Diagrammatic Algebra of First Order Logic

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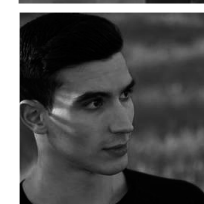
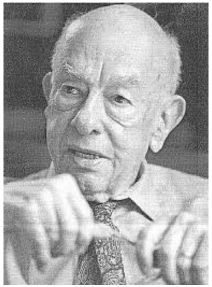
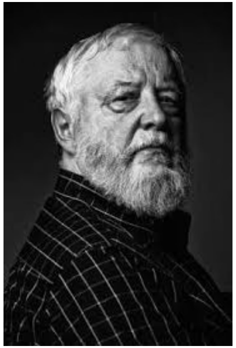


Predicate Functor Logic (Quine 1971)



"Logic in his adolescent phase was algebraic. There was Boole's algebra of classes and Peirce's algebra of relations. But in 1879 logic come of age, with Frege's quantification theory. Here the bound variables, so characteristic of analysis rather than of algebra, became central to logic."



Adjointness in Foundations (Lawvere 1969)

Boole's algebra of classes 1847

Operations: $(\vee, 0, \wedge, 1, \neg)$

Laws:

$p \vee q = q \vee p$

$p \wedge q = q \wedge p$

$(p \vee q) \vee r = p \vee (q \vee r)$

$(p \wedge q) \wedge r = p \wedge (q \wedge r)$

$p \wedge (q \vee r) = p \wedge q \vee p \wedge r$

$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

$p \vee p = p$

$p \wedge p = p$

$p \vee p \wedge q = p$

$p \wedge (p \vee q) = p$

$0 \vee p = p$

$0 \wedge p = 0$

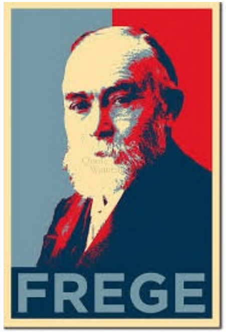
$1 \vee p = 1$

$1 \wedge p = p$

$\neg(p \vee q) = \neg p \wedge \neg q$

$\neg(p \wedge q) = \neg p \vee \neg q$

FREGE V.S. PEIRCE



Motivated by the pursuit of foundations of mathematics; Inspired by real analisys

Brings the concepts of variables and functions into the logical realm

Inspired by the work of De Morgan on relational reasoning

Introduces a calculus in which operations allow to combine relations and satisfy a set of algebraic laws

The Calculus of Relations

Beyond the boolean operators, it has

$$R \,\fatsemi\, S \stackrel{\text{def}}{=} \{(x,z) \mid \exists y \in Y \,.\, (x,y) \in R \wedge (y,z) \in S\} \subseteq X \times Z$$

$$R \,\bullet\, S \stackrel{\text{def}}{=} \{(x,z) \mid \forall y \in Y \,.\, (x,y) \in R \vee (y,z) \in S\} \subseteq X \times Z$$

$$id^{\circ}_{X} \stackrel{\text{def}}{=} \{(x,y) \mid x = y\} \subseteq X \times X$$

$$id^{\bullet}_{X} \stackrel{\text{def}}{=} \{(x,y) \mid x \neq y\} \subseteq X \times X$$

$$R^{\dagger} \stackrel{\text{def}}{=} \{(y,x) \mid (x,y) \in R\}$$

for all sets X, Y and Z

and relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$

The Calculus of Relations

Syntax

$R \in \Sigma$

$$E \quad ::= \quad R \mid id^{\circ} \mid E \,\fatsemi\, E \mid id^{\bullet} \mid E \,\bullet\, E \mid E^{\dagger} \mid \top \mid E \cap E \mid \bot \mid E \cup E \mid \overline{E}$$

Semantics

given $\mathcal{I} = (X, \rho)$ where X is a set and $\rho : \Sigma \to \mathcal{P}(X \times X)$

$$\langle R \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \rho(R) \qquad \langle id^{\circ} \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} id^{\circ}_{X} \qquad \langle E_1 \fatsemi E_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \fatsemi \langle E_2 \rangle_{\mathcal{I}}$$

$$\langle E^{\dagger} \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E \rangle_{\mathcal{I}}^{\dagger} \qquad \langle id^{\bullet} \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} id^{\bullet}_{X} \qquad \langle E_1 \bullet E_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \bullet \langle E_2 \rangle_{\mathcal{I}}$$

$$\langle \overline{E} \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \overline{\langle E \rangle_{\mathcal{I}}} \qquad \langle \bot \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \varnothing \qquad \langle E_1 \cup E_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \cup \langle E_2 \rangle_{\mathcal{I}}$$

$$\langle \top \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} X \times X \qquad \langle E_1 \cap E_2 \rangle_{\mathcal{I}} \stackrel{\text{def}}{=} \langle E_1 \rangle_{\mathcal{I}} \cap \langle E_2 \rangle_{\mathcal{I}}$$

Some Laws

$$R \fatsemi (S \bullet T) \leq_{\mathsf{CR}} (R \fatsemi S) \bullet T \qquad (R \bullet S) \fatsemi T \leq_{\mathsf{CR}} R \bullet (S \fatsemi T)$$

LINEAR DISTRIBUTIVITY

$$id^{\circ}_{Y} \leq_{\mathsf{CR}} \overline{R^{\dagger}} \bullet R \qquad R \fatsemi \overline{R^{\dagger}} \leq_{\mathsf{CR}} id^{\bullet}_{X}$$

LINEAR ADJUNCTION

TARSKI’S QUESTION

The calculus was shown to be
strictly less expressive than First Order Logic
(Lowenheim 1915)

It was forgotten until in 1941
Tarski fall in love with it

Does
a complete axiomatisation for the calculus of relations
exist?

The question has a negative answer (Monk 1961)

The Calculus of Relations

IN COMPUTER SCIENCE

Rewriting (e.g. Gavazzo 2023)

Relational databases (Codd 1970)

Proof Assistants (e.g. Pous 2013) $\Leftarrow$ Lack of variable and quantifies

Foundations of program logics (Prat 1976) (Hoare,He, 1986)

Left Residual:

if $c ; a \leq b$

then $c \leq b ; \bar{a}^{\dagger}$

$$\begin{tikzcd} X \arrow[r, "a"] & Y \\ Z \arrow[u, "b;\bar{a}^{\dagger}"] \arrow[ur, "b"'] & \end{tikzcd}$$

"The calculus of relations has an intrinsic charm and beauty which makes it a source of intellectual delight to all who become acquainted with it."

