Physics GRE:

Classical Mechanics

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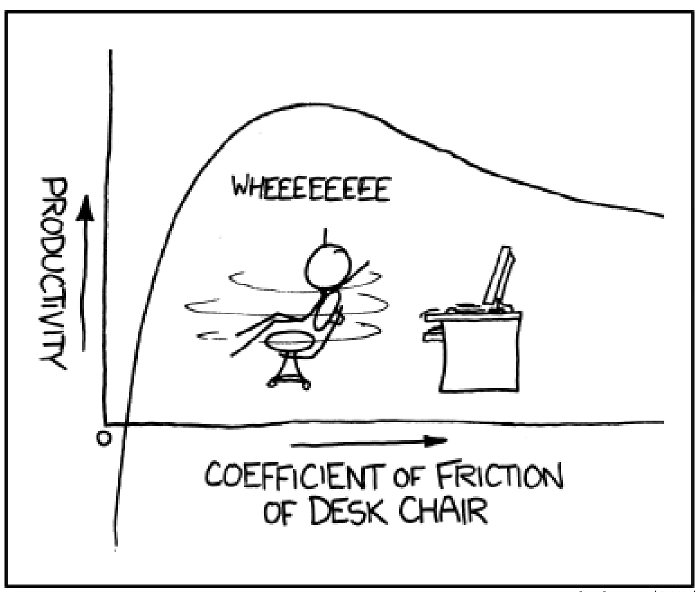

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1 Newtonian Mechanics

1.1 Kinematics The position of a particle is written as $\boldsymbol{x}(t)$ or as $\boldsymbol{r}(t)$. We define the first and second derivatives to be the velocity and acceleration of the particle:

$$\boldsymbol{v}(t) = \frac{\mathrm{d}\boldsymbol{x}(t)}{\mathrm{d}t} \qquad \boldsymbol{a}(t) = \frac{\mathrm{d}\boldsymbol{v}(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2\boldsymbol{x}(t)}{\mathrm{d}t^2} \tag{1}$$

Inverting these relations gives

$$\boldsymbol{v}(t) = \int \boldsymbol{a}(t)\,\mathrm{d}t \qquad \boldsymbol{x}(t) = \int \boldsymbol{v}(t)\,\mathrm{d}t \tag{2}$$

In the special case where the acceleration is time-independent, these reduce to the following:

$$\boldsymbol{x}(t) = \boldsymbol{x}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{a}t^2 \qquad \boldsymbol{v}(t) = \boldsymbol{v}_0 + \boldsymbol{a}t \tag{3}$$

The constants $\boldsymbol{x}_0$ and $\boldsymbol{v}_0$ must be determined by initial conditions. Often we further restrict to the case of free-fall near the surface of the Earth, in which case the constant acceleration is given by

$$\boldsymbol{a} = -g\hat{\boldsymbol{\jmath}} \qquad g \approx 9.81\,{}^{\mathrm{m}}/_{\mathrm{s}^2} \approx 10\,{}^{\mathrm{m}}/_{\mathrm{s}^2} \tag{4}$$

where $\hat{\boldsymbol{\jmath}}$ points vertically away from the center of the Earth and motion is typically taken to be restricted to a plane. The equations of motion are then

$$x(t) = x_0 + v_{0x}t \qquad v_x(t) = v_{0x} \tag{5}$$

$$y(t) = y_0 + v_{0y}t - \frac{1}{2}gt^2 \qquad v_y(t) = v_{0y} - gt \tag{6}$$

From these one may show that the maximum height reached is $y_{\mathrm{max}} = y_0 + \frac{v_{0y}^2}{2g}$ and the horizontal distance traveled before falling to the original height is $R = \frac{2v_{0x}v_{0y}}{g} = \frac{v^2}{g}\sin 2\theta$, where v and θ are the initial speed and angle.

Objects undergoing uniform circular motion have constant speed but are still accelerating, as the direction of their velocity is changing. It may be shown that the object must be accelerating towards the center of their circular path:

$$\boldsymbol{a}_c = -\frac{v^2}{r}\hat{\boldsymbol{r}} \tag{7}$$

1.2 Newton's Laws

1. An object at rest will remain at rest unless a force acts upon it; an object in motion will not change its motion unless a force acts upon it.
2. $\boldsymbol{F} = \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\boldsymbol{v})}{\mathrm{d}t}$.
3. To every action there is always opposed an equal reaction: $\boldsymbol{F}_{\mathrm{AB}} = -\boldsymbol{F}_{\mathrm{BA}}$.

Of course, the first law is a special case of the second: $\sum \boldsymbol{F} = 0 \Leftrightarrow \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = 0$. In the cases where the mass is constant, Newton's second law reduces to $\boldsymbol{F} = m\boldsymbol{a}$. Due to Newton's third law all internal forces in an object cancel when considering the motion of the object as a whole.

Consider an object in free-fall experiencing a drag force proportional to its velocity. Newton's 2[d] law gives

$$m\boldsymbol{a} = -mg\hat{\boldsymbol{\jmath}} - b\boldsymbol{v} \qquad \Longrightarrow \qquad \dot{\boldsymbol{v}} = -(g\hat{\boldsymbol{\jmath}} + \tfrac{b}{m}\boldsymbol{v}) \tag{8}$$

In general this is not so easy to solve, so restrict the motion to the vertical:

$$\dot{v}_y = -(g + \tfrac{b}{m}v_y) \tag{9}$$

Solving this through separation of variables gives

$$v_y(t) = -\frac{mg}{b} + \left(v_{0y} + \frac{mg}{b}\right)e^{-\frac{bt}{m}} \tag{10}$$

In the limit of large t, v_y approaches a constant value called the object's terminal velocity: $|v_\mathrm{t}| = \frac{mg}{b}$. Alternatively, this may be found quickly by requiring $\boldsymbol{a} = 0$ in Newton's 2[d] law above.

