = C_\mu(\mathcal{T}) = C(\Gamma)$. ⊣

Thus for a countable group Γ with $C_\mu(\Gamma) < \infty$, Γ is treeable iff $\inf\{d_\mu : \mu$ a random connected graphing of $\Gamma\}$ is attained. Moreover, for every treeable group Γ and every random treeing ν of Γ and random graphing μ of Γ, $d_\nu \leq d_\mu$, i.e., a random treeing has the smallest possible average degree of any random connected graphing.

Finally note the following.

Proposition 30.7. *If Γ is a treeable countable group and $\Delta \leq \Gamma$ is a subgroup, then Δ is treeable.*

Proof. Immediate from the fact that a subequivalence relation of a treeable equivalence relation is treeable. ⊣

Problem 30.8. Is 30.7 true for strong treeability?

31 Cost and Amenable Groups

A countable group Γ is *amenable* iff it admits a left-invariant finitely additive probability measure defined on all subsets of Γ. We have

Theorem 31.1. *Let Γ be a countable group and let Γ act in a Borel way on a standard Borel space.*

i) **(Ornstein-Weiss [OW])** *If Γ is amenable, then for any probability measure μ on X, E_X^Γ is hyperfinite μ-a.e.*
ii) **(Folklore)** *If the action is free, μ is a Γ-invariant probability measure and E_X^Γ is hyperfinite μ-a.e., then Γ is amenable.*

Corollary 31.2. *Let Γ be an infinite countable group. Then the following are equivalent:*

(i) *Γ is amenable.*
(ii) *Γ is strongly treeable and $C(\Gamma) = 1$.*
(iii) *$C(\Gamma) = 1 = \inf\{d_\mu : \mu$ is a random connected graphing of $\Gamma\}$ and this inf is attained.*
(iv) *Γ is treeable and $C(\Gamma) = 1$.*

Moreover, every amenable group has fixed price.

Proof. (i)⇒(ii) follows from 31.1, and 26.1 i). (ii) ⇒(iii) follows from 30.6. (iii)⇒(iv) follows from 30.4, and (iv)⇒(i) from 30.5, 27.12, and 31.1. The last assertion follows from 30.5. ⊣

Corollary 31.3. *Let Γ be an infinite countable amenable group. Let μ be a random treeing of Γ. Then every tree on Γ has at most two ends, μ-a.e.*

Proof. From 31.1, 22.3, and the proof of 29.5. ⊣

We conclude this section by giving an alternative proof that the cost of an infinite amenable group is 1, which avoids the use of 31.1 i). It is based on the results and methods of Benjamini-Lyons-Peres-Schramm [BLPS].

Let Γ be an infinite countable amenable group. We want to show that $C(\Gamma) = 1$. First, by 32.1 below (which follows immediately from the results in Section 23), we can assume that Γ is finitely generated. Fix a finite set of generators for $\Gamma, \{\gamma_1, \dots, \gamma_k\}$, and consider the corresponding Cayley graph C. Applying 5.3 of [BLPS] to the action of Γ on C we see that there is a random treeing μ of Γ such that for μ-a.e. G, G is a spanning tree of C (i.e., a tree on the set of vertices Γ with edges contained in those in C). In particular, Γ is treeable. It remains to show that $d_\mu = 1$.

Let us note first the following general fact about trees:

Let G be an infinite locally finite tree on the set of vertices V and let $X \subseteq V$ be a finite set. Denote by $\partial_G X$ the set of edges $(a, b) \in G$ with $a \in X$ and $b \notin X$. Then there is $0 \leq \epsilon_X \leq 1$ such that

$$\frac{\sum_{v\in X} d_G(v)}{|X|} = 2 - \frac{|\partial_G X|}{|X|}(1-2\epsilon_X).$$

To see this, consider the graph $G|X$ and let us denote its connected components by $X_1, \dots, X_n$. We have

$$\sum_{v\in X_i} d_{G_X}(v) = \sum_{v\in X_i} d_{G|X_i}(v) = 2|X_i| - 2$$

(since $G|X_i$ is a tree), so adding over i,

$$\sum_{v\in X} d_{G|X}(v) = 2|X| - 2n.$$

But also

$$\sum_{v\in X} d_G(v) = \sum_{v\in X} d_{G|X}(v) + |\partial X|,$$

so

$$\frac{\sum_{v\in X} d_G(v)}{|X|} = 2 - \frac{2n}{|X|} + \frac{|\partial_G X|}{|X|}.$$

But $n \leq |\partial_G X|$, so if $\epsilon_X = \frac{n}{|\partial_G X|} \leq 1$, we have

$$\frac{\sum_{v\in X} d_G(v)}{|X|} = 2 - \frac{|\partial_G X|}{|X|}(1-2\epsilon_X).$$

In particular,

$$\left|\frac{\sum_{v\in X} d_G(v)}{|X|} - 2\right| \leq 3\frac{|\partial_G X|}{|X|}.$$

Since Γ is amenable, fix a Følner sequence $\{X_n\}$ for Γ, i.e., a sequence of finite subsets $X_n \subseteq \Gamma$ such that for each $g \in \Gamma$, $\frac{|X_n g \Delta X_n|}{|X|} \to 0$. In particular, applying this to $g \in \{\gamma_1, \dots, \gamma_k\}$, we see that if we view X_n as a set of vertices in the Cayley graph C, we have

$$|\partial_G X_n|/|X_n| \to 0$$

for every spanning tree G of the Cayley graph C.

Now

$$\begin{aligned} 2d_\mu - 2 &= \frac{\sum_{\gamma\in X_n} \int d_G(\gamma)\ d\mu(G)}{|X_n|} - 2 \\ &= \int \left(\frac{\sum_{\gamma\in X_n} d_G(\gamma)}{|X_n|} - 2\right)\ d\mu(G). \end{aligned}$$

But

$$\left|\frac{\sum_{\gamma\in X_n} d_G(\gamma)}{|X_n|} - 2\right| \leq 3\frac{|\partial_G X_n|}{|X_n|} \to 0,$$

as $n \to \infty$, so $d_\mu = 1$.

32 Generating Subgroups

Let Γ be a countable group, $\Gamma_i \leq \Gamma,\ i \in I$, a family of subgroups of Γ and let $\langle \Gamma_i : i \in I\rangle$ be the subgroup generated by $\bigcup_i \Gamma_i$. If $\Gamma = \langle \Gamma_i : i \in I\rangle$, for any Borel action of Γ on a standard Borel space X, if $E = E^X_\Gamma,\ E_i = E^X_{\Gamma_i}$, then

$$E = \bigvee_{i \in I} E_i.$$

Thus from Section 23 we have the following

Proposition 32.1 (Gaboriau).

