A Skew Approach to Enrichment for Gray-Categories

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PSSL 107

Athens, 2 April 2023

INTRO & MOTIVATION

- Given $\mathcal{V}$ symm. monoidal cat $\Rightarrow$ $\mathcal{V}$-Cat symm. monoidal as well!

$\rightsquigarrow$ ITERATED ENRICHMENT!

- Strict higher Cats: $0\text{-}Cat := Set$
  $(n+1)\text{-}Cat := (n\text{-}Cat)\text{-}Cat$

- This is nice, but what about... weak higher Cats?

1^{st} PROBLEM & 1^{st} SOLUTION

- $\boxed{\text{1-d}}$ 1-Cat vs Cat : strict = weak ok
- $\boxed{\text{2-d}}$ 2-Cat vs Bicat : strict $\neq$ weak ... BUT any bicat ok is equivalent to a 2-Cat !
- $\boxed{\text{3-d}}$ 3-Cat vs Tricat : NOT every tricat is equivalent to a 3-Cat :(

... BUT $(2\text{-Cat}, \otimes, \mathbb{1})$-Cat are enough !

GRAY tensor product

Rmk We couldn't use the CARTESIAN product, but $\otimes$ is still monoidal !

... The Open Problem

- $\boxed{\text{4-d}}$? vs Tetracats
- AIM: We want a "nice" tensor product on Gray-Cat to go on

  enriched in $(2\text{-Cat}, \otimes, \mathbb{1})$

  ↳ Is it possible?
- PROBLEM: There is NO monoidal biclosed str. on Gray-Cat
  1. Capturing weak transformations (Crans, 1999)
  2. Interacting well with Lack's model structure on Gray-Cat (Bourke & Gurski, 2015)

LET'S LOOK AT PART 1.

- If $[A,\mathcal{B}]$ is an internal hom w/ 1-cells $\eta : F \Rightarrow G$ weak transf.

$\Leftrightarrow \quad \eta : \mathcal{2} \longrightarrow [A,\mathcal{B}]$ Gray-funct. $\overset{\text{BICLOSED}}{\Leftrightarrow}$ $\eta : A \longrightarrow [\mathcal{2},\mathcal{B}]$

$x \longmapsto \eta_x : Fx \to Gx$

$$\begin{tikzcd} x \arrow[d, "f"'] \\ y \end{tikzcd} \longmapsto \begin{tikzcd} Fx \arrow[r, "\eta_x"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & Gx \arrow[d, "Gf"] \\ Fy \arrow[r, "\eta_y"'] & Gy \end{tikzcd}$$

- PROBLEM: Functoriality in A force the equality

$$A \qquad \begin{tikzcd} x \arrow[d, "f"'] \\ y \arrow[d, "g"'] \\ z \end{tikzcd} \text{1-cells} \qquad \begin{tikzcd} Fx \arrow[r, "\eta_x"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & Gx \arrow[d, "Gf"] \\ Fy \arrow[r, "\eta_y"] \arrow[d, "Fg"'] \arrow[dr, phantom, "\Downarrow\eta_g"] & Gy \arrow[d, "Gg"] \\ Fz \arrow[r, "\eta_z"'] & Gz \end{tikzcd} = \begin{tikzcd} Fx \arrow[r, "\eta_x"] \arrow[d, "F(gf)"'] \arrow[dr, phantom, "\Downarrow\eta_{gf}"] & Gx \arrow[d, "G(gf)"] \\ Fz \arrow[r, "\eta_z"'] & Gz \end{tikzcd}$$

- This is quite unnatural

... In fact this would make weak transf.

NOT composable!

Let's look at Part 1.

