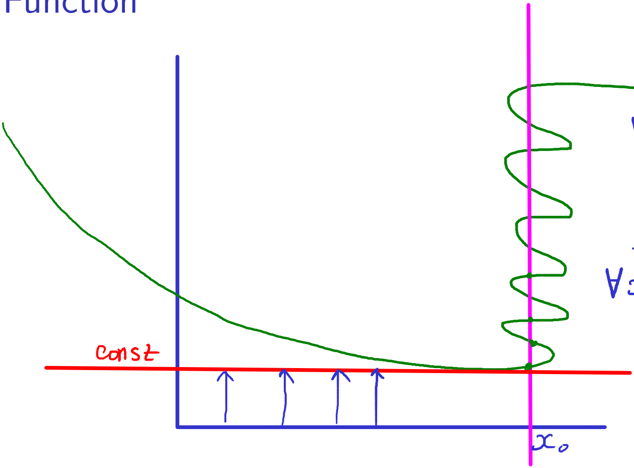
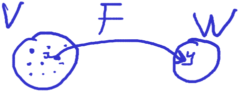
FCS HSE, MDS, Linear Algebra

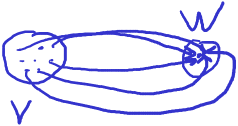
Function





<u>law</u> / <u>rule</u>:

$\forall x \in V \ \exists ! \ y \in W$



Functions in Linear Algebra

V, W - Vector space

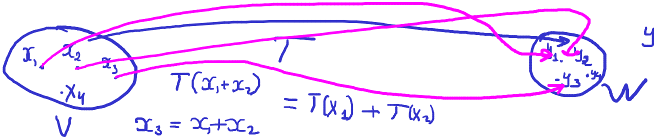
Functions between vector spaces

Definition. Let V, W be vector spaces. A transformation $T : V \to W$ is called linear if

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \forall \mathbf{u}, \mathbf{v} \in V$
2. $T(\alpha\mathbf{v}) = \alpha T(\mathbf{v})$ for all $\mathbf{v} \in V$ and for all scalars $\alpha \in \mathbb{R}$.

Properties 1 and 2 together are sometimes combined into the following one:

$$T(\alpha\mathbf{u}+\beta\mathbf{v}) = \alpha \underline{T(\mathbf{u})} + \beta \underline{T(\mathbf{v})}, \quad \forall \mathbf{u}, \mathbf{v} \in V, \quad \forall \alpha, \beta \in \mathbb{R}.$$



$y_3 = y_1 + y_2$

$T(x_1+x_2) = T(X_1) + T(X_2)$

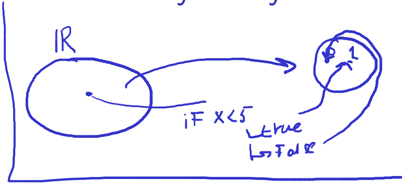
$x_3 = x_1 + x_2$

$X_4 = \alpha \cdot X_1 \qquad T(X_4) = \alpha \cdot T(X_1) = y_4$

We need a way to write a transformation analytically

$\mathbb{R} \to \mathbb{R}$
$x_1 \to y_1$
$x_2 \to y_2$
$\vdots$

$y = x^2 + 1$
$0 \to 1$
$1 \to 2$
$\vdots \to \vdots$



$y = \dots x$, $x \in V$
$y \in W$

Matrix representation of a linear transformation

$\forall x \in V = x_1 \cdot e_1 + \ldots + x_n e_n$

$\forall y \in W = y_1 \cdot \tilde{e}_1 + \ldots + y_m \cdot \tilde{e}_m$

Let us assume $T : V \to W$, vectors $e_1, \ldots, e_n$ be a standard basis in V, and vectors $\tilde{e}_1, \ldots, \tilde{e}_m$ be a basis in W.

We would like to investigate how T acts on any $x \in V$.

$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\qquad x = (x_1, \ldots, x_n)^T$

$$x = x_1 e_1 + \ldots + x_n e_n,$$

$$T(x) = T(x_1 e_1 + \ldots + x_n e_n) = x_1 T(e_1) + \ldots + x_n T(e_n).$$

$T(x) \in W$ $\qquad = T(x_1 e_1 + \ldots + x_n e_n) =$

Keep in mind, that $T(e_1), \ldots, T(e_n)$ are *vectors*, *i.e.* abstract citizens of the vector space W.

