

# PERIODIC PERMUTATIONS AND THE SUCCESSOR

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ABSTRACT. We investigate pairs of conjugacy classes of periodic permutations of  $\mathbb{Z}$  whose product contains the successor function.

## INTRODUCTION

Let  $\text{Sym}(X)$  denote the *symmetric group* of all permutations of  $X$ . The *orbit* of a point  $x \in X$  under a permutation  $\tau$  of  $X$  is given by  $[x]_\tau = \{\tau^i(x) \mid i \in \mathbb{Z}\}$ . We say that  $\tau$  is *almost trivial* if  $\tau(x) = x$  for cofinitely many  $x \in X$ , an *almost involution* if  $\tau^2$  is almost trivial, and  $(\sigma)$ -*periodic* if every orbit is finite. Define  $C(\tau) = \sum_{x \in X} 1 - 2/[x]_\tau|$  and  $\text{Cl}(\tau) = \{\sigma \circ \tau \circ \sigma^{-1} \mid \sigma \in \text{Sym}(X)\}$ . The *successor function* on  $\mathbb{Z}$  is given by  $S^\mathbb{Z}(i) = i + 1$  for all  $i \in \mathbb{Z}$ . Here we prove the following:

**Theorem A.** *Suppose that  $\rho, \sigma \in \text{Sym}(\mathbb{Z})$  are almost involutions and  $S^\mathbb{Z} \in \text{Cl}(\rho)\text{Cl}(\sigma)$ . Then  $C(\rho) + C(\sigma) \geq -1$ .*

**Theorem B.** *Suppose that  $\rho, \sigma \in \text{Sym}(\mathbb{Z})$  are periodic but not almost trivial and  $\rho$  or  $\sigma$  is not an almost involution. Then  $S^\mathbb{Z} \in \text{Cl}(\rho)\text{Cl}(\sigma)$ .*

The special case of Theorem B where neither  $\rho$  nor  $\sigma$  is an almost involution follows from [Mor89, Theorem A]. As far as I am aware, however, the special case where  $\rho$  or  $\sigma$  is an almost involution was not previously known. Regardless, the real purpose of this paper is to introduce ideas and language—in the simplest possible context—that can be used to investigate the finite-order elements  $R$  and  $S$  of the full group of an aperiodic Borel automorphism  $T$  with the property that  $T \in \text{Cl}(R)\text{Cl}(S)$ . This topic will be explored in a future paper.

In §1, we prove Theorem A. In §2, we note a symmetry that removes the need to repeat arguments. In §3, we establish a fact concerning elimination of fixed points. In §4, we describe the simplest finite approximations to pairs  $(\rho, \sigma)$  for which  $S^\mathbb{Z} \in \text{Cl}(\rho)\text{Cl}(\sigma)$ . In §5, we use these as building blocks to construct extensions of more general finite approximations. And in §6, we prove the special case of Theorem B where  $\rho$  or  $\sigma$  has finite order.

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## 1. THE PROOF OF THEOREM A

For all  $R \subseteq X^2$ , define  $\text{graph}_R(\tau) = \text{graph}(\tau) \cap R$ .

**Proposition 1.1.** *Suppose that  $\preceq$  is a linear ordering of a finite set  $F$  and  $\tau \in \text{Sym}(F)$ . Then  $|\text{graph}_{\preceq}(\tau)| \geq 1$  and  $|\text{graph}_{\succ}(\tau)| \leq |F| - 1$ .*

*Proof.* Let  $x$  be the  $\preceq$ -maximal element of  $F$ . Then  $x \succeq \tau(x)$ , so  $|\text{graph}_{\preceq}(\tau)| \geq 1$ . But  $|\text{graph}(\tau)| = |F|$ , thus  $|\text{graph}_{\succ}(\tau)| \leq |F| - 1$ .  $\square$

Define  $\mathcal{O}(\tau) = \{[x]_\tau \mid x \in X\}$ . For all sets  $K$  of cardinals, define  $\text{Per}_K(\tau) = \{x \in X \mid |[x]_\tau| \in K\}$  and  $\mathcal{O}_K(\tau) = \mathcal{O}(\tau \upharpoonright \text{Per}_K(\tau))$ . Put  $\text{graph}'_R(\tau) = \text{graph}(\tau \upharpoonright \sim \text{Per}_2(\tau)) \cap R$ .

**Proposition 1.2.** *Suppose that  $\tau$  is an almost involution of a set  $X$  and  $\preceq$  is a binary relation on  $X$  whose restriction to each orbit of  $\tau$  is a linear order. Then  $C(\tau) \geq |\text{graph}'_{\preceq}(\tau)| - |\text{graph}'_{\succ}(\tau)|$ .*

*Proof.* As  $|\text{graph}'_{\preceq}(\tau)| \leq \sum_{O \in \mathcal{O}_{\geq 3}(\tau)} (|O| - 1)$  and  $|\text{graph}'_{\succ}(\tau)| \geq |\mathcal{O}_{\geq 3}(\tau)|$  by Proposition 1.1, the desired result follows from the fact that  $C(\tau) = \sum_{O \in \mathcal{O}(\tau)} (|O| - 2) = \sum_{O \in \mathcal{O}_{\geq 3}(\tau)} (|O| - 1) - |\mathcal{O}_{\geq 3}(\tau)| - |\mathcal{O}_1(\tau)|$ .  $\square$

Given  $\tau_0, \tau_1 \in \text{Sym}(X)$ , define  $\tau_0 \amalg \tau_1 \in \text{Sym}(X \times 2)$  by  $(\tau_0 \amalg \tau_1)(x, k) = (\tau_k(x), k)$  for all  $k < 2$  and  $x \in X$ . Let  $\preceq$  denote any binary relation on  $\mathbb{Z} \times 2$  with the property that  $(i, k) \preceq (j, k) \iff i \leq j$  for all  $i, j \in \mathbb{Z}$  and  $k < 2$ . Theorem A follows from Proposition 1.2 and:

**Proposition 1.3.** *Suppose that  $\tau_0, \tau_1 \in \text{Sym}(\mathbb{Z})$  and  $S^{\mathbb{Z}} = \tau_0 \circ \tau_1$ . Then  $|\text{graph}'_{\preceq}(\tau_0 \amalg \tau_1)| \leq |\text{graph}'_{\preceq}(\tau_0 \amalg \tau_1)| + 1$ .*

*Proof.* Define  $I, J: \text{graph}(\tau_0 \amalg \tau_1) \rightarrow (\mathbb{Z} \times 2)^2$  by

$$I((i, k), (j, k)) = \begin{cases} ((j, k), (i, k)) & \text{if } i, j \in \text{Per}_2(\tau_k) \text{ and} \\ ((i, k), (j, k)) & \text{otherwise} \end{cases}$$

and

$$J((i, k), (j, k)) = ((j - (1 - k), 1 - k), (i + k, 1 - k))$$

for all  $i, j \in \mathbb{Z}$  and  $k < 2$ .

