PCH Winter Summary

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1 Trigonometry

1.1 Pythagorean Identity

$$\sin^2 x + \cos^2 x = 1$$

1.2 Even/Odd Identities

Cosine is even. $\cos(-\theta) = \cos\theta$
Sine is odd. $\sin(-\theta) = -\sin\theta$
Tangent is odd. $\tan(-\theta) = -\tan\theta$

1.3 Angle Addition and Subtraction Identities

$$\begin{aligned}
\sin(\alpha + \beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
\sin(\alpha - \beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\cos(\alpha - \beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\tan(\alpha + \beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\
\tan(\alpha - \beta) &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}
\end{aligned}$$

1.4 Double Angle Identites (PS #29)

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta \\
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
&= 1 - 2\sin^2\theta \\
&= 2\cos^2\theta - 1 \\
\tan 2\theta &= \frac{2\tan\theta}{1 - \tan^2\theta}
\end{aligned}$$

1.5 Half Angle Identites (PS #30)

$$\begin{aligned}
\sin\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{2}} \\
\cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}} \\
\tan\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\
&= \frac{1-\cos\theta}{\sin\theta} \\
&= \frac{\sin\theta}{1+\cos\theta}
\end{aligned}$$

1.6 Product-to-Sum Identities (PS #31)

$$\begin{aligned}
\sin\alpha\cos\beta &= \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2} \\
\cos\alpha\sin\beta &= \frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2} \\
\cos\alpha\cos\beta &= \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2} \\
\sin\alpha\sin\beta &= \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}
\end{aligned}$$

1.7 Sum-to-Product Identities (PS #31)

$$\begin{aligned}
\sin\alpha+\sin\beta &= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\sin\alpha-\sin\beta &= 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\
\cos\alpha+\cos\beta &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\cos\alpha-\cos\beta &= -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}
\end{aligned}$$

You don't really need to memorize these identities. They are very easy to prove and you can do it in the exam if you need them.

1.8 Finding Range

1.8.1 Quadratic with Trig Functions

Complete the square and isolate the trig function. Example (Koch CYU #1, Q4):

$$\begin{aligned} y &= 3\cos x + 2\sin^2 x + 1 \\ &= 3\cos x + 2\left(1-\cos^2 x\right) + 1 \\ &= -2\cos^2 x + 3\cos x + 3 \\ &= -2\left(\cos^2 x - \frac{3}{2}\cos x - \frac{3}{2}\right) \\ &= -2\left(\left(\cos x - \frac{3}{4}\right)^2 - \frac{9}{16} - \frac{3}{2}\right) \\ &= -2\left(\cos x - \frac{3}{4}\right)^2 + \frac{33}{8} \end{aligned}$$

$$-1 \le \cos x \le 1$$

$$-\frac{7}{4} \le \cos x - \frac{3}{4} \le \frac{1}{4}$$

$$0 \le \left(\cos x - \frac{3}{4}\right)^2 \le \frac{49}{16}$$

$$-\frac{49}{8} \le -2\left(\cos x - \frac{3}{4}\right)^2 \le 0$$

$$-\frac{16}{8} \le -2\left(\cos x - \frac{3}{4}\right)^2 + \frac{33}{8} \le \frac{33}{8}$$

$$-2 \le y \le \frac{33}{8}$$

$$\boxed{\text{Range: } \left[-2, \frac{33}{8}\right]}$$

1.8.2 Addition of Sine and Cosine

Rewrite $y = a\sin x + b\cos x$ in the form $y = r\cos(x-a)$. Example (PS #31, Q8a):

$$\begin{aligned} y &= \sqrt{3}\sin x - \cos x \\ y &= r\cos(x-a) \\ y &= r\cos x\cos a + r\sin x\sin a \end{aligned}$$

By matching coefficients:

$$\begin{aligned} r\sin a &= -1 \\ r\cos a &= \sqrt{3} \end{aligned}$$

Square both equations:

$$\begin{aligned} r^2\sin^2 a &= 1 \\ r^2\cos^2 a &= 3 \end{aligned}$$

Apply Pythagorean Identity:

$$\begin{aligned} r^2\sin^2 a &= 1 \\ r^2 - r^2\sin^2 a &= 3 \end{aligned}$$

Add the two equations:

$$\begin{aligned} r^2 &= 4 \\ r &= \pm 2 \end{aligned}$$

Because the range of the cosine function is always $[-1, 1]$, the range of $r\cos(x-a)$ is $[-r, r]$. No matter what the sign of r is, the range is $\boxed{[-2, 2]}$.

