Time evolution of dense multigraph limits under edge-conservative preferential attachment dynamics

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Abstract

We define the edge reconnecting model, a random multigraph evolving in time. At each time step we change one endpoint of a uniformly chosen edge: the new endpoint is chosen by linear preferential attachment. We consider a sequence of edge reconnecting models where the sequence of initial multigraphs is convergent in a sense which is a natural generalization of the notion of convergence of dense graph sequences, defined by Lovász and Szegedy in [11]. We investigate how the limit object evolves under the edge reconnecting dynamics if we rescale time properly: we give the complete characterization of the time evolution of the limit object from its initial state up to the stationary state, which is described in the companion paper [13]. In our proofs we use the theory of exchangeable arrays, queuing and diffusion processes. The number of parallel edges and the degrees evolve on different timescales and because of this the model exhibits subaging.

1 Introduction

We introduce the *edge reconnecting model*, a random multigraph (undirected graph with multiple and loop edges) evolving in time. Denote the multigraph at time T by $\mathcal{G}_n(T)$, where $T = 0, 1, 2, \dots$ and $n = |V(\mathcal{G}_n(T))|$ is the number of vertices. We denote by $m = |E(\mathcal{G}_n(T))|$ the number of edges (the number of vertices and edges does not change over time). Given the multigraph $\mathcal{G}_n(T)$ we get $\mathcal{G}_n(T+1)$ by uniformly choosing an edge in $E(\mathcal{G}_n(T))$, choosing one of the endpoints of that edge with a coin flip and reconnecting the edge to a new endpoint which is chosen using the rule of linear preferential attachment:

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a vertex v is chosen with probability $\frac{d(v)+\kappa}{2m+n\kappa}$, where $d(v)$ is the degree of vertex v in $\mathcal{G}_n(T)$ and $\kappa \in (0, +\infty)$ is a fixed parameter of the model.

Our aim is to describe the time evolution of the edge reconnecting model $\mathcal{G}_n(T)$ when $1 \ll n$ using the terminology of *dense graph limits*. The notion of convergence of simple graph sequences was defined and several equivalent characterizations of *graphons* (limit objects of convergent simple graph sequences) were given in [11]. In [10] we give a natural generalization of the theory of dense graph limits to multigraphs (see also [12] for similar results in a more general setting), which we briefly recall now.

Denote by $\mathcal{M}$ the set of multigraphs. If $G \in \mathcal{M}$ and $v, w \in V(G)$, denote by $E(v, w)$ the number of edges between v and w in F (loop edges count twice). For $F, G \in \mathcal{M}$ we define the density of copies of F in G by the formula

$$t_=(F, G) = \frac{1}{|V(G)|^{|V(F)|}} \sum_{\varphi: V(F) \to V(G)} \mathbb{1}[\forall\, v, w \in V(F):\ E(v, w) = E(\varphi(v), \varphi(w))].$$

We say that a sequence of multigraphs $(G_n)_{n=1}^{\infty}$ is convergent if for every $F \in \mathcal{M}$ the limit $g(F) = \lim_{n\to\infty} t_=(F, G_n)$ exists and $g(\cdot)$ is a "non-defective probability distribution" on the set of multigraphs (see Subsection 2.2 for details). In plain words: the sequence $\big(G_n\big)_{n=1}^{\infty}$ is convergent if the density of every fixed graph F in G_n converges as $n \to \infty$, and "no mass escapes to infinity" during this limiting procedure.

The definition of the limit objects of convergent multigraph sequences is slightly more complicated than that of graphons. A measurable function $W : [0, 1] \times [0, 1] \times \mathbb{N}_0 \to [0, 1]$ satisfying

$$W(x, y, l) \equiv W(y, x, l), \quad \sum_{l=0}^{\infty} W(x, y, l) \equiv 1, \quad W(x, x, 2l + 1) \equiv 0 \tag{1}$$

is called a *multigraphon*. Note that $(W(x, y, l))_{l=0}^{\infty}$ is a probability distribution on $\mathbb{N}_0$ for each $x, y \in [0, 1]$. We say that $G_n \to W$ if for every $F \in \mathcal{M}$ with $V(F) = \{1, \dots, k\}$ we have $\lim_{n\to\infty} t_=(F, G_n) = t_=(F, W)$ where

$$t_=(F, W) := \int_{[0,1]^k} \prod_{v \le w \le k} W(x_v, x_w, E(v, w))\, \mathrm{d}x_1\, \mathrm{d}x_2\, \dots\, \mathrm{d}x_k.$$

[10, Theorem 1] states that if a sequence of multigraphs $(G_n)_{n=1}^{\infty}$ is convergent then $G_n \to W$ for some multigraphon W and conversely, every multigraphon W arises this way. We say that a sequence of random multigraphs $\big(\mathcal{G}_n\big)_{n=1}^{\infty}$ converges in probability to a multigraphon W (or briefly write $\mathcal{G}_n \xrightarrow{\text{p}} W$) if for every simple graph F we have $t_=(F, \mathcal{G}_n) \xrightarrow{\text{p}} t_=(F, W)$, i.e.

$$\forall\, F \in \mathcal{M}\ \forall\, \varepsilon > 0 :\ \lim_{n\to\infty} \mathbf{P}\left(|t_=(F, \mathcal{G}_n) - t_=(F, W)| > \varepsilon\right) = 0. \tag{2}$$

In [13, Lemma 2.1] we build on methods and models of [5, Section 3.4] to explicitly describe the unique stationary distribution $\mathcal{G}_n(\infty)$ of the edge reconnecting model and in [13, Theorem 2] we prove that there is a multigraphon $\hat{W}_\infty$ such that

$$\mathcal{G}_n(\infty) \xrightarrow{\mathrm{p}} \hat{W}_\infty, \qquad n \to \infty \tag{3}$$

under the condition that $m \approx \frac{1}{2}\rho n^2$, where $\rho \in (0, +\infty)$ is a fixed parameter of the model called the *edge density.* The form of the limiting multigraphon $\hat{W}_\infty$ depends on ρ and the linear preferential attachment parameter κ.

Now we describe the main results of this paper: if we consider a sequence of edge reconnecting models with a convergent sequence of initial multigraphs $\mathcal{G}_n(0) \to W$ (satisfying some extra regularity conditions), then for every $t \in (0, +\infty)$ we have

$$\mathcal{G}_n(t \cdot n^2) \xrightarrow{\mathrm{p}} \breve{W}_t \qquad \text{and} \qquad \mathcal{G}_n(t \cdot n^3) \xrightarrow{\mathrm{p}} \tilde{W}_t, \qquad n \to \infty \tag{4}$$

where the multigraphons $\breve{W}_t$ and $\tilde{W}_t$ are explicit, continuous functions of t, the initial multigraphon W and κ. Moreover we have

$$\lim_{t\to 0_+} \breve{W}_t = W, \qquad \lim_{t\to\infty} \breve{W}_t = \lim_{t\to 0_+} \tilde{W}_t, \qquad \lim_{t\to\infty} \tilde{W}_t = \hat{W}_\infty \tag{5}$$

where $\hat{W}_\infty$ is the multigraphon in (3). Thus by (5) the convergence theorems (4) give the full characterization of the time evolution of the multigraphons arising as the graph limits of the edge reconnecting model.

Although our theorems are stated using the "multigraphon" formalism, in their proofs we use the correspondence between the theory of graph limits and that of exchangeable arrays, a connection first observed in [7]. The basic idea of the proof of our main theorems is to relate the time evolution of the edge reconnecting model to certain continuous-time stochastic processes using an appropriate rescaling of time:

- If we fix a vertex $v \in V(\mathcal{G}_n(0))$ and denote by $d(T, v)$ the degree of v in $\mathcal{G}_n(T)$ then the evolution of the $\mathbb{R}_+$-valued continuous-time stochastic process $\frac{1}{n}d(n^3 \cdot t, v)$ "almost looks like" that of a Cox-Ingersoll-Ross process (a diffusion process that is commonly used in financial mathemathics to model the evolution of interest rates). This fact is rigorously proved using the theory of stochastic differential equations and is used in the proof of $\mathcal{G}_n(t \cdot n^3) \xrightarrow{\mathrm{p}} \tilde{W}_t$.

- If we fix two vertices $v, w \in V(\mathcal{G}_n(0))$ and denote by $E(T, v, w)$ the number of parallel/loop edges connecting v and w in $\mathcal{G}_n(T)$ then the evolution of the $\mathbb{N}_0$-valued continuous-time stochastic process $E(n^2 \cdot t, v, w)$ "almost looks like" that of the queue length of an M/M/∞-queue. This fact is rigorously proved using a coupling argument and is used in the proof of $\mathcal{G}_n(t \cdot n^2) \xrightarrow{\mathrm{p}} \breve{W}_t$.

The most interesting property of the edge reconnecting model is the separation of *two different timescales* in (4) and (5): the degrees of the vertices only change significantly on the n^3 timescale, whereas the number of parallel (or loop) edges between two vertices evolves on the much faster n^2 timescale. The arrival rate of the M/M/∞-queue describing the evolution of $E(n^2 \cdot t, v, w)$ depends on the current degrees of v and w (if their degrees are high then edges appear between them with higher probability due to preferential attachment), but since the degrees evolve on the much slower n^3 timescale, they may be treated as constant background parameters on the n^2 timescale. The stochastic process $E(n^3 \cdot t + n^2 \cdot s, v, w)$ looks stationary in the time variable $s \in \mathbb{R}$ if $t \in (0, +\infty)$ is fixed and $1 \ll n$, but different values of t yield distinct pseudo-stationary distributions since $n^3 \cdot (t_2 - t_1)$ steps are enough for the background variables (degrees) to significantly change. This phenomenon is called *subaging* in [1].

A similar dynamical random graph model where no subaging occurs is studied in the context of equation-free numerical methods in [2] and [3].

The rest of this paper is organized as follows:

In Section 2 we introduce some notation, precisely formulate the above stated results, and give some heuristic hints on their proofs.

In Section 3 we relate the theory of multigraph limits to exchangeable arrays.

In Section 4 we prove some technical lemmas showing that degrees and multiple edges in the edge reconnecting model are well-behaved.

In Section 5 we prove the rigorous version of $\mathcal{G}_n(t \cdot n^2) \xrightarrow{\text{p}} \breve{W}_t$, Theorem 1.

In Section 6 we prove the rigorous version of $\mathcal{G}_n(t \cdot n^3) \xrightarrow{\text{p}} \tilde{W}_t$, Theorem 2.

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2 Notations, definitions, theorems

This section is organized as follows:

In Subsection 2.1 we precisely define the edge reconnecting model.

In Subsection 2.2 we give a probabilistic meaning to $t_=(F, W)$ by introducing W-random multigraphs and also define the average degree $D(W, x)$ of W at point x.

In Subsection 2.3 we recall some relevant properties of the M/M/∞-queue and the Cox-Ingersoll-Ross process.

In Subsection 2.4 we state Theorem 1 and Theorem 2, and we also give some heuristic comments on ideas behind their proofs.

In Subsection 2.5 we derive (5) and relate some properties of the multigraphons from our main theorems to the *configuration model*.

Denote by $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and $[n] := \{1, \dots, n\}$. Denote by $\mathcal{M}$ the set of undirected multigraphs (graphs with multiple and loop edges) and by $\mathcal{M}_n$ the set of multigraphs on n vertices. Let $G \in \mathcal{M}_n$. The adjacency matrix of a labeling of the multigraph G with $[n]$ is denoted by $(B(i,j))_{i,j=1}^n$, where $B(i,j) \in \mathbb{N}_0$ is the number of edges connecting the vertices labeled by i and j. $B(i,j) = B(j,i)$ since the graph is undirected and $B(i,i)$ is two times the number of loop edges at vertex i (thus $B(i,i)$ is an even number). An unlabeled multigraph is the equivalence class of labeled multigraphs where two labeled graphs are equivalent if one can be obtained by relabeling the other. Thus $\mathcal{M}$ is the set of these equivalence classes of labeled multigraphs, which are also called isomorphism types. We denote the set of adjacency matrices of multigraphs on n nodes by $\mathcal{A}_n$, thus

$$\mathcal{A}_n = \left\{ B \in \mathbb{N}_0^{n \times n} \, : \, B^T = B, \, \forall \, i \in [n] \;\; 2 \,|\, B(i,i) \right\}.$$

The degree of the vertex labeled by i in G with adjacency matrix $B \in \mathcal{A}_n$ is defined by $d(B,i) := \sum_{j=1}^n B(i,j)$, thus $d(B,i)$ is the number of edge-endpoints at i (loop edges count twice). Let $m = \frac{1}{2}\sum_{i,j=1}^n B(i,j) = \frac{1}{2}\sum_{i=1}^n d(B,i)$ denote the number of edges. Denote by $\mathcal{A}_n^m$ the set of adjacency matrices on n vertices with m edges.

We denote a random element of $\mathcal{A}_n$ by $\mathbf{X}_n$. We may associate a random multigraph $\mathcal{G}_n$ to $\mathbf{X}_n$ by taking the isomorphism class of $\mathbf{X}_n$.

