Непрерывные случайные величины

$$\boxed{\underline{2}\,\underline{2}\,\underline{1}.\underline{1}\,\underline{5}\,\underline{4}\,\underline{8}\,\underline{2}\,\underline{0}\,\underline{1}\,\underline{0}\,\underline{2}\,\underline{4}}$$

$\underline{2}\,\underline{2}\,\underline{0}.\underline{0}\,\underline{0}$

$\vdots$

$.\underline{9}\,\underline{9}$

$|\Omega_x| = 100$

$P(X = 222.\underline{0}\,\underline{0}\,\_\,\_\,\_\,\_\,\_\,\_\,\_\,\_\,\underline{0}) \to 0$

$P(A)$

$|\Omega_x| \to \infty$

$P(A) = \dfrac{1}{|\Omega_x|} \to 0$

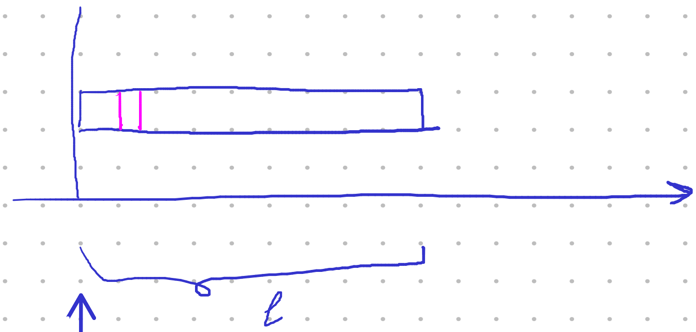
$P(219.5 < X < 220.\underline{1})$

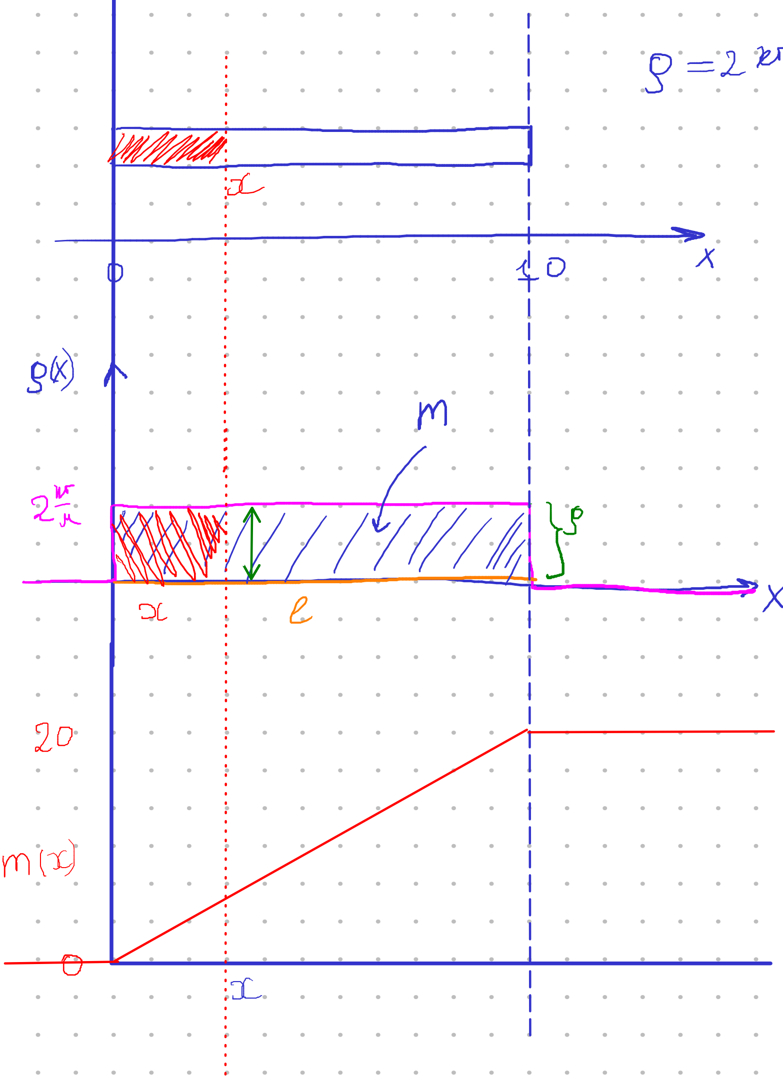
$$m = \rho \cdot V \quad \left[\frac{кг}{м^3}\right]\cdot\left[м^3\right] = [кг]$$

