PROBLEM 1.2 (Guillot, 2018, p. 21)

Let F be a field of characteristic $p \neq 2$.

Let $a \in F^{\times} \backslash F^{\times 2}$, and $K = F[\sqrt{a}]$.

PART 1

Suppose there exists an extension L/F with $K \subset L$.

Suppose that L/F is cyclic with $\mathbf{Gal}(L/F) \cong C_4$.

Show that there exists an $\alpha \in K$ such that $N_{K/F}(\alpha) = -1$.

Hint: try $\alpha = \dfrac{\theta - \sigma^2(\theta)}{\sigma(\theta) - \sigma^3(\theta)}$.

SOLUTION 1.2.1

We investigate this field extension, as we often do, by:

- analyzing the (mercifully linear) action of $\mathbf{Gal}(L/F)$ on L, and
- using norms to construct field elements (or, to prove they don't exist).

Taking Guillot's hint: let α be of the form $\dfrac{\theta - \sigma^2(\theta)}{\sigma(\theta) - \sigma^3(\theta)}$.

We take θ to be an element of $L \setminus K$. Such an element must exist by dimension.

We take σ to be a generator of $\mathbf{Gal}(L/F) \cong C_4$. Since σ fixes L, α lies in L.

Since σ acts linearly on L, we have:

$$\sigma(\alpha) = \frac{\sigma(\theta) - \sigma^3(\theta)}{\sigma^2(\theta) - \theta} = -1/\alpha. \ (*)$$

The norm of α with respect to L/F is given by $N_{L/F}(\alpha) = \alpha\, \sigma(\alpha)\, \sigma^2(\alpha)\, \sigma^3(\alpha)$.

But $N_{L/K}(\alpha)$ involves only those powers of σ that lie in $\mathbf{Gal}(L/K)$.

Clearly, K/F is Galois of degree 2 with $\mathbf{Gal}(K/F) \cong C_2$.

Since $[L : F] = [L : K][K : F] = 4$, $[L : K]$ must be 2.

So $\mathbf{Gal}(L/K)$ is a subgroup of degree 2 in $G = \mathbf{Gal}(L/F)$.

The only such subgroup is given by $\{1_G, \sigma^2\}$.

So $N_{L/K}(\alpha) = \alpha\, \sigma^2(\alpha)$, and this is equal to α^2 by $(*)$ above.

Since the image of $N_{L/K}$ is K, we now know that $\alpha^2 \in K$.

This implies that $\alpha \in K$ and we may now ask for $N_{K/F}(\alpha)$.

The non-identity element in $\mathbf{Gal}(K/F)$ – conjugation – sends $\sqrt{a}$ to $-\sqrt{a}$.

This automorphism is the same as the restriction of σ to K, denoted by $\sigma|_K$.

Indeed, σ is in the preimage of conjugation via $\dfrac{\mathbf{Gal}(L/F)}{\mathbf{Gal}(L/K)} \cong \mathbf{Gal}(K/F)$.

Therefore, the norm of α with respect to K/F is $\alpha\,\sigma|_K(\alpha)$.

By $(*)$ above, $\alpha\,\sigma|_K(\alpha) = N_{K/F}(\alpha) = -1$. $\square$

PART 2

Let $\alpha \in K$ be an element with norm $N_{K/F}(\alpha) = -1$.

Show that there exists a cyclic extension L/F of degree 4.

Hint: The case when $\alpha \in F$ must be treated separately.

When $\alpha \notin F$, try $L = K[\sqrt{1+\alpha^2}]$.

SOLUTION 1.2.2

The element α is of the form $x + \sqrt{a}y$ for some $x, y \in F$. First, suppose $\alpha \in F$:

We have $y = 0$. Since $N_{K/F}(\alpha) = -1$, we also have $x^2 = -1$.

This implies that $i = \sqrt{-1} \in F$. So F contains a primitive 4^{th} root of unity:

Therefore we may apply *The Fundamental Theorem of Kummer Theory*:

Subgroups of order $n \geq 1$ in $F^\times/F^{\times n}$ give rise to Galois extensions of degree n.

Let $[b]$ be such a subgroup. Let L/F be such an extension.

Then L is generated as $F[\sqrt[n]{b}]$ for any b in $[b]$.

See **Theorem 1.25** in (Guillot, 2018, p. 14) for the full theorem.

It suffices to show that such a subgroup and a generating element exist for $n = 4$.

Before we do so, observe that the exponent of $F^\times/F^{\times 4}$ is 4. $(*)$

Every element of $[b]$ is of the form bx^4 for some $b \in F^\times \backslash F^{\times 4}$ and $x \in F^\times$.

Furthermore, bx^4 is written in "lowest terms", so that b does not contain a 4^{th} power.

Let m be the order of $[b]$ in $F^\times/F^{\times 4}$. If $m > 4$, there exists some $c = (bx^4)^m \in F^{\times 4}$.

This implies that $b^m = b^{m-4}b^4 = 1$, contradicting that c is given in lowest terms.

By group theory, the order of any subgroup $[b]$ in $F^\times/F^{\times 4}$ must divide the exponent 4.

Now it is clear that we may choose $b = a$ and $[b] = [a]$:

Since a is not a square or a 4^{th} power in $F^\times$, the order of $[a]$ cannot be 2 or 1. $(**)$

So the order of $[a]$ is 4, and a cyclic extension L/K of degree 4 exists.