**Lemma 1.4.**  $J(\text{graph}(\tau_0 \amalg \tau_1)) \subseteq \text{graph}(\tau_0 \amalg \tau_1)$ .

*Proof.* Suppose that  $((i, k), (j, k)) \in \text{graph}(\tau_0 \amalg \tau_1)$ . If  $k = 0$ , then  $\tau_0(i) = j = S^{\mathbb{Z}}(j - 1) = (\tau_0 \circ \tau_1)(j - 1)$ , so  $i = \tau_1(j - 1)$ , thus  $J((i, 0), (j, 0)) = ((j - 1, 1), (i, 1)) \in \text{graph}(\tau_0 \amalg \tau_1)$ . If  $k = 1$ , then  $\tau_1(i) = j$ , so  $\tau_0(j) = (\tau_0 \circ \tau_1)(i) = S^{\mathbb{Z}}(i) = i + 1$ , thus  $J((i, 1), (j, 1)) = ((j, 0), (i + 1, 0)) \in \text{graph}(\tau_0 \amalg \tau_1)$ .  $\square$

**Lemma 1.5.**  $J(\text{graph}_{\preceq}(\tau_0 \amalg \tau_1)) = \text{graph}_{\preceq}(\tau_0 \amalg \tau_1)$ .

*Proof.* Note that  $((i, k), (j, k)) \in \text{graph}_{\prec}(\tau_0 \amalg \tau_1) \iff i < j \iff j - (1 - k) \geq i + k \iff J((i, k), (j, k)) \in \text{graph}_{\succeq}(\tau_0 \amalg \tau_1)$ .  $\square$

Define  $\mathbb{1}_R: R \rightarrow 2$  by  $\mathbb{1}_R(x, y) = 1 \iff x R y$ . The length of  $((i, k), (j, k)) \in \text{graph}(\tau_0 \amalg \tau_1)$  is given by  $|((i, k), (j, k))| = |i - j|$ .

**Lemma 1.6.** *If  $((i, k), (j, k)) \in \text{graph}(\tau_0 \amalg \tau_1)$ , then  $|J((i, k), (j, k))| = |((i, k), (j, k))| + (-1)^{\mathbb{1}_{<}(i, j)}$ .*

*Proof.* Note that  $i < j \iff i + k \leq j - (1 - k)$ , so

$$\begin{aligned} |(j - (1 - k)) - (i + k)| &= (-1)^{\mathbb{1}_{<}(i, j)}((i + k) - (j - (1 - k))) \\ &= (-1)^{\mathbb{1}_{<}(i, j)}((i - j) + 1) \\ &= |i - j| + (-1)^{\mathbb{1}_{<}(i, j)}, \end{aligned}$$

from which the desired result immediately follows.  $\square$

Let  $G$  be the group generated by  $I$  and  $J$ . The orbit of  $((i, k), (j, k))$  under  $G$  is given by  $[((i, k), (j, k))]_G = \{g \cdot ((i, k), (j, k)) \mid g \in G\}$ . Set  $\mathcal{O}(G) = \{[((i, k), (j, k))]_G \mid ((i, k), (j, k)) \in \text{graph}(\tau_0 \amalg \tau_1)\}$ .

**Lemma 1.7.** *Suppose that  $O \in \mathcal{O}(G)$ . Then  $\text{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O \neq \emptyset$ .*

*Proof.* Fix  $((i, k), (j, k)) \in O$ . We can assume that  $((i, k), (j, k)) \notin \text{graph}'_{\succeq}(\tau_0 \amalg \tau_1)$ . By replacing  $((i, k), (j, k))$  with  $I((i, k), (j, k))$  if necessary, we can therefore assume that  $((i, k), (j, k)) \in \text{graph}_{\prec}(\tau_0 \amalg \tau_1)$ . For all  $n \in \mathbb{N}$ , note that if  $((i_n, k_n), (j_n, k_n)) = (I \circ J)^n((i, k), (j, k))$  is in  $\text{graph}_{\prec}(\tau_0 \amalg \tau_1)$ , then  $J((i_n, k_n), (j_n, k_n)) \in \text{graph}_{\succeq}(\tau_0 \amalg \tau_1)$  and  $|J((i_n, k_n), (j_n, k_n))| = |((i_n, k_n), (j_n, k_n))| - 1$  by Lemmas 1.5 and 1.6. If  $J((i_n, k_n), (j_n, k_n)) \notin \text{graph}'_{\succeq}(\tau_0 \amalg \tau_1)$ , then  $((i_{n+1}, k_{n+1}), (j_{n+1}, k_{n+1})) \in \text{graph}_{\prec}(\tau_0 \amalg \tau_1)$ . Set  $n = |i - j| - 1$  and note that if  $J((i_m, k_m), (j_m, k_m)) \notin \text{graph}'_{\succeq}(\tau_0 \amalg \tau_1)$  for all  $m < n$ , then  $J((i_n, k_n), (j_n, k_n)) = 0$ , in which case  $J((i_n, k_n), (j_n, k_n)) \in \text{graph}'_{\succeq}(\tau_0 \amalg \tau_1)$ .  $\square$

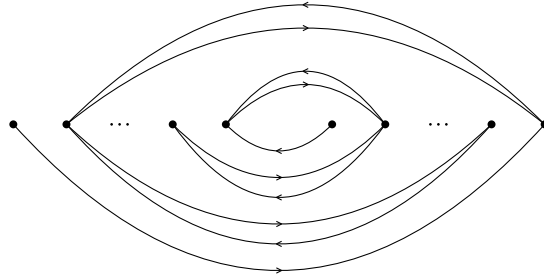


FIGURE 1. A finite orbit of  $G$ .

**Lemma 1.8.** *Suppose that  $O \in \mathcal{O}(G)$ .*

- (1) If  $|O| < \aleph_0$ , then  $|\text{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O| = |\text{graph}'_{\prec}(\tau_0 \amalg \tau_1) \cap O| = 1$ .
- (2) If  $|O| = \aleph_0$ , then  $O$  is a cofinite subset of  $\text{graph}(\tau_0 \amalg \tau_1)$ ,  $|\text{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O| = 1$ , and  $\text{graph}'_{\prec}(\tau_0 \amalg \tau_1) \cap O = \emptyset$ .