We use the standard notation $\mathbf{X}_n \sim \mathbf{X}'_n$ if $\mathbf{X}_n$ and $\mathbf{X}'_n$ are identically distributed, i.e.

$$\forall \, A \in \mathcal{A}_n : \; \mathbf{P}\left(\mathbf{X}_n = A\right) = \mathbf{P}\left(\mathbf{X}'_n = A\right).$$

If $\mathbf{X}_n$ is a random element of $\mathcal{A}_n$ then

$$\mathbf{X}_n^{[k]} := (X_n(i,j))_{i,j=1}^k$$

is a random element of $\mathcal{A}_k$.

2.1 The edge reconnecting model

Now we describe the dynamics of the edge reconnecting model, which is a discrete time Markov chain with state space $\mathcal{A}_n^m$: neither the number of vertices, nor the number of edges is changed by the dynamics. $\mathbf{X}(T) = (X(T,i,j))_{i,j=1}^n$ is the state of our Markov chain at time T.

Given the adjacency matrix $\mathbf{X}(T)$ we get $\mathbf{X}(T+1)$ in the following way: let $\kappa \in (0, +\infty)$. We choose a random vertex $\mathcal{V}_{old}(T)$ with distribution

$$\mathbf{P}\left(\mathcal{V}_{old}(T) = i \,\middle|\, \mathbf{X}(T)\right) = \frac{d(\mathbf{X}(T), i)}{2m} \tag{6}$$

Then we choose a uniform edge $\mathcal{E}_{old}(T) = \{\mathcal{V}_{old}(T), \mathcal{W}(T)\}$ going out of $\mathcal{V}_{old}(T)$:

$$\mathbf{P}\left(\mathcal{W}(T) = i \,\middle|\, \mathbf{X}(T), \mathcal{V}_{old}(T)\right) = \frac{X(T, \mathcal{V}_{old}(T), i)}{d(\mathbf{X}(T), \mathcal{V}_{old})}$$

Note that $\mathcal{E}_{old}(T)$ is uniformly distributed over all edges of the graph at time T and given $\mathcal{E}_{old}(T)$, $\mathcal{V}_{old}(T)$ is uniformly chosen from the endvertices of $\mathcal{E}_{old}(T)$. Moreover

$$\mathbf{P}\left(\mathcal{W}(T)=i \,\middle|\, \mathbf{X}(T)\right)=\frac{d(\mathbf{X}(T),i)}{2m}. \tag{7}$$

Given $\mathbf{X}(T)$, choose $\mathcal{V}_{new}(T)$ according to the rules of linear preferential attachment:

$$\mathbf{P}\left(\mathcal{V}_{new}(T)=i \,\middle|\, \mathbf{X}(T),\mathcal{V}_{old}(T),\mathcal{W}(T)\right)=\frac{d(\mathbf{X}(T),i)+\kappa}{2m+n\kappa}. \tag{8}$$

Thus $\mathcal{V}_{new}(T)$ is conditionally independent from $\mathcal{V}_{old}(T)$ and $\mathcal{W}(T)$ given $\mathbf{X}(T)$.

Let $\mathcal{E}_{new}(T) := \{\mathcal{V}_{new}(T), \mathcal{W}(T)\}$.

One step of the Markov chain consists of replacing the edge $\mathcal{E}_{old}(T)$ with $\mathcal{E}_{new}(T)$:

$$\begin{aligned} X(T+1,i,j) = X(T,i,j) - \mathbb{1}[\mathcal{V}_{old}(T)=i,\mathcal{W}(T)=j] - \mathbb{1}[\mathcal{V}_{old}(T)=j,\mathcal{W}(T)=i]+ \\ \mathbb{1}[\mathcal{V}_{new}(T)=i,\mathcal{W}(T)=j] + \mathbb{1}[\mathcal{V}_{new}(T)=j,\mathcal{W}(T)=i] \end{aligned} \tag{9}$$

This Markov chain is easily seen to be irreducible and aperiodic on $\mathcal{A}_n^m$. Note that for any $k \le n$ the $\mathbb{N}_0^{[k]}$-valued stochastic process $(d(\mathbf{X}(T),i))_{i=1}^k$, $T=0,1,\dots$ is itself a Markov chain.

2.2 Multigraphons and W-random multigraphs

In this subsection we give a probabilistic meaning to $t_=(F,W)$ by introducing W-random multigraphs and also define the average degree $D(W,x)$ of W at point x. Note that the notion of the W-random graph (see Definition 2.1) is already present in [11].

Suppose $F \in \mathcal{M}_k$, $G \in \mathcal{M}_n$ and denote by $A \in \mathcal{A}_k$ and $B \in \mathcal{A}_n$ the adjacency matrices of F and G. If $g : \mathcal{M} \to \mathbb{R}$ then we say that g is a multigraph parameter. Let $g(A) := g(F)$. Conversely, if $g : \bigcup_{k=1}^{\infty} \mathcal{A}_k \to \mathbb{R}$ is constant on isomorphism classes, then g defines a multigraph parameter.

We define the *induced homomorphism density* of F into G by

$$t_=(F,G) := t_=(A,B) := \frac{1}{n^k} \sum_{\varphi:[k]\to[n]} \mathbb{1}\left[\forall i,j\in[k]:\; A(i,j)=B(\varphi(i),\varphi(j))\right].$$

We say that a sequence of multigraphs $(G_n)_{n=1}^{\infty}$ is convergent if for every $k \in \mathbb{N}$ and every multigraph $F \in \mathcal{M}_k$ the limit $g(F) = \lim_{n\to\infty} t_=(F,G_n)$ exists, and we have $\sum_{A\in\mathcal{A}_k} g(A) = 1$. For every multigraphon W (see (1)) and multigraph $F \in \mathcal{M}_k$ with adjacency matrix $A \in \mathcal{A}_k$ we define

$$t_=(F,W) := t_=(A,W) := \int_{[0,1]^k} \prod_{i\le j\le k} W(x_i,x_j,A(i,j))\,\mathrm{d}x_1\,\mathrm{d}x_2\,\dots\,\mathrm{d}x_k \tag{10}$$

We say that $G_n \to W$ if for every $F \in \mathcal{M}$ we have $\lim_{n\to\infty} t_=(F, G_n) = t_=(F, W)$. By [10, Theorem 1] we have that a sequence of multigraphs $(G_n)_{n=1}^{\infty}$ is convergent then $G_n \to W$ for some multigraphon W. The limiting multigraphon of a convergent sequence is not unique, but if we define the equivalence relation

$$W_1 \cong W_2 \quad \iff \quad \forall\, F \in \mathcal{M}:\ t_=(F, W_1) = t_=(F, W_2) \tag{11}$$

then obviously $G_n \to W_1 \iff G_n \to W_2$. For other characterisations of the equivalence relation $\cong$ for graphons, see [4].

If $\mathbf{X}_n$ is a random element of $\mathcal{A}_n$ for each $n \in \mathbb{N}$, $\mathcal{G}_n$ is the isomorphism class of $\mathbf{X}_n$ and W is a multigraphon, then we say that $\mathbf{X}_n \xrightarrow{\mathrm{P}} W$ if $\mathcal{G}_n \xrightarrow{\mathrm{P}} W$, see (2).

For a multigraphon W and $x \in [0,1]$ we define the *average degree* of W at x and the *edge density* of W by

$$D(W, x) := \int_0^1 \sum_{l=0}^{\infty} l \cdot W(x, y, l)\, \mathrm{d}y \tag{12}$$

$$\rho(W) := \int_0^1 \int_0^1 \sum_{l=0}^{\infty} l \cdot W(x, y, l)\, \mathrm{d}y\, \mathrm{d}x \tag{13}$$

If $\rho(W) < +\infty$ then $D(W, x) < +\infty$ for Lebesgue-almost all x.

We say that a $[0,1]$-valued random variable U is uniformly distributed on $[0,1]$ (or briefly denote $U \sim \mathcal{U}[0,1]$) if $\mathbf{P}\left(U \le x\right) = x$ for all $x \in [0,1]$.

Definition 2.1 (W-random multigraphons).

Fix $k \in \mathbb{N}$. Let $(U_i)_{i=1}^k$ be i.i.d., $U_i \sim \mathcal{U}[0,1]$. Given a multigraphon W we define the $\mathcal{A}_k$-valued random variable $\mathbf{X}_W^{[k]} = (X_W(i,j))_{i,j=1}^k$ as follows:

Given the background variables $(U_i)_{i=1}^k$ the random variables $(X_W(i,j))_{i \le j \le k}$ are conditionally independent and $\mathbf{P}\left(X_W(i,j) = l \,\middle|\, (U_i)_{i=1}^k\right) = W(U_i, U_j, l)$, that is

$$\forall\, A \in \mathcal{A}_k: \quad \mathbf{P}\left(\mathbf{X}_W^{[k]} = A \,\middle|\, (U_i)_{i=1}^k\right) := \prod_{i \le j \le k} W(U_i, U_j, A(i,j)). \tag{14}$$

In plain words: if $i \neq j$ and $U_i = x$, $U_j = y$ then the number of multiple edges between the vertices labeled by i and j in $\mathbf{X}_W$ has distribution $\big(W(x,y,l)\big)_{l=1}^{\infty}$ and the number of loop edges at vertex i has distribution $\big(W(x,x,2l)\big)_{l=1}^{\infty}$.

For every multigraphon W and $A \in \mathcal{A}_k$ we have

$$t_=(A, W) \overset{(10),(14)}{=} \mathbf{P}\left(\mathbf{X}_W^{[k]} = A\right). \tag{15}$$

Recalling (11) it follows that $W_1 \cong W_2$ if and only if $\forall\, k \in \mathbb{N}:\ \mathbf{X}_{W_1}^{[k]} \sim \mathbf{X}_{W_2}^{[k]}$, thus the distribution of the W-random multigraphons determine the multigraphon up to $\cong$ equivalence. Recalling (12) and (13) we have

$$D(W, x) = \mathbf{E}\left(X_W(1,2) \,\middle|\, U_1 = x\right), \qquad \rho(W) = \mathbf{E}\left(X_W(1,2)\right). \tag{16}$$

Note that the weak law of large numbers (heuristically) implies that

$$\frac{1}{n} d(\mathbf{X}_W^{[n]}, i) \approx D(W, U_i), \qquad 1 \ll n \tag{17}$$

This relation is the reason why we gave the name *average degree* to $D(W, x)$.

2.3 Auxiliary stochastic processes

In this subsection we recall the definition and some properties of two stochastic processes: the M/M/∞-queue and the C.I.R. process.

First recall the formulas defining the Poisson, binomial and gamma distributions:

$$\mathbf{p}(k, \lambda) := e^{-\lambda} \frac{\lambda^k}{k!} \tag{18}$$

$$\mathbf{b}(k, n, p) := \binom{n}{k} p^k (1-p)^{n-k} \tag{19}$$

$$\mathbf{g}(x, \alpha, \beta) := x^{\alpha-1} \frac{\beta^\alpha e^{-\beta x}}{\Gamma(\alpha)} \mathbb{1}[x > 0] \tag{20}$$

We say that a nonnegative integer-valued random variable X has Poisson distribution with parameter λ (or briefly denote $X \sim \mathrm{POI}(\lambda)$) if $\mathbf{P}(X = k) = \mathbf{p}(k, \lambda)$ for all $k \in \mathbb{N}$. We say that a $\{0, 1, \ldots, n\}$-valued random variable Y has binomial distribution with parameters n and p (or briefly denote $Y \sim \mathrm{BIN}(n, p)$) if $\mathbf{P}(Y = k) = \mathbf{b}(k, n, p)$ for all $k \in \{0, 1, \ldots, n\}$. We say that a nonnegative real-valued random variable Z has gamma distribution with parameters α and β (or briefly denote $Z \sim \mathrm{Gamma}(\alpha, \beta)$) if $\mathbf{P}(Z \le z) = \int_0^z \mathbf{g}(x, \alpha, \beta)\,\mathrm{d}x$.

