A CHARACTERIZATION OF THE EXISTENCE OF INVARIANT MEASURES, I: INVOLUTIONS

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Abstract. We show that the existence of an invariant probability measure for a countable Borel equivalence relation with no singleton classes is equivalent to a first-order property of its full group.

Introduction

Let $\mathcal{P}(X)$ denote the family of all subsets of X. A *Borel space* is a set X equipped with a σ-algebra $\mathscr{B} \subseteq \mathcal{P}(X)$. Such a space is *standard* if $\mathscr{B}$ is the σ-algebra generated by a completely-metrizable separable topology on X. A set $B \subseteq X$ is *Borel* if $B \in \mathscr{B}$. A function between Borel spaces is *Borel* if preimages of Borel sets are Borel. A *Borel automorphism* of X is a Borel permutation of X whose inverse is also Borel. A *Borel probability measure* on X is a probability measure μ on $\mathscr{B}$. Let $\equiv_\mu$ denote the equivalence relation on the Borel automorphisms of X given by $S \equiv_\mu T \iff \{x \in X \mid S(x) \neq T(x)\}$ is μ-null. An equivalence relation E on X is *aperiodic* if all of its classes are infinite and *countable* if all of its classes are countable. A *partial transversal* of E is a set $Y \subseteq X$ that intersects every E-class in at most one point. The *full group* of E is the group $[E]$ of all Borel automorphisms of X whose graphs are contained in E. We say that μ is *E-conservative* if it concentrates off of Borel partial transversals of E, *E-invariant* if $\mu = T_*\mu$ for all $T \in [E]$, and *E-quasi-invariant* if $\mu \sim T_*\mu$ for all $T \in [E]$. An element g of a group G is an *involution* if $g^2 = 1_G$. Let $\mathrm{Inv}(G)$ denote the set of all such elements. For all $g \in G$, set $g^h = hgh^{-1}$ for all $h \in G$ and define $\mathrm{Cl}_G(g) = \{g^h \mid h \in G\}$.

Here we characterize the aperiodic countable Borel equivalence relations on standard Borel spaces that admit an invariant Borel probability measure in terms of a first-order property of their full groups:

Theorem 1. *Suppose that X is a standard Borel space, E is an aperiodic countable Borel equivalence relation on X, and $n \geq 4$. Then the following are equivalent:*

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(1) *There is an E-invariant Borel probability measure.*
(2) *There exists $I \in \mathrm{Inv}([E])$ with the property that n is the least natural number for which $\mathrm{Inv}([E]) \subseteq \mathrm{Cl}_{[E]}(I)^n$.*

We also establish an analogous result in the measure-theoretic milieu:

Theorem 2. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, μ is an E-conservative E-quasi-invariant Borel probability measure on X, and $n \geq 3$. Then the following are equivalent:*

(1) *There is an E-invariant Borel probability measure $\nu \ll \mu$.*
(2) *There exists $I \in \mathrm{Inv}([E])$ with the property that n is the least natural number for which $\mathrm{Inv}([E]) \subseteq [\mathrm{Cl}_{[E]_\mu}(I)^n]_{\equiv_\mu}$.*

Finally, we note the following fact, which can be combined with Theorem 1 to obtain the characterization promised in the abstract:

Theorem 3. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation with no singleton classes, and $n \geq 2$. Then the following are equivalent:*

(1) *The equivalence relation E is aperiodic.*
(2) *There exists $I \in \mathrm{Inv}([E])$ for which $\mathrm{Inv}([E]) \subseteq \mathrm{Cl}_{[E]}(I)^n$.*

The lower bounds on n are optimal in all three results. The E-conservativity of μ in Theorem 2 can be weakened to μ-almost-everywhere aperiodicity of E when $n \geq 4$. The almost everywhere analog of Theorem 3 holds for any E-quasi-invariant Borel probability measure, as do the analogs of all three results where involutions are replaced with automorphisms whose orbits all have cardinality 1 or k, for $k \geq 3$. These results also generalize to Borel actions of Polish groups.

In §1, we review several basic facts concerning countable Borel equivalence relations. In §2, we characterize the involutions that are products of a given number of conjugates of a given involution in $[E]$. And in §3, we establish our primary results.

1. Preliminaries

We will take the most basic facts of descriptive set theory for granted. These include Souslin's Theorem and its corollary that a function between standard Borel spaces is Borel if and only if its graph is Borel (see, for example, [**?**, Theorems 14.11 and 14.12]). They also include the Lusin–Novikov uniformization theorem (see, for example, [**?**, Theorem 18.10]). We will not give explicit proofs of straightforward consequences of these results.

