A FIRST-ORDER CHARACTERIZATION OF THE EXISTENCE OF INVARIANT MEASURES

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Abstract. We give first-order properties of the full group of an aperiodic countable Borel equivalence relation that characterize the existence of an invariant probability measure.

A *Polish space* is a second-countable topological space that admits a compatible complete metric. A *Borel space* is a set X equipped with a σ-algebra of subsets of X, referred to as the *Borel (sub)sets* of X. Such a space is *standard* if its Borel sets are generated by a Polish topology on X. A function $f \colon X \to Y$ between Borel spaces is *Borel* if preimages of Borel sets are Borel. A *Borel automorphism* of X is a Borel bijection $T \colon X \to X$ for which T^{-1} is also Borel. A *Borel probability measure* on X is a probability measure μ on the Borel subsets of X. Define an equivalence relation $\sim_\mu$ on the group of Borel automorphisms of X by $S \sim T \iff \mu(\{x \in X \mid S(x) \neq T(x)\}) = 0$.

Following the usual abuse of language, we say that an equivalence relation E on X is *countable* if all of its classes are countable. Such an equivalence relation is *aperiodic* if all of its classes are infinite. The *E-saturation* of a set $Y \subseteq X$ is given by $[Y]_E = \{x \in X \mid \exists y \in Y \; x \mathrel{E} y\}$. We say that Y is *E-complete* if $X = [Y]_E$. A *transversal* of E is a set $Y \subseteq X$ that intersects every E-class in exactly one point. We say that E is *smooth* if it admits a Borel transversal. Given Borel sets $A, B \subseteq X$ and $m, n \in \mathbb{Z}^+$, we write $mA \preceq_E nB$ if there is a Borel injection $\phi \colon m \times A \to n \times B$ for which $\mathrm{proj}_{A \times B}(\mathrm{graph}(\phi)) \subseteq E$. We write $mA \prec_E nB$ if there is such a map ϕ with the further property that $\mathrm{proj}_B((n \times B) \setminus \phi(m \times A))$ is $(E \restriction [A]_E)$-complete. We also write A and B instead of $1A$ and $1B$ and say that E is *compressible* if $X \prec_E X$.

The *full group* of E is the group $[E]$ of Borel automorphisms $T \colon X \to X$ whose graphs are contained in E. The measure-theoretic analog is given by $[E]_\mu = [E]/{\sim_\mu}$. By [MR07], two aperiodic countable Borel equivalence relations on standard Borel spaces are Borel isomorphic if and only if their full groups are isomorphic; Dye's reconstruction

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theorem (see, for example, [Kec10, Theorem 4.1]) yields the analogous result in the measure-theoretic context. Still, one can ask whether a given natural property of countable Borel equivalence relations corresponds to a natural property of full groups.

We say that μ is *E-invariant* if $\mu = T_*\mu$ for all $T \in [E]$ and *E-quasi-invariant* if $\mu \sim T_*\mu$ for all $T \in [E]$. Understanding the circumstances under which there is an E-invariant Borel probability measure is a basic problem going back to the roots of ergodic theory. The generalization of the first result in this direction (see [Hop32]) from Borel automorphisms to countable Borel equivalence relations ensures that there is an E-invariant Borel probability measure that is absolutely continuous with respect to a given E-quasi-invariant Borel probability measure μ if and only if there is no μ-conull Borel set $C \subseteq X$ for which $E \upharpoonright C$ is compressible. This eventually led to the stronger result that there is an E-invariant Borel probability measure if and only if E is not compressible (see [Nad90]).

Given a group G and $g \in G$, we use $\mathrm{Cl}(g)$ to denote the conjugacy class of g, we say that g is an *involution* if $g^2 = 1_G$, and we use $\mathrm{Inv}(G)$ to denote the set of all such involutions. A characterization of the class of countable Borel equivalence relations on standard Borel spaces that admit an invariant Borel probability measure in terms of a second-order property of full groups (a strong version of the Bergman property) appeared in [Mil21, Theorems 9 and 10]. Here we note several first-order properties that serve the same purpose:

Theorem 1. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, μ is an E-quasi-invariant Borel probability measure on X that concentrates off of Borel sets on which E is smooth, $m \geq 3$, and $n \geq 5$. Then the following are equivalent:*

(0) *There is an E-invariant Borel probability measure $\nu \ll \mu$.*
(1) *There exists $I \in \mathrm{Inv}([E]_\mu)$ with the property that m is the least natural number for which $\mathrm{Inv}([E]_\mu) \subseteq \mathrm{Cl}(I)^m$.*
(2) *There exists $I \in \mathrm{Inv}([E]_\mu)$ with the property that n is the least natural number for which $[E]_\mu = \mathrm{Cl}(I)^n$.*
(3) *There exists $T \in [E]_\mu$ with the property that n is the least natural number for which $[E]_\mu = \mathrm{Cl}(T)^n$.*