- If $[A, B]$ is an internal hom w/ 1-cells $\eta : F \Rightarrow G$ weak transf.

$$\iff \quad \eta : \mathbf{2} \longrightarrow [A, B] \text{ Gray-funct.} \quad \overset{\text{BICLOSED}}{\iff} \quad \eta : A \longrightarrow [\mathbf{2}, B]$$

$$x \longmapsto \eta_x : Fx \longrightarrow Gx$$

$$\begin{tikzcd} x \arrow[d, "f"'] \\ y \end{tikzcd} \longmapsto \begin{tikzcd} Fx \arrow[r, "\eta_x"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & Gx \arrow[d, "Gf"] \\ Fy \arrow[r, "\eta_y"'] & Gy \end{tikzcd}$$

- PROBLEM: Functoriality in A force the equality

A

$$\begin{tikzcd} x \arrow[d, "f"'] \\ y \arrow[d, "g"'] \\ z \end{tikzcd}$$

1-cells

$$\begin{tikzcd} Fx \arrow[r, "\eta_x"] \arrow[d, "Ff"'] \arrow[dr, phantom, "\Downarrow\eta_f"] & Gx \arrow[d, "Gf"] \\ Fy \arrow[r, "\eta_y"] \arrow[d, "Fg"'] \arrow[dr, phantom, "\Downarrow\eta_g"] & Gy \arrow[d, "Gg"] \\ Fz \arrow[r, "\eta_z"'] & Gz \end{tikzcd} \overset{\eta_{g,f}}{\Rrightarrow} \begin{tikzcd} Fx \arrow[r, "\eta_x"] \arrow[d, "F(gf)"'] \arrow[dr, phantom, "\Downarrow\eta_{gf}"] & Gx \arrow[d, "G(gf)"] \\ Fz \arrow[r, "\eta_z"'] & Gz \end{tikzcd}$$

We want something like an inv. 3-cell $\eta_{g,f}$

Hint to the Solution for 1.

- We need a notion of weak map $A \rightsquigarrow [\mathcal{Z}, \mathcal{B}]$
- (Gohla, 2012) $\mathrm{Lax}(A, \mathcal{B})$ = Gray-Cat of pseudo maps of Gray-Cats

  $F : A \overset{\in}{\rightsquigarrow} \mathcal{B}$ consists of a 2-functor $F : A(x,y) \longrightarrow \mathcal{B}(Fx, Fy)$

  \+ $\forall\, x \xrightarrow{f} y \xrightarrow{g} z$, an invertible 2-cell

$$
\begin{tikzcd}
 & Fy \arrow[dr, bend left, "Fg"] \arrow[d, phantom, "\Downarrow F^2_{g,f}"] & \\
Fx \arrow[ur, bend left, "Ff"] \arrow[rr, bend right, "F(gf)"'] & {} & Fz
\end{tikzcd}
$$

Cocycle!

- WARNING: It's unlikely that we could find a representing tensor !

  ↳ In general cats of weak maps are poorly behaved...

If Not Monoidal ... What?

- Aim: Structure on Gray-Cat encoding a notion of weak map $A \rightsquigarrow B$ together with a tensor product $A \otimes B$ & internal hom $[A,B]$ interacting well.

- Solution: closed skew monoidal!

$$(a\otimes b)\otimes c \xrightarrow{\alpha} a\otimes(b\otimes c)\,,\quad i\otimes a \xrightarrow{\rho} a\,,\quad a \xrightarrow{\lambda} a\otimes i \quad \text{NOT inv.!}$$

- Weak maps $\dfrac{a \rightsquigarrow b}{i\otimes a \to b \in \mathcal{C}}$

(Strict maps $a \to b \in \mathcal{C}$)

- $\mathcal{C}(a\otimes b, c) \cong \mathcal{C}(a, [b,c])$

- What's the trick?

$$\mathcal{C}(a,b) \longrightarrow \mathcal{C}(i,[a,b]) \quad \text{NOT inv.!}$$

$i\otimes a \to b$, i.e. weak maps $a \rightsquigarrow b$

The Main Result

Theorem (Bourke & L.) There are closed skew monoidal structures:

- $(\text{Gray-Cat}, \otimes_\ell, \mathbb{1})$ with internal hom $\mathrm{Lax}(A, B)$ ⇝ LAX GRAY PRODUCT

- $(\text{Gray-Cat}, \otimes_p, \mathbb{1})$ with internal hom $\mathrm{Psd}(A, B)$ ⇝ PSEUDO GRAY PRODUCT

  ↪ Also symmetric!