ρ — погонная плотность

$$\rho = \left[\frac{кг}{м}\right]$$

$$m = \left[\frac{кг}{м}\right]\cdot[м] = \rho \cdot l$$





$\rho = 2$ кг/м

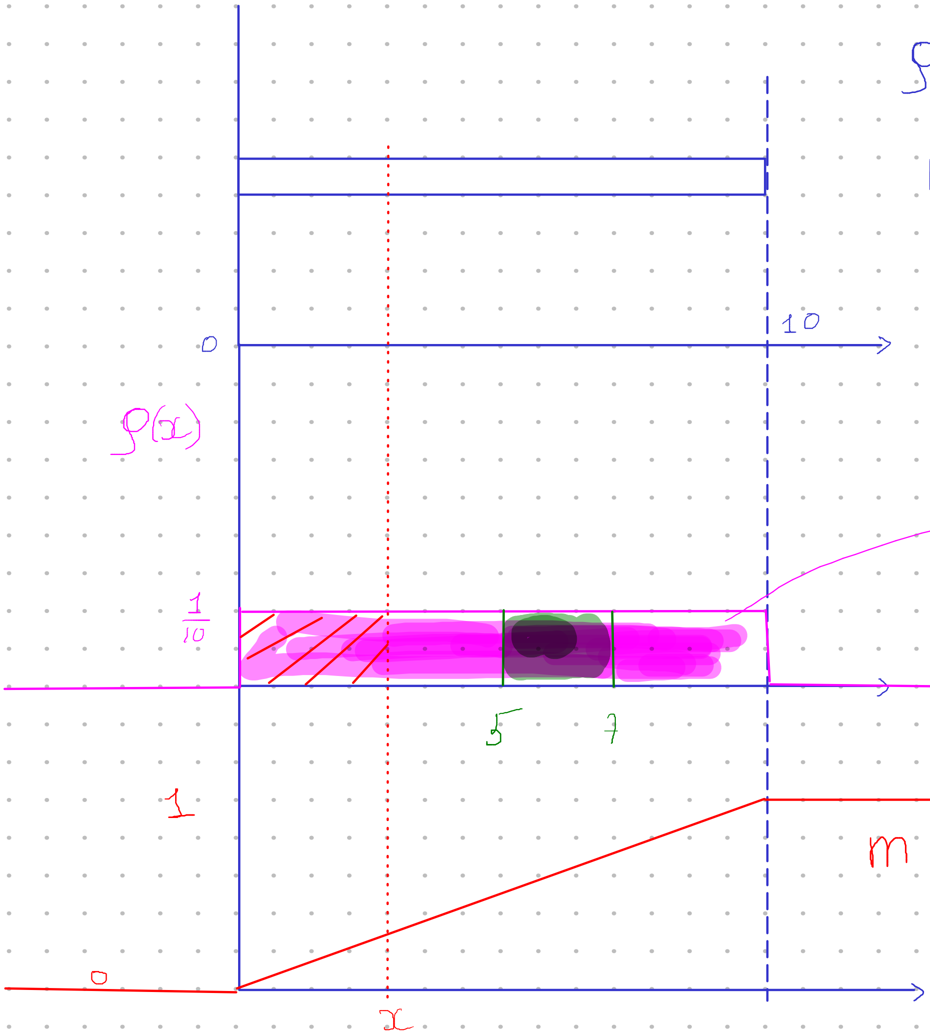
$$m = \rho \cdot l = 2\ кг/м \cdot 10\ м = 20\ кг$$

$$m(x) = \rho \cdot x, \quad x \in [0, 10]$$

$$m(0) = 0$$

$$m(10) = \rho \cdot 10 = 20$$

$$m(x) = \begin{cases} 0, & x < 0 \\ 2 \cdot x, & x \in [0, 10] \\ 20, & x > 10 \end{cases}$$



$$\rho = \frac{1}{10}\ \frac{\text{кг}}{\text{м}}$$

$$m = \rho \cdot \ell = \frac{1}{10} \cdot 10 = 1 \text{ кг}$$

$$S = \frac{1}{10} \cdot 10 = 1$$

$$= m$$

$$m(x) = \begin{cases} 0, & x < 0 \\ \rho \cdot x, & 0 \le x \le 10 \\ 1, & x > 10 \end{cases}$$

