

# Linear Algebra

Matrices and Vectors. First Introduction, Basic Operations.

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# Matrix

- A **matrix** is an ordered array of numbers arranged in  $n$  rows and  $m$  columns.

$$A_{n \times m} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

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- We usually denote it as  $A_{n \times m}$  or, to emphasize the nature of numbers in the matrix, we write  $A \in \mathbb{R}^{n \times m}$ .
- If  $n = m$ , the matrix is called square; if  $n \neq m$ , it's called rectangular

## Basic operations: transpose

**Transpose** of a matrix is an operation where rows and columns are swapped. If  $A \in \mathbb{R}^{n \times m}$ , then  $B = A^T \in \mathbb{R}^{m \times n}$ , where  $b_{ij} = a_{ji}$

$$\begin{array}{c} \left[ \begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{array} \right] \xrightarrow{A^T} \left[ \begin{array}{cc} a_{11} & a_{21} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{array} \right] \\ A_{2 \times 3} \qquad \qquad \qquad B_{3 \times 2} = A^T \end{array}$$

## Basic operations: matrix addition

- **Matrix addition** is only possible for matrices of the same size. The result is obtained by adding corresponding elements. If  $A, B \in \mathbb{R}^{n \times m}$ , then  $C = A + B$ , where  $c_{ij} = a_{ij} + b_{ij}$

**Example:**

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

## Basic operations: scalar multiplication

- **Scalar multiplication** - each element of the matrix is multiplied by the given number. If  $A \in \mathbb{R}^{n \times m}$  and  $\alpha \in \mathbb{R}$ , then  $C = \alpha A$ , where  $c_{ij} = \alpha \cdot a_{ij}$

**Example:**

$$3 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 \cdot 1 & 3 \cdot 2 \\ 3 \cdot 3 & 3 \cdot 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

**Another example - Linear combination:**

$$2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$$

# Vector

In the simplest representation, we treat a **vector** as a special case of a matrix:

Column vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^{n \times 1}$$

Row vector

$$\mathbf{y}^T = [y_1 \quad y_2 \quad \cdots \quad y_m] \in \mathbb{R}^{1 \times m}$$

**Dimension:**  $1 \times m$  (matrix with one row)

**Dimension:**  $n \times 1$  (matrix with one column)

Notation

- **Vectors:** usually denoted by lowercase letters  $x, v$  or  $\mathbf{u}$

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- **Matrices:** usually denoted by uppercase letters  $A, B, C$
- **By default:** vector is considered a **column vector**
- **Transpose:**  $\mathbf{x}^\top$  converts column to row

## Matrix-by-vector multiplication (matvec)

Standard approach

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,m-1} & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2,m-1} & a_{2m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,m-1} & a_{nm} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{m-1} \\ x_m \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$A$                      $\mathbf{x}$                      $\mathbf{y}$

$$y_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1,m-1}x_{m-1} + a_{1m}x_m$$

$$y_j = \sum_{k=1}^m a_{jk}x_k$$

## Matrix-by-vector multiplication (matvec)

Computational complexity and a brief look at parallel computing

Recall the general formula:

$$y_j = \sum_{k=1}^m a_{jk} x_k, \quad j = 1, 2, \dots, n$$

Operations analysis:

- For computing one element  $y_j$ :  $m$  multiplications (each  $a_{jk} \cdot x_k$ ),  $m - 1$  additions (summing  $m$  products)
- For the entire vector  $\mathbf{y}$  ( $n$  elements):  $n \cdot (2m - 1)$  total operations.

Time complexity:

- For square matrix  $n \times n$ :  $\mathcal{O}(2n^2 - n) = \mathcal{O}(n^2)$
- For rectangular matrix  $n \times m$ :  $\mathcal{O}(nm)$

 Natural parallelism

Computing each element  $y_j$  is **independent** of other elements!

- **Row-wise parallelization:** each processor computes its own  $y_j$

## Matrix-by-vector multiplication (matvec)

## Guru approach

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

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$$\left[ \begin{array}{cccc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array} \right] A = \left[ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \right] \mathbf{x} = \left[ \begin{array}{c} y_1 \\ y_2 \\ y_3 \end{array} \right] \mathbf{y}$$

$$y_1 = \color{red}{a_{11}x_1} + \color{blue}{a_{12}x_2} + \color{green}{a_{13}x_3} + \color{orange}{a_{14}x_4}$$

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$A$                                      $\mathbf{x}$                                      $\mathbf{y}$

$$y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4$$

$$y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4$$

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$$y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4$$

$$y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4$$

$$y_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4$$

## Matrix-by-vector multiplication (matvec)

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$$\mathbf{y} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} + x_4 \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix}$$