(Tarski 1941) (Hoare,He, 1986)

The Calculus of Neo-Peircean Relations

- OVERCOME THE MAIN LIMITATIONS OF THE CALCULUS OF RELATIONS
  - → IT HAS THE SAME EXPRESSIVITY OF FIRST-ORDER-LOGIC
  - → IT COMES WITH A COMPLETE SYSTEM OF AXIOMS
- WHILE MANTAINING ITS USUAL BENEFITS
  - → POINT-FREE REASONING: NO VARIABLES, NO QUANTIFIERS
  - → PURELY EQUATIONAL REASONING

The Calculus of Neo-Peircean Relations

- THIS IS OBTAINED BY MOVING

SYNTACTICALLY

FROM TRADITIONAL (CARTESIAN) SYNTAX

TO RESOURCE-AWARE (MONOIDAL) SYNTAX

SEMANTICALLY

FROM BINARY RELATIONS $R \subseteq X \times X$

TO "MONOIDAL" RELATIONS $R \subseteq X^n \times X^m$ for some $n,m \in \mathbb{N}$

The Calculus of Neo-Peircean Relations

- WITH THIS MOVE, ONE IMMEDIATELY IDENTIFIES NOVEL CONSTANTS

COPIER: FOR ALL SETS X, $\blacktriangleleft^{\circ}_{X} = \langle id_X, id_X \rangle : X \longrightarrow X \times X$

$$\blacktriangleleft^{\circ}_{X} \stackrel{\text{def}}{=} \{(x, (y, z)) \mid x = y \wedge x = z\}$$

DISCARD: FOR ALL SETS X, $!^{\circ}_{X} : X \longrightarrow 1 \stackrel{\text{def}}{=} \{*\}$

$$!^{\circ}_{X} \stackrel{\text{def}}{=} \{(x, \star) \mid x \in X\}$$

BY COMBINING THESE WITH OPPOSITE AND COMPLEMENT, ONE OBTAINS

$$\begin{array}{ll}
\blacktriangleleft^{\circ}_{X} \stackrel{\text{def}}{=} \{(x, (y, z)) \mid x = y \wedge x = z\} & !^{\circ}_{X} \stackrel{\text{def}}{=} \{(x, \star) \mid x \in X\} \\
\blacktriangleright^{\circ}_{X} \stackrel{\text{def}}{=} \{((y, z), x) \mid x = y \wedge x = z\} & ¡^{\circ}_{X} \stackrel{\text{def}}{=} \{(\star, x) \mid x \in X\} \\
\hline
\blacktriangleleft^{\bullet}_{X} \stackrel{\text{def}}{=} \{(x, (y, z)) \mid x \neq y \vee x \neq z\} & !^{\bullet}_{X} \stackrel{\text{def}}{=} \varnothing \\
\blacktriangleright^{\bullet}_{X} \stackrel{\text{def}}{=} \{((y, z), x) \mid x \neq y \vee x \neq z\} & ¡^{\bullet}_{X} \stackrel{\text{def}}{=} \varnothing
\end{array}$$

The Calculus of Neo-Peircean Relations

- WITH THIS MOVE, ONE IMMEDIATELY IDENTIFIES NOVEL OPERATIONS

  MONOIDAL PRODUCT

$$R \otimes S \stackrel{\text{def}}{=} \{(\,(x,v),(y,w)\,) \mid (x,y) \in R \wedge (v,w) \in S\}$$
$$R \mathbin{\otimes\!\!\!\bullet} S \stackrel{\text{def}}{=} \{(\,(x,v),(y,w)\,) \mid (x,y) \in R \vee (v,w) \in S\}$$

- BOOLEAN OPERATIONS AS WELL AS OPPOSITE CAN BE DERIVED BY COPIERS, DISCARDS AND MONOIDAL PRODUCTS

The Calculus of Neo-Peircean Relations

SYNTAX

$R \in \Sigma$: A MONOIDAL SIGNATURE, NAMELY $R : n \to m$

$$c ::= \blacktriangleleft_1^{\circ} \mid !_1^{\circ} \mid R^{\circ} \mid ¡_1^{\circ} \mid \blacktriangleright_1^{\circ} \mid id_0^{\circ} \mid id_1^{\circ} \mid \sigma_{1,1}^{\circ} \mid c \,\,\mathring{;}\,\, c \mid c \otimes c \mid$$
$$\blacktriangleleft_1^{\bullet} \mid !_1^{\bullet} \mid R^{\bullet} \mid ¡_1^{\bullet} \mid \blacktriangleright_1^{\bullet} \mid id_0^{\bullet} \mid id_1^{\bullet} \mid \sigma_{1,1}^{\bullet} \mid c \,\,\bullet\!\!;\,\, c \mid c \,\,\bullet\!\!\!\otimes\,\, c$$

SEMANTICS GIVEN $\mathcal{I} = \langle X, \rho \rangle$ X IS A SET, $\rho : \Sigma_{n,m} \to \mathcal{P}(X^n \times X^m)$

$$\mathcal{I}^\sharp(\blacktriangleleft_1^{\circ}) \stackrel{\text{def}}{=} \blacktriangleleft_X^{\circ} \qquad \mathcal{I}^\sharp(!_1^{\circ}) \stackrel{\text{def}}{=} !_X^{\circ} \qquad \mathcal{I}^\sharp(\blacktriangleright_1^{\circ}) \stackrel{\text{def}}{=} \blacktriangleright_X^{\circ} \qquad \mathcal{I}^\sharp(¡_1^{\circ}) \stackrel{\text{def}}{=} ¡_X^{\circ}$$
$$\mathcal{I}^\sharp(id_0^{\circ}) \stackrel{\text{def}}{=} id_{\mathbb{1}}^{\circ} \qquad \mathcal{I}^\sharp(id_1^{\circ}) \stackrel{\text{def}}{=} id_X^{\circ} \qquad \mathcal{I}^\sharp(\sigma_{1,1}^{\circ}) \stackrel{\text{def}}{=} \sigma_{X,X}^{\circ} \qquad \mathcal{I}^\sharp(R^{\circ}) \stackrel{\text{def}}{=} \rho(R)$$
$$\mathcal{I}^\sharp(c \,\mathring{;}\, d) \stackrel{\text{def}}{=} \mathcal{I}^\sharp(c) \,\mathring{;}\, \mathcal{I}^\sharp(d) \qquad \mathcal{I}^\sharp(c \otimes d) \stackrel{\text{def}}{=} \mathcal{I}^\sharp(c) \otimes \mathcal{I}^\sharp(d) \qquad \mathcal{I}^\sharp(R^{\bullet}) \stackrel{\text{def}}{=} \overline{\rho(R)}^{\dagger}$$

$\bullet$ STANDS FOR EITHER $\circ$ OR $\bullet$

The Calculus of Neo-Peircean Relations

COMPLETE AXIOMATISATION

Axioms of strict symmetric monoidal categories

$$a \mathbin{\mathring{;}} (b \mathbin{\mathring{;}} c) = (a \mathbin{\mathring{;}} b) \mathbin{\mathring{;}} c \qquad id^{\circ}_{n} \mathbin{\mathring{;}} c = c = c \mathbin{\mathring{;}} id^{\circ}_{m} \qquad (a \otimes b) \otimes c = a \otimes (b \otimes c) \qquad id^{\circ}_{0} \otimes c = c = id^{\circ}_{0} \otimes c$$

$$(a \otimes b) \mathbin{\mathring{;}} (c \otimes d) = (a \mathbin{\mathring{;}} c) \otimes (b \mathbin{\mathring{;}} d) \qquad \sigma^{\circ}_{1,1} \mathbin{\mathring{;}} \sigma^{\circ}_{1,1} = id^{\circ}_{2} \qquad (c \otimes id^{\circ}_{o}) \mathbin{\mathring{;}} \sigma^{\circ}_{m,o} = \sigma^{\circ}_{n,o} \mathbin{\mathring{;}} (id^{\circ}_{o} \otimes c)$$