Newton's laws may be used to analyze the motion of a rocket with no external forces. Suppose that the rocket is expelling fuel at a rate ξ in the negative-x direction with a relative speed u. That is, if the rocket has velocity $\boldsymbol{v} = v\hat{\boldsymbol{\imath}}$, then the velocity of the ejected fuel is $(v-u)\hat{\boldsymbol{\imath}}$. Applying conservation of momentum for the rocket–fuel system over a time interval $\mathrm{d}t$ gives

$$M(t)v(t) = [M(t) - \mathrm{d}m][v(t) + \mathrm{d}v] + \mathrm{d}m[v(t) - u] \tag{11}$$

where $\mathrm{d}m = \xi\,\mathrm{d}t$ is the amount of fuel released. Simplifying this expression, ignoring the term $\mathrm{d}m\,\mathrm{d}v$, yields

$$0 = M(t)\,\mathrm{d}v - u\,\mathrm{d}m = (M_0 - \xi t)\,\mathrm{d}v - u\xi\,\mathrm{d}t \tag{12}$$

This differential equation lends itself well to an exact solution:

$$v(t) = v_0 + \int_0^t \frac{u\xi}{M_0 - \xi t'}\,\mathrm{d}t' = v_0 - \Big[u\log\left(M_0 - \xi t'\right)\Big]_{t'=0}^{t} \tag{13}$$

$$= v_0 + u\log\left(\frac{M_0}{M_0 - \xi t}\right) = v_0 + u\log\left[\frac{M_0}{M(t)}\right] \tag{14}$$

1.3 Harmonic Oscillators

A system with a restoring force proportional to its displacement is known as a harmonic oscillator. The free system is described by

$$\ddot{x} + \omega_0^2 x = 0 \tag{15}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural frequency of the system. Due to the conservative nature of the linear restoring force, the total energy is conserved and at any point is given by

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}mv_\mathrm{max}^2 = \frac{1}{2}kx_\mathrm{max}^2 \tag{16}$$

Considering now a general system with a drag force proportional to the velocity, $f_\mathrm{drag} = -bv$, and a time-dependent driving force, F_dr, the equation of motion is

$$\ddot{x} + 2\Gamma\dot{x} + \omega_0^2 x = \frac{F_\mathrm{dr}(t)}{m} \qquad \Gamma = \frac{b}{2m} \tag{17}$$

In the case where the oscillator is not driven, i.e. $F_\mathrm{dr}(t) = 0$, there are three possible behaviors based on the relative magnitudes of Γ and ω_0:

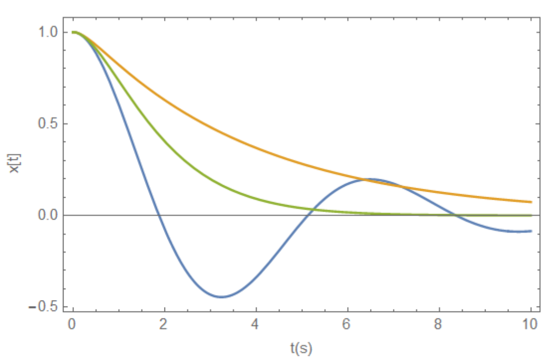

Figure 1: Damped harmonic oscillator with $\omega_0 = 1$, initial conditions $x_0 = v_0 = 0$ and $\Gamma = \frac{1}{4}, 1, 2$.

- $\Gamma^2 - \omega_0^2 < 0$ (Underdamped):

$$x(t) = e^{-\Gamma t}\left(A\cos\omega_1 t + B\sin\omega_1 t\right) \qquad \omega_1 = \sqrt{\omega_0^2 - \Gamma^2} \tag{18}$$

  The coefficients A and B are determined by the initial conditions.

- $\Gamma^2 - \omega_0^2 > 0$ (Overdamped):

$$x(t) = e^{-\Gamma t}\left(Ae^{\omega_1 t} + Be^{-\omega_1 t}\right) \qquad \omega_1 = \sqrt{\Gamma^2 - \omega_0^2} \tag{19}$$

  The coefficients A and B are again determined by the initial conditions.

- $\Gamma^2 - \omega_0^2 = 0$ (Critically damped):

$$x(t) = e^{-\Gamma t}\left[x_0 + (v_0 + \Gamma x_0)t\right] \tag{20}$$

Notice that in the underdamped case the frequency of oscillation is less than the natural frequency. In the overdamped and critically damped cases the position approaches zero exponentially, but never cross it. Critical damping represents a limiting case where the equilibrium position is approached as quickly as is possible.

For non-zero driving it is best to assume a sinusoidal driving force: if needed any motion can be taken as a superposition of these. Write $F_{\mathrm{dr}} = F_0\cos(\omega t)$, where F_0 is the amplitude and ω is the driving frequency. We look for a particular solution to the equation

$$\ddot{x} + 2\Gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t \tag{21}$$

Assuming that the system will settle into motion with the same frequency as the driving force suggests looking for solutions of the form

$$x_{\mathrm{p}}(t) = A\cos(\omega t - \phi) \tag{22}$$

Plugging this into the equation of motion gives the following conditions on A and ϕ:

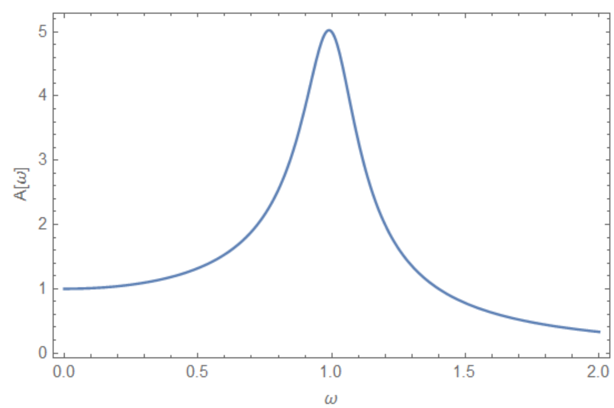

Figure 2: Amplitude for $F_{\mathrm{dr}} = m = \omega_0 = 1$ and $\Gamma = \frac{1}{10}$ as a function of the driving frequency, ω.

$$A = \frac{F_0/m}{\sqrt{4\Gamma^2\omega^2 + (\omega_0^2 - \omega^2)^2}} \qquad \phi = \arctan\left(\frac{2\Gamma\omega}{\omega_0^2 - \omega^2}\right) \tag{23}$$

When the driving frequency is near the natural frequency, $\omega \approx \omega_0$, we see that the amplitude is maximized; this is known as resonance.