OK... SO SKEW... WHAT?

$$\begin{tikzcd}
\text{SKEW MON.} \arrow[rr, leftrightarrow, squiggly] \arrow[dr, leftrightarrow, squiggly] & & \text{SKEW CLOSED} \arrow[dl, leftrightarrow, squiggly] \\
& \text{SKEW MULTI} &
\end{tikzcd}$$

- Skew Mon.? Defining ⊗ is hard...
- Skew Closed? $[-,-]$ easier to define!

  ... BUT $L : [B,C] \longrightarrow [[A,B],[A,C]]$ hard

- Skew Multicategories!

PROBLEM: Do we have to define n-ary multimaps $\forall n \in \mathbb{N}$?!

see L.'s PhD Thesis (Bourke & L.) Nope! The 4-ary structure is enough!

Skew Multicategories

(Bourke & Lack, 2018)

A skew multicategory $\mathbb{C}$ consists of:

- objects $\mathbb{C}_0$
- nullary maps $\mathbb{C}_0^{\ell}(-;a)$
- tight/loose n-multimaps $\mathbb{C}_n^{t}(a_1,..,a_n;b) \subseteq \mathbb{C}_n^{\ell}(a_1,..,a_n;b)$
- $1_a \in \mathbb{C}_1^{t}(a;a)$ identity
- substitutions $\circ_i : \mathbb{C}_m^{\ell}(\bar{b};c) \times \mathbb{C}_n^{\ell}(\bar{a};b_i) \longrightarrow \mathbb{C}_{n+m-1}^{\ell}(\bar{b}_{<i},\bar{a},\bar{b}_{>i};c)$

s.t.

- unit and associativity laws
- $g \circ_i f$ tight whenever:
  - $i=1$, g and f tight
  - $i \neq 1$ and g tight

K-ARY SKEW MULTICATEGORIES

A K-ary skew multicategory $\mathbb{C}$ consists of :

- objects $\mathbb{C}_0$
- nullary maps $\mathbb{C}^{\ell}_0(-;a)$
- tight/loose n-multimaps $\mathbb{C}^t_n(a_1,..,a_n;b) \subseteq \mathbb{C}^{\ell}_n(a_1,..,a_n;b)$ $\quad n \leq K$
- $1_a \in \mathbb{C}^t_1(a;a)$ identity
- substitutions $\circ_i : \mathbb{C}^{\ell}_m(\bar{b};c) \times \mathbb{C}^{\ell}_n(\bar{a};b_i) \longrightarrow \mathbb{C}^{\ell}_{n+m-1}(\bar{b}_{<i},\bar{a},\bar{b}_{>i};c)$ $\quad n,m,\ n+m-1 \leq K$

s.t.

- unit and associativity laws
- $g \circ_i f$ tight whenever:
  - $i=1$, g and f tight
  - $i \neq 1$ and g tight