*Proof.* By Lemma 1.7, there exists  $((i, k), (j, k)) \in \text{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O$ . For all  $n \in \mathbb{N}$ , note that if  $((i_n, k_n), (j_n, k_n)) = (I \circ J)^n((i, k), (j, k))$  is in  $\text{graph}_{\succeq}(\tau_0 \amalg \tau_1)$ , then Lemma 1.5 ensures that  $J((i_n, k_n), (j_n, k_n)) \in \text{graph}_{\prec}(\tau_0 \amalg \tau_1)$ . And if  $J((i_n, k_n), (j_n, k_n)) \notin \text{graph}'_{\prec}(\tau_0 \amalg \tau_1)$ , then  $((i_{n+1}, k_{n+1}), (j_{n+1}, k_{n+1})) \in \text{graph}_{\succeq}(\tau_0 \amalg \tau_1)$ .

Suppose now that  $n$  is the least natural number with the property that  $J((i_n, k_n), (j_n, k_n)) \in \text{graph}'_{\prec}(\tau_0 \amalg \tau_1)$ . Then  $((i, k), (j, k))$  is in  $\text{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O$ ,  $J((i_n, k_n), (j_n, k_n))$  is in  $\text{graph}'_{\prec}(\tau_0 \amalg \tau_1) \cap O$ , and the pairs of the form  $J((i_m, k_m), (j_m, k_m))$  and  $((i_{m+1}, k_{m+1}), (j_{m+1}, k_{m+1}))$ , for  $m < n$ , are in  $\text{Per}_2(\tau_0 \amalg \tau_1)^2$  and make up the rest of  $O$ , so (1) holds.

If there is no  $n \in \mathbb{N}$  for which  $J((i_n, k_n), (j_n, k_n)) \in \text{graph}'_{\prec}(\tau_0 \amalg \tau_1)$ , then  $((i, k), (j, k))$  is in  $\text{graph}'_{\succeq}(\tau_0 \amalg \tau_1) \cap O$  and the pairs of the form  $J((i_n, k_n), (j_n, k_n))$  and  $((i_{n+1}, k_{n+1}), (j_{n+1}, k_{n+1}))$ , for  $n \in \mathbb{N}$ , are in  $\text{Per}_2(\tau_0 \amalg \tau_1)^2$  and make up the rest of  $O$ , so  $\text{graph}'_{\prec}(\tau_0 \amalg \tau_1) \cap O = \emptyset$ . And a straightforward induction shows that  $i_{2n} = i + n$ ,  $j_{2n} = j - n$ ,  $i_{2n+1} = i + (n + k)$ ,  $j_{2n+1} = j - (n + (1 - k))$ , and  $k_n = k$  for all  $n \in \mathbb{N}$ , so  $\text{graph}(\tau_0 \amalg \tau_1) \setminus O \subseteq (\{i, i + 1, \dots, j\} \times 2)^2$ , thus (2) holds.  $\square$

Observe that at most one orbit of  $G$  is cofinite, so Lemma 1.8 ensures that  $|\text{graph}'_{\succeq}(\tau_0 \amalg \tau_1)| = |\mathcal{O}(G)| \leq |\text{graph}'_{\prec}(\tau_0 \amalg \tau_1)| + 1$ .  $\square$

## 2. DUALS

We use  $f: X \hookrightarrow Y$  to denote a partial injection of  $X$  into  $Y$ . For all  $\sigma: \mathbb{Z} \hookrightarrow \mathbb{Z}$ , define  $\bar{\sigma}: \mathbb{Z} \hookrightarrow \mathbb{Z}$  by  $\bar{\sigma}(i) = -\sigma^{-1}(-i)$  for all  $i \in \mathbb{Z}$ .

**Proposition 2.1.** *Suppose that  $\sigma: \mathbb{Z} \hookrightarrow \mathbb{Z}$ . Then  $\sigma = \bar{\bar{\sigma}}$ .*

*Proof.* If  $i \in \mathbb{Z}$ , then  $\bar{\bar{\sigma}}(i) = -(\bar{\sigma})^{-1}(-i)$ , so  $\bar{\sigma}(-\bar{\bar{\sigma}}(i)) = -i$ . But  $\bar{\sigma}(-\bar{\bar{\sigma}}(i)) = -\sigma^{-1}(\bar{\bar{\sigma}}(i))$ , so  $i = \sigma^{-1}(\bar{\bar{\sigma}}(i))$ , thus  $\sigma(i) = \bar{\bar{\sigma}}(i)$ .  $\square$

**Proposition 2.2.** *Suppose that  $\rho, \sigma: \mathbb{Z} \hookrightarrow \mathbb{Z}$ . Then  $\overline{\rho \circ \sigma} = \bar{\sigma} \circ \bar{\rho}$ .*

*Proof.* Observe that

$$\begin{aligned}
 (\bar{\sigma} \circ \bar{\rho})(i) &= -\sigma^{-1}(-(-\rho^{-1}(-i))) \\
 &= -(\sigma^{-1} \circ \rho^{-1})(-i) \\
 &= -(\rho \circ \sigma)^{-1}(-i) \\
 &= \overline{\rho \circ \sigma}(i)
 \end{aligned}$$

for all  $i \in \mathbb{Z}$ .  $\square$

Define  $\mathcal{F} = \{(\rho: \mathbb{Z} \hookrightarrow \mathbb{Z}, \sigma: \mathbb{Z} \hookrightarrow \mathbb{Z}) \mid \rho \circ \sigma = S^{\mathbb{Z}} \upharpoonright \text{dom}(\rho \circ \sigma)\}$ .

**Proposition 2.3.**  $(\rho, \sigma) \in \mathcal{F} \iff (\bar{\sigma}, \bar{\rho}) \in \mathcal{F}$ .

*Proof.* Note that if  $i \in \mathbb{Z}$  and  $\rho, \sigma: \mathbb{Z} \hookrightarrow \mathbb{Z}$ , then  $(\rho \circ \sigma)(i) = i+1 \iff (\rho \circ \sigma)^{-1}(i+1) = i \iff \bar{\rho} \circ \bar{\sigma}(-i-1) = -i$ , so the desired result follows from Proposition 2.2.  $\square$

Let  $(i_0 \ i_1 \ \dots \ i_n)$  denote the permutation of  $\{i_m \mid m \leq n\}$  sending  $i_m$  to  $i_{m+1}$  for all  $m < n$ .

**Proposition 2.4.** Suppose that  $n \geq 1$ ,  $(i_m)_{m \leq n}$  is strictly increasing,  $\rho = (i_0 \ i_1 \ \dots \ i_n)$ , and  $\sigma = (-i_n \ -i_{n-1} \ \dots \ -i_0)$ . Then  $\rho = \bar{\sigma}$ .