1.4 Work & Energy

The work done by a force on a moving object is

$$W = \int \boldsymbol{F}\cdot \mathrm{d}\boldsymbol{s} \tag{24}$$

The work energy theorem states that the change in energy of a system is given by the total work done on the system:

$$\sum W = \Delta E \tag{25}$$

By Newton's third law, we need only consider external forces.

So-called conservative forces may be associated with a potential energy. That is, if $\boldsymbol{F} = -\boldsymbol{\nabla} U$ for some U, then the work done, and thus change in energy, moving between two points is independent of the path taken. Then U is the potential energy associated with the conservative force, and is single-valued at every point. By its definition, we have

$$U = -\int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} \tag{26}$$

The two forms of mechanical energy are kinetic, T, and potential, U. Kinetic energy is further divided into translational and rotational kinetic energy. In the nonrelativistic limit these are

$$T_t = \frac{1}{2}mv^2 = \frac{p^2}{2m} \qquad T_r = \frac{1}{2}I\omega^2 = \frac{L^2}{2I} \tag{27}$$

Non-conservative forces need not conserve energy. A prime example is friction, which always opposes the motion of an object. Static and kinetic friction are given by

$$f_\mathrm{s} \leq \mu_\mathrm{s} N \qquad f_\mathrm{k} = \mu_\mathrm{k} N \tag{28}$$

where N is the normal force.

Impulse is defined to be

$$\boldsymbol{J} = \Delta\boldsymbol{p} = \int \boldsymbol{F}(t)\,\mathrm{d}t = \overline{\boldsymbol{F}}\,\Delta t \tag{29}$$

where $\overline{\boldsymbol{F}}$ is the average force over the time interval Δt.

1.5 Collisions Consider first a one-dimensional collision between two massive bodies. We may assume that one of the objects is initially at rest (if not, make a Galilean transformation), so that $p_{1i} = p_0$ and $p_{2i} = 0$. Conservation of linear momentum results in

$$p_{1f} + p_{2f} = p_0 \tag{30}$$

This is all we may say without a further restriction. There are two limiting cases which are of particular interest: inelastic and elastic collisions.

- **Inelastic** Here the two particles stick together after the collision so that they have the same final velocity. This condition gives

$$v_{1f} = v_{2f} = \frac{p_0}{m_1 + m_2} = \left(\frac{m_1}{m_1 + m_2}\right) v_{1i} \tag{31}$$

  The change in kinetic energy is thus

$$\Delta T = T_f - T_i = \left[\frac{1}{2}\left(\frac{m_1^2}{m_1 + m_2}\right) v_{1i}^2\right] - \left(\frac{1}{2}m_1 v_{1i}^2\right) = -\frac{1}{2}\left(\frac{m_1 m_2}{m_1 + m_2}\right) v_{1i}^2 = -\frac{1}{2}\mu v_{1i}^2 \tag{32}$$

  The energy lost is proportional to the reduced mass of the system.

- **Elastic** Here energy is conserved, allowing for an exact solution of the final velocities.

$$v_{1f} = \left(\frac{m_1 - m_2}{m_1 + m_2}\right) v_{1i} \qquad v_{2f} = \left(\frac{2m_1}{m_1 + m_2}\right) v_{1i} \tag{33}$$

  There are three limiting behaviors: $m_1 \ll m_2$, $m_1 = m_2$ and $m_1 \gg m_2$. In the case where object one is very light, $v_{1f} \approx -v_{1i}$ and $v_{2f} \approx 0$: complete rebound. When they have the same mass, $v_{1f} = 0$ and $v_{2f} = v_{1i}$: they are exactly interchanged. When object one is very massive, $v_{1f} \approx v_{1i}$ and $v_{2f} \approx 2v_{1i}$: object one barrels through giving object two a big kick.

For two-dimensional collisions again assume that one of the objects is initially at rest. Inelastic collisions are identical to those in one-dimension, so consider an elastic collision. Momentum and energy conservation give

$$\boldsymbol{p}_{1i} = \boldsymbol{p}_{1f} + \boldsymbol{p}_{2f} \qquad \frac{p_{1i}^2}{2m_1} = \frac{p_{1f}^2}{2m_1} + \frac{p_{2f}^2}{2m_2} \tag{34}$$

Taking the dot product of each side with itself in the momentum conservation vector equation gives

$$p_{1i}^2 = p_{1f}^2 + p_{2f}^2 + 2\boldsymbol{p}_{1f} \cdot \boldsymbol{p}_{2f} = p_{1f}^2 + p_{2f}^2 + 2p_{1f}p_{2f}\cos(\theta_1 - \theta_2) \tag{35}$$

This are not terribly enlightening, but if the two masses are equal then these reduce to

$$0 = 2p_{1f}p_{2f}\cos(\theta_1 - \theta_2) \qquad \Longrightarrow \qquad |\theta_1 - \theta_2| = \frac{\pi}{2} \tag{36}$$

That is, the two objects exit the collision at $90°$ from each other.

