Probability Theory and Mathematical Statistics

Probability Space. Classical Probability.

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Description of a Random Experiment

Sample Space

- A set that contains all possible *unique* outcomes of a random experiment:

$$\Omega = \{\omega_1, \omega_2, \omega_n, ...\}$$

- May be finite or infinite
- Critically depends on the setup of the random experiment and what we want to observe and describe

Description of a Random Experiment

Sample Space: Examples

Let us imagine that we toss:

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- Two identical coins: $\Omega = \{2H, 2T, Diff\}$
- Identical coins with artifically introduced distinction: $\Omega = \{Hh, Tt, Ht, Th\}$
- A die with letter faces: $\Omega = \{a, b, c, d, e, f\}$

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- An important distinction from a simple outcome: an outcome occurs *one and only one time*, but many events can occur simultaneously!
- > **Important!**
  >
  > We say that an *event has occurred* if one of the simple outcomes of that event has been realized.

Description of a Random Experiment

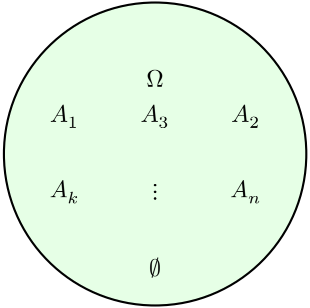
Event Space

- A set that contains all possible *events* that can be constructed from the set of outcomes Ω
- In other words, the set of all possible subsets of Ω:

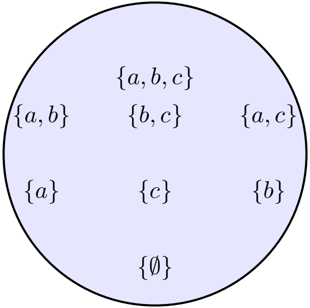
$$\mathcal{F} = \{A_1,\, A_2,\, A_n,\, ...\},\; \forall A_i \subset \Omega$$

- The sample space itself is also an event: $\Omega \subset \Omega$, therefore $\Omega \in \mathcal{F}$

($\mathcal{F}$, in general)



(Example $\mathcal{F}$: all subsets of $\{a, b, c\}$)



Description of a Random Experiment

Event Space

Question

If the cardinality of the sample space is $|\Omega| = n$, what will be the cardinality of the event space $|\mathcal{F}|$?

Description of a Random Experiment

Probability

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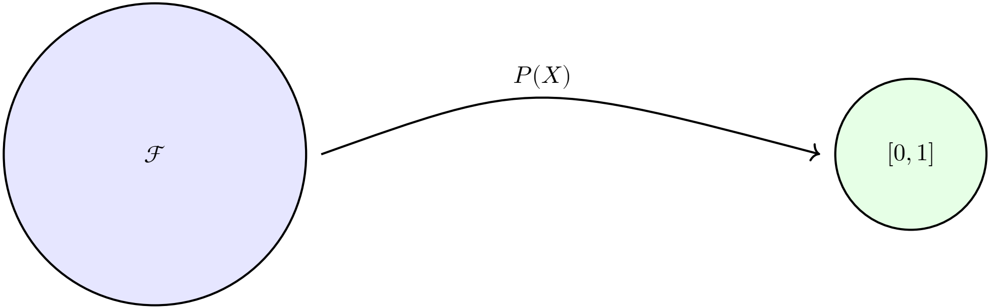

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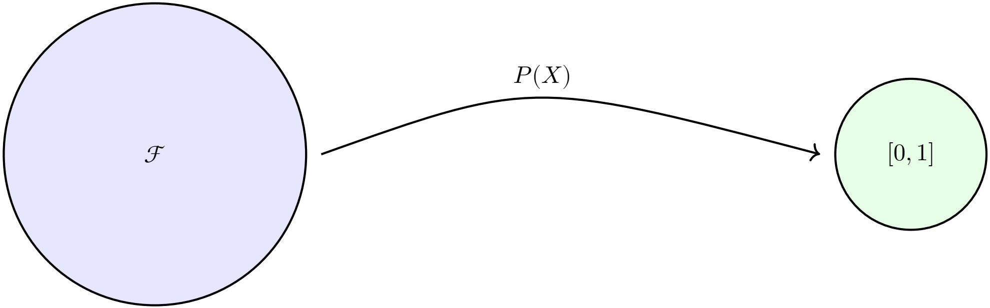

