PERIODIC PERMUTATIONS AND THE SUCCESSOR

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Abstract. We investigate pairs of conjugacy classes of periodic permutations of $\mathbb{Z}$ whose product contains the successor function.

Introduction

Let $\mathrm{Sym}(X)$ denote the *symmetric group* of all permutations of X. The *orbit* of a point $x \in X$ under a permutation τ of X is given by $[x]_\tau = \{\tau^i(x) \mid i \in \mathbb{Z}\}$. We say that τ is *almost trivial* if $\tau(x) = x$ for cofinitely many $x \in X$, an *almost involution* if τ^2 is almost trivial, and (σ-)*periodic* if every orbit is finite. Define $\mathrm{C}(\tau) = \sum_{x \in X} 1 - 2/|[x]_\tau|$ and $\mathrm{Cl}(\tau) = \{\sigma \circ \tau \circ \sigma^{-1} \mid \sigma \in \mathrm{Sym}(X)\}$. The *successor function* on $\mathbb{Z}$ is given by $S^{\mathbb{Z}}(i) = i + 1$ for all $i \in \mathbb{Z}$. Here we prove the following:

Theorem A. *Suppose that $\rho, \sigma \in \mathrm{Sym}(\mathbb{Z})$ are almost involutions and $S^{\mathbb{Z}} \in \mathrm{Cl}(\rho)\mathrm{Cl}(\sigma)$. Then $\mathrm{C}(\rho) + \mathrm{C}(\sigma) \geq -1$.*

Theorem B. *Suppose that $\rho, \sigma \in \mathrm{Sym}(\mathbb{Z})$ are periodic but not almost trivial and at most one is an almost involution. Then $S^{\mathbb{Z}} \in \mathrm{Cl}(\rho)\mathrm{Cl}(\sigma)$.*

The special case of Theorem B where neither ρ nor σ is an almost involution follows from [Mor89, Theorem A]. As far as I am aware, however, the special case where ρ or σ is an almost involution was not previously known. Regardless, the real purpose of this paper is to introduce ideas and language—in the simplest possible context—that can be used to investigate the finite-order elements R and S of the full group of an aperiodic Borel automorphism T with the property that $T \in \mathrm{Cl}(R)\mathrm{Cl}(S)$. This topic will be explored in a future paper.

In §1, we prove Theorem A. In §2, we note a symmetry that eliminates the need to repeat several arguments. In §3, we establish a fact concerning the removal of fixed points. In §4, we describe the simplest finite approximations to pairs (ρ, σ) for which $S^{\mathbb{Z}} \in \mathrm{Cl}(\rho)\mathrm{Cl}(\sigma)$. In §5, we use these as building blocks to construct extensions of more general finite approximations. And in §6, we prove the special case of Theorem B where ρ or σ has finite order.

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1. The case of two almost involutions

For all $R \subseteq X^2$, define $\mathrm{graph}_R(\tau) = \mathrm{graph}(\tau) \cap R$.

Proposition 1.1. *Suppose that $\leq$ is a linear ordering of a finite set F and $\tau \in \mathrm{Sym}(F)$. Then $|\mathrm{graph}_{\geq}(\tau)| \geq 1$ and $|\mathrm{graph}_{<}(\tau)| \leq |F| - 1$.*

Proof. Let x be the $\leq$-maximal element of F. Then $x \geq \tau(x)$, so $|\mathrm{graph}_{\geq}(\tau)| \geq 1$. But $|\mathrm{graph}(\tau)| = |F|$, thus $|\mathrm{graph}_{<}(\tau)| \leq |F| - 1$. ⊠

Define $\mathcal{O}(\tau) = \{[x]_\tau \mid x \in X\}$, $\mathrm{Per}_K(\tau) = \{x \in X \mid |[x]_\tau| \in K\}$, and $\mathcal{O}_K(\tau) = \mathcal{O}(\tau \upharpoonright \mathrm{Per}_K(\tau))$ for all sets K of cardinals.

Define $\mathrm{graph}'_R(\tau) = \mathrm{graph}(\tau \upharpoonright {\sim}\mathrm{Per}_2(\tau)) \cap R$.

Proposition 1.2. *Suppose that τ is an almost involution of a set X and $\preceq$ is a binary relation on X whose restriction to each orbit of τ is a linear order. Then $\mathrm{C}(\tau) \geq |\mathrm{graph}'_{\prec}(\tau)| - |\mathrm{graph}'_{\succeq}(\tau)|$.*

Proof. As $|\mathrm{graph}'_{\prec}(\tau)| \leq \sum_{O \in \mathcal{O}_{\geq 3}(\tau)}(|O|-1)$ and $|\mathrm{graph}'_{\succ}(\tau)| \geq |\mathcal{O}_{\geq 3}(\tau)|$ by Proposition 1.1, the desired result follows from the fact that $\mathrm{C}(\tau) = \sum_{O \in \mathcal{O}(\tau)}(|O| - 2) = \sum_{O \in \mathcal{O}_{\geq 3}(\tau)}(|O| - 1) - |\mathcal{O}_{\geq 3}(\tau)| - |\mathcal{O}_1(\tau)|$. ⊠

The *disjoint union* of $\tau_0, \tau_1 \in \mathrm{Sym}(\mathbb{Z})$ is given by $(\tau_0 \amalg \tau_1)(i, k) = (\tau_k(i), k)$ for all $i \in \mathbb{Z}$ and $k < 2$. Let $\preceq$ denote any binary relation on $\mathbb{Z} \times 2$ such that $(i, k) \preceq (j, k) \iff i \leq j$ for all $i, j \in \mathbb{Z}$ and $k < 2$. For all $i, j \in \mathbb{Z}$, we slightly abuse the usual notation by using (i, j), $[i, j)$, $(i, j]$, and $[i, j]$ to denote the corresponding intervals of integers. Theorem A follows from Proposition 1.2 and:

Proposition 1.3. *Suppose that $\tau_0, \tau_1 \in \mathrm{Sym}(\mathbb{Z})$ and $S^{\mathbb{Z}} = \tau_0 \circ \tau_1$. Then $|\mathrm{graph}'_{\prec}(\tau_0 \amalg \tau_1)| - |\mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1)| \geq -1$.*

Proof. Define $I, J \colon \mathrm{graph}(\tau_0 \amalg \tau_1) \to (\mathbb{Z} \times 2)^2$ by

$$I((i, k), (j, k)) = \begin{cases} ((j, k), (i, k)) & \text{if } i, j \in \mathrm{Per}_2(\tau_k) \text{ and} \\ ((i, k), (j, k)) & \text{otherwise} \end{cases}$$

and

$$J((i, k), (j, k)) = ((j - (1 - k), 1 - k), (i + k, 1 - k))$$

for all $i, j \in \mathbb{Z}$ and $k < 2$.

