Continuous random variables.

$$\boxed{\underline{2}\ \underline{2}\ \underline{1}\ .\ \underline{1}\ \underline{5}\ \underline{7}\ \underline{1}\ \underline{2}\ \underline{3}\ \underline{1}\ \underline{8}}$$

$P(X = 221.5) = \longrightarrow 0$ $\qquad |\Omega_X| \longrightarrow \infty$

$\underline{2}\ \underline{2}\ \underline{1}\ .\ \underline{0}\ \underline{0}$

$|\Omega_X| = 100$

$P(X = 221.15) = \frac{1}{100}$ $\qquad \underline{9}\ \underline{9}$

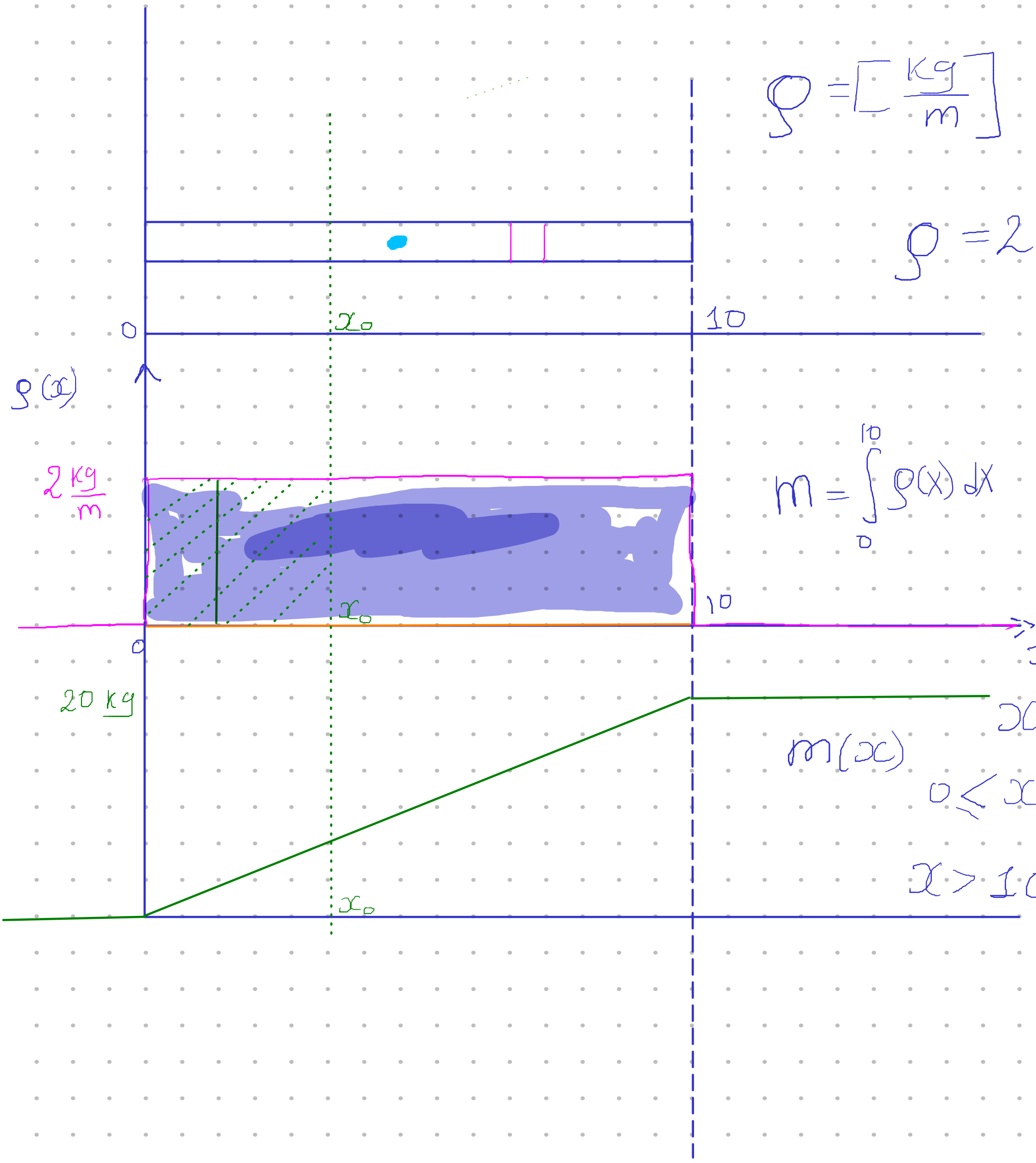
$P(X = 221.52000 \ldots 0) = 0$

Question about probability of a precise value has no sense

$$\boxed{P(a < X < b)}, \quad b \geq a$$

$P(220 < X < 221.1)$

$$m = V \cdot \rho \qquad [kg] = [m^3] \cdot \left[\frac{kg}{m^3}\right]$$



$$\rho = \left[\frac{kg}{m}\right] \qquad m = \rho \cdot l = \left[\frac{kg}{m}\right][m]$$

$$\rho = 2\,\frac{kg}{m}\ ; \quad m = 2\,\frac{kg}{m} \cdot 10m = $$
$$= 20\,kg$$