$T(e_1) \in W$ $\qquad \ldots \qquad T(e_n) \in W$

$T(e_2) \in W$

W: $T(e_1)$, $T(e_2)$

Matrix representation of a linear transformation

Let us look at them from standard basis in W:

$(x = \alpha_1 v_1 + \ldots + \alpha_n v_n)$

$\boxed{T(e_1) \in W}$

$$\begin{aligned}
\boxed{T(e_1)} &= a_{11}\tilde{e}_1 + a_{21}\tilde{e}_2 + \ldots + a_{m1}\tilde{e}_m, \\
T(e_2) &= a_{12}\tilde{e}_1 + a_{22}\tilde{e}_2 + \ldots + a_{m2}\tilde{e}_m, \\
&\vdots \\
T(e_n) &= a_{1n}\tilde{e}_1 + a_{2n}\tilde{e}_2 + \ldots + a_{mn}\tilde{e}_m
\end{aligned}$$

$\tilde{e}$ -basis in W

Then get back to $T(x) = x_1 T(e_1) + \ldots + x_n T(e_n)$.

$$\begin{aligned}
T(x) = {} & x_1\left(a_{11}\underline{\tilde{e}_1} + \ldots + a_{m1}\tilde{e}_m\right) + x_2\left(a_{12}\underline{\tilde{e}_1} + \ldots + a_{m2}\tilde{e}_m\right) \\
& + \ldots + x_n\left(a_{1n}\underline{\tilde{e}_1} + \ldots + a_{mn}\tilde{e}_m\right)
\end{aligned}$$

Matrix representation of a linear transformation

$T(x) \in W$

$$\begin{aligned} T(x) = & \; x_1 \left(a_{11}\tilde{e}_1 + \ldots + a_{m1}\tilde{e}_m\right) + \boxed{x_2 \left(a_{12}\right.}\tilde{e}_1 + \ldots + a_{m2}\tilde{e}_m) \\ & + x_n \left(a_{1n}\tilde{e}_1 + \ldots + a_{mn}\tilde{e}_m\right) \end{aligned}$$

$$= \underbrace{\boxed{\left(a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n\right)}}_{\gamma_1}\tilde{e}_1 + \overbrace{\boxed{\left(a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n\right)}}^{\gamma_2}\tilde{e}_2 + \underbrace{\boxed{\left(a_{m1}x_1 + a_{12}x_2 + \ldots + a_{mn}x_n\right)}}_{\gamma_m}\tilde{e}_m$$

$T(x) = \gamma_1 \cdot \tilde{e}_1 + \gamma_2 \tilde{e}_2 + \ldots + \gamma_m \tilde{e}_m$

$T(x) \in W$ $\qquad$ $\tilde{e}$ - basis in W

Matvec... again...

Finally:

$$[T(x)]_{\tilde{e}} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = A_T\,[x]_e\,.$$

Huh!

If indeed e and $\tilde{e}$ are standard bases then

To construct matrix A_T of a linear transformation T we need just to know images of basis vectors: $T(e_1), \dots, T(e_n)$, *i.e.*

$$e_1 \xrightarrow{T} \underset{T(e_1)}{a_1} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, e_2 \xrightarrow{T} \underset{T(e_2)}{a_2} = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \dots, e_n \xrightarrow{T} \underset{T(e_n)}{a_n} = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Matrix representation of a linear transformation

Sweet and easy for any $x \in V$, $x = (x_1, \dots, x_n)^\top$:

$$x = x_1 e_1 + \dots + x_n e_n,$$
$$T(x) = x_1 T(e_1) + \dots + x_n T(e_n).$$

Then we want to observe $T(x)$ in standard basis $\tilde{e}$ as well:

$$[T(x)] = x_1 a_1 + \dots + x_n a_n$$
$$= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

$A - ?$

$$A = \left[a_1 \middle| a_2 \middle| \ldots \middle| a_n \right]$$

$a_1 = T(e_1)$

$a_2 = T(e_2)$

Ex $T : \mathbb{R}^2 \to \mathbb{R}^2$

$$\boxed{T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 2x_1 - 5x_2 \end{pmatrix}}$$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix} \to \begin{pmatrix} 5 \\ -8 \end{pmatrix}$ $\qquad$ $\begin{pmatrix} 0 \\ 10 \end{pmatrix} \to \begin{pmatrix} 20 \\ -50 \end{pmatrix}$