The M/M/∞-queue with arrival rate μ and service rate 1 is an $\mathbb{N}_0$-valued continuous-time Markov chain Y_t, $t \in [0, +\infty)$ with infinitesimal jump rates

$$\mathbf{P}\left(Y_{t+\mathrm{d}t} = k+1 \,\middle|\, Y_t = k\right) = \mu\,\mathrm{d}t + o(\mathrm{d}t) \tag{21}$$

$$\mathbf{P}\left(Y_{t+\mathrm{d}t} = k-1 \,\middle|\, Y_t = k\right) = k\,\mathrm{d}t + o(\mathrm{d}t) \tag{22}$$

$$\mathbf{P}\left(Y_{t+\mathrm{d}t} = k \,\middle|\, Y_t = k\right) = 1 - (\mu + k)\mathrm{d}t + o(\mathrm{d}t) \tag{23}$$

Heuristically, Y_t is the length of a queue at time t, where customers arrive according to a Poisson process with rate μ, customers are served parallelly and each customer is served with rate 1. It is well-known (see [9, Exercise 5.8]) that if $Y_0 = h \in \mathbb{N}_0$ then

$$\mathbf{P}\left(Y_t = k \,\middle|\, Y_0 = h\right) = \mathbf{q}(t, h, k, \mu) := \sum_{l=0}^{k} \mathbf{b}(l, h, e^{-t}) \cdot \mathbf{p}(k-l, (1-e^{-t})\mu), \tag{24}$$

i.e. Y_t has the same distribution as the sum of two independent random variables with $\mathrm{BIN}(h, e^{-t})$ and $\mathrm{POI}((1-e^{-t})\mu)$ distributions. From (24) we get that indeed $Y_t \xrightarrow{\mathrm{p}} h$ as $t \to 0$ and the stationary distribution of the queue is $\mathrm{POI}(\mu)$:

$$\lim_{t \to 0} \mathbf{q}(t, h, k, \mu) = \mathbb{1}[k = h], \qquad \lim_{t \to \infty} \mathbf{q}(t, h, k, \mu) = \mathbf{p}(k, \mu). \tag{25}$$

Fix $\kappa, \rho \in (0, +\infty)$. The Cox–Ingersoll–Ross (C.I.R.) process is a diffusion process with stochastic differential equation

$$\mathrm{d}Z_t = \left(\kappa - \frac{\kappa}{\rho} Z_t\right) \mathrm{d}t + \sqrt{2Z_t}\,\mathrm{d}B_t, \tag{26}$$

where B_t denotes the standard Brownian motion (for an introduction to SDE, see [14]).

Heuristically the SDE (26) tells us the mean and variance of small incerements of the continuous-time $\mathbb{R}_+$-valued Markov process $(Z_t)_{t\geq 0}$ given the present value of Z_t:

$$\mathbf{E}\left(Z_{t+\mathrm{d}t} - z \,\middle|\, Z_t = z\right) \approx \left(\kappa - \frac{\kappa}{\rho} z\right) \mathrm{d}t, \qquad \mathbf{Var}\left(Z_{t+\mathrm{d}t} - z \,\middle|\, Z_t = z\right) \approx 2z\,\mathrm{d}t \tag{27}$$

It is well-known (see [6, Chapter 4.6]) that if we denote

$$\alpha := \frac{\kappa}{\rho} \qquad \text{and} \qquad \tau(\alpha, t) := \frac{\alpha}{\exp(\alpha t) - 1}$$

and if we start the process $(Z_t)_{t\geq 0}$ from the initial value $Z_0 = z$ then $2(\tau(\alpha, t) + \alpha) \cdot Z_t$ follows a noncentral chi-square distribution with 2κ degrees of freedom and non-centrality parameter $2z \cdot \tau(\alpha, t)$, thus we have $\mathbf{P}\left(Z_t \leq x \,\middle|\, Z_0 = z\right) = \int_0^x f(t, z, y)\,\mathrm{d}y$ where

$$f(t, z, y) = \sum_{i=0}^{\infty} \mathbf{p}(i, z \cdot \tau(\alpha, t))\mathbf{g}(y, \kappa + i, \tau(\alpha, t) + \alpha). \tag{28}$$

Note that using (28) one can derive that indeed $Z_t \xrightarrow{\mathrm{p}} z$ as $t \to 0$ and the stationary distribution of $(Z_t)_{t\geq 0}$ is Gamma$(\kappa, \frac{\kappa}{\rho})$:

$$\lim_{t\to 0} f(t, z, y) = \delta_{z,y}, \qquad \lim_{t\to\infty} f(t, z, y) = \mathbf{g}(y, \kappa, \frac{\kappa}{\rho}). \tag{29}$$

2.4 Statements of Theorem 1 and Theorem 2

In this subsection we state the main results of this paper describing the time evolution of the limiting multigraphons of a sequence of edge reconnecting models $\mathbf{X}_n(\cdot)$, $n \to \infty$.

In Theorem 1 we precisely formulate $\mathbf{X}_n(t \cdot n^2) \xrightarrow{\mathrm{p}} \breve{W}_t$.

In Theorem 2 we precisely formulate $\mathbf{X}_n(t \cdot n^3) \xrightarrow{\mathrm{p}} \tilde{W}_t$.

Note that in (4), (5) and above we used the notations $\breve{W}_t$ and $\tilde{W}_t$ in order to give the most simple formulations of these results, nevertheless our real notations are going to be slightly different.

Now we describe the evolution of the edge reconnecting model by describing the evolution of the limiting multigraphons. We consider a sequence of initial multigraphs

$(G_n)_{n=1}^{\infty}$ which converge to a multigraphon W. We assume $|V(G_n)| = n$. We denote the adjacency matrix of G_n by $B_n \in \mathcal{A}_n$. We assume that the technical condition

$$\exists\, \lambda > 0,\ C < +\infty\ \ \forall\, n: \quad \frac{1}{\binom{n}{2}} \sum_{i<j\le n} e^{\lambda B_n(i,j)} \le C, \quad \frac{1}{n}\sum_{i=1}^{n} e^{\lambda B_n(i,i)} \le C \tag{30}$$

holds.

First we state Theorem 1 about the evolution of the edge reconnecting model on the $T = \mathcal{O}(n^2)$ timescale. In the Introduction this result was referred to as $\mathcal{G}_n(t \cdot n^2) \xrightarrow{\text{p}} \breve{W}_t$ in order to make the notation as simple as possible. In fact we are going to prove $\mathcal{G}_n(t \cdot \frac{\rho(W)}{2} \cdot n^2) \xrightarrow{\text{p}} W_t$, thus $W_t = \breve{W}_{\frac{\varrho}{2}t}$. The notation $\breve{W}_t$ will no longer be used.

Theorem 1. *Let us fix $\kappa \in (0, +\infty)$. We consider the edge reconnecting model $\mathbf{X}_n(T)$, $T = 0, 1, \dots$ on the state space $\mathcal{A}_n^{m(n)}$ and initial state $\mathbf{X}_n(0) = B_n \in \mathcal{A}_n^{m(n)}$ for $n = 1, 2, \dots$. We assume $B_n \to W$ for some multigraphon W and that (30) holds.*

Then for all $t \in [0, +\infty)$ we have

$$\mathbf{X}_n\left(\left\lfloor t \cdot \frac{\rho(W)\cdot n^2}{2} \right\rfloor\right) \xrightarrow{p} W_t \qquad \text{as} \qquad n \to \infty \tag{31}$$

where (recall (24))

$$W_t(x,y,k) = \begin{cases} \sum_{h=0}^{\infty} W(x,y,h)\mathbf{q}(t,h,k,\frac{D(W,x)\cdot D(W,y)}{\rho(W)}) & \text{if } x \ne y \\ \mathbb{1}[2\,|\,k] \cdot \sum_{h=0}^{\infty} W(x,y,h)\mathbf{q}(t,\frac{h}{2},\frac{k}{2},\frac{D(W,x)\cdot D(W,y)}{2\rho(W)}) & \text{if } x = y \end{cases} \tag{32}$$

We prove Theorem 1 in Section 5. Before stating further theorems, we devote a few paragraphs to the heuristics behind Theorem 1.

In order to give some insight about (32), we now give a probabilistic way to generate a random element of $\mathcal{A}_k$ with the same distribution as $\mathbf{X}_{W_t}^{[k]}$ (see Definition 2.1):

We first generate $\mathbf{X}_W^{[k]}$ using the background variables $(U_i)_{i=1}^k$, then we get $\mathbf{X}_{W_t}^{[k]}$ by letting the entries $(\mathbf{X}_{W_t}(i,j))_{i,j=1}^k$ evolve in time:

- if $i < j$, we run an M/M/∞-queue Y_t with initial value $Y_0 = \mathbf{X}_W(i,j)$, arrival rate $\frac{D(W,U_i)\cdot D(W,U_j)}{\rho(W)}$ and service rate 1 and let $\mathbf{X}_{W_t}(i,j) := Y_t$
- if $i = j$, we do the same thing with the only exception being that the queue describing the evolution of the number of loop edges has arrival rate $\frac{D(W,U_i)\cdot D(W,U_j)}{2\rho(W)}$.

Now we give a heuristic argument explaining why do M/M/∞-queues enter the picture:

We look at the evolution of $X_n(T,i,j)$ for some $i \ne j$ (the case of loop edges is analogous). We denote by $\mathcal{D}_n(T,i) := \frac{1}{n} d(\mathbf{X}_n(T), i)$. From (7) and (8) it follows that

$$\mathbf{P}\left(X_n(T+1,i,j) = X_n(T,i,j) + 1 \,\middle|\, \mathbf{X}_n\right) \approx \frac{\mathcal{D}_n(T,i)\cdot n}{2m} \cdot \frac{\mathcal{D}_n(T,j)\cdot n + \kappa}{2m + n\kappa} + \frac{\mathcal{D}_n(T,j)\cdot n}{2m} \cdot \frac{\mathcal{D}_n(T,i)\cdot n + \kappa}{2m + n\kappa} \approx \frac{1}{m}\frac{\mathcal{D}_n(T,i)\mathcal{D}_n(T,j)}{\rho} \tag{33}$$

$$\mathbf{P}\left(X_n(T+1,i,j)=X_n(T,i,j)-1 \,\middle|\, \mathbf{X}_n\right) \approx \frac{1}{m} X_n(T,i,j) \tag{34}$$

In the statement of Theorem 1 we used the time scaling $T = \lfloor t \cdot \frac{\rho \cdot n^2}{2} \rfloor \approx t \cdot m$, thus if we denote $\mathrm{d}t := \frac{1}{m}$ then $T+1$ corresponds to $t + \mathrm{d}t$. If we define $Y_t := X(t \cdot m, i, j)$ and compare (33), (34) to (21), (22) then we see that the time evolution of Y_t approximates that of an M/M/∞-queue with arrival rate $\mu = \frac{\mathcal{D}_n(T,i)\mathcal{D}_n(T,j)}{\rho}$ and service rate 1. We will later see that on the $T \asymp n^2$ timescale $\mathcal{D}_n(T,i)$ does not change significantly, so that we have

$$\mathcal{D}_n(T,i) \approx \mathcal{D}_n(0,i) \overset{(17)}{\approx} D(W,U_i).$$

We note here that the identity $D(W,x) \equiv D(W_t,x)$ can be formally derived from (32).

Now we look at the evolution of the edge reconnecting model on the $T = \mathcal{O}(n^3)$ timescale. In the Introduction this result was referred to as $\mathcal{G}_n(t \cdot n^3) \xrightarrow{\mathrm{p}} \tilde{W}_t$ in order to make the notation as simple as possible. In fact we are going to prove $\mathcal{G}_n(t \cdot \rho(W) \cdot n^3) \xrightarrow{\mathrm{p}} \hat{W}_t$, thus $\hat{W}_t = \tilde{W}_{\rho t}$. The notation $\tilde{W}_t$ will no longer be used.

Theorem 2. *Let us fix $\kappa \in (0,+\infty)$. We consider the edge reconnecting model $\mathbf{X}_n(T)$, $T = 0,1,\ldots$ on the state space $\mathcal{A}_n^{m(n)}$ and initial state $\mathbf{X}_n(0) = B_n \in \mathcal{A}_n^{m(n)}$ for $n = 1,2,\ldots$. We assume $B_n \to W$ for some multigraphon W and that (30) holds.*

Then for all $t \in (0,+\infty)$ (but not for t=0) we have

$$\mathbf{X}_n\left(\lfloor t \cdot \rho(W) \cdot n^3 \rfloor\right) \xrightarrow{p} \hat{W}_t \qquad \textit{as} \qquad n \to \infty \tag{35}$$

where

$$\hat{W}_t(x,y,k) = \begin{cases} \mathbf{p}(k, \frac{F_t^{-1}(x)F_t^{-1}(y)}{\rho(W)}) & \textit{if } x \neq y \\ 1\!\!1[2|k] \cdot \mathbf{p}\left(\frac{k}{2}, \frac{F_t^{-1}(x)F_t^{-1}(y)}{2\rho(W)}\right) & \textit{if } x = y \end{cases} \tag{36}$$

and F_t^{-1} is the inverse function of $F_t(x) = \int_0^x f(t,y)\,\mathrm{d}y$ where (recall (28) and note that $f(t,z,y)$ also depends on the parameters κ and ρ)

$$f(t,y) = \int_0^\infty f(t,z,y)\,\mathrm{d}F_0(z), \tag{37}$$

and $F_0(x) = \int_0^1 1\!\!1[D(W,y) \leq x]\,\mathrm{d}y$, $x \in [0,+\infty)$.

We prove Theorem 2 in Section 6. Now we devote a few paragraphs to the heuristics behind Theorem 2. In order to give some insight about (36), we now give a probabilistic way to generate a random element of $\mathcal{A}_k$ with the same distribution as $\mathbf{X}_{\hat{W}_t}^{[k]}$:

Let us first generate $\mathbf{X}_W^{[k]}$ using the background variables $(U_i)_{i=1}^k$. Let us define $Z_i(0) := D(W,U_i)$. Now $(Z_i(0))_{i=1}^k$ are i.i.d. with probability distribution function F_0. We let $(Z_i(t))_{t \geq 0}$ evolve in time according to the SDE (26), so that $(Z_i(t))_{i=1}^k$ are i.i.d.

with probability distribution function F_t. Given the background variables $(Z_i(t))_{i=1}^k$ let $\mathbf{X}_{\hat{W}_t}(i,j) \sim \mathrm{POI}(\frac{Z_i(t)Z_j(t)}{\rho(W)})$ if $i \neq j$, and let $\frac{1}{2}\mathbf{X}_{\hat{W}_t}(i,i) \sim \mathrm{POI}(\frac{Z_i(t)Z_i(t)}{2\rho(W)})$.