$$m = \int_0^{10} \rho(x)\,dx$$

$m(x)$

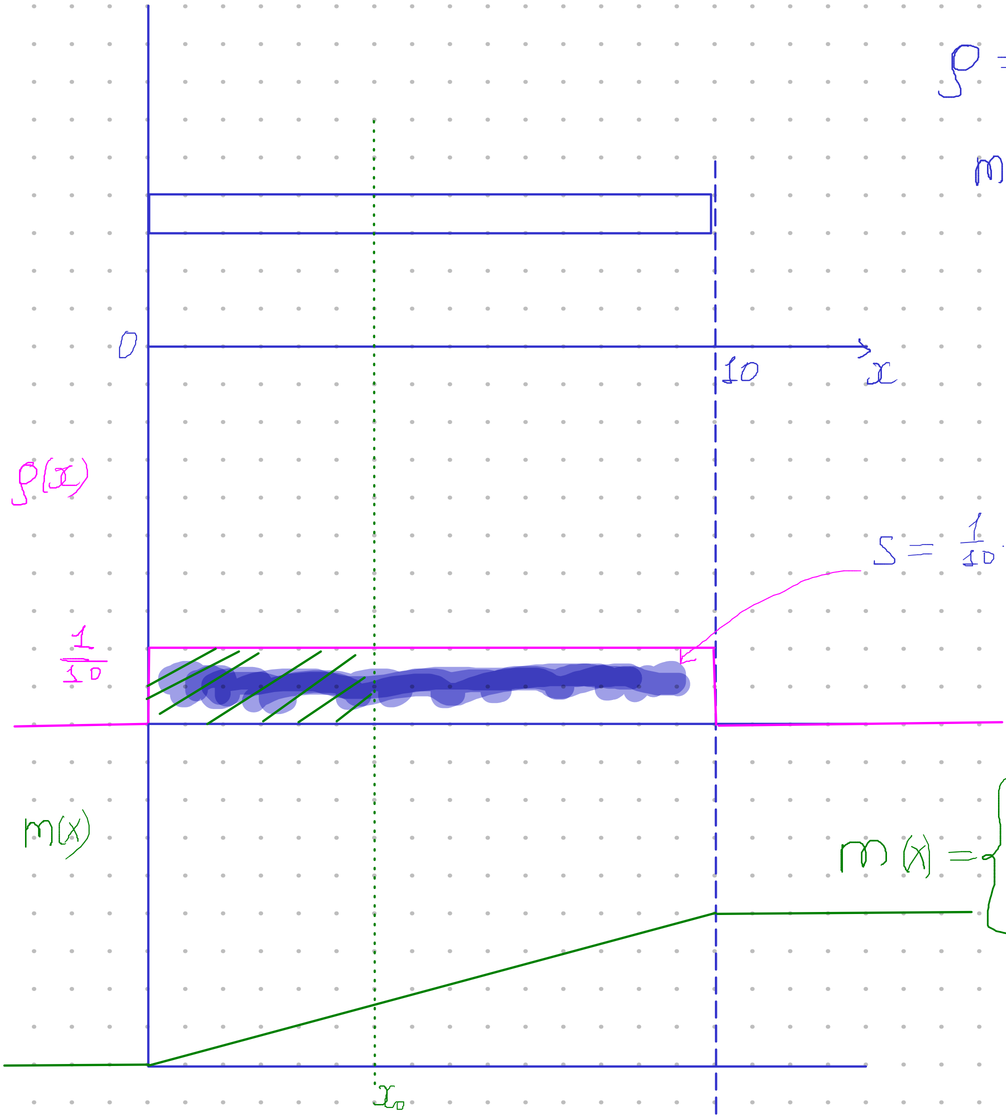
$x < 0 \qquad m(x) = 0$

$0 \leq x \leq 10 \qquad m(x) = \rho \cdot x$

$x > 10 \qquad m(x) = 20$

$$\rho = \frac{1}{10}\left[\frac{\text{кг}}{\text{м}}\right]$$

$$m = \rho \cdot \ell = \frac{1}{10} \cdot 10 = 1 \text{ kg}$$



$$S = \frac{1}{10} \cdot 10 = 1$$

$$m(x) = \begin{cases} 0, & x < 0 \\ \rho \cdot x, & 0 \leqslant x \leqslant 10 \\ 1, & x > 10 \end{cases}$$

Mass $\longrightarrow$ Probability

Probability Theory

Continuous Random Variables

Gleb Karpov

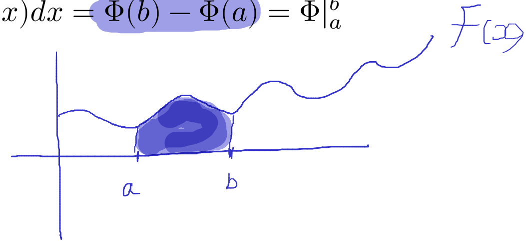
HSE FCS

Fundamental Theorem of Calculus

- **Def.** If function $f(x)$ is continuous on the segment $[a, b]$, and $\Phi(x)$ is any of its antiderivatives on its, then:

$$\int_a^b f(x)dx = \Phi(b) - \Phi(a) = \Phi|_a^b$$

$$= \Phi\Big|_a^b$$



$$F(x) = \Phi'(x)$$

$$\Phi \xrightarrow{\text{der.}} F$$