1.6 Rotational Motion For rotating objects we introduce angular variables θ, ω and α, analogous to x, v and a:

$$\omega(t) = \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} \qquad \alpha(t) = \frac{\mathrm{d}\omega(t)}{\mathrm{d}t} = \frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} \tag{37}$$

In the case of constant angular acceleration we have the corresponding kinematic equations:

$$\theta(t) = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \qquad \omega(t) = \omega_0 + \alpha t \tag{38}$$

For objects that are "rolling without slipping" the angular velocity and the linear velocity are related by

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r} \qquad v = r\omega \tag{39}$$

The period, frequency and angular frequency are related by

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \tag{40}$$

There is also the rotational form of Newton's second law,

$$\boldsymbol{\tau} = \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \frac{\mathrm{d}(\boldsymbol{r} \times \boldsymbol{p})}{\mathrm{d}t} = \frac{\mathrm{d}(I\boldsymbol{\omega})}{\mathrm{d}t} \tag{41}$$

where $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$ is the torque and $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ is the angular momentum. In both of these definitions the vector $\boldsymbol{r}$ is the displacement from the chosen axis of rotation. In the case of a rigid, rotating object the angular momentum may also be expressed at $\boldsymbol{L} = I\boldsymbol{\omega}$, where I is the momentum of inertia, defined as

$$I = \sum_i m_i r_i^2 = \int r^2 \,\mathrm{d}m \tag{42}$$

where here r is the distance to the axis of rotation. In the common case where the density of the object is a function of position, i.e. $\frac{\mathrm{d}m}{\mathrm{d}\boldsymbol{r}} = \rho(\boldsymbol{r})$, this becomes

$$I = \int r^2 \rho(\boldsymbol{r}) \,\mathrm{d}\boldsymbol{r} \tag{43}$$

Geometry	I
Hoop (about axis)	MR^2
Cylinder (about axis)	$\frac{1}{2}MR^2$
Rod (about center)	$\frac{1}{12}ML^2$
Rod (about end)	$\frac{1}{3}ML^2$
Rectangular Slab (about center)	$\frac{1}{12}M(W^2+H^2)$
Hollow Sphere (about center)	$\frac{2}{3}MR^2$
Solid Sphere (about center)	$\frac{2}{5}MR^2$

Table 1: Moments of inertia for common constant-density geometries.

In general the moment of inertia is of the form $I = \beta MR^2$, with R denoting some geometric "size" of the object and with $\beta \geq 0$. Often it is easiest to calculate the moment of inertia for an object when considering rotations about its center of mass. The parallel axis theorem is useful for finding moments of inertia about other axis that may not lend themselves well to direct integration.

$$I = I_{\text{cm}} + Md^2 \tag{44}$$

The axis of rotation in question must be parallel to that for I_{cm} and d denotes the perpendicular distance between these two axes. Note that this shows that the moment of inertia for any object is smallest when about an axis through its center of mass.

Consider a physical pendulum fixed at one end of length L. Its center of mass lies a distance d from the pivot, and the momentum of inertia about that end is I. The coordinate in question is the angle measured from the vertical. The torque due to gravity with the pendulum lying in the xy-plane is

$$\boldsymbol{\tau} = \boldsymbol{d} \times \boldsymbol{F} = -mgd(\hat{\boldsymbol{d}} \times \hat{\boldsymbol{\jmath}}) = -mgd\sin\theta\hat{\boldsymbol{k}} \tag{45}$$

where $\hat{\boldsymbol{\jmath}}$ is "up" and $\hat{\boldsymbol{k}}$ is "out of the board". Applying the angular form of Newton's 2^{d} law gives

$$-mgd\sin\theta\hat{\boldsymbol{k}} = I\boldsymbol{\alpha} \tag{46}$$

Since the pendulum stays in the xy-plane, $\boldsymbol{\alpha} = \alpha\hat{\boldsymbol{k}}$ and this becomes simply a scalar equation:

$$\ddot{\theta} + \left(\frac{mgd}{I}\right)\sin\theta = 0 \tag{47}$$

This has no closed-form solution, but for small amplitudes, $|\theta| \ll 1$, it reduces to simple harmonic motion with frequency

$$\omega = \sqrt{\frac{mgd}{I}} \tag{48}$$

For example a uniform rod of length l fixed at one end will oscillate with frequency $\omega = \sqrt{\frac{3g}{2l}}$.

1.7 Noninertial Reference Frames

Accelerating reference frames lead to fictitious forces. Two common ones that arise when considering rotating reference frames are the centrifugal force and the Coriolis force.

The centrifugal force arises as

$$\boldsymbol{F}_{\text{cf}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \tag{49}$$

where $\boldsymbol{\omega}$ is the angular velocity of the frame and $\boldsymbol{r}$ is the position in the rotating frame coordinates. The right-hand rule shows that this force is direct outward from the axis of rotation: this is the tendency for rotating bodies to "fling things away".

The Coriolis force is

$$\boldsymbol{F}_{\text{Co}} = -2m\boldsymbol{\omega} \times \boldsymbol{v} \tag{50}$$

where $\boldsymbol{\omega}$ is the angular velocity of the frame and $\boldsymbol{v}$ is the velocity in the rotating frame coordinates. This force is responsible for the swirling of hurricanes in the north and south hemispheres.

1.8 Dynamics of Systems of Particles For a system of particles we may define the center of mass to be

$$\boldsymbol{r}_{\text{cm}} = \frac{1}{M}\sum_i m_i \boldsymbol{r}_i = \frac{1}{M}\int \boldsymbol{r}\,\mathrm{d}m \qquad M = \sum_i m_i = \int \mathrm{d}m \tag{51}$$

This is a weighted average of the positions, and shows its use through the following calculation,

$$\ddot{\boldsymbol{r}}_{\text{cm}} = \boldsymbol{a}_{\text{cm}} = \frac{1}{M}\sum_i m_i \boldsymbol{a}_i = \frac{1}{M}\sum_i \boldsymbol{F}_i^{\text{net}} = \frac{1}{M}\sum \boldsymbol{F}^{\text{ext}} \tag{52}$$

where the sum over i is over all particles in the system and $\boldsymbol{F}_i^{\text{net}}$ denotes the net force on the i^{th} particle. By Newton's third law all internal forces come in pairs which cancel in this sum, leaving only a sum over forces external to the system. The conclusion is that the center of mass moves as a point object would under the influence of these external forces. This validates the unspoken assumption that we may ignore things such as inter-molecular forces when considering the motions of macroscopic, composite objects.