We say that an equivalence relation E on X is *finite* if all of its classes are finite. An *fsr* of E is a finite subequivalence relation of the restriction of E to a subset of X. We will also take for granted the existence of Borel maximal fsrs and the immediate corollary that aperiodic countable Borel equivalence relations have Borel subequivalence relations whose classes all have a given finite cardinality (see, for example, [**?**, Lemma 7.3 and Proposition 7.4]).

The *support* of $T\colon X \to X$ is $\mathrm{supp}(T) = \{x \in X \mid x \neq T(x)\}$.

Proposition 4. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $I \in \mathrm{Inv}([E])$, and $E \restriction {\sim}\mathrm{supp}(I)$ is aperiodic. Then there is an extension of $I \restriction \mathrm{supp}(I)$ to a fixed-point-free element of $\mathrm{Inv}([E])$.*

Proof. Fix a Borel subequivalence relation F of $E \restriction {\sim}\mathrm{supp}(I)$ whose classes all have size two, let J be the unique fixed-point-free element of $[F]$, and observe that the extension of $I \restriction \mathrm{supp}(I)$ by J is as desired. □

The *E-saturation* of a set $Y \subseteq X$ is $[Y]_E = \{x \in X \mid \exists y \in Y \; x \mathrel{E} y\}$. We say that Y is *E-complete* if $X = [Y]_E$. Given Borel sets $A, B \subseteq X$ and $m, n \geq 1$, we write $mA \sim_E nB$ if there is a Borel bijection $\phi\colon m \times A \to n \times B$ with the property that $\mathrm{proj}_{A \times B}(\mathrm{graph}(\phi)) \subseteq E$. We write $mA \preccurlyeq_E nB$ if there is a Borel injection $\phi\colon m \times A \to n \times B$ for which $\mathrm{proj}_{A \times B}(\mathrm{graph}(\phi)) \subseteq E$ and $mA \lll_E nB$ if there is such an injection ϕ with the further property that $\mathrm{proj}_B((n \times B) \setminus \phi(m \times A))$ is $(E \restriction B)$-complete. We also write A and B instead of $1A$ and $1B$.

Proposition 5. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, and $A, B \subseteq X$ are Borel. Then there is a partition of X into E-invariant Borel sets $X_\lll$, $X_\sim$, and $X_\ggg$ with the property that $A \cap X_\lll \lll_E B$, $A \cap X_\sim \sim_E B \cap X_\sim$, and $B \cap X_\ggg \lll_E A$.*

Proof. Set $A' = A \setminus B$ and $B' = B \setminus A$, fix a Borel maximal fsr F of E whose classes are all sets of size two that intersect both A' and B', and let I be the unique fixed-point-free element of $[F]$. Then the sets $X_\lll = [B' \setminus \mathrm{proj}_X(F)]_E$, $X_\ggg = [A' \setminus \mathrm{proj}_X(F)]_E$, and $X_\sim = {\sim}(X_\lll \cup X_\ggg)$ are as desired, as witnessed by the corresponding restrictions of $I \cup \mathrm{id}_{A \cap B}$. □

A Borel set $B \subseteq X$ is *E-large* if $X \preccurlyeq_E nB$ for some $n \geq 1$.

Proposition 6. *Suppose that X is a standard Borel space and E is an aperiodic countable Borel equivalence relation on X. Then there is an E-large Borel set $B \subseteq X$ whose complement is also E-large.*

Proof. Fix a Borel subequivalence relation F of E whose classes all have cardinality two, fix a Borel transversal B of F, and observe that $B \sim_F {\sim}B$, thus $X \sim_F 2B \sim_F 2({\sim}B)$. $\square$

An E-*injection* of a set $Y \subseteq X$ into a set $Z \subseteq X$ is an injection of Y into Z whose graph is contained in E. An E-*bijection* is a surjective E-injection. A *compression* of E is an E-injection $\phi\colon X \to X$ for which ${\sim}\phi(X)$ is E-complete and E is *compressible* if there is a Borel compression of E. We say that a Borel set $B \subseteq X$ is E-*compressible* if $E \restriction B$ is compressible. Given $\phi\colon X \to X$ and $Y \subseteq X$ for which $Y \subseteq \bigcup_{n \geq 1} \phi^{-n}(Y)$, define $\nu_{\phi,Y}\colon Y \to \mathbb{N}$ and $\phi_Y\colon Y \to Y$ by $\nu_{\phi,Y}(y) = \min\{n \geq 1 \mid \phi^n(y) \in Y\}$ and $\phi_Y(y) = \phi^{\nu_{\phi,Y}(y)}(y)$ for all $y \in Y$.