$$\int F(x)dx = \boxed{\Phi(x) + C}$$

$$\int c\,dx = c\cdot x + b,\quad b - \text{const}$$
$$c - \text{const}$$

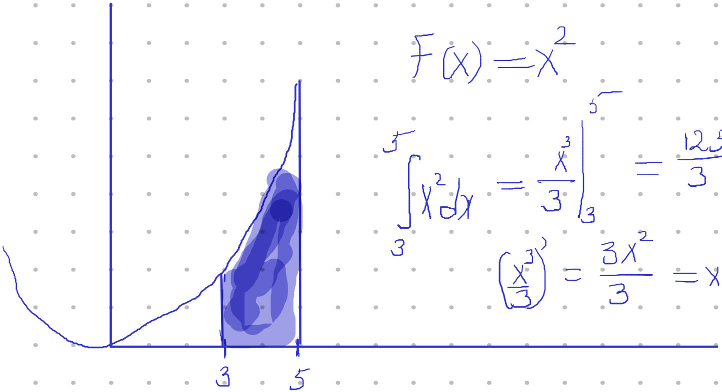
$$F(x) = c \qquad \Phi = cx + b$$

$$\boxed{\begin{gathered}\int (F+g)\,dx = \int F\,dx + \int g\,dx \\ \int c\cdot F(x)\,dx = c\cdot\int F(x)\,dx\end{gathered}}$$

$$\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C$$

$$F(x) = x^{\alpha} \qquad \Phi = \frac{x^{\alpha+1}}{\alpha+1} + C$$

$$\Phi'(x) = \frac{(\alpha+1)\cdot x^{\alpha}}{(\alpha+1)} + 0 = x^{\alpha} = F(x)$$



$$F(x) = x^2$$

$$\int_3^5 x^2\,dx = \frac{x^3}{3}\Bigg|_3^5 = \frac{125}{3} - \frac{27}{3}$$

$$\left(\frac{x^3}{3}\right)' = \frac{3x^2}{3} = x^2$$

Probability Density Function

- By definition: X is a continuous random variable if there exists a nonnegative function $f_X(x)$ defined for $\forall x \in \mathbb{R}$, such that any probability of the form $P(a \leq X \leq b)$ can be found by:

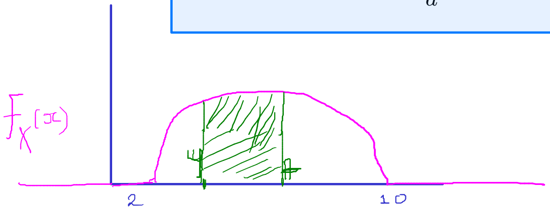
f_Y

f_W

$$\boxed{P(a \leq X \leq b) = \int_a^b f_X(x)dx}$$

$, \forall a, \forall b ;$

$a \leq b$



Probability Density Function

P.d.F.

- By definition: X is a continuous random variable if there exists a nonnegative function $f_X(x)$ defined for $\forall x \in \mathbb{R}$, such that any probability of the form $P(a \leq X \leq b)$ can be found by:

$$P(a \leq X \leq b) = \int_a^b f_X(x)dx$$

- Normalisation property. Since $P(\Omega_X)$ should be equal to 1, here we have:

$$1 = P\{X \in (-\infty, +\infty)\} = \int_{-\infty}^{+\infty} f_X(x)dx$$

Cumulative Distribution Function

- **Def.** CDF of random variable X is a non-decreasing function $F_X(x)$ defined for $\forall x \in \mathbb{R}$, such that:

$$F_X(x) = P\{X \in (-\infty, x)\}$$

Cumulative Distribution Function

- **Def.** CDF of random variable X is a non-decreasing function $F_X(x)$ defined for $\forall x \in \mathbb{R}$, such that:

$$F_X(x) = P\{X \in (-\infty, x)\}$$

$F_X(x)$

- For CRV we can rewrite it as:

$$F_X(x) = P\{X \in (-\infty, x)\} = \int_{-\infty}^{x} f_X(u)du$$

$u = x$

$$F_X(t) = P(-\infty < X < t) = \int_{-\infty}^{t} f_X(x)dx$$

Cumulative Distribution Function CDF

- **Def.** CDF of random variable X is a non-decreasing function $F_X(x)$ defined for $\forall x \in \mathbb{R}$, such that:

$$F_X(x) = P\{X \in (-\infty, x)\}$$

- For CRV we can rewrite it as:

$$F_X(t) = \int_{-\infty}^{t} f_X(x)\, dx$$

$$F_X(x) = P\{X \in (-\infty, x)\} = \int_{-\infty}^{\cancel{u}\, x} f_X(u) du$$

- Basic properties (for self-check also):

$$F_X(-\infty) = 0, \qquad F_X(\infty) = 1$$

CDF as a way to avoid intergration

$$\int_a^b f_X(x)\,dx$$

- Suppose we are interested in $P(a < X < b)$. One way - to compute integral according to the defition of CRV.
- Consider interval $\mathcal{D} = (-\infty, b)$. It can be decoupled in a union of two complementary sets:

$$\mathcal{D} = (-\infty,\ a] \cup (a,\ b)$$

CDF as a way to avoid intergration

- Suppose we are interested in $P(a < X < b)$. One way - to compute integral according to the defition of CRV.
- Consider interval $\mathcal{D} = (-\infty, b)$. It can be decoupled in a union of two complementary sets:

$$\mathcal{D} = (-\infty,\ a] \cup (a,\ b)$$

- Then according to additivity principle of probability:

$$P\{\mathcal{D}\} = P\{(-\infty,\ a]\} + P\{(a,\ b)\}$$

$$P(X \in \mathcal{D}) = P(X \in (-\infty; a]) + P(X \in (a, b))$$

CDF as a way to avoid intergration

- Return to the CDF formulation:

$$\int_{-\infty}^{b} f_X(u)du = \int_{-\infty}^{a} f_X(u)du + \int_{a}^{b} f_X(u)du$$