(i) *If $\Gamma = \langle \Gamma_1, \Gamma_2\rangle$ and Γ_1, Γ_2 have fixed price, then $C(\Gamma) \leq C(\Gamma_1) + C(\Gamma_2)$.*

(ii) *If $\Gamma = \langle \Gamma_1, \Gamma_2, \dots\rangle$ (a finite or infinite list), $\bigcap_n \Gamma_n$ is infinite and each Γ_n has fixed price, then*

$$C(\Gamma) - 1 \leq \sum_n (C(\Gamma_n) - 1).$$

In particular, if $C(\Gamma_n) = 1$ for all n, then $C(\Gamma) = 1$ and Γ has fixed price. Also, if $\Gamma_n \subseteq \Gamma_{n+1}$, so that $\Gamma = \bigcup_n \Gamma_n$, and $C(\Gamma_n) \to 1$, then $C(\Gamma) = 1$ and Γ has fixed price.

(iii) *If $\Gamma = \langle \Gamma_1, \Gamma_2, \dots\rangle$, $\bigcap_n \Gamma_n = \Gamma_0$ is amenable, and each Γ_n has fixed price, then*

$$C(\Gamma) - C(\Gamma_0) \leq \sum_n (C(\Gamma_n) - C(\Gamma_0)).$$

33 Products

Let Γ be a countable group and $\Gamma_1, \Gamma_2 \leq \Gamma$ two subgroups. We say that Γ_1, Γ_2 *commute* (setwise), in symbols $\Gamma_1 \Box \Gamma_2$, if $\Gamma_1\Gamma_2 = \Gamma_2\Gamma_1$, or equivalently $\Gamma_1\Gamma_2$ is a subgroup of Γ, namely $\langle \Gamma_1, \Gamma_2\rangle$ (here $\Gamma_1\Gamma_2 = \{g_1 g_2 : g_1 \in \Gamma_1,\ g_2 \in \Gamma_2\}$, and $\langle \Gamma_1, \Gamma_2\rangle$ is the subgroup generated by Γ_1, Γ_2). If $\Gamma = \Gamma_1\Gamma_2$ (so that $\Gamma_1 \Box \Gamma_2$), then for any Borel action of Γ on a standard Borel space, if $E_i = E^X_{\Gamma_i}$, then $E_1 \Box E_2$. So from Section 24 we have:

Theorem 33.1 (Pavelich [P], Solecki).

i) If $\Gamma = \Gamma_1\Gamma_2$ and Γ_1, Γ_2 are infinite and have fixed price, then

$$C(\Gamma) \leq C(\Gamma_1) + C(\Gamma_2) - 1.$$

In particular, if $C(\Gamma_1) = C(\Gamma_2) = 1$, then $C(\Gamma) = 1$ and Γ has fixed price.

ii) If $\Gamma = \langle \Gamma_1, \Gamma_2, \dots \rangle$ (a finite or infinite list), where each Γ_i is infinite, has fixed price, $C(\Gamma_i) = 1$, and $\Gamma_i \square \Gamma_{i+1}$, then $C(\Gamma) = 1$ and Γ has fixed price.

There are examples of groups $\Gamma_1, \Gamma_2, \Gamma$ with $\Gamma = \Gamma_1\Gamma_2$, $[\Gamma : \Gamma_i] = \infty$, for $i = 1, 2$, such that $C(\Gamma_1) = 1$ and $C(\Gamma) > 1$. Take for instance $\Gamma = F_2 = \langle a, b \rangle$. Consider the homomorphism $\varphi : F_2 \to \mathbb{Z}$ which sends a to 0 and b to 1. Let $\Gamma_1 = \langle b \rangle$ and $\Gamma_2 = \ker(\varphi)$. Then $\Gamma_1\Gamma_2 = \Gamma$, but as we will see in Section 36, $C(\Gamma) = 2$. Note however that, as Γ_2 is a normal subgroup of infinite index in Γ, it is isomorphic to F_∞ so, again, as we will see in Section 36, $C(\Gamma_2) = \infty$.

The following remains open.

Problem 33.2. Let $\Gamma = \Gamma_1\Gamma_2$ and assume that $[\Gamma : \Gamma_i] = \infty$, and $C(\Gamma_i) < \infty$, $i = 1, 2$. Is $C(\Gamma) \leq \min\{C(\Gamma_1), C(\Gamma_2)\}$? Is it even true that $C(\Gamma) = 1$?

The next result is a somewhat stronger version of a result of Gaboriau, who proved the second assertion assuming that Γ contains an element of infinite order.

Theorem 33.3 (Gaboriau). *Let Γ, Δ be infinite countable groups. Then $C(\Gamma \times \Delta) = 1$. If moreover Γ contains an infinite subgroup which has cost 1 and fixed price, then $\Gamma \times \Delta$ has fixed price.*

Proof. If E is generated by a free action of Γ and F by a free action of Δ, then $E \times F$ is generated by a free action of $\Gamma \times \Delta$, so $C(\Gamma \times \Delta) = 1$ follows from 24.9.

We now prove the last assertion. Consider a free action of $\Gamma \times \Delta$ on a standard Borel space with invariant probability measure μ. Let $F = E^X_{\Gamma\times\Delta}$. We have to show that $C_\mu(F) = 1$. Fix an infinite subgroup $\Sigma \subseteq \Gamma$ which has cost 1 and has fixed price. We will view Γ, Δ as subgroups of $\Gamma \times \Delta$.

Let $E = E^X_\Delta$, so that clearly $E \lhd F$. Let also $E_1 = E^X_\Sigma$, so that $C_\mu(E_1) = 1$ and E_1 is aperiodic. Now $E \vee E_1 = E^X_{\Sigma\times\Delta}$, so $E_1 = E^X_\Sigma \lhd E^X_{\Sigma\times\Delta} = E \vee E_1$. Thus, by 24.11, $C_\mu(F) = 1$. ⊣

Corollary 33.4 (Pemantle-Peres [$\mathbf{P}^2$]). *Let Γ and Δ be infinite countable groups. Then $\Gamma \times \Delta$ is treeable iff both Γ and Δ are amenable.*

Problem 33.5. Let Γ, Δ be countable infinite groups. Does $\Gamma \times \Delta$ have fixed price?

34 Subgroups of Finite Index

The main result is the following:

Theorem 34.1 (Gaboriau). *Let Γ, Δ be countable groups with $\Gamma \leq \Delta$. Then if $[\Delta : \Gamma]$ is finite,*

$$C(\Gamma) - 1 = [\Delta : \Gamma](C(\Delta) - 1).$$

Proof. The inequality $C(\Gamma) - 1 \leq [\Delta : \Gamma](C(\Delta) - 1]$ follows from 25.3 (when Γ, Δ are infinite - the finite case is obvious).