1.9 The Virial Theorem For a system bound by a power-law force the virial theorem states that the time-average of the kinetic and potential energies are related by

$$U_{12} \propto r_{12}^n \qquad \Longrightarrow \qquad 2\langle T_{\text{TOT}}\rangle = n\,\langle U_{\text{TOT}}\rangle \tag{53}$$

where T_{TOT} is the total kinetic energy of all substituent parts and U_{TOT} is the total potential energy arising from all pairs. In the case of a familiar inverse-square force this becomes

$$\langle T_{\text{TOT}}\rangle = -\frac{1}{2}\langle U_{\text{TOT}}\rangle \tag{54}$$

For a linear restoring force, $n = 2$, the assertion is that

$$\langle T_{\text{TOT}}\rangle = \langle U_{\text{TOT}}\rangle \tag{55}$$

1.10 Central Forces & Celestial Mechanics Of special interest are those forces which are "central", in that they result from potentials which are spherically symmetric. Newtonian gravity and the Coulomb force are both inverse square laws. In general we will consider forces arising from potentials of the form

$$U_n(\boldsymbol{r}) = kr^n \tag{56}$$

All central forces are conservative forces, and so the energy of an object in such a potential is conserved. In addition, since the force acts radially, the torque about the origin is zero

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}_n(\boldsymbol{r}) = -nkr^{n-1}(\boldsymbol{r} \times \hat{\boldsymbol{r}}) = 0 \tag{57}$$

This implies that the object's angular momentum about the origin is conserved.

An important result is Bertrand's theorem, which shows that closed, bound orbits are only possible for an inverse-square law and isotropic harmonic oscillator: $n = -1$ and $n = 2$. They correspond to the following potentials:

$$U_{-1}(\boldsymbol{r}) = \frac{k}{r} \qquad U_2(\boldsymbol{r}) = kr^2 \tag{58}$$

It was observed that the orbit of Mercury was not closed, but was rather precessing. Newtonian gravity was not able to explain this process, even considering the interactions with the other planets, but the drift discrepancy is predicted by the general theory of relativity and agrees quite well with observation.

For the inverse-square law of Newtonian gravity there are four types of orbits, corresponding to different conic sections:

Orbit	Eccentricity	Total Energy	Orbital Speed
Circle	$\epsilon = 0$	$E = -\frac{GMm}{2r}$	$v = \sqrt{\frac{GM}{2r}}$
Ellipse	$0 < \epsilon < 1$	$-\frac{GMm}{2r} < E < 0$	$\sqrt{\frac{GM}{2r}} < v < \sqrt{\frac{2GM}{r}}$
Parabola	$\epsilon = 1$	$E = 0$	$v = \sqrt{\frac{2GM}{r}}$
Hyperbola	$\epsilon > 1$	$E > 0$	$v > \sqrt{\frac{2GM}{r}}$

Kepler's laws of planetary motion are:

1. The orbit of a planet is an ellipse with the Sun at one focus.
2. A line segment joining a planet and the Sun sweeps out equal areas during equal time intervals.
3. The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit: $\frac{T^2}{a^3} = \frac{4\pi^2}{G(M+m)} \approx \frac{4\pi^2}{GM}$.

In addition, for Newtonian gravity we have the shell-theorem, which is analogous to Gauss' law in electromagnetism:

$$\boldsymbol{\nabla} \cdot \boldsymbol{g} = -4\pi G\rho \qquad \oint_{\partial V} \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{A} = -4\pi G \int_V \rho \, \mathrm{d}V \tag{59}$$

The quantity $\boldsymbol{g}$ is the gravitational field and is given by $\frac{\boldsymbol{F}}{m}$, similar to how the electric field is defined. In practice this means that for spherically symmetric mass distributions one needs only consider the gravitational force from mass closer to the origin than the object in question. Outside of the surface of the Earth the force of gravity may be found to be the familiar

$$\boldsymbol{F}_{\text{out}}(\boldsymbol{r}) = -\frac{GMm}{r^2}\hat{\boldsymbol{r}} \tag{60}$$

However, within the Earth at a radius $r < R$, let V be the ball of radius r so that we have

$$\oint_{\partial V} \boldsymbol{g} \cdot \mathrm{d}\boldsymbol{A} = -4\pi G \int_V \rho \, \mathrm{d}V \tag{61}$$

$$\frac{F}{m} \oint_{\partial V} \mathrm{d}A = -\frac{4\pi GM}{\frac{4}{3}\pi R^3} \int_V \mathrm{d}V \tag{62}$$

$$\frac{F}{m}\left(4\pi r^2\right) = -\frac{4\pi GM}{\frac{4}{3}\pi R^3}\left(\frac{4}{3}\pi r^3\right) \tag{63}$$

$$F = -\frac{GMmr}{R^3} \tag{64}$$

where we leverage the spherical symmetry to simplify $\boldsymbol{g}\cdot \mathrm{d}\boldsymbol{A} = |\boldsymbol{g}|\,|\mathrm{d}\boldsymbol{A}| = g\,\mathrm{d}A$ and conclude that F is constant everywhere on the closed surface. The final result is

$$\boldsymbol{F}(\boldsymbol{r}) = \begin{cases} -\frac{GMm}{R^2}\frac{r}{R}\hat{\boldsymbol{r}} & r < R \\ -\frac{GMm}{r^2}\hat{\boldsymbol{r}} & r > R \end{cases} \tag{65}$$

Of course these agree at the boundary $r = R$, as expected. Near the surface of the Earth, $r \approx R$, the force of gravity is nearly constant with the form

$$\boldsymbol{F} \approx -mg\hat{\boldsymbol{r}} \qquad g = \frac{GM_\oplus}{R_\oplus^2} \approx 9.81\,{}^{\mathrm{N}}\!/\!_{\mathrm{kg}} \tag{66}$$

with $M_\oplus$ and $R_\oplus$ being the mass and radius of the Earth. This justifies the constant acceleration assumption used in the kinematic equations.