Now we give a heuristic argument explaining why do C.I.R. processes enter the picture: Pick $i \in [n]$ and denote by $\mathcal{D}_n(T) := \frac{1}{n}d(\mathbf{X}_n(T), i)$. It follows from (6) and (8) that

$$\mathbf{E}\left(\mathcal{D}_n(T+1) - \mathcal{D}_n(T) \,\middle|\, \mathbf{X}_n(T)\right) = \frac{\mathcal{D}_n(T) + \frac{\kappa}{n}}{2m + n\kappa} - \frac{\mathcal{D}_n(T)}{2m} \approx \frac{1}{2mn}\left(\kappa - \frac{\kappa}{\rho}\mathcal{D}_n(T)\right) \quad (38)$$

$$\mathbf{Var}\left(\mathcal{D}_n(T+1) - \mathcal{D}_n(T) \,\middle|\, \mathbf{X}_n(T)\right) \approx \frac{\mathcal{D}_n(T) + \frac{\kappa}{n}}{2mn + n^2\kappa} + \frac{\mathcal{D}_n(T)}{2mn} \approx \frac{1}{2mn}2\mathcal{D}_n(T) \quad (39)$$

We look at the time evolution of the stochastic process $Z_t := \mathcal{D}_n\left(\lfloor t \cdot 2nm \rfloor\right)$. In the statement of Theorem 2 we used the time scaling $T = \lfloor t \cdot \rho \cdot n^3 \rfloor \approx t \cdot 2mn$. If we let $\mathrm{d}t = \frac{1}{2nm}$ then $T+1$ corresponds to $t + \mathrm{d}t$. Let $\mathrm{d}Z_t := Z_{t+\mathrm{d}t} - Z_t$. From (38) and (39) we get

$$\mathbf{E}\left(\mathrm{d}Z_t \,\middle|\, Z_t\right) \approx \left(\kappa - \frac{\kappa}{\rho}Z_t\right)\mathrm{d}t \qquad \mathbf{Var}\left(\mathrm{d}Z_t \,\middle|\, Z_t\right) \approx 2Z_t\mathrm{d}t$$

Thus the process Z_t approximates the solution of the SDE of the C.I.R. process (27).

2.5 Properties of W_t and $\hat{W}_t$

In this subsection we relate some properties of the multigraphons W_t and $\hat{W}_t$ that appear in Theorem 1 and Theorem 2 to the *configuration model*. But first, we show

$$\lim_{t\to 0_+} W_t \cong W, \qquad \lim_{t\to\infty} W_t \cong \lim_{t\to 0_+} \hat{W}_t, \qquad \lim_{t\to\infty} \hat{W}_t \cong \hat{W}_\infty \quad (40)$$

where $\hat{W}_\infty$ is the multigraphon defined by

$$\hat{W}_\infty(x,y,k) = \begin{cases} \mathbf{p}(k, \frac{F^{-1}(x)F^{-1}(y)}{\rho}) & \text{if } x \neq y \\ \mathbb{1}[2|k] \cdot \mathbf{p}\left(\frac{k}{2}, \frac{F^{-1}(x)F^{-1}(y)}{2\rho}\right) & \text{if } x = y \end{cases} \quad (41)$$

and F^{-1} is the inverse function of $F(x) = \int_0^x \mathbf{g}(y, \kappa, \frac{\kappa}{\rho})\mathrm{d}y$, see (20).

In order to make sense of (40) we define convergence on the space of multigraphons: we say that $\lim_{n\to\infty} W_n \cong W$ if $\lim_{n\to\infty} t_=(F, W_n) \to t_=(F, W)$ for all $F \in \mathcal{M}$. The limit of a convergent sequence is only determined up to the $\cong$ equivalence, see (11).

One can see by looking at (32), (24), (19), (18) that W_t is a continuous function of t.

Similarly, (36), (37), (28), (18), (20) imply that $\hat{W}_t$ is a continuous function of t.

If we substitute $t \to 0_+$ into (32) we indeed get $W_t \to W$ by (25).

If we substitute $t \to \infty$ into (36) we get $\hat{W}_t \to \hat{W}_\infty$ by (29).

If we let $t \to \infty$ in (32) and $t \to 0_+$ in (36) we get

$$\lim_{t\to\infty} W_t(x,y,k) \overset{(25)}{=} \begin{cases} \mathbf{p}(k, \frac{D(W,x)D(W,y)}{\rho(W)}) & \text{if } x \neq y \\ \mathbb{1}[2|k] \cdot \mathbf{p}\left(\frac{k}{2}, \frac{D(W,x)D(W,y)}{2\rho(W)}\right) & \text{if } x = y \end{cases} \tag{42}$$

$$\lim_{t\to 0_+} \hat{W}_t(x,y,k) \overset{(29)}{=} \begin{cases} \mathbf{p}(k, \frac{F_0^{-1}(x)F_0^{-1}(y)}{\rho(W)}) & \text{if } x \neq y \\ \mathbb{1}[2|k] \cdot \mathbf{p}\left(\frac{k}{2}, \frac{F_0^{-1}(x)F_0^{-1}(y)}{2\rho(W)}\right) & \text{if } x = y \end{cases} \tag{43}$$

where $F_0(z) = \int_0^1 \mathbb{1}[D(W,y) \leq z]\,\mathrm{d}y$ and $F_0^{-1}(x) := \min\{z : F_0(z) \geq x\}$. We have $\lim_{t\to\infty} W_t \cong \lim_{t\to 0_+} \hat{W}_t$, because the corresponding W-random multigraphons have the same distribution, since $D(W,U_i) \sim F_0^{-1}(U_i)$. Thus we have seen that (40) holds.

A well-known way to generate a random multigraph with a prescribed degree sequence is called the *configuration model*: we draw $d(v)$ stubs (half-edges) at each vertex v and then we uniformly choose one from the the set of possible matchings of these stubs. In [13] we call such random multigraphs *edge stationary* and in [13, Theorem 1] we characterize the special form of limiting multigraphons that arise as the limit of edge stationary dense multigraph sequences: these multigraphons are of form (41) where F is a generic probability distribution function on $\mathbb{R}_+$ and $F^{-1}(x) := \min\{z : F(z) \geq x\}$ is the generalized inverse of F. The name of edge stationarity comes from the fact that the space of edge stationary distributions is invariant under the edge reconnecting dynamics, see [13, Section 4]. The stationary distribution of the edge reconnecting model is an example of an edge stationary multigraph, see (3),(41).

The heuristic explanation of the fact that the limiting multigraphon $\lim_{t\to\infty} W_t$ from (42) has the special form that appears in [13, Theorem 1] is as follows: If $T \approx t \cdot n^2$ where $1 \ll t$ then the degrees of vertices in $\mathcal{G}_n(0)$ and $\mathcal{G}_n(T)$ are very close to each other (c.f. $D(W,x) \equiv D(W_t,x)$), whereas T steps are enough for the model to rearrange and mix the edges, so $\mathcal{G}_n(T)$ looks edge stationary. Similarly, for any $t > 0$, $\hat{W}_t$ from (36) also looks edge stationary.

Roughly speaking, if we start the edge reconnecting model from an arbitrary initial multigraph, then we have to run our process for $n^2 \ll T$ steps until $\mathcal{G}_n(T)$ becomes "edge stationary" and run it for $n^3 \ll T$ steps until $\mathcal{G}_n(T)$ becomes "stationary".

3 Vertex exchangeable random adjacency matrices

In this section we define the notion of vertex exchangeability of random adjacency matrices and recall two lemmas from [13]: in Lemma 3.1 we relate convergence of dense random multigraphs to convergence of the probability measures of the corresponding vertex exchangeable random arrays and in Lemma 3.2 we give sufficient conditions under which convergence of dense random multigraphs imply convergence of the degree distribution of these graphs, see (17).

Let $\mathbf{X} = (X(i,j))_{i,j=1}^n$ denote a random element of $\mathcal{A}_n$. We say that the distribution $\mathbf{X}$ is *vertex exchangeable* if for all permutations $\tau : [n] \to [n]$ the $\mathcal{A}_n$-valued random variables $(X(i,j))_{i,j=1}^n$ and $(X(\tau(i),\tau(j)))_{i,j=1}^n$ have the same distribution:

$$(X(i,j))_{i,j=1}^n \sim (X(\tau(i),\tau(j)))_{i,j=1}^n . \tag{44}$$

In graph theoretic terms (44) means that the distribution of the random graph is invariant under the relabeling. It follows from Definition 2.1 that $\mathbf{X}_W^{[k]}$ is vertex exchangeable.

In the statements of Theorem 1 and Theorem 2 the initial state of the Markov chain $\mathbf{X}_n(T) = (X_n(T,i,j))_{i,j=1}^n$ was the deterministic adjacency matrix $\mathbf{X}_n(0) = B_n$, but if we define

$$\hat{X}_n(0,i,j) := B_n(\pi(i),\pi(j)) . \tag{45}$$

where π denotes a uniformly chosen random permutation of $[n]$ and denote the edge reconnecting Markov chain with this initial distribution by $\hat{\mathbf{X}}_n(T)$, $T = 1,2,\ldots$, then

$$\left(\hat{X}_n(T,i,j)\right)_{i,j=1}^n \sim \left(X_n(T,\pi(i),\pi(j))\right)_{i,j=1}^n, \quad t_=(A,\mathbf{X}_n) \sim t_=(A,\hat{\mathbf{X}}_n), \tag{46}$$

thus we get that the assertion of Theorem 1 and Theorem 2 holds for $\mathbf{X}_n(T)$ if and only if it holds for $\hat{\mathbf{X}}_n(T)$. From now on we are going to use this trick to replace $\mathbf{X}_n(T)$ by $\hat{\mathbf{X}}_n(T)$ and assume that the distribution of $\mathbf{X}_n(T)$ is vertex exchangeable.

If $\mathbf{X}_n$ is a random element of $\mathcal{A}_n$ for each $n \in \mathbb{N}$ and W is a multigraphon then we say that $\mathbf{X}_n$ converges in distribution to $\mathbf{X}_W$ as $n \to \infty$ (or briefly denote $\mathbf{X}_n \xrightarrow{\mathrm{d}} \mathbf{X}_W$) if for all $k \in \mathbb{N}$ we have $\mathbf{X}_n^{[k]} \xrightarrow{\mathrm{d}} \mathbf{X}_W^{[k]}$, i.e.

$$\forall\, k \in \mathbb{N},\ A \in \mathcal{A}_n : \lim_{n\to\infty} \mathbf{P}\left(\mathbf{X}_n^{[k]} = A\right) = \mathbf{P}\left(\mathbf{X}_W^{[k]} = A\right) \overset{(15)}{=} t_=(A,W)$$

Recall that we say that $\mathbf{X}_n \xrightarrow{\mathrm{p}} W$ if

$$\forall\, k \in \mathbb{N}\ \ \forall\, A \in \mathcal{A}_k\ \ \forall\, \varepsilon > 0 : \lim_{n\to\infty} \mathbf{P}\left(|t_=(A,\mathbf{X}_n) - t_=(A,W)| > \varepsilon\right) = 0.$$

We state here [13, Lemma 3.1] without proof:

Lemma 3.1. *Let $\mathbf{X}_n = (X_n(i,j))_{i,j=1}^n$ be a random, vertex exchangeable element of $\mathcal{A}_n$ for all $n \in \mathbb{N}$. The following statements are equivalent:*

$$\mathbf{X}_n \xrightarrow{p} W \quad \iff \quad \mathbf{X}_n \xrightarrow{d} \mathbf{X}_W. \tag{47}$$

For a real-valued nonnegative random variable X define

$$\mathbf{E}(X;m) := \mathbf{E}(X \cdot \mathbb{1}[X \geq m]) .$$

A sequence of real-valued nonnegative random variables $(X_n)_{n=1}^\infty$ is uniformly integrable (see [15, Chapter 13]) if

$$\lim_{m\to\infty} \max_n \mathbf{E}(X_n;m) = 0.$$

We state here a special case of [13, Lemma 3.2/(ii)] without proof:

Lemma 3.2. *If $\mathbf{X}_n$ is a random vertex exchangeable element of $\mathcal{A}_n$ for each $n \in \mathbb{N}$, $\mathbf{X}_n \xrightarrow{d} \mathbf{X}_W$ holds for some multigraphon W and the sequences*

$$(X_n(1,1))_{n=1}^{\infty} \quad \textit{and} \quad (X_n(1,2))_{n=1}^{\infty}$$

are uniformly integrable then for all $k \in \mathbb{N}$ we have

$$\left(\mathbf{X}_n^{[k]}, \left(\frac{1}{n} d(\mathbf{X}_n, i) \right)_{i=1}^{k} \right) \xrightarrow{d} \left(\mathbf{X}_W^{[k]}, (D(W, U_i))_{i=1}^{k} \right),$$

where $\mathbf{X}_W^{[k]}$ is generated using the background variables $(U_i)_{i=1}^k$ according to Definition 2.1.