Lemma 1.4. $J(\mathrm{graph}(\tau_0 \amalg \tau_1)) \subseteq \mathrm{graph}(\tau_0 \amalg \tau_1)$.

Proof. Suppose that $((i, k), (j, k)) \in \mathrm{graph}(\tau_0 \amalg \tau_1)$.

If $k = 0$, then $\tau_0(i) = j = S^{\mathbb{Z}}(j-1) = (\tau_0 \circ \tau_1)(j-1)$, so $i = \tau_1(j-1)$, thus $J((i, 0), (j, 0)) = ((j - 1, 1), (i, 1)) \in \mathrm{graph}(\tau_0 \amalg \tau_1)$.

If $k = 1$, then $\tau_1(i) = j$, so $\tau_0(j) = (\tau_0 \circ \tau_1)(i) = S^{\mathbb{Z}}(i) = i + 1$, thus $J((i, 1), (j, 1)) = ((j, 0), (i + 1, 0)) \in \mathrm{graph}(\tau_0 \amalg \tau_1)$. ⊠

Lemma 1.5. $J(\mathrm{graph}_{\prec}(\tau_0 \amalg \tau_1)) = \mathrm{graph}_{\succeq}(\tau_0 \amalg \tau_1)$.

Proof. Note that $((i,k),(j,k)) \in \mathrm{graph}_{\prec}(\tau_0 \amalg \tau_1) \iff i < j \iff j-(1-k) \geq i+k \iff J((i,k),(j,k)) \in \mathrm{graph}_{\succeq}(\tau_0 \amalg \tau_1)$. ⊠

The *length* of $((i,k),(j,k)) \in \mathrm{graph}(\tau_0 \amalg \tau_1)$ is $|((i,k),(j,k))| = |i-j|$.

Lemma 1.6. *Suppose that* $((i,k),(j,k)) \in \mathrm{graph}(\tau_0 \amalg \tau_1)$.

(1) *If* $i < j$, *then* $|J((i,k),(j,k))| = |((i,k),(j,k))| - 1$.
(2) *If* $i \geq j$, *then* $|J((i,k),(j,k))| = |((i,k),(j,k))| + 1$.

Proof. If $i < j$, then $i+1 \leq j$, so

$$|(j-(1-k))-(i+k)| = j-(i+1) = (j-i)-1 = |i-j|-1,$$

thus $|J((i,k),(j,k))| = |((i,k),(\pi,j))| - 1$. If $i \geq j$, then $i+1 > j$, so

$$|(j-(1-k))-(i+k)| = (i+1)-j = (i-j)+1 = |i-j|+1,$$

thus $|J((i,k),(j,k))| = |((i,k),(\pi,j))| + 1$. ⊠

Let G be the group generated by I and J. The *orbit* of $((i,k),(j,k))$ under G is given by $[((i,k),(j,k))]_G = \{g \cdot ((i,k),(j,k)) \mid g \in G\}$. Set $\mathcal{O}(G) = \{[((i,k),(j,k))]_G \mid ((i,k),(j,k)) \in \mathrm{graph}(\tau_0 \amalg \tau_1)\}$.

Lemma 1.7. *Suppose that* $O \in \mathcal{O}(G)$. *Then* $\mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O \neq \emptyset$.

Proof. Fix $((i,k),(j,k)) \in O$. We can assume that $((i,k),(j,k)) \notin \mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1)$. By replacing $((i,k),(j,k))$ with $I((i,k),(j,k))$ if necessary, we can therefore assume that $((i,k),(j,k)) \in \mathrm{graph}_{\prec}(\tau_0 \amalg \tau_1)$. For all $n \in \mathbb{N}$, note that if $((i_n,k),(j_n,k)) = (I \circ J)^n((i,k),(j,k))$ is in $\mathrm{graph}_{\prec}(\tau_0 \amalg \tau_1)$, then $J((i_n,k),(j_n,k)) \in \mathrm{graph}_{\succeq}(\tau_0 \amalg \tau_1)$ and $|J((i_n,k),(j_n,k))| = |((i_n,k),(j_n,k))| - 1$ by Lemmas 1.5 and 1.6. And if $J((i_n,k),(j_n,k)) \notin \mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1)$, then $((i_{n+1},k),(j_{n+1},k)) \in \mathrm{graph}_{\prec}(\tau_0 \amalg \tau_1)$. Set $n = |i-j|-1$ and note that if $J((i_m,k),(j_m,k)) \notin \mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1)$ for all $m < n$, then $J((i_n,k),(j_n,k)) = 0$, in which case $J((i_n,k),(j_n,k)) \in \mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1)$. ⊠

Lemma 1.8. *Suppose that* $O \in \mathcal{O}(G)$.

(1) *If* $|O| < \aleph_0$, *then* $|\mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O| = |\mathrm{graph}'_{<}(\tau_0 \amalg \tau_1) \cap O| = 1$.
(2) *If* $|O| = \aleph_0$, *then* O *is a cofinite subset of* $\mathrm{graph}(\tau_0 \amalg \tau_1)$, $|\mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O| = 1$, *and* $\mathrm{graph}'_{<}(\tau_0 \amalg \tau_1) \cap O = \emptyset$.

Proof. By Lemma 1.7, there exists $((i,k),(j,k)) \in \mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O$. For all $n \in \mathbb{N}$, note that if $((i_n,k),(j_n,k)) = (I \circ J)^n((i,k),(j,k))$ is in $\mathrm{graph}_{\succeq}(\tau_0 \amalg \tau_1)$, then Lemma 1.5 ensures that $J((i_n,k),(j_n,k)) \in \mathrm{graph}_{\prec}(\tau_0 \amalg \tau_1)$. Moreover, if $J((i_n,k),(j_n,k)) \notin \mathrm{graph}'_{\prec}(\tau_0 \amalg \tau_1)$, then $((i_{n+1},k),(j_{n+1},k)) \in \mathrm{graph}_{\succeq}(\tau_0 \amalg \tau_1)$.

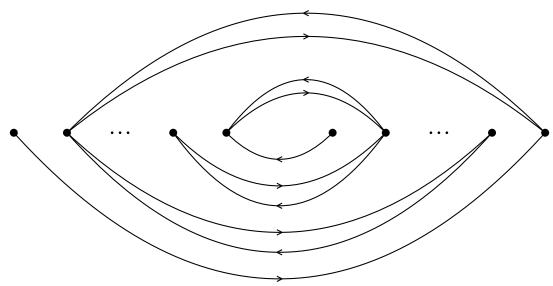

Figure 1. A finite orbit of G.