*Proof.* If  $m < n$ , then  $\bar{\sigma}(i_m) = -\sigma^{-1}(-i_m) = -(-i_{m+1}) = i_{m+1}$ .  $\square$

### 3. ELIMINATING FIXED POINTS

For all  $k \in \mathbb{N}$ , let  $\text{par}(k)$  denote the remainder when  $k$  is divided by two. For all  $\rho, \sigma \in \text{Sym}(X)$ , set  $\delta(\rho, \sigma) = \{x \in X \mid \rho(x) \neq \sigma(x)\}$  and

$$\text{Mal}(\rho, \sigma) = \{x \in \text{Per}_{\mathbb{N}+3}(\sigma) \mid |[x]_{\sigma} \setminus \text{Per}_1(\rho)| = 1\}.$$

**Proposition 3.1.** Suppose that  $m \geq 1$ ,  $\rho$  and  $\sigma$  are permutations of a set  $X$ , and  $\forall n \geq 3 \ 0 < |\text{Mal}(\rho, \sigma) \cap \text{Per}_{2\mathbb{N}+n}(\sigma)| < \aleph_0$ . Then there are permutations  $\rho'$  and  $\sigma'$  of  $X$  such that:

- (1)  $\rho \circ \sigma = \rho' \circ \sigma'$ ,
- (2)  $\delta(\rho, \rho') = \delta(\sigma^{-1}, (\sigma')^{-1}) = \text{Mal}(\rho, \sigma) \cap \text{Per}_1(\rho)$ ,
- (3)  $\text{Mal}(\rho, \sigma) \cap \text{Per}_1(\rho) \subseteq \text{Per}_m(\rho')$ , and
- (4)  $\forall n \geq 3 \ \text{Mal}(\rho, \sigma) \cap \text{Per}_n(\sigma) \subseteq \text{Per}_n(\sigma')$ .

*Proof.* Define  $Y = \text{Mal}(\rho, \sigma)$  and  $Z = Y \setminus \text{Per}_1(\rho)$ . For all  $n \geq 3$ , set  $Y_n = \text{Per}_{2\mathbb{N}+n}(\sigma) \cap Y$  and  $Z_n = \text{Per}_{2\mathbb{N}+n}(\sigma) \cap Z$ . Fix an equivalence relation  $F_4$  on  $Z_4$  whose classes all have cardinality  $m^2$ , as well as  $\pi_{0,1}, \pi_{0,2} \in \text{Sym}(Z_4)$ , whose graphs are contained in  $F_4$ , such that the orbits of  $\pi_{0,1}$ ,  $\pi_{0,2}$ , and  $\pi_{0,3} = (\pi_{0,1} \circ \pi_{0,2})^{-1}$  all have cardinality  $m$ . For all  $n \in (\mathbb{N}+3) \setminus \{4\}$ , fix an equivalence relation  $F_n$  on  $Z_n$  whose classes all have cardinality  $m$ , fix  $\pi_{\text{par}(n), n-2} \in \text{Sym}(Z_n)$  whose orbits coincide with the equivalence classes of  $F_n$ , and set  $\pi_{\text{par}(n), n-1} = \pi_{\text{par}(n), n-2}^{-1}$ . Then the support of  $\pi = \text{id}_{X \setminus (Y \setminus Z)} \cup \bigcup_{n \geq 1, p < 2} \sigma^n \circ \pi_{p,n} \circ \sigma^{-n}$  is  $Y \setminus Z$ , so  $\rho' = \rho \circ \pi$  and  $\sigma' = \pi^{-1} \circ \sigma$  satisfy conditions (1)–(3).

**Lemma 3.2.** Suppose that  $\ell < n$ . Then

$$(\sigma')^{\ell} \upharpoonright Z_n = (\sigma^{\ell} \circ \pi_{\text{par}(n), \ell}^{-1} \circ \dots \circ \pi_{\text{par}(n), 1}^{-1}) \upharpoonright Z_n. \quad (*)$$

*Proof.* The case  $\ell = 0$  is trivial. If  $\ell > 0$  and  $(*)$  holds at  $\ell - 1$ , then

$$\begin{aligned}
(\sigma')^\ell \upharpoonright Z_n &= (\sigma' \circ (\sigma')^{\ell-1}) \upharpoonright Z_n \\
&= (\sigma' \circ \sigma^{\ell-1} \circ \pi_{\text{par}(n), \ell-1}^{-1} \circ \cdots \circ \pi_{\text{par}(n), 1}^{-1}) \upharpoonright Z_n \\
&= (\pi^{-1} \circ \sigma^\ell \circ \pi_{\text{par}(n), \ell-1}^{-1} \circ \cdots \circ \pi_{\text{par}(n), 1}^{-1}) \upharpoonright Z_n \\
&= (\sigma^\ell \circ \pi_{\text{par}(n), \ell}^{-1} \circ \sigma^{-\ell} \circ \sigma^\ell \circ \pi_{\text{par}(n), \ell-1}^{-1} \circ \cdots \circ \pi_{\text{par}(n), 1}^{-1}) \upharpoonright Z_n \\
&= (\sigma^\ell \circ \pi_{\text{par}(n), \ell}^{-1} \circ \cdots \circ \pi_{\text{par}(n), 1}^{-1}) \upharpoonright Z_n,
\end{aligned}$$

so  $(*)$  also holds at  $\ell$ .  $\square$

For all  $n \geq 3$ , set  $Y'_n = \text{Per}_n(\sigma) \cap Y$  and  $Z'_n = \text{Per}_n(\sigma) \cap Z$ . Lemma 3.2 ensures that  $Y'_n = \bigcup_{\ell < n} \sigma^\ell(Z'_n) = \bigcup_{\ell < n} (\sigma')^\ell(Z'_n)$  and

$$\begin{aligned}
(\sigma')^n \upharpoonright Z'_n &= (\sigma' \circ (\sigma')^{n-1}) \upharpoonright Z'_n \\
&= (\sigma' \circ \sigma^{n-1} \circ \pi_{\text{par}(n), n-1}^{-1} \circ \cdots \circ \pi_{\text{par}(n), 1}^{-1}) \upharpoonright Z'_n \\
&= (\sigma' \circ \sigma^{n-1}) \upharpoonright Z'_n \\
&= (\sigma' \circ \sigma^{-1}) \upharpoonright Z'_n \\
&= \text{id}_{Z'_n},
\end{aligned}$$

so condition (4) also holds.  $\square$

We write  $\rho \cong \sigma$  to indicate that  $\rho$  and  $\sigma$  are isomorphic.

