Rpis

Lista 1 29.02.24

Zad 1 Sprawdzić:

(a) $\sum_{k=0}^{n}\binom{n}{k}p^k(1-p)^{n-k}=1$

Wzór na dwumian Newtona:
$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$
W naszym przypadku:
$x=1-p \quad y=p$

$$\sum_{k=0}^{n}\binom{n}{k}\underbrace{p^k}_{y}\underbrace{(1-p)}_{x}^{n-k}=$$

$$=((1-p)+p)^n=(1)^n=1 \blacksquare$$

(b) $\sum_{k=0}^{n}k\binom{n}{k}p^k(1-p)^{n-k}=np$

$k\binom{n}{k}=n\binom{n-1}{k-1}$

$$\sum_{\substack{k=0}}^{n}k\binom{n}{k}p^k(1-p)^{n-k}=\sum_{k=1}^{n}\underbrace{k\binom{n}{k}}p^k(1-p)^{n-k}=\sum_{k=1}^{n}\underbrace{n\binom{n-1}{k-1}}p^k(1-p)^{n-k}=$$

dla $k=0$ to suma $=0$

$$=\sum_{k=1}^{n}np\binom{n-1}{k-1}p^{k-1}(1-p)^{n-k}=np\cdot\sum_{k=1}^{n}\binom{n-1}{k-1}p^{k-1}(1-p)^{n-k}=np\cdot((1-p)+p)^n=np\cdot1^n=np$$ $\blacksquare$

Szereg Maclaurina

$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\dots+\frac{f^{(n)}(0)}{n!}x^n+R_n(x)$

a dla e^x wszystkie pochodne są 1 w zerze, więc

Zad 2 Sprawdzić:

(a) $\sum_{k=0}^{\infty}e^{-\lambda}\frac{\lambda^k}{k!}=1$

$$e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}=e^{-\lambda}\cdot\left(\lambda^0\cdot\frac{1}{0!}+\lambda^1\cdot\frac{1}{1!}+\lambda^2\cdot\frac{1}{2!}+\lambda^3\cdot\frac{1}{3!}+\dots\right)$$ → to jest ten szereg dla e^λ

$$=e^{-\lambda}\cdot e^{\lambda}=e^0=1 \blacksquare$$

(b) $\sum_{k=0}^{\infty}k\cdot e^{-\lambda}\frac{\lambda^k}{k!}=\lambda$

$$\sum_{k=0}^{\infty}k\cdot e^{-\lambda}\frac{\lambda^k}{k!}=e^{-\lambda}\cdot\overbrace{\sum_{k=0}^{\infty}\frac{k\cdot\lambda^k}{k!}}^{\text{to dla } k=0 \text{ równa się } 0}=e^{-\lambda}\cdot\sum_{k=1}^{\infty}\frac{\lambda^k}{(k-1)!}=e^{-\lambda}\cdot\lambda\cdot\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}=$$

liczymy od 1

Niech $\begin{cases}x=k-1\\x+1=k\end{cases}$

$$=e^{-\lambda}\cdot\lambda\cdot\underbrace{\sum_{x=0}^{\infty}\frac{\lambda^x}{x!}}=e^{-\lambda}\cdot e^{\lambda}\cdot\lambda=\lambda$$

z poprzedniego zadania to $=e^\lambda$ $\blacksquare$

$\boxed{\text{Zad 3}}$

Funkcja Γ-Eulera - to wartość całki:

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\, dt \quad , \; p > 0$$

Cel: $\Gamma(p) = (p-1)\Gamma(p-1) \qquad p \in \mathbb{R}_+$

w szczególności $\Gamma(n) = (n-1)! \qquad n \in \mathbb{N}$

$\boxed{1^\circ}$ $\Gamma(p) = (p-1)\Gamma(p-1)$

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\, dt = \left\{ \begin{array}{ll} u = t^{p-1} \xrightarrow{\text{pochodna}} & du = (p-1)t^{p-2} \\ dv = e^{-t} \xrightarrow{\text{całka}} & v = -e^{-t} \end{array} \right\} =$$

$$u \cdot v - \int v \cdot du$$

$$= -t^{p-1} e^{-t} - \int_0^\infty -e^{-t} \cdot (p-1) t^{p-2}\, dt = -t^{p-1} e^{-t} + \int_0^\infty e^{-t} \cdot (p-1) t^{p-2}\, dt =$$

$$\left\{ \lim_{t \to \infty} \frac{-t^{p-1}}{e^t} = \frac{-(p-1) \cdot t^{p-2}}{e^t} = \ldots = \lim_{t \to \infty} \frac{-(p-1)!\, t^0}{\underbrace{e^t}_{\to \infty}} = 0 \right.$$

