PERIODIC PERMUTATIONS AND THE SUCCESSOR

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Abstract. We investigate pairs of conjugacy classes of periodic permutations of $\mathbb{Z}$ whose product contains the successor function.

Introduction

Let $\mathrm{Sym}(X)$ denote the *symmetric group* of all permutations of X. The *orbit* of a point $x \in X$ under a permutation τ of X is given by $[x]_\tau = \{\tau^i(x) \mid i \in \mathbb{Z}\}$. We say that τ is *almost trivial* if $\tau(x) = x$ for cofinitely many $x \in X$, an *almost involution* if τ^2 is almost trivial, and (σ-)*periodic* if every orbit is finite. Define $\mathrm{C}(\tau) = \sum_{x \in X} 1 - 2/|[x]_\tau|$ and $\mathrm{Cl}(\tau) = \{\sigma \circ \tau \circ \sigma^{-1} \mid \sigma \in \mathrm{Sym}(X)\}$. The *successor function* on $\mathbb{Z}$ is given by $S^{\mathbb{Z}}(i) = i + 1$ for all $i \in \mathbb{Z}$. Here we prove the following:

Theorem A. *Suppose that $\rho, \sigma \in \mathrm{Sym}(\mathbb{Z})$ are almost involutions and $S^{\mathbb{Z}} \in \mathrm{Cl}(\rho)\mathrm{Cl}(\sigma)$. Then $\mathrm{C}(\rho) + \mathrm{C}(\sigma) \geq -1$.*

Theorem B. *Suppose that $\rho, \sigma \in \mathrm{Sym}(\mathbb{Z})$ are periodic but not almost trivial and ρ or σ is not an almost involution. Then $S^{\mathbb{Z}} \in \mathrm{Cl}(\rho)\mathrm{Cl}(\sigma)$.*

The special case of Theorem B where neither ρ nor σ is an almost involution follows from [Mor89, Theorem A]. As far as I am aware, however, the special case where ρ or σ is an almost involution was not previously known. Regardless, the real purpose of this paper is to introduce ideas and language—in the simplest possible context—that can be used to investigate the finite-order elements R and S of the full group of an aperiodic Borel automorphism T with the property that $T \in \mathrm{Cl}(R)\mathrm{Cl}(S)$. This topic will be explored in a future paper.

In §1, we prove Theorem A. In §2, we note a symmetry that removes the need to repeat arguments. In §3, we establish a fact concerning elimination of fixed points. In §4, we describe the simplest finite approximations to pairs (ρ, σ) for which $S^{\mathbb{Z}} \in \mathrm{Cl}(\rho)\mathrm{Cl}(\sigma)$. In §5, we use these as building blocks to construct extensions of more general finite approximations. And in §6, we prove the special case of Theorem B where ρ or σ has finite order.

2010 *Mathematics Subject Classification.* Primary 03E15, 28A05, 37B05.

Key words and phrases. Conjugate, symmetric group, permutation, successor.

1. The proof of Theorem A

For all $R \subseteq X^2$, define $\mathrm{graph}_R(\tau) = \mathrm{graph}(\tau) \cap R$.

Proposition 1.1. *Suppose that $\preceq$ is a linear ordering of a finite set F and $\tau \in \mathrm{Sym}(F)$. Then $|\mathrm{graph}_{\succeq}(\tau)| \geq 1$ and $|\mathrm{graph}_{\prec}(\tau)| \leq |F| - 1$.*

Proof. Let x be the $\preceq$-maximal element of F. Then $x \succeq \tau(x)$, so $|\mathrm{graph}_{\succeq}(\tau)| \geq 1$. But $|\mathrm{graph}(\tau)| = |F|$, thus $|\mathrm{graph}_{\prec}(\tau)| \leq |F| - 1$. ⊠

Define $\mathcal{O}(\tau) = \{[x]_\tau \mid x \in X\}$. For all sets K of cardinals, define $\mathrm{Per}_K(\tau) = \{x \in X \mid |[x]_\tau| \in K\}$ and $\mathcal{O}_K(\tau) = \mathcal{O}(\tau \restriction \mathrm{Per}_K(\tau))$. In this context, we use k and $\geq k$ as shorthand for $\{k\}$ and $\{k, k+1, \ldots, \aleph_0\}$. Put $\mathrm{graph}'_R(\tau) = \mathrm{graph}(\tau \restriction {\sim}\mathrm{Per}_2(\tau)) \cap R$.

Proposition 1.2. *Suppose that τ is an almost involution of a set X and $\preceq$ is a binary relation on X whose restriction to each orbit of τ is a linear order. Then $\mathrm{C}(\tau) \geq |\mathrm{graph}'_{\prec}(\tau)| - |\mathrm{graph}'_{\succeq}(\tau)|$.*

Proof. As $|\mathrm{graph}'_{\prec}(\tau)| \leq \sum_{O \in \mathcal{O}_{\geq 3}(\tau)} (|O| - 1)$ and $|\mathrm{graph}'_{\succ}(\tau)| \geq |\mathcal{O}_{\geq 3}(\tau)|$ by Proposition 1.1, the desired result follows from the fact that $\mathrm{C}(\tau) = \sum_{O \in \mathcal{O}(\tau)} (|O| - 2) = \sum_{O \in \mathcal{O}_{\geq 3}(\tau)} (|O| - 1) - |\mathcal{O}_{\geq 3}(\tau)| - |\mathcal{O}_1(\tau)|$. ⊠

Given $\tau_0, \tau_1 \in \mathrm{Sym}(X)$, define $\tau_0 \amalg \tau_1 \in \mathrm{Sym}(X \times 2)$ by $(\tau_0 \amalg \tau_1)(x, k) = (\tau_k(x), k)$ for all $k < 2$ and $x \in X$. Let $\preceq$ denote any binary relation on $\mathbb{Z} \times 2$ with the property that $(i, k) \preceq (j, k) \iff i \leq j$ for all $i, j \in \mathbb{Z}$ and $k < 2$. Theorem A follows from Proposition 1.2 and:

Proposition 1.3. *Suppose that $\tau_0, \tau_1 \in \mathrm{Sym}(\mathbb{Z})$ and $S^{\mathbb{Z}} = \tau_0 \circ \tau_1$. Then $|\mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1)| \leq |\mathrm{graph}'_{\prec}(\tau_0 \amalg \tau_1)| + 1$.*

Proof. Define $I, J \colon \mathrm{graph}(\tau_0 \amalg \tau_1) \to (\mathbb{Z} \times 2)^2$ by

$$I((i,k),(j,k)) = \begin{cases} ((j,k),(i,k)) & \text{if } i, j \in \mathrm{Per}_2(\tau_k) \text{ and} \\ ((i,k),(j,k)) & \text{otherwise} \end{cases}$$

and

$$J((i,k),(j,k)) = ((j - (1-k), 1-k), (i+k, 1-k))$$

for all $i, j \in \mathbb{Z}$ and $k < 2$.