**Proposition 3.3.** *Suppose that  $m \geq 1$ ,  $\rho$  and  $\sigma$  are permutations of an infinite set  $X$ ,  $\forall n \geq 3 \neg 0 < |\text{Mal}(\rho, \sigma) \cap \text{Per}_{2\mathbb{N}+n}(\sigma)| < \aleph_0$ , and  $|\text{Per}_m(\rho)| \geq \max(\aleph_0, |\text{Mal}(\rho, \sigma) \cap \text{Per}_1(\rho)|)$ . Then there are permutations  $\rho' \cong \rho \upharpoonright \sim(\text{Mal}(\rho, \sigma) \cap \text{Per}_1(\rho))$  and  $\sigma' \cong \sigma$  of  $X$  with  $\rho \circ \sigma = \rho' \circ \sigma'$ .*

*Proof.* Proposition 3.1 yields  $\rho', \sigma' \in \text{Sym}(X)$  such that  $\rho \circ \sigma = \rho' \circ \sigma'$  and  $|\mathcal{O}_\kappa(\rho \upharpoonright \sim(\text{Mal}(\rho, \sigma) \cap \text{Per}_1(\rho)))| = |\mathcal{O}_\kappa(\rho')|$  and  $|\mathcal{O}_\kappa(\sigma)| = |\mathcal{O}_\kappa(\sigma')|$  for all cardinals  $\kappa$ .  $\square$

#### 4. BUILDING BLOCKS

For all  $i, j \in \mathbb{Z}$ , we slightly abuse the usual notation by using  $(i, j)$ ,  $[i, j)$ ,  $(i, j]$ , and  $[i, j]$  to denote the corresponding intervals of integers. Set  $\mathcal{F}(i, j] = \{(\rho, \sigma) \in \mathcal{F} \mid \rho: (i, j] \hookrightarrow (i, j] \text{ and } \sigma: (i, j) \hookrightarrow (i, j)\}$ , noting that  $\forall(\rho, \sigma) \in \mathcal{F}(i, j] \text{ dom}(\rho \circ \sigma) = (i, j - 1]$ .

**Proposition 4.1.** *If  $i < j$  and  $(\rho, \sigma) \in \mathcal{F}(i, j]$ , then  $\rho(j) = i + 1$ .*

*Proof.* Observe that  $\rho((i, j - 1]) = (\rho \circ \sigma)((i, j - 1]) = (i + 1, j]$ .  $\square$

Set  $\mathcal{F}[i, j) = \{(\rho, \sigma) \in \mathcal{F} \mid \rho: (i, j) \hookrightarrow (i, j) \text{ and } \sigma: [i, j) \hookrightarrow [i, j)\}$ , this time noting that  $\forall(\rho, \sigma) \in \mathcal{F}(i, j) S^{\mathbb{Z}}(j - 1) \notin \text{rng}(\rho)$ , and therefore  $\forall(\rho, \sigma) \in \mathcal{F}(i, j) \text{ dom}(\rho \circ \sigma) = [i, j - 1)$ .





FIGURE 4. Building blocks from Propositions 4.5 and 4.6.

**Proposition 4.5.** *Suppose that  $i < j$  are integers. Then the pair  $(\text{id}_{(i,j)}, (i \ i+1 \ \dots \ j-1))$  is in  $\mathcal{F}[i, j]$ .*

*Proof.* If  $i+1 = j$ , then this follows from Proposition 4.4. Otherwise, Proposition 4.4 ensures that  $(\text{id}_{\{k\}}, \emptyset) \in \mathcal{F}(k-1, k]$  for all  $k \in (i, j)$ , so Proposition 4.2 yields the desired result.  $\boxtimes$

**Proposition 4.6.** *Suppose that  $m \geq 1$  and  $(i_k)_{k < m}$  is a strictly increasing sequence of integers. Then the pair*

$$((i_0 \ i_1 \ \dots \ i_{m-1}) \cup \bigcup_{k < m-1} \text{id}_{(i_k, i_{k+1})}, \bigcup_{k < m-1} (i_k \ i_k + 1 \ \dots \ i_{k+1} - 1))$$

*is in  $\mathcal{F}(i_0 - 1, i_{m-1}]$ .*

*Proof.* If  $m = 1$ , then this follows from Proposition 4.4. Otherwise,  $(\text{id}_{(i_k, i_{k+1})}, (i_k \ i_k + 1 \ \dots \ i_{k+1} - 1)) \in \mathcal{F}[i_k, i_{k+1})$  for all  $k < m-1$  by Proposition 4.5, so Proposition 4.3 yields the desired result.  $\boxtimes$

## 5. EXTENSION

Given  $n \geq 3$  and  $\rho, \sigma: X \hookrightarrow X$ , we say that a fixed point  $x$  of  $\rho$  is *n-malleable* if  $x \in \text{Per}_n(\sigma)$ ,  $[x]_\sigma \subseteq \text{dom}(\rho)$ , and  $|[x]_\sigma \setminus \text{Per}_1(\rho)| = 1$ .

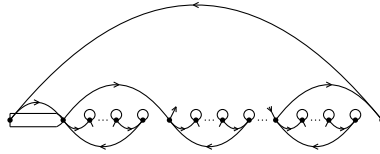


FIGURE 5. The extension provided by Proposition 5.1.

**Proposition 5.1.** *Suppose that  $i < j$ ,  $m \geq 2$ ,  $n_k \geq 3$  for all  $k < m-2$ , and  $(\rho, \sigma) \in \mathcal{F}[i, j]$ . Then there exists  $(\rho', \sigma') \in \mathcal{F}(i-1, j + \sum_{k < m-2} n_k]$  such that:*

- $\rho'$  is obtained from  $\rho$  by adding a single cycle of length  $m$  and  $n_k - 1$   $n_k$ -malleable fixed points for all  $k < m-2$ .
- $\sigma'$  is obtained from  $\sigma$  by adding a cycle of length  $n_k$  for all  $k < m-2$ .



*Proof.* Recursively define  $i_0 = i$ ,  $i_1 = j$ , and  $i_k = i_{k-1} + n_{k-2}$  for all  $2 \leq k \leq m-1$ . Set  $(\rho_0, \sigma_0) = (\rho, \sigma)$ . For all  $1 \leq k \leq m-2$ , Proposition 4.5 ensures that  $(\rho_k, \sigma_k) = (\text{id}_{(i_k, i_{k+1})}, (i_k \ i_k + 1 \ \cdots \ i_{k+1} - 1))$  is in  $\mathcal{F}[i_k, i_{k+1}]$ . So  $(\rho', \sigma') = ((i_0 \ i_1 \ \cdots \ i_{m-1}) \cup \bigcup_{k \leq m-2} \rho_k, \bigcup_{k \leq m-2} \sigma_k)$  is in  $\mathcal{F}(i_0 - 1, i_{m-1}]$  by Proposition 4.3. But  $i_0 - 1 = i - 1$  and  $i_{m-1} = j + \sum_{k < m-2} n_k$ .  $\square$

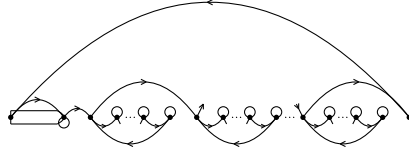


FIGURE 6. The extension provided by Proposition 5.2.