4 Bounds on multiple edges and degrees

In this section we state and prove Lemma 4.1 which, roughly speaking, states that degrees and multiple edges in the edge reconnecting model remain well-behaved.

If we replace the initial matrix B_n with its vertex exchangeable version $(X_n(0,i,j))_{i,j=1}^n$ using the trick (45) then the technical condition (30) becomes

$$\exists\, \lambda > 0,\ C < +\infty \ \ \forall n \, \forall i,j \in [n]: \quad \mathbf{E}\left(e^{\lambda X_n(0,i,j)}\right) \leq C \tag{48}$$

It is easy to see that (48) implies that the sequences $(X_n(0,1,2))_{n=1}^{\infty}$ and $(X_n(0,1,1))_{n=1}^{\infty}$ are uniformly integrable. If we assume $B_n \to W$ and define $\rho := \rho(W)$ then

$$\lim_{n\to\infty} \frac{2m(n)}{n^2} = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j=1}^{n} X_n(0,i,j) = \lim_{n\to\infty} \frac{1}{n^2} \sum_{i,j=1}^{n} \mathbf{E}\left(X_n(0,i,j)\right) \overset{(44)}{=}$$
$$\lim_{n\to\infty} \left(\frac{n-1}{n} \mathbf{E}\left(X_n(0,1,2)\right) + \frac{1}{n}\mathbf{E}\left(X_n(0,1,1)\right) \right) \overset{(47),(48)}{=} \mathbf{E}\left(X_W(1,2)\right) \overset{(16)}{=} \rho \tag{49}$$

Now we state and prove a lemma which says that if the initial state $\mathbf{X}_n(0)$ of the edge reconnecting model is well-behaved (i.e. (48) holds) then model remains well-behaved at later times T as well:

(i) For all $T = \mathcal{O}(n^3)$ the normalized degree $D(T) = \frac{1}{n} d(\mathbf{X}_n(T), i)$ of a vertex $i \in [n]$ satisfies $D(T) = \mathcal{O}(1)$ uniformly in n, a bit more precisely: $\mathbf{P}\left(D(T) \geq z\right)$ decays exponentially as $z \to \infty$.

(ii) For all $T = \mathcal{O}(n^3)$ the number of parallel/loop edges $X_n(T,i,j)$ between vertices $i,j \in [n]$ satisfies $X_n(T,i,j) = \mathcal{O}(1)$ uniformly in n, a bit more precisely: $X_n(T,i,j)$ has finite moments.

(iii) if $T_1 \leq T_2 = \mathcal{O}(n^3)$ and $T_2 - T_1 \ll n^3$ then $D(T_1) \approx D(T_2)$, a bit more precisely: the second moment of $D(T_1) - D(T_2)$ is $\mathcal{O}(n^{-3}(T_2 - T_1))$.

Lemma 4.1. *Let us fix $\kappa, \rho \in (0, +\infty)$. We consider the edge reconnecting model $\mathbf{X}_n(T)$, $T = 0, 1, \dots$ on the state space $\mathcal{A}_n^{m(n)}$ with a vertex exchangeable initial state $\mathbf{X}_n(0)$ for $n = 1, 2, \dots$ satisfying $\lim_{n\to\infty} \frac{2m(n)}{n^2} = \rho$ and (48) for some $\lambda < \frac{\kappa}{\rho}$. Then*

(i) There exists an $n' \in \mathbb{N}$ such that for every $t, z \in [0, +\infty)$ and $n \geq n'$ we have

$$\mathbf{P}\left(\max_{0\leq T\leq 2mnt} \frac{1}{n} d(\mathbf{X}_n(T), i) \geq z\right) \leq C \cdot e^{2\lambda\kappa t} \cdot e^{-\lambda z} \tag{50}$$

with the C of (48).

(ii) For every $p > 1$ and $t \in [0, +\infty)$ there exists a $C' = C'(\kappa, \rho, \lambda, C, p, t)$ (where C is the constant from (48)) such that for all $n \in \mathbb{N}$, $i, j \in [n]$ and $T \leq 2mnt$ we have

$$\mathbf{E}\left(X_n(T, i, j)^p\right) \leq C'. \tag{51}$$

(iii) There exists a constant $C'' = C''(\kappa, \rho, \lambda, C, t)$ such that for all $n \in \mathbb{N}$, all $T_1 \leq T_2 \leq t \cdot n^3$ and $i \in [n]$ we have

$$\mathbf{E}\left(\left(\frac{1}{n} d(\mathbf{X}_n(T_2), i) - \frac{1}{n} d(\mathbf{X}_n(T_1), i)\right)^2\right) \leq \frac{C'' \cdot (T_2 - T_1)}{n^3} \tag{52}$$

Proof of Lemma 4.1 (i). Fix $i \in [n]$ and denote

$$d(T) := d(\mathbf{X}_n(T), i), \qquad D(T) := \frac{1}{n} d(\mathbf{X}_n(T), i).$$

Denote by $(\mathcal{F}_T)_{0\leq T}$ the natural filtration generated by the process.

If $a(T) := \mathbf{E}\left(e^{\lambda\cdot(D(T+1)-D(T))} - 1 \,\middle|\, \mathcal{F}_T\right)$ then $M(T) := e^{\lambda D(T)} \prod_{l=0}^{T-1}(1 + a(l))^{-1}$ is a nonnegative martingale. By Doob's submartingale inequality we have

$$\mathbf{P}\left(\max_{0\leq T\leq T'} M(T) \geq x\right) \leq \frac{\mathbf{E}\left(e^{\lambda D(0)}\right)}{x} \tag{53}$$

$$\max_{0\leq T\leq T'} M(T) < x \quad \Longrightarrow \quad \forall\, T \leq T' : \; e^{\lambda D(T)} \leq x \exp\left(\sum_{l=0}^{T-1} a(l)\right) \tag{54}$$

Now we give an upper bound on $a(T)$. Using

$$D(T+1) = D(T) + \frac{1}{n}\mathbb{1}[\mathcal{V}_{new}(T) = i] - \frac{1}{n}\mathbb{1}[\mathcal{V}_{old}(T) = i], \tag{55}$$

(6), (8) and the fact that $\mathcal{V}_{new}(T)$ and $\mathcal{V}_{old}(T)$ are conditionally independent given $\mathcal{F}_T$ we get

$$a(T) = \left(1 + \frac{d(T)+\kappa}{2m+n\kappa}(e^{\frac{\lambda}{n}}-1)\right)\left(1+\frac{d(T)}{2m}(e^{-\frac{\lambda}{n}}-1)\right) - 1 \le$$
$$\frac{d(T)+\kappa}{2m+n\kappa}\left(\frac{\lambda}{n}+\frac{1}{2}e^{\frac{\lambda}{n}}\frac{\lambda^2}{n^2}\right) + \frac{d(T)}{2m}\left(-\frac{\lambda}{n}+\frac{1}{2}e^{\frac{\lambda}{n}}\frac{\lambda^2}{n^2}\right) =$$
$$\frac{\lambda}{n}\frac{1}{4m^2+n2m\kappa}\left(d(T)\cdot\left(e^{\frac{\lambda}{n}}\lambda\cdot\left(\frac{2m}{n}+\frac{1}{2}\kappa\right)-n\kappa\right)+2m\kappa\cdot\left(1+\frac{1}{2}e^{\frac{\lambda}{n}}\frac{\lambda}{n}\right)\right) \quad (56)$$

Now $\lambda < \frac{\kappa}{\rho}$ and $\lim_{n\to\infty}\frac{2m(n)}{n^2}=\rho$, thus if n is big enough then $\lambda < e^{-\frac{\lambda}{n}}\cdot\frac{\kappa}{\rho+\frac{1}{2}\frac{\kappa}{n}}$, which implies that the coefficient of $d(T)$ is negative in the right hand side of (56), thus

$$a(T) \le \frac{1}{2mn}\lambda\kappa\frac{1+\frac{1}{2}\frac{\lambda}{n}e^{\frac{\lambda}{n}}}{1+\frac{n\kappa}{2m}} \le \frac{1}{2mn}2\lambda\kappa. \quad (57)$$

From (53), (54) and (57) it follows that

$$\mathbf{P}\left(\max_{0\le T\le 2mnt} e^{\lambda D(T)} \ge x\exp(2\lambda\kappa t)\right) \le \frac{\mathbf{E}\left(e^{\lambda D(0)}\right)}{x}$$

Substituting $x=\exp(-2\kappa\lambda t)\exp(\lambda z)$ and using

$$\mathbf{E}\left(e^{\lambda D(0)}\right) = \mathbf{E}\left(\exp\left(\frac{1}{n}\sum_{j=1}^{n}\lambda X(0,i,j)\right)\right) \le \mathbf{E}\left(\frac{1}{n}\sum_{j=1}^{n}\exp(\lambda X(0,i,j))\right) \overset{(48)}{\le} C$$

we arrive at (50). □

Proof of Lemma 4.1 (ii). Fix n and $i,j\in[n]$. We only prove the statement of the lemma if $i\ne j$, the proof of the diagonal case is similar. Denote by

$$X(T) := X_n(T,i,j), \qquad d(T,i) := d(\mathbf{X}_n(T),i), \qquad D(T,i) = \frac{1}{n}d(T,i).$$

Using (9) we get

$$\mathbf{P}\left(X(T+1)=X(T)+1 \,\middle|\, \mathcal{F}_T\right) =$$
$$\frac{d(T,i)+\kappa}{2m+n\kappa}\left(\frac{d(T,j)}{2m}\left(1-\frac{X(T)}{d(T,j)}\right)\right) + \frac{d(T,j)+\kappa}{2m+n\kappa}\left(\frac{d(T,i)}{2m}\left(1-\frac{X(T)}{d(T,i)}\right)\right) \quad (58)$$

$$\mathbf{P}\left(X(T+1)=X(T)-1 \,\middle|\, \mathcal{F}_T\right) = \frac{X(T)}{2m}\left(1-\frac{d(T,i)+\kappa}{2m+n\kappa}\right) + \frac{X(T)}{2m}\left(1-\frac{d(T,j)+\kappa}{2m+n\kappa}\right) \quad (59)$$

From this it is straightforward to derive

$$\mathbf{E}\left(e^{\lambda X(T+1)} - e^{\lambda X(T)} \,\middle|\, \mathcal{F}_T\right) \le$$
$$e^{\lambda X(T)} \left((e^{\lambda} - 1) \left(\frac{d(T,i) + \kappa}{2m + n\kappa} \frac{d(T,j)}{2m} + \frac{d(T,j) + \kappa}{2m + n\kappa} \frac{d(T,i)}{2m} \right) + (e^{-\lambda} - 1) \frac{X(T)}{m} \right)$$

Define the stopping time

$$\tau_y := \min \left\{ T \,:\, \frac{d(T,i) + \kappa}{2m + n\kappa} \frac{d(T,j)}{2m} + \frac{d(T,j) + \kappa}{2m + n\kappa} \frac{d(T,i)}{2m} > \frac{y}{m} \right\}$$

and $X_y(T) := X(T)\mathbb{1}[\tau_y > T]$. Now we prove that for all $T \in \mathbb{N}$

$$\mathbf{E}\left(e^{\lambda X_y(T)}\right) \le \max \left\{ C \,,\, \exp(y\lambda e^{\lambda}) \left(1 + \frac{(e^{\lambda} - 1)y}{m}\right) \right\}. \tag{60}$$

It is straightforward to check that

$$\mathbf{E}\left(e^{\lambda X_y(T+1)} - e^{\lambda X_y(T)} \,\middle|\, \mathcal{F}_T\right) \le e^{\lambda X_y(T)} \left((e^{\lambda} - 1)\frac{y}{m} + (e^{-\lambda} - 1)\frac{X_y(T)}{m} \right). \tag{61}$$

If we denote $E(T) := \mathbf{E}\left(e^{\lambda X_y(T)}\right)$, take the expectation of (61) and use Jensen's inequality then we get

$$E(T+1) - E(T) \le \frac{E(T)}{m} \left((e^{\lambda} - 1)y + (e^{-\lambda} - 1)\frac{\log(E(T))}{\lambda} \right). \tag{62}$$

We prove (60) by induction. For $T = 0$ we use (48). If $E(T) > \exp(y\lambda e^{\lambda})$, then by (62) $E(T+1) < E(T)$ and if $E(T) \le \exp(y\lambda e^{\lambda})$ then

$$E(T+1) \le E(T) + \exp(y\lambda e^{\lambda})\frac{(e^{\lambda} - 1)y}{m} \le \exp(y\lambda e^{\lambda}) \left(1 + \frac{(e^{\lambda} - 1)y}{m}\right).$$