Probability Theory

Continuous Random Variables

Gleb Karpov

HSE FCS

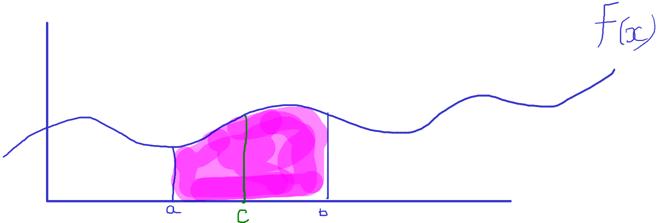
Fundamental Theorem of Calculus

$F(x) = \Phi'(x)$

$\int F(x)dx = \Phi(x) + C$; $(\Phi(x)+C)' = \Phi'(x) + C' = F(x) + 0$

- **Def.** If function $f(x)$ is continuous on the segment $[a, b]$, and $\Phi(x)$ is any of its antiderivatives on its, then:

$$\int_a^b f(x)dx = \Phi(b) - \Phi(a) = \Phi|_a^b = \Phi\Big|_a^b$$



$$\int (F+g)\,dx = \int F\,dx + \int g\,dx$$

$$\int c \cdot F(x)\,dx = c \cdot \int F(x)\,dx$$

$$\int_a^b F(x)\,dx = \int_a^c F(x)\,dx + \int_c^b F(x)\,dx, \quad \forall\, a, b, c \in \mathbb{R}!$$

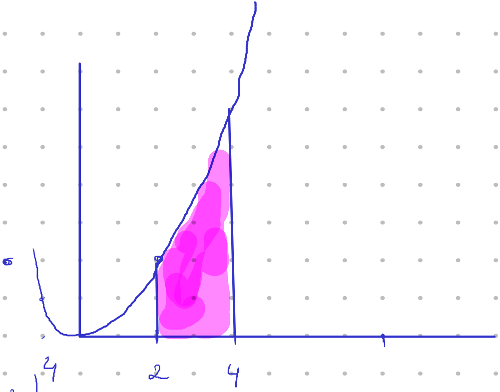
$$a \leq c \leq b$$

$$\int a\,dx = ax + C, \quad a, C - const \qquad (ax+c)' = \varphi'(x) = a = F(x)$$

$$\int x^{\alpha}\,dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \qquad \left(\frac{x^{\alpha+1}}{\alpha+1}\right)' = \frac{(\alpha+1)}{\alpha+1} x^{\alpha} = x^{\alpha} = F(x)$$

$$\int \cos x\,dx = \sin x + c \qquad \int \sin x\,dx = -\cos x + c$$

$F = x^2$



$$\int_2^4 x^2\,dx = \left.\frac{x^3}{3}\right|_2^4 = \frac{64}{3} - \frac{8}{3} = \frac{56}{3}$$

Probability Density Function

- By definition: X is a continuous random variable if there exists a nonnegative function $f_X(x)$ defined for $\forall x \in \mathbb{R}$, such that any probability of the form $P(a \leq X \leq b)$ can be found by:

$\forall a, \forall b : a \leq b$

F_Y

F_W

$$\boxed{P(a \leq X \leq b) = \int_a^b f_X(x)dx}$$

$P(\Omega_x) = 1$

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$$P(a \leq X \leq b) = \int_a^b f_X(x)dx$$

- Normalisation property. Since $P(\Omega_X)$ should be equal to 1, here we have:

$$1 = P\{X \in (-\infty, +\infty)\} = \int_{-\infty}^{+\infty} f_X(x)dx$$

Cumulative Distribution Function

- **Def.** CDF of random variable X is a non-decreasing function $F_X(x)$ defined for $\forall x \in \mathbb{R}$, such that:

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$F_X(t) = P(X < t)$

$\int_{-\infty}^{t} F_X(x)dx$

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x

$x \to u$

- Basic properties (for self-check also):

$$F_X(-\infty) = 0, \qquad F_X(\infty) = 1 = \int_{-\infty}^{+\infty} F_X(u)du = 1$$