For the inequality $C(\Gamma) - 1 \geq [\Delta : \Gamma](C(\Delta) - 1)$, consider a free Borel action of Γ on a standard Borel space X with invariant probability measure μ. We now consider the *induced action* of Δ. Let T be a transversal for the left-cosets of Γ in Δ with $1 \in T$. Thus $|T| = n = [\Delta : \Gamma]$. Say $T = \{t_1 = 1, t_2, \ldots, t_n\}$. Consider the action of Δ on T given by $\delta \cdot t =$ the unique element of T in the coset $\delta t \Gamma$. Let also $\rho(\delta, t) = (\delta \cdot t)^{-1} \delta t \in \Gamma$. Let $Y = X \times T$. The induced action of Δ is the action of Δ on Y given by

$$\delta \cdot (x, t) = (\rho(\delta, t) \cdot x, \delta \cdot t).$$

Let ν be the normalized counting measure on $T : \nu(A) = \frac{|A|}{|T|}$. Then it is easy to check that $\mu \times \nu$ is a Δ-invariant measure on $X \times T$. Also it is easy to check that the action of Δ is free.

Identify X with $X_1 = \{(x, 1) : x \in X\}$. This is consistent with the Γ-action since for $\gamma \in \Gamma,\ \gamma \cdot (x, 1) = (\gamma \cdot x, 1)$. Now X is a complete section of E^Y_Δ as $t^{-1} \cdot (x, t) = (\rho(x, t^{-1}) \cdot x, t^{-1} \cdot t) = (\rho(x, t^{-1}) \cdot x, 1)$ and the intersection of X with every E^Y_Δ-class is a single E^X_Γ-class. Thus $E^Y_\Delta | X = E^X_\Gamma$ and $(\mu \times \nu)|X = \frac{\mu}{n}$. So, by 7.1,

$$\begin{aligned} C_{\mu \times \nu}(E^Y_\Delta) &= C_{\frac{\mu}{n}}(E^X_\Gamma) + (\mu \times \nu)(Y \setminus X) \\ &= \tfrac{C_\mu(E^X_\Gamma)}{n} + \tfrac{n-1}{n}, \end{aligned}$$

or $n(C_{\mu \times \nu}(E^Y_\Delta) - 1) = C_\mu(E^X_\Gamma) - 1$. Thus $C(\Gamma) - 1 \geq n(C(\Delta) - 1)$. ⊣

The following is a consequence of the proof of 34.1.

Corollary 34.2. *Under the assumptions of 34.1, if Δ has fixed price, so does Γ. If Γ is treeable, so is Δ.*

Proof. The second assertion follows from the easy fact that if E is a countable Borel equivalence relation, S is a complete Borel section and $E|S$ is treeable, then so is E. ⊣

It is also easy to check that if $\Gamma \trianglelefteq \Delta$ is a finite normal subgroup of Δ, then $|\Gamma|(C(\Delta) - 1) = C(\Delta/\Gamma) - 1$, and that Δ has fixed price iff Δ/Γ does.

Problem 34.3. Under the assumptions of 34.1, if Γ has fixed price, does Δ have fixed price?

35 Cheap Groups

An infinite countable group Γ is *cheap* if $C(\Gamma) = 1$, i.e., has the smallest possible cost. Otherwise we call Γ *expensive*. We summarize below some basic facts concerning cheap groups that we established earlier.

Proposition 35.1. *i) Every infinite amenable group is cheap, treeable and has fixed price. Conversely, every cheap treeable group is amenable.*

ii) A group generated by a sequence of cheap fixed price subgroups with infinite intersection is cheap and has fixed price.

iii) A group generated by a sequence of cheap infinite subgroups, with the property that each member of the sequence has fixed price and commutes with the next one, is cheap and has fixed price.

iv) The product of two infinite groups is cheap.

v) Any group that has finite index over or under a cheap group is cheap.

Proof. i) is included in 31.2. ii) is part of 32.1, and iii) of 33.1. iv) is in 33.3 and finally v) in 34.1. ⊣

We additionally have the following criterion.

Proposition 35.2 (Gaboriau). *Let Γ be a countable group and $N \trianglelefteq \Gamma$ an infinite normal subgroup. Then for any Γ' with $N \leq \Gamma' \leq \Gamma$ which has fixed price, $C(\Gamma) \leq C(\Gamma')$. Any countable group that contains a cheap normal subgroup of fixed price is cheap and has fixed price.*

Proof. Immediate from 24.10. ⊣

Corollary 35.3. *Any countable group with infinite center is cheap and has fixed price.*

One can also prove the following generalization of 35.2 and an observation of Furman.

Proposition 35.4. *Let Γ be a countable group and $N \leq \Gamma$ an infinite subgroup of Γ, which is almost normal in Γ, in the sense that Γ is generated by elements γ such that $\gamma^{-1}N\gamma \cap N$ is infinite. Then for any Γ' with $N \leq \Gamma' \leq \Gamma$ which has fixed price, $C(\Gamma) \leq C(\Gamma')$. Thus if N is cheap and has fixed price, the same holds for Γ.*

Proof. This is similar to 24.10, using the full version of 24.7. Let $\Gamma = \langle \gamma_0, \gamma_1, \dots \rangle$, where $\gamma_i^{-1}N\gamma_i \cap N$ is infinite. Consider a free Borel action of Γ on a standard Borel space X with invariant probability measure μ. Let $\mathcal{G}_n$ be the union of the graphs of $x \mapsto \gamma_i \cdot x, i = 0, \dots, n$, and their inverses, and let E_n the subequivalence relation they generate. Note that if $\gamma_i \cdot a = b$, there are infinitely many $\gamma \in N$ such that if $\gamma \cdot a = a', \gamma_i \cdot a' = b'$, then for some $\gamma' \in N, \gamma' \cdot b = b'$ (any $\gamma \in \gamma_i^{-1}N\gamma_i \cap N$ satisfies this). So $\mathcal{G}_n, E_n^X$

satisfy the hypotheses of 24.7. Thus, for any $\epsilon > 0$, we can find $\mathcal{G}'_n \subseteq \mathcal{G}_n$ with $C_\mu(\mathcal{G}'_n) < \frac{\epsilon}{2^n}$, so that for any graphing $\mathcal{G}'$ of $E^X_{\Gamma'}$ we have that $\mathcal{G}'_n \cup \mathcal{G}'$ is a graphing of $E_n \vee E^X_{\Gamma'}$. Choose now $\mathcal{G}'$ with $C_\mu(\mathcal{G}') \leq C_\mu(E^X_{\Gamma'}) + \epsilon = C(\Gamma') + \epsilon$. Then $\mathcal{G}' \cup \bigcup_n \mathcal{G}'_n = \mathcal{G}$ graphs E^X_Γ with cost $\leq C(\Gamma') + \epsilon + \sum_n \frac{\epsilon}{2^n} = C(\Gamma') + 3\epsilon$. So $C(E^X_\Gamma) \leq C(\Gamma')$. ⊣

Another generalization of 35.2 is the following:

Theorem 35.5 (Gaboriau). *Let Γ be a countable group and assume Γ contains a normal subgroup $N \trianglelefteq \Gamma$ with fixed price, such that $N, \Gamma/N$ are infinite. If N has finite cost, then Γ is cheap.*

Proof. Fix a free Borel action of Γ on X with invariant probability measure μ and a free Borel action of $\Delta = \Gamma/N$ on Y with invariant probability measure ν. Let $\pi : \Gamma \to \Delta$ be the canonical homomorphism with kernel N. Consider the action of Γ on $Z = X \times Y$ given by

$$\gamma \cdot (x, y) = (\gamma \cdot x, \pi(\gamma) \cdot y).$$

It is free and preserves $\rho = \mu \times \nu$. We will show that $C_\rho(E^Z_\Gamma) = 1$.

