

# PERIODIC PERMUTATIONS AND THE SUCCESSOR

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ABSTRACT. We investigate pairs of conjugacy classes of periodic permutations of  $\mathbb{Z}$  whose product contains the successor function.

## INTRODUCTION

Given  $g, h: X \rightarrow Y$ , define  $g \amalg h: \{g, h\} \times X \rightarrow \{g, h\} \times Y$  by  $(g \amalg h)(f, x) = (f, f(x))$  for all  $f \in \{g, h\}$  and  $x \in X$ . Given an element  $g$  of a group  $G$ , set  $\text{Cl}(g) = \{hgh^{-1} \mid h \in G\}$ .

Let  $\text{Sym}(X)$  denote the *symmetric group* of all permutations of  $X$ . The *orbit* of a point  $x \in X$  under a permutation  $\tau$  of  $X$  is given by  $[x]_\tau = \{\tau^i(x) \mid i \in \mathbb{Z}\}$ . Set  $\mathcal{O}(\tau) = \{[x]_\tau \mid x \in X\}$ . For all sets  $K$  of cardinals, define  $\text{Per}_K(\tau) = \{x \in X \mid |[x]_\tau| \in K\}$  and  $\mathcal{O}_K(\tau) = \mathcal{O}(\tau \upharpoonright \text{Per}_K(\tau))$ . We will use straightforward shorthand for the set  $K$ . We say that  $\tau$  is *almost trivial* if  $\text{Per}_{\geq 2}(\tau)$  is finite, an *almost involution* if  $\text{Per}_{\geq 3}(\tau)$  is finite, and *( $\sigma$ -)periodic* if  $\text{Per}_{\aleph_0}(\tau) = \emptyset$ .

The *successor function* on  $\mathbb{Z}$  is given by  $S^{\mathbb{Z}}(i) = i + 1$  for all  $i \in \mathbb{Z}$ . Here we prove the following:

**Theorem A.** *Suppose that  $\rho, \sigma \in \text{Sym}(\mathbb{Z})$  are periodic.*

- (1) *If  $\rho$  and  $\sigma$  are almost involutions and  $S^{\mathbb{Z}} \in \text{Cl}(\rho)\text{Cl}(\sigma)$ , then  $|\text{Per}_1(\rho \amalg \sigma)| \leq |\text{Per}_{\geq 3}(\rho \amalg \sigma)| - 2|\mathcal{O}_{\geq 3}(\rho \amalg \sigma)| + 1$ .*
- (2) *If  $\rho$  or  $\sigma$  is not an almost involution and neither is almost trivial, then  $S^{\mathbb{Z}} \in \text{Cl}(\rho)\text{Cl}(\sigma)$ .*

The special case of (2) where neither  $\rho$  nor  $\sigma$  is an almost involution follows from [Mor89, Theorem A]. As far as I am aware, however, the special case of (2) where  $\rho$  or  $\sigma$  is an almost involution was not previously known. Regardless, the real purpose of this paper is to introduce ideas and language—in the simplest possible context—that can be used to investigate the finite-order elements  $R$  and  $S$  of the full group of an aperiodic Borel automorphism  $T$  for which  $T \in \text{Cl}(R)\text{Cl}(S)$ . This topic will be explored in a future paper.

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In §1, we prove (1). In §2, we note a symmetry that eliminates the need to repeat arguments at several points throughout the paper. In §3, we establish a technical fact concerning the removal of fixed points. In §4, we describe the simplest finite approximations to pairs  $(\rho, \sigma)$  for which  $S^{\mathbb{Z}} \in \text{Cl}(\rho)\text{Cl}(\sigma)$ . In §5, we use these as building blocks to construct extensions of more general finite approximations. And in §6, we prove the special case of (2) where  $\rho$  or  $\sigma$  has finite order.

### 1. THE CASE OF TWO ALMOST INVOLUTIONS

For all  $R \subseteq X^2$ , define  $\text{graph}_R(\tau) = \text{graph}(\tau) \cap R$ .

**Proposition 1.1.** *Suppose that  $\leq$  is a linear ordering of a finite set  $F$  and  $\tau \in \text{Sym}(F)$ . Then  $|\text{graph}_{<}(\tau)| \leq |F| - 1$  and  $|\text{graph}_{\geq}(\tau)| \geq 1$ .*

*Proof.* Let  $x$  be the  $\leq$ -maximal element of  $F$ . Then  $x \geq \tau(x)$ , so  $|\text{graph}_{\geq}(\tau)| \geq 1$ . But  $|\text{graph}(\tau)| = |F|$ , thus  $|\text{graph}_{<}(\tau)| \leq |F| - 1$ .  $\square$

Define  $\text{graph}'_R(\tau) = \text{graph}(\tau \upharpoonright \sim \text{Per}_2(\tau)) \cap R$ .

**Proposition 1.2.** *Suppose that  $\tau$  is an almost involution of a set  $X$ ,  $\leq$  is a binary relation on  $X$  whose restriction to each orbit of  $\tau$  is a linear order, and  $|\text{graph}'_{\geq}(\tau)| \leq |\text{graph}'_{<}(\tau)| + 1$ . Then  $|\text{Per}_1(\tau)| \leq |\text{Per}_{\geq 3}(\tau)| - 2|\mathcal{O}_{\geq 3}(\tau)| + 1$ .*

*Proof.* Note that  $|\text{Per}_1(\tau)| = |\text{graph}'_{= }(\tau)|$  and Proposition 1.1 yields that  $|\mathcal{O}_{\geq 3}(\tau)| \leq |\text{graph}'_{>}(\tau)|$  and  $|\text{graph}'_{<}(\tau)| \leq \sum_{O \in \mathcal{O}_{\geq 3}(\tau)} (|O| - 1)$ , so

$$\begin{aligned} |\text{Per}_1(\tau)| + |\mathcal{O}_{\geq 3}(\tau)| &\leq |\text{graph}'_{= }(\tau)| + |\text{graph}'_{>}(\tau)| \\ &\leq |\text{graph}'_{<}(\tau)| + 1 \\ &\leq \sum_{O \in \mathcal{O}_{\geq 3}(\tau)} (|O| - 1) + 1 \\ &= |\text{Per}_{\geq 3}(\tau)| - |\mathcal{O}_{\geq 3}(\tau)| + 1, \end{aligned}$$

thus subtracting  $|\mathcal{O}_{\geq 3}(\tau)|$  from each side yields the desired result.  $\square$

For all  $\rho, \sigma \in \text{Sym}(\mathbb{Z})$ , we will slightly abuse notation by also using  $\leq$  to denote any binary relation on  $\{\rho, \sigma\} \times \mathbb{Z}$  with the property that  $(\pi, i) \leq (\pi, j) \iff i \leq j$  for all  $\pi \in \{\rho, \sigma\}$  and  $i, j \in \mathbb{Z}$ . Part (1) of Theorem A follows from Proposition 1.2 and:

**Proposition 1.3.** *Suppose that  $\rho, \sigma \in \text{Sym}(\mathbb{Z})$  and  $S^{\mathbb{Z}} \in \text{Cl}(\rho)\text{Cl}(\sigma)$ . Then  $|\text{graph}'_{\geq}(\rho \amalg \sigma)| \leq |\text{graph}'_{<}(\rho \amalg \sigma)| + 1$ .*