CDF as a way to avoid intergration

$$\int_a^b F_X(x)\,dx$$

- Suppose we are interested in $P(a < X < b)$. One way - to compute integral according to the defition of CRV.
- Consider interval $\mathcal{D} = (-\infty, b)$. It can be decoupled in a union of two complementary sets:

$$\mathcal{D} = (-\infty,\ a] \cup (a,\ b)$$

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$$\mathcal{D} = (-\infty,\ a] \cup (a,\ b)$$

- Then according to additivity principle of probability:

$$P\{\mathcal{D}\} = P\{(-\infty,\ a]\} + P\{(a,\ b)\}$$

$$P(X \in \mathcal{D}) = P(X \in (-\infty, a]) + P(X \in (a, b))$$

CDF as a way to avoid intergration

- Return to the CDF formulation:

$$\int_{-\infty}^{b} f_X(u)du = \int_{-\infty}^{a} f_X(u)du + \int_{a}^{b} f_X(u)du$$

- And finally:

$$P(a \leq X \leq b) = \int_{a}^{b} f_X(x)dx = F_X(b) - F_X(a).$$

$$f_X(x) = [F_X(x)]'$$

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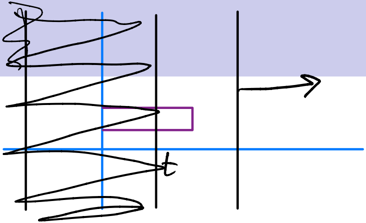
Examples

CDF

$F_X(t) = P(-\infty < X < t)$

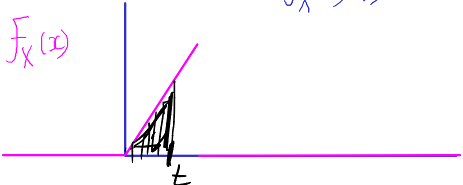


- A random variable X has density function:

$$f_X(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Check that it is valid and find the cumulative distribution function of X.

$f_X(x) \geq 0 \quad \forall x$



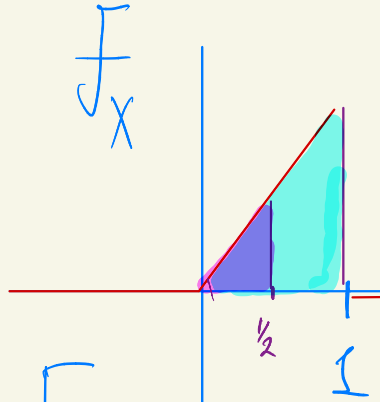
$$\int_{-\infty}^{+\infty} F_X(x)\,dx = 1 = \cancelto{0}{\int_{-\infty}^{0} F\,dx} + \int_0^1 F\,dx + \cancelto{0}{\int_1^{+\infty} F\,dx}$$

$$= \int_0^1 2x\,dx = 2\int_0^1 x\,dx = 2\frac{x^2}{2}\Big|_0^1 =$$

$$= \frac{2\cdot 1^2}{2} - \frac{2\cdot 0^2}{2} = 1$$

$$F_X(t) = \begin{cases} 0, & t<0 \\ t^2, & 0\leq t\leq 1 \\ 1, & t>1 \end{cases}$$

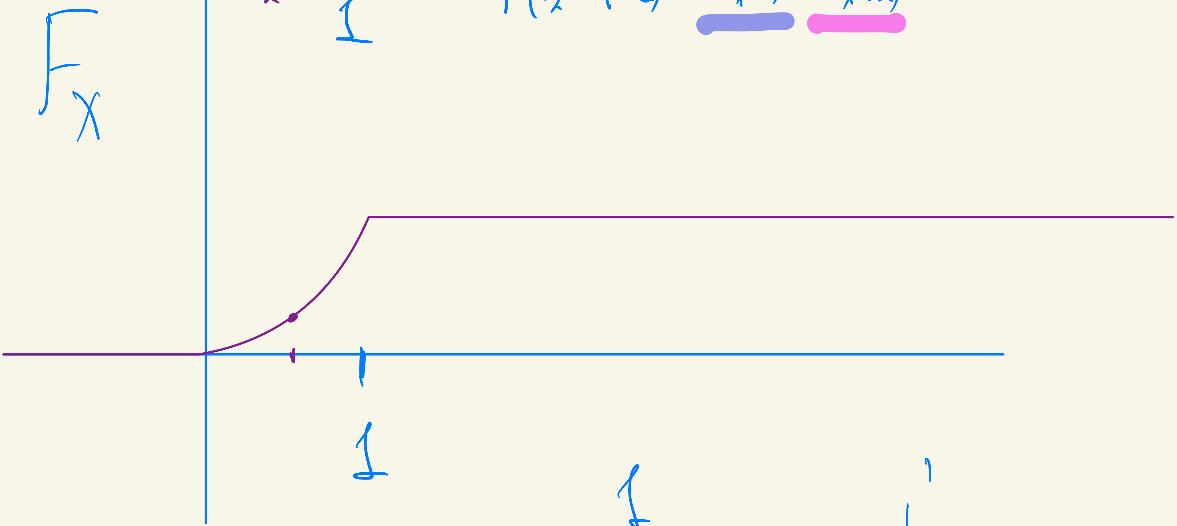
$$\int_0^t 2x\,dx = 2\frac{x^2}{2}\Big|_0^t = \frac{2t^2}{2} - \frac{2\cdot 0^2}{2} = t^2$$