Let R be an aperiodic hyperfinite subequivalence relation of E^Y_Δ, induced by a Borel automorphism T of Y. Find $\{Y_n\}$, a partition of Y into Borel sets, and $\{\gamma_n\}$ in Γ such that for $y \in Y_n, T(y) = \pi(\gamma_n) \cdot y$. Put $\theta_n = T|Y_n$. Let $Z_n = X \times Y_n$ and let $\psi_n(z) = \gamma_n \cdot z$, for $z \in Z_n$. Let R' be the equivalence relation on Z induced by $\{\psi_n\}$. Clearly $C_\rho(R') = 1$.

Now fix $\epsilon > 0$. Since $C(N) = C_\rho(E^Z_N) < \infty$, find an L-graphing $\{\varphi_i\}_{i \in \mathbb{N}}$ of E^Z_N with $\sum_i C_\rho(\{\varphi_i\}) < \infty$. Then choose M large enough so that we have $\sum_{i \geq M} C_\rho(\{\varphi_i\}) < \epsilon$.

Let $A \subseteq Y$ be a complete Borel section for R with $\mu(A) < \epsilon/M$. Then $Z' = X \times A$ is N-invariant and $\rho(Z') < \epsilon/M$. Since $\{\varphi_i|Z'\}_{i \in \mathbb{N}}$ is an L-graphing of $E^{Z'}_N$, we have

$$\begin{aligned} C_{\rho|Z'}(E^{Z'}_N) &\leq \sum_i C_{\rho|Z'}(\{\varphi_i|Z'\}) \\ &\leq M \cdot \rho(Z') + \sum_{i \geq M} C_\rho(\{\varphi_i\}) \\ &< \epsilon + \epsilon = 2\epsilon. \end{aligned}$$

Claim 35.6. $E^{Z'}_N \vee R' \supseteq E^Z_N$.

Proof. Suppose $(x, y) E^Z_N (x', y')$. Then there is $\delta \in N$ with $\delta \cdot x = x'$ and $\pi(\delta) \cdot y = y'$, so $y = y'$. Find $n_1, \dots, n_k$ and $\epsilon_1, \dots, \epsilon_k \in \{\pm 1\}$ such that $\theta^{\epsilon_1}_{n_1} \dots \theta^{\epsilon_k}_{n_k} = y_0 \in A$. Let $\gamma = \gamma^{\epsilon_i}_{n_1} \dots \gamma^{\epsilon_k}_{n_k}$, so that $\pi(\gamma) \cdot y = y_0$, since $\theta_n(u) = \pi(\gamma_n) \cdot u$ for $u \in \mathrm{dom}(\theta_n) = Y_n$. Thus $\gamma \cdot (x, y) = (\gamma \cdot x, \pi(\gamma) \cdot y) = (\gamma \cdot x, y_0) \in Z'$ and, since also $\psi^{\epsilon_1}_{n_1} \dots \psi^{\epsilon_k}_{n_k} \cdot (x, y) = \gamma^{\epsilon_1}_{n_1} \dots \gamma^{\epsilon_k}_{n_k} \cdot (x, y) = \gamma \cdot (x, y) = (\gamma \cdot x, y_0)$, we have $(x, y) R' (\gamma \cdot x, y_0)$. Similarly $(\delta \cdot x, y) R' (\gamma\delta \cdot x, y_0) \in Z'$. Finally if $\delta' \in N$

is such that $\delta'\gamma = \gamma\delta$, we have $\delta' \cdot (\gamma \cdot x, y_0) = (\delta'\gamma \cdot x, \pi(\delta') \cdot y_0) = (\gamma\delta \cdot x, y_0)$. So

$$(x, y) R' (\gamma \cdot x, y_0) E_N^{Z'} (\gamma\delta \cdot x, y_0) R' (\delta \cdot x, y) = (x', y'),$$

and we are done. ⊣

Now clearly $C_\rho(E_N^{Z'} \vee R') \leq 1 + 2\epsilon$. Since $E_N^Z \lhd E_\Gamma^Z$, it follows from 24.10 that $C_\mu(E_\Gamma^Z) \leq C_\mu(E_N^{Z'} \vee R') \leq 1 + 2\epsilon$, so $C_\mu(E_\Gamma^Z) = 1$. ⊣

Note that it is necessary in the preceding theorem to assume that N has finite cost: Take $\Gamma = F_2$ and let $\pi : F_1 \to \mathbb{Z}$ be an onto homomorphism with kernel N. Then $N, \Gamma/N$ are infinite and N has infinite cost, being isomorphic to the free group with infinitely many generators, while $C(\Gamma) = 2$ (we use 36.2 here).

We will finish this section by applying the preceding results to show that the groups $\mathrm{SL}_n(R)$, where R is an infinite countable commutative ring with identity, are cheap and have fixed price, in most *but not all* cases. These facts, noted by Pavelich, generalize results of Gaboriau, who showed that $\mathrm{SL}_n(\mathbb{Z})$, $n \geq 3$, is cheap and so is $\mathrm{SL}_2(R)$, when R is the ring of algebraic integers in a real quadratic extension $\mathbb{Q}(\sqrt{d})$, $d \in \mathbb{N}$, $d \geq 2$, d square free.