## Matrix-by-vector product

Visual comparison

$$\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

$A$                      $\mathbf{x}$                      $\mathbf{y}$

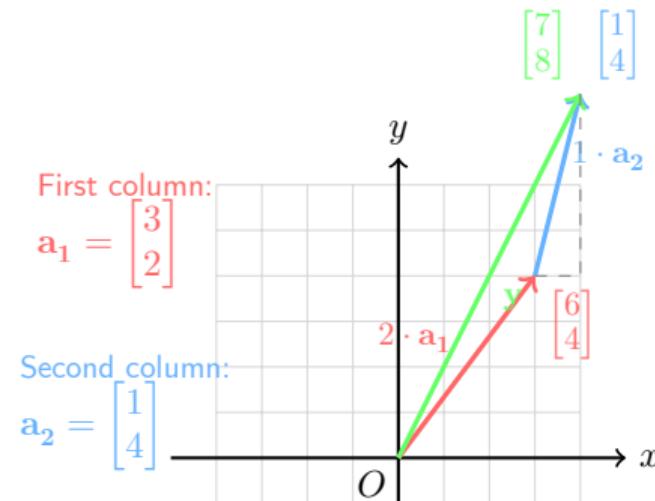
Traditional View: Row-wise

Calculations:

- $y_1 = 3 \cdot 2 + 1 \cdot 1 = 7$
- $y_2 = 2 \cdot 2 + 4 \cdot 1 = 8$

Guru View: Column Linear Combination

Linear combination:



# Matrix-by-matrix multiplication (matmul, General Matrix Multiplication - GEMM)

Standard approach

$$\begin{array}{c} \text{\color{red} i-th row} \rightarrow \\ \left[ \begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ik} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{array} \right] \end{array} \quad \begin{array}{c} \text{\color{blue} j-th column} \\ \left[ \begin{array}{cccc} b_{11} & \cdots & \color{blue}{b_{1j}} & \cdots & b_{1m} \\ b_{21} & \cdots & \color{blue}{b_{2j}} & \cdots & b_{2m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{k1} & \cdots & \color{blue}{b_{kj}} & \cdots & b_{km} \end{array} \right] \end{array} = \begin{array}{c} B \\ \left[ \begin{array}{cccc} c_{11} & \cdots & c_{1j} & \cdots & c_{1m} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{i1} & \cdots & \color{green}{c_{ij}} & \cdots & c_{im} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ c_{n1} & \cdots & c_{nj} & \cdots & c_{nm} \end{array} \right] \end{array}$$

A

C

$$c_{ij} = \color{red}{a_{i1}} \cdot \color{blue}{b_{1j}} + \color{red}{a_{i2}} \cdot \color{blue}{b_{2j}} + \cdots + \color{red}{a_{ik}} \cdot \color{blue}{b_{kj}}$$

$$c_{ij} = \sum_{l=1}^k a_{il} b_{lj}$$

## Matrix multiplication (matmul, General Matrix Multiplication - GEMM)

## Column-wise approach

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ik} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nk} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1m} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{k1} & b_{k2} & b_{k3} & \cdots & b_{km} \end{bmatrix} \quad C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{i1} & c_{i2} & c_{i3} & \cdots & c_{im} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{n1} & c_{n2} & c_{n3} & \cdots & c_{nm} \end{bmatrix}$$

Each column of  $C = A \times$  corresponding column of  $B$

$$\mathbf{C}_j = A\mathbf{B}_j, \quad j = 1, 2, \dots, m$$

## Matrix multiplication (matmul, General Matrix Multiplication - GEMM)

Properties of matrix multiplication

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$$(AB)C = A(BC)$$

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3. **Scalar multiplication:**

$$\alpha(AB) = (\alpha A)B = A(\alpha B)$$

## Matrix multiplication (matmul, General Matrix Multiplication - GEMM)

Important limitations

### 4. Non-commutativity:

$$AB \neq BA \quad (\text{in general})$$

**Example:** For  $2 \times 2$  matrices:

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

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### 5. Transpose of product:

$$(AB)^T = B^T A^T$$

**i** Note

Note the **reversed order** of matrices when transposing!

# Matrix multiplication (matmul, General Matrix Multiplication - GEMM)

Computational complexity and parallelization

Recall the general formula:

$$c_{ij} = \sum_{l=1}^k a_{il} b_{lj}, \quad i = 1, 2, \dots, n, \quad j = 1, 2, \dots, m$$

Operations analysis for matrices  $A_{n \times k}$  and  $B_{k \times m}$ :

- For computing one element  $c_{ij}$ :  $k$  multiplications,  $k - 1$  additions
- Total number of operations:  $n \times m \times (2k - 1)$

Time complexity:

- For square matrices  $n \times n$ :  $\mathcal{O}(2n^3 - n^2) = \mathcal{O}(n^3)$
- For rectangular matrices:  $\mathcal{O}(nmk)$

💡 Natural parallelism

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- **By rows:** each processor handles its rows of matrix  $A$

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## 💡 Natural parallelism

- **By elements:** each  $c_{ij}$  is computed independently
- **By rows:** each processor handles its rows of matrix  $A$
- **By columns:** each processor handles its columns of matrix  $B$
- **Block approach:** dividing matrices into blocks for efficient cache usage

## Example. Simple but important idea about matrix computations.

Suppose you have the following expression

$$b = A_1 A_2 A_3 x,$$

where  $A_1, A_2, A_3 \in \mathbb{R}^{3 \times 3}$  are random square dense (fully filled with numbers) matrices, and  $x \in \mathbb{R}^3$  is a vector. You need to compute  $b$ .

Which approach is best to use?

1.  $A_1 A_2 A_3 x$  (left to right)
2.  $(A_1 (A_2 (A_3 x)))$  (right to left)
3. It doesn't matter
4. The results of the first two options will not be the same.

Check the attached .ipynb file in the repository.