$A-?$ $a_1 = T(e_1) = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$; $\overset{a_2 \,=}{T(e_2)} = T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \end{pmatrix} \Rightarrow \boxed{A = [a_1 | a_2]} = \boxed{\begin{pmatrix} 1 & 2 \\ 2 & -5 \end{pmatrix}}$

$A \cdot x = \begin{pmatrix} 1 & 2 \\ 2 & -5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 + 2 \cdot 2 \\ 2 - 5 \cdot 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -8 \end{pmatrix}$

$x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 \\ 2 & -5 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 10 \end{pmatrix} = \begin{pmatrix} 20 \\ -50 \end{pmatrix}$

$\boxed{\text{Ex } 2}$

$$\mathbb{R}^3 \xrightarrow{\quad} \mathbb{R}^2$$

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ 2x_3 - x_1 \end{pmatrix}$$

$$T\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$$

$$T(x) = T\left(x_1 \cdot e_1 + x_2 \cdot e_2 + x_3 e_3\right) =$$

$$= x_1 \cdot T(e_1) + x_2 \cdot T(e_2) + x_3 \cdot T(e_3) =$$

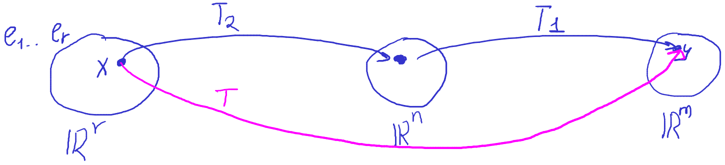
$$T(e_1) = T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad = x_1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} + x_2 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_3 \cdot \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$T(e_2) = T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 2 \end{pmatrix}$$

$$T(e_3) = T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

Matrix form of transformations' composition



$$T(x) = T_1(T_2(x))$$

T_2	B
T_1	A
T	C -?

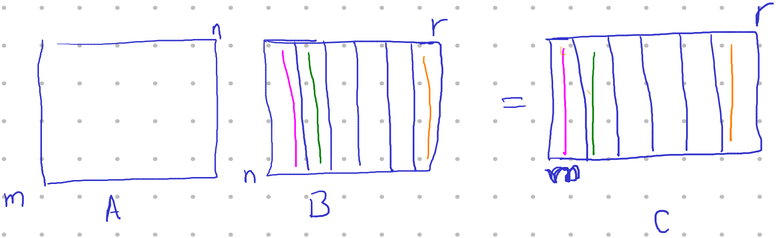
$$C = \left[T(e_1) \,|\, T(e_2) \,|\, \ldots \,|\, T(e_r)\right]$$

$$T(e_i) = T_1(T_2(e_i)) = T_1(\underbrace{B \cdot e_i}) = T_1(B_i) = A \cdot B_i;$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}$$

$$\left(\text{—//—} \right) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 0 \end{pmatrix}$$

$$C = \left[A \cdot B_1 \,|\, A \cdot B_2 \,|\, \ldots \,|\, A \cdot B_r\right]$$



m
n
A
n
r
B
=
r
m
C

1

$$\begin{array}{ccc} \text{dim } n & & \text{dim } m \\ V & \xrightarrow{\quad T \quad} & W \end{array}$$

I_W - st. basis in W

$x \in V \qquad x = (x_1, \dots, x_n)^T$

I_V - st. basis in $V \qquad I_V = {}_{n}\begin{pmatrix} 1 & & 0 \\ 0 & \ddots & \\ 0 & 0 & 1 \end{pmatrix}^{n}$

$$x = x_1 \cdot e_1 + x_2 e_2 + \dots + x_n e_n$$

$$T(x) = T(\Sigma x_i e_i) = x_1 \cdot T(e_1) + \dots + x_n \cdot T(e_n) =$$

$T(e_i)$ - vector from W

Need to know each $T(e_i)$

$T(e_1) = a_1 \in W$

$\vdots$

$T(e_n) = a_n \in W$

$$I_W = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}_{m \times m}$$

$a_1 = (a_{1_1} \; \dots \; a_{1_m})^T$

$[a_1]_{I_W} = (a_{11} \; \dots \; a_{1m})^T$

$= x_1 \cdot a_1 + x_2 \cdot a_2 + \dots + x_n \cdot a_n$

$$= \left[a_1 \middle| a_2 \middle| \dots \middle| a_n \right] \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A_T \cdot [x]$$