If R is a commutative ring with identity 1, $\mathrm{SL}_n(R)$ is the group of $n \times n$ matrices with entries in R and determinant 1. We denote by $E_n(R) = \langle e_{ij}(r) : 1 \leq i, j \leq n,\ i \neq j,\ r \in R \rangle$ the subgroup of $\mathrm{SL}_n(R)$ generated by the elementary matrices $e_{ij}(r)$, where $e_{ij}(r)$ is the $n \times n$ matrix $(a_{k\ell}) \in \mathrm{SL}_n(R)$, with $a_{kk} = 1$, $a_{ij} = r$, $a_{k\ell} = 0$ for all other (k, ℓ). Gaboriau has shown that $E_n(R)$ is cheap, when $n \geq 3$. This can be seen as follows: Note that if $\Gamma_{ij} = \langle e_{ij}(r) : r \in R \rangle$ is the subgroup of $E_n(R)$ generated by $e_{ij}(r) \in R$, then $\Gamma_{ij} \cong \langle R, + \rangle$, so Γ_{ij} is infinite abelian, thus has cost 1 and is of fixed price. Now Γ_{ij} commutes (pointwise) with $\Gamma_{k\ell}$ if $i = k$ or $j = \ell$. If $n \geq 3$, we can enumerate all $\{\Gamma_{ij}\}$ in a sequence $\Gamma_1, \Gamma_2, \ldots, \Gamma_\ell$ so that $\Gamma_i \Box \Gamma_{i+1}$. Then, by 33.1 (ii), $E_n(R)$ is cheap and $E_n(R)$ has fixed price.

Now it is a theorem of algebra that $E_n(R) \trianglelefteq SL_n(R)$, if $n \geq 3$, so $\mathrm{SL}_n(R)$ is cheap and has fixed price, if $n \geq 3$, by 35.2.

We now consider $E_2(R)$, $\mathrm{SL}_2(R)$. First, in general, $E_2(R)$ is not a normal subgroup of $\mathrm{SL}_2(R)$ but, under certain conditions, satisfied by many rings in practice, we actually have that $E_2(R) = \mathrm{SL}_2(R)$. For example, this is true if R is a Euclidean domain, a semi-local ring, or the ring of algebraic integers in a real quadratic field extension $\mathbb{Q}(\sqrt{d})$, $d \in \mathbb{N}$, $d \geq 2$, d square free. Call such rings *nice*.

Next comes the question of whether $E_2(R)$ is cheap. Again (as we will see soon for $R = \mathbb{Z}$) this may not be true in general, but it holds under the condition that R has infinitely many units. This is because $E_2(R)$ is generated by the subgroups,

$$U = \left\{ \begin{pmatrix} u & r \\ 0 & u^{-1} \end{pmatrix} : u \in R^*, \ r \in R \right\}$$

$$L = \left\{ \begin{pmatrix} u & 0 \\ r & u^{-1} \end{pmatrix} : u \in R^*, \ r \in R \right\}$$

Now U, L are solvable, thus amenable, so are cheap with fixed price, and $U \cap L$ is infinite. So by 32.1(ii) we are done.

Thus we have that if R is nice and has infinitely many units, then $\mathrm{SL}_2(R)$ is cheap and has fixed price. In particular, if $\mathrm{SL}_2(R)$ is not amenable (e.g., if $R \supseteq \mathbb{Z}$), then it is not treeable.

For example, if R is the ring of algebraic integers in a real quadratic extension $\mathbb{Q}(\sqrt{d})$, $d \in \mathbb{N}$, $d \geq 2$, d square free, then R is nice and, by a theorem of Dirichlet, R has infinitely many units, so $\mathrm{SL}_2(R)$ is cheap and has fixed price (Gaboriau). Also if $R = \mathbb{Z}[\frac{1}{2}]$, the ring of dyadic rationals, R is nice (it is a Euclidean ring) and has infinitely many units, so $\mathrm{SL}_2(R)$ is cheap and has fixed price (Pavelich [P]), thus it is also not treeable (Kechris [K1]).

Finally, we note that the hypothesis that R has infinitely many units is important, since a result of Gaboriau, that we will see in the next section, shows that $\mathrm{SL}_2(\mathbb{Z}) \cong \mathbb{Z}_6 *_{\mathbb{Z}_2} \mathbb{Z}_4$ is not cheap, having cost $(1 - 1/6) + (1 - 1/4) - (1 - 1/2) = 1 + 1/12$ ($\mathbb{Z}$ is clearly nice, being a Euclidean domain).

Problem 35.7. Let R be a countable infinite Euclidean domain. Is it true that $\mathrm{SL}_2(R)$ is cheap iff R has infinitely many units?

36 Free and Amalgamated Products

From Section 27 we have:

Theorem 36.1 (Gaboriau for finite families). *Let $\Gamma = *^i_\Delta \Gamma_i$ be the amalgamated product of $\{\Gamma_i\}_{i \in I}$ over an amenable group Δ. Assume each Γ_i has fixed price and finite cost. Then*

$$C(\Gamma) - C(\Delta) = \sum_i (C(\Gamma_i) - C(\Delta)),$$

and Γ has fixed price. In particular, if $\Gamma = \Gamma_1 *_\Delta \Gamma_2$, where Γ_1, Γ_2 have fixed price and finite cost, and Δ is amenable, then Γ has fixed price and

$$C(\Gamma) = C(\Gamma_1) + C(\Gamma_2) - C(\Delta),$$

and if $\Gamma = \Gamma_1 * \Gamma_2$, where Γ_1, Γ_2 have fixed price and finite cost, then Γ has fixed price and

$$C(\Gamma) = C(\Gamma_1) + C(\Gamma_2).$$

Proof. Immediate from 27.6. $\dashv$

Corollary 36.2 (Gaboriau). *$C(F_n) = n$, for $n = 1, 2, \ldots, \infty$, and F_n has fixed price.*

Consider free Borel actions of two groups Γ_1, Γ_2 on standard Borel spaces X_1, X_2. Let $E_{\Gamma_i}^{X_i}$ be the corresponding equivalence relations. They are *Borel isomorphic* if there is a Borel bijection $f : X_1 \to X_2$ such that $xE_{\Gamma_1}^{X_1}y \Leftrightarrow f(x)E_{\Gamma_2}^{X_2}f(y)$.

Corollary 36.3 (Gaboriau). *If $n \neq m$ $(n, m = 1, 2, \ldots, \infty)$, then no free Borel action of F_n with invariant finite measure gives an equivalence relation isomorphic to an equivalence relation given by a free Borel action of F_m.*

It should be pointed out here that each F_n has a free Borel action on some X_n with $E_{X_n}^{F_n}$ hyperfinite. (For example, one can consider any free Borel action of F_n on some space Y_n with invariant finite measure, so that $E_{Y_n}^{F_n}$ is not smooth, and use the Glimm-Effros Dichotomy (see [DJK]) to find a Borel invariant set $X_n \subseteq Y_n$ for which $E_{Y_n}^{F_n}|X_n = E_{X_n}^{F_n}$ is hyperfinite.) Then each $E_{X_n}^{F_n}$ is hyperfinite and does not admit an invariant finite measure (by 31.1 ii)). Thus all $E_{X_n}^{F_n}$ are Borel isomorphic (see [DJK], 9.1). So the hypothesis in 36.3 about invariant finite measures is crucial.

