RPIS

Lista 4

$\boxed{\text{Zad 3}}$ Funkcja $f_{xy}(x,y) = -xy + x$ $\quad 0 \leq x \leq 2$, $0 \leq y \leq 1$ $\quad$ $f_{xy}(x,y) = 0$ dla innych

Czy X, Y są niezależne? $\quad$ ppb $P(1 \leq X \leq 2, 0{,}5 \leq Y \leq 2)$ - ?

[Zmienne X, Y są niezależne $\Leftrightarrow \forall x,y \in \mathbb{R}$ $\underbrace{f(x,y)}_{\text{f. gęstości}} = \underbrace{f_1(x) \cdot f_2(y)}_{\text{rozkłady brzegowe gęstości}}$]

$$f_1(x) = \int_0^1 f(x,y)\,dy = -\int_0^1 xy\,dy + \int_0^1 x\,dy = -\frac{x}{2} \cdot y^2 \Big|_0^1 + xy\Big|_0^1 = -\frac{x}{2} + x = \frac{x}{2}$$

$$f_2(y) = \int_0^2 f(x,y)\,dx = -\int_0^2 xy\,dx + \int_0^2 x\,dx = -\frac{y}{2} \cdot x^2\Big|_0^2 + \frac{x^2}{2}\Big|_0^2 = -\frac{4}{2}y + \frac{4}{2} = -2y + 2$$

$$f_1(x) \cdot f_2(y) = \frac{x}{2} \cdot (-2y + 2) = -xy + x$$

$$f(x,y) = f_1(x) \cdot f_2(y) \Rightarrow \text{zmienne są niezależne}$$

$$P(1 \leq X \leq 2, 0{,}5 \leq Y \leq 2) = P(1 \leq X \leq 2) \cdot P(0{,}5 \leq Y \leq 2) = \int_1^2 \frac{x}{2}\,dx \cdot \int_{\frac{1}{2}}^{1} 2(1-y)\,dy =$$

$$= \frac{x^2}{4}\Big|_1^2 \cdot 2 \cdot \left(\int_{\frac{1}{2}}^1 1\,dy - \int_{\frac{1}{2}}^1 y\,dy\right) = \left(\frac{4}{4} - \frac{1}{4}\right) \cdot 2 \cdot \left(y\Big|_{\frac{1}{2}}^1 - \frac{y^2}{2}\Big|_{\frac{1}{2}}^1\right) = \frac{3}{4} \cdot 2 \cdot \left(\frac{1}{2} - \left(\frac{1}{2} - \frac{1}{8}\right)\right) =$$

$$= \frac{3}{2} \cdot \left(\frac{1}{2} - \frac{3}{8}\right) = \frac{3}{4} - \frac{9}{16} = \frac{3}{16}$$

$\boxed{\text{Zad 5}}$ Niech X - ciągła zmienna losowa

$Y = F_X(X)$ $\quad$ Cel: $Y \sim U[0;1]$

dystrybuanta X (Y - wynik)

$F_X(x)$ jest zawsze z przedziału $[0,1]$ (bo to jest funkcja prawdopodobieństwa)

$$\downarrow$$

$$Y \in [0,1]$$

$Y \sim U[0;1] \Leftrightarrow f_Y(y) = 1$

„ma rozkład jednostajny" $\quad$ „każda wartość ma prawdopodobieństwo 1"

$$F_Y(y) = P(Y < y) \overset{\text{treść}}{=} P(F_X(X) < y) = P(X < F_X^{-1}(y)) =$$

$$= F_X(F_X^{-1}(y)) = y$$

$$f_Y(y) = (F_Y(y))' = 1 \quad \blacksquare$$

Zad 2) Czy można tak dobrać stałą C, aby $f_{xy}(x,y) = Cxy + x + 2y$ dla $0 \le x \le 3$, $1 \le y \le 2$ była gęstością

$$\int_1^2\int_0^3 f(x,y)\,dx\,dy = 1$$

$$\int_1^2\int_0^3 Cxy + x + 2y\,dx\,dy = 1$$

$$\int_1^2\int_0^3 Cxy+x+2y\,dx\,dy = C\cdot\int_1^2\int_0^3 xy\,dx\,dy + \int_1^2\int_0^3 x\,dx\,dy + 2\int_1^2\int_0^3 y\,dx\,dy =$$

$$= C\int_1^2 y\cdot\frac{x^2}{2}\Big|_0^3 dy + \int_1^2 \frac{x^2}{2}\Big|_0^3 dy + 2\int_1^2 x\cdot y\Big|_0^3 = C\cdot\int_1^2 y\cdot\frac{9}{2}\,dy + \int_1^2 \frac{9}{2}\,dy + 2\int_1^2 3y =$$