1.8.3 With Reciprocals

On an interval where the denominator is never zero, just work it out normally. Example:

$$y = \frac{3}{2-\cos x}$$
$$-1 \leq \cos x \leq 1$$
$$1 \leq 2-\cos x \leq 3$$

$$2-\cos x \text{ cannot be } 0.$$

$$3 \geq \frac{3}{2-\cos x} \geq 1$$

Remember to flip the inequality sign when taking the reciprocal.

$$\boxed{\text{Range: } [1,3]}$$

On an interval where the denominator could be zero, we simply need to change the last step from an **and** condition into an **or** condition. Example (Koch CYU #2, Q3b):

$$y = \frac{1}{1-3\sqrt{1-\sin^2 x}}$$
$$y = \frac{1}{1-3\sqrt{\cos^2 x}}$$
$$y = \frac{1}{1-3\left|\cos x\right|}$$

Remember the absolute value! $\sqrt{x^2} = |x|$, not x.

$$-1 \leq \cos x \leq 1$$
$$0 \leq \left|\cos x\right| \leq 1$$
$$-3 \leq -3\left|\cos x\right| \leq 0$$
$$-2 \leq 1-3\left|\cos x\right| \leq 1$$
$$-\frac{1}{2} \leq \frac{1}{1-3\left|\cos x\right|} \geq 1$$

We have to flip the inequality when taking reciprocals, but the left side involves a negative sign, so it gets flipped back into $\leq$. However, this statement is really two statements chained together, and can be split into:

$$-\frac{1}{2} \leq \frac{1}{1-3\left|\cos x\right|} \quad \textbf{or} \quad \frac{1}{1-3\left|\cos x\right|} \geq 1$$

$$\boxed{\text{Range: } \left(-\infty, -\frac{1}{2}\right] \cup [1,\infty)}$$

1.8.4 Composition of Inverse Trig Function and Something Else

First, we see that each inverse trigonometric function, apart from arcsec and arccsc, is either increasing or decreasing.

arcsin: increasing
arccos: decreasing
arctan: increasing
arccot: decreasing

Basically, a function is decreasing if and only if it starts with "arcco." Based on this property, when finding the range of a composite function in the form $f(g(x))$ where $f(x)$ is one of the inverse trigonometric functions, we only need to find the range of the inside function $g(x)$, and plug in the min and max of g to find the range of the whole function. Example (PS #35, Q4e):

$$y = \arctan \frac{x^2 + \sqrt{3}}{x^2 + 1}$$

Range of "inside":

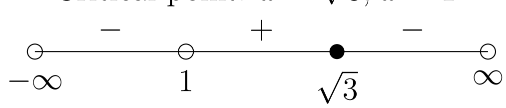
$$y = \frac{x^2 + \sqrt{3}}{x^2 + 1}$$
$$yx^2 + y = x^2 + \sqrt{3}$$
$$x^2(y-1) = \sqrt{3} - y$$
$$x = \pm\sqrt{\frac{\sqrt{3} - y}{y - 1}}$$

$$y = \pm\sqrt{\frac{\sqrt{3} - x}{x - 1}}$$

Find domain (because that would be range of original function):

$$\frac{\sqrt{3} - x}{x - 1} \geq 0$$

Critical point: $x = \sqrt{3}$, $x = 1$



$$x \in \left(1, \sqrt{3}\right]$$
$$\arctan 1 = \frac{\pi}{4}, \ \arctan \sqrt{3} = \frac{\pi}{3}$$

The interval is open on 1, so it should also be open for $\frac{\pi}{4}$.

$$\boxed{\text{Range: } \left(\frac{\pi}{4}, \frac{\pi}{3}\right]}$$

1.9 Finding Periodicity (PS #31)

Find the periodicity of each function, then find the lowest common multiple.

1.10 Graphing (PS #31)

Find the left end point and right end point of the parent function. Then find where the inner function is equal to each end point. Check for horizontal reflections, work out the vertical dilation/stretch and translation, then graph one period.

1.11 Derivatives of Trigonometric Functions (PS #32)

$$\frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\cot x = -\csc^2 x$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\sec x = \tan(x)\sec(x)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\csc x = -\cot(x)\csc(x)$$

1.12 Trigonometric Limits (PS #33)

In indeterminate forms, a common trick is to multiple by the conjugate of either the numerator or the denominator. The purpose of this is to get the limit into the form $\lim_{h\to 0}\sin(h)/h$ or $\lim_{h\to 0}(\cos h - 1)/h$, which we know evaluates to 1 and 0. We cannot use L'Hôpital's Rule, but it can be used to check the answer (secretly). If the denominator is zero but the numerator is non-zero, analyse the one-sided limits to determine if the limit is ∞, $-\infty$ or does not exist. Otherwise, plug in the value in the limit. Refer to PS #33 for examples.