Having established (60) we prove (51) by showing that

$$\mathbf{E}\left(X(T)^p\right) \le 1 + \int_1^{\infty} \mathbf{P}\left(X(T)^p \ge x\right) \,\mathrm{d}x < +\infty.$$

$$\mathbf{P}\left(X(T)^p \ge x\right) \le \mathbf{P}\left(X_y(T)^p \ge x\right) + \mathbf{P}\left(X(T) \ne X_y(T)\right) \le \frac{\mathbf{E}\left(e^{\lambda X_y(T)}\right)}{\exp(\lambda x^{1/p})} + \mathbf{P}\left(\tau_y > T\right) \overset{(60)}{\le}$$
$$\frac{C_1 \exp(C_2 y)}{\exp(\lambda x^{1/p})} + \mathbf{P}\left(\max_{T \le 2nmt} D(T,i)D(T,j) > C_3 y\right) \overset{(50)}{\le} C_1 \exp(C_2 y - \lambda x^{1/p}) + C_4 e^{-C_5\sqrt{y}}$$

Now choosing $y = x^{1/2p}$ we indeed get $\int_1^{\infty} \mathbf{P}\left(X(T)^p \ge x\right) \,\mathrm{d}x < +\infty$. □

Proof of Lemma 4.1 (iii). Fix $i \in [n]$. We use the notation $D(T) = \frac{1}{n} d(\mathbf{X}_n(T), i)$. We say that $a_n = \mathcal{O}(b_n)$ if there exists a constant c depending only on κ,ρ,λ,C and t such that $a_n \leq c \cdot b_n$ for all $n \in \mathbb{N}$. It follows from (50) that

$$\forall T \leq t \cdot n^3 : \ \mathbf{E}\left(D(T)\right) = \mathcal{O}(1) \qquad \forall T, T' \leq t \cdot n^3 : \ \mathbf{E}\left(D(T)D(T')\right) = \mathcal{O}(1). \tag{63}$$

Using (55), (6), (8) and the fact that $\mathcal{V}_{new}(T)$ and $\mathcal{V}_{old}(T)$ are conditionally independent given $\mathcal{F}_T$ we get

$$\mathbf{E}\left(D(T+1) - D(T) \,\middle|\, \mathcal{F}_T\right) = \frac{D(T) + \frac{\kappa}{n}}{2m + n\kappa} - \frac{D(T)}{2m} = \frac{\kappa}{2mn + n^2\kappa} - \frac{n\kappa D(T)}{4m^2 + 2mn\kappa},$$

$$\mathbf{E}\left((D(T+1) - D(T))^2 \,\middle|\, \mathcal{F}_T\right) = \frac{1}{n^2}\left(\frac{nD(T) + \kappa}{2m + n\kappa} + \frac{nD(T)}{2m} - 2\frac{nD(T) + \kappa}{2m + n\kappa}\frac{nD(T)}{2m}\right).$$

We prove (52) by induction on $T_2 - T_1$.

$$\mathbf{E}\left((D(T_2+1) - D(T_1))^2\right) = \mathbf{E}\left((D(T_2) - D(T_1))^2\right) +$$
$$2\mathbf{E}\left(\mathbf{E}\left(D(T_2+1) - D(T_2) \,\middle|\, \mathcal{F}_{T_2}\right)(D(T_2) - D(T_1))\right) + \mathbf{E}\left((D(T_2+1) - D(T_2))^2\right) \overset{(63)}{=}$$
$$\mathbf{E}\left((D(T_2) - D(T_1))^2\right) + \mathcal{O}\left(\frac{1}{n^3}\right)$$

□

We state a lemma about the speed of convergence of the M/M/∞-queue to its stationary distribution.

Lemma 4.2. *Let Y_t be an $\mathbb{N}_0$-valued continuous-time Markov chain with infinitesimal jump rates* (21),(22),(23) *and initial state $h \in \mathbb{N}_0$. Then for all $t \geq 0$ and $l \in \mathbb{N}_0$ we have*

$$\left|\mathbf{P}\left(Y_t = l\right) - \lim_{s\to\infty} \mathbf{P}\left(Y_s = l\right)\right| \leq e^{-t} \cdot (h + \mu) \tag{64}$$

Proof of Lemma 4.2. According to (25) $Y_s \xrightarrow{\mathrm{d}} \mathrm{POI}(\mu)$ as $s \to \infty$. Let

$$Y_t^{bin} \sim \mathrm{BIN}(h, e^{-t}), \quad Y_t^{poi} \sim \mathrm{POI}((1 - e^{-t})\mu), \quad Y_t^{\infty} \sim \mathrm{POI}(e^{-t}\mu)$$

be mutually independent random variables.

By (24) we have $Y_t^{bin} + Y_t^{poi} \sim Y_t$ and $Y_t^{poi} + Y_t^{\infty} \sim \mathrm{POI}(\mu)$.

$$\left|\mathbf{P}\left(Y_t = l\right) - \lim_{s\to\infty} \mathbf{P}\left(Y_s = l\right)\right| = \left|\mathbf{P}\left(Y_t^{bin} + Y_t^{poi} = l\right) - \mathbf{P}\left(Y_t^{poi} + Y_t^{\infty} = l\right)\right| \leq$$
$$\mathbf{P}\left(Y_t^{bin} + Y_t^{poi} \neq Y_t^{poi} + Y_t^{\infty}\right) \leq \mathbf{P}\left(Y_t^{bin} \neq 0\right) + \mathbf{P}\left(Y_t^{\infty} \neq 0\right) =$$
$$1 - (1 - e^{-t})^h + (1 - \exp(-e^{-t}\mu)) \leq e^{-t} \cdot (h + \mu)$$

□

5 Proof of Theorem 1

In this section we prove Theorem 1 by coupling the evolution of multiple edges between the vertices $1 \leq i \leq j \leq k$ to $\binom{k+1}{2}$ independent M/M/∞-queues.

Given a random element $\mathbf{X}$ of $\mathcal{A}_k$ we define the modified adjacency matrix $\mathbf{X}^*$ in the following way: let $X^*(i,j) := X(i,j)$ if $i \neq j$ and $X^*(i,i) := \frac{1}{2}X(i,i)$.

We assume that the distribution of $\mathbf{X}_n(T)$ is vertex exchangeable (see the paragraph after (46)). We are going to prove (31) using Lemma 3.1: we only need to show that for all $k \in \mathbb{N}$ and $t \geq 0$ we have

$$\mathbf{X}_n^{[k]}\left(\lfloor t \cdot \frac{\rho(W) \cdot n^2}{2} \rfloor\right) \xrightarrow{\mathrm{d}} \mathbf{X}_{W_t}^{[k]} \quad \text{as} \quad n \to \infty. \tag{65}$$

Note that the evolution of $\left(\mathbf{X}_n^{[k]}(T), (d(\mathbf{X}_n(T), i))_{i=1}^k\right)$ is itself a Markov chain under the edge reconnecting dynamics.

We are going to prove (65) by coupling the $\mathcal{A}_k$-valued discrete-time process $\mathbf{X}_n^{[k]}(T)$ to an $\mathcal{A}_k$-valued continuous-time process $\mathbf{Y}_n^{[k]}(t)$ which we define now:

- The initial states are the same: $\forall\, i,j \in [k] :\ Y_n(0,i,j) = X_n(0,i,j)$.
- Given $\mathbf{Y}_n^{[k]}(0) = \mathbf{X}_n^{[k]}(0)$, the evolution of $Y_n(t,i,j)$ is a continuous-time Markov process for each $i,j \in [k]$, the entries $(Y_n(t,i,j))_{i \leq j \leq k}$ evolve independently and $Y_n(t,i,j) \equiv Y_n(t,j,i)$, thus $\mathbf{Y}_n^{[k]}(t)$ is a random element of $\mathcal{A}_k$.
- The process $Y_n^*(t,i,j)$ is an M/M/∞-queue (see (21), (22), (23)) with service rate 1 and arrival rate
$$\mu = \mu_{i,j} := \frac{d(\mathbf{X}_n(0), i) d(\mathbf{X}_n(0), j)}{2m(n) \cdot (1 + \mathbb{1}[i=j])}. \tag{66}$$

Now we show that for all $t \geq 0$

$$\mathbf{Y}_n^{[k]}(t) \xrightarrow{\mathrm{d}} \mathbf{X}_{W_t}^{[k]} \quad \text{as} \quad n \to \infty. \tag{67}$$

From the assumptions $\mathbf{X}_n(0) \xrightarrow{\mathrm{p}} W$, (48) and Lemma 3.2 it follows that

$$\left(\mathbf{Y}_n^{[k]}(0), \left(\frac{1}{n} d(\mathbf{X}_n(0), i)\right)_{i=1}^k\right) \xrightarrow{\mathrm{d}} \left(\mathbf{X}_W^{[k]}, (D(W, U_i))_{i=1}^k\right).$$

Now (67) easily follows from this, (24), (49), Definition 2.1 and (32).

Denote by $\mathcal{D}_n(T,i) := \frac{1}{n} d(\mathbf{X}_n(T), i)$. We are going to construct a coupling (joint realization on the same probability space) of the discrete time $\mathcal{A}_k$-valued Markov chains $\mathbf{X}_n^{[k]}(T)$ and $\mathbf{Y}_n^{[k]}\left(\frac{T}{m(n)}\right)$ for $T = 0, 1, \ldots$ such that for any $\nu < \frac{5}{2}$ we have

$$\lim_{n \to \infty} \mathbf{P}\left(\forall T \leq n^\nu : \quad \mathbf{X}_n^{[k]}(T) = \mathbf{Y}_n^{[k]}\left(\frac{T}{m(n)}\right)\right) = 1. \tag{68}$$

Before proving (68) we first assume that it holds and deduce Theorem 1 from it:

Fix $t \in (0, +\infty)$. If $2 < \nu$ and n is large enough then $2t \cdot m(n) < n^{\nu}$. It is easy to see that (67), (68) and $\lim_{n\to\infty} \frac{2m(n)}{n^2} = \rho(W)$ together imply (65).

Now we start proving (68).

For $i, j \in [k]$ we define the matrix $E_{i,j} \in \mathcal{A}_k$ by

$$E_{i,j}(i', j') := \mathbb{1}[i = i', j = j'] + \mathbb{1}[i = j', j = i']$$

Fix $n \in \mathbb{N}$. We introduce the events

$$\begin{aligned} E_X^{\pm}(T, i, j) &:= \left\{ \mathbf{X}_n^{[k]}(T+1) = \mathbf{X}_n^{[k]}(T) \pm E_{i,j} \right\} \\ E_Y^{\pm}(T, i, j) &:= \left\{ Y_n^{[k]}\left(\frac{T+1}{m}\right) = Y_n^{[k]}\left(\frac{T}{m}\right) \pm E_{i,j} \right\} \\ E_X(T, \emptyset) &:= \left\{ \mathbf{X}_n^{[k]}(T+1) = \mathbf{X}_n^{[k]}(T) \right\} \\ E_Y(T, \emptyset) &:= \left\{ Y_n^{[k]}\left(\frac{T+1}{m}\right) = Y_n^{[k]}\left(\frac{T}{m}\right) \right\} \end{aligned}$$

It is straightforward to derive from (24) that there is an absolute constant $\hat{C}$ such that if we define

$$\mathrm{Err}_Y(T) := \frac{\hat{C}}{m^2} \left(1 + \sum_{i,j=1}^{k} Y_n\left(\frac{T}{m}, i, j\right) + \mu_{i,j} \right)^2 \tag{69}$$

then

$$\left| \mathbf{P}\left(E_Y^{+}(T, i, j) \,\middle|\, \mathcal{F}_T \right) - \frac{\mu_{i,j}}{m} \right| \le \mathrm{Err}_Y(T) \tag{70}$$

$$\left| \mathbf{P}\left(E_Y^{-}(T, i, j) \,\middle|\, \mathcal{F}_T \right) - \frac{Y^*\left(\frac{T}{m}, i, j\right)}{m} \right| \le \mathrm{Err}_Y(T) \tag{71}$$

$$\left| \mathbf{P}\left(E_Y(T, \emptyset) \,\middle|\, \mathcal{F}_T \right) - 1 + \frac{\sum_{i \le j \le k} Y^*\left(\frac{T}{m}, i, j\right) + \mu_{i,j}}{m} \right| \le \mathrm{Err}_Y(T) \tag{72}$$

From the definition of the edge reconnecting model it follows (similarly to (58) and (59)) that there is a constant $\tilde{C}$ depending only on κ and ρ such that if we define

$$\mathrm{Err}_X(T) := \frac{\tilde{C}}{n^3} \left(1 + \sum_{i,j=1}^{k} X_n(T, i, j) + \sum_{i=1}^{k} \mathcal{D}_n(T, i) \right)^2 + \sum_{i,j=1}^{k} \frac{1}{m} \left| \frac{d(\mathbf{X}_n(T), i)\, d(\mathbf{X}_n(T), j)}{2m \cdot (1 + \mathbb{1}[i = j])} - \mu_{i,j} \right| \tag{73}$$

then

$$\left|\mathbf{P}\left(E_X^+(T,i,j)\,\middle|\,\mathcal{F}_T\right)-\frac{\mu_{i,j}}{m}\right|\leq \mathrm{Err}_X(T) \tag{74}$$

$$\left|\mathbf{P}\left(E_X^-(T,i,j)\,\middle|\,\mathcal{F}_T\right)-\frac{X^*(T,i,j)}{m}\right|\leq \mathrm{Err}_X(T) \tag{75}$$

$$\left|\mathbf{P}\left(E_X(T,\emptyset)\,\middle|\,\mathcal{F}_T\right)-1+\frac{\sum_{i\leq j\leq k}X^*(T,i,j)+\mu_{i,j}}{m}\right|\leq \mathrm{Err}_X(T) \tag{76}$$