Suppose now that n is the least natural number with the property that $J((i_n, k), (j_n, k)) \in \mathrm{graph}'_{\prec}(\tau_0 \amalg \tau_1)$. Then $((i, k), (j, k))$ is in $\mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O$, $J((i_n, k), (j_n, k))$ is in $\mathrm{graph}'_{\prec}(\tau_0 \amalg \tau_1) \cap O$, and the pairs of the form $J((i_m, k), (j_m, k))$ and $((i_{m+1}, k), (j_{m+1}, k))$, for $m < n$, are in $\mathrm{Per}_2(\tau_0 \amalg \tau_1)^2$ and make up the rest of O, so (1) holds.

Finally, suppose that there is no $n \in \mathbb{N}$ for which $J((i_n, k), (j_n, k)) \in \mathrm{graph}'_{\prec}(\tau_0 \amalg \tau_1)$. Then $((i, k), (j, k))$ is in $\mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O$ and the pairs of the form $J((i_n, k), (j_n, k))$ and $((i_{n+1}, k), (j_{n+1}, k))$, for $n \in \mathbb{N}$, are in $\mathrm{Per}_2(\tau_0 \amalg \tau_1)^2$ and make up the rest of O, in which case $\mathrm{graph}'_{\prec}(\tau_0 \amalg \tau_1) \cap O = \emptyset$. And a straightforward induction shows that $i_{2n} = i+n$, $j_{2n} = j-n$, $i_{2n+1} = i+(n+k)$, and $j_{2n+1} = j-(n+(1-k))$ for all $n \in \mathbb{N}$, so $\mathrm{graph}(\tau_0 \amalg \tau_1) \setminus O \subseteq ([i, j] \times 2)^2$, thus (2) holds. ⊠

As at most one orbit of G can be cofinite, Lemma 1.8 ensures that $|\mathrm{graph}'_{\prec}(\tau_0 \amalg \tau_1)| - |\mathrm{graph}'_{\succeq}(\tau_0 \amalg \tau_1)| \geq (|\mathcal{O}(G)| - 1) - |\mathcal{O}(G)| = -1$. ⊠

2. Duals

We use $f \colon X \hookrightarrow Y$ to denote a partial injection of X into Y. For all $\sigma \colon \mathbb{Z} \hookrightarrow \mathbb{Z}$, define $\overline{\sigma} \colon \mathbb{Z} \hookrightarrow \mathbb{Z}$ by $\overline{\sigma}(i) = -\sigma^{-1}(-i)$ for all $i \in \mathbb{Z}$.

Proposition 2.1. *Suppose that $\sigma \colon \mathbb{Z} \hookrightarrow \mathbb{Z}$. Then $\sigma = \overline{\overline{\sigma}}$.*

Proof. If $i \in \mathbb{Z}$, then $\overline{\overline{\sigma}}(i) = -(\overline{\sigma})^{-1}(-i)$, so $\overline{\sigma}(-\overline{\overline{\sigma}}(i)) = -i$. But $\overline{\sigma}(-\overline{\overline{\sigma}}(i)) = -\sigma^{-1}(\overline{\overline{\sigma}}(i))$, so $i = \sigma^{-1}(\overline{\overline{\sigma}}(i))$, thus $\sigma(i) = \overline{\overline{\sigma}}(i)$. ⊠

Proposition 2.2. *Suppose that $\rho, \sigma \colon \mathbb{Z} \hookrightarrow \mathbb{Z}$. Then $\overline{\rho \circ \sigma} = \overline{\sigma} \circ \overline{\rho}$.*

Proof. Observe that

$$\begin{aligned}(\overline{\sigma} \circ \overline{\rho})(i) &= -\sigma^{-1}(-(-\rho^{-1}(-i))) \\ &= -(\sigma^{-1} \circ \rho^{-1})(-i) \\ &= -(\rho \circ \sigma)^{-1}(-i) \\ &= \overline{\rho \circ \sigma}(i)\end{aligned}$$

for all $i \in \mathbb{Z}$. $\boxtimes$

Define $\mathcal{F} = \{(\rho\colon \mathbb{Z} \hookrightarrow \mathbb{Z}, \sigma\colon \mathbb{Z} \hookrightarrow \mathbb{Z}) \mid \rho \circ \sigma = S^{\mathbb{Z}} \restriction \mathrm{dom}(\rho \circ \sigma)\}$.

Proposition 2.3. *$(\rho, \sigma) \in \mathcal{F} \iff (\overline{\sigma}, \overline{\rho}) \in \mathcal{F}$.*

Proof. Note that if $i \in \mathbb{Z}$ and $\rho, \sigma\colon \mathbb{Z} \hookrightarrow \mathbb{Z}$, then $(\rho \circ \sigma)(i) = i + 1 \iff (\rho \circ \sigma)^{-1}(i + 1) = i \iff \overline{\rho \circ \sigma}(-i - 1) = -i$, so the desired result follows from Proposition 2.2. $\boxtimes$

Let $(i_0\ i_1\ \cdots\ i_n)$ denote the permutation of $\{i_m \mid m \leq n\}$ sending i_m to i_{m+1} for all $m < n$.

Proposition 2.4. *Suppose that $n \geq 1$, $(i_m)_{m \leq n}$ is strictly increasing, $\rho = (i_0\ i_1\ \cdots\ i_n)$, and $\sigma = (-i_n\ \ -i_{n-1}\ \cdots\ -i_0)$. Then $\rho = \overline{\sigma}$.*

Proof. If $m < n$, then $\overline{\sigma}(i_m) = -\sigma^{-1}(-i_m) = -(-i_{m+1}) = i_{m+1}$. $\boxtimes$

3. Eliminating fixed points

For all $k \in \mathbb{N}$, let $\mathrm{par}(k)$ denote the remainder when k is divided by two. For all $\rho, \sigma \in \mathrm{Sym}(X)$, set $\delta(\rho, \sigma) = \{x \in X \mid \rho(x) \neq \sigma(x)\}$ and

$$\mathrm{Mal}(\rho, \sigma) = \{x \in \mathrm{Per}_{\mathbb{N}+3}(\sigma) \mid |[x]_\sigma \setminus \mathrm{Per}_1(\rho)| = 1\}.$$

Proposition 3.1. *Suppose that $m \geq 1$, ρ and σ are permutations of a set X, and $\forall n \geq 3\ \neg 0 < |\mathrm{Mal}(\rho, \sigma) \cap \mathrm{Per}_{2\mathbb{N}+n}(\sigma)| < \aleph_0$. Then there are permutations ρ' and σ' of X such that:*