**Proposition 5.2.** *Suppose that  $i < j$ ,  $m \geq 3$ ,  $n_k \geq 3$  for all  $k < m-3$ , and  $(\rho, \sigma) \in \mathcal{F}[i, j]$ . Then there exists  $(\rho', \sigma') \in \mathcal{F}(i - 1, j + 1 + \sum_{k < m-3} n_k]$  such that:*

- $\rho'$  is obtained from  $\rho$  by adding a single cycle of length  $m$  and  $n_k - 1$   $n_k$ -malleable fixed points for all  $k < m - 3$ .
- $\sigma'$  is obtained from  $\sigma$  by adding a fixed point and a cycle of length  $n_k$  for all  $k < m - 3$ .

*Proof.* Recursively define  $i_0 = i$ ,  $i_1 = j$ ,  $i_2 = j + 1$ , and  $i_k = i_{k-1} + n_{k-3}$  for all  $3 \leq k \leq m - 1$ . Set  $(\rho_0, \sigma_0) = (\rho, \sigma)$ . For all  $1 \leq k \leq m - 2$ , Proposition 4.5 ensures that  $(\rho_k, \sigma_k) = (\text{id}_{(i_k, i_{k+1})}, (i_k \ i_k + 1 \ \cdots \ i_{k+1} - 1))$  is in  $\mathcal{F}[i_k, i_{k+1}]$ . So  $(\rho', \sigma') = ((i_0 \ i_1 \ \cdots \ i_{m-1}) \cup \bigcup_{k \leq m-2} \rho_k, \bigcup_{k \leq m-2} \sigma_k)$  is in  $\mathcal{F}(i_0 - 1, i_{m-1}]$  by Proposition 4.3. But  $i_0 - 1 = i - 1$  and  $i_{m-1} = j + 1 + \sum_{k < m-3} n_k$ .  $\square$

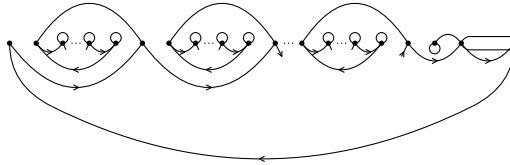


FIGURE 7. The extension provided by Proposition 5.3.

**Proposition 5.3.** *Suppose that  $i < j$ ,  $n \geq 3$ ,  $n_k \geq 3$  for all  $k < n - 3$ , and  $(\rho, \sigma) \in \mathcal{F}[i, j]$ . Then there exists  $(\rho', \sigma') \in \mathcal{F}[i - 2 - \sum_{k < n-3} (n_k + 1), j + 1]$  such that:*

- $\rho'$  is obtained from  $\rho$  by adding  $n - 2$  cycles of length two and  $n_k - 1$   $n_k$ -malleable fixed points for all  $k < n - 3$ .

- $\sigma'$  is obtained from  $\sigma$  by adding a single fixed point, a cycle of length  $n$ , and a cycle of length of  $n_k$  for all  $k < n - 3$ .

*Proof.* Recursively define  $i_{n-1} = j$ ,  $i_{n-2} = i$ ,  $i_{n-3} = i - 2$ , and  $i_k = i_{k+1} - (n_k + 1)$  for all  $k \leq n - 4$ . Set  $(\rho_{n-2}, \sigma_{n-2}) = (\rho, \sigma)$ . For all  $k \leq n - 3$ , Proposition 4.6 implies that

$$(\rho_k, \sigma_k) = (\text{id}_{(i_k+1, i_{k+1})} \cup (i_k + 1 \ i_{k+1}), (i_k + 1 \ i_k + 2 \ \cdots \ i_{k+1} - 1))$$

is in  $\mathcal{F}(i_k, i_{k+1}]$ . So  $(\rho', \sigma') = (\bigcup_{k \leq n-2} \rho_k, (i_0 \ i_1 \ \cdots \ i_{n-1}) \cup \bigcup_{k \leq n-2} \sigma_k)$  is in  $\mathcal{F}[i_0, i_{n-1} + 1]$  by Proposition 4.2. But  $i - 2 - \sum_{k < n-3} (n_k + 1) = i_0$  and  $j + 1 = i_{n-1} + 1$ .  $\square$

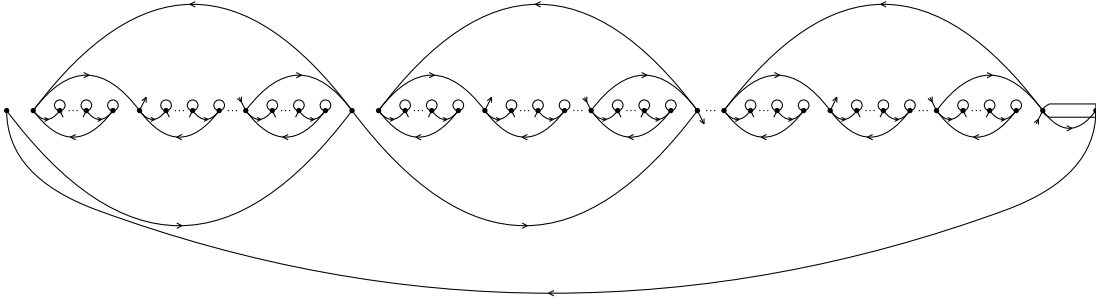


FIGURE 8. The extension provided by Proposition 5.4.

**Proposition 5.4.** *Suppose that  $i < j$ ,  $m \geq 2$ ,  $n \geq 2$ ,  $n_{k,\ell} \geq 3$  for all  $k < m - 1$  and  $\ell < n - 2$ , and  $(\rho, \sigma) \in \mathcal{F}(i, j]$ . Then there exists  $(\rho', \sigma') \in \mathcal{F}[i - \sum_{\ell < n-2} (1 + \sum_{k < m-1} n_{k,\ell}), j + 1]$  such that:*

- $\rho'$  is obtained from  $\rho$  by adding  $n - 2$  cycles of length  $m$  and  $n_{k,\ell} - 1$   $n_{k,\ell}$ -malleable fixed points for all  $k < m - 1$  and  $\ell < n - 2$ .
- $\sigma'$  is obtained from  $\sigma$  by adding a cycle of length  $n$  and a cycle of length  $n_{k,\ell}$  for all  $k < m - 1$  and  $\ell < n - 2$ .