1.13 Inverse Trigonometric Functions (PS #34-35)

The inverse trigonometric functions are defined as the inverse of their respective trigonometric functions. However, since they are periodic, an inverse would fail the vertical line test and would not be considered a function. Therefore, they are defined as **the inverse of the trigonometric function over an interval**. PS #34 is only concerned with arcsin, arccos and arctan. Below are the restricted domains of each function:

$$\text{sin: } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
$$\text{cos: } [0, \pi]$$
$$\text{tan: } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Below are their respective ranges:

$$\begin{aligned}\sin\text{:}\ & [-1,1]\\ \cos\text{:}\ & [-1,1]\\ \tan\text{:}\ & (-\infty,\infty)\end{aligned}$$

The domain and range of the inverse are inverted. Therefore, below are the domain and range of the inverse trigonometric functions:

$$\begin{aligned}\text{arcsin domain: } & [-1,1]\\ \text{arcsin range: } & \left[-\frac{\pi}{2},\frac{\pi}{2}\right]\\ \text{arccos domain: } & [-1,1]\\ \text{acccos range: } & [0,\pi]\\ \text{arctan domain: } & (-\infty,\infty)\\ \text{arctan range: } & \left(-\frac{\pi}{2},\frac{\pi}{2}\right)\end{aligned}$$

NOTE! $\arcsin(\sin x)$ does not always equal x. This applies to any other trigonometric function. For any value x, $\arcsin(\sin x)$ is a value on $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ with the same sine as x. However, it is true on $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ in this example, and is true on the domain of the restricted function/range of the inverse function for any other trigonometric function. The graph of $\arcsin(\sin x)$ and $\arccos(\cos x)$ looks like a zigzag, while $\arctan(\tan x)$ is the graph of $y = x$ on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ repeating with period π.

A good way to solve problems or proofs involving inverse trigonometric functions is to draw a right-angle triangle. Setting an angle to the arcfunction involved, you can then work out the ratio of the sides, which might help with the question.

1.13.1 Assortment of Various Identities Involving Inverse Trigonometric Functions

These are, in general, quite useless. However they might be handy for some specific questions, although there will probably be an easier way for those.

$$\begin{aligned}\arcsin(-x) &= -\arcsin x\\ \arctan(-x) &= -\arctan x\\ \arccos(-x) &= \pi-\arccos x\end{aligned}$$

These can be derived from the definitions of each function.

$$\begin{aligned}\arccos(\cos x) &= x,\ x\in[0,\pi]\\ \cos(\arccos x) &= x,\ x\in[-1,1]\\ \arcsin(\sin x) &= x,\ x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\\ \sin(\arcsin x) &= x,\ x\in[-1,1]\\ \arctan(\tan x) &= x,\ x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\\ \tan(\arctan x) &= x,\ x\in\mathbb{R}\end{aligned}$$

These can be proved by drawing triangles. That's what probably should do in CYU/exam instead of memorizing these.

$$\begin{aligned}
\sin\left(\arccos x\right) = \cos\left(\arcsin x\right) &= \sqrt{1-x^2} \\
\sin\left(\arctan x\right) &= \frac{x}{\sqrt{x^2+1}} \\
\cos\left(\arctan x\right) &= \frac{1}{\sqrt{x^2+1}} \\
\tan\left(\arcsin x\right) &= \frac{x}{\sqrt{1-x^2}} \\
\tan\left(\arccos x\right) &= \frac{\sqrt{1-x^2}}{x}
\end{aligned}$$

2 Derivations

2.1 Tangent Angle Addition

Proof.
$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\
&= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta} \\
&= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta} \Big/ \frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta} \\
&= \left(\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}\right) \Big/ \left(\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}\right) \\
&= \boxed{\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}}
\end{aligned}$$

□

2.2 Tangent Angle Subtraction

Proof.
$$\begin{aligned}
\tan(\alpha-\beta) &= \tan(\alpha+(-\beta)) \\
&= \frac{\tan\alpha+\tan(-\beta)}{1-\tan\alpha\tan(-\beta)} \\
&= \frac{\tan\alpha+(-\tan\beta)}{1-\tan\alpha\cdot(-\tan\beta)} \\
&= \boxed{\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}}
\end{aligned}$$