1. *$\rho \circ \sigma = \rho' \circ \sigma'$,*
2. *$\delta(\rho, \rho') = \delta(\sigma^{-1}, (\sigma')^{-1}) = \mathrm{Mal}(\rho, \sigma) \cap \mathrm{Per}_1(\rho)$,*
3. *$\mathrm{Mal}(\rho, \sigma) \cap \mathrm{Per}_1(\rho) \subseteq \mathrm{Per}_m(\rho')$, and*
4. *$\forall n \geq 3\ \mathrm{Mal}(\rho, \sigma) \cap \mathrm{Per}_n(\sigma) \subseteq \mathrm{Per}_n(\sigma')$.*

Proof. Define $Y = \mathrm{Mal}(\rho, \sigma)$ and $Z = Y \setminus \mathrm{Per}_1(\rho)$. For all $n \geq 3$, set $Y_n = \mathrm{Per}_{2\mathbb{N}+n}(\sigma) \cap Y$ and $Z_n = \mathrm{Per}_{2\mathbb{N}+n}(\sigma) \cap Z$. Fix an equivalence relation F_4 on Z_4 whose classes all have cardinality m^2, as well as $\pi_{0,1}, \pi_{0,2} \in \mathrm{Sym}(Z_4)$, whose graphs are contained in F_4, such that the orbits of $\pi_{0,1}$, $\pi_{0,2}$, and $\pi_{0,3} = (\pi_{0,1} \circ \pi_{0,2})^{-1}$ all have cardinality m. For all $n \in (\mathbb{N}+3) \setminus \{4\}$, fix an equivalence relation F_n on Z_n whose classes all have cardinality m, fix $\pi_{\mathrm{par}(n),n-2} \in \mathrm{Sym}(Z_n)$ whose orbits coincide with the equivalence classes of F_n, and set $\pi_{\mathrm{par}(n),n-1} = \pi^{-1}_{\mathrm{par}(n),n-2}$. Then the support of $\pi = \mathrm{id}_{X \setminus (Y \setminus Z)} \cup \bigcup_{p<2, n\geq 1} \sigma^n \circ \pi_{p,n} \circ \sigma^{-n}$ is $Y \setminus Z$, so $\rho' = \rho \circ \pi$ and $\sigma' = \pi^{-1} \circ \sigma$ satisfy conditions (1)–(3).

Lemma 3.2. *Suppose that $\ell \leq n - 1$. Then*

$$(\sigma')^\ell \restriction Z_n = (\sigma^\ell \circ \pi^{-1}_{\mathrm{par}(n),\ell} \circ \cdots \circ \pi^{-1}_{\mathrm{par}(n),1}) \restriction Z_n. \qquad (*)$$

Proof. The case $\ell = 0$ is trivial. If $\ell > 0$ and $(*)$ holds at $\ell - 1$, then

$$\begin{aligned}(\sigma')^\ell \restriction Z_n &= (\sigma' \circ (\sigma')^{\ell-1}) \restriction Z_n \\ &= (\sigma' \circ \sigma^{\ell-1} \circ \pi^{-1}_{\mathrm{par}(n),\ell-1} \circ \cdots \circ \pi^{-1}_{\mathrm{par}(n),1}) \restriction Z_n \\ &= (\pi^{-1} \circ \sigma^{\ell} \circ \pi^{-1}_{\mathrm{par}(n),\ell-1} \circ \cdots \circ \pi^{-1}_{\mathrm{par}(n),1}) \restriction Z_n \\ &= (\sigma^{\ell} \circ \pi^{-1}_{\mathrm{par}(n),\ell} \circ \sigma^{-\ell} \circ \sigma^{\ell} \circ \pi^{-1}_{\mathrm{par}(n),\ell-1} \circ \cdots \circ \pi^{-1}_{\mathrm{par}(n),1}) \restriction Z_n \\ &= (\sigma^{\ell} \circ \pi^{-1}_{\mathrm{par}(n),\ell} \circ \cdots \circ \pi^{-1}_{\mathrm{par}(n),1}) \restriction Z_n,\end{aligned}$$

so $(*)$ also holds at ℓ. ⊠

For all $n \geq 3$, set $Y'_n = \mathrm{Per}_n(\sigma) \cap Y$ and $Z'_n = \mathrm{Per}_n(\sigma) \cap Z$. Lemma 3.2 ensures that $Y'_n = \bigcup_{\ell \leq n-1} \sigma^\ell(Z'_n) = \bigcup_{\ell \leq n-1} (\sigma')^\ell(Z'_n)$ and

$$\begin{aligned}(\sigma')^n \restriction Z'_n &= (\sigma' \circ (\sigma')^{n-1}) \restriction Z'_n \\ &= (\sigma' \circ \sigma^{n-1} \circ \pi^{-1}_{\mathrm{par}(n),n-1} \circ \cdots \circ \pi^{-1}_{\mathrm{par}(n),1}) \restriction Z'_n \\ &= (\sigma' \circ \sigma^{n-1}) \restriction Z'_n \\ &= (\sigma' \circ \sigma^{-1}) \restriction Z'_n \\ &= \mathrm{id}_{Z'_n},\end{aligned}$$

so condition (4) also holds. ⊠

We write $\rho \cong \sigma$ to indicate that ρ and σ are isomorphic.

Proposition 3.3. *Suppose that $m \geq 1$, ρ and σ are permutations of a set X, $\forall n \geq 3\ \neg 0 < |\mathrm{Mal}(\rho, \sigma) \cap \mathrm{Per}_{2\mathbb{N}+n}(\sigma)| < \aleph_0$, and $\mathrm{Per}_m(\rho)$ is infinite. Then there are permutations $\rho' \cong \rho \restriction \sim(\mathrm{Mal}(\rho, \sigma) \cap \mathrm{Per}_1(\rho))$ and $\sigma' \cong \sigma$ of X for which $\rho \circ \sigma = \rho' \circ \sigma'$.*

Proof. Proposition 3.1 yields $\rho', \sigma' \in \mathrm{Sym}(X)$ such that $\rho \circ \sigma = \rho' \circ \sigma'$ and $|\mathcal{O}_\kappa(\rho \restriction \sim(\mathrm{Mal}(\rho, \sigma) \cap \mathrm{Per}_1(\rho)))| = |\mathcal{O}_\kappa(\rho')|$ and $|\mathcal{O}_\kappa(\sigma)| = |\mathcal{O}_\kappa(\sigma')|$ for all cardinals κ. ⊠

4. Building blocks

Set $\mathcal{F}(i, j] = \{(\rho, \sigma) \in \mathcal{F} \mid \rho\colon (i, j] \hookrightarrow (i, j] \text{ and } \sigma\colon (i, j) \hookrightarrow (i, j)\}$, noting that $\forall (\rho, \sigma) \in \mathcal{F}(i, j]\ \mathrm{dom}(\rho \circ \sigma) = (i, j - 1]$.