Proposition 7. *Suppose that X is a standard Borel space, E is a compressible countable Borel equivalence relation on X, and $B \subseteq X$ is an E-complete Borel set. Then the following are equivalent:*

1. *B is E-large.*
2. *B is E-compressible.*
3. *$B \sim_E X$.*

Proof. As (3) $\implies$ (1) is trivial and (2) $\implies$ (3) follows from [**?**, Proposition 2.2], we need only show (1) $\implies$ (2). Towards this end, fix a Borel compression $\phi\colon X \to X$ of E. As B is E-large, there exist $n \geq 1$, a partition of X into Borel sets $B_0, \ldots, B_{n-1}$, and Borel E-injections $\phi_0\colon B_0 \to B, \ldots, \phi_{n-1}\colon B_{n-1} \to B$. Set $C = {\sim}\phi(X)$, $D = \bigcup_{n \in \mathbb{N}} \phi^n(C)$, $D_m = D \cap \bigcap_{i \in \mathbb{N}} \bigcup_{j \geq i} \phi^{-j}(B_m)$ for all $m < n$, and $D'_m = D_m \setminus \bigcup_{\ell < m} [D_\ell]_E$ for all $m < n$. Then $\bigcup_{m<n} \phi_m \circ \phi_{B_m \cap D'_m} \circ \phi_m^{-1}$ is a compression of the restriction of E to an E-complete Borel subset of B, so B is E-compressible. $\square$

Proposition 8. *Suppose that X is a standard Borel space, E is an aperiodic countable Borel equivalence relation on X, and $n \geq 2$. Then:*

1. *There is a Borel set $B \subseteq X$ for which $X \sim_E nB$.*
2. *Suppose that $B \subseteq X$ is a Borel set for which $X \sim_E nB$ and $Y \subseteq X$ is an E-invariant Borel set. Then*

$$Y \preccurlyeq_E (n-1)B \iff Y \text{ is } E\text{-compressible.}$$

Proof. To see (1), fix a Borel subequivalence relation F of E whose classes all have cardinality n, $T \in [F]$ for which $F = E_T^X$, and a Borel transversal $B \subseteq X$ of F. Then $X = \coprod_{m<n} T^m(B)$, so $X \sim_F nB$.

To see (2), note that if Y is E-compressible, then Proposition 7 ensures that $B \cap Y$ is E-compressible (since $Y \preccurlyeq_E nB$), so $Y \preccurlyeq_E B \preccurlyeq_E (n-1)B$. Conversely, if $Y \preccurlyeq_E (n-1)B$, then $Y \preccurlyeq_E (n-1)(B \cap Y) \lll_E n(B \cap Y) \preccurlyeq_E Y$, so Y is E-compressible. $\square$

2. Generating one involution from another

A *transversal* of E is an E-complete partial transversal of E and E is *smooth* if there is a Borel transversal of E. An *embedding* of a function $S \colon X \to X$ into a function $T \colon Y \to Y$ is an injection $\phi \colon X \to Y$ with the property that $\phi \circ S = T \circ \phi$. An *isomorphism* is a surjective embedding. Let $\mathrm{Sym}(X)$ denote the group of all permutations of X.

The following observation ensures that the obvious "local" requirement is the only obstacle to writing an involution in $[E]$ as a composition of conjugates of involutions in $[E]$ when E is smooth:

Proposition 9. *Suppose that X is a standard Borel space, E is a smooth countable Borel equivalence relation on X, $n \geq 1$, $I_0, \ldots, I_n \in \mathrm{Inv}([E])$, and $\forall C \in X/E$ $I_n \restriction C \in \prod_{m<n} \mathrm{Cl}_{\mathrm{Sym}(C)}(I_m \restriction C)$. Then $I_n \in \prod_{m<n} \mathrm{Cl}_{[E]}(I_m)$.*

Proof. As there are only countably many isomorphism classes of involutions of countable sets, we can assume that there exist a countable cardinal k, $\iota_0, \ldots, \iota_n \in \mathrm{Inv}(\mathrm{Sym}(k))$, and E-invariant Borel functions $\phi_0, \ldots, \phi_n \colon X \to k$ such that $\phi_m \restriction C$ is an isomorphism of $I_m \restriction C$ with ι_m for all $C \in X/E$ and $m \leq n$. Fix $\tau_0, \ldots, \tau_{n-1} \in \mathrm{Sym}(k)$ for which $\iota_n = \circ_{m<n} \iota_m^{\tau_m}$ and define $T_0, \ldots, T_{n-1} \in [E]$ by setting $T_m(x) = ((\phi_n \restriction [x]_E)^{-1} \circ \tau_m \circ \phi_m)(x)$ for $m < n$ and $x \in X$. Then