Theorem 2. *Suppose that X is a standard Borel space, E is an aperiodic countable Borel equivalence relation on X, $m \geq 4$, and $n \geq 5$. Then the following are equivalent:*

(0) *There is an E-invariant Borel probability measure on X.*
(1) *There exists $I \in \mathrm{Inv}([E])$ with the property that m is the least natural number for which $\mathrm{Inv}([E]) \subseteq \mathrm{Cl}(I)^m$.*

(2) *There exists $I \in \mathrm{Inv}([E])$ with the property that n is the least natural number for which $[E] = \mathrm{Cl}(I)^n$.*
(3) *There exists $T \in [E]$ with the property that n is the least natural number for which $[E] = \mathrm{Cl}(T)^n$.*

The equivalence of conditions (0) and (1) in Theorem 1 is a consequence of the following two facts:

Proposition 3. *Suppose that X is a standard Borel space, E is an aperiodic countable Borel equivalence relation on X, and $n \in \mathbb{Z}^+$. Then E is compressible if and only if $X \preceq_E (n+1)B \implies X \preceq_E nB$ for all Borel sets $B \subseteq X$.*

Proposition 4. *Suppose that X is a standard Borel space, E is an aperiodic countable Borel equivalence relation on X, $I \in \mathrm{Inv}([E])$, and $n \geq 2$. Then the following are equivalent:*
(a) *$X \preceq_E n(\mathrm{supp}(I))$.*
(b) *For all $J \in \mathrm{Inv}([E])$, there is an E-invariant Borel set $B \subseteq X$ such that $E \restriction \sim B$ is smooth and $J \restriction B \in \mathrm{Cl}(I \restriction B)^n$.*

Propositions 3 and 4 follow from fairly straightforward arguments utilizing known results on compressibility and uniformization.

The equivalence of conditions (0) and (1) in Theorem 2 is a consequence of Propositions 3 and 4 and the fact that $\mathrm{Inv}(S_\infty) \subseteq \mathrm{Cl}(\iota)^3$ for all $\iota \in \mathrm{Inv}(S_\infty)$ with infinite support (see [Mor88]).

Proposition 3 and the following fact yield the equivalence of condition (0) and the strengthening of condition (1) where $\mathrm{Cl}(I)^m$ contains every element of $[E]_\mu$ of finite order in Theorem 1:

Proposition 5. *Suppose that X is a standard Borel space, E is an aperiodic countable Borel equivalence relation on X, $I \in \mathrm{Inv}([E])$, and $n \geq 2$. Then the following are equivalent:*
(a′) *$2X \preceq_E n(\mathrm{supp}(I))$.*
(b′) *For all $T \in [E]$ of finite order, there is an E-invariant Borel set $B \subseteq X$ such that $E \restriction \sim B$ is smooth and $T \restriction B \in \mathrm{Cl}(I \restriction B)^n$.*

The fact that (b′) $\implies$ (a′) follows from Levitt's formula for the cost of hyperfinite equivalence relations (see, for example, [KM04, Theorem 20.1]). The special case of (a′) $\implies$ (b′) in which n is an even number other than two follows from Propositions 3 and 4 and the following special case of [Mil21, Proposition 1.1]:

Proposition 6. *Suppose that X is a standard Borel space and $T \colon X \to X$ is a Borel automorphism. Then there there are involutions $I, J \in [E_T^X]$ for which $T = I \circ J$ if and only if E_T^X is smooth.*

The proof of the special case where n is odd can be established using the idea behind the proof of Proposition 4. The special case where $n = 2$ can be established by also considering multiple ways of writing permutations as compositions of two involutions.

