PCH Winter Summary

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1 Trigonometry

1.1 Pythagorean Identity

$$\sin^2 x + \cos^2 x = 1$$

1.2 Even/Odd Identities

Cosine is even. $\cos(-\theta) = \cos\theta$
Sine is odd. $\sin(-\theta) = -\sin\theta$
Tangent is odd. $\tan(-\theta) = -\tan\theta$

1.3 Angle Addition and Subtraction Identities

$$\begin{aligned}
\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
\sin(\alpha-\beta) &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\tan(\alpha+\beta) &= \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \\
\tan(\alpha-\beta) &= \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}
\end{aligned}$$

1.4 Double Angle Identites (PS #29)

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta \\
\cos 2\theta &= \cos^2\theta - \sin^2\theta \\
&= 1 - 2\sin^2\theta \\
&= 2\cos^2\theta - 1 \\
\tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}
\end{aligned}$$

1.5 Half Angle Identites (PS #30)

$$\begin{aligned}
\sin\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{2}} \\
\cos\frac{\theta}{2} &= \pm\sqrt{\frac{1+\cos\theta}{2}} \\
\tan\frac{\theta}{2} &= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} \\
&= \frac{1-\cos\theta}{\sin\theta} \\
&= \frac{\sin\theta}{1+\cos\theta}
\end{aligned}$$

1.6 Product-to-Sum Identities (PS #31)

$$\begin{aligned}
\sin\alpha\cos\beta &= \frac{\sin(\alpha+\beta)+\sin(\alpha-\beta)}{2} \\
\cos\alpha\sin\beta &= \frac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2} \\
\cos\alpha\cos\beta &= \frac{\cos(\alpha-\beta)+\cos(\alpha+\beta)}{2} \\
\sin\alpha\sin\beta &= \frac{\cos(\alpha-\beta)-\cos(\alpha+\beta)}{2}
\end{aligned}$$

1.7 Sum-to-Product Identities (PS #31)

$$\begin{aligned}
\sin\alpha+\sin\beta &= 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\sin\alpha-\sin\beta &= 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2} \\
\cos\alpha+\cos\beta &= 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} \\
\cos\alpha-\cos\beta &= -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}
\end{aligned}$$

1.8 Finding Range

1.8.1 Quadratic with Trig Functions

Complete the square and isolate the trig function. Example (Koch CYU #1, Q4):

$$\begin{aligned}
y &= 3\cos x + 2\sin^2 x + 1 \\
&= 3\cos x + 2\left(1-\cos^2 x\right) + 1 \\
&= -2\cos^2 x + 3\cos x + 3 \\
&= -2\left(\cos^2 x - \frac{3}{2}\cos x - \frac{3}{2}\right) \\
&= -2\left(\left(\cos x - \frac{3}{4}\right)^2 - \frac{9}{16} - \frac{3}{2}\right) \\
&= -2\left(\cos x - \frac{3}{4}\right)^2 + \frac{33}{8}
\end{aligned}$$

$$\begin{gathered}
-1 \le \cos x \le 1 \\
-\frac{7}{4} \le \cos x - \frac{3}{4} \le \frac{1}{4} \\
0 \le \left(\cos x - \frac{3}{4}\right)^2 \le \frac{49}{16} \\
-\frac{49}{8} \le -2\left(\cos x - \frac{3}{4}\right)^2 \le 0 \\
-\frac{16}{8} \le -2\left(\cos x - \frac{3}{4}\right)^2 + \frac{33}{8} \le \frac{33}{8} \\
-2 \le y \le \frac{33}{8} \\
\boxed{\text{Range: } \left[-2, \frac{33}{8}\right]}
\end{gathered}$$

1.8.2 Addition of Sine and Cosine

Rewrite $y = a\sin x + b\cos x$ in the form $y = r\cos(x-a)$. Example (PS #31, Q8a):

$$\begin{aligned}
y &= \sqrt{3}\sin x - \cos x \\
y &= r\cos(x-a) \\
y &= r\cos x\cos a + r\sin x\sin a
\end{aligned}$$

By matching coefficients:

$$\begin{aligned}
r\sin a &= -1 \\
r\cos a &= \sqrt{3}
\end{aligned}$$

Square both equations:

$$\begin{aligned}
r^2\sin^2 a &= 1 \\
r^2\cos^2 a &= 3
\end{aligned}$$

Apply Pythagorean Identity:

$$\begin{aligned}
r^2\sin^2 a &= 1 \\
r^2 - r^2\sin^2 a &= 3
\end{aligned}$$

Add the two equations:

$$\begin{aligned}
r^2 &= 4 \\
r &= \pm 2
\end{aligned}$$

Because the range of the cosine function is always $[-1,1]$, the range of $r\cos(x-a)$ is $[-r,r]$. No matter what the sign of r is, the range is $\boxed{[-2,2]}$.