2 Lagrangian & Hamiltonian Formalism

2.1 Lagrangian Mechanics Let $\boldsymbol{q} = (q_1, q_2, \ldots, q_n)$ be a set of n coordinates describing the state of a system. We would like to know how these coordinates change through time. Introduce the Lagrangian, $L(\boldsymbol{q}, \dot{\boldsymbol{q}}; t)$, which is a function of the coordinates, their first time derivatives and perhaps time; in the nonrelativistic case we have $L = T - U$. Hamilton's principle asserts that the time integral of L between two states is an extremum with respect to variations to the path. Defining the action as $S = \int L(\boldsymbol{q}, \dot{\boldsymbol{q}}; t)\,\mathrm{d}t$, Hamilton's principle reduces to $\delta S = 0$.

Let $\boldsymbol{q}(t)$ describe the evolution of a system between two times, and write an arbitrary, small variation in $\boldsymbol{q}(t)$ as $\epsilon\boldsymbol{\eta}(t)$, where ϵ is small and $\boldsymbol{\eta}(t_1) = \boldsymbol{\eta}(t_2) = 0$. Then we have

$$\boldsymbol{q}(t) \longrightarrow \boldsymbol{q}(t) + \epsilon\boldsymbol{\eta}(t) \qquad \dot{\boldsymbol{q}}(t) \longrightarrow \dot{\boldsymbol{q}}(t) + \epsilon\dot{\boldsymbol{\eta}}(t) \tag{67}$$

Hamilton's principle gives

$$\delta S = \delta \int_{t_1}^{t_2} L(\boldsymbol{q}, \dot{\boldsymbol{q}}; t)\,\mathrm{d}t \tag{68}$$

$$0 = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial \boldsymbol{q}}\delta\boldsymbol{q} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}}\delta\dot{\boldsymbol{q}} + \frac{\partial L}{\partial t}\delta t\right]\mathrm{d}t \tag{69}$$

$$0 = \int_{t_1}^{t_2} \frac{\partial L}{\partial \boldsymbol{q}}\epsilon\boldsymbol{\eta}(t)\,\mathrm{d}t + \int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{\boldsymbol{q}}}\epsilon\dot{\boldsymbol{\eta}}(t)\,\mathrm{d}t \tag{70}$$

$$0 = \int_{t_1}^{t_2} \frac{\partial L}{\partial \boldsymbol{q}}\epsilon\boldsymbol{\eta}(t)\,\mathrm{d}t + \left[\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\epsilon\boldsymbol{\eta}(t)\right]_{t=t_1}^{t_2} - \int_{t_1}^{t_2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)\epsilon\boldsymbol{\eta}(t)\,\mathrm{d}t \tag{71}$$

$$0 = \int_{t_1}^{t_2}\left[\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right]\epsilon\boldsymbol{\eta}(t)\,\mathrm{d}t \tag{72}$$

The action is an extremum for the path $\boldsymbol{q}$ iff the above vanishes for all $\boldsymbol{\eta}$. This implies that the expression in the brackets must vanish, giving the Euler-Lagrange equations:

$$\delta S = 0 \qquad \Longrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} - \frac{\partial L}{\partial \boldsymbol{q}} = 0 \tag{73}$$

Coordinate System	Lagrangian
Cartesian	$\frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)-U(x,y,z)$
Cylindrical	$\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2+\dot{z}^2\right)-U(r,\theta,z)$
Spherical	$\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2+r^2\dot{\varphi}^2\sin^2\theta\right)-U(r,\theta,\phi)$

Table 2: General Lagrangians for a single point mass in common coordinate systems.

These compose a system of n second-order differential equations in the coordinates $q_1,\ldots,q_n$. Specifying initial conditions uniquely determines the solution and thus also the time evolution of the coordinates.

Of special interest are so-called cyclic coordinates, where q_i does not appear in the Lagrangian. In this case the Euler-Lagrange equation for this coordinate reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}p_i=0 \qquad p_i\equiv\frac{\partial L}{\partial\dot{q}_i} \tag{74}$$

That is, the conjugate momentum to q_i, as defined above, is a constant of motion.

2.2 Hamiltonian Mechanics

A set of coordinates $\boldsymbol{q}$ have conjugate momenta defined by

$$\boldsymbol{p}=\frac{\partial L}{\partial\dot{\boldsymbol{q}}} \tag{75}$$

A Legendre transformation of the Lagrangian results in the Hamiltonian:

$$H(\boldsymbol{q},\boldsymbol{p};t)=\dot{\boldsymbol{q}}\cdot\boldsymbol{p}-L(\boldsymbol{q},\dot{\boldsymbol{q}};t) \tag{76}$$

In the nonrelativistic case we have $H=T+U$. Using this definition for the Hamiltonian it is clear that H and L are related by

$$\frac{\partial H}{\partial\boldsymbol{q}}=-\frac{\partial L}{\partial\boldsymbol{q}} \qquad \frac{\partial H}{\partial\boldsymbol{p}}=\dot{\boldsymbol{q}} \qquad \frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t} \tag{77}$$

The Euler-Lagrange equations simplify these to Hamilton's equations:

$$\dot{\boldsymbol{q}}=\frac{\partial H}{\partial\boldsymbol{p}} \qquad \dot{\boldsymbol{p}}=-\frac{\partial H}{\partial\boldsymbol{q}} \qquad \frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t} \tag{78}$$

In comparison to Lagrangian mechanics these give $2n$ first-order differential equations in the coordinates $q_1,\ldots,q_n$ and $p_1,\ldots,p_n$. Again, specifying initial conditions uniquely determines the time evolution of the coordinates.

