RPIS

Lista 3 14.03.24 | 3,5 - 1

$\boxed{\text{zad } 1}$

(a) X - zmienna losowa ; ma gęstość $f(x) = 2x$ dla $x \in [0,1]$

Dystrybuanta? Gęstość zmiennej $Y = X^2$?

$$F_x(t) = P(X \le t) = \int_0^t 2x\,dx = x^2\Big|_0^t = t^2 \quad \text{dla } t \in [0,1]$$

Musimy znaleźć funkcję gęstości dla $Y = X^2$

Niech $t \in [0,1]$

$$F_y(t) = P(X^2 \le t) = P(X \le \sqrt{t}) = F_x(\sqrt{t})$$

$$f_y(t) = F_y'(t) = (F_x(\sqrt{t}))' = f_x(\sqrt{t}) \cdot \frac{1}{2\sqrt{t}} = \underbrace{2\sqrt{t}}_{f(x)=2x} \cdot \frac{1}{2\sqrt{t}} = 1$$

$f(x) = 2x \to f(\sqrt{t}) = 2\sqrt{t}$

(b) X - zmienna losowa ; gęstość $f(x) = 1{,}5 \cdot \sqrt{x}$ dla $x \in [0,1]$

$F(x) - ?$ $f(x)$ of $Y = X^2$

$$F_x(t) = P(X \le t) = \int_0^t 1{,}5\sqrt{x}\,dx = t^{\frac{3}{2}} \quad t \in [0,1]$$

$$F_y(t) = P(X^2 \le t) = P(X \le \sqrt{t}) = F_x(\sqrt{t})$$

$$f_y(t) = (F_y(t))' = (F_x(\sqrt{t}))' = f_x(\sqrt{t}) \cdot (\sqrt{t})' = 1{,}5 \cdot t^{\frac{1}{4}} \cdot \frac{1}{2\sqrt{t}} = \frac{3}{4} \cdot t^{\frac{1}{4}} \cdot \frac{1}{2t^{\frac{1}{2}}} =$$

$$= \frac{3}{4} \cdot \frac{1}{2} \cdot t^{+\frac{1}{4}} \cdot t^{-\frac{1}{2}} = \frac{3}{8} \cdot t^{-\frac{1}{4}}$$

(c) Niech $X \sim U[-2,2]$. Rozkład zmiennej $Y = |X|$ - ?

$$F_y(t) = P(Y \le t) = P(|X| \le t) = P(-t \le X \le t) = P(X \le t) - P(X \le -t) =$$

$$= F_x(t) - F_x(-t)$$

$$F_y(t) = F_x(t) - F_x(-t) \quad |\ ()'$$

$U(a,b) = \frac{1}{b-a}$ dla $[a,b]$

$$f_y(t) = f_x(t) + f_x(-t) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

(d) $X \sim U[-1,1]$ $Y = X^3$, $Z = X^2$?

$$U(-1,1) = \begin{cases} 1/2 & \text{dla } x \in [-1,1] \\ 0 & \text{wpp} \end{cases}$$

- $F_y(t) = P(Y \le t) = P(X \le \sqrt[3]{t}) = F_y(\sqrt[3]{t}) = \frac{1}{2}(\sqrt[3]{t} + 1)$

  $f_y(x) = F_y'(t) = \frac{1}{2} \cdot \frac{1}{3} \cdot t^{-\frac{2}{3}} = \frac{1}{6} t^{-\frac{2}{3}}$

- $F_z(t) = P(X^2 \le t) = P(X \le t) - P(X \le -\sqrt{t}) =$

  $= F_x(\sqrt{t}) - F_x(-\sqrt{t}) = \frac{1}{2}(\sqrt{t} + 1) - \frac{1}{2}(-\sqrt{t} + 1) = \frac{1}{2}(2\sqrt{t}) = \sqrt{t}$

  $f_z(\cancel{t}) = F_z'(t) = (\sqrt{t})' = \frac{1}{2t^{\frac{1}{2}}} = \frac{1}{2\sqrt{t}}$

Zad 3 X ma rozkład $B(n_1,p)$, Y —||— $B(n_2,p)$ $\}$ są niezależne

X, Y są niezależne, gdy $P(X=x_i, Y=y_k) = P(X=x_i)\cdot P(Y=y_k)$ (dla dyskretnych)

$p_k = \binom{n}{k} p^k (1-p)^{n-k}$

Cel: $Z = X+Y$ ma rozkład $B(n_1+n_2, p)$

Dla $Z = X+Y$ musimy zsumować ppb wszystkich przypadków, w których $X+Y$ osiąga k