Understanding Tight/Loose

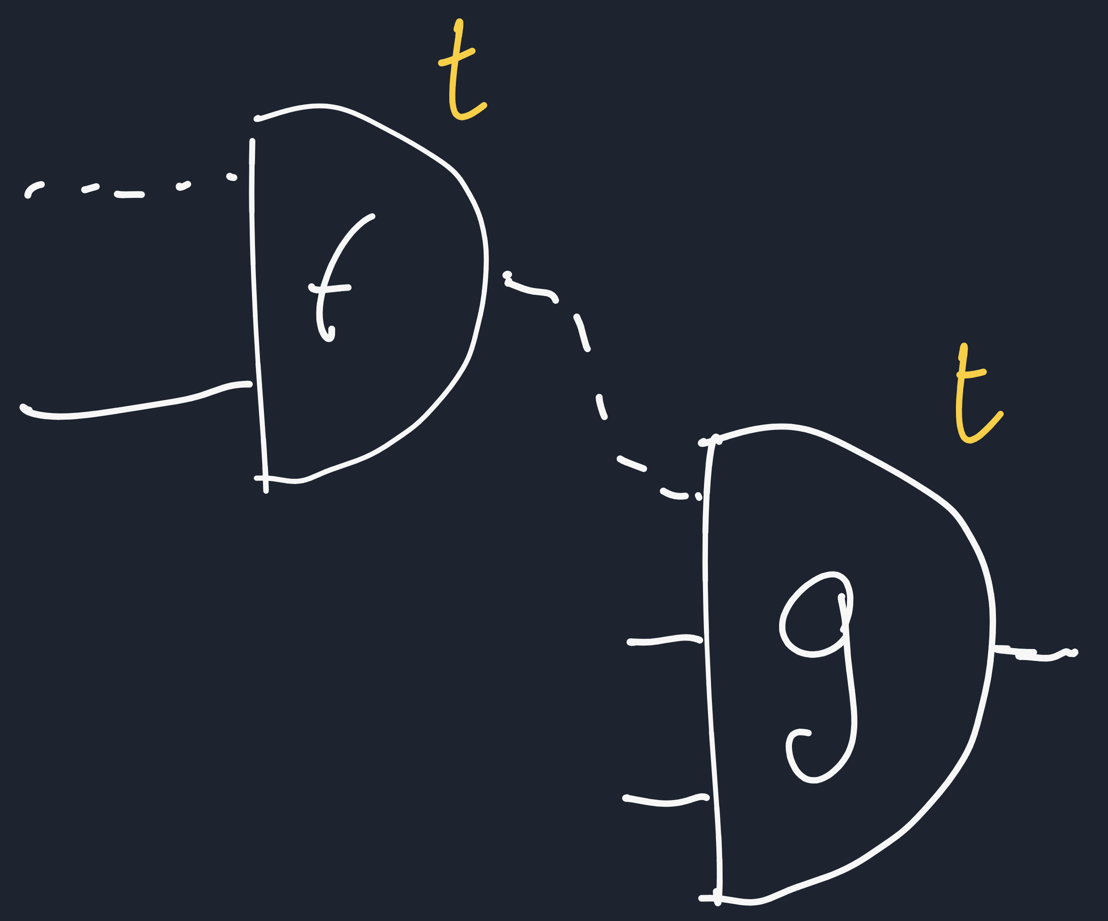
- Leading example: Cats w/ a choice of finite product

  loose n-ary $\rightsquigarrow$ funct. $A_1 \times \dots \times A_n \to B$ prod-preserving up-to-iso

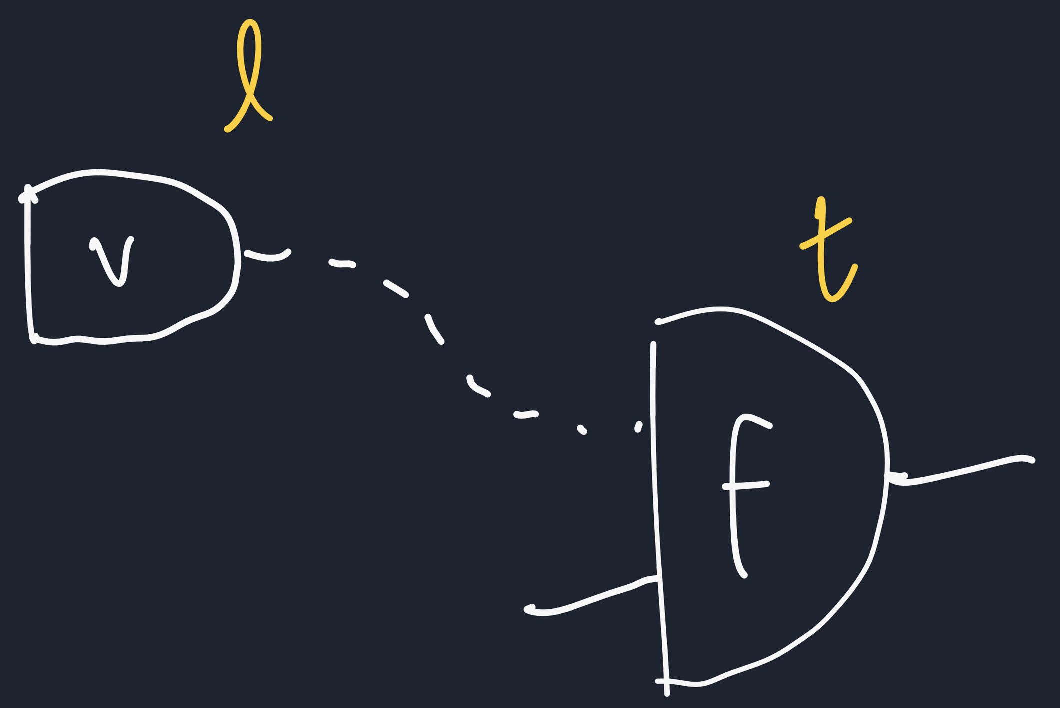
  tight n-ary $\rightsquigarrow$ " $A_1 \times \dots \times A_n \to B$ " "