Hamilton's equations may be rephrased in terms of Poisson brackets, which are defined to be

$$\{f,g\}=\frac{\partial f}{\partial\boldsymbol{q}}\cdot\frac{\partial g}{\partial\boldsymbol{p}}-\frac{\partial f}{\partial\boldsymbol{p}}\cdot\frac{\partial g}{\partial\boldsymbol{q}} \tag{79}$$

Poisson brackets satisfy the Jacobi identity, as well as the following:

$$\{f,g\}=-\{g,f\} \qquad \{f+g,h\}=\{f,h\}+\{g,h\} \qquad \{fg,h\}=\{f,h\}g+f\{g,h\} \tag{80}$$

The time evolution of any function is then given by

$$\frac{\mathrm{d}}{\mathrm{d}t}f(\boldsymbol{q},\boldsymbol{p};t)=\frac{\partial f}{\partial\boldsymbol{q}}\cdot\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t}+\frac{\partial f}{\partial\boldsymbol{p}}\cdot\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t}+\frac{\partial f}{\partial t} \tag{81}$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \left[\frac{\partial f}{\partial \boldsymbol{q}} \cdot \left(\frac{\partial H}{\partial \boldsymbol{p}}\right) + \frac{\partial f}{\partial \boldsymbol{p}} \cdot \left(-\frac{\partial H}{\partial \boldsymbol{q}}\right)\right] + \frac{\partial f}{\partial t} \tag{82}$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\} + \frac{\partial f}{\partial t} \tag{83}$$

In particular, we have

$$\frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}t} = \{\boldsymbol{q}, H\} \qquad \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \{\boldsymbol{p}, H\} \qquad \frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} \tag{84}$$

2.3 Generalized Coordinates & Normal Modes Both Lagrangian and Hamiltonian mechanics give equations of motion which couple the coordinates. There are cases where one may define a new coordinate system in which the equations of motion completely decouple. These generalized coordinates lead to an easier analysis, and any motion will be a superposition of these simple motions.

The best way to see this is through an example. Consider two penduli connected by a spring. Both have mass m and length l and the spring has a natural length equal to the separation of the penduli and spring-constant k. We take as coordinates the two angles from the vertical: θ_1 and θ_2. Assuming small angles, the Lagrangian for this system is

$$L = \frac{1}{2}ml^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) - \frac{1}{2}mgl\left(\theta_1^2 + \theta_2^2\right) - \frac{1}{2}kl^2(\theta_2 - \theta_1)^2 \tag{85}$$

The Euler-Lagrange equations give

$$ml\ddot{\theta}_1 + mg\theta_1 - kl(\theta_2 - \theta_1) = 0 \tag{86}$$

$$ml\ddot{\theta}_2 + mg\theta_2 + kl(\theta_2 - \theta_1) = 0 \tag{87}$$

These are not so clean. Introduce the generalized coordinates $\xi = \frac{1}{\sqrt{2}}(\theta_1 + \theta_2)$ and $\eta = \frac{1}{\sqrt{2}}(\theta_2 - \theta_1)$, inspired by the third term in the above Lagrangian. One way to proceed is to rewrite the Lagrangian:

$$L = \frac{1}{2}ml^2\left(\dot{\xi}^2 + \dot{\eta}^2\right) - \frac{1}{2}mgl\left(\xi^2 + \eta^2\right) - kl^2\eta^2 = \left[\frac{1}{2}ml^2\dot{\xi}^2 - \frac{1}{2}mgl\xi^2\right] + \left[\frac{1}{2}ml^2\dot{\eta}^2 - \left(\frac{1}{2}mgl + kl^2\right)\eta^2\right] \tag{88}$$

By the choice in coordinates the Lagrangian decomposes into a sum. At this point we might recognize these as the Lagrangians for simple harmonic oscillators. If not, the equations of motion are quite revealing:

$$\ddot{\xi} + \left(\frac{g}{l}\right)\xi = 0 \tag{89}$$

$$\ddot{\eta} + \left(\frac{g}{l} + \frac{2k}{m}\right)\eta = 0 \tag{90}$$

Of course one could bypass writing the Lagrangian out again by simple substituting the new coordinates into the Euler-Lagrange equations found initially. In either case, we have two normal modes with eigenfrequencies $\omega_1 = \sqrt{\frac{g}{l}}$ and $\omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}}$. The first of these corresponds to the two penduli moving together with frequency $\sqrt{\frac{g}{l}}$, as if the spring were not there at all (note that k is not present in the frequency expression). The second corresponds to the two penduli moving asymmetrically, and the presence of the spring increases the frequency from the natural frequency $\omega_0 = \sqrt{\frac{g}{l}}$. Indeed, these interpretations exactly match the definitions of ξ and η.

Such an in-depth analysis is often time consuming. One can gain much insight by considering limit cases for masses, spring constants and lengths.

3 Fluid Dynamics

Pressure is defined to be

$$P = \frac{\mathrm{d}F}{\mathrm{d}A} \tag{91}$$

The Bernoulli equation for incompressible fluids is

$$P + \frac{1}{2}\rho v^2 + \rho g h = \text{const.} \tag{92}$$

This is analogous to conservation of energy. When the fluid is static we see that the pressure increases linearly with the depth, proportional to the fluid density. Fluid flow also satisfies the continuity equation:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0 \tag{93}$$

For incompressible fluids this amounts to the condition

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \tag{94}$$

In the simple case where an incompressible fluid is moving through a tube of variable cross-sectional area, we must have

$$Av = \text{const.} \tag{95}$$

where A is the cross-sectional area at some point and v is the speed of the fluid at this point. Simply, the rate at which fluid enters the tube must be the same as the rate at which fluid exits.