Whar if bases are not standard?

$$[y] = A_T \cdot [x] \qquad x \in V \to y \in W$$

$$\underline{[y]_C = L_T \; [x]_B}$$

$$[y] = C \cdot [y]_C \qquad [x] = B \cdot [x]_B$$

$$[y] = A_T \cdot [x]$$

$$\underbrace{C^{-1} C}_{I} \cdot [y]_C = C^{-1} A_T \cdot B \cdot [x]_B$$

$$\underline{[y]_C = \underbrace{C^{-1} A_T B}_{L_T} [x]_B}$$

What if bases are not standard?

$$\boxed{L_T = C^{-1} A_T B}$$

The new form of linear transformation' matrix
If we change from standard basis to basis B
in the domain space
And from standard basis to basis C
in the target space

Examples

$\boxed{\text{Ex 1}}$ a) $\underset{V}{\mathbb{R}^3} \longrightarrow \underset{W}{\mathbb{R}^3}$

$\forall x \in V$ $\quad \varphi(x) = y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2+1 \\ x_3+2 \end{pmatrix}$

$x = (x_1, x_2, x_3)^T$

1 check the linearity

$$\varphi(\alpha x) = \alpha \cdot \varphi(x)$$

$\alpha x = (\alpha x_1, \alpha x_2, \alpha x_3)^T$

$$\varphi(\alpha x) = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 + 1 \\ \alpha x_3 + 2 \end{pmatrix} \neq \alpha \cdot \varphi(x) = \begin{pmatrix} \alpha x_1 \\ \alpha (x_2+1) \\ \alpha (x_3+2) \end{pmatrix}$$

not Linear

b.

$x = (x_1, x_2, x_3)^T$

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \varphi(x) = \begin{pmatrix} x_2 + x_3 \\ 2x_1 + x_3 \\ 3x_1 - x_2 \end{pmatrix}$$

1. Check the linearity

$\forall\, u, v \in V \;:\; \varphi(u+v) = \varphi(u) + \varphi(v)$

$u = (u_1, u_2, u_3)^T \;;\; v = (v_1, v_2, v_3)^T$

$$\varphi\begin{pmatrix} u_1+v_1 \\ u_2+v_2 \\ u_3+v_3 \end{pmatrix} = \begin{pmatrix} u_2+v_2+u_3+v_3 \\ 2(u_1+v_1)+u_3+v_3 \\ 3(u_1+v_1)-(u_2+v_2) \end{pmatrix}$$

They are equal

$$\varphi(u)+\varphi(v) = \begin{pmatrix} u_2+u_3 \\ 2u_1+u_3 \\ 3u_1-u_2 \end{pmatrix} + \begin{pmatrix} v_2+v_3 \\ 2v_1+v_3 \\ 3v_1-v_2 \end{pmatrix}$$

$$\varphi(\alpha u) = \alpha \cdot \varphi(u)$$ equality holds

$$\varphi\begin{pmatrix} \alpha u_1 \\ \alpha u_2 \\ \alpha u_3 \end{pmatrix} = \begin{pmatrix} \alpha(u_2 + u_3) \\ \alpha \cdot (2u_1 + u_3) \\ \alpha (3u_1 - u_2) \end{pmatrix} = \alpha \cdot \varphi(u)$$

$\varphi(x)$ - <u>is</u> a linear transformation

A_φ – ?

$$A_\varphi = \left[\varphi(e_1)\,\middle|\,\varphi(e_2)\,\middle|\,\varphi(e_3)\right]$$

$$\varphi(x) = \begin{pmatrix} x_2 + x_3 \\ 2x_1 + x_3 \\ 3x_1 - x_2 \end{pmatrix}$$

$$\varphi(e_1) = \varphi\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\2\\3\end{pmatrix}$$

$$\varphi(e_2) = \varphi\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$\varphi(e_3) = \varphi\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$A_\varphi = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & -1 & 0 \end{pmatrix}$$

$$x = (1, 2, 3)^T$$

$$\varphi(x) = \begin{pmatrix}5\\5\\1\end{pmatrix}$$

$$A_\varphi \cdot [x] = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix} 2+3 \\ 2+0+3 \\ 3-2+0 \end{pmatrix} = \begin{pmatrix}5\\5\\1\end{pmatrix}$$

$$x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 2 & 2 \end{pmatrix} \qquad [x]_B = \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix}$$

