FCS HSE, MDS, Linear Algebra

Function

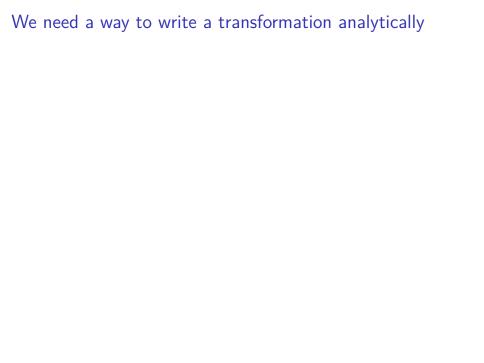
Functions between vector spaces

Definition. Let V,W be vector spaces. A transformation $T:V\to W$ is called linear if

- 1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \forall \mathbf{u}, \mathbf{v} \in V$
- 2. $T(\alpha \mathbf{v}) = \alpha T(\mathbf{v})$ for all $\mathbf{v} \in V$ and for all scalars $\alpha \in \mathbb{R}$.

Properties 1 and 2 together are sometimes combined into the following one:

$$T(\alpha \mathbf{u} + \beta \mathbf{v}) = \alpha T(\mathbf{u}) + \beta T(\mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in V, \quad \forall \alpha, \beta \in \mathbb{R}.$$



Let us assume $T:V\to W$, vectors e_1,\dots,e_m be a standard basis in V, and vectors $\tilde{e}_1,\dots,\tilde{e}_m$ be a basis in W.

We would like to investigate how T acts on any $x \in V$.

$$x=x_1e_1+\ldots+x_ne_n,$$

$$T(x)=T(x_1e_1+\ldots+x_ne_n)=x_1T(e_1)+\ldots+x_nT(e_n).$$

Keep in mind, that $T(e_1),\dots,T(e_n)$ are vectors, i.e. abstract citizens of the vector space W.

Let us look at them from standard basis in W:

$$\begin{split} T(e_{\mathbf{1}}) &= a_{1\mathbf{1}}\tilde{e}_1 + a_{21}\tilde{e}_2 + \ldots + a_{m1}\tilde{e}_m, \\ T(e_{\mathbf{2}}) &= a_{1\mathbf{2}}\tilde{e}_1 + a_{2\mathbf{2}}\tilde{e}_2 + \ldots + a_{m2}\tilde{e}_m, \\ &\vdots \\ T(e_{\mathbf{n}}) &= a_{1\mathbf{n}}\tilde{e}_1 + a_{2\mathbf{n}}\tilde{e}_2 + \ldots + a_{m\mathbf{n}}\tilde{e}_m \end{split}$$

Then get back to $T(x) = x_1 T(e_1) + \ldots + x_n T(e_n)$.

$$\begin{split} T(x) &= x_{\mathbf{1}} \left(a_{1\mathbf{1}} \tilde{e}_1 + \ldots + a_{m\mathbf{1}} \tilde{e}_m \right) + x_{\mathbf{2}} \left(a_{1\mathbf{2}} \tilde{e}_1 + \ldots + a_{m\mathbf{2}} \tilde{e}_m \right) \\ &+ x_{n} \left(a_{1n} \tilde{e}_1 + \ldots + a_{mn} \tilde{e}_m \right) \end{split}$$

$$\begin{split} T(x) &= x_{\mathbf{1}} \left(a_{11} \tilde{e}_1 + \ldots + a_{m\mathbf{1}} \tilde{e}_m \right) + x_{\mathbf{2}} \left(a_{12} \tilde{e}_1 + \ldots + a_{m\mathbf{2}} \tilde{e}_m \right) \\ &+ x_{n} \left(a_{1n} \tilde{e}_1 + \ldots + a_{mn} \tilde{e}_m \right) \\ &= \left(a_{11} x_{\mathbf{1}} + a_{12} x_{\mathbf{2}} + \ldots + a_{1n} x_{\mathbf{n}} \right) \tilde{e}_1 + \left(a_{21} x_{\mathbf{1}} + a_{22} x_{\mathbf{2}} + \ldots + a_{2n} x_{\mathbf{n}} \right) \tilde{e}_2 \\ &\quad + \left(a_{m\mathbf{1}} x_{\mathbf{1}} + a_{12} x_{\mathbf{2}} + \ldots + a_{mn} x_{\mathbf{n}} \right) \tilde{e}_m \end{split}$$

Matvec... again...

Finally:

$$[T(x)]_{\tilde{e}} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = A_T [x]_e .$$

Huh!

If indeed e and \tilde{e} are standard bases then

To construct matrix A_T of a linear transformation T we need just to know images of basis vectors: $T(e_1), \ldots, T(e_n)$, i.e.

$$e_1 \overset{T}{\rightarrow} a_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, e_2 \overset{T}{\rightarrow} a_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \dots, e_n \overset{T}{\rightarrow} a_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

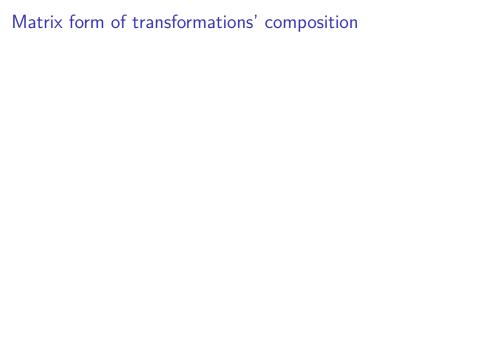
Sweet and easy for any $x \in V$, $x = (x_1, \dots, x_n)^\top$:

$$x = x_1e_1 + \ldots + x_ne_n,$$

$$T(x) = x_1T(e_1) + \ldots + x_nT(e_n).$$

Then we want to observe T(x) in standard basis \tilde{e} as well:

$$\begin{split} [T(x)] &= x_1 a_1 + \ldots + x_n a_n \\ &= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \ldots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}. \end{split}$$







Examples