For any joint realization (coupling) of the discrete time processes $\mathbf{X}_n^{[k]}(T)$ and $\mathbf{Y}_n^{[k]}\left(\frac{T}{m}\right)$, $T=0,1,\dots$ define $E(T)$ to be the event that the $\mathcal{A}_k$-valued increment from T to $T+1$ of these two $\mathcal{A}_k$-valued processes is the same:

$$E(T):=\left\{(E_X(T,\emptyset)\cap E_Y(T,\emptyset))\cup\bigcup_{\epsilon\in\{+,-\}}\bigcup_{i\leq j\leq k}(E_X^\epsilon(T,i,j)\cap E_Y^\epsilon(T,i,j))\right\}.$$

For any coupling the inclusion

$$\left\{\mathbf{X}_n^{[k]}(T)=\mathbf{Y}_n^{[k]}\left(\frac{T}{m}\right)\right\}\cap E(T)\subseteq\left\{\mathbf{X}_n^{[k]}(T+1)=\mathbf{Y}_n^{[k]}\left(\frac{T+1}{m}\right)\right\} \tag{77}$$

holds. Let

$$\mathrm{Err}(T):=2k^2(\mathrm{Err}_X(T)+\mathrm{Err}_Y(T)). \tag{78}$$

Now if we compare (70) to (74), (71) to (75) and (72) to (76), it easily follows that there exists a coupling for which

$$\mathbf{P}\left(E(T)\,\middle|\,\mathcal{F}_T\right)\geq \mathbb{1}[\mathbf{X}_n^{[k]}(T)=\mathbf{Y}_n^{[k]}\left(\frac{T}{m}\right)]\cdot(1-\mathrm{Err}(T))$$

Putting this inequality together with (77), multiplying both sides by

$$\mathbb{1}[\forall\, T'\leq T-1:\;\; \mathbf{X}_n^{[k]}(T')=\mathbf{Y}_n^{[k]}\left(\frac{T'}{m}\right)]$$

and taking the expectation of both sides of the inequality we get

$$\mathbf{P}\left(\forall\, T'\leq T+1:\;\; \mathbf{X}_n^{[k]}(T')=\mathbf{Y}_n^{[k]}\left(\frac{T'}{m}\right)\right)\geq$$
$$\mathbf{P}\left(\forall\, T'\leq T:\;\; \mathbf{X}_n^{[k]}(T')=\mathbf{Y}_n^{[k]}\left(\frac{T'}{m}\right)\right)-\mathbf{E}\left(\mathrm{Err}(T)\right).$$

Thus in order to prove (68) we only need to show

$$\lim_{n\to\infty}\sum_{T=0}^{n^\nu}\mathbf{E}\left(\mathrm{Err}(T)\right)=0. \tag{79}$$

In the remaning part of this section we prove (79).

First we show that if $T \le n^\nu$ then

$$\mathbf{E}\left(\mathrm{Err}_X(T)\right) = \mathcal{O}\left(n^{-5/2}\right). \tag{80}$$

Since $\nu < \frac{5}{2} < 3$, we have $n^\nu \le nm$ if n is large enough, thus

$$\mathbf{E}\left(\mathcal{D}_n(T,i)^2\right) = \mathcal{O}(1), \qquad \mathbf{E}\left(X_n(T,i,j)^2\right) = \mathcal{O}(1)$$

follow from Lemma 4.1 (i) and and Lemma 4.1 (ii), respectively.

$$\begin{aligned}
\mathbf{E}\left(\frac{1}{m}\left|\frac{d(\mathbf{X}_n(T),i)d(\mathbf{X}_n(T),j)}{2m\cdot(1+\mathbb{1}[i=j])} - \mu_{i,j}\right|\right) &\overset{(66)}{=} \\
\mathcal{O}\left(\frac{1}{n^2}\mathbf{E}\left(|\mathcal{D}_n(T,i)\mathcal{D}_n(T,j) - \mathcal{D}_n(0,i)\mathcal{D}_n(0,j)|\right)\right) &= \\
\frac{1}{n^2}\mathcal{O}\left(\mathbf{E}\left(|\mathcal{D}_n(T,i)-\mathcal{D}_n(0,i)|\cdot\mathcal{D}_n(T,j)\right) + \mathbf{E}\left(|\mathcal{D}_n(T,j)-\mathcal{D}_n(0,j)|\cdot\mathcal{D}_n(0,i)\right)\right) &\overset{(*)}{=} \\
\frac{1}{n^2}\mathcal{O}\Big(\sqrt{\mathbf{E}\left((\mathcal{D}_n(T,i)-\mathcal{D}_n(0,i))^2\right)}\sqrt{\mathbf{E}\left(\mathcal{D}_n(T,j)^2\right)} + & \\
\sqrt{\mathbf{E}\left((\mathcal{D}_n(T,j)-\mathcal{D}_n(0,j))^2\right)}\sqrt{\mathbf{E}\left(\mathcal{D}_n(0,i)^2\right)}\Big) &\overset{(52)}{=} \\
\mathcal{O}\left(\frac{1}{n^2}\sqrt{\frac{n^2}{n^3}}\right)\mathcal{O}(1) = \mathcal{O}\left(n^{-5/2}\right) &
\end{aligned} \tag{81}$$

The equation marked by $(*)$ follows from the Cauchy-Schwartz inequality.

Taking the expectation of (73) and using (80), (81) we indeed get (80).

Now we show that if $T \le n^\nu$ then

$$\mathbf{E}\left(\mathrm{Err}_Y(T)\right) = \mathcal{O}\left(n^{-4}\right). \tag{82}$$

The proof of $\mathbf{E}\left(Y_n\left(\frac{T}{m},i,j\right)^2\right) = \mathcal{O}(1)$ is similar to that of Lemma 4.1 (ii) and we omit it, $\mathbf{E}\left(\mu_{i,j}^2\right) = \mathcal{O}(1)$ follows from Lemma 4.1 (i). Taking the expectation of (69) we get (82).

Now if we substitute (82) and (80) into (78) we get $\mathbf{E}\left(\mathrm{Err}(T)\right) = \mathcal{O}\left(n^{-5/2}\right)$ from which (79) follows using $\nu < \frac{5}{2}$.

6 Proof of Theorem 2

In this section we prove Theorem 2 in two stages:

In Subsection 6.1 we prove that the joint evolution of the (normed, rescaled) degrees of the vertices $1, 2, \ldots, k$ behave like independent C.I.R. processes if $1 \ll n$. Given this result we prove (using the results of Section 5) that after $n^2 \ll T$ steps the state of the edge reconnecting model is essentially edge stationary in Subsection 6.2.

6.1 Evolution of degrees

Lemma 6.1. *Let us fix $\kappa \in (0, +\infty)$. We consider the edge reconnecting model $\mathbf{X}_n(T)$, $T = 0, 1, \dots$ on the state space $\mathcal{A}_n^{m(n)}$ with a vertex exchangeable initial state $\mathbf{X}_n(0)$ for $n = 1, 2, \dots$ satisfying and (48). We assume $\mathbf{X}_n(0) \xrightarrow{p} W$ for some multigraphon W.*

Then for all $t \in [0, +\infty)$ and $k \in \mathbb{N}$ we have

$$\left(\mathcal{D}_n\left(\lfloor t \cdot \rho(W) \cdot n^3 \rfloor, i\right)\right)_{i\in[k]} \xrightarrow{d} (Z_{t,i})_{i\in[k]} \qquad \text{as } n \to \infty \tag{83}$$

where $(Z_{t,i})_{i\in[k]}$ are i.i.d. with distribution function $F_t(x) = \int_0^x f(t, y)\, \mathrm{d}y$ where $f(t, x)$ is defined by (37).

In order to prove this lemma, we are going to apply a special case of [8, Corollary 2.2], which we reformulate to fit our needs and notation:

Theorem 3. *Let $\beta : \mathbb{R} \to \mathbb{R}$ and $\gamma : \mathbb{R} \to \mathbb{R}$ be continuous functions. Assume that the stochastic differential equation*

$$\mathrm{d}Z_t = \beta(Z_t)\mathrm{d}t + \gamma(Z_t)\, \mathrm{d}B_t \tag{84}$$

has a unique weak solution with $Z_0 = z_0$ for all $z_0 \in \mathbb{R}$. Let $F_0(x)$ be a probability distribution function on $\mathbb{R}$.

Fix $k \in \mathbb{N}$. For each $n \in \mathbb{N}$ let $(\mathcal{D}_n(T, i))_{i\in[k],T\in\mathbb{N}}$ be a discrete time $\mathbb{R}^k$-valued stochastic process adapted to the filtration $(\mathcal{F}_{n,T})_{T\in\mathbb{N}}$. Let

$$d\mathcal{D}_n(T, i) := \mathcal{D}_n(T + 1, i) - \mathcal{D}_n(T, i).$$

Suppose

$$(\mathcal{D}_n(0, i))_{i=1}^k \xrightarrow{d} (Z_{0,i})_{i=1}^k \qquad \text{as} \qquad n \to \infty \tag{85}$$

where $(Z_{0,i})_{i=1}^k$ are i.i.d. with distribution function F_0. Let $m : \mathbb{N} \to \mathbb{N}$ and suppose that for each $t^* \in [0, +\infty)$ and each $1 \le i, j \le k$ we have

$$\sup_{t\in[0,t^*]} \left| \sum_{T=0}^{\lfloor 2\cdot m(n)\cdot n\cdot t\rfloor} \mathbf{E}\left(d\mathcal{D}_n(T, i) \,\middle|\, \mathcal{F}_{n,T}\right) - \frac{1}{2m(n)\cdot n} \cdot \sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t\rfloor} \beta(\mathcal{D}_n(T, i)) \right| \xrightarrow{p} 0 \tag{86}$$

$$\sup_{t\in[0,t^*]} \left| \sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t\rfloor} \mathbf{Cov}\left(d\mathcal{D}_n(T, i), d\mathcal{D}_n(T, j) \,\middle|\, \mathcal{F}_{n,T}\right) - \frac{1}{2m(n)\cdot n} \cdot \sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t\rfloor} \mathbb{1}[i = j] \cdot \gamma^2(\mathcal{D}_n(t, i)) \right| \xrightarrow{p} 0 \tag{87}$$

$$\sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t^* \rfloor} \mathbf{E}\left((d\mathcal{D}_n(T,i))^2 \mathbb{1}[|d\mathcal{D}_n(T,i)| > \varepsilon] \,\middle|\, \mathcal{F}_{n,T} \right) \xrightarrow{p} 0 \text{ for all } \varepsilon > 0 \tag{88}$$

as $n \to \infty$.

Then the distributions of the $\mathbb{R}^k$-valued continuous-time stochastic processes

$$\left(\mathcal{D}_n\left(\lfloor 2m(n)\cdot n\cdot t\rfloor, i\right)\right)_{i\in[k], t\geq 0}$$

converge weakly to the distribution of $(Z_{t,i})_{i\in[k],t\geq 0}$ as $n \to \infty$ in the Skorohod space $\mathbb{D}(\mathbb{R}^k)$, where $(Z_{t,i})_{i\in[k],t\geq 0}$ are i.i.d. solutions of (84), or briefly:

$$\left(\mathcal{D}_n\left(\lfloor 2m(n)\cdot n\cdot t\rfloor, i\right)\right)_{i\in[k], t\geq 0} \xrightarrow{\mathcal{L}} (Z_{t,i})_{i\in[k],t\geq 0} \tag{89}$$

Proof of Lemma 6.1.

We are going to use Theorem 3 to prove that for all k we have (89) where $(Z_{t,i})_{i\in[k],t\geq 0}$ are i.i.d. solutions of (26) with initial distribution functions $\mathbf{P}\left(Z_{0,i} \leq x\right) = F_0(x)$, where $F_0(x)$ is defined as in Theorem 2. From this the claim of Lemma 6.1 indeed follows, since by (49) we have $\lim_{n\to\infty} \frac{2m(n)}{n^2} = \rho(W)$, thus

$$\left(\mathcal{D}_n\left(\lfloor 2m(n)\cdot n\cdot t\rfloor, i\right)\right)_{i\in[k], t\geq 0} - \left(\mathcal{D}_n\left(\lfloor t\cdot\rho(W)\cdot n^3\rfloor, i\right)\right)_{i\in[k], t\geq 0} \xrightarrow{\mathcal{L}} \left(\, 0 \,\right)_{i\in[k],t\geq 0},$$

from which it follows that for each $t \geq 0$ the relation (83) holds, where $(Z_{t,i})_{i\in[k],t\geq 0}$ are i.i.d. solutions of (26), and using (28) we get that $(Z_{t,i})_{i\in[k]}$ are i.i.d. with distribution function $F_t(x) = \int_0^x f(t,y)\,\mathrm{d}y$ where $f(t,x)$ is defined by (37).