*Proof.* Let  $I$  denote the involution of  $\text{graph}(\rho \amalg \sigma)$  given by

$$I((\pi, i), (\pi, j)) = \begin{cases} ((\pi, j), (\pi, i)) & \text{if } i, j \in \text{Per}_2(\pi) \text{ and} \\ ((\pi, i), (\pi, j)) & \text{otherwise.} \end{cases}$$

Define  $J: \text{graph}(\rho \amalg \sigma) \rightarrow (\{\rho, \sigma\} \times \mathbb{Z})^2$  by

$$J((\pi, i), (\pi, j)) = \begin{cases} ((\sigma, j-1), (\sigma, i)) & \text{if } \pi = \rho \text{ and} \\ ((\rho, j), (\rho, i+1)) & \text{if } \pi = \sigma. \end{cases}$$

**Lemma 1.4.**  $J$  is an involution of  $\text{graph}(\rho \amalg \sigma)$ .

*Proof.* It is sufficient to show that if  $((\pi, i), (\pi, j)) \in \text{graph}(\rho \amalg \sigma)$ , then  $J((\pi, i), (\pi, j)) \in \text{graph}(\rho \amalg \sigma)$ .

If  $\pi = \rho$ , then  $\rho(i) = j = S^{\mathbb{Z}}(j-1) = (\rho \circ \sigma)(j-1)$ , so  $i = \sigma(j-1)$ , thus  $J((\pi, i), (\pi, j)) = ((\sigma, j-1), (\sigma, i)) \in \text{graph}(\rho \amalg \sigma)$ .

If  $\pi = \sigma$ , then  $\sigma(i) = j$ , so  $\rho(j) = (\rho \circ \sigma)(i) = S^{\mathbb{Z}}(i) = i+1$ , thus  $J((\pi, i), (\pi, j)) = ((\rho, j), (\rho, i+1)) \in \text{graph}(\rho \amalg \sigma)$ .  $\square$

**Lemma 1.5.**  $J(\text{graph}_{<}(\rho \amalg \sigma)) = \text{graph}_{\geq}(\rho \amalg \sigma)$ .

*Proof.* Note that  $((\pi, i), (\pi, j)) \in \text{graph}_{<}(\rho \amalg \sigma) \iff i < j \iff j-1 \geq i \iff j \geq i+1 \iff J((\pi, i), (\pi, j)) \in \text{graph}_{\geq}(\rho \amalg \sigma)$ .  $\square$

The *length* of  $((\pi, i), (\pi, j)) \in \text{graph}(\rho \amalg \sigma)$  is  $|((\pi, i), (\pi, j))| = |i-j|$ .

**Lemma 1.6.** Suppose that  $((\pi, i), (\pi, j)) \in \text{graph}(\rho \amalg \sigma)$ .

- (1) If  $i < j$ , then  $|J((\pi, i), (\pi, j))| = |((\pi, i), (\pi, j))| - 1$ .
- (2) If  $i \geq j$ , then  $|J((\pi, i), (\pi, j))| = |((\pi, i), (\pi, j))| + 1$ .

*Proof.* If  $i < j$ , then  $i+1 \leq j$ , so

$$|(j-1) - i| = |j - (i+1)| = j - (i+1) = (j-i) - 1 = |i-j| - 1,$$

thus  $|J((\pi, i), (\pi, j))| = |((\pi, i), (\pi, j))| - 1$ . If  $i \geq j$ , then  $i+1 > j$ , so

$$|(j-1) - i| = |j - (i+1)| = (i+1) - j = (i-j) + 1 = |i-j| + 1,$$

thus  $|J((\pi, i), (\pi, j))| = |((\pi, i), (\pi, j))| + 1$ .  $\square$

Let  $G$  be the group generated by  $I$  and  $J$ . The *orbit* of  $((\pi, i), (\pi, j))$  under  $G$  is given by  $[((\pi, i), (\pi, j))]_G = \{g \cdot ((\pi, i), (\pi, j)) \mid g \in G\}$ . Set  $\mathcal{O}(G) = \{[((\pi, i), (\pi, j))]_G \mid ((\pi, i), (\pi, j)) \in \text{graph}(\rho \amalg \sigma)\}$ .

**Lemma 1.7.** Suppose that  $O \in \mathcal{O}(G)$ . Then  $\text{graph}'_{\geq}(\rho \amalg \sigma) \cap O \neq \emptyset$ .

*Proof.* Fix  $((\pi, i), (\pi, j)) \in O$ . We can assume that  $((\pi, i), (\pi, j)) \notin \text{graph}'_{\geq}(\rho \amalg \sigma)$ . By replacing  $((\pi, i), (\pi, j))$  with  $I((\pi, i), (\pi, j))$  if necessary, we can therefore assume that  $((\pi, i), (\pi, j)) \in \text{graph}_{<}(\rho \amalg \sigma)$ . For all  $n \in \mathbb{N}$ , define  $((\pi, i_n), (\pi, j_n)) = (I \circ J)^n((\pi, i), (\pi, j))$ . Note that if  $((\pi, i_n), (\pi, j_n)) \in \text{graph}_{<}(\rho \amalg \sigma)$ , then  $J((\pi, i_n), (\pi, j_n)) \in \text{graph}_{\geq}(\rho \amalg \sigma)$  and  $|J((\pi, i_n), (\pi, j_n))| = |((\pi, i_n), (\pi, j_n))| - 1$  by Lemmas 1.5 and 1.6. Observe further that if  $J((\pi, i_n), (\pi, j_n)) \notin \text{graph}'_{\geq}(\rho \amalg \sigma)$ , then

$((\pi, i_{n+1}), (\pi, j_{n+1})) \in \text{graph}_{<}(\rho \amalg \sigma)$ . Setting  $n = |i - j| - 1$ , it follows that if  $J((\pi, i_m), (\pi, j_m)) \notin \text{graph}'_{<}(\rho \amalg \sigma)$  for all  $m < n$ , then  $J((\pi, i_n), (\pi, j_n)) = 0$ , thus  $J((\pi, i_n), (\pi, j_n)) \in \text{graph}'_{\geq}(\rho \amalg \sigma)$ .  $\square$

For all  $i, j \in \mathbb{Z}$ , we slightly abuse the usual notation by using  $(i, j)$ ,  $[i, j)$ ,  $(i, j]$ , and  $[i, j]$  to denote the corresponding intervals of integers.

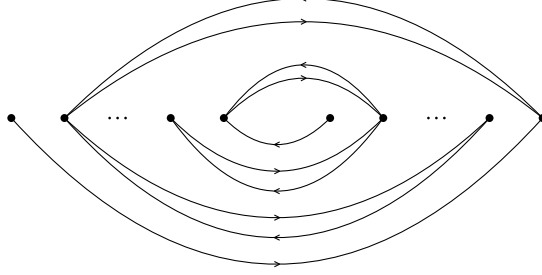


FIGURE 1. A finite orbit of  $G$ .