Proposition 4.1. *If $i < j$ and $(\rho, \sigma) \in \mathcal{F}(i, j]$, then $\rho(j) = i + 1$.*

Proof. Observe that $\rho((i, j - 1]) = (\rho \circ \sigma)((i, j - 1]) = (i + 1, j]$. ⊠

Set $\mathcal{F}[i, j) = \{(\rho, \sigma) \in \mathcal{F} \mid \rho\colon (i, j) \hookrightarrow (i, j) \text{ and } \sigma\colon [i, j) \hookrightarrow [i, j)\}$, this time noting that $\forall (\rho, \sigma) \in \mathcal{F}(i, j]\ S^{\mathbb{Z}}(j-1) \notin \mathrm{rng}(\rho)$, and therefore $\forall (\rho, \sigma) \in \mathcal{F}(i, j]\ \mathrm{dom}(\rho \circ \sigma) = [i, j - 1)$.

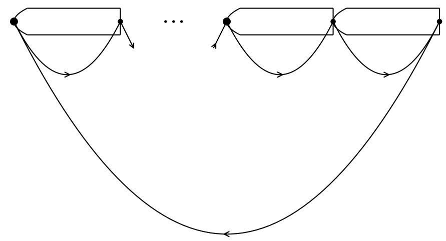

FIGURE 2. The extension provided by Proposition 4.2.

Proposition 4.2. *Suppose that* $n \geq 1$, $(i_m)_{m \leq n}$ *is strictly increasing,* $\forall m < n\ (\rho_m, \sigma_m) \in \mathcal{F}(i_m, i_{m+1}]$, $\rho = \bigcup_{m<n} \rho_m$, *and* $\sigma = (i_0\ i_1\ \cdots\ i_n) \cup \bigcup_{m<n} \sigma_m$. *Then* $(\rho, \sigma) \in \mathcal{F}[i_0, i_n + 1)$.

Proof. As $[i_0, i_n) = \{i_m \mid m < n\} \cup \bigcup_{m<n}(i_m, i_{m+1} - 1]$, it follows that

$$\begin{aligned}
(\rho, \sigma) \in \mathcal{F}[i_0, i_n + 1) &\iff \forall k \in [i_0, i_n)\ (\rho \circ \sigma)(k) = k + 1 \\
&\iff \forall m < n\ (\rho \circ \sigma)(i_m) = i_m + 1 \\
&\iff \forall m < n\ \rho(i_{m+1}) = i_m + 1 \\
&\iff \forall m < n\ \rho_m(i_{m+1}) = i_m + 1,
\end{aligned}$$

so Proposition 4.1 yields the desired result. ⊠

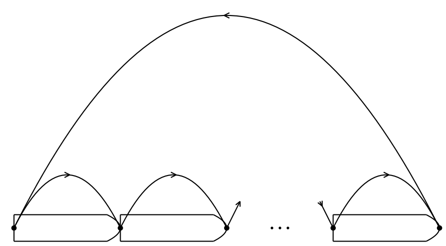

FIGURE 3. The extension provided by Proposition 4.3.

Proposition 4.3. *Suppose that* $n \geq 1$, $(i_m)_{m \leq n}$ *is strictly increasing,* $\forall m < n\ (\rho_m, \sigma_m) \in \mathcal{F}[i_m, i_{m+1})$, $\rho = (i_0\ i_1\ \cdots\ i_n) \cup \bigcup_{m<n} \rho_m$, *and* $\sigma = \bigcup_{m<n} \sigma_m$. *Then* $(\rho, \sigma) \in \mathcal{F}(i_0 - 1, i_n]$.

Proof. By Propositions 2.1, 2.3, 2.4, and 4.2. ⊠

Proposition 4.4. *Suppose that* $i \in \mathbb{Z}$. *Then* $(\emptyset, \mathrm{id}_{\{i\}}) \in \mathcal{F}[i, i+1)$ *and* $(\mathrm{id}_{\{i\}}, \emptyset) \in \mathcal{F}(i - 1, i]$.

Proof. As $[i, i) = (i - 1, i - 1] = \emptyset$, the definitions of $\mathcal{F}[i, i + 1)$ and $\mathcal{F}(i-1, i]$ yield that $(\rho, \sigma) \in \mathcal{F}[i, i+1) \iff (\rho = \emptyset$ and $\mathrm{dom}(\sigma) = \{i\})$ and $(\rho, \sigma) \in \mathcal{F}(i - 1, i] \iff (\mathrm{dom}(\rho) = \{i\}$ and $\sigma = \emptyset)$. ⊠

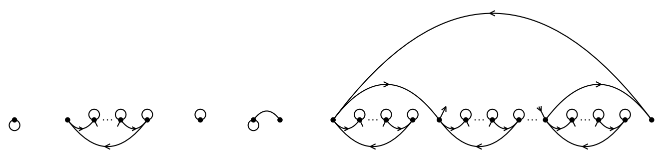

FIGURE 4. Building blocks from Propositions 4.5 and 4.6.

Proposition 4.5. *Suppose that $i < j$ are integers. Then the pair $(\mathrm{id}_{(i,j)}, (i\ i+1\ \cdots\ j-1))$ is in $\mathcal{F}[i,j)$.*

Proof. If $i+1 = j$, then this follows from Proposition 4.4. Otherwise, Proposition 4.4 ensures that $(\mathrm{id}_{\{k\}}, \emptyset) \in \mathcal{F}(k-1,k]$ for all $k \in (i,j)$, so Proposition 4.2 yields the desired result. ⊠

Proposition 4.6. *Suppose that $m \geq 1$ and $(i_k)_{k<m}$ is a strictly increasing sequence of integers. Then the pair*

$$((i_0\ i_1\ \cdots\ i_{m-1}) \cup \bigcup_{k<m-1} \mathrm{id}_{(i_k,i_{k+1})}, \bigcup_{k<m-1}(i_k\ i_k+1\ \cdots\ i_{k+1}-1))$$

is in $\mathcal{F}(i_0-1, i_{m-1}]$.

Proof. If $m = 1$, then this follows from Proposition 4.4. Otherwise, $(\mathrm{id}_{(i_k,i_{k+1})}, (i_k\ i_k+1\ \cdots\ i_{k+1}-1)) \in \mathcal{F}[i_k, i_{k+1})$ for all $k < m-1$ by Proposition 4.5, so Proposition 4.3 yields the desired result. ⊠

5. Extension

Given $n \geq 3$ and $\rho, \sigma \colon X \rightharpoonup X$, we say that a fixed point x of ρ is *n-malleable* if $x \in \mathrm{Per}_n(\sigma)$, $[x]_\sigma \subseteq \mathrm{dom}(\rho)$, and $|[x]_\sigma \setminus \mathrm{Per}_1(\rho)| = 1$.