Theorem 36.4 (Gaboriau). *For each $c \geq 1$, there is a countable fixed price group G_c such that $C(G_c) = c$. Such a G_c can be taken to be either strongly treeable or not treeable.*

Proof. Consider F_2 and the products $F_2 \times \mathbb{Z}_n$. By 20.1, $C(F_2 \times \mathbb{Z}_n) = 1 + \frac{1}{n}(2-1) = 1 + \frac{1}{n}$. Thus $C(F_2 \times \mathbb{Z}_{2^n}) = 1 + 2^{-n}$.

If $c = 1$, take $G_1 = \mathbb{Z}$. If $2 > c > 1$, write $c = 1 + 2^{-k_0} + 2^{-k_1} + \ldots$, where $1 \leq k_0 < k_1 < k_2 \ldots$ Let $G_c = *_{\mathbb{Z}}^i (F_2 \times \mathbb{Z}_{2^{k_i}})$ (here $\mathbb{Z} = \langle a \rangle$ is viewed as a subgroup of $F_2 = \langle a, b \rangle$). Then $C(G_c) = 1 + \sum_i ((1 + 2^{-k_i}) - 1) = c$. If $c \geq 2$ and $1 \leq c - n < 2$, let $G_c = F_n * G_{c-n}$.

We will now find groups H_c with $C(H_c) = c$, which are strongly treeable. For $c = 1$, we again take $H_1 = \mathbb{Z}$. If we have found H_c for $1 < c < 2$, then we take for $c \geq 2$, $1 \leq c - n < 2$, $H_c = F_n * H_{c-n}$. So assume $1 < c < 2$ and let $c = 1 + \sum_{i=1}^{\infty} \frac{\epsilon_i}{2^i}$, $\epsilon_i \in \{0, 1\}$. We will find two sequences of groups $\Gamma_0, \Gamma_1, \ldots, \Delta_0, \Delta_1, \ldots$ such that all Γ_i, $i \geq 1$, and Δ_i, $i \geq 0$, are finite, Γ_0 is strongly treeable of fixed price, and if we let $H_0 = \Gamma_0$, $H_{n+1} = H_n \vee \Gamma_{n+1}$, then $\Delta_n \subseteq H_n \cap \Gamma_{n+1}$, $H_{n+1} = H_n *_{\Delta_n} \Gamma_{n+1}$, for $n \geq 0$, and $C(H_n) = 1 + \sum_{i \leq n} \frac{\epsilon_i}{2^i}$, $\forall n \geq 1$. Now $H_0 \subseteq H_1 \subseteq \ldots$ and we let $H_c = \bigcup_n H_n$. Then, by 27.17, H_c is strongly treeable, has fixed price, and $C(H_c) = \lim C(H_n) = c$. (Note that each H_n has fixed price by 36.1.)

We take $\Gamma_0 = \bigoplus_n \mathbb{Z}_{2^n}$ and $\Gamma_n = \mathbb{Z}_{2^n}$, if $n \geq 1$. We also take $\Delta_n = \mathbb{Z}_{2^{(n+1)-\epsilon_{n+1}}}$. Then $C(H_1) = 1 + \frac{\epsilon_1}{2}$, and assuming $C(H_n) = 1 + \sum_{i \leq n} \frac{\epsilon_i}{2^i}$, we have $C(H_{n+1}) = C(H_n) + C(\Gamma_{n+1}) - C(\Delta_n) = 1 + \sum_{i \leq n} \frac{\epsilon_i}{2^i} + (1 - \frac{1}{2^{n+1}}) - (1 - \frac{2^{\epsilon_{n+1}}}{2^{n+1}}) = 1 + \sum_{i \leq n} \frac{\epsilon_i}{2_i} + \frac{\epsilon_{n+1}}{2^{n+1}} = 1 + \sum_{i \leq n+1} \frac{\epsilon_i}{2^i}$.

We finally find not treeable groups K_c with $C(K_c) = c$. It is enough to consider $1 \leq c < 2$. Let $K_1 = F_2 \times \mathbb{Z}$. For $c > 1$, fix n with $d = c - \frac{1}{2^n} > 1$ and let $K_c = (F_2 \times \mathbb{Z} \times \mathbb{Z}_{2^n}) *_{\mathbb{Z}_{2^n}} G_d$. $\dashv$

Gaboriau [G2], VI.16, has also shown that for each $c \geq 1$ we can find a fixed price group of cost c, which is moreover finitely generated.

We also have a converse of 36.1.

Theorem 36.5 (Gaboriau). *Let $\Gamma = \langle \Gamma_1, \Gamma_2, \dots \rangle$, $\Delta = \Gamma_{i_1} \cap \Gamma_{i_2}$ for any $i_1 \neq i_2$, and assume Δ is finite, each Γ_i has fixed price, and $\sum_i [C(\Gamma_i) - C(\Delta)] < \infty$. If $C(\Gamma) - C(\Delta) = \sum_i (C(\Gamma_i) - C(\Delta))$, then $\Gamma = *^i_\Delta \Gamma_i$.*

Proof. This follows from 27.16. $\dashv$

Gaboriau has given a counterexample to show that 36.5 fails if Δ is not finite (in his example Δ is actually $\mathbb{Z}^2$, hence amenable). This is as follows: $\Gamma = \mathrm{SL}_3(\mathbb{Z})$, $\Gamma_1 =$ all matrices in $\mathrm{SL}_3(\mathbb{Z})$ that fix $(1, 0, 0)$, $\Gamma_2 =$ all matrices in $\mathrm{SL}_3(\mathbb{Z})$ that fix $(0, 1, 0)$, and $\Delta = \Gamma_1 \cap \Gamma_2 \cong \mathbb{Z}^2$. Then as in Section 35 we can see that $\Gamma_1 = \langle \Gamma_{1,1}, \Gamma_{1,2}, \Gamma_{1,3}, \Gamma_{1,4} \rangle$, where $\Gamma_{1,i} \square \Gamma_{1,i+1}$, and each $\Gamma_{1,i}$ is infinite, with fixed price, and $C(\Gamma_{1,i}) = 1$, so, by 33.1 (ii), Γ_1 is cheap and so is Γ_2. Thus $C(\mathrm{SL}_3(\mathbb{Z})) = 1 = C(\Gamma_1) + C(\Gamma_2) - C(\Delta)$. However $\mathrm{SL}_3(\mathbb{Z})$ cannot be written in a non-trivial way as $A *_C B$ (i.e., with $A \neq C \neq B$), by a result in group theory, see Serre [S], 6.6.

37 HNN-extensions

Let Γ be a countable group, $\Delta \subseteq \Gamma$ a subgroup and $\varphi : \Delta \to \Gamma$ an injective homomorphism. Let

$$\Gamma_{*\Delta} = \Gamma *_{\varphi, \Delta} = \langle \Gamma, t | t^{-1} \delta t = \varphi(\delta),\ \delta \in \Delta \},$$

be the HNN-extension of Γ (relative to φ).