$$= C\cdot\frac{9}{2}\cdot\frac{y^2}{2}\Big|_1^2 + \frac{9}{2}\cdot y\Big|_1^2 + \cancel{2}\cdot 3\cdot\frac{y^2}{\cancel{2}}\Big|_1^2 = C\cdot\frac{9}{2}\cdot\frac{3}{2} + \frac{9}{2} + 9 =$$

$$= C\cdot\frac{27}{4} + \frac{9}{2} + 9 = \frac{27C+18+36}{4} = \frac{27C+54}{4} = 1$$

$$27C + 54 = 4$$
$$27C = -50$$
$$C = -\frac{50}{27}$$

$$f(3,1) = -\frac{50}{27}\cdot 3\cdot 1 + 3 + 2 = -\frac{150}{27} + 5 < 0$$

Więc f nie jest gęstością

↓

nie można dobrać C :(

Zad 8) Niech X podlega stan. rozkł. Cauchy'ego

$$f_X(x) = \frac{1}{\pi(1+x^2)}, \quad x\in\mathbb{R}$$

Cel: $Y = \frac{1}{X}$ ma również ten rozkład $\left(f_Y(y) = \frac{1}{\pi(1+y^2)}\right)$

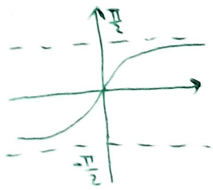
1) $F_X(t) = \int_{-\infty}^{t} \frac{1}{\pi(1+x^2)}\,dx =$

Czyli musimy obliczyć gęstość Y

$$\boxed{F_Y(y)} = P(Y\le y) = P\left(\frac{1}{X}\le y\right) = P\left(\frac{1}{y}\le X\right) = \underbrace{1}_{\text{wszystkie opcje}} - \underbrace{\overbrace{P\left(X<\frac{1}{y}\right)}^{F_X\left(\frac{1}{y}\right)}}_{\text{te, które nigdy się nie spełnią}} = 1 - F_X\left(\frac{1}{y}\right) =$$

$$= 1 - \int_{-\infty}^{\frac{1}{y}} \frac{1}{\pi(1+x^2)}\,dx = 1 - \frac{1}{\pi}\left(\text{arctg}\left(\frac{1}{y}\right) - \text{arctg}(-\infty)\right) =$$

$\lim_{x\to-\infty} \text{arctg}(x) = -\frac{\pi}{2}$



$$= 1 - \frac{1}{\pi}\left(\text{arctg}\left(\frac{1}{y}\right) - \left(-\frac{\pi}{2}\right)\right) = 1 - \frac{\text{arctg}\left(\frac{1}{y}\right)}{\pi} - \frac{1}{2} = \boxed{\frac{1}{2} - \frac{\text{arctg}\left(\frac{1}{y}\right)}{\pi}}$$

$$f_Y(y) = F'(y) = \left(\frac{1}{2} - \frac{\text{arctg}\left(\frac{1}{y}\right)}{\pi}\right)' = 0 - \frac{1}{\pi}\left(\text{arctg}\left(\frac{1}{y}\right)\right)' =$$

$$= -\frac{1}{\pi}\cdot\frac{1}{\left(1+\frac{1}{y^2}\right)}\cdot\left(\frac{1}{y}\right)' = +\frac{1}{\pi}\cdot\frac{1}{\left(1+\frac{1}{y^2}\right)\cdot y^2} = \frac{1}{\pi}\cdot\frac{1}{y^2+1} = \frac{1}{\pi\cdot(1+y^2)}$$

Czyli y ma ten rozkład

Niech $Y = X^2$ (X na $\mathbb{R}$)

Cel: $f_Y(y) = \dfrac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}}$ $\quad y > 0$

$F_X(t) = \int_{-\infty}^{t} f_X(x)\,dx$

$F_Y(t) = P(Y \le t) = P(X^2 \le t) =$
$= P(-\sqrt{t} \le X \le \sqrt{t}) = P(X \le \sqrt{t}) - P(X \le -\sqrt{t}) =$
$= F_X(\sqrt{t}) - F_X(-\sqrt{t})$

$f_Y(t) = F_Y'(t) = \left(F_X(\sqrt{t}) - F_X(-\sqrt{t})\right)' = \left(F_X(\sqrt{t})\right)' - \left(F_X(-\sqrt{t})\right)' =$

$= (\sqrt{t})' \cdot \underbrace{F_X'(\sqrt{t})}_{f_X(\sqrt{t})} - (-\sqrt{t})' \cdot \underbrace{F_X'(-\sqrt{t})}_{f_X(-\sqrt{t})} = \frac{1}{2\sqrt{t}} \cdot f_X(\sqrt{t}) + \frac{1}{2\sqrt{t}} \cdot f_X(-\sqrt{t}) =$

$= \dfrac{f_X(\sqrt{t}) + f_X(-\sqrt{t})}{2\sqrt{t}}$ ■

Zad 7 (X, Y) ma gęstość $f(x,y) = \frac{1}{2\pi}$ dla $\frac{x^2}{4} + y^2 \le 1$

EX, EY, $E(X \cdot Y)$ - ? Czy X, Y są niezależne?