Lemma 1.4. $J(\mathrm{graph}(\tau_0 \amalg \tau_1)) \subseteq \mathrm{graph}(\tau_0 \amalg \tau_1)$.

Proof. Suppose that $((i,k),(j,k)) \in \mathrm{graph}(\tau_0 \amalg \tau_1)$. If $k = 0$, then $\tau_0(i) = j = S^{\mathbb{Z}}(j-1) = (\tau_0 \circ \tau_1)(j-1)$, so $i = \tau_1(j-1)$, thus $J((i,0),(j,0)) = ((j-1,1),(i,1)) \in \mathrm{graph}(\tau_0 \amalg \tau_1)$. If $k = 1$, then $\tau_1(i) = j$, so $\tau_0(j) = (\tau_0 \circ \tau_1)(i) = S^{\mathbb{Z}}(i) = i+1$, thus $J((i,1),(j,1)) = ((j,0),(i+1,0)) \in \mathrm{graph}(\tau_0 \amalg \tau_1)$. ⊠

Lemma 1.5. $J(\mathrm{graph}_{\prec}(\tau_0 \amalg \tau_1)) = \mathrm{graph}_{\succeq}(\tau_0 \amalg \tau_1)$.

Proof. Note that $((i,k),(j,k)) \in \mathrm{graph}_{\prec}(\tau_0 \amalg \tau_1) \iff i < j \iff j-(1-k) \geq i+k \iff J((i,k),(j,k)) \in \mathrm{graph}_{\succeq}(\tau_0 \amalg \tau_1)$. ⊠

Define $\mathbb{1}_R \colon R \to 2$ by $\mathbb{1}_R(x,y) = 1 \iff x \mathrel{R} y$. The *length* of $((i,k),(j,k)) \in \mathrm{graph}(\tau_0 \amalg \tau_1)$ is given by $|((i,k),(j,k))| = |i-j|$.

Lemma 1.6. *If* $((i,k),(j,k)) \in \mathrm{graph}(\tau_0 \amalg \tau_1)$*, then* $|J((i,k),(j,k))| = |((i,k),(j,k))| + (-1)^{\mathbb{1}_<(i,j)}$.

Proof. As $i < j \iff i+k \leq j-(1-k)$, it follows that

$$\begin{aligned}|(j-(1-k))-(i+k)| &= (-1)^{\mathbb{1}_<(i,j)}((i+k)-(j-(1-k))) \\ &= (-1)^{\mathbb{1}_<(i,j)}((i-j)+1) \\ &= |i-j| + (-1)^{\mathbb{1}_<(i,j)},\end{aligned}$$

which immediately yields the desired result. ⊠

Let G be the group generated by I and J. The *orbit* of $((i,k),(j,k))$ under G is given by $[((i,k),(j,k))]_G = \{g \cdot ((i,k),(j,k)) \mid g \in G\}$. Set $\mathcal{O}(G) = \{[((i,k),(j,k))]_G \mid ((i,k),(j,k)) \in \mathrm{graph}(\tau_0 \amalg \tau_1)\}$.

Lemma 1.7. *Suppose that* $O \in \mathcal{O}(G)$*. Then* $\mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O \neq \emptyset$.

Proof. Fix $((i,k),(j,k)) \in O$. We can assume that $((i,k),(j,k)) \notin \mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1)$. By replacing $((i,k),(j,k))$ with $I((i,k),(j,k))$ if necessary, we can therefore assume that $((i,k),(j,k)) \in \mathrm{graph}_{\prec}(\tau_0 \amalg \tau_1)$. For all $n \in \mathbb{N}$, note that if $((i_n,k_n),(j_n,k_n)) = (I \circ J)^n((i,k),(j,k))$ is in $\mathrm{graph}_{\prec}(\tau_0 \amalg \tau_1)$, then $J((i_n,k_n),(j_n,k_n)) \in \mathrm{graph}_{\succeq}(\tau_0 \amalg \tau_1)$ and $|J((i_n,k_n),(j_n,k_n))| = |((i_n,k_n),(j_n,k_n))| - 1$ by Lemmas 1.5 and 1.6. If $J((i_n,k_n),(j_n,k_n)) \notin \mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1)$, then $((i_{n+1},k_{n+1}),(j_{n+1},k_{n+1})) \in \mathrm{graph}_{\prec}(\tau_0 \amalg \tau_1)$. Set $n = |i-j|-1$ and note that if $J((i_m,k_m),(j_m,k_m)) \notin \mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1)$ for all $m < n$, then $J((i_n,k_n),(j_n,k_n)) = 0$, in which case $J((i_n,k_n),(j_n,k_n)) \in \mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1)$. ⊠

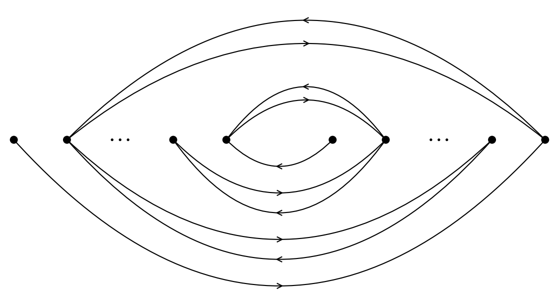

FIGURE 1. A finite orbit of G.

Lemma 1.8. *Suppose that $O \in \mathcal{O}(G)$.*

(1) *If $|O| < \aleph_0$, then $|\mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O| = |\mathrm{graph}'_{\prec}(\tau_0 \amalg \tau_1) \cap O| = 1$.*
(2) *If $|O| = \aleph_0$, then O is a cofinite subset of $\mathrm{graph}(\tau_0 \amalg \tau_1)$, $|\mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O| = 1$, and $\mathrm{graph}'_{\prec}(\tau_0 \amalg \tau_1) \cap O = \emptyset$.*

Proof. By Lemma 1.7, there exists $((i,k),(j,k)) \in \mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O$. For all $n \in \mathbb{N}$, note that if $((i_n,k_n),(j_n,k_n)) = (I \circ J)^n((i,k),(j,k))$ is in $\mathrm{graph}_{\succeq}(\tau_0 \amalg \tau_1)$, then Lemma 1.5 ensures that $J((i_n,k_n),(j_n,k_n)) \in \mathrm{graph}_{\prec}(\tau_0 \amalg \tau_1)$. And if $J((i_n,k_n),(j_n,k_n)) \notin \mathrm{graph}'_{\prec}(\tau_0 \amalg \tau_1)$, then $((i_{n+1},k_{n+1}),(j_{n+1},k_{n+1})) \in \mathrm{graph}_{\succeq}(\tau_0 \amalg \tau_1)$.