From the Bernoulli equation it is clear that for static fluids the pressure increases with depth. An object submerged in the fluid will have unequal pressures on its top and bottom, leading to a buoyancy force. This buoyant force on an object is equal to the weight of the fluid which it displaced; this is Archimedes' principle.

$$F_{\text{buoy}} = \rho_{\text{fl}} V g \tag{96}$$

A Summary

Kinematics

$$
\begin{aligned}
\boldsymbol{x}(t) &= \boldsymbol{x}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{a}t^2 && \text{(Constant Acceleration)} \\
\boldsymbol{v}(t) &= \boldsymbol{v}_0 + \boldsymbol{a}t \\
y_{\max} &= y_0 + \frac{v_{0y}^2}{2g} && \text{(Max Height)} \\
R &= \frac{v^2}{g}\sin 2\theta && \text{(Range Equation)}
\end{aligned}
$$

Newton's Laws

$$
\begin{aligned}
\sum \boldsymbol{F} &= 0 \quad \iff \quad \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = 0 && (1^{\text{st}} \text{ Law}) \\
\boldsymbol{F} &= \frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = \frac{\mathrm{d}(m\boldsymbol{v})}{\mathrm{d}t} && (2^{\text{d}} \text{ Law}) \\
\boldsymbol{F}_{\text{AB}} &= -\boldsymbol{F}_{\text{BA}} && (3^{\text{d}} \text{ Law})
\end{aligned}
$$

Work & Energy

$$
\begin{aligned}
W &= \int \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{s} && \text{(Work)} \\
\sum W &= \Delta E && \text{(Work-Energy Theorem)} \\
\boldsymbol{F} &= -\boldsymbol{\nabla} U && \text{(Conservative Force)} \\
T &= T_t + T_r = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{p^2}{2m} + \frac{L^2}{2I} && \text{(Kinetic Energy)} \\
f_{\text{s}} &\leq \mu_{\text{s}} N && \text{(Static Friction)} \\
f_{\text{k}} &= \mu_{\text{k}} N && \text{(Kinetic Friction)} \\
\boldsymbol{J} &= \Delta \boldsymbol{p} = \int \boldsymbol{F}(t)\,\mathrm{d}t && \text{(Impulse)}
\end{aligned}
$$

Rotational Motion

$$
\begin{aligned}
\theta(t) &= \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 && \text{(Constant Angular Acceleration)} \\
\omega(t) &= \omega_0 + \alpha t \\
T &= \frac{1}{f} = \frac{2\pi}{\omega} && \text{(Period} \leftrightarrow \text{Frequency)} \\
\boldsymbol{\tau} &= \boldsymbol{r} \times \boldsymbol{F} && \text{(Torque)} \\
\boldsymbol{L} &= \boldsymbol{r} \times \boldsymbol{p} = I\boldsymbol{\omega} && \text{(Angular Momentum)} \\
I &= \sum_i m_i r_i^2 = \int r^2\,\mathrm{d}m = \int r^2 \rho(\boldsymbol{r})\,\mathrm{d}\boldsymbol{r} && \text{(Momentum of Inertia)}
\end{aligned}
$$

$$\boldsymbol{\tau} = \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} = \frac{\mathrm{d}(\boldsymbol{r} \times \boldsymbol{p})}{\mathrm{d}t} = \frac{\mathrm{d}(I\boldsymbol{\omega})}{\mathrm{d}t} \qquad \text{(Angular } 2^{\mathrm{d}} \text{ Law)}$$

Noninertial Reference Frames

$$\boldsymbol{F}_{\mathrm{cf}} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \qquad \text{(Centrifugal Force)}$$

$$\boldsymbol{F}_{\mathrm{Co}} = -2m\boldsymbol{\omega} \times \boldsymbol{v} \qquad \text{(Coriolis Force)}$$

Dynamics of Systems of Particles

$$\boldsymbol{r}_{\mathrm{cm}} = \frac{1}{M}\sum_i m_i \boldsymbol{r}_i = \frac{1}{M}\int \boldsymbol{r}\,\mathrm{d}m \qquad \text{(Center of Mass)}$$

Central Forces & Celestial Mechanics

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \text{const.} \qquad \text{(Kepler's } 2^{\mathrm{d}} \text{ Law)}$$

$$\frac{T^2}{a^3} = \frac{4\pi^2}{G(M+m)} \approx \frac{4\pi^2}{GM} \qquad \text{(Kepler's } 3^{\mathrm{d}} \text{ Law)}$$

$$g = \frac{GM_{\oplus}}{R_{\oplus}^2} \approx 9.81\,{}^{\mathrm{N}}\!/\!{}_{\mathrm{kg}} \qquad \text{(Earth Surface Gravity)}$$

Lagrangian Mechanics

$$S = \int L(\boldsymbol{q}, \dot{\boldsymbol{q}}; t)\,\mathrm{d}t \qquad \text{(Action)}$$

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} - \frac{\partial L}{\partial \boldsymbol{q}} \qquad \text{(Euler-Lagrange Equations)}$$

Hamiltonian Mechanics

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \qquad \text{(Conjugate Momentum)}$$

$$H(\boldsymbol{q}, \boldsymbol{p}; t) = \dot{\boldsymbol{q}} \cdot \boldsymbol{p} - L(\boldsymbol{q}, \dot{\boldsymbol{q}}; t) \qquad \text{(Hamiltonian)}$$

$$\begin{aligned} \dot{\boldsymbol{q}} &= \frac{\partial H}{\partial \boldsymbol{p}} = \{\boldsymbol{q}, H\} \qquad \text{(Hamilton's Equations)} \\ \dot{\boldsymbol{p}} &= -\frac{\partial H}{\partial \boldsymbol{q}} = \{\boldsymbol{p}, H\} \\ \frac{\mathrm{d}H}{\mathrm{d}t} &= \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \end{aligned}$$

Fluid Dynamics

$$P = \frac{\mathrm{d}F}{\mathrm{d}A} \qquad \text{(Pressure)}$$

$$\text{const.} = P + \frac{1}{2}\rho v^2 + \rho g h \qquad \text{(Bernoulli Equation)}$$

$$0 = \frac{\mathrm{d}\rho}{\mathrm{d}t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) \qquad \text{(Continuity Equation)}$$

$$Av = \text{const.} \qquad \text{(Incompressible Tube Flow)}$$

$$F_{\mathrm{buoy}} = \rho_{\mathrm{fl}} V g \qquad \text{(Archimedes' Principle)}$$