1.9 Derivatives of Trigonometric Functions (PS #32)

1.10 Trigonometric Limits (PS #33)

1.11 Inverse Trigonometric Functions (PS #34-35)

2 Derivations

2.1 Tangent Angle Addition

Proof.
$$\begin{aligned}
\tan(\alpha+\beta) &= \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} \\
&= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta} \\
&= \frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}\Big/\frac{\cos\alpha\cos\beta-\sin\alpha\sin\beta}{\cos\alpha\cos\beta} \\
&= \left(\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}\right)\Big/\left(\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}\right) \\
&= \boxed{\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}}
\end{aligned}$$

□

2.2 Tangent Angle Subtraction

Proof.
$$\begin{aligned}
\tan(\alpha-\beta) &= \tan(\alpha+(-\beta)) \\
&= \frac{\tan\alpha+\tan(-\beta)}{1-\tan\alpha\tan(-\beta)} \\
&= \frac{\tan\alpha+(-\tan\beta)}{1-\tan\alpha\cdot(-\tan\beta)} \\
&= \boxed{\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}}
\end{aligned}$$

□

2.3 Sine Double Angle

Proof.
$$\begin{aligned}
\sin 2\theta &= \sin(\theta+\theta) \\
&= \sin\theta\cos\theta+\cos\theta\sin\theta \\
&= \boxed{2\sin\theta\cos\theta}
\end{aligned}$$

□

2.4 Cosine Double Angle

Proof.
$$\begin{aligned}
\cos 2\theta &= \cos(\theta + \theta) \\
&= \cos\theta\cos\theta - \sin\theta\sin\theta \\
&= \boxed{\cos^2\theta - \sin^2\theta} \\
&= \cos^2\theta - \left(1 - \cos^2\theta\right) \\
&= \boxed{2\cos^2\theta - 1} \\
&= 2\left(1 - \sin^2\theta\right) - 1 \\
&= \boxed{1 - 2\sin^2\theta}
\end{aligned}$$

□

2.5 Tangent Double Angle

Proof.
$$\begin{aligned}
\tan 2\theta &= \tan(\theta + \theta) \\
&= \frac{\tan\theta + \tan\theta}{1 - \tan\theta\tan\theta} \\
&= \boxed{\frac{2\tan\theta}{1 - \tan^2\theta}}
\end{aligned}$$

□

2.6 Sine Half Angle

Proof.
$$\begin{aligned}
\cos\theta &= \cos\left(2 \cdot \frac{\theta}{2}\right) \\
\cos\theta &= 1 - 2\sin^2\frac{\theta}{2} \\
2\sin^2\frac{\theta}{2} &= 1 - \cos\theta \\
&\boxed{\sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos\theta}{2}}}
\end{aligned}$$

□

2.7 Cosine Half Angle

Proof.
$$\begin{aligned}
\cos\theta &= \cos\left(2 \cdot \frac{\theta}{2}\right) \\
\cos\theta &= 2\cos^2\frac{\theta}{2} - 1 \\
2\cos^2\frac{\theta}{2} &= 1 + \cos\theta \\
&\boxed{\cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos\theta}{2}}}
\end{aligned}$$

□

2.8 Tangent Half Angle

Proof.
$$
\begin{aligned}
\tan\frac{\theta}{2} &= \sin\frac{\theta}{2}/\cos\frac{\theta}{2} \\
&= \sqrt{\frac{1-\cos\theta}{2}}/\sqrt{\frac{1+\cos\theta}{2}} \\
&= \sqrt{\frac{1-\cos\theta}{2}/\frac{1+\cos\theta}{2}} \\
&= \boxed{\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}} \\
&= \sqrt{\frac{1-\cos\theta}{1+\cos\theta}\cdot\frac{1-\cos\theta}{1-\cos\theta}} \\
&= \sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}} \\
&= \sqrt{\frac{(1-\cos\theta)^2}{\sin^2\theta}} \\
&= \boxed{\frac{1-\cos\theta}{\sin\theta}} \\
&= \sqrt{\frac{1-\cos\theta}{1+\cos\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}} \\
&= \sqrt{\frac{1-\cos^2\theta}{(1-\cos\theta)^2}} \\
&= \sqrt{\frac{\sin^2\theta}{(1+\cos\theta)^2}} \\
&= \boxed{\frac{\sin\theta}{1+\cos\theta}}
\end{aligned}
$$