**Lemma 1.8.** *Suppose that  $O \in \mathcal{O}(G)$ .*

- (1) *If  $O$  is finite, then  $|\text{graph}'_{\geq}(\rho \amalg \sigma) \cap O| = |\text{graph}'_{<}(\rho \amalg \sigma) \cap O| = 1$ .*
- (2) *If  $O$  is infinite, then  $O$  is a cofinite subset of  $\text{graph}(\rho \amalg \sigma)$ ,  $|\text{graph}'_{\geq}(\rho \amalg \sigma) \cap O| = 1$ , and  $\text{graph}'_{<}(\rho \amalg \sigma) \cap O = \emptyset$ .*

*Proof.* By Lemma 1.7, there exists  $((\pi, i), (\pi, j)) \in \text{graph}'_{\geq}(\rho \amalg \sigma) \cap O$ . For all  $n \in \mathbb{N}$ , define  $((\pi, i_n), (\pi, j_n)) = (I \circ J)^n((\pi, i), (\pi, j))$ . If  $n \in \mathbb{N}$  and  $((\pi, i_n), (\pi, j_n)) \in \text{graph}_{\geq}(\rho \amalg \sigma)$ , then Lemma 1.5 ensures that  $J((\pi, i_n), (\pi, j_n)) \in \text{graph}_{<}(\rho \amalg \sigma)$ . If  $J((\pi, i_n), (\pi, j_n)) \notin \text{graph}'_{<}(\rho \amalg \sigma)$ , then  $((\pi, i_{n+1}), (\pi, j_{n+1})) \in \text{graph}_{\geq}(\rho \amalg \sigma)$ .

Suppose now that  $n$  is the least natural number with the property that  $J((\pi, i_n), (\pi, j_n)) \in \text{graph}'_{<}(\rho \amalg \sigma)$ . Then  $((\pi, i), (\pi, j))$  is in  $\text{graph}'_{\geq}(\rho \amalg \sigma) \cap O$ ,  $J((\pi, i_n), (\pi, j_n))$  is in  $\text{graph}'_{<}(\rho \amalg \sigma) \cap O$ , and the pairs  $J((\pi, i_m), (\pi, j_m)), ((\pi, i_{m+1}), (\pi, j_{m+1})) \in \text{Per}_2(\rho \amalg \sigma)^2$ , for  $m < n$ , make up the rest of  $O$ , so condition (1) holds.

Finally, suppose that there is no  $n \in \mathbb{N}$  for which  $J((\pi, i_n), (\pi, j_n)) \in \text{graph}'_{<}(\rho \amalg \sigma)$ . Then  $((\pi, i), (\pi, j))$  is in  $\text{graph}'_{\geq}(\rho \amalg \sigma) \cap O$  and the pairs  $J((\pi, i_n), (\pi, j_n)), ((\pi, i_{n+1}), (\pi, j_{n+1})) \in \text{Per}_2(\rho \amalg \sigma)^2$ , for  $n \in \mathbb{N}$ , make up the rest of  $O$ , in which case  $\text{graph}'_{<}(\rho \amalg \sigma) \cap O = \emptyset$ . Moreover, a straightforward induction shows that, for all  $n \in \mathbb{N}$ , the following hold:

- $i_{2n} = i + n$  and  $j_{2n} = j - n$ .
- $\pi = \rho \implies i_{2n+1} = i + n$  and  $j_{2n+1} = j - (n + 1)$ .
- $\pi = \sigma \implies i_{2n+1} = i + (n + 1)$  and  $j_{2n+1} = j - n$ .

Then  $\text{graph}(\rho \amalg \sigma) \setminus O \subseteq (\{\rho, \sigma\} \times [i, j])^2$ , so condition (2) holds.  $\square$

As at most one orbit of  $G$  can be cofinite, Lemma 1.8 ensures that  $|\text{graph}'_{\geq}(\rho \amalg \sigma)| = |\mathcal{O}(G)| \leq |\text{graph}'_{<}(\rho \amalg \sigma)| + 1$ .  $\square$

## 2. DUALS

We use  $f: X \hookrightarrow Y$  to denote a partial injection of  $X$  into  $Y$ . For all  $\sigma: \mathbb{Z} \hookrightarrow \mathbb{Z}$ , define  $\bar{\sigma}: \mathbb{Z} \hookrightarrow \mathbb{Z}$  by  $\bar{\sigma}(i) = -\sigma^{-1}(-i)$  for all  $i \in \mathbb{Z}$ .

**Proposition 2.1.** *Suppose that  $\sigma: \mathbb{Z} \hookrightarrow \mathbb{Z}$ . Then  $\sigma = \bar{\bar{\sigma}}$ .*

*Proof.* If  $i \in \mathbb{Z}$ , then  $\bar{\bar{\sigma}}(i) = -(\bar{\sigma})^{-1}(-i)$ , so  $\bar{\sigma}(-\bar{\bar{\sigma}}(i)) = -i$ . But  $\bar{\sigma}(-\bar{\bar{\sigma}}(i)) = -\sigma^{-1}(\bar{\bar{\sigma}}(i))$ , so  $i = \sigma^{-1}(\bar{\bar{\sigma}}(i))$ , thus  $\sigma(i) = \bar{\bar{\sigma}}(i)$ .  $\square$

**Proposition 2.2.** *Suppose that  $\rho, \sigma: \mathbb{Z} \hookrightarrow \mathbb{Z}$ . Then  $\bar{\rho} \circ \bar{\sigma} = \bar{\sigma} \circ \bar{\rho}$ .*

*Proof.* Observe that

$$\begin{aligned} (\bar{\sigma} \circ \bar{\rho})(i) &= -\sigma^{-1}(-(-\rho^{-1}(-i))) \\ &= -(\sigma^{-1} \circ \rho^{-1})(-i) \\ &= -(\rho \circ \sigma)^{-1}(-i) \\ &= \bar{\rho \circ \sigma}(i) \end{aligned}$$

for all  $i \in \mathbb{Z}$ .  $\square$

Define  $\mathcal{F} = \{(\rho: \mathbb{Z} \hookrightarrow \mathbb{Z}, \sigma: \mathbb{Z} \hookrightarrow \mathbb{Z}) \mid \rho \circ \sigma = S^{\mathbb{Z}} \upharpoonright \text{dom}(\rho \circ \sigma)\}$ .

**Proposition 2.3.**  $(\rho, \sigma) \in \mathcal{F} \iff (\bar{\sigma}, \bar{\rho}) \in \mathcal{F}$ .

*Proof.* Note that if  $i \in \mathbb{Z}$  and  $\rho, \sigma: \mathbb{Z} \hookrightarrow \mathbb{Z}$ , then  $(\rho \circ \sigma)(i) = i + 1 \iff (\rho \circ \sigma)^{-1}(i + 1) = i \iff \bar{\rho} \circ \bar{\sigma}(-i - 1) = -i$ , so the desired result follows from Proposition 2.2.  $\square$

Let  $(i_0 \ i_1 \ \dots \ i_n)$  denote the permutation of  $\{i_m \mid m \leq n\}$  sending  $i_m$  to  $i_{m+1}$  for all  $m < n$ .