Axioms of cartesian bicategories

$$\blacktriangleleft^{\circ}_{n} \mathbin{\mathring{;}} (id^{\circ}_{n} \otimes \blacktriangleleft^{\circ}_{n}) \overset{(\blacktriangleleft^{\circ}\text{-as})}{=} \blacktriangleleft^{\circ}_{n} \mathbin{\mathring{;}} (\blacktriangleleft^{\circ}_{n} \otimes id^{\circ}_{n}) \qquad (id^{\circ}_{n} \otimes \blacktriangleright^{\circ}_{n}) \mathbin{\mathring{;}} \blacktriangleright^{\circ}_{n} \overset{(\blacktriangleright^{\circ}\text{-as})}{=} (\blacktriangleright^{\circ}_{n} \otimes id^{\circ}_{n}) \mathbin{\mathring{;}} \blacktriangleright^{\circ}_{n}$$

$$\blacktriangleleft^{\circ}_{n} \mathbin{\mathring{;}} (id^{\circ}_{n} \otimes !^{\circ}_{n}) \overset{(\blacktriangleleft^{\circ}\text{-un})}{=} id^{\circ}_{n} \qquad (id^{\circ}_{n} \otimes ¡^{\circ}_{n}) \mathbin{\mathring{;}} \blacktriangleright^{\circ}_{n} \overset{(\blacktriangleright^{\circ}\text{-un})}{=} id^{\circ}_{n}$$

$$\blacktriangleleft^{\circ}_{n} \mathbin{\mathring{;}} \sigma^{\circ}_{n,n} \overset{(\blacktriangleleft^{\circ}\text{-co})}{=} \blacktriangleleft^{\circ}_{n} \qquad \sigma^{\circ}_{n,n} \mathbin{\mathring{;}} \blacktriangleright^{\circ}_{n} \overset{(\blacktriangleright^{\circ}\text{-co})}{=} \blacktriangleright^{\circ}_{n}$$

$$(\blacktriangleleft^{\circ}_{n} \otimes id^{\circ}_{n}) \mathbin{\mathring{;}} (id^{\circ}_{n} \otimes \blacktriangleright^{\circ}_{n}) \overset{(\mathrm{F}^{\circ})}{=} (id^{\circ}_{n} \otimes \blacktriangleleft^{\circ}_{n}) \mathbin{\mathring{;}} (\blacktriangleright^{\circ}_{n} \otimes id^{\circ}_{n}) \qquad \blacktriangleleft^{\circ}_{n} \mathbin{\mathring{;}} \blacktriangleright^{\circ}_{n} \overset{(\mathrm{S}^{\circ})}{=} id^{\circ}_{n}$$

$$¡^{\circ}_{n} \mathbin{\mathring{;}} !^{\circ}_{n} \overset{(\epsilon !^{\circ})}{\leq} id^{\circ}_{0} \qquad \blacktriangleright^{\circ}_{n} \mathbin{\mathring{;}} \blacktriangleleft^{\circ}_{n} \overset{(\epsilon \blacktriangleleft^{\circ})}{\leq} (id^{\circ}_{n} \otimes id^{\circ}_{n}) \qquad c \mathbin{\mathring{;}} \blacktriangleleft^{\circ}_{m} \overset{(\blacktriangleleft^{\circ}\text{-nat})}{\leq} \blacktriangleleft^{\circ}_{n} \mathbin{\mathring{;}} (c \otimes c)$$

$$id^{\circ}_{n} \overset{(\eta !^{\circ})}{\leq} !^{\circ}_{n} \mathbin{\mathring{;}} ¡^{\circ}_{n} \qquad id^{\circ}_{n} \overset{(\eta \blacktriangleleft^{\circ})}{\leq} \blacktriangleleft^{\circ}_{n} \mathbin{\mathring{;}} \blacktriangleright^{\circ}_{n} \qquad c \mathbin{\mathring{;}} !^{\circ}_{m} \overset{(!^{\circ}\text{-nat})}{\leq} !^{\circ}_{n}$$

Axioms of cocartesian bicategories

$$\blacktriangleleft^{\bullet}_{n} \mathbin{\dot{;}} (id^{\bullet}_{n} \circledast \blacktriangleleft^{\bullet}_{n}) \overset{(\blacktriangleleft^{\bullet}\text{-as})}{=} \blacktriangleleft^{\bullet}_{n} \mathbin{\dot{;}} (\blacktriangleleft^{\bullet}_{n} \circledast id^{\bullet}_{n}) \qquad (id^{\bullet}_{n} \circledast \blacktriangleright^{\bullet}_{n}) \mathbin{\dot{;}} \blacktriangleright^{\bullet}_{n} \overset{(\blacktriangleright^{\bullet}\text{-as})}{=} (\blacktriangleright^{\bullet}_{n} \circledast id^{\bullet}_{n}) \mathbin{\dot{;}} \blacktriangleright^{\bullet}_{n}$$

$$\blacktriangleleft^{\bullet}_{n} \mathbin{\dot{;}} (id^{\bullet}_{n} \circledast !^{\bullet}_{n}) \overset{(\blacktriangleleft^{\bullet}\text{-un})}{=} id^{\bullet}_{n} \qquad (id^{\bullet}_{n} \circledast ¡^{\bullet}_{n}) \mathbin{\dot{;}} \blacktriangleright^{\bullet}_{n} \overset{(\blacktriangleright^{\bullet}\text{-un})}{=} id^{\bullet}_{n}$$

$$\blacktriangleleft^{\bullet}_{n} \mathbin{\dot{;}} \sigma^{\bullet}_{n,n} \overset{(\blacktriangleleft^{\bullet}\text{-co})}{=} \blacktriangleleft^{\bullet}_{n} \qquad \sigma^{\bullet}_{n,n} \mathbin{\dot{;}} \blacktriangleright^{\bullet}_{n} \overset{(\blacktriangleright^{\bullet}\text{-co})}{=} \blacktriangleright^{\bullet}_{n}$$

$$(\blacktriangleleft^{\bullet}_{n} \circledast id^{\bullet}_{n}) \mathbin{\dot{;}} (id^{\bullet}_{n} \circledast \blacktriangleright^{\bullet}_{n}) \overset{(\mathrm{F}^{\bullet})}{=} (id^{\bullet}_{n} \circledast \blacktriangleleft^{\bullet}_{n}) \mathbin{\dot{;}} (\blacktriangleright^{\bullet}_{n} \circledast id^{\bullet}_{n}) \qquad \blacktriangleleft^{\bullet}_{n} \mathbin{\dot{;}} \blacktriangleright^{\bullet}_{n} \overset{(\mathrm{S}^{\bullet})}{=} id^{\bullet}_{n}$$