Suppose now that n is the least natural number with the property that $J((i_n,k_n),(j_n,k_n)) \in \mathrm{graph}'_{\prec}(\tau_0 \amalg \tau_1)$. Then $((i,k),(j,k))$ is in $\mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O$, $J((i_n,k_n),(j_n,k_n))$ is in $\mathrm{graph}'_{\prec}(\tau_0 \amalg \tau_1) \cap O$, and the pairs of the form $J((i_m,k_m),(j_m,k_m))$ and $((i_{m+1},k_{m+1}),(j_{m+1},k_{m+1}))$, for $m < n$, are in $\mathrm{Per}_2(\tau_0 \amalg \tau_1)^2$ and make up the rest of O, so (1) holds.

If there is no $n \in \mathbb{N}$ for which $J((i_n,k_n),(j_n,k_n)) \in \mathrm{graph}'_{\prec}(\tau_0 \amalg \tau_1)$, then $((i,k),(j,k))$ is in $\mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O$ and the pairs of the form $J((i_n,k_n),(j_n,k_n))$ and $((i_{n+1},k_{n+1}),(j_{n+1},k_{n+1}))$, for $n \in \mathbb{N}$, are in $\mathrm{Per}_2(\tau_0 \amalg \tau_1)^2$ and make up the rest of O, so $\mathrm{graph}'_{\prec}(\tau_0 \amalg \tau_1) \cap O = \emptyset$. A straightforward induction shows that $(i_{2n}, j_{2n}, k_{2n}) = (i+n, j-n, k)$ and $(i_{2n+1}, j_{2n+1}, k_{2n+1}) = (i+(n+k), j-(n+(1-k)), 1-k)$ for all $n \in \mathbb{N}$, so $\mathrm{graph}(\tau_0 \amalg \tau_1) \setminus O \subseteq (\{i, i+1, \ldots, j\} \times 2)^2$, thus (2) holds. ⊠

As at most one orbit of G can be cofinite, Lemma 1.8 ensures that $|\mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1)| = |\mathcal{O}(G)| \leq |\mathrm{graph}'_{\prec}(\tau_0 \amalg \tau_1)| + 1$. ⊠

2. Symmetry

We use $f \colon X \hookrightarrow Y$ to denote a partial injection of X into Y. For all $\sigma \colon \mathbb{Z} \hookrightarrow \mathbb{Z}$, define $\overline{\sigma} \colon \mathbb{Z} \hookrightarrow \mathbb{Z}$ by $\overline{\sigma}(i) = -\sigma^{-1}(-i)$ for all $i \in \mathbb{Z}$.

Proposition 2.1. *Suppose that $\sigma \colon \mathbb{Z} \hookrightarrow \mathbb{Z}$. Then $\sigma = \overline{\overline{\sigma}}$.*

Proof. If $i \in \mathbb{Z}$, then $\overline{\overline{\sigma}}(i) = -(\overline{\sigma})^{-1}(-i)$, so $\overline{\sigma}(-\overline{\overline{\sigma}}(i)) = -i$. But $\overline{\sigma}(-\overline{\overline{\sigma}}(i)) = -\sigma^{-1}(\overline{\overline{\sigma}}(i))$, so $i = \sigma^{-1}(\overline{\overline{\sigma}}(i))$, thus $\sigma(i) = \overline{\overline{\sigma}}(i)$. ⊠

Proposition 2.2. *Suppose that $\rho, \sigma \colon \mathbb{Z} \hookrightarrow \mathbb{Z}$. Then $\overline{\rho \circ \sigma} = \overline{\sigma} \circ \overline{\rho}$.*

Proof. Observe that

$$\begin{aligned}(\overline{\sigma} \circ \overline{\rho})(i) &= -\sigma^{-1}(-(-\rho^{-1}(-i))) \\ &= -(\sigma^{-1} \circ \rho^{-1})(-i) \\ &= -(\rho \circ \sigma)^{-1}(-i) \\ &= \overline{\rho \circ \sigma}(i)\end{aligned}$$

for all $i \in \mathbb{Z}$. ⊠

Define $\mathcal{F} = \{(\rho\colon \mathbb{Z} \rightharpoonup \mathbb{Z}, \sigma\colon \mathbb{Z} \rightharpoonup \mathbb{Z}) \mid \rho \circ \sigma = S^{\mathbb{Z}} \restriction \operatorname{dom}(\rho \circ \sigma)\}$.

Proposition 2.3. $(\rho, \sigma) \in \mathcal{F} \iff (\overline{\sigma}, \overline{\rho}) \in \mathcal{F}$.

Proof. Note that if $i \in \mathbb{Z}$ and $\rho, \sigma\colon \mathbb{Z} \rightharpoonup \mathbb{Z}$, then $(\rho \circ \sigma)(i) = i + 1 \iff (\rho \circ \sigma)^{-1}(i + 1) = i \iff \overline{\rho \circ \sigma}(-i - 1) = -i$, so the desired result follows from Proposition 2.2. ⊠

Let $(i_0 \; i_1 \; \cdots \; i_n)$ denote the permutation of $\{i_m \mid m \leq n\}$ sending i_m to i_{m+1} for all $m < n$.

Proposition 2.4. *Suppose that $n \geq 1$, $(i_m)_{m \leq n}$ is strictly increasing, $\rho = (i_0 \; i_1 \; \cdots \; i_n)$, and $\sigma = (-i_n \; -i_{n-1} \; \cdots \; -i_0)$. Then $\rho = \overline{\sigma}$.*

Proof. If $m < n$, then $\overline{\sigma}(i_m) = -\sigma^{-1}(-i_m) = -(-i_{m+1}) = i_{m+1}$. ⊠

3. Eliminating fixed points

For all $k \in \mathbb{N}$, let $\operatorname{par}(k)$ denote the remainder when k is divided by two. For all $\rho, \sigma \in \operatorname{Sym}(X)$, set $\delta(\rho, \sigma) = \{x \in X \mid \rho(x) \neq \sigma(x)\}$ and

$$\operatorname{Mal}(\rho, \sigma) = \{x \in \operatorname{Per}_{\mathbb{N}+3}(\sigma) \mid |[x]_\sigma \setminus \operatorname{Per}_1(\rho)| = 1\}.$$

Proposition 3.1. *Suppose that $m \geq 1$, ρ and σ are permutations of a set X, and $\operatorname{Mal}(\rho, \sigma) \cap \operatorname{Per}_{2\mathbb{N}+n}(\sigma)$ is empty or infinite for all $n \geq 3$. Then there exist $\rho', \sigma' \in \operatorname{Sym}(X)$ such that:*

1. $\rho \circ \sigma = \rho' \circ \sigma'$,
2. $\delta(\rho, \rho') = \delta(\sigma^{-1}, (\sigma')^{-1}) = \operatorname{Mal}(\rho, \sigma) \cap \operatorname{Per}_1(\rho)$,
3. $\operatorname{Mal}(\rho, \sigma) \cap \operatorname{Per}_1(\rho) \subseteq \operatorname{Per}_m(\rho')$, *and*
4. $\forall n \geq 3 \; \operatorname{Mal}(\rho, \sigma) \cap \operatorname{Per}_n(\sigma) \subseteq \operatorname{Per}_n(\sigma')$.