**Proposition 2.4.** *Suppose that  $n \geq 1$ ,  $(i_m)_{m \leq n}$  is strictly increasing,  $\rho = (i_0 \ i_1 \ \dots \ i_n)$ , and  $\sigma = (-i_n \ -i_{n-1} \ \dots \ -i_0)$ . Then  $\rho = \bar{\sigma}$ .*

*Proof.* If  $m < n$ , then  $\bar{\sigma}(i_m) = -\sigma^{-1}(-i_m) = -(-i_{m+1}) = i_{m+1}$ .  $\square$

## 3. ELIMINATING FIXED POINTS

For all  $k \in \mathbb{N}$ , let  $\text{par}(k)$  denote the remainder when  $k$  is divided by two. For all  $\rho, \sigma \in \text{Sym}(X)$ , set  $\delta(\rho, \sigma) = \{x \in X \mid \rho(x) \neq \sigma(x)\}$  and

$$\text{Mal}(\rho, \sigma) = \{x \in \text{Per}_{\mathbb{N}+3}(\sigma) \mid |[x]_{\sigma} \setminus \text{Per}_1(\rho)| = 1\}.$$

**Proposition 3.1.** *Suppose that  $m \geq 1$ ,  $\rho$  and  $\sigma$  are permutations of a set  $X$ , and  $\forall n \geq 3 \ \neg 0 < |\text{Mal}(\rho, \sigma) \cap \text{Per}_{2\mathbb{N}+n}(\sigma)| < \aleph_0$ . Then there are permutations  $\rho'$  and  $\sigma'$  of  $X$  such that:*

- (1)  $\rho \circ \sigma = \rho' \circ \sigma'$ ,
- (2)  $\delta(\rho, \rho') = \delta(\sigma^{-1}, (\sigma')^{-1}) = \text{Mal}(\rho, \sigma) \cap \text{Per}_1(\rho)$ ,
- (3)  $\text{Mal}(\rho, \sigma) \cap \text{Per}_1(\rho) \subseteq \text{Per}_m(\rho')$ , and
- (4)  $\forall n \geq 3 \text{ Mal}(\rho, \sigma) \cap \text{Per}_n(\sigma) \subseteq \text{Per}_n(\sigma')$ .

*Proof.* Define  $Y = \text{Mal}(\rho, \sigma)$  and  $Z = Y \setminus \text{Per}_1(\rho)$ . For all  $n \geq 3$ , set  $Y_n = \text{Per}_{2\mathbb{N}+n}(\sigma) \cap Y$  and  $Z_n = \text{Per}_{2\mathbb{N}+n}(\sigma) \cap Z$ . Fix an equivalence relation  $F_4$  on  $Z_4$  whose classes all have cardinality  $m^2$ , as well as  $\pi_{0,1}, \pi_{0,2} \in \text{Sym}(Z_4)$ , whose graphs are contained in  $F_4$ , such that the orbits of  $\pi_{0,1}$ ,  $\pi_{0,2}$ , and  $\pi_{0,3} = (\pi_{0,1} \circ \pi_{0,2})^{-1}$  all have cardinality  $m$ . For all  $n \in (\mathbb{N}+3) \setminus \{4\}$ , fix an equivalence relation  $F_n$  on  $Z_n$  whose classes all have cardinality  $m$ , fix  $\pi_{\text{par}(n),n-2} \in \text{Sym}(Z_n)$  whose orbits coincide with the equivalence classes of  $F_n$ , and set  $\pi_{\text{par}(n),n-1} = \pi_{\text{par}(n),n-2}^{-1}$ . Then the support of  $\pi = \text{id}_{X \setminus (Y \setminus Z)} \cup \bigcup_{p < 2, n \geq 1} \sigma^n \circ \pi_{p,n} \circ \sigma^{-n}$  is  $Y \setminus Z$ , so  $\rho' = \rho \circ \pi$  and  $\sigma' = \pi^{-1} \circ \sigma$  satisfy conditions (1)–(3).

**Lemma 3.2.** *Suppose that  $\ell \leq n - 1$ . Then*

$$(\sigma')^\ell \upharpoonright Z_n = (\sigma^\ell \circ \pi_{\text{par}(n),\ell}^{-1} \circ \cdots \circ \pi_{\text{par}(n),1}^{-1}) \upharpoonright Z_n. \quad (*)$$

*Proof.* The case  $\ell = 0$  is trivial. If  $\ell > 0$  and  $(*)$  holds at  $\ell - 1$ , then

$$\begin{aligned} (\sigma')^\ell \upharpoonright Z_n &= (\sigma' \circ (\sigma')^{\ell-1}) \upharpoonright Z_n \\ &= (\sigma' \circ \sigma^{\ell-1} \circ \pi_{\text{par}(n),\ell-1}^{-1} \circ \cdots \circ \pi_{\text{par}(n),1}^{-1}) \upharpoonright Z_n \\ &= (\pi^{-1} \circ \sigma^\ell \circ \pi_{\text{par}(n),\ell-1}^{-1} \circ \cdots \circ \pi_{\text{par}(n),1}^{-1}) \upharpoonright Z_n \\ &= (\sigma^\ell \circ \pi_{\text{par}(n),\ell}^{-1} \circ \sigma^{-\ell} \circ \sigma^\ell \circ \pi_{\text{par}(n),\ell-1}^{-1} \circ \cdots \circ \pi_{\text{par}(n),1}^{-1}) \upharpoonright Z_n \\ &= (\sigma^\ell \circ \pi_{\text{par}(n),\ell}^{-1} \circ \cdots \circ \pi_{\text{par}(n),1}^{-1}) \upharpoonright Z_n, \end{aligned}$$

so  $(*)$  also holds at  $\ell$ . \(\square\)

For all  $n \geq 3$ , set  $Y'_n = \text{Per}_n(\sigma) \cap Y$  and  $Z'_n = \text{Per}_n(\sigma) \cap Z$ . Lemma 3.2 ensures that  $Y'_n = \bigcup_{\ell \leq n-1} \sigma^\ell(Z'_n) = \bigcup_{\ell \leq n-1} (\sigma')^\ell(Z'_n)$  and

$$\begin{aligned} (\sigma')^n \upharpoonright Z'_n &= (\sigma' \circ (\sigma')^{n-1}) \upharpoonright Z'_n \\ &= (\sigma' \circ \sigma^{n-1} \circ \pi_{\text{par}(n),n-1}^{-1} \circ \cdots \circ \pi_{\text{par}(n),1}^{-1}) \upharpoonright Z'_n \\ &= (\sigma' \circ \sigma^{n-1}) \upharpoonright Z'_n \\ &= (\sigma' \circ \sigma^{-1}) \upharpoonright Z'_n \\ &= \text{id}_{Z'_n}, \end{aligned}$$

so condition (4) also holds. \(\square\)

We write  $\rho \cong \sigma$  to indicate that  $\rho$  and  $\sigma$  are isomorphic.