$$!^{\bullet}_{n} \mathbin{\dot{;}} ¡^{\bullet}_{n} \overset{(\epsilon ¡^{\bullet})}{\leq} id^{\bullet}_{n} \qquad \blacktriangleleft^{\bullet}_{n} \mathbin{\dot{;}} \blacktriangleright^{\bullet}_{n} \overset{(\epsilon \blacktriangleright^{\bullet})}{\leq} id^{\bullet}_{n} \qquad \blacktriangleleft^{\bullet}_{n} \mathbin{\dot{;}} (c \circledast c) \overset{(\blacktriangleleft^{\bullet}\text{-nat})}{\leq} c \mathbin{\dot{;}} \blacktriangleleft^{\bullet}_{m}$$

$$id^{\bullet}_{0} \overset{(\eta ¡^{\bullet})}{\leq} ¡^{\bullet}_{n} \mathbin{\dot{;}} !^{\bullet}_{n} \qquad \blacktriangleright^{\bullet}_{n} \mathbin{\dot{;}} \blacktriangleleft^{\bullet}_{n} \overset{(\eta \blacktriangleright^{\bullet})}{\leq} (id^{\bullet}_{n} \circledast id^{\bullet}_{n}) \qquad !^{\bullet}_{n} \overset{(!^{\bullet}\text{-nat})}{\leq} c \mathbin{\dot{;}} !^{\bullet}_{m}$$

Axioms of closed symmetric monoidal linear bicategories

$$a \mathbin{\mathring{;}} (b \mathbin{\dot{;}} c) \overset{(\delta_l)}{\leq} (a \mathbin{\mathring{;}} b) \mathbin{\dot{;}} c \qquad (a \mathbin{\dot{;}} b) \mathbin{\mathring{;}} c \overset{(\delta_r)}{\leq} a \mathbin{\dot{;}} (b \mathbin{\mathring{;}} c)$$

$$id^{\circ}_{n+m} \overset{(\tau\sigma^{\circ})}{\leq} \sigma^{\circ}_{n,m} \mathbin{\dot{;}} \sigma^{\bullet}_{m,n} \qquad \sigma^{\bullet}_{n,m} \mathbin{\mathring{;}} \sigma^{\circ}_{m,n} \overset{(\gamma\sigma^{\circ})}{\leq} id^{\bullet}_{n+m} \qquad id^{\circ}_{n} \overset{(\tau R^{\circ})}{\leq} R^{\circ} \mathbin{\dot{;}} R^{\bullet} \qquad R^{\bullet} \mathbin{\mathring{;}} R^{\circ} \overset{(\gamma R^{\circ})}{\leq} id^{\bullet}_{m}$$

$$id^{\circ}_{n+m} \overset{(\tau\sigma^{\bullet})}{\leq} \sigma^{\bullet}_{n,m} \mathbin{\dot{;}} \sigma^{\circ}_{m,n} \qquad \sigma^{\circ}_{n,m} \mathbin{\mathring{;}} \sigma^{\bullet}_{m,n} \overset{(\gamma\sigma^{\bullet})}{\leq} id^{\bullet}_{n+m} \qquad id^{\circ}_{m} \overset{(\tau R^{\bullet})}{\leq} R^{\bullet} \mathbin{\dot{;}} R^{\circ} \qquad R^{\circ} \mathbin{\mathring{;}} R^{\bullet} \overset{(\gamma R^{\bullet})}{\leq} id^{\bullet}_{n}$$

$$id^{\circ}_{n+m} \overset{(\circledast^{\circ})}{\leq} id^{\circ}_{n} \circledast id^{\circ}_{m} \qquad id^{\bullet}_{n} \otimes id^{\bullet}_{m} \overset{(\otimes^{\bullet})}{\leq} id^{\bullet}_{n+m}$$

$$(a \mathbin{\dot{;}} b) \otimes (c \mathbin{\dot{;}} d) \overset{(\nu^{\circ}_{l})}{\leq} (a \otimes c) \mathbin{\dot{;}} (b \circledast d) \qquad (a \circledast c) \mathbin{\mathring{;}} (b \otimes d) \overset{(\nu^{\bullet}_{l})}{\leq} (a \mathbin{\mathring{;}} b) \circledast (c \mathbin{\mathring{;}} d)$$

$$(a \mathbin{\dot{;}} b) \otimes (c \mathbin{\dot{;}} d) \overset{(\nu^{\circ}_{r})}{\leq} (a \circledast c) \mathbin{\dot{;}} (b \otimes d) \qquad (a \otimes c) \mathbin{\mathring{;}} (b \circledast d) \overset{(\nu^{\bullet}_{r})}{\leq} (a \mathbin{\mathring{;}} b) \circledast (c \mathbin{\mathring{;}} d)$$

Additional axioms for fo-bicategories

$$id^{\circ}_{n} \overset{(\tau \blacktriangleleft^{\circ})}{\leq} \blacktriangleleft^{\circ}_{n} \mathbin{\dot{;}} \blacktriangleright^{\bullet}_{n} \qquad \blacktriangleright^{\bullet}_{n} \mathbin{\mathring{;}} \blacktriangleleft^{\circ}_{n} \overset{(\gamma \blacktriangleleft^{\circ})}{\leq} id^{\bullet}_{n+n} \qquad id^{\circ}_{n+n} \overset{(\tau \blacktriangleright^{\circ})}{\leq} \blacktriangleright^{\circ}_{n} \mathbin{\dot{;}} \blacktriangleleft^{\bullet}_{n} \qquad \blacktriangleleft^{\bullet}_{n} \mathbin{\mathring{;}} \blacktriangleright^{\circ}_{n} \overset{(\gamma \blacktriangleright^{\circ})}{\leq} id^{\bullet}_{n}$$

$$id^{\circ}_{n} \overset{(\tau !^{\circ})}{\leq} !^{\circ}_{n} \mathbin{\dot{;}} ¡^{\bullet}_{n} \qquad ¡^{\bullet}_{n} \mathbin{\mathring{;}} !^{\circ}_{n} \overset{(\gamma !^{\circ})}{\leq} id^{\bullet}_{0} \qquad id^{\circ}_{0} \overset{(\tau ¡^{\circ})}{\leq} ¡^{\circ}_{n} \mathbin{\dot{;}} !^{\bullet}_{n} \qquad !^{\bullet}_{n} \mathbin{\mathring{;}} ¡^{\circ}_{n} \overset{(\gamma ¡^{\circ})}{\leq} id^{\bullet}_{n}$$