- And finally:

$$P(a<X<b) = \int_{a}^{b} f_X(x)dx = F_X(b) - F_X(a).$$

$$f_X(x) = F_X'(x)$$

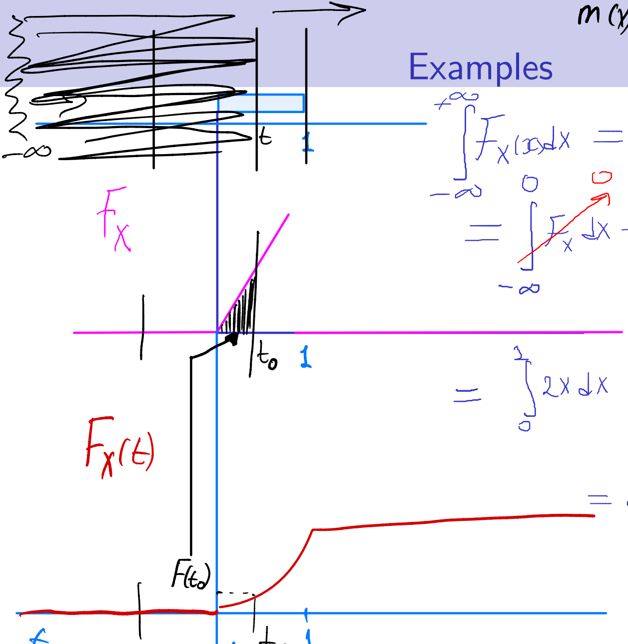
Examples

- A random variable X has density function:

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Check that it is valid and find the cumulative distribution function of X.

Examples



$$\int_{-\infty}^{+\infty} f_X(x)\,dx = 1 =$$

$$= \int_{-\infty}^{0} f_X\,dx + \int_{0}^{1} f_X\,dx + \int_{1}^{+\infty} f_X\,dx =$$

$$= \int_0^1 2x\,dx = 2\frac{x^2}{2}\Big|_0^1 =$$

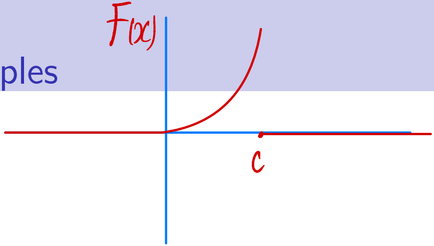
$$= \frac{2\cdot 1^2}{2} - \frac{2\cdot 0^2}{2} = 1.$$

$$\int_0^t 2x\,dx = 2\frac{x^2}{2}\Big|_0^t \quad t^2$$

$$F_X(t) = \begin{cases} 0, & t<0 \\ t^2, & 0\le t\le 1 \\ 1, & t>1 \end{cases}$$

Examples

$$F = \begin{cases} 0, & x \leq 0 \\ x^3/4, & 0 < x < c \\ 0, & x \geq c \end{cases}$$

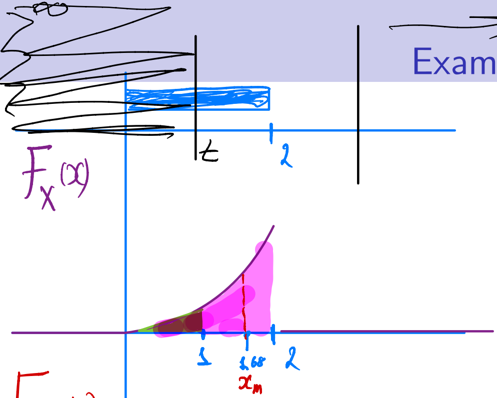


Find the constant c such that $f_X(x)$ is a valid p.d.f of a random variable X, find the cumulative distribution function, $F_X(x)$, and find a **median** of the density function:

- $f(x) = \frac{x^3}{4}$, $0 < x < c$
- $f(x) = \frac{3x^2}{16}$, $-c < x < c$

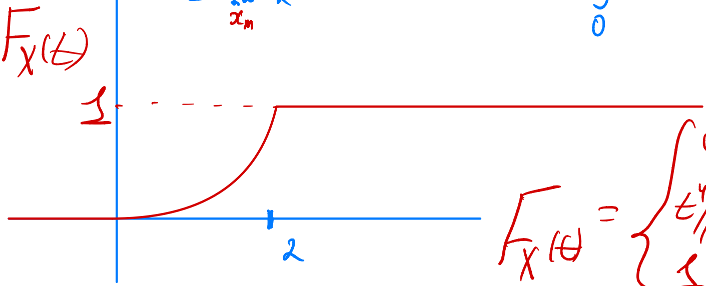
$$\int_{-\infty}^{+\infty} F_X(x)\,dx = \int_{-\infty}^{0} \cancel{0}\,dx + \int_0^c \frac{x^3}{4}\,dx + \int_c^{+\infty} \cancel{0}\,dx = \left.\frac{x^4}{16}\right|_0^c = \frac{c^4}{16} - \frac{0^4}{16} = 1 \Rightarrow \boxed{c = 2}$$

Examples



$$F_X(t) = P(-\infty < X < t)$$

$$\int_0^t f_X(x)\,dx = \int_0^t \frac{x^3}{4}\,dx = \frac{t^4}{16}$$



$$F_X(t) = \begin{cases} 0, & t < 0 \\ t^4/16, & 0 \le t \le 2 \\ 1, & t > 2 \end{cases}$$

Examples

$$P(1<X<2) = \int_1^2 F_X(x)\,dx = \frac{15}{16}$$

$\boxed{P(X<1)}$

$\boxed{P(X<2)}$

$$P(X<2) - P(X<1) =$$

$$= \frac{2^4}{16} - \frac{1^4}{16} = \boxed{\frac{15}{16}}$$

$$x_m : P(X \leq x_m) = P(X \geq x_m) = 1/2$$

$$F_X(x_m) = 1/2 \qquad \frac{x_m^4}{16} = 1/2 \rightarrow x_m^4 = 8$$

$$x_m = \sqrt[4]{8} \approx 1.68$$

Examples

$$f_X(x) = \begin{cases} 0, & X < 10 \\ \frac{10}{X^2}, & X \geq 10 \end{cases}$$

check valid pdf

$$\int_{-\infty}^{+\infty} F_X \, dx = 1 \rightarrow \int_{10}^{+\infty} \frac{10}{X^2} \, dX = -10 \left[\frac{1}{X} \Big|_{10}^{+\infty} \right] =$$

$$= -10 \cdot \left[0 - \frac{1}{10} \right] = 1$$

$$\int X^{\alpha} = \frac{X^{\alpha+1}}{\alpha+1}$$

Examples

$$F_X(x) = \begin{cases} 0, & x<0 \\ \frac{1}{2} \sin x, & 0\le x\le \pi \\ 0, & x>\pi \end{cases}$$

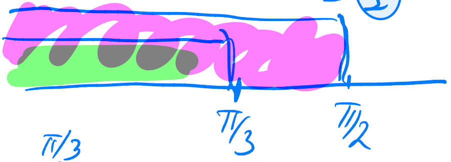
$$\int_0^{\pi} \frac{1}{2} \sin x \, dx = 1$$

$$-\frac{1}{2}\left[\cos x\Big|_0^{\pi}\right] = -\frac{1}{2}\left[-1-1\right] = 1$$

$$P\left(X<\pi/3 \mid X<\pi/2\right) = 1/2$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

$$= \frac{P(X<\pi/3)}{P(X<\pi/2)} = \frac{1/4}{1/2} = \frac{1}{2}$$



$$\frac{1}{2}\int_0^{\pi/3} \sin x \, dx = -\frac{1}{2}\left[\cos x\Big|_0^{\pi/3}\right] = -\frac{1}{2}\cdot\left(+\frac{1}{2}-1\right) = 1/4$$

$$-\frac{1}{2}\cos\Big|_0^{\pi/2} = -\frac{1}{2}(0-1) = 1/2$$