Proof. Define $Y = \operatorname{Mal}(\rho, \sigma)$ and $Z = Y \setminus \operatorname{Per}_1(\rho)$. For all $n \geq 3$, set $Y_n = \operatorname{Per}_{2\mathbb{N}+n}(\sigma) \cap Y$ and $Z_n = \operatorname{Per}_{2\mathbb{N}+n}(\sigma) \cap Z$. Fix an equivalence relation F_4 on Z_4 whose classes all have cardinality m^2, as well as $\pi_{0,1}, \pi_{0,2} \in \operatorname{Sym}(Z_4)$, whose graphs are contained in F_4, such that the orbits of $\pi_{0,1}$, $\pi_{0,2}$, and $\pi_{0,3} = (\pi_{0,1} \circ \pi_{0,2})^{-1}$ all have cardinality m. For all $n \in (\mathbb{N}+3) \setminus \{4\}$, fix an equivalence relation F_n on Z_n whose classes all have cardinality m, fix $\pi_{\operatorname{par}(n),n-2} \in \operatorname{Sym}(Z_n)$ whose orbits coincide with the equivalence classes of F_n, and set $\pi_{\operatorname{par}(n),n-1} = \pi^{-1}_{\operatorname{par}(n),n-2}$. Then the support of $\pi = \operatorname{id}_{X \setminus (Y \setminus Z)} \cup \bigcup_{n \geq 1, p < 2} \sigma^n \circ \pi_{p,n} \circ \sigma^{-n}$ is $Y \setminus Z$, so $\rho' = \rho \circ \pi$ and $\sigma' = \pi^{-1} \circ \sigma$ satisfy conditions (1)–(3).

Lemma 3.2. *Suppose that $\ell < n$. Then*

$$(\sigma')^\ell \restriction Z_n = (\sigma^\ell \circ \pi^{-1}_{\operatorname{par}(n),\ell} \circ \cdots \circ \pi^{-1}_{\operatorname{par}(n),1}) \restriction Z_n. \qquad (*)$$

Proof. The case $\ell = 0$ is trivial. If $\ell > 0$ and $(*)$ holds at $\ell - 1$, then

$$\begin{aligned}(\sigma')^\ell \restriction Z_n &= (\sigma' \circ (\sigma')^{\ell-1}) \restriction Z_n \\ &= (\sigma' \circ \sigma^{\ell-1} \circ \pi^{-1}_{\mathrm{par}(n),\ell-1} \circ \cdots \circ \pi^{-1}_{\mathrm{par}(n),1}) \restriction Z_n \\ &= (\pi^{-1} \circ \sigma^\ell \circ \pi^{-1}_{\mathrm{par}(n),\ell-1} \circ \cdots \circ \pi^{-1}_{\mathrm{par}(n),1}) \restriction Z_n \\ &= (\sigma^\ell \circ \pi^{-1}_{\mathrm{par}(n),\ell} \circ \sigma^{-\ell} \circ \sigma^\ell \circ \pi^{-1}_{\mathrm{par}(n),\ell-1} \circ \cdots \circ \pi^{-1}_{\mathrm{par}(n),1}) \restriction Z_n \\ &= (\sigma^\ell \circ \pi^{-1}_{\mathrm{par}(n),\ell} \circ \cdots \circ \pi^{-1}_{\mathrm{par}(n),1}) \restriction Z_n,\end{aligned}$$

so $(*)$ also holds at ℓ. ⊠

For all $n \geq 3$, set $Y'_n = \mathrm{Per}_n(\sigma) \cap Y$ and $Z'_n = \mathrm{Per}_n(\sigma) \cap Z$. Lemma 3.2 ensures that $Y'_n = \bigcup_{\ell<n} \sigma^\ell(Z'_n) = \bigcup_{\ell<n} (\sigma')^\ell(Z'_n)$ and

$$\begin{aligned}(\sigma')^n \restriction Z'_n &= (\sigma' \circ (\sigma')^{n-1}) \restriction Z'_n \\ &= (\sigma' \circ \sigma^{n-1} \circ \pi^{-1}_{\mathrm{par}(n),n-1} \circ \cdots \circ \pi^{-1}_{\mathrm{par}(n),1}) \restriction Z'_n \\ &= (\sigma' \circ \sigma^{n-1}) \restriction Z'_n \\ &= (\sigma' \circ \sigma^{-1}) \restriction Z'_n \\ &= \mathrm{id}_{Z'_n},\end{aligned}$$

so condition (4) also holds. ⊠

We write $\rho \cong \sigma$ to indicate that ρ and σ are isomorphic.

Proposition 3.3. *Suppose that $m \geq 1$, ρ and σ are permutations of a set X, $\mathrm{Mal}(\rho, \sigma) \cap \mathrm{Per}_{2\mathbb{N}+n}(\sigma)$ is empty or infinite for all $n \geq 3$, and $|\mathrm{Per}_m(\rho)| \geq |\mathrm{Mal}(\rho, \sigma)|$. Then there exist $\rho', \sigma' \in \mathrm{Sym}(X)$ such that $\rho \circ \sigma = \rho' \circ \sigma'$, $\rho' \cong \rho \restriction {\sim}(\mathrm{Mal}(\rho, \sigma) \cap \mathrm{Per}_1(\rho))$, and $\sigma' \cong \sigma$.*

Proof. Proposition 3.1 yields $\rho', \sigma' \in \mathrm{Sym}(X)$ such that $\rho \circ \sigma = \rho' \circ \sigma'$ and $|\mathcal{O}_\kappa(\rho \restriction {\sim}(\mathrm{Mal}(\rho, \sigma) \cap \mathrm{Per}_1(\rho)))| = |\mathcal{O}_\kappa(\rho')|$ and $|\mathcal{O}_\kappa(\sigma)| = |\mathcal{O}_\kappa(\sigma')|$ for all cardinals κ. ⊠

4. Building blocks

For all $i, j \in \mathbb{Z}$, we slightly abuse the usual notation by using (i, j), $[i, j)$, $(i, j]$, and $[i, j]$ to denote the corresponding intervals of integers. Set $\mathcal{F}(i, j] = \{(\rho, \sigma) \in \mathcal{F} \mid \rho \colon (i, j] \hookrightarrow (i, j] \text{ and } \sigma \colon (i, j) \hookrightarrow (i, j)\}$, noting that $\forall (\rho, \sigma) \in \mathcal{F}(i, j]\ \mathrm{dom}(\rho \circ \sigma) = (i, j - 1]$.