$$\boxed{F_X(1) = P(-\infty < X < 1) = 1}$$

$$\boxed{F_X(1/2) = P(-\infty < X < 1/2) = 1/16}$$

$$P(1/2 < X < 1) = \underline{F_X(1)} - \underline{F_X(1/2)}$$

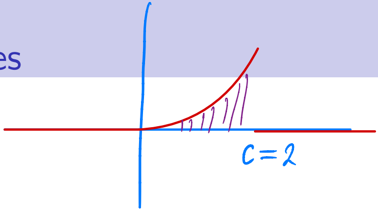


$$P\left(\frac{1}{2} < X < 1\right) = \int_{1/2}^{1} 2x\,dx = x^2\Big|_{1/2}^{1} =$$

$$= 1 - 1/4 = 3/4$$

$1.\ F = \begin{cases} 0, & x \le 0 \\ x^3/4, & 0 < x < c \\ 0, & x \geq c \end{cases}$

Examples



Find the constant c such that $f_X(x)$ is a valid p.d.f of a random variable X, find the cumulative distribution function, $F_X(x)$, and find a **median** of the density function:

- $f(x) = \frac{x^3}{4},\ 0 < x < c$
- $f(x) = \frac{3x^2}{16},\ -c < x < c$

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$$

$$= \int_{-\infty}^{0} \cancel{0}\,dx + \int_{0}^{c} \frac{x^3}{4}\,dx + \int_{c}^{+\infty} \cancel{0}\,dx =$$

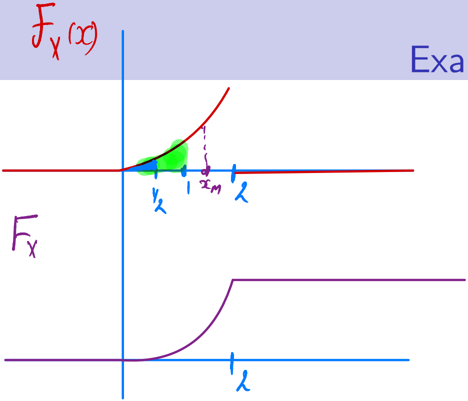
0 0

$$\Rightarrow \int_0^c \frac{x^3}{4}\,dx = 1$$

$$\frac{x^4}{16}\Big|_0^c = \frac{c^4}{16} - 0 = 1$$

$$\mapsto c = 2$$

Examples



$$P(1/2 < X < 1)$$

$$F_X(1) - F_X(1/2) = \frac{1}{16} - \frac{1}{16 \cdot 16}$$

$$F_X(t) = P(-\infty < X < t) =$$

$$= \int_{-\infty}^{t} f_X(x)\,dx$$

$$F_X(t) = \begin{cases} 0, & t \leq 0 \\ t^4/16, & 0 < t < 1 \\ 1, & t \geq 1 \end{cases}$$