*Proof.* Recursively define  $i_{n-1} = j$ ,  $i_{n-2} = i$ ,  $i_{m-1,\ell} = i_{\ell+1}$ ,  $i_{k,\ell} = i_{k+1,\ell} - n_{k,\ell}$ , and  $i_\ell = i_{0,\ell} - 1$  for all  $k \leq m - 2$  and  $\ell \leq n - 3$ . Set  $(\rho_{n-2}, \sigma_{n-2}) = (\rho, \sigma)$ . For all  $\ell \leq n - 3$ , Proposition 4.6 implies that the pair  $(\rho_\ell, \sigma_\ell)$ , given by  $\rho_\ell = (i_{0,\ell} \ i_{1,\ell} \ \cdots \ i_{m-1,\ell}) \cup \bigcup_{k < m-1} \text{id}_{(i_{k,\ell}, i_{k+1,\ell})}$  and  $\sigma_\ell = \bigcup_{k < m-1} (i_{k,\ell} \ i_{k,\ell} + 1 \ \cdots \ i_{k+1,\ell} - 1)$ , is in  $\mathcal{F}(i_\ell, i_{\ell+1}]$ . So Proposition 4.2 yields that  $(\rho', \sigma') = (\bigcup_{\ell \leq n-2} \rho_\ell, (i_0 \ i_1 \ \cdots \ i_{n-1}) \cup \bigcup_{\ell \leq n-2} \sigma_\ell)$  is in  $\mathcal{F}[i_0, i_{n-1} + 1]$ . But  $i - \sum_{\ell < n-2} (1 + \sum_{k < m-1} n_{k,\ell}) = i_0$  and  $j + 1 = i_{n-1} + 1$ .  $\square$

We say that a fixed point  $x$  of  $\rho$  is *anti-malleable* if  $x \in \text{Per}_2(\sigma)$ ,  $[x]_\sigma \subseteq \text{dom}(\rho)$ , and  $|[x]_\sigma \setminus \text{Per}_1(\rho)| = 1$ .

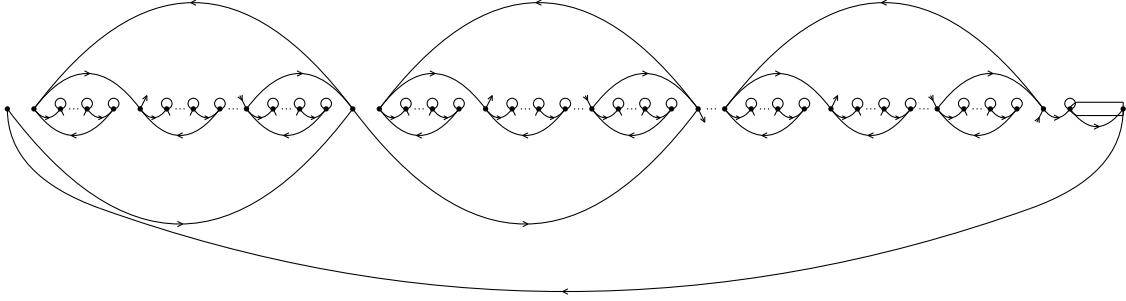


FIGURE 9. The extension provided by Proposition 5.5.

**Proposition 5.5.** *Suppose that  $i < j$ ,  $m \geq 2$ ,  $n \geq 3$ ,  $n_{k,\ell} \geq 3$  for all  $k < m - 1$  and  $\ell < n - 3$ , and  $(\rho, \sigma) \in \mathcal{F}(i, j]$ . Then there exists  $(\rho', \sigma') \in \mathcal{F}[i - 1 - \sum_{\ell < n-3} (1 + \sum_{k < m-1} n_{k,\ell}), j + 1)$  such that:*

- $\rho'$  is obtained from  $\rho$  by adding a single anti-malleable fixed point,  $n - 3$  cycles of length  $m$ , and  $n_{k,\ell} - 1$   $n_{k,\ell}$ -malleable fixed points for all  $k < m - 1$  and  $\ell < n - 3$ .
- $\sigma'$  is obtained from  $\sigma$  by adding a cycle of length  $n$  and a cycle of length  $n_{k,\ell}$  for all  $k < m - 1$  and  $\ell < n - 3$ .

*Proof.* Recursively define  $i_{n-1} = j$ ,  $i_{n-2} = i$ ,  $i_{n-3} = i - 1$ ,  $i_{m-1,\ell} = i_{\ell+1}$ ,  $i_{k,\ell} = i_{k+1,\ell} - n_{k,\ell}$ , and  $i_\ell = i_{0,\ell} - 1$  for all  $k \leq m - 2$  and  $\ell \leq n - 4$ . Set  $(\rho_{n-2}, \sigma_{n-2}) = (\rho, \sigma)$ . For all  $\ell \leq n - 3$ , Proposition 4.6 implies that the pair  $(\rho_\ell, \sigma_\ell)$ , given by  $\rho_\ell = (i_{0,\ell} \ i_{1,\ell} \ \cdots \ i_{m-1,\ell}) \cup \bigcup_{k < m-1} \text{id}_{(i_{k,\ell}, i_{k+1,\ell})}$  and  $\sigma_\ell = \bigcup_{k < m-1} (i_{k,\ell} \ i_{k,\ell} + 1 \ \cdots \ i_{k+1,\ell} - 1)$ , is in  $\mathcal{F}(i_\ell, i_{\ell+1}]$ . So  $(\rho', \sigma') = (\bigcup_{\ell \leq n-2} \rho_\ell, (i_0 \ i_1 \ \cdots \ i_{n-1}) \cup \bigcup_{\ell \leq n-2} \sigma_\ell)$  is in  $\mathcal{F}[i_0, i_{n-1} + 1)$  by Proposition 4.2. But  $i - 1 - \sum_{\ell < n-3} (1 + \sum_{k < m-1} n_{k,\ell}) = i_0$  and  $j + 1 = i_{n-1} + 1$ .  $\square$

For all integers  $i < j$ , set  $\mathcal{F}_0(i, j) = \mathcal{F}[i, j]$  and  $\mathcal{F}_1(i, j) = \mathcal{F}(i, j]$ .