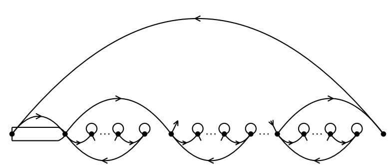

FIGURE 5. The extension provided by Proposition 5.1.

Proposition 5.1. *Suppose that $i < j$, $m \geq 2$, $n_k \geq 3$ for all $k < m-2$, and $(\rho, \sigma) \in \mathcal{F}[i,j)$. Then there exists $(\rho', \sigma') \in \mathcal{F}(i-1, j+\sum_{k<m-2} n_k]$ such that:*

- *ρ' is obtained from ρ by adding a single cycle of length m and $n_k - 1$ n_k-malleable fixed points for all $k < m-2$.*
- *σ' is obtained from σ by adding a cycle of length n_k for all $k < m-2$.*

Proof. Recursively define $i_0 = i$, $i_1 = j$, and $i_k = i_{k-1} + n_{k-2}$ for all $2 \leq k \leq m-1$. Set $(\rho_0, \sigma_0) = (\rho, \sigma)$. For all $1 \leq k \leq m-2$, Proposition 4.5 ensures that $(\rho_k, \sigma_k) = (\mathrm{id}_{(i_k, i_{k+1})}, (i_k \ i_k + 1 \ \cdots \ i_{k+1} - 1))$ is in $\mathcal{F}[i_k, i_{k+1})$. So $(\rho', \sigma') = ((i_0 \ i_1 \ \cdots \ i_{m-1}) \cup \bigcup_{k \leq m-2} \rho_k, \bigcup_{k \leq m-2} \sigma_k)$ is in $\mathcal{F}(i_0 - 1, i_{m-1}]$ by Proposition 4.3. But $i_0 - 1 = i - 1$ and $i_{m-1} = j + \sum_{k<m-2} n_k$. ⊠

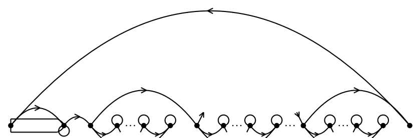

FIGURE 6. The extension provided by Proposition 5.2.

Proposition 5.2. *Suppose that $i < j$, $m \geq 3$, $n_k \geq 3$ for all $k < m-3$, and $(\rho, \sigma) \in \mathcal{F}[i, j)$. Then there exists $(\rho', \sigma') \in \mathcal{F}(i-1, j+1+\sum_{k<m-3} n_k]$ such that:*

- *ρ' is obtained from ρ by adding a single cycle of length m and $n_k - 1$ n_k-malleable fixed points for all $k < m-3$.*
- *σ' is obtained from σ by adding a fixed point and a cycle of length n_k for all $k < m-3$.*

Proof. Recursively define $i_0 = i$, $i_1 = j$, $i_2 = j+1$, and $i_k = i_{k-1} + n_{k-3}$ for all $3 \leq k \leq m-1$. Set $(\rho_0, \sigma_0) = (\rho, \sigma)$. For all $1 \leq k \leq m-2$, Proposition 4.5 ensures that $(\rho_k, \sigma_k) = (\mathrm{id}_{(i_k, i_{k+1})}, (i_k \ i_k{+}1 \ \cdots \ i_{k+1}{-}1))$ is in $\mathcal{F}[i_k, i_{k+1})$. So $(\rho', \sigma') = ((i_0 \ i_1 \ \cdots \ i_{m-1}) \cup \bigcup_{k \leq m-2} \rho_k, \bigcup_{k \leq m-2} \sigma_k)$ is in $\mathcal{F}(i_0 - 1, i_{m-1}]$ by Proposition 4.3. But $i_0 - 1 = i - 1$ and $i_{m-1} = j + 1 + \sum_{k<m-3} n_k$. ⊠

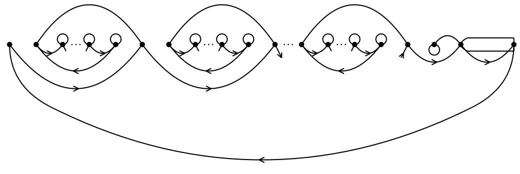

FIGURE 7. The extension provided by Proposition 5.3.

Proposition 5.3. *Suppose that $i < j$, $n \geq 3$, $n_k \geq 3$ for all $k < n-3$, and $(\rho, \sigma) \in \mathcal{F}(i, j]$. Then there exists $(\rho', \sigma') \in \mathcal{F}[i - 2 - \sum_{k<n-3}(n_k + 1), j+1)$ such that:*

- *ρ' is obtained from ρ by adding $n-2$ cycles of length two and $n_k - 1$ n_k-malleable fixed points for all $k < n-3$.*

- *σ' is obtained from σ by adding a single fixed point, a cycle of length n, and a cycle of length of n_k for all $k < n-3$.*

Proof. Recursively define $i_{n-1} = j$, $i_{n-2} = i$, $i_{n-3} = i-2$, and $i_k = i_{k+1} - (n_k + 1)$ for all $k \leq n-4$. Set $(\rho_{n-2}, \sigma_{n-2}) = (\rho, \sigma)$. For all $k \leq n-3$, Proposition 4.6 implies that

$$(\rho_k, \sigma_k) = (\mathrm{id}_{(i_k+1, i_{k+1})} \cup (i_k + 1 \ i_{k+1}), (i_k + 1 \ i_k + 2 \ \cdots \ i_{k+1} - 1))$$

is in $\mathcal{F}(i_k, i_{k+1}]$. So $(\rho', \sigma') = (\bigcup_{k \leq n-2} \rho_k, (i_0 \ i_1 \ \cdots \ i_{n-1}) \cup \bigcup_{k \leq n-2} \sigma_k)$ is in $\mathcal{F}[i_0, i_{n-1}+1)$ by Proposition 4.2. But $i - 2 - \sum_{k<n-3}(n_k+1) = i_0$ and $j + 1 = i_{n-1} + 1$. ⊠

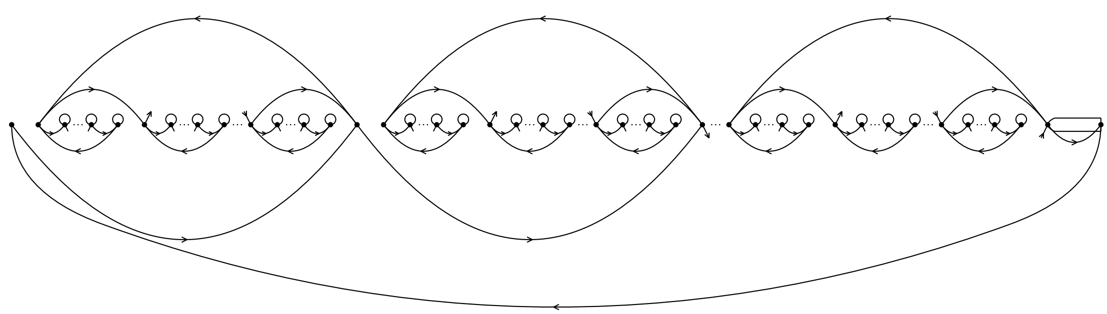

FIGURE 8. The extension provided by Proposition 5.4.