$$id^{\circ}_{n} \overset{(\tau \blacktriangleleft^{\bullet})}{\leq} \blacktriangleleft^{\bullet}_{n} \mathbin{\dot{;}} \blacktriangleright^{\circ}_{n} \qquad \blacktriangleright^{\circ}_{n} \mathbin{\mathring{;}} \blacktriangleleft^{\bullet}_{n} \overset{(\gamma \blacktriangleleft^{\bullet})}{\leq} id^{\bullet}_{n+n} \qquad id^{\circ}_{n+n} \overset{(\tau \blacktriangleright^{\bullet})}{\leq} \blacktriangleright^{\bullet}_{n} \mathbin{\dot{;}} \blacktriangleleft^{\circ}_{n} \qquad \blacktriangleleft^{\circ}_{n} \mathbin{\mathring{;}} \blacktriangleright^{\bullet}_{n} \overset{(\gamma \blacktriangleright^{\bullet})}{\leq} id^{\bullet}_{n}$$

$$id^{\circ}_{n} \overset{(\tau !^{\bullet})}{\leq} !^{\bullet}_{n} \mathbin{\dot{;}} ¡^{\circ}_{n} \qquad ¡^{\circ}_{n} \mathbin{\mathring{;}} !^{\bullet}_{n} \overset{(\gamma !^{\bullet})}{\leq} id^{\bullet}_{0} \qquad id^{\circ}_{0} \overset{(\tau ¡^{\bullet})}{\leq} ¡^{\bullet}_{n} \mathbin{\dot{;}} !^{\circ}_{n} \qquad !^{\circ}_{n} \mathbin{\mathring{;}} ¡^{\bullet}_{n} \overset{(\gamma ¡^{\bullet})}{\leq} id^{\bullet}_{0}$$

$$(\blacktriangleleft^{\bullet}_{n} \otimes id^{\circ}_{n}) \mathbin{\mathring{;}} (id^{\circ}_{n} \otimes \blacktriangleright^{\circ}_{n}) \overset{(\mathrm{F}^{\bullet}_{\circ})}{=} (id^{\circ}_{n} \otimes \blacktriangleleft^{\circ}_{n}) \mathbin{\mathring{;}} (\blacktriangleright^{\bullet}_{n} \otimes id^{\circ}_{n}) \qquad (\blacktriangleleft^{\circ}_{n} \otimes id^{\circ}_{n}) \mathbin{\mathring{;}} (id^{\circ}_{n} \otimes \blacktriangleright^{\bullet}_{n}) \overset{(\mathrm{F}^{\circ}_{\bullet})}{=} (id^{\circ}_{n} \otimes \blacktriangleleft^{\bullet}_{n}) \mathbin{\mathring{;}} (\blacktriangleright^{\circ}_{n} \otimes id^{\circ}_{n})$$

$$(\blacktriangleleft^{\circ}_{n} \circledast id^{\bullet}_{n}) \mathbin{\dot{;}} (id^{\bullet}_{n} \circledast \blacktriangleright^{\bullet}_{n}) \overset{(\mathrm{F}_{\bullet}^{\circ})}{=} (id^{\bullet}_{n} \circledast \blacktriangleleft^{\bullet}_{n}) \mathbin{\dot{;}} (\blacktriangleright^{\circ}_{n} \circledast id^{\bullet}_{n}) \qquad (\blacktriangleleft^{\bullet}_{n} \circledast id^{\bullet}_{n}) \mathbin{\dot{;}} (id^{\bullet}_{n} \circledast \blacktriangleright^{\circ}_{n}) \overset{(\mathrm{F}_{\circ}^{\bullet})}{=} (id^{\bullet}_{n} \circledast \blacktriangleleft^{\circ}_{n}) \mathbin{\dot{;}} (\blacktriangleright^{\bullet}_{n} \circledast id^{\bullet}_{n})$$

MAIN THEOREM

Let E, F be two expressions.

$E \boxed{\leq} F$ iff $\forall \mathcal{I}, \ \mathcal{I}^{\#}(E) \leq \mathcal{I}^{\#}(E)$