$$[y] = \begin{pmatrix} 5 \\ 5 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/2 \\ 2 \\ 1 \end{pmatrix}$$

$$\boxed{L_T = C^{-1} A_T B}$$

$$C = I_W$$

basis in target space
was not changed

$$L_\varphi = A_\varphi \cdot B$$

$$L_\varphi = \begin{pmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 3 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 6 & 2 \\ 4 & 2 & 2 \\ 6 & -4 & 0 \end{pmatrix}$$

$$\begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix} = L_\varphi \cdot [x]_B = \begin{pmatrix} 0 & 6 & 2 \\ 4 & 2 & 2 \\ 6 & -4 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ 1 \end{pmatrix}$$

Derivative as Linear Transformation

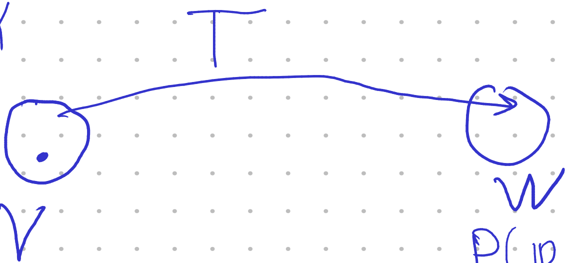
$$(x^\alpha)' = \alpha \cdot x^{\alpha-1}$$

x^2 $\quad$ $2x = (x^2)'$

$$(f+g)' = f' + g'$$

$$(\alpha f)' = \alpha \cdot f'$$

Ex



$P(\mathbb{R}, 3)$ $\qquad$ $P(\mathbb{R}, 2)$

$f(x) = x^3 + x^2 - x + 1$

$T(f(x)) = 3x^2 + 2x - 1$

lives in W

basis = $\{1, x, x^2, x^3\}$ Standard.
in V

$F(x) = 3x^3 + 2x^2 - 2x - 2$

$[F(x)] = \begin{pmatrix} -2 \\ -2 \\ 2 \\ 3 \end{pmatrix}$

Standard basis in W $= \boxed{\{1, x, x^2\}}$

$A_T = \begin{bmatrix} T(e_1) & T(e_2) & T(e_3) & T(e_4) \end{bmatrix}$

$T(e_1) = T(1) = (1)' = 0$

$[T(e_1)] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$T(e_2) = T(x) = x' = 1$

$[T(e_2)] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$T(e_3) = T(x^2) = (x^2)' = 2x \quad \Big| \quad T(e_4) = (x^3)' = 3x^2$$

$$[T(e_3)] = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \quad \Big| \quad [T(e_4)] = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$F(x) = 3x^3 + 2x^2 - 2x - 2 \quad \Big| \quad T(F(x)) = F'(x) = 9x^2 + 4x - 2$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ -2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ 9 \end{bmatrix}$$

$$I_2 = \{1, x, x^2\}$$

$$\boxed{L_T = C^{-1} A_T B}$$

$$C_2 = (1,\ 1+x,\ 1-x+x^2)$$

$$B = I_3 = \{1, x, x^2, x^3\}$$

$$\underset{g(x)}{1+x+x^2} = -2 \cdot 1 + 2(1+x) + 1 \cdot (1-x+x^2)$$

$$[g(x)]_{C_2} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$C_2 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = [g(x)]$$

$$L_T = C_2^{-1} \cdot A_T \cdot I$$

$$C_2^{-1} = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$L_T = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} =$$

$$= \begin{pmatrix} 0 & 1 & -2 & -6 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$2\cdot 1 + 2\cdot(1+x) + 2(1-x+x^2) =$$

$$= 6 + 2x^2 = g(x)$$

$$[g(x)]_{C_2} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \qquad [g(x)] = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}$$

$$F(x) \in V, \quad F(x) = 6\cdot x + \frac{2}{3}\cdot x^3 - \pi \cdot 10^6$$

$$F'(x) = 6 + 2x^2$$

$$\underset{P(\mathbb{R},3)}{F(x)} \longrightarrow \underset{P(\mathbb{R},2)}{g(x)}$$

$$[g]_{C_2} = L_T \cdot [F(x)]$$

$$\begin{pmatrix} 0 & 1 & -2 & -6 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix} \cdot \begin{bmatrix} -\pi \cdot 10^6 \\ 6 \\ 0 \\ 2/3 \end{bmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = [g(x)]_{C_2}$$

V