$x^2 + y^2 \le 1$
$y^2 \le 1 - \frac{x^2}{4}$
$-\sqrt{1 - \frac{x^2}{4}} \le y \le \sqrt{1 - \frac{x^2}{4}}$

$\frac{x^2}{4} + y^2 \le 1$
$x^2 \le 4 - 4y^2$
$-\sqrt{4 - 4y^2} \le x \le \sqrt{4 - 4y^2}$

$\frac{x^2}{4} + y^2 \le 1 \Rightarrow \frac{x^2}{4} \le 1 \wedge y^2 \le 1$

$x^2 \le 4$ $\qquad -1 \le y \le 1$
$-2 \le x \le 2$ $\qquad \Downarrow$
$\Downarrow$ $\qquad y \in [-1, 1]$
$x \in [-2, 2]$

$f(0,0)$

$\frac{2 \cdot \sqrt{1-y^2}}{\pi} \cdot \frac{\sqrt{1 - \frac{x^2}{4}}}{\pi} = \frac{2}{\pi} \cdot \frac{1}{\pi} = \frac{2}{\pi^2}$

są zależne $\neq \frac{1}{2\pi}$

$f_1(x) = f_X(x) = \int_{\mathbb{R}} f(x,y)\,dy =$

$= \int_{-\sqrt{1-\frac{x^2}{4}}}^{\sqrt{1-\frac{x^2}{4}}} \frac{1}{2\pi}\,dy = \frac{1}{2\pi} \cdot y \Big|_{-\sqrt{\dots}}^{\sqrt{\dots}} =$

$= \frac{1}{2\pi} \cdot \sqrt{1 - \frac{x^2}{4}} + \frac{1}{2\pi} \cdot \sqrt{1 - \frac{x^2}{4}} = 2 \cdot \frac{1}{2\pi} \sqrt{1 - \frac{x^2}{4}} =$

$= \boxed{\dfrac{\sqrt{1 - \frac{x^2}{4}}}{\pi}}$

$f_2(y) = f_Y(y) = \int_{\mathbb{R}} f(x,y)\,dx =$

$= \int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}} \frac{1}{2\pi}\,dx = \frac{1}{2\pi} \cdot x \Big|_{-\sqrt{\dots}}^{\sqrt{\dots}} =$

$= \frac{1}{2\pi} \cdot 2\sqrt{4 - 4y^2} = \frac{\sqrt{4-4y^2}}{\pi} = \boxed{\dfrac{2 \cdot \sqrt{1-y^2}}{\pi}}$

$EX = \int_{\mathbb{R}} f_1(x) \cdot x\,dx =$

$= \int_{-2}^{2} \frac{1}{\pi} \cdot \sqrt{1 - \frac{x^2}{4}} \cdot x\,dx$

$\int \sqrt{1 - \frac{x^2}{4}} \cdot x\,dx = \begin{bmatrix} u = 1 - \frac{x^2}{4} \\ du = -\frac{x}{2}\,dx \\ dx = -\frac{2}{x}\,du \end{bmatrix} = \int -2 \cdot \sqrt{u}\,du =$

$= -2 \cdot \frac{2}{3} \cdot u^{\frac{3}{2}} = -\frac{4}{3} \cdot \left(1 - \frac{x^2}{4}\right)^{\frac{3}{2}}$ *

$= \frac{1}{\pi} \cdot \left(-\frac{4}{3}\left(1 - \frac{x^2}{4}\right)^{\frac{3}{2}}\right) \Big|_{-2}^{2} = 0$

$EY = \int_{-1}^{1} \frac{1}{\pi} \cdot 2\sqrt{1-y^2} \cdot y \cdot dy$

$\int \sqrt{1-y^2} \cdot y\,dy = \begin{bmatrix} 1 - y^2 = u \\ -2y\,dy = du \\ y\,dy = -\frac{du}{2} \end{bmatrix} = \int \sqrt{u} \cdot \left(-\frac{1}{2}\right) du =$

$= -\frac{1}{2} \cdot \frac{2}{3} \cdot u^{\frac{3}{2}} = -\frac{1}{3} \cdot (1 - y^2)^{\frac{3}{2}}$

$= \frac{1}{\pi} \cdot \left(-\frac{2}{3} \cdot (1-y^2)^{\frac{3}{2}}\right) \Big|_{-1}^{1} = 0$

$E(X \cdot Y) = \int_{-1}^{1} \int_{-2}^{2} x \cdot y \cdot f_1(x) \cdot f_2(y)\,dx\,dy =$

$= \int_{-1}^{1} y \cdot f_2(y) \cdot \underbrace{\left(-\frac{4}{3}\left(1 - \frac{x^2}{4}\right)^{\frac{3}{2}}\right) \Big|_{-2}^{2}}_{0}\,dy = 0$ *