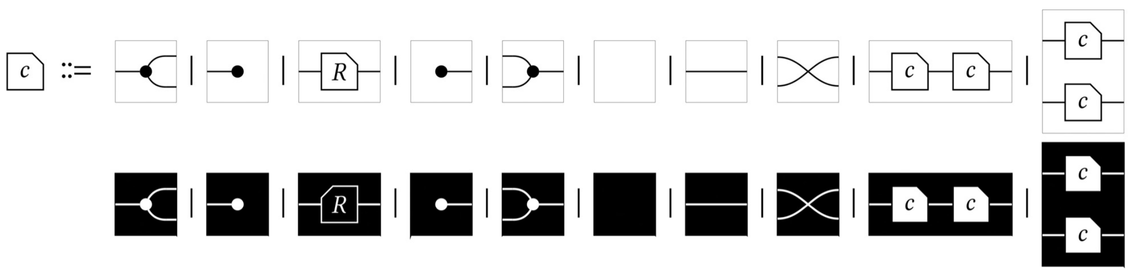
GENERATED BY

Diagrammatic Syntax

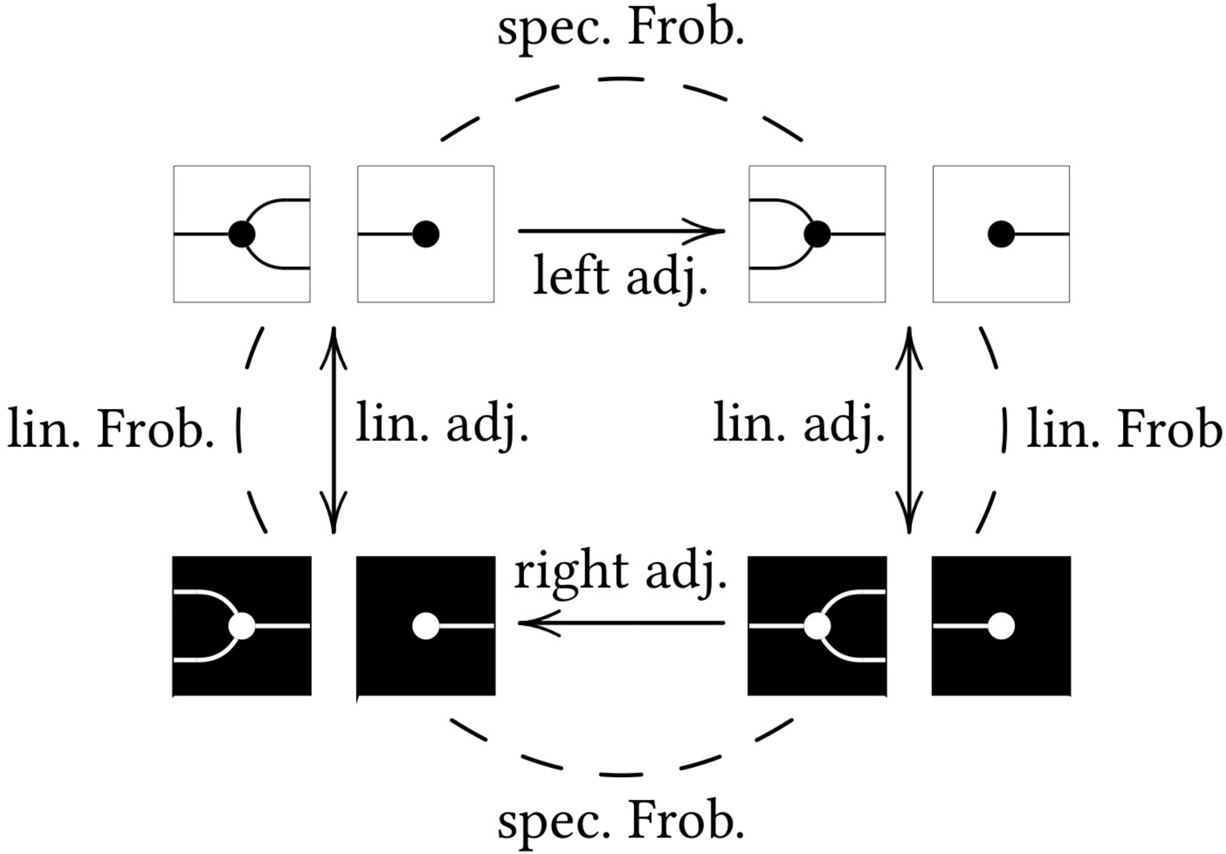
TERMS

$$
\begin{aligned}
c \;::=\; & \blacktriangleleft^{\circ}_{1} \mid !^{\circ}_{1} \mid R^{\circ} \mid \text{¡}^{\circ}_{1} \mid \blacktriangleright^{\circ}_{1} \mid id^{\circ}_{0} \mid id^{\circ}_{1} \mid \sigma^{\circ}_{1,1} \mid c \mathbin{\overset{\circ}{;}} c \mid c \otimes c \mid \\
& \blacktriangleleft^{\bullet}_{1} \mid !^{\bullet}_{1} \mid R^{\bullet} \mid \text{¡}^{\bullet}_{1} \mid \blacktriangleright^{\bullet}_{1} \mid id^{\bullet}_{0} \mid id^{\bullet}_{1} \mid \sigma^{\bullet}_{1,1} \mid c \mathbin{\overset{\bullet}{;}} c \mid c \mathbin{\otimes^{\bullet}} c
\end{aligned}
$$

DIAGRAMS



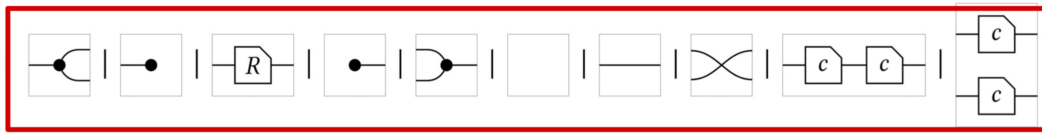
THE TAO OF LOGIC



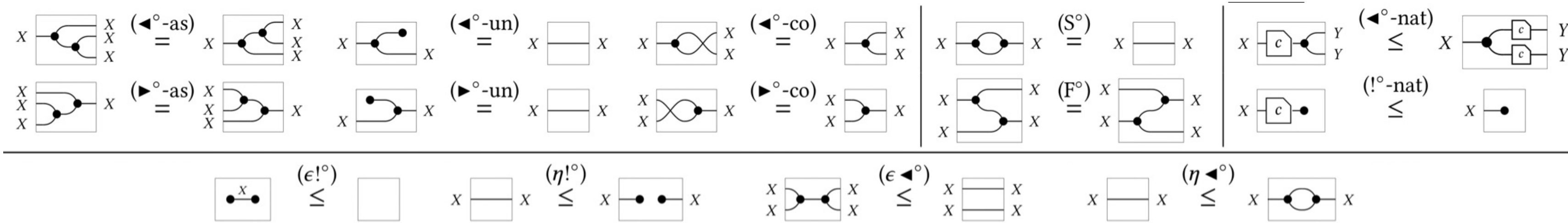
CARTESIAN BICATEGORIES

(CARBONI, WALTERS 1987)

Operations



Laws

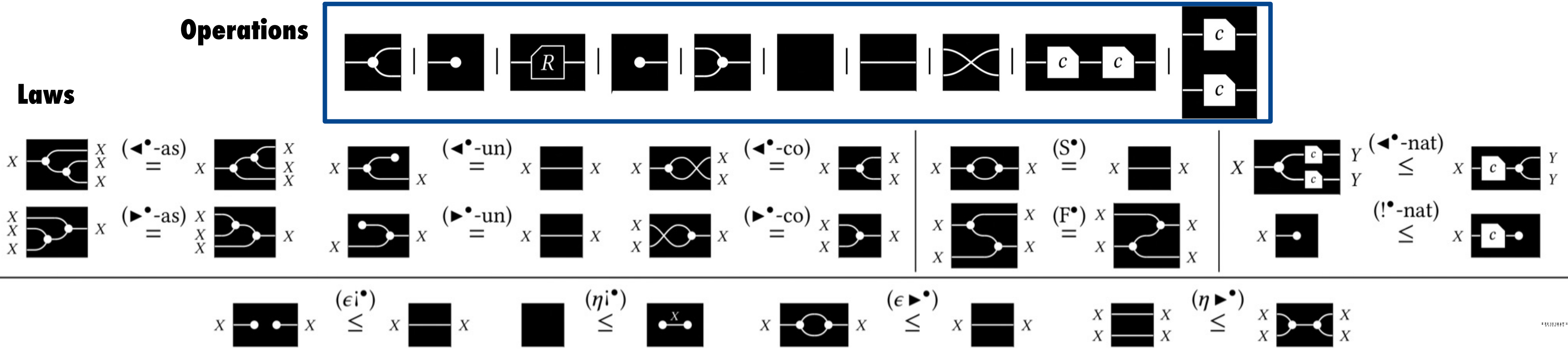


MAIN EXAMPLE: $\mathbf{Rel}^\circ$, sets and relations with the white structure

INTERNAL LANGUAGE: The regular fragment of FOL

$(\exists, \wedge, \top, =)$

COCARTESIAN BICATEGORIES



MAIN EXAMPLE: $\mathsf{Rel}^{\bullet}$, sets and relations with the black structure