$$
\begin{aligned}
&I_n(x)\\
&\quad = ((\phi_n \restriction [x]_E)^{-1} \circ \iota_n \circ \phi_n)(x)\\
&\quad = ((\phi_n \restriction [x]_E)^{-1} \circ (\circ_{m<n} \iota_m^{\tau_m}) \circ \phi_n)(x)\\
&\quad = (\circ_{m<n} (\phi_n \restriction [x]_E)^{-1} \circ \iota_m^{\tau_m} \circ \phi_n)(x)\\
&\quad = (\circ_{m<n} (\phi_n \restriction [x]_E)^{-1} \circ \tau_m \circ \iota_m \circ \tau_m^{-1} \circ \phi_n)(x)\\
&\quad = (\circ_{m<n} (\phi_n \restriction [x]_E)^{-1} \circ \tau_m \circ \phi_m \circ I_m \circ (\phi_m \restriction [x]_E)^{-1} \circ \tau_m^{-1} \circ \phi_n)(x)\\
&\quad = (\circ_{m<n} T_m \circ I_m \circ T_m^{-1})(x)\\
&\quad = (\circ_{m<n} I_m^{T_m})(x)
\end{aligned}
$$

for all $x \in X$. □

Let $\equiv_E$ denote the equivalence relation on the Borel automorphisms of X given by $S \equiv_E T \iff E \restriction \{x \in X \mid S(x) \neq T(x)\}$ is smooth.

Proposition 10. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $n \geq 2$, and $I_0, \ldots, I_{n-1} \in \mathrm{Inv}([E])$ have the following properties:*

1. $\forall m < n \ \mathrm{supp}(I_m) \subseteq \bigcup_{k \in n \setminus \{m\}} \mathrm{supp}(I_k)$.
2. $\forall j, k < n \ I_j \restriction (\mathrm{supp}(I_j) \cap \mathrm{supp}(I_k)) = I_k \restriction (\mathrm{supp}(I_j) \cap \mathrm{supp}(I_k))$.

Then $\mathrm{id}_X \in [\prod_{m<n} \mathrm{Cl}_{[E]}(I_m)]_{\equiv_E}$.

Proof. For all $K \subseteq n$, set $X_K = \bigcap_{k \in K} \mathrm{supp}(I_k) \setminus \bigcup_{k \in \sim K} \mathrm{supp}(I_k)$. By focusing separately on each of these sets, we need only establish the special case of the proposition where each I_k is fixed-point free (thus they are all the same). If n is even, then this special case is trivial. So it only remains to check the case that $n = 3$.

Set $I = I_0 = I_1 = I_2$ and fix a Borel maximal fsr F of E whose classes are all I-invariant sets of cardinality four. As $\sim\mathrm{proj}_X(F)$ intersects every E-class in at most one orbit of I, we can assume that it is empty. But the product of the three fixed-point-free involutions in $\mathrm{Sym}(4)$ is the identity, so $\mathrm{id}_4 \in \mathrm{Cl}_{\mathrm{Sym}(4)}(\iota)^3$ for all fixed-point free $\iota \in \mathrm{Inv}(\mathrm{Sym}(4))$, thus $\mathrm{id}_X \in \mathrm{Cl}_{[F]}(I)^3$ by Proposition 9. $\square$

We will use Proposition 10 in conjunction with the following fact, which is the main technical observation underlying our primary results:

Proposition 11. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $I, J \in \mathrm{Inv}([E])$, $n \geq 2$, $E \restriction \sim\mathrm{supp}(I)$ and $E \restriction \sim\mathrm{supp}(J)$ are aperiodic, and $\mathrm{supp}(I) \preccurlyeq_E n\mathrm{supp}(J)$. Then X is the union of E-invariant Borel sets Y and Z for which:*

1. *There exists $T \in [E \restriction Y]$ such that $T \restriction \mathrm{supp}(I \restriction Y)$ is an embedding of $I \restriction \mathrm{supp}(I \restriction Y)$ into J.*
2. *There exist $T_0, \ldots, T_{n-1} \in [E \restriction Z]$ such that $T_m \restriction \mathrm{supp}(J \restriction Z)$ is an embedding of $J \restriction \mathrm{supp}(J \restriction Z)$ into I for all $m < n$ and $\mathrm{supp}(I \restriction Z) = \bigcup_{m<n} T_m(\mathrm{supp}(J \restriction Z))$.*

Proof. As $\mathrm{supp}(I) \subseteq [\mathrm{supp}(J)]_E$, we can assume that $\mathrm{supp}(J)$ is E-complete. By Proposition 4, there are extensions of $I \restriction \mathrm{supp}(I)$ and $J \restriction \mathrm{supp}(J)$ to fixed-point-free elements I' and J' of $\mathrm{Inv}([E])$. Fix Borel transversals A' and B' of $E^X_{I'}$ and $E^X_{J'}$.