Proposition 4.1. *If $i < j$ and $(\rho, \sigma) \in \mathcal{F}(i, j]$, then $\rho(j) = i + 1$.*

Proof. Observe that $\rho((i, j - 1]) = (\rho \circ \sigma)((i, j - 1]) = (i + 1, j]$. ⊠

Set $\mathcal{F}[i, j) = \{(\rho, \sigma) \in \mathcal{F} \mid \rho \colon (i, j) \hookrightarrow (i, j) \text{ and } \sigma \colon [i, j) \hookrightarrow [i, j)\}$, this time noting that $\forall (\rho, \sigma) \in \mathcal{F}(i, j]\ S^{\mathbb{Z}}(j-1) \notin \mathrm{rng}(\rho)$, and therefore $\forall (\rho, \sigma) \in \mathcal{F}(i, j]\ \mathrm{dom}(\rho \circ \sigma) = [i, j - 1)$.

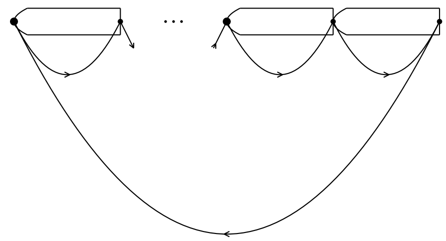

FIGURE 2. The extension provided by Proposition 4.2.

Proposition 4.2. *Suppose that* $n \geq 1$, $(i_m)_{m \leq n}$ *is strictly increasing,* $\forall m < n\ (\rho_m, \sigma_m) \in \mathcal{F}(i_m, i_{m+1}]$, $\rho = \bigcup_{m<n} \rho_m$, *and* $\sigma = (i_0\ i_1\ \cdots\ i_n) \cup \bigcup_{m<n} \sigma_m$. *Then* $(\rho, \sigma) \in \mathcal{F}[i_0, i_n + 1)$.

Proof. As $[i_0, i_n) = \{i_m \mid m < n\} \cup \bigcup_{m<n}(i_m, i_{m+1} - 1]$, it follows that

$$\begin{aligned}(\rho, \sigma) \in \mathcal{F}[i_0, i_n + 1) &\iff \forall i \in [i_0, i_n)\ (\rho \circ \sigma)(i) = i + 1 \\ &\iff \forall m < n\ (\rho \circ \sigma)(i_m) = i_m + 1 \\ &\iff \forall m < n\ \rho(i_{m+1}) = i_m + 1 \\ &\iff \forall m < n\ \rho_m(i_{m+1}) = i_m + 1,\end{aligned}$$

so Proposition 4.1 yields the desired result. ⊠

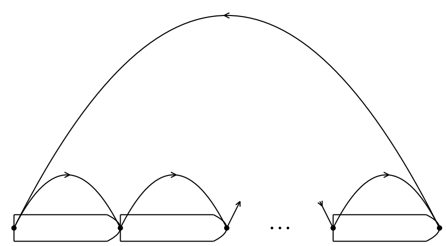

FIGURE 3. The extension provided by Proposition 4.3.

Proposition 4.3. *Suppose that* $n \geq 1$, $(i_m)_{m \leq n}$ *is strictly increasing,* $\forall m < n\ (\rho_m, \sigma_m) \in \mathcal{F}[i_m, i_{m+1})$, $\rho = (i_0\ i_1\ \cdots\ i_n) \cup \bigcup_{m<n} \rho_m$, *and* $\sigma = \bigcup_{m<n} \sigma_m$. *Then* $(\rho, \sigma) \in \mathcal{F}(i_0 - 1, i_n]$.

Proof. By Propositions 2.1, 2.3, 2.4, and 4.2. ⊠

Proposition 4.4. *Suppose that* $i \in \mathbb{Z}$. *Then* $(\emptyset, \mathrm{id}_{\{i\}}) \in \mathcal{F}[i, i+1)$ *and* $(\mathrm{id}_{\{i\}}, \emptyset) \in \mathcal{F}(i-1, i]$.

Proof. As $[i, i) = (i-1, i-1] = \emptyset$, the definitions of $\mathcal{F}[i, i+1)$ and $\mathcal{F}(i-1, i]$ yield that $(\rho, \sigma) \in \mathcal{F}[i, i+1) \iff (\rho = \emptyset$ and $\mathrm{dom}(\sigma) = \{i\})$ and $(\rho, \sigma) \in \mathcal{F}(i-1, i] \iff (\mathrm{dom}(\rho) = \{i\}$ and $\sigma = \emptyset)$. ⊠

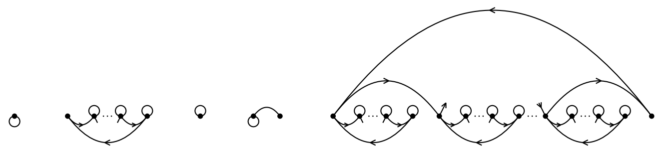

FIGURE 4. Building blocks from Propositions 4.5 and 4.6.

Proposition 4.5. *Suppose that $i < j$ are integers. Then the pair $(\mathrm{id}_{(i,j)}, (i\ i+1\ \cdots\ j-1))$ is in $\mathcal{F}[i,j)$.*

Proof. If $i+1=j$, then this follows from Proposition 4.4. Otherwise, Proposition 4.4 ensures that $(\mathrm{id}_{\{k\}}, \emptyset) \in \mathcal{F}(k-1,k]$ for all $k \in (i,j)$, so Proposition 4.2 yields the desired result. ⊠

Proposition 4.6. *Suppose that $m \geq 1$ and $(i_k)_{k<m}$ is a strictly increasing sequence of integers. Then the pair*

$$((i_0\ i_1\ \cdots\ i_{m-1}) \cup \bigcup_{k<m-1} \mathrm{id}_{(i_k,i_{k+1})}, \bigcup_{k<m-1} (i_k\ i_k+1\ \cdots\ i_{k+1}-1))$$

is in $\mathcal{F}(i_0-1, i_{m-1}]$.