The equivalence of conditions (0) and (2) in Theorem 1 is a consequence of Propositions 3 and 5 and:

Proposition 7. *Suppose that X is a standard Borel space, E is an aperiodic countable Borel equivalence relation on X, $I \in \mathrm{Inv}([E])$, $n \geq 3$, $2X \prec_E n(\mathrm{supp}(I))$, and $T \in [E]$. Then there is an E-invariant Borel set $B \subseteq X$ such that $E \restriction \sim B$ is smooth and $T \restriction B \in \mathrm{Cl}(I \restriction B)^n$.*

We say that a set $Y \subseteq X$ is $T^{\pm 1}$-*complete* if $X = \bigcup_{n \in \mathbb{N}} T^n(Y) = \bigcup_{n \in \mathbb{N}} T^{-n}(Y)$. One can establish Proposition 7 using the proof of Proposition 5 and the following special case of [Mil21, Proposition 1.18]:

Proposition 8. *Suppose that X is a standard Borel space, $T \colon X \to X$ is Borel, and $B \subseteq X$ is a $T^{\pm 1}$-complete Borel set. Then there exists $I \in \mathrm{Inv}([E_T^X])$ for which $\mathrm{supp}(I) \subseteq B$ and $I \circ T$ is periodic.*

The equivalence of conditions (0) and (2) in Theorem 2 is a consequence of Propositions 3, 5, and 7 and the fact that $S_\infty = \mathrm{Cl}(\tau)^4$ for all $\tau \in S_\infty$ with infinite support (see [Ber73]).

The equivalence of conditions (0) and (2) and the following generalization of [Ber73] yield the equivalence of conditions (0) and (3) in Theorems 1 and 2:

Proposition 9. *Suppose that X is a standard Borel space, E is an aperiodic countable Borel equivalence relation on X, $T \in [E]$, and $2X \prec_E 3(\mathrm{supp}(T))$. Then there exists $S \in \mathrm{Cl}(T)^2$ with the property that $[E] = \mathrm{Cl}(S)^2$.*

In addition to relying upon the main result of [Mor89], the proof of Proposition 9 breaks naturally into two pieces. The first is the following consequence of Proposition 8:

Proposition 10. *Suppose that X is a standard Borel space, E is an aperiodic countable Borel equivalence relation, $T \in [E]$, and $2X \prec_E 3(\mathrm{supp}(T))$. Then there exists $S \in \mathrm{Cl}(T)^2$ of finite order such that every orbit of S has cardinality at least three and*

$$\forall x \in X \exists n \in \mathbb{N} \forall m \in \{n, 2n\} \ |\{y \in [x]_E \mid |[y]_S| = m\}| = \aleph_0.$$

The second is a special case of a generalization of [Mil21, Theorem 3]) whose proof is quite involved and has not been written up:

Theorem 11. *Suppose that X is a standard Borel space, E is an aperiodic countable Borel equivalence relation on X, $R, S, T \in [E]$, R and S have finite order, and R and S^2 are fixed-point free. Then there is an E-invariant Borel set $B \subseteq X$ such that $E \restriction {\sim}B$ is smooth and $T \restriction B \in \mathrm{Cl}(R \restriction B)\mathrm{Cl}(S \restriction B)$.*

I do not know whether Theorem 1 goes through when $n = 4$, but the following consequence of Propositions 4, 6, and 7 and Theorem 11 ensure that the bounds in Theorem 1 are otherwise optimal:

Theorem 12. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, μ is an E-quasi-invariant Borel probability measure on X that concentrates off of Borel sets on which E is smooth, $m \leq 2$, and $n \leq 3$. Then:*

1. *There exists $I \in \mathrm{Inv}([E]_\mu)$ for which m is the least natural number such that $\mathrm{Inv}([E]_\mu) \subseteq \mathrm{Cl}(I)^m$ if and only if $m = 2$.*
2. *There exists $I \in \mathrm{Inv}([E]_\mu)$ for which n is the least natural number such that $\mathrm{Inv}([E]_\mu) \subseteq \mathrm{Cl}(I)^n$ if and only if $n = 3$.*
3. *There exists $T \in [E]_\mu$ for which n is the least natural number such that $\mathrm{Inv}([E]_\mu) \subseteq \mathrm{Cl}(T)^n$ if and only if $n \geq 2$.*

The following consequence of Propositions 4, 6, and 7, Theorem 11, [Mor88], and [Mor89] ensure that the bounds in Theorem 2 are optimal:

Theorem 13. *Suppose that X is a standard Borel space, E is an aperiodic countable Borel equivalence relation on X, $m \leq 3$, and $n \leq 4$. Then:*

1. *There exists $I \in \mathrm{Inv}([E])$ for which m is the least natural number such that $\mathrm{Inv}([E]) \subseteq \mathrm{Cl}(I)^m$ if and only if $m \geq 2$.*
2. *There exists $I \in \mathrm{Inv}([E])$ for which n is the least natural number such that $\mathrm{Inv}([E]) \subseteq \mathrm{Cl}(I)^n$ if and only if $n \geq 3$.*
3. *There exists $T \in [E]$ for which n is the least natural number such that $\mathrm{Inv}([E]) \subseteq \mathrm{Cl}(T)^n$ if and only if $n \geq 2$.*

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