(robimy de l'Hospitala $(p-1)$ razy)

$$= 0 + \int_0^\infty e^{-t} \cdot (p-1) t^{p-2}\, dt = (p-1) \underbrace{\int_0^\infty e^{-t} \cdot t^{p-2}\, dt}_{\Gamma(p-1)} = (p-1)\Gamma(p-1) \quad \blacksquare$$

$\boxed{2^\circ}$ $\Gamma(n) = (n-1)! \qquad n \in \mathbb{N}$

Dowód: indukcja po n.

Podstawa: Niech $n = 1$

$$\Gamma(1) = \int_0^\infty t^0 e^{-t}\, dt = \int_0^\infty e^{-t}\, dt = -e^{-t} \Big|_0^\infty = 0 - (-1) = 1 \;\; ?= (1-1)! = 0! \quad ✓$$

Krok: $n \to n+1$ Założenie: $\Gamma(n) = (n-1)!$

$$\Gamma(n+1) = \int_0^\infty t^n e^{-t}\, dt = \left\{ \begin{array}{ll} u = t^n & du = n \cdot t^{n-1} \\ dv = e^{-t} & v = -e^{-t} \end{array} \right\} = \underbrace{t^n \cdot (-e^{-t})}_{0} + \int_0^\infty e^{-t} \cdot n \cdot t^{n-1}\, dt =$$

$$u \cdot v - \int v \cdot du \qquad \text{(z poprzedniego pkt)}$$

$$= \int_0^\infty e^{-t} \cdot n \cdot t^{n-1}\, dt = n \cdot \underbrace{\int_0^\infty e^{-t} t^{n-1}}_{\Gamma(n)} =$$

$$= n \cdot \Gamma(n) \overset{\text{zał}}{=} n \cdot (n-1)! = n! \quad \blacksquare$$

Obliczyć: $f(x) = \lambda \cdot e^{-\lambda x}$ $\boxed{\text{zad 4}}$

a) $\int_0^\infty f(x)\,dx$

$$\int_0^\infty \lambda \cdot e^{-\lambda \cdot x}\,dx = \lambda \cdot \int_0^\infty e^{-\lambda \cdot x}\,dx = \left\{ \begin{array}{l} v = \lambda x \to dv = \lambda dx \\ dx = \frac{dv}{\lambda} \\ \text{(podstawienie} \end{array} \right\} =$$

$$= \lambda \cdot \int_0^\infty e^v \cdot \frac{1}{\lambda} \cdot dv = \frac{\cancel{\lambda}}{\cancel{\lambda}} \cdot \int_0^\infty e^v\,dv = -e^{-v} \Big|_0^\infty = 0 - (-1) = 1$$

b) $\int_0^\infty x \cdot f(x)\,dx$

$$\int_0^\infty x \cdot \lambda \cdot e^{-\lambda x} dx = \lambda \cdot \int_0^\infty x \cdot e^{-\lambda x} dx = \left\{ \begin{array}{ll} u = x & du = 1 \\ dv = e^{-\lambda x} & v = \int e^{-\lambda x} dx = \underbrace{\frac{1}{\lambda} \cdot \int e^{-v} dv}_{\text{z poprzedniego}} = -\frac{1}{\lambda} e^{-\lambda x} \\ u \cdot v - \int v\,du & \end{array} \right\}$$

$$= \lambda \cdot \left[\frac{-x}{\lambda} \cdot e^{-\lambda x} \Big|_0^\infty + \int_0^\infty \frac{1}{\lambda} \cdot e^{-\lambda x} \cdot 1 \right] =$$