Description of a Random Experiment

Probability

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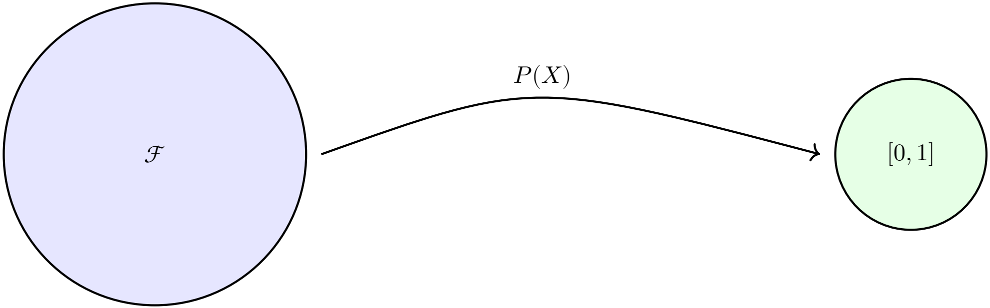
- This coefficient of our confidence is measured from 0 to 1, and we call it *probability*
- Formally, probability is a function that assigns a number to each event:

$$P(X) : \mathcal{F} \longrightarrow [0,1] \subset \mathbb{R}$$



Properties of the Probability Function

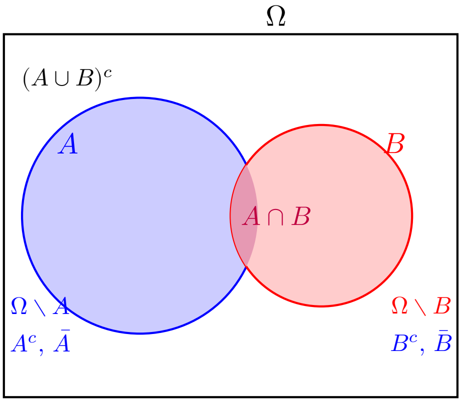
- $0 \leq P(X) \leq 1, \ \forall X \in \mathcal{F}$
- $P(\Omega) = 1, \ P(\emptyset) = 0$
- *Additivity property of probability*:

$$\forall X, Y \in \mathcal{F} : X \cap Y = \emptyset, \ P(X \cup Y) = P(X) + P(Y)$$

More generally: If a collection of events A_1, A_2, ... are pairwise disjoint, i.e., $A_i \cap A_j = \emptyset$, $i \neq j$, then:

$$P\left(\bigcup_{i=1}^{m} A_i\right) = \sum_{i=1}^{m} P\left(A_i\right)$$

Complementary Events and the Inclusion-Exclusion Formula



- **Property**: If $A \in \mathcal{F}$, then $P(A) + P(\Omega \backslash A) = 1$.

Events A and $\bar{A} = \Omega \backslash A$ are such that $A \cap \bar{A} = \emptyset$, and $\Omega = A \cup \bar{A}$, from which using the properties of probability we obtain: $P(\Omega) = P(A \cup \bar{A}) = P(A) + P(\Omega \backslash A) = 1$

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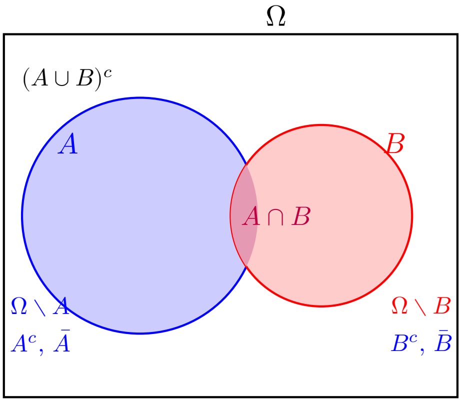

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- **Inclusion-Exclusion Formula**: If $A, B \in \mathcal{F}$, then