Now suppose $\alpha \notin F$. We take Guillot's hint and try $L = K[\sqrt{1+\alpha^2}]$.

Clearly, K/F is Galois of degree 2 with $\mathbf{Gal}(K/F) \cong C_2$.

Let σ be the non-identity (conjugation) and $\overline{\alpha} = \sigma(\alpha)$.

We have: $N_{K/F}(\beta) = (1+\alpha^2)\,(1+\sigma(\alpha)^2) = (1+\alpha^2)\,(1+\overline{\alpha}^2)$.

Recall that $N_{K/F}(\alpha) = \alpha\,\sigma(\alpha) = -1$, and so $\alpha^2\,\sigma(\alpha)^2 = 1$.

Applying these formulae, we get $N_{K/F}(\beta) = (\alpha - \overline{\alpha})^2$.

If $1+\alpha^2$ lies in $K^{\times 2}$, then $N_{K/F}(1+\alpha^2) = N_{K/F}(\beta)^2$ for some $\beta \in K$.

But $N_{K/F}(\beta) \in F$, which implies that $N_{K/F}(1+\alpha^2)$ also lies in $F^{\times 2}$.

This cannot be the case since $(\alpha - \sigma(\alpha))^2 = 4y^2a$ does not lie in $F^{\times 2}$.

While $4y^2$ is a square, recall that we have assumed $a \notin F^{\times 2}$.

Therefore, $1+\alpha^2$ is not a square in $K^\times$.

We may now apply the so-called *Equivariant Kummer Theory*:

Clearly, L/K is Galois of degree 2 with $\mathbf{Gal}(L/K) \cong C_2$.

Suppose that K contains a primitive n^{th} root of unity.

Consider subgroups of order $n \geq 1$ in $K^\times/K^{\times n}$ fixed by the action of $\mathbf{Gal}(K/F)$.

These give rise to Galois extensions L/K of degree n such that L/F is also Galois.

Let $[b]$ be such a subgroup of $K^\times/K^{\times n}$. Then L is generated as $K[\sqrt[n]{b}]$ for any b in $[b]$.

See **Theorem 1.26** in (Guillot, 2018, p. 14) for the full theorem.

Certainly, K contains the primitive *square* root of unity, namely -1.

Using $(*)$ and $(**)$ above, the order of $[\beta]$ in $K^\times/K^{\times 2}$ is 2.

It remains to show that $[\beta]$ is preserved by the action of $\mathbf{Gal}(K/F)$:

$$\sigma(1+\alpha^2) = 1+\overline{\alpha}^2 = \frac{N_{K/F}(1+\alpha^2)}{1+\alpha^2} = \frac{4y^2\alpha}{1+\alpha^2} = (1+\alpha^2)\cdot\left(\frac{2y\sqrt{\alpha}}{1+\alpha^2}\right)^2.$$

Thus, $\sigma(1+\alpha^2) = (1+\alpha^2)\cdot k^2$ for $k = \left(\frac{2y\sqrt{\alpha}}{1+\alpha^2}\right)^2 \in K^\times$, and σ fixes α. $\square$

PART 3

Show that the following are equivalent:

(a) There exists an $\alpha \in K$ with $N_{K/F}(\alpha) = -1$, and

(b) a is a sum of two squares in F

SOLUTION 1.2.3

Recall that F is a field of characteristic $p \neq 2$ and $K = F[\sqrt{a}]$.

Clearly, K/F is Galois of degree 2 with $\mathbf{Gal}(K/F) \cong C_2$.

Let σ be the non-identity (conjugation) which sends $\sqrt{a}$ to $\sqrt{-a}$.

If $a = x^2 + y^2$ for some $y \neq 0$ and x in F, then $\frac{x^2-a}{y^2} = -1$. Also,

$$\frac{x+\sqrt{a}}{y}\cdot\frac{x-\sqrt{a}}{y} = \frac{x+\sqrt{a}}{y}\cdot\sigma(\frac{x+\sqrt{a}}{y}) = -1.$$

Suppose $\alpha = \frac{x+\sqrt{a}}{y} \in K$. The equation above reduces to $N_{K/F}(\alpha) = -1$.

On the other hand, if $y = 0$, then $a = x^2$. This cannot be the case because $a \notin F^{\times 2}$.

Therefore (b) $\implies$ (a).

Every element of K is of the form $x + \sqrt{a}y$ for some $x, y \in F$.

If $N_{K/F}(\alpha) = -1$, then $(x+\sqrt{a}y)\cdot(x-\sqrt{a}y) = x^2 - ay^2 = -1$.

If $y = 0$, then $a = x^2$. This cannot be the case because $a \notin F^{\times 2}$.

So we can safely rearrange $x^2 - ay^2 = -1$ as $a = \left(\frac{1}{y}\right)^2 + \left(\frac{x}{y}\right)^2$.

Therefore (a) $\implies$ (b). $\square$

REFERENCES

Guillot, Pierre. (2018). *A Gentle Course in Local Class Field Theory: Local Number Fields, Brauer Groups, Galois Cohomology.* Cambridge: Cambridge University Press.

Morandi, Patrick. (1996). *Field and Galois Theory.* Graduate Texts in Mathematics, vol 167. Springer, New York, NY.