We first consider the case where E is compressible and $\mathrm{supp}(J)$ is E-large. As A' is E-large and Proposition 5 allows us to assume that $A' \cap \mathrm{supp}(I) \preccurlyeq_E A' \setminus \mathrm{supp}(I)$ or $A' \setminus \mathrm{supp}(I) \preccurlyeq_E A' \cap \mathrm{supp}(I)$, we can therefore assume that $A' \setminus \mathrm{supp}(I)$ or $A' \cap \mathrm{supp}(I)$ is E-large.

If $A' \setminus \mathrm{supp}(I)$ is E-large, then appeal to Proposition 6 to obtain an E-large Borel set $C' \subseteq B' \cap \mathrm{supp}(J)$ for which $B' \setminus C'$ is E-large, as well as to Proposition 7 to obtain a Borel E-injection $\phi \colon A' \cap \mathrm{supp}(I) \to C'$ and a Borel E-bijection $\psi \colon A' \setminus \mathrm{supp}(I) \to B' \setminus \phi(A' \cap \mathrm{supp}(I))$. Then the function $T = \phi \cup (J' \circ \phi \circ I') \cup \psi \cup (J' \circ \psi \circ I')$ is as desired.

If $A' \cap \mathrm{supp}(I)$ is E-large, then we can assume that $B' \setminus \mathrm{supp}(J) \preccurlyeq_E A' \setminus \mathrm{supp}(I)$ or $A' \setminus \mathrm{supp}(I) \preccurlyeq_E B' \setminus \mathrm{supp}(J)$ by Proposition 5. If there is a Borel E-injection $\phi \colon B' \setminus \mathrm{supp}(J) \to A' \setminus \mathrm{supp}(I)$, then Proposition 7 yields a Borel E-bijection $\psi \colon B' \cap \mathrm{supp}(J) \to A' \setminus \phi(B' \setminus \mathrm{supp}(J))$, so the inverse of the function $T = \phi \cup (I' \circ \phi \circ J') \cup \psi \cup (I' \circ \psi \circ J')$ is as desired. If there is a Borel E-injection $\phi \colon A' \setminus \mathrm{supp}(I) \to B' \setminus \mathrm{supp}(J)$, then appeal to Proposition 6 to obtain E-large Borel sets $A'' \subseteq A' \cap \mathrm{supp}(I)$ and $B'' \subseteq B' \cap \mathrm{supp}(J)$ for which $(A' \cap \mathrm{supp}(I)) \setminus A''$ and $(B' \cap \mathrm{supp}(J)) \setminus B''$ are E-large, as well as to Proposition 7 to obtain Borel E-bijections $\psi_0, \psi_1 \colon A' \cap \mathrm{supp}(I) \to B' \setminus \phi(A' \setminus \mathrm{supp}(I))$ for which $A'' = \psi_0^{-1}(B'')$ and $(A' \cap \mathrm{supp}(I)) \setminus A'' = \psi_1^{-1}(B'')$. Then the inverses of the functions $T_m = \phi \cup (J' \circ \phi \circ I') \cup \psi_m \cup (J' \circ \psi_m \circ I')$, for $m < 2$, are as desired.

We now consider the general case. By Proposition 5, we can assume that $A' \cap \mathrm{supp}(I) \preccurlyeq_E B' \cap \mathrm{supp}(J)$, $B' \cap \mathrm{supp}(J) \lll_E A' \cap \mathrm{supp}(I) \preccurlyeq_E n(B' \cap \mathrm{supp}(J))$, or $n(B' \cap \mathrm{supp}(J)) \lll_E A' \cap \mathrm{supp}(I)$.

Suppose first that $n(B' \cap \mathrm{supp}(J)) \lll_E A' \cap \mathrm{supp}(I)$. Then $\mathrm{supp}(I) \preccurlyeq_E n\mathrm{supp}(J) \lll_E \mathrm{supp}(I)$ and $\mathrm{supp}(I) \cup \mathrm{supp}(J) \preccurlyeq_E (n+1)\mathrm{supp}(J)$, so $\mathrm{supp}(I) \cup \mathrm{supp}(J)$ is E-compressible and $\mathrm{supp}(J)$ is $(E \restriction (\mathrm{supp}(I) \cup \mathrm{supp}(J)))$-large, thus Proposition 7 ensures that $\mathrm{supp}(J)$ is E-compressible, hence E is compressible and $\mathrm{supp}(J)$ is E-large.