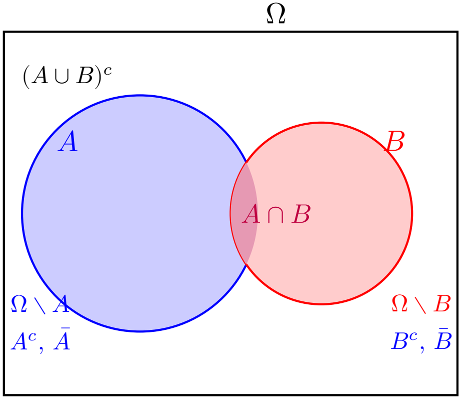
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Event A is the union of disjoint sets $A \backslash B$ and $A \cap B$, hence $P(A) = P(A \backslash B) + P(A \cap B)$ and symmetrically $P(B) = P(B \backslash A) + P(A \cap B)$.

$$\begin{aligned} P(A) + P(B) &= P(A \backslash B) + 2P(A \cap B) + P(B \backslash A) \\ &= P((A \backslash B) \cup (A \cap B) \cup (B \backslash A)) + P(A \cap B) \\ &= P(A \cup B) + P(A \cap B). \end{aligned}$$

Probability

Complementary Events and the Inclusion-Exclusion Formula



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- **Property**: If $A, B \in \mathcal{F}$ and $A \subseteq B$, then $P(A) \leq P(B)$. Easy to show: $P(B) = P(A) + P(B \backslash A) \geq P(A)$.

Classical Probability

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- Let us formalize: $\Omega = \{\omega_1, \ldots, \omega_n\}$, $P(\{\omega_1\}) = \ldots = P(\{\omega_n\}) = p$.
- Let us apply the properties of probability to obtain an intuitively clear result about the probability of a simple outcome:

$$\Omega = \{\omega_1\} \cup \cdots \cup \{\omega_n\} = \bigcup_{i=1}^{n} \{\omega_i\}$$

$$P(\Omega) = P\left(\bigcup_{i=1}^{n} \{\omega_i\}\right) = \sum_{i=1}^{n} P(\{\omega_i\}) = n \cdot p = 1 \to p = \frac{1}{n}$$

Classical Probability

First Important Formula

- Without loss of generality, assume that some event A consists of $k \leq n$ simple outcomes, $A = \{\omega_1, \ldots, \omega_k\}$. We are interested in $P(A)$.

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- Let us derive the formula for the probability of A:

$$A = \{\omega_1\} \cup \cdots \cup \{\omega_k\} = \bigcup_{i=1}^{k} \{\omega_i\}$$

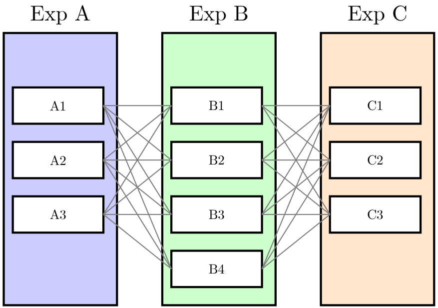
$$P(A) = P\left(\bigcup_{i=1}^{k} \{\omega_i\}\right) = \sum_{i=1}^{k} P(\{\omega_i\}) = k \cdot p = \frac{k}{n} = \frac{|A|}{|\Omega|}$$

Combinatorial Probability

Basic Principle of Counting

Definition

If r experiments are to be performed such that the first can result in any of n_1 possible outcomes; and if for each of these n_1 possible outcomes there are n_2 possible outcomes of the second experiment; and if for each of the possible outcomes of the first two experiments there are n_3 possible outcomes of the third experiment; and if ..., then there are a total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the r experiments.



Permutations

- **Motivation**: How many different ordered sequences can be formed from the set $\{a, b, c\}$

Combinatorial Probability

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- $P_n^r = n \cdot (n-1) \cdot (n-2) \cdot ... \cdot (n-r+1)$

Permutations: Examples

- **Question**

  Random number sampling. Let the population consist of ten digits 0, 1, ..., 9. Each sequence of five digits represents a sample of size $r = 5$. What is the probability that five consecutive random digits are all different?

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- $P = \frac{P_{10}^5}{10^5} = 0.3024.$

Combinatorial Probability

Combinations

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- In general, since $n(n-1)\cdots(n-k+1)$ represents the number of different ways in which a group of k elements can be chosen from n elements when the order of selection is important, and since each group of k elements will be counted $k!$ times in this count, it follows that the number of different groups of k elements that can be formed from a set of n elements is

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- **Definition**

  We define C_n^k, for $k \leq n$, as

  $$C_n^k = \frac{n!}{(n-k)!k!},$$

  and say that C_n^k represents the number of possible combinations (collections) of size k obtained from n objects. The order of elements in this case is not considered important.

