FCS HSE, MDS, Linear Algebra

Function

Functions between vector spaces

Definition. Let V, W be vector spaces. A transformation $T : V \to W$ is called linear if

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}) \forall \mathbf{u}, \mathbf{v} \in V$
2. $T(\alpha \mathbf{v}) = \alpha T(\mathbf{v})$ for all $\mathbf{v} \in V$ and for all scalars $\alpha \in \mathbb{R}$.

Properties 1 and 2 together are sometimes combined into the following one:

$$T(\alpha \mathbf{u} + \beta \mathbf{v}) = \alpha T(\mathbf{u}) + \beta T(\mathbf{v}), \quad \forall \mathbf{u}, \mathbf{v} \in V, \quad \forall \alpha, \beta \in \mathbb{R}.$$

We need a way to write a transformation analytically

Matrix representation of a linear transformation

Let us assume $T : V \to W$, vectors $e_1, \dots, e_m$ be a standard basis in V, and vectors $\tilde{e}_1, \dots, \tilde{e}_m$ be a basis in W.

We would like to investigate how T acts on any $x \in V$.

$$x = x_1 e_1 + \dots + x_n e_n,$$

$$T(x) = T(x_1 e_1 + \dots + x_n e_n) = x_1 T(e_1) + \dots + x_n T(e_n).$$

Keep in mind, that $T(e_1), \dots, T(e_n)$ are *vectors*, *i.e.* abstract citizens of the vector space W.

Matrix representation of a linear transformation

Let us look at them from standard basis in W:

$$\begin{gathered}
T(e_1) = a_{11}\tilde{e}_1 + a_{21}\tilde{e}_2 + \ldots + a_{m1}\tilde{e}_m, \\
T(e_2) = a_{12}\tilde{e}_1 + a_{22}\tilde{e}_2 + \ldots + a_{m2}\tilde{e}_m, \\
\vdots \\
T(e_n) = a_{1n}\tilde{e}_1 + a_{2n}\tilde{e}_2 + \ldots + a_{mn}\tilde{e}_m
\end{gathered}$$

Then get back to $T(x) = x_1 T(e_1) + \ldots + x_n T(e_n)$.

$$\begin{gathered}
T(x) = x_1\left(a_{11}\tilde{e}_1 + \ldots + a_{m1}\tilde{e}_m\right) + x_2\left(a_{12}\tilde{e}_1 + \ldots + a_{m2}\tilde{e}_m\right) \\
+ x_n\left(a_{1n}\tilde{e}_1 + \ldots + a_{mn}\tilde{e}_m\right)
\end{gathered}$$

Matrix representation of a linear transformation

$$
\begin{aligned}
T(x) = {} & x_1 \left(a_{11}\tilde{e}_1 + ... + a_{m1}\tilde{e}_m\right) + x_2 \left(a_{12}\tilde{e}_1 + ... + a_{m2}\tilde{e}_m\right) \\
& + x_n \left(a_{1n}\tilde{e}_1 + ... + a_{mn}\tilde{e}_m\right) \\
= {} & \left(a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n\right)\tilde{e}_1 + \left(a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n\right)\tilde{e}_2 \\
& + \left(a_{m1}x_1 + a_{12}x_2 + ... + a_{mn}x_n\right)\tilde{e}_m
\end{aligned}
$$

Matvec... again...

Finally:

$$[T(x)]_{\tilde{e}} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = A_T \, [x]_e \, .$$

Huh!

If indeed e and $\tilde{e}$ are standard bases then

To construct matrix A_T of a linear transformation T we need just to know images of basis vectors: $T(e_1), \dots, T(e_n)$, *i.e.*

$$e_1 \xrightarrow{T} a_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, e_2 \xrightarrow{T} a_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \dots, e_n \xrightarrow{T} a_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Matrix representation of a linear transformation

Sweet and easy for any $x \in V$, $x = (x_1, \dots, x_n)^\top$:

$$x = x_1 e_1 + \dots + x_n e_n,$$

$$T(x) = x_1 T(e_1) + \dots + x_n T(e_n).$$

Then we want to observe $T(x)$ in standard basis $\tilde{e}$ as well:

$$[T(x)] = x_1 a_1 + \dots + x_n a_n$$

$$= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

Matrix form of transformations' composition

Whar if bases are not standard?

What if bases are not standard?

Examples