Suppose next that there is a Borel E-injection $\phi' \colon A' \cap \mathrm{supp}(I) \to B' \cap \mathrm{supp}(J)$. By Proposition 5, we can assume that $A' \setminus \mathrm{supp}(I) \lll_E B' \setminus \phi'(A' \cap \mathrm{supp}(I))$, $A' \setminus \mathrm{supp}(I) \sim_E B' \setminus \phi'(A' \cap \mathrm{supp}(I))$, or $B' \setminus \phi'(A' \cap \mathrm{supp}(I)) \lll_E A' \setminus \mathrm{supp}(I)$. In the middle case, there is an extension of ϕ' to a Borel E-bijection $\phi \colon A' \to B'$, in which case the function $T = \phi \cup (J' \circ \phi \circ I')$ is as desired. In the other cases, there is either an extension of ϕ' to a Borel E-injection $\phi \colon A' \to B'$ for which $B' \setminus \phi(A')$ is $(E \restriction B')$-complete or an extension of $(\phi')^{-1}$ to a Borel E-injection $\psi \colon B' \to A'$ for which $A' \setminus \psi(B')$ is $(E \restriction A')$-complete, in which case $\phi \cup (J' \circ \phi \circ I')$ or $\psi \cup (I' \circ \psi \circ J')$ is a compression of E. By Proposition 5, we can assume that $\mathrm{supp}(K) \preccurlyeq_E {\sim}\mathrm{supp}(K)$ or ${\sim}\mathrm{supp}(K) \preccurlyeq_E \mathrm{supp}(K)$ and therefore that ${\sim}\mathrm{supp}(K)$ or $\mathrm{supp}(K)$ is E-large for all $K \in \{I, J\}$. But if $\mathrm{supp}(J)$ is not E-large, then $\mathrm{supp}(I)$ is not E-large, so both ${\sim}\mathrm{supp}(I)$ and ${\sim}\mathrm{supp}(J)$ are E-large, thus so too are $A' \setminus \mathrm{supp}(I)$ and $B' \setminus \phi'(A' \cap \mathrm{supp}(I))$. Proposition 7 therefore ensures that ϕ' extends to a Borel E-bijection $\phi \colon A' \to B'$, so the function $T = \phi \cup (J' \circ \phi \circ I')$ is as desired.

Suppose finally that $B' \cap \mathrm{supp}(J) \lll_E A' \cap \mathrm{supp}(I)$ but there are Borel sets $B'_0, \dots, B'_{n-1} \subseteq B' \cap \mathrm{supp}(J)$ and Borel E-injections $\phi''_m \colon B'_m \to A' \cap \mathrm{supp}(I)$ for which $(\phi''_m(B'_m))_{m<n}$ partitions $A' \cap \mathrm{supp}(I)$. By Proposition 5, we can assume that $(A' \cap \mathrm{supp}(I)) \setminus \phi''_m(B'_m) \preccurlyeq_E (B' \cap \mathrm{supp}(J)) \setminus B'_m$ for some $m < n$ or $(B' \cap \mathrm{supp}(J)) \setminus B'_m \preccurlyeq_E (A' \cap \mathrm{supp}(I)) \setminus \phi''_m(B'_m)$ for all $m < n$. In the former case, it follows that $B' \cap \mathrm{supp}(J) \lll_E A' \cap \mathrm{supp}(I) \preccurlyeq_E B' \cap \mathrm{supp}(J)$, so $\mathrm{supp}(J)$ is E-compressible, thus E is compressible and $\mathrm{supp}(J)$ is E-large by Proposition 7. In the latter case, there are extensions of $\phi''_0, \dots, \phi''_{n-1}$ to Borel E-injections $\phi'_0, \dots, \phi'_{n-1} \colon B' \cap \operatorname{supp}(J) \to A' \cap \operatorname{supp}(I)$. By Proposition 5, we can assume that $B' \setminus \operatorname{supp}(J) \ll_E A' \setminus \phi'_m(B' \cap \operatorname{supp}(J))$ for some $m < n$, $B' \setminus \operatorname{supp}(J) \sim_E A' \setminus \phi'_m(B' \cap \operatorname{supp}(J))$ for all $m < n$, or $A' \setminus \phi'_m(B' \cap \operatorname{supp}(J)) \ll_E B' \setminus \operatorname{supp}(J)$ for some $m < n$. In the middle case, there are extensions of $\phi'_0, \dots, \phi'_{n-1}$ to Borel E-bijections $\phi_0, \dots, \phi_{n-1} \colon B' \to A'$, so the functions $T_m = \phi_m \cup (I' \circ \phi_m \circ J')$, for $m < n$, are as desired. In the other cases, there exists $m < n$ for which there is either an extension of ϕ'_m to a Borel E-injection $\phi_m \colon B' \to A'$ such that $A' \setminus \phi_m(B')$ is $(E \restriction A')$-complete or an extension of $(\phi'_m)^{-1}$ to a Borel E-injection $\psi_m \colon A' \to B'$ such that $B' \setminus \psi_m(A')$ is $(E \restriction B')$-complete, in which case $\phi_m \cup (I' \circ \phi_m \circ J')$ or $\psi_m \cup (J' \circ \psi_m \circ I')$ is a compression of E. By Proposition 5, we can assume that $\operatorname{supp}(K) \preccurlyeq_E {\sim}\operatorname{supp}(K)$ or ${\sim}\operatorname{supp}(K) \preccurlyeq_E \operatorname{supp}(K)$ and therefore that ${\sim}\operatorname{supp}(K)$ or $\operatorname{supp}(K)$ is E-large for all $K \in \{I, J\}$. But if $\operatorname{supp}(J)$ is not E-large, then $\operatorname{supp}(I)$ is not E-large, so both ${\sim}\operatorname{supp}(I)$ and ${\sim}\operatorname{supp}(J)$ are E-large, thus so too are $B' \setminus \operatorname{supp}(J)$ and $A' \setminus \phi'_m(B' \cap \operatorname{supp}(J))$ for all $m < n$. Proposition 7 therefore ensures that $\phi'_0, \dots, \phi'_{n-1}$ extend to Borel E-bijections $\phi_0, \dots, \phi_{n-1} \colon B' \to A'$, so the functions $T_m = \phi_m \cup (I' \circ \phi_m \circ J')$, for $m < n$, are as desired. □

