Homework 12

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Exercise 12.1: Complete lattices.

- $\mathbb{N}$, the set of natural numbers $\{0,1,2\ldots\}$ with the usual order.

→ This is not a complete lattice. In order to be a complete lattice a greatest lower bound must exist for every subset of the initial set.

If we take the subset $A=\{\}$ there exists no $\sqcap A$. Since $\sqcap A$ should be the greatest element of the set $\mathbb{N}$. Since $\mathbb{N}$ is an infinite set and has no upper bound such element doesn't exist.

- $\mathbb{N} \cup \{\infty\}$ the set of natural numbers plus infinity, with the usual order and $n < \infty$ for all $n \in \mathbb{N}$

→ This is a complete lattice. Proof:

We fix $A \subseteq \mathbb{N} \cup \{\infty\}$ and consider the cases:

- case $A=\{\}$:

$\sqcap A$ must be the greatest element in $\mathbb{N} \cup \{\infty\}$. this element will be ∞. Since $n < \infty \; \forall n \in \mathbb{N}$

$\sqcap A = \infty$.

- case $A \neq \{\}$:

We take the subset $A' = \{x \mid x \in \mathbb{N} \cup \{\infty\}.\ x \leq a \; \forall a \in A\}$.

A' corresponds to all the lower bounds of A.

A' is a finite set, therefore it has a maximum element.

$\sqcap A = \max A'$.

- Finite set A with a total order $\leq$ on it.

→ This is a complete lattice. Proof:
We fix $B \subseteq A$ and consider the cases:

- case $B = \{\}$:

$\sqcap B$ must be the greatest element in A.
Since A is a finite set, it has a maximum element such that $\sqcap B = \max A$.

- case $B \neq \{\}$:

We take the subset $B' = \{x \mid x \in A.\ x \leq b\ \forall b \in B\}$.
B' corresponds to all the lower bounds of B.
B' is a finite set since it is a subset of a finite set. B' being finite means that it has a maximum element. $\sqcap B = \max B'$.

- $\mathbb{B}^*$, the set of all lists of booleans $\{[], [True], [False], ...\}$, with prefix order: $a \leq b \iff \exists c.\ b = a @ c$.

→ This is not a complete lattice.
In order to be a complete lattice a greatest lower bound must exist for every subset of the initial set.

If we take the subset $A = \{\}$ there exists no $\sqcap A$.
$\sqcap A$ should be the greatest element of the set $\mathbb{B}^*$. Since $\mathbb{B}^*$ is an infinite set and has no upper bound, such element doesn't exist.

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Exercise 12.2: Collecting Semantics

X := 0 ; Y := 2 {A0} ;
{A1}
WHILE 0 < Y
DO {A2} (X := X + Y ; Y := Y − 1 {A3})
{A4}

	0	1	2	3	4	5	6	7	8	9
A_0	{}	02								
A_1	{}		02			02, 21			02, 21, 30	
A_2	{}			02			02, 21			
A_3	{}				21			21, 30		
A_4	{}									30