Theorem 37.1 (Gaboriau). *If Γ has fixed price and finite cost and Δ is amenable, then $\Gamma_{*\Delta}$ has fixed price and*

$$C(\Gamma_{*\Delta}) = C(\Gamma) + 1 - C(\Delta).$$

Proof. Consider a free Borel action of $\Gamma_{*\Delta}$ on X with invariant probability measure μ. Put $f(x) = t \cdot x$. Let $\bar{X}$ be another (disjoint) copy of X, with $x \mapsto \bar{x}$ be the identification map and let $\bar{\mu}$ be the copy of μ in $\bar{X}$. There is also a copy of the action of $\Gamma_{*\Delta}$ on $\bar{X}$: $\overline{g \cdot x} = g \cdot \bar{x}$. Now let $Y = X \sqcup \bar{X}$, let $\nu = \mu + \bar{\mu}$ and consider the following equivalence relations on Y:

$R_1 =$ the equivalence relation generated by E^X_Γ and $x \mapsto \bar{x}$.

$R_2 =$ the equivalence relation generated by $E^{\bar{X}}_\Delta$ and $x \mapsto \overline{f(x)}$.

$R_3 = E^{\bar{X}}_\Delta \sqcup E^X_{\varphi(\Delta)} = E^{\bar{X}}_\Delta \sqcup E^X_{t^{-1}\Delta t}$.

The characteristic property of $\Gamma_{*\Delta}$ (Britton's Lemma) is that if $1 = \gamma_0 t^{\epsilon_1}\gamma_1 t^{\epsilon_2}\dots t^{\epsilon_n}\gamma_n$, with $n > 0$, $\epsilon_i \in \{\pm 1\}$, $\gamma_i \in \Gamma$, we must have some $1 \leq m \leq n-1$ with $\epsilon_m = -1$, $\gamma_m \in \Delta$, $\epsilon_{m+1} = +1$ or else $\epsilon_m = +1$, $\gamma_m \in \varphi(\Delta)$, $\epsilon_{m+1} = -1$. Using this it is easy to check that

$$R = R_1 *_{R_3} R_2$$

so, as R_3 is clearly hyperfinite and R_1, R_2 have finite cost (see below),

$$C_\nu(R) = C_\nu(R_1) + C_\nu(R_2) - C_\nu(R_3).$$

Notice that $X, \bar{X}$ are complete sections for R, R_1, R_2, $\nu|X = \mu, \nu|\bar{X} = \bar{\mu}$, $R|X = E^X_{\Gamma_{*\Delta}}$, $R_1|X = E^X_\Gamma$, $R_2|\bar{X} = E^{\bar{X}}_\Delta$, so

$$\begin{aligned} C_\nu(R) &= C_\mu(E^X_{\Gamma_{*\Delta}}) + 1 \\ C_\nu(R_1) &= C_\mu(E^X_\Gamma) + 1 \\ C_\mu(R_2) &= C_\mu(E^X_\Delta) + 1. \end{aligned}$$

Also $X, \bar{X}$ are R_3-invariant, so

$$\begin{aligned} C_\nu(R_3) &= C_{\nu|X}(R_3|X) + C_{\nu|\bar{X}}(R_3|\bar{X}) \\ &= 2C_\mu(E^X_\Delta). \end{aligned}$$

So

$$C_\mu(E^X_{\Gamma_{*\Delta}}) + 1 = C_\mu(E^X_\Gamma) + 1 + C_\mu(E^X_\Delta) + 1 - 2C_\mu(E^X_\Delta)$$

or

$$\begin{aligned} C_\mu(E^X_{\Gamma_{*\Delta}}) &= C_\mu(E^X_\Gamma) + 1 - C_\mu(E^X_\Delta) \\ &= C(\Gamma) + 1 - C(\Delta). \end{aligned}$$ ⊣

By using the same method one can also prove a converse to 37.1.

Theorem 37.2 (Gaboriau). *Let $G = \langle \Gamma, t\rangle$ be a countable group, $\Delta \leq \Gamma$, $t^{-1}\Delta t \subseteq \Gamma$. If Γ has fixed price and finite cost, Δ is finite, and $C(G) = C(\Gamma) + 1 - C(\Delta)$, then $G = \Gamma_{*\Delta}$.*

38 A List of Open Problems

We collect here the open problems that were mentioned earlier in this chapter:

18.2 *Let E be a countable Borel equivalence relation on X, $\mathcal{I}_E$ the standard Borel space of E-invariant probability measures on X. Is the function*

$$\mu \in \mathcal{I}_E \mapsto C_\mu(E)$$

Borel?

24.4, 24.5 *Let E_1, E_2 be commuting aperiodic countable Borel equivalence relations on X, let $E = E_1 \vee E_2$, and let μ be an E-invariant probability measure on X.*

If $C_\mu(E_1),\ C_\mu(E_2) < \infty$ and each E-class contains infinitely many E_1- and infinitely many E_2-classes, is it true that $C_\mu(E) \leq \min\{C_\mu(E_i), C_\mu(E_2)\}$ or even that $C_\mu(E) = 1$?

25.5 *Let $E \subseteq F$ be aperiodic countable Borel equivalence relations on X and let μ be a finite F-invariant measure. If $[F : E] = n$ is it true that*

$$C_\mu(E) - \mu(X) = n(C_\mu(F) - \mu(X))?$$

27.3 (Gaboriau) *Let E_1, E_2 be countable Borel equivalence relations, let $E = E_1 \vee E_2,\ E_3 = E_1 \cap E_2$ and assume that $E_1 \perp_{E_3} E_2$. Let μ be a finite E-invariant measure. If E_3 is hyperfinite is it true that*

$$C_\mu(E_1 *_{E_3} E_2) = C_\mu(E_1) + C_\mu(E_2) - C_\mu(E_3),$$

even if one of $C_\mu(E_1),\ C_\mu(E_2)$ is ∞?