We now show that the obvious "global" requirement is the only obstacle to writing an involution in $[E]$ as a composition of conjugates of another involution in $[E]$ off of a Borel set where E is smooth:

Theorem 12. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $I, J \in \operatorname{Inv}([E])$, $n \geq 2$, and $\operatorname{supp}(I) \preccurlyeq_E n\operatorname{supp}(J)$. Then $I \in [\operatorname{Cl}_{[E]}(J)^n]_{\equiv_E}$.*

Proof. By throwing out an E-invariant Borel set on which E is smooth, we can assume that both $E \restriction {\sim}\operatorname{supp}(I)$ and $E \restriction {\sim}\operatorname{supp}(J)$ are aperiodic. By Proposition 11, we can therefore assume that either there exists $T \in [E]$ whose restriction to the support of I is an embedding of $I \restriction \operatorname{supp}(I)$ into J or there exist $T_0, \dots, T_{n-1} \in [E]$, whose restrictions to the support of J are embeddings of $J \restriction \operatorname{supp}(J)$ into I, with the property that $\operatorname{supp}(I) = \bigcup_{m<n} T_m(\operatorname{supp}(J))$. In the former case, Proposition 10 ensures that $\mathrm{id}_X \in [\operatorname{Cl}_{[E]}(I^T)\operatorname{Cl}_{[E]}(J)^n]_{\equiv_E}$. In the latter, Proposition 10 implies that $\mathrm{id}_X \in [\operatorname{Cl}_{[E]}(I)\prod_{m<n}\operatorname{Cl}_{[E]}(J^{T_m})]_{\equiv_E}$. □

In particular, it follows that the obvious "global" and "local" requirements are the only obstacles to writing an involution in $[E]$ as a composition of conjugates of another involution in $[E]$:

Theorem 13. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, $I, J \in \operatorname{Inv}([E])$, $n \geq 2$,* $\operatorname{supp}(I)$

$\preccurlyeq_E$ $n\mathrm{supp}(J)$, and $\forall C \in X/E$ $I \upharpoonright C \in \mathrm{Cl}_{\mathrm{Sym}(C)}(J \upharpoonright C)^n$. Then $I \in \mathrm{Cl}_{[E]}(J)^n$.

Proof. By Proposition 9 and Theorem 12. □

3. Main results

Along with the natural generalization of [?] to countable Borel equivalence relations, the following fact yields Theorem 1:

Theorem 14. *Suppose that X is a standard Borel space, E is an aperiodic countable Borel equivalence relation on X, and $n \geq 4$. Then exactly one of the following holds:*

1. *The equivalence relation E is compressible.*
2. *There exists $I \in \mathrm{Inv}([E])$ with the property that n is the least natural number for which $\mathrm{Inv}([E]) \subseteq \mathrm{Cl}_{[E]}(I)^n$.*

Proof. To see (1) $\implies \neg$(2), suppose that $I \in \mathrm{Inv}([E])$ and $\mathrm{Inv}([E]) \subseteq \mathrm{Cl}_{[E]}(I)^n$. As there is a fixed-point-free element of $\mathrm{Inv}([E])$, it follows that $X \preccurlyeq_E n\mathrm{supp}(I)$, so Proposition 7 implies that $X \preccurlyeq_E \mathrm{supp}(I)$. As $\mathrm{Inv}(\mathrm{Sym}(\mathbb{N})) \subseteq \mathrm{Cl}_{\mathrm{Sym}(\mathbb{N})}(\iota)^3$ for all $\iota \in \mathrm{Inv}(\mathrm{Sym}(\mathbb{N}))$ with infinite support (see [?]), Theorem 13 ensures that $\mathrm{Inv}([E]) \subseteq \mathrm{Cl}_{[E]}(I)^3$.