  & STRICTLY in A_1

- Tight ~ strict in the 1st variable



= TIGHT

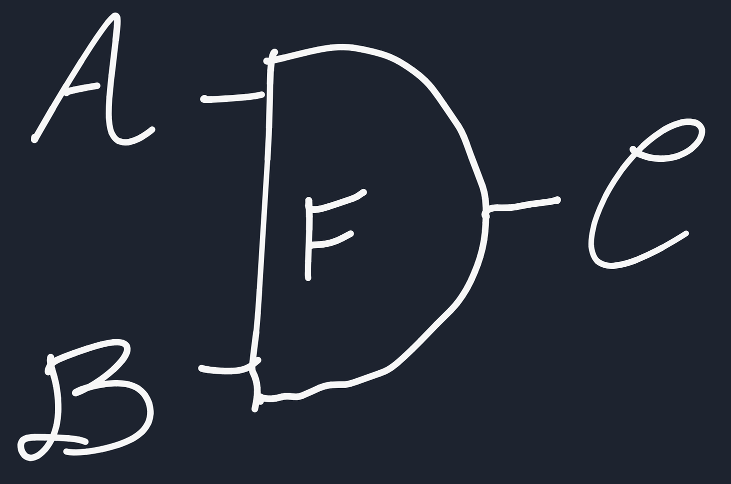


= LOOSE

↳ All nullary maps have to be LOOSE

THE SKEW MULTICATS $\mathbb{L}\mathrm{ax}$ / $\mathbb{P}\mathrm{sd}$

- $\boxed{\text{0-ary}}$ $\bullet \xrightarrow{x} A$, are objects $x \in A$
- $\boxed{\text{1-ary}}$ loose $A \rightsquigarrow B$ pseudo-maps
  tight $A \to B$ Gray-functors
- $\boxed{\text{n-ary}}$ "Inductively", e.g. binary loose



$\forall a \quad F^a : B \to C$ pseudo funct

$\forall b \quad F_b : A \to C$ pseudo funct

If F tight
F_b Gray-functor + ...

$+$

$\forall a \xrightarrow{f} a' \quad F^f : F^a \to F^{a'}$ lax transf

$\forall b \xrightarrow{g} b' \quad F_g : F^b \to F^{b'}$ oplax transf

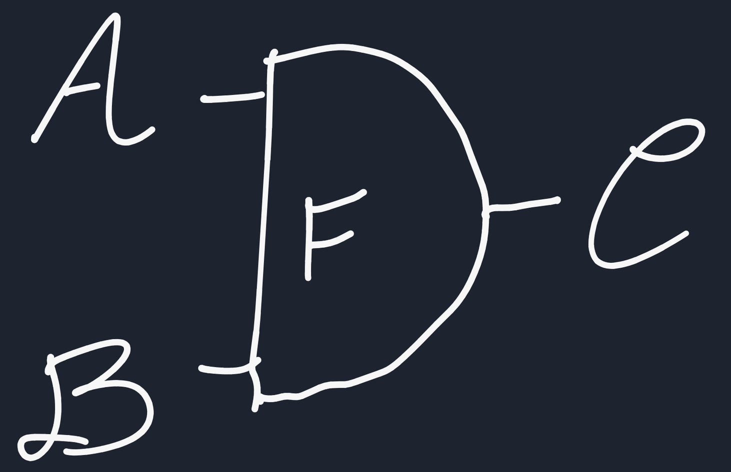
$\leftrightsquigarrow$ $A \rightsquigarrow \mathcal{L}ax(B, C)$