INTERNAL LANGUAGE: The coregular fragment of FOL

$(\forall, \vee, F, =)$

LINEAR BICATEGORIES

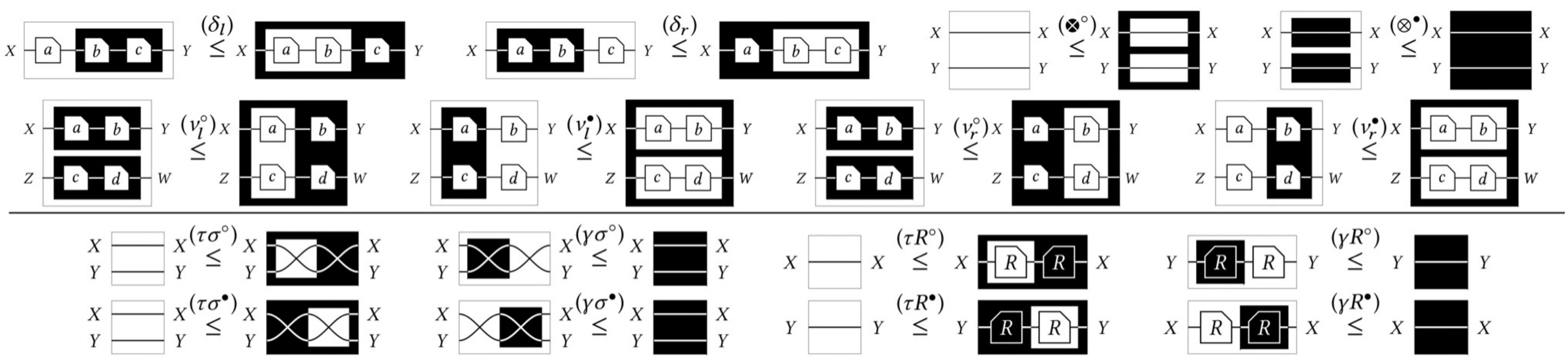
(Cockett, Seely, 1997)

CARTESIAN BICATEGORIES
$\updownarrow$
REGULAR LOGIC

$\longleftrightarrow$

COCARTESIAN BICATEGORIES
$\updownarrow$
COREGULAR LOGIC

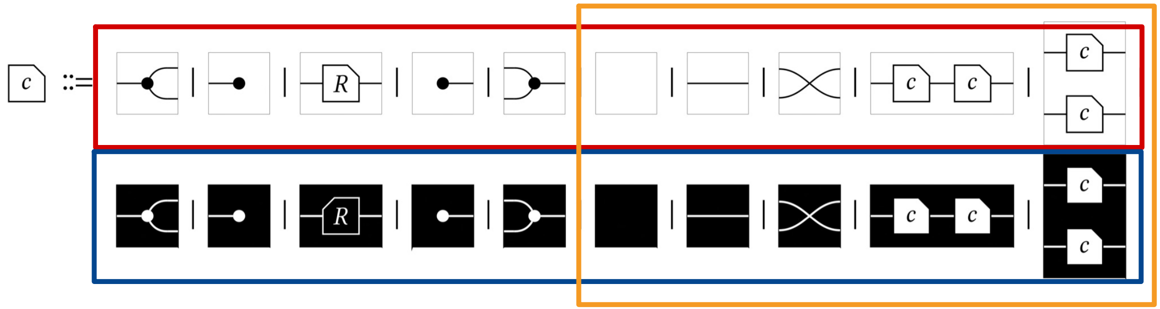
To make them interact one needs some categorical structure dealing with two compositions $(\,;\,,\,;\,)$



FIRST ORDER BICATEGORIES main example Rel

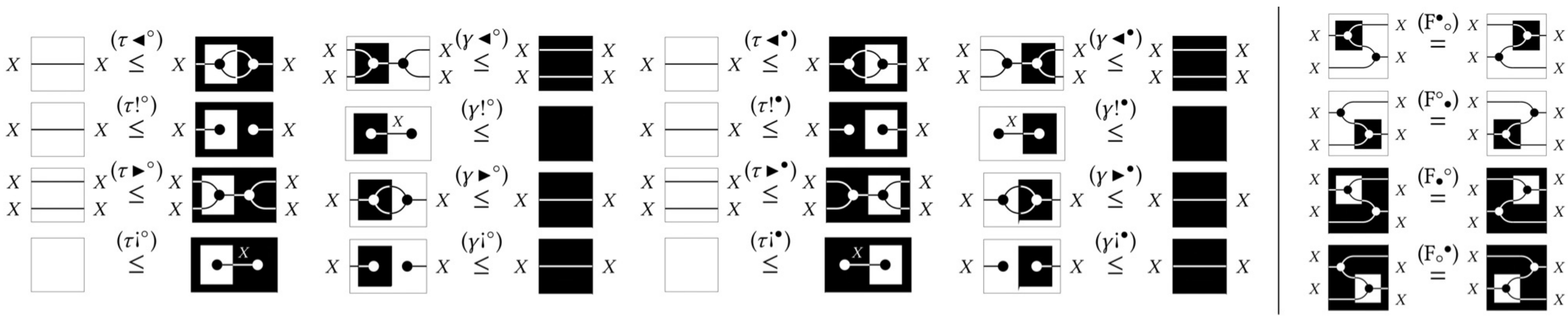
CARTESIAN BICATEGORY

CO CARTESIAN BICATEGORY



LINEAR BICATEGORY

SUCH THAT



WHITE COMONOID IS LINEAR ADJOINT TO BLACK MONOID

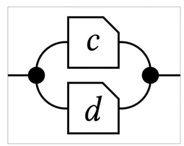
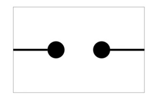
BLACK COMONOID IS LINEAR ADJOINT TO WHITE MONOID

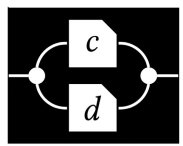
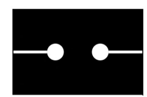
LINEAR FROBENIUS

FIRST ORDER BICATEGORIES: PROPERTIES

PROPOSITION A

In a first-order bicategory $\mathcal{C}$, each homset $\mathcal{C}[X,Y]$ carries a boolean algebra

$c \sqcap d \overset{\text{def}}{=}$ X  Y $\qquad \top \overset{\text{def}}{=}$ X  Y

$c \sqcup d \overset{\text{def}}{=}$ X  Y $\qquad \bot \overset{\text{def}}{=}$ X  Y

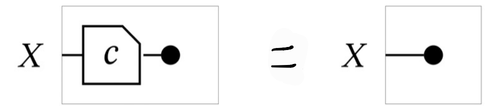
$\neg : (\mathcal{C}, \mathring{;}, \dot{;}) \xrightarrow{\cong} (\mathcal{C}^{co}, \dot{;}, \mathring{;})$ swaps black and white

$(\cdot)^{\dagger} : (\mathcal{C}, \mathring{;}, \dot{;}) \xrightarrow{\cong} (\mathcal{C}^{op}, \mathring{;}, \dot{;})$ swaps left and right

FIRST ORDER BICATEGORIES: PROPERTIES

DEFINITION

An arrow $c: X \to Y$ is said to be a map if:



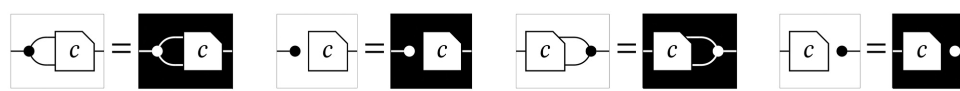
EXAMPLE In Rel maps coincide with functions

SYNTAX: To make a symbol in Σ be interpreted as a function is enough to impose these axioms.