↓

uwzględniamy wszystkie możliwości dla $X(i)$ i $Y(k-i)$

(bo one mogą pochodzić jak z X, tak i z Y

$$P(Z=k) = P(X+Y=k) = \sum_{i=0}^{k} P(X=i, Y=k-i) \overset{\text{niezal.}}{=} \sum_{i=0}^{k} P(X=i)\cdot P(Y=k-i) =$$

z rozkładu (def.)

$$= \sum_{i=0}^{k} \underbrace{\binom{n_1}{i} p^i (1-p)^{n_1-i}}\cdot\underbrace{\binom{n_2}{k-i} p^{k-i} (1-p)^{n_2-(k-i)}} = \sum_{i=0}^{k}\binom{n_1}{i}\binom{n_2}{k-i} p^k (1-p)^{(n_1+n_2)-k} =$$

$$p^k (1-p)^{(n_1+n_2)-k}\cdot\sum_{i=0}^{k}\binom{n_1}{i}\binom{n_2}{k-i} = p^k (1-p)^{n_1+n_2-k}\cdot\binom{n_1+n_2}{k} \Rightarrow \text{ma rozkład } B(n_1+n_2, p)$$

(tożsamość Cauchy'ego)

Zad 4 Niezależne X, Y mają rozkład Poissona z parametrami λ_1, λ_2

Cel: $Z = X+Y$ ma rozkład Poissona z parametrami $\lambda_1+\lambda_2$

Rozkład Poissona: $p_k = P(X=k) = e^{-\lambda}\cdot\frac{\lambda^k}{k!}$

$$P(Z=k) = P(X+Y=k) = \sum_{i=0}^{k} P(X=i)P(Y=k-i) =$$

$$= \sum_{i=0}^{k} \underbrace{e^{-\lambda_1}\cdot\frac{\lambda_1^i}{i!}}\cdot\underbrace{e^{-\lambda_2}\cdot\frac{\lambda_2^{k-i}}{(k-i)!}} = \sum_{i=0}^{k} e^{-(\lambda_1+\lambda_2)}\cdot\frac{\lambda_1^i\lambda_2^{k-i}}{i!\,(k-i)!} =$$

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}\cdot x^{n-k}\cdot y^k$$

$$(\lambda_1+\lambda_2)^k = \sum\binom{k}{i}\cdot\lambda_2^{k-i}\cdot\lambda_1^i$$

$$= e^{-(\lambda_1+\lambda_2)}\cdot\sum_{i=0}^{k}\frac{\lambda_1^i\lambda_2^{k-i}}{i!\cdot(k-i)!}\cdot\frac{k!}{k!} = e^{-(\lambda_1+\lambda_2)}\cdot\sum_{i=0}^{k}\frac{1}{k!}\cdot\underbrace{\frac{k!}{i!(k-i)!}}_{\binom{k}{i}}\cdot\lambda_1^i\cdot\lambda_2^{k-i} =$$

$$= e^{-(\lambda_1+\lambda_2)}\cdot\frac{1}{k!}\cdot\underbrace{(\lambda_1+\lambda_2)^k} = e^{-(\lambda_1+\lambda_2)}\cdot\frac{(\lambda_1+\lambda_2)^k}{k!} = \text{Poisson}(\lambda_1+\lambda_2)$$

zad 7

$f(x,y) = 6xy$

$0 < x < 2$
$0 < y < 1 - \frac{1}{2}x$

$f_x(x) - ?$
$f_y(y) - ?$

$$f_x(x) = \int_0^{1-\frac{1}{2}x} 6xy\,dy = \cancel{6}^{3} \cdot x \cdot \frac{y^2}{\cancel{2}} \Big|_0^{1-\frac{1}{2}x} = 3x \cdot \left(1 - \frac{1}{2}x\right)^2 =$$

$$= 3x \cdot \left(1 - x + \frac{1}{4}x^2\right) = 3x - 3x^2 + \frac{3x^3}{4}$$

$$f_y(y) = \int_0^2 6xy\,dx = 3x^2y \Big|_0^2 = 3 \cdot 4 \cdot y = 12y$$

zad 8

alfa	A	α		
beta	B	β		
(d)zeta	~~Δ δ~~	Z	ζ	
eta	H	η		
lambda	Λ	λ		
chi	X	χ		
ksi	Ξ	ξ		
fi	Φ	ψ		
rho	P	ρ		

A	α
B	β
Δ	δ
H	η
Λ	λ
X	χ
Ξ	ξ
Φ	ψ
P	ρ

zad 10

$|A_1| = 40$

$|A_2| = 32$

$|A_3| = 18$

$|A_4| = 10$

wartość X - zbiór, z którego pochodzi element

wartość Y - moc losowo wybranego zbioru

X

x_i	40	32	18	10
p_i	$\frac{40}{100}$	$\frac{32}{100}$	$\frac{18}{100}$	$\frac{10}{100}$