□

2.3 Sine Double Angle

Proof.
$$\begin{aligned}
\sin 2\theta &= \sin(\theta+\theta) \\
&= \sin\theta\cos\theta+\cos\theta\sin\theta \\
&= \boxed{2\sin\theta\cos\theta}
\end{aligned}$$

□

2.4 Cosine Double Angle

Proof.
$$\begin{aligned}\cos 2\theta &= \cos(\theta+\theta)\\ &= \cos\theta\cos\theta - \sin\theta\sin\theta\\ &= \boxed{\cos^2\theta - \sin^2\theta}\\ &= \cos^2\theta - \left(1-\cos^2\theta\right)\\ &= \boxed{2\cos^2\theta - 1}\\ &= 2\left(1-\sin^2\theta\right) - 1\\ &= \boxed{1 - 2\sin^2\theta}\end{aligned}$$

□

2.5 Tangent Double Angle

Proof.
$$\begin{aligned}\tan 2\theta &= \tan(\theta+\theta)\\ &= \frac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta}\\ &= \boxed{\frac{2\tan\theta}{1-\tan^2\theta}}\end{aligned}$$

□

2.6 Sine Half Angle

Proof.
$$\begin{aligned}\cos\theta &= \cos\left(2\cdot\frac{\theta}{2}\right)\\ \cos\theta &= 1 - 2\sin^2\frac{\theta}{2}\\ 2\sin^2\frac{\theta}{2} &= 1-\cos\theta\\ &\boxed{\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}}\end{aligned}$$

□

2.7 Cosine Half Angle

Proof.
$$\begin{aligned}\cos\theta &= \cos\left(2\cdot\frac{\theta}{2}\right)\\ \cos\theta &= 2\cos^2\frac{\theta}{2} - 1\\ 2\cos^2\frac{\theta}{2} &= 1+\cos\theta\\ &\boxed{\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}}\end{aligned}$$

□

2.8 Tangent Half Angle

Proof. $\tan\dfrac{\theta}{2} = \sin\dfrac{\theta}{2}/\cos\dfrac{\theta}{2}$

$$\begin{aligned}
&= \sqrt{\frac{1-\cos\theta}{2}}/\sqrt{\frac{1+\cos\theta}{2}} \\
&= \sqrt{\frac{1-\cos\theta}{2}/\frac{1+\cos\theta}{2}} \\
&= \boxed{\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}} \\
&= \sqrt{\frac{1-\cos\theta}{1+\cos\theta}\cdot\frac{1-\cos\theta}{1-\cos\theta}} \\
&= \sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}} \\
&= \sqrt{\frac{(1-\cos\theta)^2}{\sin^2\theta}} \\
&= \boxed{\frac{1-\cos\theta}{\sin\theta}} \\
&= \sqrt{\frac{1-\cos\theta}{1+\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}} \\
&= \sqrt{\frac{1-\cos^2\theta}{(1-\cos\theta)^2}} \\
&= \sqrt{\frac{\sin^2\theta}{(1+\cos\theta)^2}} \\
&= \boxed{\frac{\sin\theta}{1+\cos\theta}}
\end{aligned}$$

□

2.9 Product-to-Sum Identities

Proof.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \quad (1)$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \quad (2)$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \quad (3)$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \quad (4)$$

$$(1) + (2)$$

$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta + \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta$$
$$\boxed{\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2}}$$

$$(1) - (2)$$

$$\sin(\alpha+\beta) - \sin(\alpha-\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta - \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta$$
$$\boxed{\cos\alpha\sin\beta = \frac{\sin(\alpha+\beta) - \sin(\alpha-\beta)}{2}}$$

$$(3) + (4)$$

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\alpha\cos\beta + \sin\alpha\sin\beta$$
$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$$
$$\boxed{\cos\alpha\cos\beta = \frac{\cos(\alpha-\beta) + \cos(\alpha+\beta)}{2}}$$

$$(3) - (4)$$

$$\cos(\alpha+\beta) - \cos(\alpha-\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta - \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta$$
$$\sin\alpha\sin\beta = -\frac{\cos(\alpha+\beta) - \cos(\alpha-\beta)}{2}$$
$$\boxed{\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}}$$

$\square$

2.10 Sum-to-Product Identities

Proof.