**Proposition 3.3.** *Suppose that  $m \geq 1$ ,  $\rho$  and  $\sigma$  are permutations of a set  $X$ ,  $\forall n \geq 3 \neg 0 < |\text{Mal}(\rho, \sigma) \cap \text{Per}_{2\mathbb{N}+n}(\sigma)| < \aleph_0$ , and  $\text{Per}_m(\rho)$  is infinite. Then there are permutations  $\rho' \cong \rho \upharpoonright \sim(\text{Mal}(\rho, \sigma) \cap \text{Per}_1(\rho))$  and  $\sigma' \cong \sigma$  of  $X$  for which  $\rho \circ \sigma = \rho' \circ \sigma'$ .*

*Proof.* Proposition 3.1 yields  $\rho', \sigma' \in \text{Sym}(X)$  such that  $\rho \circ \sigma = \rho' \circ \sigma'$  and  $|\mathcal{O}_\kappa(\rho \upharpoonright \sim(\text{Mal}(\rho, \sigma) \cap \text{Per}_1(\rho)))| = |\mathcal{O}_\kappa(\rho')|$  and  $|\mathcal{O}_\kappa(\sigma)| = |\mathcal{O}_\kappa(\sigma')|$  for all cardinals  $\kappa$ .  $\square$

#### 4. BUILDING BLOCKS

Set  $\mathcal{F}(i, j] = \{(\rho, \sigma) \in \mathcal{F} \mid \rho: (i, j] \hookrightarrow (i, j] \text{ and } \sigma: (i, j] \hookrightarrow (i, j]\}$ , noting that  $\forall (\rho, \sigma) \in \mathcal{F}(i, j] \text{ dom}(\rho \circ \sigma) = (i, j - 1]$ .

**Proposition 4.1.** *If  $i < j$  and  $(\rho, \sigma) \in \mathcal{F}(i, j]$ , then  $\rho(j) = i + 1$ .*

*Proof.* Observe that  $\rho((i, j - 1]) = (\rho \circ \sigma)((i, j - 1]) = (i + 1, j]$ .  $\square$

Set  $\mathcal{F}[i, j] = \{(\rho, \sigma) \in \mathcal{F} \mid \rho: (i, j) \hookrightarrow (i, j) \text{ and } \sigma: [i, j] \hookrightarrow [i, j]\}$ , this time noting that  $\forall (\rho, \sigma) \in \mathcal{F}(i, j] \ S^{\mathbb{Z}}(j - 1) \notin \text{rng}(\rho)$ , and therefore  $\forall (\rho, \sigma) \in \mathcal{F}(i, j] \text{ dom}(\rho \circ \sigma) = [i, j - 1]$ .

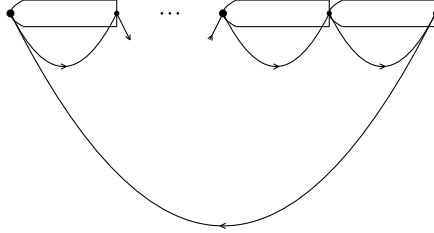


FIGURE 2. The extension provided by Proposition 4.2.

**Proposition 4.2.** *Suppose that  $n \geq 1$ ,  $(i_m)_{m \leq n}$  is strictly increasing,  $\forall m < n \ (\rho_m, \sigma_m) \in \mathcal{F}(i_m, i_{m+1}]$ ,  $\rho = \bigcup_{m < n} \rho_m$ , and  $\sigma = (i_0 \ i_1 \ \dots \ i_n) \cup \bigcup_{m < n} \sigma_m$ . Then  $(\rho, \sigma) \in \mathcal{F}[i_0, i_n + 1]$ .*

*Proof.* As  $[i_0, i_n] = \{i_m \mid m < n\} \cup \bigcup_{m < n} (i_m, i_{m+1} - 1]$ , it follows that

$$\begin{aligned} (\rho, \sigma) \in \mathcal{F}[i_0, i_n + 1] &\iff \forall k \in [i_0, i_n] \ (\rho \circ \sigma)(k) = k + 1 \\ &\iff \forall m < n \ (\rho \circ \sigma)(i_m) = i_m + 1 \\ &\iff \forall m < n \ \rho(i_{m+1}) = i_m + 1 \\ &\iff \forall m < n \ \rho_m(i_{m+1}) = i_m + 1, \end{aligned}$$

so Proposition 4.1 yields the desired result.  $\square$

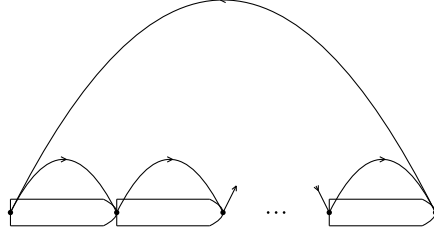


FIGURE 3. The extension provided by Proposition 4.3.

**Proposition 4.3.** *Suppose that  $n \geq 1$ ,  $(i_m)_{m \leq n}$  is strictly increasing,  $\forall m < n$   $(\rho_m, \sigma_m) \in \mathcal{F}[i_m, i_{m+1})$ ,  $\rho = (i_0 \ i_1 \ \cdots \ i_n) \cup \bigcup_{m < n} \rho_m$ , and  $\sigma = \bigcup_{m < n} \sigma_m$ . Then  $(\rho, \sigma) \in \mathcal{F}(i_0 - 1, i_n]$ .*

*Proof.* By Propositions 2.1, 2.3, 2.4, and 4.2. ⊠

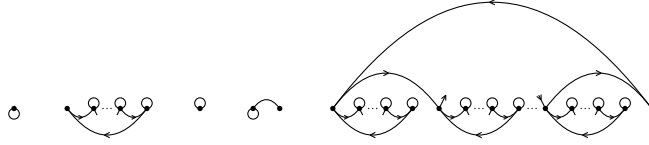


FIGURE 4. Building blocks from Propositions 4.5 and 4.6.

**Proposition 4.4.** *Suppose that  $i \in \mathbb{Z}$ . Then  $(\emptyset, \text{id}_{\{i\}}) \in \mathcal{F}[i, i+1)$  and  $(\text{id}_{\{i\}}, \emptyset) \in \mathcal{F}(i-1, i]$ .*

*Proof.* As  $[i, i) = (i-1, i-1] = \emptyset$ , the definitions of  $\mathcal{F}[i, i+1)$  and  $\mathcal{F}(i-1, i]$  yield that  $(\rho, \sigma) \in \mathcal{F}[i, i+1) \iff (\rho = \emptyset \text{ and } \text{dom}(\sigma) = \{i\})$  and  $(\rho, \sigma) \in \mathcal{F}(i-1, i] \iff (\text{dom}(\rho) = \{i\} \text{ and } \sigma = \emptyset)$ . ⊠

**Proposition 4.5.** *Suppose that  $i < j$  are integers. Then the pair  $(\text{id}_{(i,j)}, (i \ i+1 \ \cdots \ j-1))$  is in  $\mathcal{F}[i, j)$ .*

*Proof.* If  $i+1 = j$ , then this follows from Proposition 4.4. Otherwise, Proposition 4.4 ensures that  $(\text{id}_{\{k\}}, \emptyset) \in \mathcal{F}(k-1, k]$  for all  $k \in (i, j)$ , so Proposition 4.2 yields the desired result. ⊠

**Proposition 4.6.** *Suppose that  $m \geq 1$  and  $(i_k)_{k < m}$  is a strictly increasing sequence of integers. Then the pair*

*$((i_0 \ i_1 \ \cdots \ i_{m-1}) \cup \bigcup_{k < m-1} \text{id}_{(i_k, i_{k+1})}, \bigcup_{k < m-1} (i_k \ i_k+1 \ \cdots \ i_{k+1}-1))$*   
*is in  $\mathcal{F}(i_0 - 1, i_{m-1}]$ .*

*Proof.* If  $m = 1$ , then this follows from Proposition 4.4. Otherwise,  $(\text{id}_{(i_k, i_{k+1})}, (i_k \ i_k+1 \ \cdots \ i_{k+1}-1)) \in \mathcal{F}[i_k, i_{k+1})$  for all  $k < m-1$  by Proposition 4.5, so Proposition 4.3 yields the desired result. ⊠



## 5. EXTENSION

Given  $n \geq 3$  and  $\rho, \sigma: X \hookrightarrow X$ , we say that a fixed point  $x$  of  $\rho$  is  $n$ -malleable if  $x \in \text{Per}_n(\sigma)$ ,  $[x]_\sigma \subseteq \text{dom}(\rho)$ , and  $|[x]_\sigma \setminus \text{Per}_1(\rho)| = 1$ .