PROPOSITION B

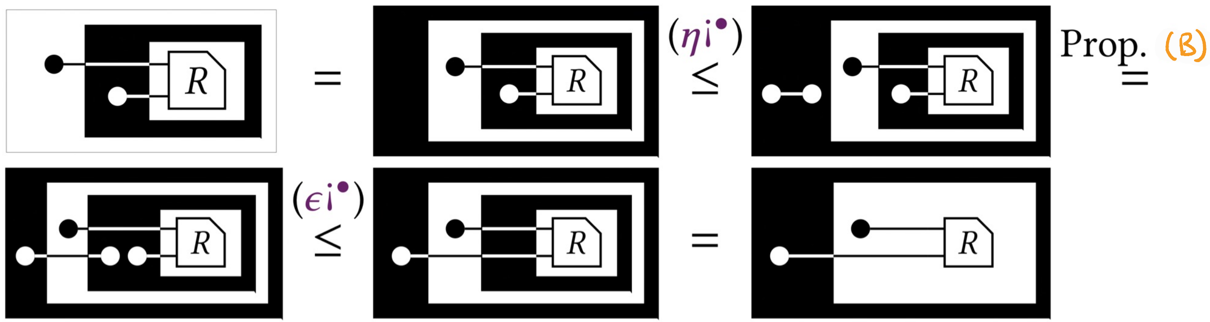
For all maps $f: X \to Y$ and arrows $c: Y \to Z$, $f ; c = \bar{f} ; c$

and thus



Proofs as diagram rewrites

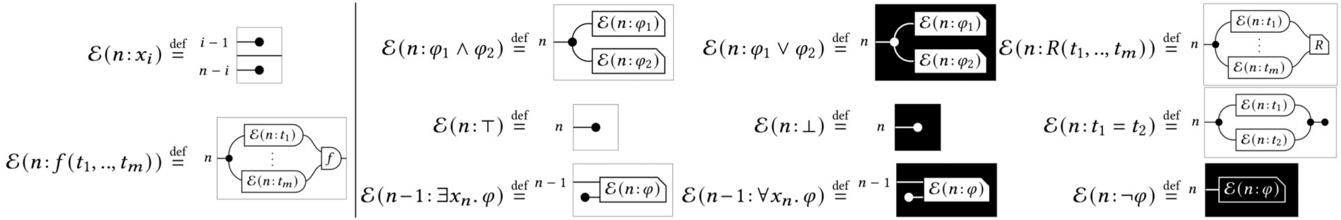
$\exists x.\forall y.\, R(x,y)$



$\forall y.\exists x.\, R(x,y)$

(IN THE PROPOSITIONAL CASE, THE AXIOMS OF FO-BICATEGORIES GIVE RISE TO THE DEEP-INFERENCE SYSTEM SKS_g (Brunnler 2003))

Compositional encodings



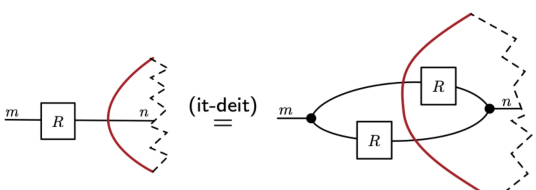
FIRST-ORDER LOGIC

CALCULUS OF RELATIONS

CALCULUS OF NEO PEIRCEAN RELATIONS

EXISTENTIAL GRAPHS

PREDICATE FUNCTOR LOGIC BY QUINE



A FOX THEOREM OF CLASSICAL LOGIC

$$
\begin{tikzcd}
\mathbb{CB} \arrow[rr, bend left, "\mathrm{HmI}"] \arrow[rr, phantom, "\perp"] & & \mathbb{EED} \arrow[ll, bend left, "\mathrm{Rel}"] \\
\mathbb{F}\mathrm{ob} \arrow[u, "U"] \arrow[rr, bend left, "\mathrm{HmI}"] \arrow[rr, phantom, "\perp"] & & \mathbb{BHD} \arrow[ll, bend left, "\mathrm{Rel}"] \arrow[u, "U"']
\end{tikzcd}
$$

$$\mathrm{HmI}(\mathcal{C}) \overset{\mathrm{def}}{:=} \mathrm{Hom}_{\mathcal{C}}[-, I] : \mathrm{Map}(\mathcal{C}) \longrightarrow \mathrm{Bool}$$

Both the adjunctions become equivalence when restricting to doctrines with comprehensive diagonal and rule of unique choice introduced in (Maietti, Pasquali, Rosolini 2017).

PEIRCEAN BICATEGORIES

DEFINITION

A Peircean bicategory consists of a Cartesian bicategory such that PROPOSITIONS A and B hold. Namely,

A) every homset carry a boolean algebra

B) for all maps $f\colon X \to Y$ and arrows $c\colon Y \to Z$, $f;c = \overline{f;\overline{c}}$

THEOREM $\mathbb{F}\mathbb{ob} \cong \mathbb{PB}$

The Calculus of Relations IN COMPUTER SCIENCE

Rewriting (e.g. Gavazzo 2023)

Relational databases (Codd 1970)

Proof Assistants (e.g. Pous 2013) $\Longleftarrow$ Lack of variable and quantifies

Foundations of program logics (Prat 1976) (Hoare,He, 1986)

$$P, Q \subseteq S^0 \times S^1$$

HL $\{P\}\, c\, \{Q\}$ $\boxed{P ; C \subseteq Q}$

NC $(P)\ c\ (Q)$ $\boxed{P \supseteq Q ; C^{op}}$

IL $[P]\, c\, [Q]$ $\boxed{P ; C \supseteq Q}$

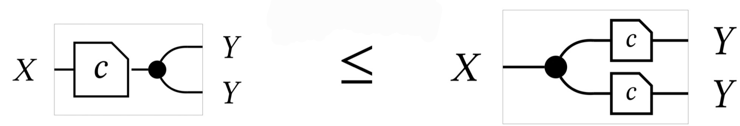
SIL $\langle\!\langle P \rangle\!\rangle\ c\ \langle\!\langle Q \rangle\!\rangle$ $\boxed{P \subseteq Q ; C^{op}}$

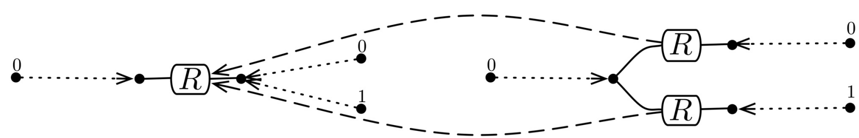
Conclusion

- Calculus of Neo Peircean relations as FO bicategory freely generated by a monoidal signature Σ
- For a first-order theory (Σ, E) freely generate a FO bicategory: Models are functors from such category to Rel
- The empty model does not require a special treatment
- Function symbols are not separate syntactic entities.
- No variables, No quantifiers
- Purely equational reasoning: Complete axiomatisation
- Proofs as rewrites (link to deep inference)

Future Work

- Beyond FOL: Higher order? Intuitionistic? Linear?
- Corresponding allegorical notions?
- Combinatorial characterisation by means of Hypergraphs?





(nested application conditions)

- Develop a proof theory and investigate the link with deep inference