$$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2}$$
$$\cos\alpha\sin\beta = \frac{\sin(\alpha+\beta) - \sin(\alpha-\beta)}{2}$$
$$\cos\alpha\cos\beta = \frac{\cos(\alpha-\beta) + \cos(\alpha+\beta)}{2}$$
$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}$$

$$\text{Let } \theta = \alpha + \beta,\ \varphi = \alpha - \beta$$
$$\alpha = \frac{\theta + \varphi}{2},\ \beta = \frac{\theta - \varphi}{2}$$

$$\sin\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2} = \frac{\sin\theta + \sin\varphi}{2}$$
$$\cos\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2} = \frac{\sin\theta - \sin\varphi}{2}$$
$$\cos\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2} = \frac{\cos\varphi + \cos\theta}{2}$$
$$\sin\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2} = \frac{\cos\varphi - \cos\theta}{2}$$

$$\boxed{\sin\theta + \sin\varphi = 2\sin\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}}$$
$$\boxed{\sin\theta - \sin\varphi = 2\cos\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2}}$$
$$\cos\varphi + \cos\theta = 2\cos\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}$$
$$\cos\varphi - \cos\theta = 2\sin\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2}$$

$$\boxed{\cos\theta + \cos\varphi = 2\cos\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}}$$
$$\boxed{\cos\theta - \cos\varphi = -2\sin\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2}}$$

□

2.11 Limit of $\sin(h)/h$ as $h \to 0$

Proof. By calculator: $\lim_{h\to 0}\frac{\sin h}{h} = \boxed{1}$ □

2.12 Limit of $(\cos h - 1)/h$ as $h \to 0$

Proof.
$$\begin{aligned}
\lim_{h\to 0} \frac{\cos h - 1}{h} &= \lim_{h\to 0} \left(\frac{\cos h - 1}{h} \cdot \frac{\cos h + 1}{\cos h + 1} \right) \\
&= \lim_{h\to 0} \left(\frac{1 - \cos^2 h}{\cos h + 1} \cdot \frac{1}{h} \right) \\
&= \lim_{h\to 0} \frac{\sin h}{\cos h + 1} \cdot \lim_{h\to 0} \frac{\sin h}{h} \\
&= \frac{\sin 0}{\cos 0 + 1} \cdot 1 \\
&= \boxed{0}
\end{aligned}$$

□

2.13 Sine Derivative

Proof.
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \sin x &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h\to 0} \left(\sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h} \right) \\
&= \sin(x) \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} + \cos(x) \cdot \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin(x) \cdot 0 + \cos(x) \cdot 1 \\
&= \boxed{\cos x}
\end{aligned}$$

□

2.14 Cosine Derivative

Proof.
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} \cos x &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)(\cos h - 1) - \sin x \sin h}{h} \\
&= \lim_{h\to 0} \left(\cos x \cdot \frac{\cos h - 1}{h} - \sin x \cdot \frac{\sin h}{h} \right) \\
&= \cos(x) \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin(x) \cdot \lim_{h\to 0} \frac{\sin h}{h} \\
&= \cos(x) \cdot 0 - \sin(x) \cdot 1 \\
&= \boxed{-\sin x}
\end{aligned}$$

□

2.15 Tangent Derivative

Proof.
$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\tan x &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin x}{\cos x}\right] \\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}x}[\sin x]\cos x - \sin(x)\frac{\mathrm{d}}{\mathrm{d}x}[\cos x]}{\cos^2 x} \\
&= \frac{\cos x \cdot \cos x - \sin(x)(-\cos x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \boxed{\sec^2 x}
\end{aligned}
$$

□

2.16 Cotangent Derivative

Proof.
$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\cot x &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\tan x}\right] \\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}x}[1]\tan x - 1 \cdot \frac{\mathrm{d}}{\mathrm{d}x}[\tan x]}{\tan^2 x} \\
&= \frac{-\sec^2 x \cdot \cos^2 x}{\tan^2 x \cdot \cos^2 x} \\
&= -\frac{1}{\sin^2 x} \\
&= \boxed{-\csc^2 x}
\end{aligned}
$$

□

2.17 Secant Derivative

Proof.
$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sec x &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\cos x}\right] \\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}x}[1]\cos x - 1 \cdot \frac{\mathrm{d}}{\mathrm{d}x}[\cos x]}{\cos^2 x} \\
&= \frac{\sin x}{\cos^2 x} \\
&= \boxed{\tan(x)\sec(x)}
\end{aligned}
$$

□

2.18 Cosecant Derivative

Proof.
$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\csc x &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{\sin x}\right] \\ &= \frac{\frac{\mathrm{d}}{\mathrm{d}x}[1]\sin x - 1\cdot\frac{\mathrm{d}}{\mathrm{d}x}[\sin x]}{\sin^2 x} \\ &= \frac{-\cos x}{\sin^2 x} \\ &= \boxed{-\cot(x)\csc(x)}\end{aligned}$$

$\square$