Proof. If $m=1$, then this follows from Proposition 4.4. Otherwise, $(\mathrm{id}_{(i_k,i_{k+1})}, (i_k\ i_k+1\ \cdots\ i_{k+1}-1)) \in \mathcal{F}[i_k, i_{k+1})$ for all $k < m-1$ by Proposition 4.5, so Proposition 4.3 yields the desired result. ⊠

5. Extension

Given $n \geq 3$ and $\rho, \sigma \colon X \rightharpoonup X$, we say that a fixed point x of ρ is *n-malleable* if $x \in \mathrm{Per}_n(\sigma)$, $[x]_\sigma \subseteq \mathrm{dom}(\rho)$, and $|[x]_\sigma \setminus \mathrm{Per}_1(\rho)| = 1$.

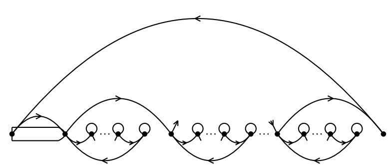

FIGURE 5. The extension provided by Proposition 5.1.

Proposition 5.1. *Suppose that $i < j$, $m \geq 2$, $n_k \geq 3$ for all $k < m-2$, and $(\rho, \sigma) \in \mathcal{F}[i,j)$. Then there exists $(\rho', \sigma') \in \mathcal{F}(i-1, j+\sum_{k<m-2} n_k]$ such that:*

- *ρ' is obtained from ρ by adding a single cycle of length m and $n_k - 1$ n_k-malleable fixed points for all $k < m-2$.*
- *σ' is obtained from σ by adding a cycle of length n_k for all $k < m-2$.*

Proof. Recursively define $i_0 = i$, $i_1 = j$, and $i_k = i_{k-1} + n_{k-2}$ for all $2 \leq k \leq m-1$. Set $(\rho_0, \sigma_0) = (\rho, \sigma)$. For all $1 \leq k \leq m-2$, Proposition 4.5 ensures that $(\rho_k, \sigma_k) = (\mathrm{id}_{(i_k, i_{k+1})}, (i_k \ i_k + 1 \ \cdots \ i_{k+1} - 1))$ is in $\mathcal{F}[i_k, i_{k+1})$. So $(\rho', \sigma') = ((i_0 \ i_1 \ \cdots \ i_{m-1}) \cup \bigcup_{k \leq m-2} \rho_k, \bigcup_{k \leq m-2} \sigma_k)$ is in $\mathcal{F}(i_0 - 1, i_{m-1}]$ by Proposition 4.3. But $i_0 - 1 = i - 1$ and $i_{m-1} = j + \sum_{k<m-2} n_k$. ⊠

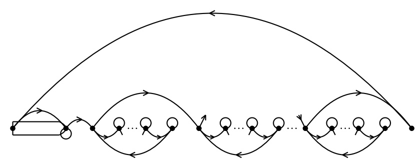

FIGURE 6. The extension provided by Proposition 5.2.

Proposition 5.2. *Suppose that $i < j$, $m \geq 3$, $n_k \geq 3$ for all $k < m-3$, and $(\rho, \sigma) \in \mathcal{F}[i, j)$. Then there exists $(\rho', \sigma') \in \mathcal{F}(i - 1, j + 1 + \sum_{k<m-3} n_k]$ such that:*

- *ρ' is obtained from ρ by adding a single cycle of length m and $n_k - 1$ n_k-malleable fixed points for all $k < m-3$.*
- *σ' is obtained from σ by adding a fixed point and a cycle of length n_k for all $k < m-3$.*

Proof. Recursively define $i_0 = i$, $i_1 = j$, $i_2 = j+1$, and $i_k = i_{k-1} + n_{k-3}$ for all $3 \leq k \leq m-1$. Set $(\rho_0, \sigma_0) = (\rho, \sigma)$. For all $1 \leq k \leq m-2$, Proposition 4.5 ensures that $(\rho_k, \sigma_k) = (\mathrm{id}_{(i_k, i_{k+1})}, (i_k \ i_k{+}1 \ \cdots \ i_{k+1}{-}1))$ is in $\mathcal{F}[i_k, i_{k+1})$. So $(\rho', \sigma') = ((i_0 \ i_1 \ \cdots \ i_{m-1}) \cup \bigcup_{k \leq m-2} \rho_k, \bigcup_{k \leq m-2} \sigma_k)$ is in $\mathcal{F}(i_0 - 1, i_{m-1}]$ by Proposition 4.3. But $i_0 - 1 = i - 1$ and $i_{m-1} = j + 1 + \sum_{k<m-3} n_k$. ⊠

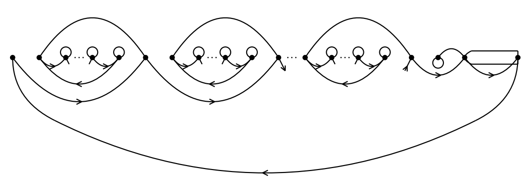

FIGURE 7. The extension provided by Proposition 5.3.

Proposition 5.3. *Suppose that $i < j$, $n \geq 3$, $n_k \geq 3$ for all $k < n-3$, and $(\rho, \sigma) \in \mathcal{F}(i, j]$. Then there exists $(\rho', \sigma') \in \mathcal{F}[i - 2 - \sum_{k<n-3}(n_k + 1), j + 1)$ such that:*

- *ρ' is obtained from ρ by adding $n-2$ cycles of length two and $n_k - 1$ n_k-malleable fixed points for all $k < n-3$.*

- *σ' is obtained from σ by adding a single fixed point, a cycle of length n, and a cycle of length of n_k for all $k < n - 3$.*

Proof. Recursively define $i_{n-1} = j$, $i_{n-2} = i$, $i_{n-3} = i - 2$, and $i_k = i_{k+1} - (n_k + 1)$ for all $k \leq n - 4$. Set $(\rho_{n-2}, \sigma_{n-2}) = (\rho, \sigma)$. For all $k \leq n - 3$, Proposition 4.6 implies that

$$(\rho_k, \sigma_k) = (\mathrm{id}_{(i_k+1, i_{k+1})} \cup (i_k + 1 \ i_{k+1}), (i_k + 1 \ i_k + 2 \ \cdots \ i_{k+1} - 1))$$

is in $\mathcal{F}(i_k, i_{k+1}]$. So $(\rho', \sigma') = (\bigcup_{k \leq n-2} \rho_k, (i_0 \ i_1 \ \cdots \ i_{n-1}) \cup \bigcup_{k \leq n-2} \sigma_k)$ is in $\mathcal{F}[i_0, i_{n-1}+1)$ by Proposition 4.2. But $i - 2 - \sum_{k<n-3}(n_k + 1) = i_0$ and $j + 1 = i_{n-1} + 1$. ⊠