28.7 *Prove or disprove the following dichotomy:*

For any aperiodic countable Borel equivalence relation E on X, exactly one of the following holds:

- *(I) There is a Borel E-invariant set $A \subseteq X$ with $E_0 \cong_B E|A$.*
- *(II) There is a free Borel action of F_2 on X such that $E^X_{F_2} \subseteq E$.*

28.14 (Gaboriau) *Prove or disprove the following dichotomy:*

Let E be a countable Borel equivalence relation and μ an E-invariant, ergodic Borel probability measure. Then exactly one of the following holds:

- *(I) E is hyperfinite, μ-a.e.*
- *(II) There is a Borel equivalence relation $F \subseteq E$ which is induced by a free Borel action of F_2, μ-a.e.*

29.4 *Is the function $\Gamma \in \mathrm{GROUP} \mapsto C(\Gamma)$ Borel? Is the set $\{\Gamma \in \mathrm{GROUP} : C(\Gamma) = 1\}$ Borel?*

29.6 (Gaboriau) *Are there countable groups that do not have fixed price?*

30.3 (Gaboriau) *Is every treeable countable group strongly treeable?*

30.8 (Gaboriau) *If Γ is a strongly treeable countable group and $\Delta \leq \Gamma$ is a subgroup, is Δ strongly treeable?*

33.2 *Let Γ be a countable group $\Gamma_1, \Gamma_2 \leq \Gamma$ and $\Gamma = \Gamma_1\Gamma_2$. Assume that $[\Gamma : \Gamma_i] = \infty$ and $C(\Gamma_i) < \infty$, for $i = 1, 2$. Is $C(\Gamma) \leq \min\{C(\Gamma_1), C(\Gamma_2)\}$ or even $C(\Gamma) = 1$?*

33.5 (Gaboriau) *Let Γ, Δ be countable infinite groups. Does $\Gamma \times \Delta$ have fixed price?*

34.3 (Gaboriau) *Let Γ, Δ be countable groups with $\Gamma \leq \Delta$. If $[\Delta : \Gamma]$ is finite and Γ has fixed price, does Δ have fixed price?*

35.7 *Let R be a countable infinite Euclidean domain. Is it true that* $\mathrm{SL}_2(R)$ *is cheap iff R has infinitely many units?*

References

[A] Adams, S.: Indecomposability of treed equivalence relations, Israel J. Math., **64(3)**, 362–380 (1988)

[A1] Adams, S.: Trees and amenable equivalence relations. Erg. Theory and Dynam. Systems, **10**, 1–14 (1990)

[BLPS] Benjamini, I., Lyons, R., Peres, Y., Schramm, O.: Group-invariant percolation on graphs. Geom. Funct. Anal., **9**, 29–66 (1999)

[CFW] Connes, A., Feldman, J., Weiss, B.: An amenable equivalence relation is generated by a single transformation, Ergod. Th. & Dynam. Sys., **1**, 431–450 (1981)

[DJK] Dougherty, R., Jackson, S., Kechris, A.S.: The structure of hyperfinite Borel equivalence relations. Trans. Amer. Math. Soc., **341**, 193–225 (1994)

[D] Dye, H.A.: On groups of measure preserving transformations, I, Amer. J. Math., **81**, 119–159 (1959), II, Amer. J. Math., **85**, 551–576 (1963)

[F] Farrell, R.H.: Representation of invariant measures. Ill.J. Math., **6**, 447–467 (1962)

[FM] Feldman, J., Moore, C.C.: Ergodic equivalence relations and von Neumann algebras, I. Trans. Amer. Math. Soc., **234**, 289–324 (1977)

[G1] Gaboriau, D.: Mercuriale de groupes et de relations. C.R. Acad. Sci. Paris, Sér. I Math., **326**, 219-222 (1998)

[G2] Gaboriau, D.: Coût des relations d'equivalence et des groupes. Inv. Math., **139**, 41–98 (2000)

[G3] Gaboriau, D.: On orbit equivalence of measure preserving actions. In: Burger, M., Iozzi, A. (ed) Rigidity in dynamics and geometry (Cambridge, 2000). Springer-Verlag, Berlin (2002)

[H] Hjorth, G.: A lemma for cost attained. Preprint (2002) (available at: www.math.ucla.edu/∼greg/research.html)

[H1] Hjorth, G.: A converse to Dye's Theorem. Preprint (2002) (available at: www.math.ucla.edu/∼greg/research.html)

[HK] Hjorth, G., Kechris, A.S.: Borel equivalence relat ions and classifications of countable models, Ann. Pure and Appl. Logic, **82**, 221–272 (1996)

[JKL] Jackson, S., Kechris, A.S., and Louveau, A.: Countable Borel equivalence relations. J. Math. Logic, **2**, 1–80 (2002)

[Ka] Kaimanovich, V.A.: Amenability, hyperfiniteness, and isoperimetric inequalities. C. R. Acad. Sci. Paris Sér. I Math., **325**, 999–1004 (1997)

[K] Kechris, A.S.: Classical Descriptive Set Theory. Springer-Verlag, New York (1995)

[K1] Kechris, A.S.: On the classification problem for rank 2 torsion-free abelian groups, J. London Math. Soc., **62**, 437–450 (2000)

[Kr] Krieger, W.: On non-singular transformations of a measure space, I, Z. Wahrschein. veew. Geb., **11**, 83–97 (1969), II, Z. Wahrschein. veew. Geb., **11**, 98–119 (1969)

[L] Levitt, G.: On the cost of generating an equivalence relation. Ergod. Th. and Dynam. Syst., **15**, 1173–1181 (1995)

[M] Miller, B.D.: Full groups, classification, and equivalence relations, Ph.D. Thesis, Berkeley (2004)

[N] Nadkarni, M.: On the existence of a finite invariant measure. Proc. Indian Acad. Sci. Math. Sci., **100**, 203–220 (1990)

[OW] Ornstein, D., Weiss, B.: Ergodic theory and amenable group actions, I: The Rohlin lemma. Bull. Amer. Math. Soc., **2**, 161–164 (1980)

[P] Pavelich, J.: Commuting equivalence relations and scales for differentiable functions, Ph.D. Thesis, Caltech (2002)

[P^2] Pemantle R., Peres, Y.: Nonamenable products are not treeable. Israel J. Math., **118**, 147–155 (2000)

[S] Serre, J.P.: Trees. Springer-Verlag, Berlin (2003)

[S^2] Slaman, T., Steel, J.: Definable functions on degrees. In: Kechris, A.S., Martin, D.A., Steel, J.R. (ed) Cabal Seminar, 81–85 (Lecture Notes in Mathematics 1333). Springer-Verlag, Berlin (1988)

[SW^2] Sullivan, D., Weiss, B., Wright, J.D.M.: Generic dynamics and monotone complete C^*-algebras. Trans. Amer. Math. Soc., **295(2)**, 795–809 (1986)

[Su] Sutherland, C.E.: Notes on orbit equivalence: Krieger's Theorem. Lecture Note Series, **23**, Mathematical Institute, University of Oslo (1976)

[V] Varadarajan, V.S.: Groups of automorphisms of Borel spaces. Trans. Amer. Math. Soc., **109**, 191–220 (1963)

[W] Wagon, S.: The Banach-Tarski Paradox. Cambridge Univ. Press, Cambridge (1993)

[We] Weiss, B.: Measurable Dynamics. Contemp. Math., **26**, 395–421 (1984)

[Wr] Wright, D.: Generic countable equidecomposability. Quart. J. Math. Oxford, **42**, 125–128, (1991)

[Z] Zimmer, R.: Ergodic Theory and Semisimple Groups. Birkhäuser-Verlag, Basel (1984)

Index