Proposition 5.4. *Suppose that $i < j$, $m \geq 2$, $n \geq 2$, $n_{k,\ell} \geq 3$ for all $k < m-1$ and $\ell < n-2$, and $(\rho, \sigma) \in \mathcal{F}(i, j]$. Then there exists $(\rho', \sigma') \in \mathcal{F}[i - \sum_{k<m-1, \ell<n-2} n_{k,\ell}, j+1)$ such that:*

- *ρ' is obtained from ρ by adding $n-2$ cycles of length m and $n_{k,\ell} - 1$ $n_{k,\ell}$-malleable fixed points for all $k < m-1$ and $\ell < n-2$.*
- *σ' is obtained from σ by adding a cycle of length n and a cycle of length $n_{k,\ell}$ for all $k < m-1$ and $\ell < n-2$.*

Proof. Recursively define $i_{n-1} = j$, $i_{n-2} = i$, $i_\ell = i_{\ell+1} - \sum_{k<m-1} n_{k,\ell}$, $i_{0,\ell} = i_\ell + 1$, and $i_{k,\ell} = i_{k-1,\ell} + n_{k-1,\ell}$ for $k \leq m-1$ and $\ell \leq n-3$. Set $(\rho_{n-2}, \sigma_{n-2}) = (\rho, \sigma)$. For all $\ell \leq n-3$, Proposition 4.6 implies that the pair (ρ_ℓ, σ_ℓ), given by $\rho_\ell = (i_{0,\ell} \ i_{1,\ell} \ \cdots \ i_{m-1,\ell}) \cup \bigcup_{k<m-1} \mathrm{id}_{(i_{k,\ell}, i_{k+1,\ell})}$ and $\sigma_\ell = \bigcup_{k<m_\ell - 1} (i_{k,\ell} \ i_{k,\ell}+1 \ \cdots \ i_{k+1,\ell} - 1)$, is in $\mathcal{F}(i_\ell, i_{\ell+1}]$. So Proposition 4.2 yields that $(\rho', \sigma') = (\bigcup_{\ell \leq n-2} \rho_\ell, (i_0 \ i_1 \ \cdots \ i_{n-1}) \cup \bigcup_{\ell \leq n-2} \sigma_\ell)$ is in $\mathcal{F}[i_0, i_{n-1}+1)$. But $i - \sum_{k<m-1, \ell<n-2} n_{k,\ell} = i_0$ and $j + 1 = i_{n-1} + 1$. ⊠

We say that a fixed point x of ρ is *anti-malleable* if $x \in \mathrm{Per}_2(\sigma)$, $[x]_\sigma \subseteq \mathrm{dom}(\rho)$, and $|[x]_\sigma \setminus \mathrm{Per}_1(\rho)| = 1$.

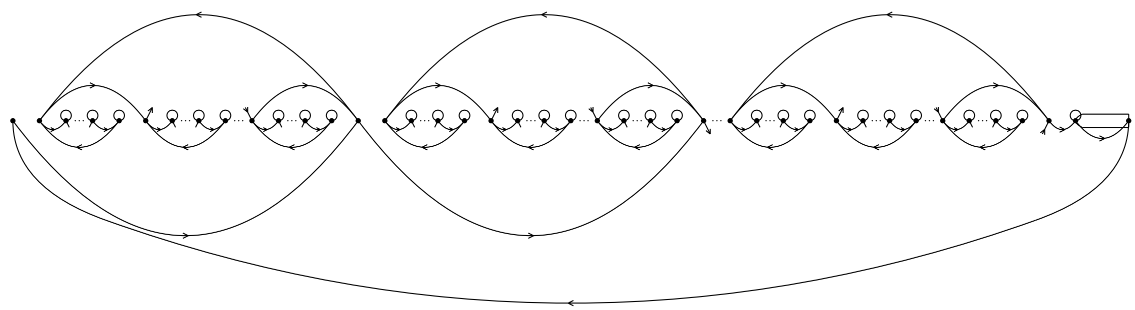

FIGURE 9. The extension provided by Proposition 5.5.

Proposition 5.5. *Suppose that $i < j$, $m \geq 2$, $n \geq 3$, $n_{k,\ell} \geq 3$ for all $k < m-1$ and $\ell < n-3$, and $(\rho, \sigma) \in \mathcal{F}(i, j]$. Then there exists $(\rho', \sigma') \in \mathcal{F}[i - 1 - \sum_{k<m-1,\ell<n-3} n_{k,\ell}, j + 1)$ such that:*

- *ρ' is obtained from ρ by adding a single anti-malleable fixed point, $n-3$ cycles of length m, and $n_{k,\ell} - 1$ $n_{k,\ell}$-malleable fixed points for all $k < m-1$ and $\ell < n-3$.*
- *σ' is obtained from σ by adding a cycle of length n and a cycle of length $n_{k,\ell}$ for all $k < m-1$ and $\ell < n-3$.*

Proof. Recursively define $i_{n-1} = j$, $i_{n-2} = i$, $i_{n-3} = i - 1$, $i_\ell = i_{\ell+1} - \sum_{k<m-1} n_{k,\ell}$, $i_{0,\ell} = i_\ell + 1$, and $i_{k,\ell} = i_{k-1,\ell} + n_{k-1,\ell}$ for $k \leq m-1$ and $\ell \leq n-4$. Set $(\rho_{n-2}, \sigma_{n-2}) = (\rho, \sigma)$. For all $\ell \leq n-3$, Proposition 4.6 implies that the pair (ρ_ℓ, σ_ℓ), given by $\rho_\ell = (i_{0,\ell}\ i_{1,\ell}\ \cdots\ i_{m-1,\ell}) \cup \bigcup_{k<m-1} \mathrm{id}_{(i_{k,\ell}, i_{k+1,\ell})}$ and $\sigma_\ell = \bigcup_{k<m-1} (i_{k,\ell}\ i_{k,\ell}+1\ \cdots\ i_{k+1,\ell}-1)$, is in $\mathcal{F}(i_\ell, i_{\ell+1}]$. So $(\rho', \sigma') = (\bigcup_{\ell \leq n-2} \rho_\ell, (i_0\ i_1\ \cdots\ i_{n-1}) \cup \bigcup_{\ell \leq n-2} \sigma_\ell)$ is in $\mathcal{F}[i_0, i_{n-1} + 1)$ by Proposition 4.2. But $i - 1 - \sum_{k<m-1,\ell<n-3} n_{k,\ell} = i_0$ and $j + 1 = i_{n-1} + 1$. ⊠