Combinations: Example

Problems often fit the model of drawing balls from a bag.

Example: A bag contains 15 balls, of which 10 are red and 5 are white. Four balls are selected from the bag. There is an ambiguity here: for example, if in one draw I select four red balls, and in another draw I select four different red balls, are these samples considered the same or not?

We will assume that these are not the same samples. For example, we can imagine that the balls are numbered, so we can distinguish balls of the same color. This way of thinking is very useful for calculating probabilities in the classical scheme.

Combinatorial Probability

Combinations: Example

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- > **Question**
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  > How many samples (of size 4) consist entirely of red balls?
- Order is not important, but numbers are important, we choose 4 elements from the set of red balls. Therefore, the answer is: $C_{10}^{4} = 210$.

Combinatorial Probability

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  How many samples contain 2 red and 2 white balls?
- We can choose 2 numbered red balls in C_{10}^2 ways and 2 numbered white balls in C_5^2 ways. Neither choice affects the other, so the answer is: $C_{10}^2 \cdot C_5^2 = 45 \cdot 10 = 450$.

Combinatorial Probability

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  > How many samples contain 2 red and 2 white balls?
- We can choose 2 numbered red balls in C_{10}^{2} ways and 2 numbered white balls in C_{5}^{2} ways. Neither choice affects the other, so the answer is: $C_{10}^{2} \cdot C_{5}^{2} = 45 \cdot 10 = 450$.
- > **Question**
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  > How many samples (of size 4) contain exactly 3 red balls?
- We can choose 3 numbered red balls in C_{10}^{3} ways and 1 numbered white ball in C_{5}^{1} ways. Neither choice affects the other, so the answer is: $C_{10}^{3} \cdot C_{5}^{1} = 120 \cdot 5 = 600$.

Combinations: Example

- **Question**

  How many samples (of size 4) contain at least 3 red balls?

Combinatorial Probability

Combinations: Example

- **Question**

  How many samples (of size 4) contain at least 3 red balls?

- This is the number of samples with 3 red plus the number of samples with 4 red. We can choose 4 numbered red balls in C_{10}^{4} ways and 0 numbered white balls in C_{5}^{0} ways. Neither choice affects the other, so the answer is: $C_{10}^{4} \cdot C_{5}^{0} = 210 \cdot 1 = 210$. From the previous example, there are 600 ways to choose samples with exactly 3 red balls, so our answer is: $600 + 210 = 810$.

Combinatorial Probability

Mixed Counting Problems

- 10 red, 5 white, numbered balls

This is also the total number of samples (1365) minus the number of samples without red balls, which equals $C_{10}^{0} \cdot C_{5}^{4} = 5$.

Combinatorial Probability

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Mixed Counting Problems

- 10 red, 5 white, numbered balls
- > **Question**
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  > How many samples (of size 4) contain at least one red ball?
- This is $C_{10}^{1} \cdot C_{5}^{3} + C_{10}^{2} \cdot C_{5}^{2} + C_{10}^{3} \cdot C_{5}^{1} + C_{10}^{4} \cdot C_{5}^{0}$, which equals $10 \cdot 10 + 45 \cdot 10 + 120 \cdot 5 + 210 \cdot 1 = 100 + 450 + 600 + 210 = 1360$.

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Geometric Probability

Definition

Geometric probability applies to experiments where the sample space Ω is a subset of Euclidean space (line segment, area, volume).

Basic Formulas

For the one-dimensional case (line segment):

$$P(A) = \frac{\text{length of set } A}{\text{length of set } \Omega}$$

For the two-dimensional case (area):

$$P(A) = \frac{\text{area of set } A}{\text{area of set } \Omega}$$

For the three-dimensional case (volume):

$$P(A) = \frac{\text{volume of set } A}{\text{volume of set } \Omega}$$

Geometric Probability

Classical Geometric Probability Problems

1. **Meeting Problem**: Two people agree to meet at a certain place between 12:00 and 13:00. Each arrives at a random time and waits 15 minutes. What is the probability of meeting?

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3. **Random Point in a Circle Problem**: A point is chosen randomly inside a circle of radius R. What is the probability that the distance from the point to the center is less than r?