(Gray if F tight)

s.t. ooo

THE SKEW MULTICATS $\mathbb{L}\mathrm{ax}$ / $\mathbb{P}\mathrm{sd}$

- $\boxed{\text{0-ary}}$ $\bullet \xrightarrow{x} A$, are objects $x \in A$
- $\boxed{\text{1-ary}}$ loose $A \rightsquigarrow B$ pseudo-maps
  tight $A \longrightarrow B$ Gray-functors
- $\boxed{\text{n-ary}}$ "Inductively", e.g. binary loose



$\forall a \quad F^a : \mathcal{B} \longrightarrow \mathcal{C}$ pseudo funct

$\forall b \quad F_b : \mathcal{A} \longrightarrow \mathcal{C}$ pseudo funct

$\forall a \xrightarrow{f} a' \quad F^f : F^a \longrightarrow F^{a'}$ pseudo transf

$+$

$\forall b \xrightarrow{g} b' \quad F_g : F^b \longrightarrow F^{b'}$ pseudo transf

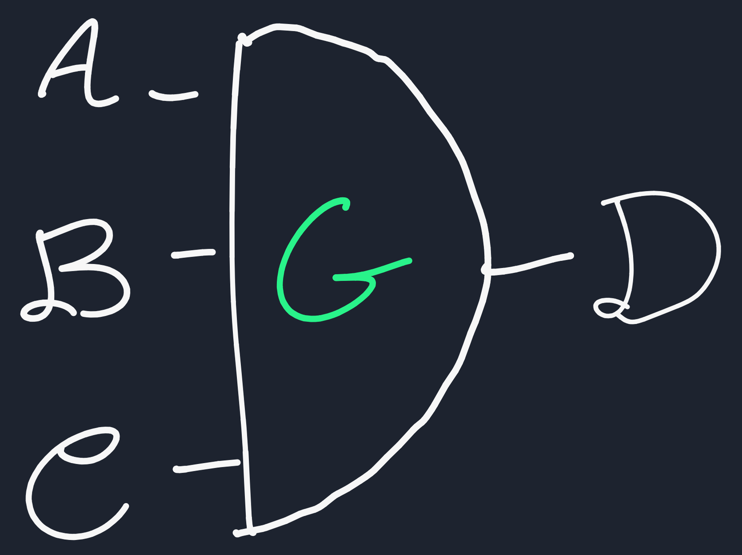
s.t. ooo

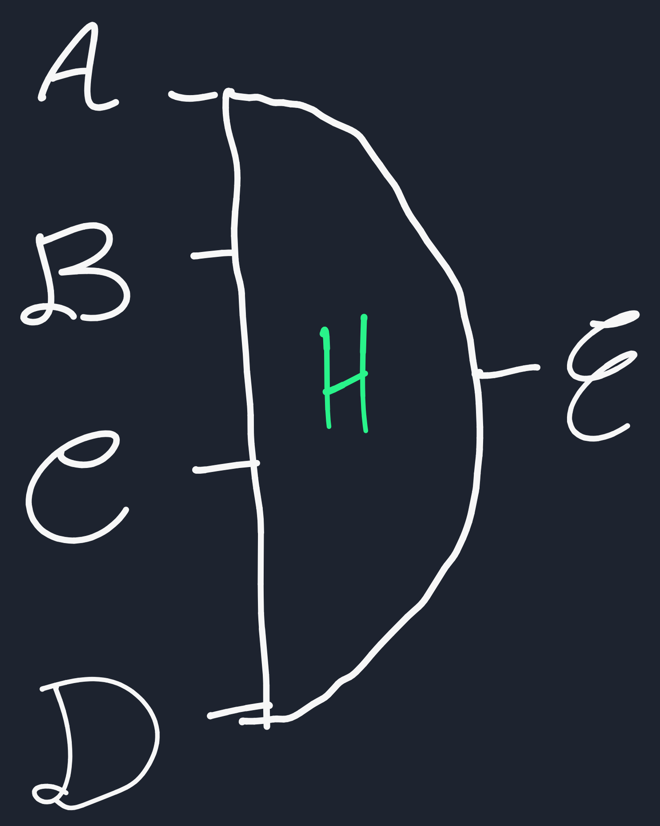
If F tight
F_b Gray-functor + ...