To see $\neg$(1) $\implies$ (2), apply Proposition 8 to obtain a Borel set $B \subseteq X$ with the property that n is the least natural number for which $X \preccurlyeq_E nB$ and fix $I \in \mathrm{Inv}([E])$ whose support is B. As $\mathrm{Sym}(\mathbb{N}) \subseteq \mathrm{Cl}_{\mathrm{Sym}(\mathbb{N})}(\iota)^n$ for all $\iota \in \mathrm{Inv}(\mathrm{Sym}(\mathbb{N}))$ with infinite support (see [?]), Theorem 13 ensures that $\mathrm{Inv}([E]) \subseteq \mathrm{Cl}_{[E]}(I)^n$. But $\mathrm{Cl}_{[E]}(I)^{<n}$ does not contain any fixed-point-free element of $\mathrm{Inv}([E])$. □

Along with the natural generalization of [?] to countable Borel equivalence relations, the following fact yields Theorem 2:

Theorem 15. *Suppose that X is a standard Borel space, E is a countable Borel equivalence relation on X, μ is an E-conservative E-quasi-invariant Borel probability measure on X, and $n \geq 3$. Then exactly one of the following holds:*

1. *There is an E-compressible μ-conull Borel set.*
2. *There exists $I \in \mathrm{Inv}([E])$ with the property that n is the least natural number for which $\mathrm{Inv}([E]) \subseteq [\mathrm{Cl}_{[E]}(I)^n]_{\equiv_\mu}$.*

Proof. By throwing out an E-invariant μ-null Borel set, we can assume that E is aperiodic.

To see (1) $\implies \neg$(2), suppose that $I \in \mathrm{Inv}([E])$ and $\mathrm{Inv}([E]) \subseteq [\mathrm{Cl}_{[E]}(I)^n]_{\equiv_\mu}$. Fix a fixed-point free $K \in \mathrm{Inv}([E])$. By throwing out an E-invariant μ-null Borel set, we can assume that E is compressible and

$K \in \mathrm{Cl}_{[E]}(I)^n$. Then $X \preccurlyeq_E n\mathrm{supp}(I)$, so $X \preccurlyeq_E \mathrm{supp}(I)$ by Proposition 7, thus $\mathrm{Inv}([E]) \subseteq [\mathrm{Cl}_{[E]}(I)^2]_{\equiv_\mu}$ by Theorem 12.

It remains to see $\neg(2) \implies (1)$. By Proposition 8, there is a Borel set $B \subseteq X$ such that $X \preccurlyeq_E nB$ but the only E-invariant Borel sets $Y \subseteq X$ for which $Y \preccurlyeq_E (n-1)B$ are E-compressible. Fix $I \in \mathrm{Inv}([E])$ whose support is B. Then $\mathrm{Inv}([E]) \subseteq [\mathrm{Cl}_{[E]}(I)^n]_{\equiv_\mu}$ by Theorem 12. Fix a fixed-point free $J \in \mathrm{Inv}([E])$. Then there is an E-invariant μ-conull Borel set $Y \subseteq X$ for which $J \upharpoonright Y \in \mathrm{Cl}_{[E]}(I \upharpoonright Y)^{<n}$, so $Y \preccurlyeq_E (n-1)B$, thus Y is E-compressible. $\square$

Finally, we have the following:

Proof of Theorem 3. To see $\neg(1) \implies \neg(2)$, fix a finite equivalence class C of E and observe that $\mathrm{parity}(I^n \upharpoonright C) = \mathrm{parity}(J \upharpoonright C)$ for all $I \in \mathrm{Inv}([E])$ and $J \in \mathrm{Cl}_{[E]}(I)^n$. To see $(1) \implies (2)$, fix a Borel subequivalence relation F of E whose classes all have cardinality three and $I \in \mathrm{Inv}([F])$ whose support is F-complete. Then $X \ll_F 2\mathrm{supp}(I)$. As $\mathrm{Inv}(\mathrm{Sym}(\mathbb{N})) \subseteq \mathrm{Cl}_{\mathrm{Sym}(\mathbb{N})}(\iota)^n$ for all $\iota \in \mathrm{Inv}(\mathrm{Sym}(\mathbb{N}))$ such that $\mathrm{supp}(\iota)$ and $\sim\mathrm{supp}(\iota)$ are both infinite (see [**?**, Corollary 2.4] and [**?**]), Theorem 13 ensures that $\mathrm{Inv}([E]) \subseteq \mathrm{Cl}_{[E]}(I)^n$. $\square$

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