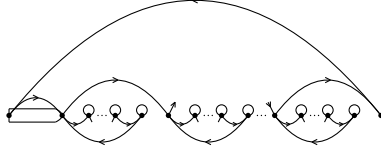


FIGURE 5. The extension provided by Proposition 5.1.

**Proposition 5.1.** *Suppose that  $i < j$ ,  $m \geq 2$ ,  $n_k \geq 3$  for all  $k < m-2$ , and  $(\rho, \sigma) \in \mathcal{F}[i, j]$ . Then there exists  $(\rho', \sigma') \in \mathcal{F}(i-1, j + \sum_{k < m-2} n_k)$  such that:*

- $\rho'$  is obtained from  $\rho$  by adding a single cycle of length  $m$  and  $n_k - 1$   $n_k$ -malleable fixed points for all  $k < m-2$ .
- $\sigma'$  is obtained from  $\sigma$  by adding a cycle of length  $n_k$  for all  $k < m-2$ .

*Proof.* Recursively define  $i_0 = i$ ,  $i_1 = j$ , and  $i_k = i_{k-1} + n_{k-2}$  for all  $2 \leq k \leq m-1$ . Set  $(\rho_0, \sigma_0) = (\rho, \sigma)$ . For all  $1 \leq k \leq m-2$ , Proposition 4.5 ensures that  $(\rho_k, \sigma_k) = (\text{id}_{(i_k, i_{k+1})}, (i_k \ i_k + 1 \ \cdots \ i_{k+1} - 1))$  is in  $\mathcal{F}[i_k, i_{k+1}]$ . So  $(\rho', \sigma') = ((i_0 \ i_1 \ \cdots \ i_{m-1}) \cup \bigcup_{k \leq m-2} \rho_k, \bigcup_{k \leq m-2} \sigma_k)$  is in  $\mathcal{F}(i_0 - 1, i_{m-1}]$  by Proposition 4.3. But  $i_0 - 1 = i - 1$  and  $i_{m-1} = j + \sum_{k < m-2} n_k$ .  $\square$

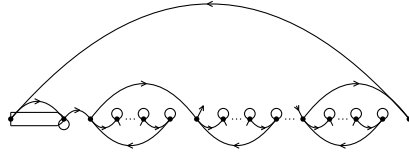


FIGURE 6. The extension provided by Proposition 5.2.

**Proposition 5.2.** *Suppose that  $i < j$ ,  $m \geq 3$ ,  $n_k \geq 3$  for all  $k < m-3$ , and  $(\rho, \sigma) \in \mathcal{F}[i, j]$ . Then there exists  $(\rho', \sigma') \in \mathcal{F}(i-1, j+1 + \sum_{k < m-3} n_k)$  such that:*

- $\rho'$  is obtained from  $\rho$  by adding a single cycle of length  $m$  and  $n_k - 1$   $n_k$ -malleable fixed points for all  $k < m-3$ .
- $\sigma'$  is obtained from  $\sigma$  by adding a fixed point and a cycle of length  $n_k$  for all  $k < m-3$ .

*Proof.* Recursively define  $i_0 = i$ ,  $i_1 = j$ ,  $i_2 = j + 1$ , and  $i_k = i_{k-1} + n_{k-3}$  for all  $3 \leq k \leq m - 1$ . Set  $(\rho_0, \sigma_0) = (\rho, \sigma)$ . For all  $1 \leq k \leq m - 2$ , Proposition 4.5 ensures that  $(\rho_k, \sigma_k) = (\text{id}_{(i_k, i_{k+1})}, (i_k \ i_{k+1} \cdots i_{k+1} - 1))$  is in  $\mathcal{F}[i_k, i_{k+1}]$ . So  $(\rho', \sigma') = ((i_0 \ i_1 \cdots i_{m-1}) \cup \bigcup_{k \leq m-2} \rho_k, \bigcup_{k \leq m-2} \sigma_k)$  is in  $\mathcal{F}(i_0 - 1, i_{m-1})$  by Proposition 4.3. But  $i_0 - 1 = i - 1$  and  $i_{m-1} = j + 1 + \sum_{k < m-3} n_k$ .  $\square$

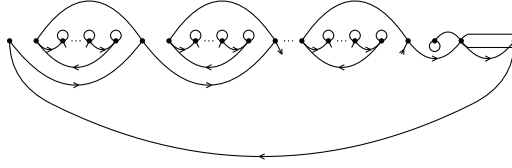


FIGURE 7. The extension provided by Proposition 5.3.

**Proposition 5.3.** *Suppose that  $i < j$ ,  $n \geq 3$ ,  $n_k \geq 3$  for all  $k < n - 3$ , and  $(\rho, \sigma) \in \mathcal{F}(i, j]$ . Then there exists  $(\rho', \sigma') \in \mathcal{F}[i - 2 - \sum_{k < n-3} (n_k + 1), j + 1)$  such that:*

- $\rho'$  is obtained from  $\rho$  by adding  $n - 2$  cycles of length two and  $n_k - 1$   $n_k$ -malleable fixed points for all  $k < n - 3$ .
- $\sigma'$  is obtained from  $\sigma$  by adding a single fixed point, a cycle of length  $n$ , and a cycle of length of  $n_k$  for all  $k < n - 3$ .