$$\int_0^t \frac{x^3}{4}\,dx = t^4/16$$

$$X_m : P(X \leq x_m) = P(X \geq x_m) = 1/2$$

$$F_X(x_m) = 1/2$$

$$x_m^4/16 = 1/2 \rightarrow x_m^4 = 8$$

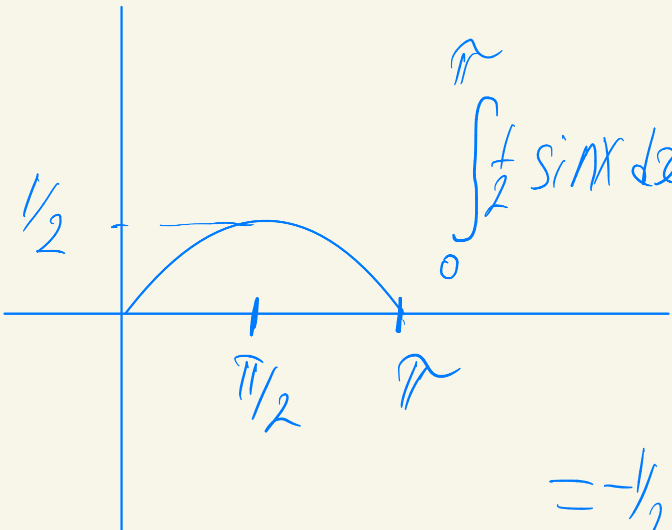
$$x_m = \sqrt[4]{8}$$

$F = 0.5 \sin x \,, \quad x \in [0, \pi]$

$P(X < \pi/3 \mid X < \pi/2)$

$\quad A \qquad\quad B$

$$f_X = \begin{cases} 0, & x < 0 \\ 1/2 \sin x, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$$



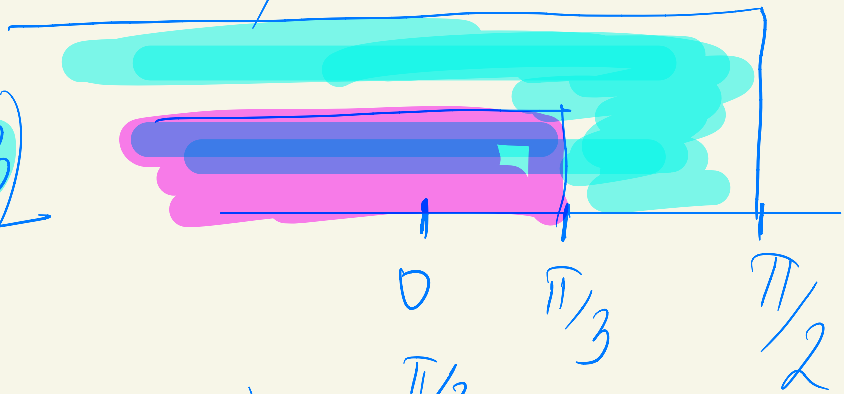
$$\int_0^\pi \frac{1}{2} \sin x \, dx = \frac{1}{2} \cdot (-\cos x) \Big|_0^\pi =$$

$$= -1/2 \left[\cos x \Big|_0^\pi \right] =$$

$$= -1/2 [-1 - 1] = 1.$$

$$P(X \mid Y) = \frac{P(X \cap Y)}{P(Y)}$$

$$\frac{P(A \cap B)}{P(B)}$$



$$= \frac{P(X < \pi/3)}{P(X < \pi/2)} = \frac{\int_{-\infty}^{\pi/3} F_X \, dx}{\int_{-\infty}^{\pi/2} F_X \, dx} = \frac{1/4}{1/2} = \frac{1}{2}$$

$$\int_0^{\pi/3} \tfrac{1}{2} \sin x = -\frac{1}{2}\left[\cos x \Big|_0^{\pi/3}\right] = -\tfrac{1}{2}\left(\tfrac{1}{2} - 1\right) = \tfrac{1}{4}$$

$$\hookrightarrow -\tfrac{1}{2}\left[\cos x \Big|_0^{\pi/2}\right] = -\tfrac{1}{2}\left[0 - 1\right] = \tfrac{1}{2}$$

$$F_X(x) = \begin{cases} \dfrac{10}{x^2}, & x \geqslant 10 \\ 0, & x < 10 \end{cases}$$

$$\int_{-\infty}^{+\infty} F_X(x)\,dx = \int_{-\infty}^{10} 0\,dx \quad 0$$

$$+ \int_{10}^{+\infty} \frac{10}{x^2}\,dx = -10\left[\frac{1}{x}\Big|_{10}^{+\infty}\right] =$$

$$\int x^{\alpha} = \frac{x^{\alpha+1}}{\alpha+1}$$

$$= -10\left[0 - \frac{1}{10}\right] =$$

$$= \frac{10}{10} = 1.$$