**Proposition 5.6.** *Suppose that  $i < j$ ,  $m \geq 2$ ,  $n \geq 1$ ,  $p < 2$ ,  $(\rho, \sigma) \in \mathcal{F}_p(i, j)$ , and  $\pi \in \{\rho, \sigma\}$ . Then, for some  $\ell \in \mathbb{N}$  and all  $n_0, \dots, n_{\ell-1} \geq 3$ , there exist  $i' < i$ ,  $j' > j$ , and  $(\rho', \sigma') \in \mathcal{F}_{1-p}(i', j')$  such that:*

- $\rho'$  is obtained from  $\rho$  by adding cycles of length  $m$  and  $n_k - 1$   $n_k$ -malleable fixed points for all  $k < \ell$ , as well as a cycle of length  $n$  if  $\pi = \rho$  (which is an anti-malleable fixed point if  $n = 1$ ).
- $\sigma'$  is obtained from  $\sigma$  by adding a cycle of length  $n_k$  for all  $k < \ell$ , as well as a cycle of length  $n$  if  $\pi = \sigma$ .

Moreover, if  $n \geq 3$ , then  $\ell > 0$ .

*Proof.* If  $n = 1$ , then this follows Propositions 5.2, 5.3, and 5.5. If  $n \geq 2$ , then it follows from Propositions 5.1 and 5.4.  $\square$

## 6. THE PROOF OF THEOREM B

The special case of Theorem B where  $\rho$  or  $\sigma$  has finite order is a consequence of Propositions 2.1 and 2.3 and:

**Theorem 6.1.** *Suppose that  $m \geq 2$ ,  $\rho, \sigma \in \text{Sym}(\mathbb{Z})$  are periodic, and  $\text{Per}_m(\rho)$  and  $\text{Per}_{\geq 3}(\sigma)$  are infinite. Then  $S^{\mathbb{Z}} \in \text{Cl}(\rho)\text{Cl}(\sigma)$ .*

*Proof.* Fix an enumeration  $(\pi_n, O_n)_{n \in \mathbb{N}}$  of the pairs of the form  $(\pi, O)$ , where  $\pi \in \{\rho, \sigma\}$  and  $O \in \mathcal{O}(\pi)$ . Then there exist an infinite set  $N \subseteq \mathbb{N}$  and  $p < 2$  such that  $\pi_n = \sigma$ ,  $\text{par}(|O_n|) = p$ , and  $3 \leq |O_n| \leq |O_{n+1}|$  for all  $n \in N$ . Fix  $n_{-1} \in N$ , set  $N_0 = \mathbb{N} \setminus \{n_{-1}\}$ , and apply Proposition 4.5 to find  $i_0 < j_0$  and  $(\rho_0, \sigma_0) \in \mathcal{F}_0(i_0, j_0)$  such that every point of  $\text{dom}(\rho_0)$  is an  $n_{-1}$ -malleable fixed point and  $\sigma_0$  is a cycle of length  $|O_{n_{-1}}|$ .

Suppose that  $k$  is a natural number for which we have found  $i_k < j_k$ , a cofinite set  $N_k \subseteq \mathbb{N}$ , and  $(\rho_k, \sigma_k) \in \mathcal{F}_{\text{par}(k)}(i_k, j_k)$ . If  $k \in 2\mathbb{N}$ , then let  $n_k$  be the least element of  $N_k$  for which  $(\pi_{n_k} = \rho$  and  $|O_{n_k}| \geq 2)$  or  $(\pi_{n_k} = \sigma, |O_{n_k}| = 1, \text{ and } m \geq 3)$ . If  $k \in 4\mathbb{N} + 1$ , then let  $n_k$  be the least element of  $N_k$  for which  $(\pi_{n_k} = \sigma, |O_{n_k}| = 1, \text{ and } m = 2)$ ,  $(\pi_{n_k} = \sigma \text{ and } |O_{n_k}| = 2)$ , or  $(\pi_{n_k} = \rho \text{ and } |O_{n_k}| = 1)$ . And if  $k \in 4\mathbb{N} + 3$ , then let  $n_k$  be the least element of  $N_k$  for which  $\pi_{n_k} = \sigma$  and  $|O_{n_k}| \geq 3$ . Proposition 5.6 ensures that, for some  $\ell_k \in \mathbb{N}$  and all  $F_k \subseteq N \cap (N_k \setminus \{n\})$  of cardinality  $\ell_k$ , there exist  $i_{k+1} < i_k$ ,  $j_{k+1} > j_k$ , and  $(\rho_{k+1}, \sigma_{k+1}) \in \mathcal{F}(i_{k+1}, j_{k+1})$  such that:

- (1)  $\rho_{k+1}$  is obtained from  $\rho_k$  by adding cycles of length  $m$  and  $|O_n| - 1$   $|O_n|$ -malleable fixed points for all  $n \in F_k$ , as well as a cycle of length  $|O_{n_k}|$  if  $\pi_{n_k} = \rho$  (which is an anti-malleable fixed point if  $|O_{n_k}| = 1$ ).
- (2)  $\sigma_{k+1}$  is obtained from  $\sigma_k$  by adding a cycle of length  $m$  for all  $n \in F_k$ , as well as a cycle of length  $|O_n|$  if  $\pi_{n_k} = \sigma$ .

Set  $N_{k+1} = N_k \setminus (F_k \cup \{n_k\})$ .

Define  $\rho_\infty = \bigcup_{k \in \mathbb{N}} \rho_k$  and  $\sigma_\infty = \bigcup_{k \in \mathbb{N}} \sigma_k$ . As  $(i_k)_{k \in \mathbb{N}}$  is strictly decreasing and  $(j_k)_{k \in \mathbb{N}}$  is strictly increasing, these are permutations of  $\mathbb{Z}$  whose composition is  $S^{\mathbb{Z}}$ . As  $F_k \neq \emptyset$  for all  $k \in 4\mathbb{N} + 3$ , it follows that  $|\text{Mal}(\rho_\infty, \sigma_\infty) \cap \text{Per}_{2\mathbb{N}+n}(\sigma_\infty)| \in \{0, \aleph_0\}$  for all  $n \in 2\mathbb{N} + p$ . And clearly  $\text{Mal}(\rho_\infty, \sigma_\infty) \cap \text{Per}_{2\mathbb{N}+(1-p)}(\sigma_\infty) = \emptyset$ . As the fact that  $\bigcap_{k \in \mathbb{N}} N_k = \emptyset$  ensures that  $\rho_\infty \upharpoonright \sim (\text{Mal}(\rho_\infty, \sigma_\infty) \cap \text{Per}_1(\rho_\infty)) \cong \rho$  and  $\sigma_\infty \cong \sigma$ , Proposition 3.3 yields conjugates  $\rho'$  of  $\rho$  and  $\sigma'$  of  $\sigma$  for which  $\rho' \circ \sigma' = \rho_\infty \circ \sigma_\infty = S^{\mathbb{Z}}$ .  $\square$

The fact that every almost involution has finite order and [Mor89, Theorem A] therefore yield Theorem B.

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