$E(x) = \sum x_i p_i = \cancel{40} \cdot \frac{\cancel{40}}{\cancel{100}} + 32 \cdot \frac{32}{100} + 18 \cdot \frac{18}{100} + 10 \cdot \frac{10}{100} =$

$= 16 + 10,24 + 3,24 + 1 = 30,48$

Y

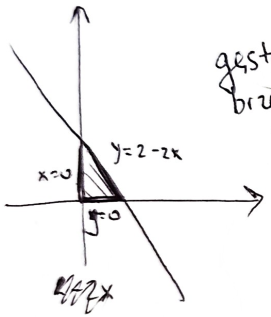
y_i	40	32	18	10
p_i	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

$E(y) = \sum y_i p_i = \frac{40}{4} + \frac{32}{4} + \frac{18}{4} + \frac{10}{4} =$

$= 10 + 8 + 4,5 + 2,5 = 25$

zad 6

gęstość $f(x,y) = 15x^2 y$



gęstość brzegowa $- f_1(x) = \int_0^{2-2x} 15x^2 y \, dy = \frac{15}{2} x^2 y \Big|_0^{2-2x} = 30x^2(1-x)^2$ całka po y

wart. oczekiwana $EX = \int_0^1 x \, 30x^2(1-x)^2 \, dx =$

$= 30 \int_0^1 x^3 (x^2 - 2x + 1) \, dx = 30 \int_0^1 x^5 - 2x^4 + x^3 \, dx =$

$= 30 \left(\frac{x^6}{6} \Big|_0^1 - \frac{2x^5}{5} \Big|_0^1 + \frac{x^4}{4} \Big|_0^1 \right) = 30 \left(\frac{1}{6} - \frac{2}{5} + \frac{1}{4} \right) =$

$= 30 \left(\frac{5}{30} - \frac{12}{30} + \frac{7,5}{30} \right) = \frac{1}{2}$

$\boxed{\text{zad 9}}$ z ćwiczeń

1) wersja

$$E(X) = p\sum_{k=1} k\cdot(1-p)^{k-1} = p\left(1 + 2(1-p) + 3(1-p)^2 + \dots\right) =$$

$$= p\left(\sum_{k=0}(1-p)^k + \sum_{k=1}(1-p)^k + \dots\right) = p\left(\sum_{k=0}(1-p)^k + (1-p)\sum_{k=0}(1-p)^k + (1-p)^2\sum_{k=0}(1-p)^k + \dots\right)$$

$$= p\left(\frac{1}{1-(1-p)} + \frac{1-p}{1-(1-p)} + \dots\right) = 1 + 1 - p + \dots = \frac{1}{1-(1-p)} = \frac{1}{p} \quad \blacksquare$$

2) wersja

$$E(X) = \sum_{k=1} kp(1-p)^{k-1} = -p\sum_{k=1}\underbrace{-k(1-p)^{k-1}}_{[(1-p)^k]'} = -p\left[\sum_{k=1}(1-p)^k\right]' =$$

$$= -p\cdot\left[\frac{(1-p)}{1-(1-p)}\right]' = -p\left[\frac{1}{p} - 1\right]' = -p\cdot\frac{-1}{p^2} = \frac{1}{p}$$

$$V(X) = \sum_{k\geq 1}\left(k - \frac{1}{p}\right)^2 p(1-p)^{k-1} =$$

$$= \sum_{k\geq 1}\left(k^2 - k + \underbrace{k\left(1 - \frac{2}{p}\right)}_{+k - \frac{2k}{p}} + \frac{1}{p^2}\right)p(1-p)^{k-1} =$$

$$= p\sum_{k\geq 1}\left[\left((1-p)^k\right)''_p + p\left(1 - \frac{2}{p}\right) - \left((1-p)^k\right)' + \frac{1}{p^2}\,p\right](1-p)^{k-1} =$$

$$= (1-p)p\left[\sum_{k\geq 1}(1-p)^k\right]'' + (2-p)\left[\sum_{k\geq 1}(1-p)\right]' + \frac{1}{p^2} =$$

$$= -(1-p)\cdot p\cdot\underbrace{\left(\frac{1}{p^2}\right)'}_{-\frac{2\cdot 1}{p^3}} + (2-p)\cdot\frac{1}{p^2} + \frac{1}{p^2} = 2\cdot(1-p)\cdot\frac{1}{p^2} + \frac{(p-2)}{p^2} + \frac{1}{p^2} =$$

$$= \frac{1}{p^2}\left[2(1-p) + p - 2 + 1\right] = (1-p)\frac{1}{p^2}$$

$$\left[(1-p)^k\right]'' = \left[-k(1-p)^{k-1}\right]' = k\cdot(k-1)(1-p)^{k-2}$$

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