6. The main result

The special case of Theorem B where ρ or σ has finite order is a consequence of Propositions 2.1 and 2.3 and:

Theorem 6.1. *Suppose that $m \geq 2$, $\rho, \sigma \in \mathrm{Sym}(\mathbb{Z})$ are periodic, and $\mathrm{Per}_m(\rho)$ and $\mathrm{Per}_{\geq 3}(\sigma)$ are infinite. Then $S^{\mathbb{Z}} \in \mathrm{Cl}(\rho)\mathrm{Cl}(\sigma)$.*

Proof. For all integers $i < j$, set $\mathcal{F}_0(i, j) = \mathcal{F}[i, j)$ and $\mathcal{F}_1(i, j) = \mathcal{F}(i, j]$. Fix an enumeration $(\pi_n, O_n)_{n\in\mathbb{N}}$ of the pairs of the form (π, O), where $\pi \in \{\rho, \sigma\}$ and $O \in \mathcal{O}(\pi)$. Then there is an infinite set $N \subseteq \mathbb{N}$ and $p < 2$ such that $\pi_n = \sigma$, $\mathrm{par}(|O_n|) = p$, and $3 \leq |O_n| \leq |O_{n+1}|$ for all $n \in N$. Fix $n_{-1} \in N$, set $N_0 = \mathbb{N} \setminus \{n_{-1}\}$, and apply Proposition 4.5 to find $i_0 < j_0$ and $(\rho_0, \sigma_0) \in \mathcal{F}_0(i_0, j_0)$ such that every point of $\mathrm{dom}(\rho_0)$ is a malleable fixed point and the lone orbit of σ_0 has cardinality $|O_{n_{-1}}|$.

Suppose that k is a natural number for which we have found $i_k < j_k$, a cofinite set $N_k \subseteq \mathbb{N}$, and $(\rho_k, \sigma_k) \in \mathcal{F}_{\mathrm{par}(k)}(i_k, j_k)$. If $k \in 2\mathbb{N}$, then let n_k be the least element of N_k for which ($\pi_{n_k} = \rho$ and $|O_{n_k}| \geq 2$) or ($\pi_{n_k} = \sigma$, $|O_{n_k}| = 1$, and $m \geq 3$). If $k \in 4\mathbb{N} + 1$, then let n_k be the least element of N_k for which ($\pi_{n_k} = \sigma$, $|O_{n_k}| = 1$, and $m = 2$), ($\pi_{n_k} = \sigma$ and $|O_{n_k}| = 2$), or ($\pi_{n_k} = \rho$ and $|O_{n_k}| = 1$). And if $k \in 4\mathbb{N} + 3$, then let n_k be the least element of N_k for which $\pi_{n_k} = \sigma$ and $|O_{n_k}| \geq 3$.

Lemma 6.2. *For some $\ell_k \in \mathbb{N}$ and any set $F_k \subseteq N \cap (N_k \setminus \{n\})$ of cardinality ℓ_k, there exist $i_{k+1} < i_k$, $j_{k+1} > j_k$, and $(\rho_{k+1}, \sigma_{k+1}) \in \mathcal{F}_{\mathrm{par}(k+1)}(i_{k+1}, j_{k+1})$ such that:*

- *ρ_{k+1} is obtained from ρ_k by adding a set of cycles of length k and $|O_n| - 1$ $|O_n|$-malleable fixed points for all $n \in F_k$, as well as a cycle of length $|O_{n_k}|$ if ($\pi_{n_k} = \rho$ and $|O_{n_k}| \geq 2$) and an anti-malleable fixed point if ($\pi_{n_k} = \rho$ and $|O_{n_k}| = 1$).*
- *σ_{k+1} is obtained from σ_k by adding a cycle of length $|O_n|$ for all $n \in F_k$, as well as a cycle of length $|O_n|$ if $\pi_{n_k} = \sigma$.*

Proof. If $k \in 2\mathbb{N}$, then the desired result follows from Propositions 5.1 and 5.2. Otherwise, it follows from Propositions 5.3–5.5. ⊠

Set $N_{k+1} = N_k \setminus (F_k \cup \{n_k\})$.

Define $\rho_\infty = \bigcup_{k \in \mathbb{N}} \rho_k$ and $\sigma_\infty = \bigcup_{k \in \mathbb{N}} \sigma_k$. As $(i_k)_{k \in \mathbb{N}}$ is strictly decreasing and $(j_k)_{k \in \mathbb{N}}$ is strictly increasing, these are permutations of $\mathbb{Z}$ whose composition is $S^{\mathbb{Z}}$. As $F_k \neq \emptyset$ for all $k \in 4\mathbb{N} + 3$, it follows that $\neg 0 < |\mathrm{Mal}(\rho_\infty, \sigma_\infty) \cap \mathrm{Per}_{2\mathbb{N}+n}(\sigma_\infty)| < \aleph_0$ for all $n \in 2\mathbb{N} + p$. And clearly $\mathrm{Mal}(\rho_\infty, \sigma_\infty) \cap \mathrm{Per}_{2\mathbb{N}+(1-p)}(\sigma_\infty) = \emptyset$. As the fact that $\bigcap_{k \in \mathbb{N}} N_k = \emptyset$ ensures that $\rho_\infty \upharpoonright {\sim}(\mathrm{Mal}(\rho_\infty, \sigma_\infty) \cap \mathrm{Per}_1(\rho_\infty)) \cong \rho$ and $\sigma_\infty \cong \sigma$, Proposition 3.3 yields conjugates ρ' of ρ and σ' of σ for which $\rho' \circ \sigma' = \rho_\infty \circ \sigma_\infty = S^{\mathbb{Z}}$. ⊠

The fact that every almost involution has finite order and [Mor89, Theorem A] therefore yield Theorem B.

References

[Mor89] G. Moran, *Conjugacy classes whose square is an infinite symmetric group*, Trans. Amer. Math. Soc. **316** (1989), no. 2, 493–522.

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