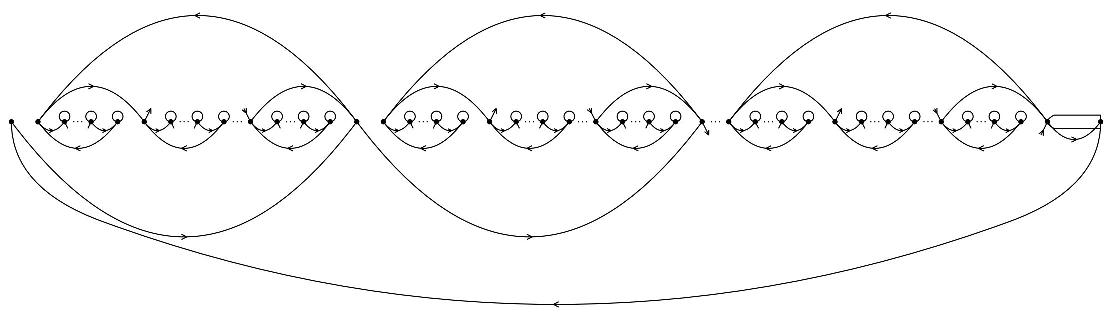

FIGURE 8. The extension provided by Proposition 5.4.

Proposition 5.4. *Suppose that $i < j$, $m \geq 2$, $n \geq 2$, $n_{k,\ell} \geq 3$ for all $k < m - 1$ and $\ell < n - 2$, and $(\rho, \sigma) \in \mathcal{F}(i, j]$. Then there exists $(\rho', \sigma') \in \mathcal{F}[i - \sum_{\ell<n-2}(1 + \sum_{k<m-1} n_{k,\ell}), j + 1)$ such that:*

- *ρ' is obtained from ρ by adding $n - 2$ cycles of length m and $n_{k,\ell} - 1$ $n_{k,\ell}$-malleable fixed points for all $k < m - 1$ and $\ell < n - 2$.*
- *σ' is obtained from σ by adding a cycle of length n and a cycle of length $n_{k,\ell}$ for all $k < m - 1$ and $\ell < n - 2$.*

Proof. Recursively define $i_{n-1} = j$, $i_{n-2} = i$, $i_{m-1,\ell} = i_{\ell+1}$, $i_{k,\ell} = i_{k+1,\ell} - n_{k,\ell}$, and $i_\ell = i_{0,\ell} - 1$ for all $k \leq m - 2$ and $\ell \leq n - 3$. Set $(\rho_{n-2}, \sigma_{n-2}) = (\rho, \sigma)$. For all $\ell \leq n-3$, Proposition 4.6 implies that the pair (ρ_ℓ, σ_ℓ), given by $\rho_\ell = (i_{0,\ell} \ i_{1,\ell} \ \cdots \ i_{m-1,\ell}) \cup \bigcup_{k<m-1} \mathrm{id}_{(i_{k,\ell}, i_{k+1,\ell})}$ and $\sigma_\ell = \bigcup_{k<m_\ell-1}(i_{k,\ell} \ i_{k,\ell}+1 \ \cdots \ i_{k+1,\ell}-1)$, is in $\mathcal{F}(i_\ell, i_{\ell+1}]$. So Proposition 4.2 yields that $(\rho', \sigma') = (\bigcup_{\ell \leq n-2} \rho_\ell, (i_0 \ i_1 \ \cdots \ i_{n-1}) \cup \bigcup_{\ell \leq n-2} \sigma_\ell)$ is in $\mathcal{F}[i_0, i_{n-1} + 1)$. But $i - \sum_{\ell<n-2}(1 + \sum_{k<m-1} n_{k,\ell}) = i_0$ and $j + 1 = i_{n-1} + 1$. ⊠

We say that a fixed point x of ρ is *anti-malleable* if $x \in \mathrm{Per}_2(\sigma)$, $[x]_\sigma \subseteq \mathrm{dom}(\rho)$, and $|[x]_\sigma \setminus \mathrm{Per}_1(\rho)| = 1$.

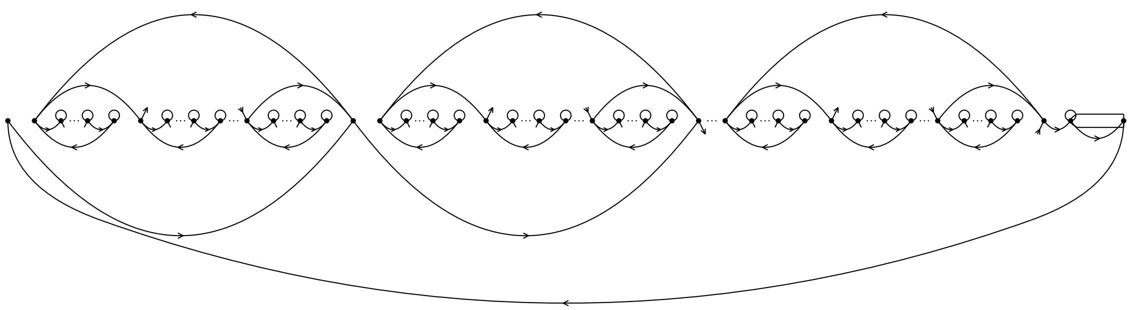

FIGURE 9. The extension provided by Proposition 5.5.

Proposition 5.5. *Suppose that* $i < j$, $m \geq 2$, $n \geq 3$, $n_{k,\ell} \geq 3$ *for all* $k < m-1$ *and* $\ell < n-3$, *and* $(\rho, \sigma) \in \mathcal{F}(i, j]$. *Then there exists* $(\rho', \sigma') \in \mathcal{F}[i - 1 - \sum_{\ell < n-3}(1 + \sum_{k<m-1} n_{k,\ell}), j + 1)$ *such that:*

- ρ' *is obtained from* ρ *by adding a single anti-malleable fixed point,* $n-3$ *cycles of length* m, *and* $n_{k,\ell} - 1$ $n_{k,\ell}$*-malleable fixed points for all* $k < m-1$ *and* $\ell < n-3$.
- σ' *is obtained from* σ *by adding a cycle of length* n *and a cycle of length* $n_{k,\ell}$ *for all* $k < m-1$ *and* $\ell < n-3$.