$$= \lambda \cdot \left[-\frac{x}{\lambda} \cdot e^{-\lambda x} \Big|_0^\infty + \frac{1}{\lambda} \cdot \int_0^\infty e^{-\lambda x} \right] = \overbrace{-x \cdot e^{-\lambda x} \Big|_0^\infty}^{0} + \int_0^\infty e^{-\lambda x} =$$

$$\left\{ \lim_{x \to \infty} (-x \cdot e^{-\lambda x}) = \lim_{x \to \infty} \frac{-x}{e^{\lambda x}} \overset{d\text{-}H}{=} \lim_{x \to \infty} \frac{-1}{\underbrace{\lambda e^{\lambda x}}_{\to \infty}} = 0 \right.$$

$$= \int_0^\infty e^{-\lambda x} = \frac{1}{\lambda} \quad \cancel{= -\frac{1}{\lambda} \cdot e^{-\lambda x}} \quad \frac{1}{\lambda} \quad ? \quad \text{z a)}$$

$\boxed{\text{zad 5}}$ Wykazać, że $D_n = n$

$$D_n = \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ 1 & 1 & & & 0 \\ 1 & & 1 & & \\ \vdots & 0 & & \ddots & \\ 1 & & & & 1 \end{bmatrix} \quad \begin{array}{l} \text{dodajemy} \\ \text{do 1-go wiersza} \\ \text{wszystkie kolejne} \end{array} = \begin{bmatrix} n & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & & & & 0 \\ 1 & & 1 & & & \\ \vdots & & 0 & & \ddots & \\ 1 & & & & & 1 \end{bmatrix}$$

↓

to jest macierz $n \times n$

↓

więc jedynek jest n

Otrzymujemy, że D_n - ~~dolnotrójkątna~~ diagonalna → $\det(D_n) = n \cdot 1 \cdot 1 \cdot \ldots \cdot 1 = n$

zad 7 $\bar{s}$ – średnia ciągu $s_1, \dots, s_n$

Udowodnić:

a) $\sum_{k=1}^{n} (x_k - \bar{x})^2 = \sum_{k=1}^{n} x_k^2 - n \cdot \bar{x}^2$

$$\sum_{k=1}^{n} (x_k - \bar{x})^2 = \sum_{k=1}^{n} (x_k^2 - 2x_k\bar{x} + \bar{x}^2) =$$

$$= \sum_{k=1}^{n} x_k^2 - \sum_{k=1}^{n} 2x_k\bar{x} + \bar{x}^2 = \sum_{k=1}^{n} x_k^2 - (2x_1\bar{x} + 2x_2\bar{x} + \dots + 2x_n\bar{x}) + \bar{x}^2 \cdot n$$

$$= \sum_{k=1}^{n} x_k^2 - 2\bar{x}(x_1 + x_2 + \dots + x_n) + \bar{x}^2 n =$$

$$= \sum_{k=1}^{n} x_k^2 - 2\bar{x} \underbrace{\frac{(x_1 + x_2 + \dots + x_n) \cdot n}{n}}_{\text{średnia ciągu } x_1 \dots x_n} + \bar{x}^2 \cdot n =$$

$$= \sum_{k=1}^{n} x_k^2 - 2\bar{x}^2 \cdot n + \bar{x}^2 \cdot n = \sum_{k=1}^{n} x_k^2 - n \cdot \bar{x}^2 \quad \blacksquare$$

(b) $\sum_{k=1}^{n} (x_k - \bar{x})(y_k - \bar{y}) = \sum_{k=1}^{n} x_k y_k - n\bar{x}\bar{y}$

$$\sum_{k=1}^{n} (x_k - \bar{x})(y_k - \bar{y}) = \sum_{k=1}^{n} (x_k y_k - \bar{x} y_k - \bar{y} x_k + \bar{x}\bar{y}) =$$

$$= \sum_{k=1}^{n} x_k y_k - \sum_{k=1}^{n} \bar{x} y_k - \sum_{k=1}^{n} \bar{y} x_k + \sum_{k=1}^{n} \bar{x}\bar{y} =$$

