PCH Winter Summary

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1 Trigonometry

1.1 Pythagorean Identity

$$\sin^2 x + \cos^2 x = 1$$

1.2 Even/Odd Identities

$$\text{Cosine is even. } \cos(-\theta) = \cos\theta$$
$$\text{Sine is odd. } \sin(-\theta) = -\sin\theta$$
$$\text{Tangent is odd. } \tan(-\theta) = -\tan\theta$$

1.3 Angle Addition and Subtraction Identities

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\tan(\alpha+\beta) &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} \\
\tan(\alpha-\beta) &= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}
\end{aligned}$$

1.4 Double Angle Identites

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta \\
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
&= 1 - 2\sin^2\theta \\
&= 2\cos^2\theta - 1 \\
\tan 2\theta &= \frac{2\tan\theta}{1 - \tan^2\theta}
\end{aligned}$$

1.5 Product-to-Sum Identities

$$\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}$$
$$\cos\alpha\sin\beta = \frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2}$$
$$\cos\alpha\cos\beta = \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2}$$
$$\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}$$

2 Derivations

2.1 Tangent Angle Addition

Proof.
$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\
&= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta} \\
&= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}\Big/\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta} \\
&= \left(\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}\right)\Big/\left(\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}\right) \\
&= \boxed{\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}}
\end{aligned}$$

□

2.2 Tangent Angle Subtraction

Proof.
$$\begin{aligned}
\tan(\alpha-\beta) &= \tan(\alpha+(-\beta)) \\
&= \frac{\tan\alpha+\tan(-\beta)}{1-\tan\alpha\tan(-\beta)} \\
&= \frac{\tan\alpha+(-\tan\beta)}{1-\tan\alpha\cdot(-\tan\beta)} \\
&= \boxed{\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}}
\end{aligned}$$

□

2.3 Sine Double Angle

Proof.
$$\begin{aligned}
\sin 2\theta &= \sin(\theta+\theta) \\
&= \sin\theta\cos\theta+\cos\theta\sin\theta \\
&= \boxed{2\sin\theta\cos\theta}
\end{aligned}$$

□

2.4 Cosine Double Angle

Proof.
$$\begin{aligned}\cos 2\theta &= \cos(\theta+\theta)\\ &= \cos\theta\cos\theta - \sin\theta\sin\theta\\ &= \boxed{\cos^2\theta - \sin^2\theta}\\ &= \cos^2\theta - \left(1-\cos^2\theta\right)\\ &= \boxed{2\cos^2\theta - 1}\\ &= 2\left(1-\sin^2\theta\right) - 1\\ &= \boxed{1-2\sin^2\theta}\end{aligned}$$

□

2.5 Tangent Double Angle

Proof.
$$\begin{aligned}\tan 2\theta &= \tan(\theta+\theta)\\ &= \frac{\tan\theta+\tan\theta}{1-\tan\theta\tan\theta}\\ &= \boxed{\frac{2\tan\theta}{1-\tan^2\theta}}\end{aligned}$$

□

2.6 Product-to-Sum Identities

Proof.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{1}$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \tag{2}$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{3}$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \tag{4}$$

$$(1)+(2)$$

$$\begin{aligned}\sin(\alpha+\beta)+\sin(\alpha-\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta+\sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \sin(\alpha+\beta)+\sin(\alpha-\beta) &= 2\sin\alpha\cos\beta\end{aligned}$$
$$\boxed{\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2}}$$

$$(1)-(2)$$

$$\begin{aligned}\sin(\alpha+\beta)-\sin(\alpha-\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta-\sin\alpha\cos\beta+\cos\alpha\sin\beta\\ \sin(\alpha+\beta)-\sin(\alpha-\beta) &= 2\cos\alpha\sin\beta\end{aligned}$$
$$\boxed{\cos\alpha\sin\beta = \frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2}}$$

$$(3) + (4)$$

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta$$

$$\boxed{\cos\alpha\cos\beta = \frac{\cos(\alpha-\beta) + \cos(\alpha+\beta)}{2}}$$

$$(3) - (4)$$

$$\cos(\alpha+\beta) - \cos(\alpha-\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta - \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta$$

$$\sin\alpha\sin\beta = -\frac{\cos(\alpha+\beta) - \cos(\alpha-\beta)}{2}$$

$$\boxed{\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}}$$

$\square$