Proof. Recursively define $i_{n-1} = j$, $i_{n-2} = i$, $i_{n-3} = i - 1$, $i_{m-1,\ell} = i_{\ell+1}$, $i_{k,\ell} = i_{k+1,\ell} - n_{k,\ell}$, and $i_\ell = i_{0,\ell} - 1$ for all $k \leq m-2$ and $\ell \leq n-4$. Set $(\rho_{n-2}, \sigma_{n-2}) = (\rho, \sigma)$. For all $\ell \leq n-3$, Proposition 4.6 implies that the pair (ρ_ℓ, σ_ℓ), given by $\rho_\ell = (i_{0,\ell}\ i_{1,\ell}\ \cdots\ i_{m-1,\ell}) \cup \bigcup_{k<m-1} \mathrm{id}_{(i_{k,\ell}, i_{k+1,\ell})}$ and $\sigma_\ell = \bigcup_{k<m-1}(i_{k,\ell}\ i_{k,\ell}+1\ \cdots\ i_{k+1,\ell}-1)$, is in $\mathcal{F}(i_\ell, i_{\ell+1}]$. So $(\rho', \sigma') = (\bigcup_{\ell \leq n-2} \rho_\ell, (i_0\ i_1\ \cdots\ i_{n-1}) \cup \bigcup_{\ell \leq n-2} \sigma_\ell)$ is in $\mathcal{F}[i_0, i_{n-1}+1)$ by Proposition 4.2. But $i - 1 - \sum_{\ell<n-3}(1 + \sum_{k<m-1} n_{k,\ell}) = i_0$ and $j + 1 = i_{n-1} + 1$. ⊠

6. THE PROOF OF THEOREM B

For all integers $i < j$, set $\mathcal{F}_0(i, j) = \mathcal{F}[i, j)$ and $\mathcal{F}_1(i, j) = \mathcal{F}(i, j]$.

Theorem 6.1. *Suppose that* $m \geq 2$, $\rho, \sigma \in \mathrm{Sym}(\mathbb{Z})$ *are periodic, and* $\mathrm{Per}_m(\rho)$ *and* $\mathrm{Per}_{\geq 3}(\sigma)$ *are infinite. Then* $S^{\mathbb{Z}} \in \mathrm{Cl}(\rho)\mathrm{Cl}(\sigma)$.

Proof. Fix an enumeration $(\pi_n, O_n)_{n\in\mathbb{N}}$ of the pairs of the form (π, O), where $\pi \in \{\rho, \sigma\}$ and $O \in \mathcal{O}(\pi)$. Then there exist an infinite set $N \subseteq \mathbb{N}$ and $p < 2$ such that $\pi_n = \sigma$, $\mathrm{par}(|O_n|) = p$, and $3 \leq |O_n| \leq |O_{n+1}|$ for all $n \in N$. Fix $n_{-1} \in N$, set $N_0 = \mathbb{N} \setminus \{n_{-1}\}$, and apply Proposition 4.5 to find $i_0 < j_0$ and $(\rho_0, \sigma_0) \in \mathcal{F}[i_0, j_0)$ such that every point of $\mathrm{dom}(\rho_0)$ is an n_{-1}-malleable fixed point and σ_0 is a cycle of length $|O_{n_{-1}}|$.

Suppose that k is a natural number for which we have found $i_k < j_k$, a cofinite set $N_k \subseteq \mathbb{N}$, and $(\rho_k, \sigma_k) \in \mathcal{F}_{\mathrm{par}(k)}(i_k, j_k)$. If $k \in 2\mathbb{N}$, then let n_k be the least element of N_k for which ($\pi_{n_k} = \rho$ and $|O_{n_k}| \geq 2$) or ($\pi_{n_k} = \sigma$, $|O_{n_k}| = 1$, and $m \geq 3$). If $k \in 4\mathbb{N}+1$, then let n_k be the least element of N_k for which ($\pi_{n_k} = \sigma$, $|O_{n_k}| = 1$, and $m = 2$), ($\pi_{n_k} = \sigma$ and $|O_{n_k}| = 2$), or ($\pi_{n_k} = \rho$ and $|O_{n_k}| = 1$). And if $k \in 4\mathbb{N}+3$, then let n_k be the least element of N_k for which $\pi_{n_k} = \sigma$ and $|O_{n_k}| \geq 3$. Propositions 5.1–5.5 ensure that, for some $\ell_k \in \mathbb{N}$ and all $F_k \subseteq N \cap (N_k \setminus \{n\})$ of cardinality ℓ_k, there exist $i_{k+1} < i_k$, $j_{k+1} > j_k$, and $(\rho_{k+1}, \sigma_{k+1}) \in \mathcal{F}_{\mathrm{par}(k+1)}(i_{k+1}, j_{k+1})$ such that:

- ρ_{k+1} is obtained from ρ_k by adding cycles of length m and $|O_n|-1$ $|O_n|$-malleable fixed points for all $n \in F_k$, as well as a cycle of length $|O_{n_k}|$ if $\pi_{n_k} = \rho$ (which is an anti-malleable fixed point if $|O_{n_k}| = 1$).
- σ_{k+1} is obtained from σ_k by adding a cycle of length m for all $n \in F_k$, as well as a cycle of length $|O_n|$ if $\pi_{n_k} = \sigma$.

Set $N_{k+1} = N_k \setminus (F_k \cup \{n_k\})$.

Define $\rho_\infty = \bigcup_{k\in\mathbb{N}} \rho_k$ and $\sigma_\infty = \bigcup_{k\in\mathbb{N}} \sigma_k$. As $(i_k)_{k\in\mathbb{N}}$ is strictly decreasing and $(j_k)_{k\in\mathbb{N}}$ is strictly increasing, these are permutations of $\mathbb{Z}$ whose composition is $S^{\mathbb{Z}}$. As $\ell_k \geq 1$ for all $k \in 4\mathbb{N}+3$, it follows that $|\mathrm{Mal}(\rho_\infty, \sigma_\infty) \cap \mathrm{Per}_{2\mathbb{N}+n}(\sigma_\infty)| \in \{0, \aleph_0\}$ for all $n \in 2\mathbb{N}+p$. And clearly $\mathrm{Mal}(\rho_\infty, \sigma_\infty) \cap \mathrm{Per}_{2\mathbb{N}+(1-p)}(\sigma_\infty) = \emptyset$. As the fact that $\bigcap_{k\in\mathbb{N}} N_k = \emptyset$ ensures that $\rho_\infty \restriction {\sim}(\mathrm{Mal}(\rho_\infty, \sigma_\infty) \cap \mathrm{Per}_1(\rho_\infty)) \cong \rho$ and $\sigma_\infty \cong \sigma$, Proposition 3.3 yields conjugates ρ' of ρ and σ' of σ for which $\rho' \circ \sigma' = \rho_\infty \circ \sigma_\infty = S^{\mathbb{Z}}$. ⊠

The special case of Theorem B where ρ or σ has finite order follows from Propositions 2.1 and 2.3 and Theorem 6.1. As almost involutions have finite order, the full version follows from [Mor89, Theorem A].

References

[Mor89] G. Moran, *Conjugacy classes whose square is an infinite symmetric group*, Trans. Amer. Math. Soc. **316** (1989), no. 2, 493–522.

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