$\rightsquigarrow$

$\mathcal{A} \rightsquigarrow \mathrm{Psd}(\mathcal{B}, \mathcal{C})$
(Gray if F tight)

THE SKEW MULTICATS $\mathbb{L}ax$ / $\mathbb{P}sd$

- $\boxed{\text{0-ary}}$ $\bullet \xrightarrow{x} A$, are objects $x \in A$
- $\boxed{\text{1-ary}}$ loose $A \rightsquigarrow B$ pseudo-maps
  tight $A \longrightarrow B$ Gray-functors
- $\boxed{\text{n-ary}}$ "Inductively"
  - F binary $\leftrightsquigarrow$ $F^a : \mathcal{B} \longrightarrow \mathcal{C}$, $F_b : \mathcal{A} \longrightarrow \mathcal{C}$ unary + ... s.t. ...
  - 

    $\leftrightsquigarrow$ $G^a : \mathcal{B}, \mathcal{C} \longrightarrow \mathcal{D}$ $G_b : \mathcal{A}, \mathcal{C} \longrightarrow \mathcal{D}$
    $G_c : \mathcal{A}, \mathcal{B} \longrightarrow \mathcal{D}$ bim. + ... s.t. ...
  - 

    $\leftrightsquigarrow$ $H^a : \mathcal{B}, \mathcal{C}, \mathcal{D} \longrightarrow \mathcal{E}$ $H_b : \mathcal{A}, \mathcal{C}, \mathcal{D} \longrightarrow \mathcal{E}$
    $H_c : \mathcal{A}, \mathcal{B}, \mathcal{D} \longrightarrow \mathcal{E}$ $H_d : \mathcal{A}, \mathcal{B}, \mathcal{C} \longrightarrow \mathcal{E}$
    term. s.t. ...

SEMI STRICT 4-CATS (NOT AS BAD AS THEY SOUND)

• MEMO: 2-Cat $\rightsquigarrow$ Gray-Cat

$$\begin{tikzcd} x \arrow[d, bend right, "f"'] \arrow[d, bend left, "f'"] \arrow[d, phantom, "\overset{\alpha}{\Rightarrow}"] \\ y \arrow[d, bend right, "g"'] \arrow[d, bend left, "g'"] \arrow[d, phantom, "\overset{\beta}{\Rightarrow}"] \\ z \end{tikzcd} \qquad \begin{aligned} &(\beta\circ f')\cdot(g\circ\alpha) \\ &\qquad\quad \| \\ &(g'\circ\alpha)\cdot(\beta\circ f) \end{aligned} \quad \rightsquigarrow \quad \begin{tikzcd} gf \arrow[r, "g\circ\alpha"] \arrow[d, "\beta\circ f"'] \arrow[dr, phantom, "\Downarrow \beta{:}\alpha"] & gf' \arrow[d, "\beta\circ f'"] \\ g'f \arrow[r, "g'\circ\alpha"'] & g'f' \end{tikzcd}$$

$\boxed{\text{Interchange}}$

INVERTIBLE 3-CELL satisfying axioms (G1), (G2), (G3), (G4)

• IDEA: Gray-Cat $\rightsquigarrow$ Semi-strict 4-cat $:= (\text{Gray-Cat}, \otimes_p, 1\!\!1)$-Cat

Axioms (G1) — (G4) $\rightsquigarrow$ INVERTIBLE 4-CELLS (G1) — (G4)

true ... modulo cocycles

The End

Thanks for listening!

Post-credit: Homotopical Aspects

- Thm (Bourke, 2017) $\mathcal{C}$ model cat, $(\mathcal{C}, \otimes, \alpha, \rho, \lambda, I)$ skew mon.
  1. AXIOM (M) $\Rightarrow$ $\exists\ (Ho(\mathcal{C}), \otimes_\ell, I)$ a skew mon. str.
  2. Given 1., $\otimes_\ell$ is monoidal $\Leftrightarrow$ $\forall\ X, Y, Z$ cofibrant obj. α, ρ, λ are weak equivalences
- In our case we already know that $\forall\ A$ (NOT necc. cofib.) λ & ρ are weak equivalences!
- There is also a similar Thm. for mon. skew closed & for symmetry