*Proof.* Recursively define  $i_{n-1} = j$ ,  $i_{n-2} = i$ ,  $i_{n-3} = i - 2$ , and  $i_k = i_{k+1} - (n_k + 1)$  for all  $k \leq n - 4$ . Set  $(\rho_{n-2}, \sigma_{n-2}) = (\rho, \sigma)$ . For all  $k \leq n - 3$ , Proposition 4.6 implies that

$$(\rho_k, \sigma_k) = (\text{id}_{(i_k+1, i_{k+1})} \cup (i_k + 1 \ i_{k+1}), (i_k + 1 \ i_k + 2 \cdots i_{k+1} - 1))$$

is in  $\mathcal{F}(i_k, i_{k+1}]$ . So  $(\rho', \sigma') = (\bigcup_{k \leq n-2} \rho_k, (i_0 \ i_1 \cdots i_{n-1}) \cup \bigcup_{k \leq n-2} \sigma_k)$  is in  $\mathcal{F}[i_0, i_{n-1} + 1)$  by Proposition 4.2. But  $i - 2 - \sum_{k < n-3} (n_k + 1) = i_0$  and  $j + 1 = i_{n-1} + 1$ .  $\square$

**Proposition 5.4.** *Suppose that  $i < j$ ,  $m \geq 2$ ,  $n \geq 2$ ,  $n_{k,\ell} \geq 3$  for all  $k < m - 1$  and  $\ell < n - 2$ , and  $(\rho, \sigma) \in \mathcal{F}(i, j]$ . Then there exists  $(\rho', \sigma') \in \mathcal{F}[i - \sum_{k < m-1, \ell < n-2} n_{k,\ell}, j + 1)$  such that:*

- $\rho'$  is obtained from  $\rho$  by adding  $n - 2$  cycles of length  $m$  and  $n_{k,\ell} - 1$   $n_{k,\ell}$ -malleable fixed points for all  $k < m - 1$  and  $\ell < n - 2$ .
- $\sigma'$  is obtained from  $\sigma$  by adding a cycle of length  $n$  and a cycle of length  $n_{k,\ell}$  for all  $k < m - 1$  and  $\ell < n - 2$ .

*Proof.* Recursively define  $i_{n-1} = j$ ,  $i_{n-2} = i$ ,  $i_\ell = i_{\ell+1} - \sum_{k < m-1} n_{k,\ell}$ ,  $i_{0,\ell} = i_\ell + 1$ , and  $i_{k,\ell} = i_{k-1,\ell} + n_{k-1,\ell}$  for  $k \leq m - 1$  and  $\ell \leq n - 3$ . Set  $(\rho_{n-2}, \sigma_{n-2}) = (\rho, \sigma)$ . For all  $\ell \leq n - 3$ , Proposition 4.6 implies that the

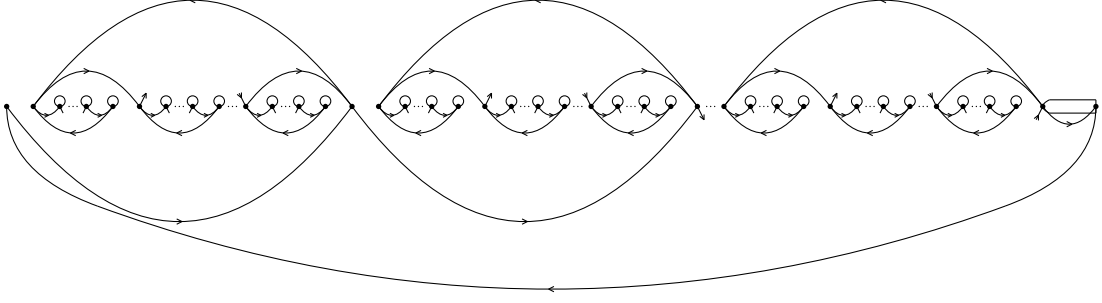


FIGURE 8. The extension provided by Proposition 5.4.

pair  $(\rho_\ell, \sigma_\ell)$ , given by  $\rho_\ell = (i_{0,\ell} \ i_{1,\ell} \ \cdots \ i_{m-1,\ell}) \cup \bigcup_{k < m-1} \text{id}_{(i_{k,\ell}, i_{k+1,\ell})}$  and  $\sigma_\ell = \bigcup_{k < m-1} (i_{k,\ell} \ i_{k,\ell}+1 \ \cdots \ i_{k+1,\ell}-1)$ , is in  $\mathcal{F}(i_\ell, i_{\ell+1}]$ . So Proposition 4.2 yields that  $(\rho', \sigma') = (\bigcup_{\ell \leq n-2} \rho_\ell, (i_0 \ i_1 \ \cdots \ i_{n-1}) \cup \bigcup_{\ell \leq n-2} \sigma_\ell)$  is in  $\mathcal{F}[i_0, i_{n-1}+1)$ . But  $i - \sum_{k < m-1, \ell < n-2} n_{k,\ell} = i_0$  and  $j+1 = i_{n-1}+1$ .  $\square$

We say that a fixed point  $x$  of  $\rho$  is *anti-malleable* if  $x \in \text{Per}_2(\sigma)$ ,  $[x]_\sigma \subseteq \text{dom}(\rho)$ , and  $|[x]_\sigma \setminus \text{Per}_1(\rho)| = 1$ .

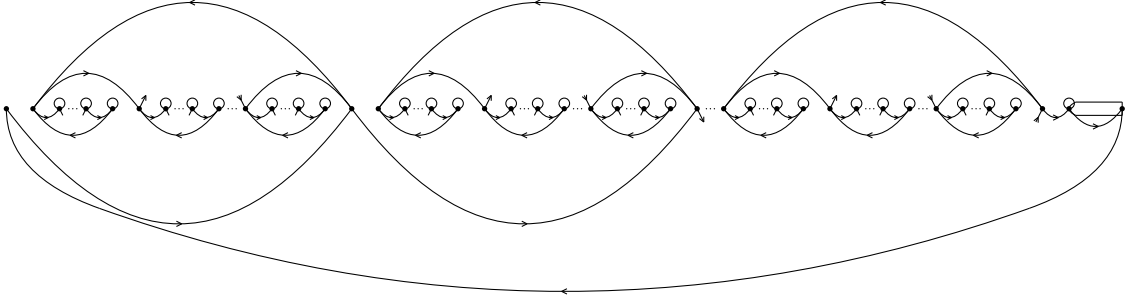


FIGURE 9. The extension provided by Proposition 5.5.

**Proposition 5.5.** *Suppose that  $i < j$ ,  $m \geq 2$ ,  $n \geq 3$ ,  $n_{k,\ell} \geq 3$  for all  $k < m-1$  and  $\ell < n-3$ , and  $(\rho, \sigma) \in \mathcal{F}(i, j]$ . Then there exists  $(\rho', \sigma') \in \mathcal{F}[i-1 - \sum_{k < m-1, \ell < n-3} n_{k,\ell}, j+1)$  such that:*

- $\rho'$  is obtained from  $\rho$  by adding a single anti-malleable fixed point,  $n-3$  cycles of length  $m$ , and  $n_{k,\ell}-1$   $n_{k,\ell}$ -malleable fixed points for all  $k < m-1$  and  $\ell < n-3$ .
- $\sigma'$  is obtained from  $\sigma$  by adding a cycle of length  $n$  and a cycle of length  $n_{k,\ell}$  for all  $k < m-1$  and  $\ell < n-3$ .