We need to check that (85), (86), (87) and (88) holds with

$$\beta(z) = \kappa - \frac{\kappa}{\rho} z, \qquad \gamma(z) = \sqrt{2z}.$$

From the assumptions $\mathbf{X}_n(0) \xrightarrow{\mathrm{p}} W$, (48) and Lemma 3.2 it follows that

$$\left(\mathcal{D}_n(0,i)\right)_{i\in[k]} \xrightarrow{\mathrm{d}} \left(D(W,U_i)\right)_{i\in[k]},$$

thus by (16) and the definition of F_0 in Theorem 2 we get that the probability distribution function of $D(W,U_i)$ is F_0 and (85) holds.

Now we check that (86) holds:

$$\sup_{t\in[0,t^*]} \left| \sum_{T=0}^{\lfloor 2\cdot m(n)\cdot n\cdot t\rfloor} \mathbf{E}\left(d\mathcal{D}_n(T,i) \,\middle|\, \mathcal{F}_{n,T}\right) - \frac{1}{2m(n)\cdot n} \cdot \sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t\rfloor} \left(\kappa - \frac{\kappa}{\rho}\mathcal{D}_n(T,i)\right) \right|$$

$$\stackrel{(6),(8),(55)}{\leq} \sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t^*\rfloor} \left| \left(\frac{\mathcal{D}_n(T,i) + \frac{\kappa}{n}}{2m(n) + n\kappa} - \frac{\mathcal{D}_n(T,i)}{2m(n)} \right) - \frac{1}{2m(n)\cdot n} \left(\kappa - \frac{\kappa}{\rho}\mathcal{D}_n(T,i)\right) \right| =$$

$$\sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t^*\rfloor} \frac{1}{2m(n)\cdot n} \left(\left(\mathcal{O}\left(\frac{1}{n}\right) + \left(\frac{\kappa}{\rho} - \frac{\kappa}{\frac{2m(n)}{n^2}}\right)\right) \mathcal{D}_n(T,i) + \mathcal{O}\left(\frac{n}{m(n)}\right)\right) \tag{90}$$

By Lemma 4.1 (i) we have $\mathbf{E}\left(\mathcal{D}_n(T,i)\right) = \mathcal{O}(1)$, thus $\mathbf{E}\left((90)\right) \to 0$ as $n \to \infty$ which implies (86).

We prove (87) by treating the cases $i = j$ and $i \neq j$ separately.

First we prove (87) when $i = j$. Using (6), (8), (55) and the fact that $\mathcal{V}_{new}(T)$ and $\mathcal{V}_{old}(T)$ are conditionally independent given $\mathcal{F}_{n,T}$ we get

$$\sup_{t\in[0,t^*]}\left|\sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t\rfloor}\mathbf{Var}\left(d\mathcal{D}_n(T,i)\,\middle|\,\mathcal{F}_{n,T}\right) - \frac{1}{2m(n)\cdot n}\cdot\sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t\rfloor}2\mathcal{D}_n(T,i)\right| \le$$

$$\sum_{T=0}^{\lfloor 2m(n)\cdot nt^*\rfloor}\left|\frac{1}{n^2}\left(\frac{n\mathcal{D}_n(T,i)}{2m}\left(1-\frac{n\mathcal{D}_n(T,i)}{2m}\right)+\right.\right.$$
$$\left.\left.\frac{n\mathcal{D}_n(T,i)+\kappa}{2m+n\kappa}\left(1-\frac{n\mathcal{D}_n(T,i)+\kappa}{2m+n\kappa}\right)\right)-\frac{1}{2m(n)\cdot n}2\mathcal{D}_n(T,i)\right| =$$

$$\sum_{T=0}^{\lfloor 2m(n)\cdot nt^*\rfloor}\frac{1}{2m(n)\cdot n}\left(\mathcal{O}\left(\frac{n}{m(n)}\right)\mathcal{D}_n(T,i)^2+\mathcal{O}\left(\frac{1}{n}\right)+\mathcal{O}\left(\frac{n}{m(n)}\right)\mathcal{D}_n(T,i)\right) \quad (91)$$

By Lemma 4.1 (i) we have $\mathbf{E}\left(\mathcal{D}_n(T,i)^2\right) = \mathcal{O}(1)$, thus $\mathbf{E}\left((91)\right) \to 0$ as $n \to \infty$ which implies (87) for $i = j$.

Now we prove (87) when $i \neq j$:

$$\sup_{t\in[0,t^*]}\left|\sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t\rfloor}\mathbf{Cov}\left(d\mathcal{D}_n(T,i), d\mathcal{D}_n(T,j)\,\middle|\,\mathcal{F}_{n,T}\right)\right| \le$$

$$\sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t^*\rfloor}\left|-\left(\frac{\mathcal{D}_n(T,i)}{2m(n)}\cdot\frac{\mathcal{D}_n(T,j)+\frac{\kappa}{n}}{2m(n)+n\kappa}+\frac{\mathcal{D}_n(T,j)}{2m(n)}\cdot\frac{\mathcal{D}_n(T,i)+\frac{\kappa}{n}}{2m(n)+n\kappa}\right)-\right.$$
$$\left.\left(\frac{\mathcal{D}_n(T,i)+\frac{\kappa}{n}}{2m(n)+n\kappa}-\frac{\mathcal{D}_n(T,i)}{2m(n)}\right)\cdot\left(\frac{\mathcal{D}_n(T,j)+\frac{\kappa}{n}}{2m(n)+n\kappa}-\frac{\mathcal{D}_n(T,j)}{2m(n)}\right)\right| =$$

$$\sum_{T=0}^{\lfloor 2m(n)\cdot n\cdot t^*\rfloor}\frac{1}{2m(n)\cdot n}\left(\mathcal{O}\left(\frac{n}{m(n)}\right)\left(\mathcal{D}_n(T,i)+\mathcal{D}_n(T,j)\right)^2\right) \quad (92)$$

By Lemma 4.1 (i) we have $\mathbf{E}\left(\mathcal{D}_n(T,i)^2\right) = \mathcal{O}(1)$ and $\mathbf{E}\left(\mathcal{D}_n(T,j)^2\right) = \mathcal{O}(1)$, which implies $\mathbf{E}\left((92)\right) \to 0$ as $n \to \infty$ which in turn implies (87) for $i \neq j$. (88) is trivial since $\mathbf{P}\left(|d\mathcal{D}_n(T,i)| \le \frac{1}{n}\right) = 1$.

Having checked that (85), (86), (87) and (88) holds, we can use Theorem 3 to prove that we have (89) where $\left(Z_{t,i}\right)_{i\in[k],t\ge 0}$ are i.i.d. solutions of (26) with initial distribution functions $F_0(x)$, which finishes the proof of Lemma 6.1, as described in the beginning of the proof. □

6.2 Asymptotic edge-stationarity

Similarly to Section 5 we assume that the distribution of $\mathbf{X}_n(T)$ is vertex exchangeable. We are going to prove (35) using Lemma 3.1: we only need to show that for all $k \in \mathbb{N}$ and $t > 0$ we have

$$\mathbf{X}_n^{[k]}\left(\lfloor t \cdot \rho(W) \cdot n^3 \rfloor\right) \xrightarrow{\mathrm{d}} \mathbf{X}^{[k]}_{\hat{W}_t}. \tag{93}$$

Proof of Theorem 2. Let $(Z_{t,i})_{i\in[k]}$ denote i.i.d. random variables with distribution function $F_t(x) = \int_0^x f(t,y)\,\mathrm{d}y$ where $f(t,x)$ is defined by (37). Recall the notion of $\mathbf{p}(k,\lambda)$ from (18). Define the function $\mathbf{p}(A,(z_i)_{i=1}^k)$ for $A \in \mathcal{A}_k$ and $z_i \in [0,+\infty)$, $i \in [k]$ by

$$\mathbf{p}(A,(z_i)_{i=1}^k) := \prod_{i=1}^k \prod_{j=i}^k \mathbf{p}\left(A^*(i,j), \frac{z_i \cdot z_j}{\rho \cdot (1 + \mathbb{1}[i=j])}\right). \tag{94}$$

By (14) and (36), in order to prove (93) we only need to check that for all $A \in \mathcal{A}_k$

$$\lim_{n\to\infty} \mathbf{P}\left(\mathbf{X}_n^{[k]}\left(\lfloor t \cdot \rho(W) \cdot n^3 \rfloor\right) = A\right) = \mathbf{E}\left(\mathbf{p}(A,(Z_{t,i})_{i=1}^k)\right).$$

We (somewhat arbitrarily) fix $2 < \nu < \frac{5}{2}$. Let

$$T_0^n := \lfloor t \cdot \rho(W) \cdot n^3 \rfloor - \lfloor n^\nu \rfloor.$$

It easily follows from $\nu < \frac{5}{2} < 3$, Lemma 4.1 (iii) and Lemma 6.1 that

$$(\mathcal{D}_n(T_0^n, i))_{i\in[k]} \xrightarrow{\mathrm{d}} (Z_{t,i})_{i\in[k]} \quad \text{as} \quad n \to \infty. \tag{95}$$

Now we couple $\mathbf{X}_n^{[k]}(T_0^n + T)$ to $\mathbf{Y}_n^{[k]}\left(\frac{T}{m(n)}\right)$ in a similar fashion as in Section 5:

- The initial state of $\mathbf{Y}_n^{[k]}$ is $\forall\, i,j \in [k]:\ Y_n(0,i,j) = X_n(T_0^n,i,j)$.
- Given $\mathbf{X}_n(T_0^n)$, the entries $(Y_n(t,i,j))_{i\le j\le k}$ evolve independently and $Y_n(t,i,j) \equiv Y_n(t,j,i)$.
- Given $\mathbf{X}_n(T_0^n)$, the evolution of $Y_n^*(t,i,j)$ is is an M/M/∞-queue with service rate 1 and arrival rate

$$\mu = \mu_{i,j} := \frac{d(\mathbf{X}_n(T_0^n),i)d(\mathbf{X}_n(T_0^n),j)}{2m(n)\cdot(1+\mathbb{1}[i=j])} = \frac{\mathcal{D}_n(T_0^n,i)\mathcal{D}_n(T_0^n,j)}{\frac{2m(n)}{n^2}\cdot(1+\mathbb{1}[i=j])}. \tag{96}$$

Now we show that

$$\mathbf{Y}_n^{[k]}\left(\frac{\lfloor n^\nu \rfloor}{m(n)}\right) \xrightarrow{\mathrm{d}} \mathbf{X}^{[k]}_{\hat{W}_t} \quad \text{as} \quad n \to \infty. \tag{97}$$

First note that

$$\lim_{n\to\infty}\lim_{s\to\infty}\mathbf{P}\left(\mathbf{Y}_n^{[k]}(s)=A\right)\overset{(25)}{=}\lim_{n\to\infty}\mathbf{E}\left(\prod_{i=1}^k\prod_{j=i}^k\mathbf{p}\left(A^*(i,j),\frac{\mathcal{D}_n(T_0^n,i)\mathcal{D}_n(T_0^n,j)}{\frac{2m(n)}{n^2}\cdot(1+\mathbb{1}[i=j])}\right)\right)\overset{(95)}{=}$$
$$\mathbf{E}\left(\prod_{i=1}^k\prod_{j=i}^k\mathbf{p}\left(A^*(i,j),\frac{Z_{t,i}\cdot Z_{t,j}}{\rho\cdot(1+\mathbb{1}[i=j])}\right)\right)\overset{(94)}{=}\mathbf{E}\left(\mathbf{p}(A,(Z_{t,i})_{i=1}^k)\right) \quad (98)$$

Let $t_n:=\frac{\lfloor n^\nu\rfloor}{m(n)}$. $\lim_{n\to\infty}t_n=+\infty$ follows from $2<\nu$. We have

$$\left|\mathbf{P}\left(\mathbf{Y}_n^{[k]}(t_n)=A\right)-\lim_{s\to\infty}\mathbf{P}\left(\mathbf{Y}_n^{[k]}(s)=A\right)\right|\overset{(64)}{\leq}$$
$$\exp(-t_n)\cdot\sum_{i=1}^k\sum_{j=i}^k\left(\mathbf{E}\left(X_n(T_0^n,i,j)\right)+\mathbf{E}\left(\mu_{i,j}\right)\right)\overset{(96),(50),(51)}{=}\exp(-t_n)\mathcal{O}(1). \quad (99)$$

Thus (97) follows from (98) and (99).

Using the proof of (68) we can construct a coupling such that we have

$$\lim_{n\to\infty}\mathbf{P}\left(\forall\, 0\leq T\leq n^\nu: \quad \mathbf{X}_n^{[k]}(T_0^n+T)=\mathbf{Y}_n^{[k]}\left(\frac{T}{m(n)}\right)\right)=1.$$

Now (93) follows from this, $T_0^n+\lfloor n^\nu\rfloor=\lfloor t\cdot\rho(W)\cdot n^3\rfloor$ and (97). □

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