$$= \sum_{k=1}^{n} x_k y_k - \bar{x} \underbrace{\sum_{k=1}^{n} y_k}_{\underbrace{\frac{y_1 + y_2 + \dots + y_n}{n} \cdot n}_{\bar{y} \Downarrow \bar{y} \cdot n}} - \bar{y} \sum_{k=1}^{n} x_k + n \cdot \bar{x}\bar{y} = \sum_{k=1}^{n} x_k y_k - \cancel{\bar{x}\bar{y} \cdot n} - \cancel{\bar{y}\bar{x} \cdot n} + n \cdot \bar{x}\bar{y} =$$

tak samo

$$= \sum_{k=1}^{n} x_k y_k + n \cdot \bar{x} \cdot \bar{y} \quad \blacksquare$$

Scanned with ACE Scanner

Udowodnić:

zad 9

$$\int_0^\infty x^k \lambda \cdot e^{-\lambda x} dx = \frac{k!}{\lambda^k} \qquad k=0,1,\dots, \qquad \lambda>0$$

Indukcja po k

Podstawa $k=0$

$$L = \int_0^\infty \lambda x^k \cdot e^{-\lambda x} dx = \lambda \int_0^\infty x^0 \cdot e^{-\lambda x} dx = \lambda \int e^{-\lambda x} dx = 1 \quad (\text{zad } 4-a)$$

$$P = \frac{0!}{\lambda^0} = \frac{1}{1} = 1 \qquad L = P \ ✓$$

Krok: $k \to k+1$ założenie:

$$\int_0^\infty x^k \lambda \cdot e^{-\lambda x} dx = \frac{k!}{\lambda^k}$$

$$\lambda \int_0^\infty x^k \cdot e^{-\lambda x} dx = \frac{k!}{\lambda^k}$$

$$\int_0^\infty x^k \cdot e^{-\lambda x} dx = \frac{k!}{\lambda^{k+1}}$$

Cel: $\int_0^\infty x^{k+1} \cdot \lambda \cdot e^{-\lambda x} dx = \frac{(k+1)!}{\lambda^{k+1}}$

$$\int_0^\infty x^{k+1} \lambda e^{-\lambda x} dx = \left\{ \begin{array}{ll} u = x^{k+1} & du = (k+1)x^k \\ dv = \lambda e^{-\lambda x} & v = \lambda \int e^{-\lambda x} dx = \lambda \int e^{-v} \frac{1}{\lambda} \cdot dv = \int e^{-v} = -e^{-v} = -e^{-\lambda x} \end{array} \right\}$$

$$\left(\begin{array}{l} u = \lambda x \\ du = \lambda dx \Rightarrow dx = \frac{1}{\lambda} \cdot dv \end{array} \right)$$

$$= \overbrace{-e^{-\lambda x} \cdot x^{k+1} \Big|_0^\infty}^{0} + \int_0^\infty e^{-\lambda x} \cdot (k+1)x^k dx$$

Robimy de-Hosp. k razy

$$\left\{ \lim_{x\to\infty} \frac{-x^{k+1}}{e^{\lambda x}} = \lim_{x\to\infty} \frac{-(k+1)x^k}{\lambda e^{\lambda x}} = \dots = \lim_{x\to\infty} \frac{-(k+1)!}{\lambda^k e^{\lambda x}} = 0 \right\}$$

$$= \int_0^\infty e^{-\lambda x} \cdot (k+1) x^k dx = (k+1) \underbrace{\int_0^\infty e^{-\lambda x} x^k dx}_{\text{z założenia to } = \frac{k!}{\lambda^{k+1}}} = \frac{(k+1)\cdot k!}{\lambda^{k+1}} = \frac{(k+1)!}{\lambda^{k+1}} \quad \blacksquare$$

zad 10 Obliczyć całkę nieoznaczoną:

$$\int x \cdot e^{-\frac{x^2}{2}} dx = \left\{ \begin{array}{ll} v = \frac{x^2}{2} & dv = \frac{2x}{2} dx = x \cdot dx \\ & dx = \frac{1}{x} \cdot dv \end{array} \right\} =$$

$$= \int x \cdot e^{-v} \cdot \frac{1}{x} dv = \int e^{-v} dv = -e^{-v} + C = -e^{-\frac{x^2}{2}} + C$$