*Proof.* Recursively define  $i_{n-1} = j$ ,  $i_{n-2} = i$ ,  $i_{n-3} = i-1$ ,  $i_\ell = i_{\ell+1} - \sum_{k < m-1} n_{k,\ell}$ ,  $i_{0,\ell} = i_\ell + 1$ , and  $i_{k,\ell} = i_{k-1,\ell} + n_{k-1,\ell}$  for  $k \leq m-1$  and

$\ell \leq n - 4$ . Set  $(\rho_{n-2}, \sigma_{n-2}) = (\rho, \sigma)$ . For all  $\ell \leq n - 3$ , Proposition 4.6 implies that the pair  $(\rho_\ell, \sigma_\ell)$ , given by  $\rho_\ell = (i_{0,\ell} \ i_{1,\ell} \ \cdots \ i_{m-1,\ell}) \cup \bigcup_{k < m-1} \text{id}_{(i_{k,\ell}, i_{k+1,\ell})}$  and  $\sigma_\ell = \bigcup_{k < m-1} (i_{k,\ell} \ i_{k,\ell} + 1 \ \cdots \ i_{k+1,\ell} - 1)$ , is in  $\mathcal{F}(i_\ell, i_{\ell+1}]$ . So  $(\rho', \sigma') = (\bigcup_{\ell \leq n-2} \rho_\ell, (i_0 \ i_1 \ \cdots \ i_{n-1}) \cup \bigcup_{\ell \leq n-2} \sigma_\ell)$  is in  $\mathcal{F}[i_0, i_{n-1} + 1)$  by Proposition 4.2. But  $i - 1 - \sum_{k < m-1, \ell < n-3} n_{k,\ell} = i_0$  and  $j + 1 = i_{n-1} + 1$ .  $\square$

## 6. THE MAIN RESULT

The special case of part (2) of Theorem A where  $\rho$  or  $\sigma$  has finite order is a consequence of Proposition 2.3 and:

**Theorem 6.1.** *Suppose that  $m \geq 2$ ,  $\rho, \sigma \in \text{Sym}(\mathbb{Z})$  are periodic, and  $\text{Per}_m(\rho)$  and  $\text{Per}_{\geq 3}(\sigma)$  are infinite. Then  $S^\mathbb{Z} \in \text{Cl}(\rho)\text{Cl}(\sigma)$ .*

*Proof.* For all integers  $i < j$ , set  $\mathcal{F}_0(i, j) = \mathcal{F}[i, j)$  and  $\mathcal{F}_1(i, j) = \mathcal{F}(i, j]$ . Fix an enumeration  $(\pi_n, O_n)_{n \in \mathbb{N}}$  of the pairs of the form  $(\pi, O)$ , where  $\pi \in \{\rho, \sigma\}$  and  $O \in \mathcal{O}(\pi)$ . Then there is an infinite set  $N \subseteq \mathbb{N}$  and  $p < 2$  such that  $\pi_n = \sigma$ ,  $\text{par}(|O_n|) = p$ , and  $3 \leq |O_n| \leq |O_{n+1}|$  for all  $n \in N$ . Fix  $n_{-1} \in N$ , set  $N_0 = \mathbb{N} \setminus \{n_{-1}\}$ , and apply Proposition 4.5 to find  $i_0 < j_0$  and  $(\rho_0, \sigma_0) \in \mathcal{F}_0(i_0, j_0)$  such that every point of  $\text{dom}(\rho_0)$  is a malleable fixed point and the lone orbit of  $\sigma_0$  has cardinality  $|O_{n_{-1}}|$ .

Suppose that  $k$  is a natural number for which we have found  $i_k < j_k$ , a cofinite set  $N_k \subseteq \mathbb{N}$ , and  $(\rho_k, \sigma_k) \in \mathcal{F}_{\text{par}(k)}(i_k, j_k)$ . If  $k \in 2\mathbb{N}$ , then let  $n_k$  be the least element of  $N_k$  for which  $(\pi_{n_k} = \rho$  and  $|O_{n_k}| \geq 2)$  or  $(\pi_{n_k} = \sigma, |O_{n_k}| = 1, \text{ and } m \geq 3)$ . If  $k \in 4\mathbb{N} + 1$ , then let  $n_k$  be the least element of  $N_k$  for which  $(\pi_{n_k} = \sigma, |O_{n_k}| = 1, \text{ and } m = 2)$ ,  $(\pi_{n_k} = \sigma \text{ and } |O_{n_k}| = 2)$ , or  $(\pi_{n_k} = \rho \text{ and } |O_{n_k}| = 1)$ . And if  $k \in 4\mathbb{N} + 3$ , then let  $n_k$  be the least element of  $N_k$  for which  $\pi_{n_k} = \sigma$  and  $|O_{n_k}| \geq 3$ .

**Lemma 6.2.** *For some finite cardinal  $\kappa_k$  and any subset  $F_k$  of  $N \cap (N_k \setminus \{n\})$  of cardinality  $\kappa_k$ , there exist  $i_{k+1} < i_k$ ,  $j_{k+1} > j_k$ , and  $(\rho_{k+1}, \sigma_{k+1}) \in \mathcal{F}_{\text{par}(k+1)}(i_{k+1}, j_{k+1})$  such that:*

- $\rho_{k+1}$  is obtained from  $\rho_k$  by adding a set of cycles of length  $k$  and  $|O_n| - 1$   $|O_n|$ -malleable fixed points for all  $n \in F_k$ , as well as a cycle of length  $|O_{n_k}|$  if  $(\pi_{n_k} = \rho \text{ and } |O_{n_k}| \geq 2)$  and an anti-malleable fixed point if  $(\pi_{n_k} = \rho \text{ and } |O_{n_k}| = 1)$ .
- $\sigma_{k+1}$  is obtained from  $\sigma_k$  by adding a cycle of length  $|O_n|$  for all  $n \in F_k$ , as well as a cycle of length  $|O_n|$  if  $\pi_{n_k} = \sigma$ .

*Proof.* If  $k \in 2\mathbb{N}$ , then the desired result follows from Propositions 5.1 and 5.2. Otherwise, it follows from Propositions 5.3–5.5.  $\square$

Set  $N_{k+1} = N_k \setminus (F_k \cup \{n_k\})$ .

Define  $\rho_\infty = \bigcup_{k \in \mathbb{N}} \rho_k$  and  $\sigma_\infty = \bigcup_{k \in \mathbb{N}} \sigma_k$ . As  $(i_k)_{k \in \mathbb{N}}$  is strictly decreasing and  $(j_k)_{k \in \mathbb{N}}$  is strictly increasing, these are permutations of  $\mathbb{Z}$  whose composition is  $S^{\mathbb{Z}}$ . Note that  $\text{Mal}(\rho_\infty, \sigma_\infty) \cap \text{Per}_{n+2\mathbb{N}}(\sigma_\infty) = \emptyset$  for all  $n \in 2\mathbb{N} + (1-p)$ . As  $F_k \neq \emptyset$  for all  $k \in 4\mathbb{N} + 3$ , it follows that  $\neg 0 < |\text{Mal}(\rho_\infty, \sigma_\infty) \cap \text{Per}_{n+2\mathbb{N}}(\sigma_\infty)| < \aleph_0$  for all  $n \in 2\mathbb{N} + p$ . As the fact that  $\bigcap_{k \in \mathbb{N}} N_k = \emptyset$  ensures that  $\rho_\infty \upharpoonright \sim(\text{Mal}(\rho_\infty, \sigma_\infty) \cap \text{Per}_1(\rho_\infty)) \cong \rho$  and  $\sigma_\infty \cong \sigma$ , Proposition 3.3 yields conjugates  $\rho'$  of  $\rho$  and  $\sigma'$  of  $\sigma$  for which  $\rho' \circ \sigma' = \rho_\infty \circ \sigma_\infty = S^{\mathbb{Z}}$ .  $\square$

As every almost involution has finite order, part (2) of Theorem A now follows from [Mor89, Theorem A].

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