□

2.9 Product-to-Sum Identities

Proof.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{1}$$
$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \tag{2}$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{3}$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \tag{4}$$

$$(1) + (2)$$

$$\begin{aligned}
\sin(\alpha+\beta) + \sin(\alpha-\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta + \sin\alpha\cos\beta - \cos\alpha\sin\beta \\
\sin(\alpha+\beta) + \sin(\alpha-\beta) &= 2\sin\alpha\cos\beta
\end{aligned}$$

$$\boxed{\sin\alpha\cos\beta = \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2}}$$

$$(1) - (2)$$

$$\begin{aligned}
\sin(\alpha+\beta) - \sin(\alpha-\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta - \sin\alpha\cos\beta + \cos\alpha\sin\beta \\
\sin(\alpha+\beta) - \sin(\alpha-\beta) &= 2\cos\alpha\sin\beta
\end{aligned}$$

$$\boxed{\cos\alpha\sin\beta = \frac{\sin(\alpha+\beta) - \sin(\alpha-\beta)}{2}}$$

$$(3) + (4)$$

$$\begin{aligned}
\cos(\alpha+\beta) + \cos(\alpha-\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta + \cos\alpha\cos\beta + \sin\alpha\sin\beta \\
\cos(\alpha+\beta) + \cos(\alpha-\beta) &= 2\cos\alpha\cos\beta
\end{aligned}$$

$$\boxed{\cos\alpha\cos\beta = \frac{\cos(\alpha-\beta) + \cos(\alpha+\beta)}{2}}$$

$$(3) - (4)$$

$$\begin{aligned}
\cos(\alpha+\beta) - \cos(\alpha-\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta - \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\cos(\alpha+\beta) - \cos(\alpha-\beta) &= -2\sin\alpha\sin\beta
\end{aligned}$$

$$\sin\alpha\sin\beta = -\frac{\cos(\alpha+\beta) - \cos(\alpha-\beta)}{2}$$

$$\boxed{\sin\alpha\sin\beta = \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}}$$

□

2.10 Sum-to-Product Identities

Proof.

$$\begin{aligned}
\sin\alpha\cos\beta &= \frac{\sin(\alpha+\beta) + \sin(\alpha-\beta)}{2} \\
\cos\alpha\sin\beta &= \frac{\sin(\alpha+\beta) - \sin(\alpha-\beta)}{2} \\
\cos\alpha\cos\beta &= \frac{\cos(\alpha-\beta) + \cos(\alpha+\beta)}{2} \\
\sin\alpha\sin\beta &= \frac{\cos(\alpha-\beta) - \cos(\alpha+\beta)}{2}
\end{aligned}$$

$$\text{Let } \theta = \alpha + \beta,\ \varphi = \alpha - \beta$$
$$\alpha = \frac{\theta + \varphi}{2},\ \beta = \frac{\theta - \varphi}{2}$$

$$\begin{aligned}
\sin\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2} &= \frac{\sin\theta + \sin\varphi}{2} \\
\cos\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2} &= \frac{\sin\theta - \sin\varphi}{2} \\
\cos\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2} &= \frac{\cos\varphi + \cos\theta}{2} \\
\sin\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2} &= \frac{\cos\varphi - \cos\theta}{2}
\end{aligned}$$

$$\boxed{\sin\theta + \sin\varphi = 2\sin\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}}$$
$$\boxed{\sin\theta - \sin\varphi = 2\cos\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2}}$$
$$\cos\varphi + \cos\theta = 2\cos\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}$$
$$\cos\varphi - \cos\theta = 2\sin\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2}$$

$$\boxed{\cos\theta + \cos\varphi = 2\cos\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}}$$
$$\boxed{\cos\theta - \cos\varphi = -2\sin\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2}}$$

$\square$