

# PCH Winter Summary

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## 1 Trigonometry

### 1.1 Pythagorean Identity

$$\sin^2 x + \cos^2 x = 1$$

### 1.2 Even/Odd Identities

Cosine is even.  $\cos(-\theta) = \cos \theta$

Sine is odd.  $\sin(-\theta) = -\sin \theta$

Tangent is odd.  $\tan(-\theta) = -\tan \theta$

### 1.3 Angle Addition and Subtraction Identities

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

$$\tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

### 1.4 Double Angle Identities (PS #29)

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$= 1 - 2 \sin^2 \theta$$

$$= 2 \cos^2 \theta - 1$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

## 1.5 Half Angle Identities (PS #30)

$$\begin{aligned}\sin \frac{\theta}{2} &= \pm \sqrt{\frac{1 - \cos \theta}{2}} \\ \cos \frac{\theta}{2} &= \pm \sqrt{\frac{1 + \cos \theta}{2}} \\ \tan \frac{\theta}{2} &= \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} \\ &= \frac{1 - \cos \theta}{\sin \theta} \\ &= \frac{\sin \theta}{1 + \cos \theta}\end{aligned}$$

## 1.6 Product-to-Sum Identities (PS #31)

$$\begin{aligned}\sin \alpha \cos \beta &= \frac{\sin (\alpha + \beta) + \sin (\alpha - \beta)}{2} \\ \cos \alpha \sin \beta &= \frac{\sin (\alpha + \beta) - \sin (\alpha - \beta)}{2} \\ \cos \alpha \cos \beta &= \frac{\cos (\alpha - \beta) + \cos (\alpha + \beta)}{2} \\ \sin \alpha \sin \beta &= \frac{\cos (\alpha - \beta) - \cos (\alpha + \beta)}{2}\end{aligned}$$

## 1.7 Sum-to-Product Identities (PS #31)

$$\begin{aligned}\sin \alpha + \sin \beta &= 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \\ \sin \alpha - \sin \beta &= 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2} \\ \cos \alpha + \cos \beta &= 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} \\ \cos \alpha - \cos \beta &= -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}\end{aligned}$$

## 1.8 Finding Range

### 1.8.1 Quadratic with Trig Functions

Complete the square and isolate the trig function. Example (Koch CYU #1, Q4):

$$\begin{aligned}
 y &= 3 \cos x + 2 \sin^2 x + 1 \\
 &= 3 \cos x + 2(1 - \cos^2 x) + 1 \\
 &= -2 \cos^2 x + 3 \cos x + 3 \\
 &= -2 \left( \cos^2 x - \frac{3}{2} \cos x - \frac{3}{2} \right) \\
 &= -2 \left( \left( \cos x - \frac{3}{4} \right)^2 - \frac{9}{16} - \frac{3}{2} \right) \\
 &= -2 \left( \cos x - \frac{3}{4} \right)^2 + \frac{33}{8}
 \end{aligned}$$

$$\begin{aligned}
 -1 &\leq \cos x \leq 1 \\
 -\frac{7}{4} &\leq \cos x - \frac{3}{4} \leq \frac{1}{4} \\
 0 &\leq \left( \cos x - \frac{3}{4} \right)^2 \leq \frac{49}{16} \\
 -\frac{49}{8} &\leq -2 \left( \cos x - \frac{3}{4} \right)^2 \leq 0 \\
 -\frac{16}{8} &\leq -2 \left( \cos x - \frac{3}{4} \right)^2 + \frac{33}{8} \leq \frac{33}{8} \\
 -2 &\leq y \leq \frac{33}{8}
 \end{aligned}$$

Range:  $\left[ -2, \frac{33}{8} \right]$

### 1.8.2 Addition of Sine and Cosine

Rewrite  $y = a \sin x + b \cos x$  in the form  $y = r \cos(x - a)$ . Example (PS #31, Q8a):

$$\begin{aligned}
 y &= \sqrt{3} \sin x - \cos x \\
 y &= r \cos(x - a) \\
 y &= r \cos x \cos a + r \sin x \sin a
 \end{aligned}$$

Apply Pythagorean Identity:

$$\begin{aligned}
 r^2 \sin^2 a &= 1 \\
 r^2 - r^2 \sin^2 a &= 3
 \end{aligned}$$

Add the two equations:

$$r^2 = 4$$

$$r = \pm 2$$

By matching coefficients:

$$\begin{aligned}
 r \sin a &= -1 \\
 r \cos a &= \sqrt{3}
 \end{aligned}$$

Square both equations:

$$\begin{aligned}
 r^2 \sin^2 a &= 1 \\
 r^2 \cos^2 a &= 3
 \end{aligned}$$

Because the range of the cosine function is always  $[-1, 1]$ , the range of  $r \cos(x - a)$  is  $[-r, r]$ . No matter what the sign of  $r$  is, the range is  $\boxed{[-2, 2]}$ .

- 1.9 Finding Periodicity (PS #31)
- 1.10 Graphing (PS #31)
- 1.11 Derivatives of Trigonometric Functions (PS #32)
- 1.12 Trigonometric Limits (PS #33)
- 1.13 Inverse Trigonometric Functions (PS #34-35)

## 2 Derivations

### 2.1 Tangent Angle Addition

$$\begin{aligned} \text{Proof. } \tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \\ &= \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta} \\ &= \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta} / \frac{\cos \alpha \cos \beta - \sin \alpha \sin \beta}{\cos \alpha \cos \beta} \\ &= \left( \frac{\sin \alpha \cos \beta}{\cos \alpha \cos \beta} + \frac{\cos \alpha \sin \beta}{\cos \alpha \cos \beta} \right) / \left( \frac{\cos \alpha \cos \beta}{\cos \alpha \cos \beta} - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta} \right) \\ &= \boxed{\frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}} \end{aligned}$$

□

### 2.2 Tangent Angle Subtraction

$$\begin{aligned} \text{Proof. } \tan(\alpha - \beta) &= \tan(\alpha + (-\beta)) \\ &= \frac{\tan \alpha + \tan(-\beta)}{1 - \tan \alpha \tan(-\beta)} \\ &= \frac{\tan \alpha + (-\tan \beta)}{1 - \tan \alpha \cdot (-\tan \beta)} \\ &= \boxed{\frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}} \end{aligned}$$

□

### 2.3 Sine Double Angle

$$\begin{aligned} \text{Proof. } \sin 2\theta &= \sin(\theta + \theta) \\ &= \sin \theta \cos \theta + \cos \theta \sin \theta \\ &= \boxed{2 \sin \theta \cos \theta} \end{aligned}$$

□

## 2.4 Cosine Double Angle

$$\begin{aligned} \text{Proof. } \cos 2\theta &= \cos(\theta + \theta) \\ &= \cos \theta \cos \theta - \sin \theta \sin \theta \\ &= \boxed{\cos^2 \theta - \sin^2 \theta} \\ &= \cos^2 \theta - (1 - \cos^2 \theta) \\ &= \boxed{2 \cos^2 \theta - 1} \\ &= 2(1 - \sin^2 \theta) - 1 \\ &= \boxed{1 - 2 \sin^2 \theta} \end{aligned}$$

□

## 2.5 Tangent Double Angle

$$\begin{aligned} \text{Proof. } \tan 2\theta &= \tan(\theta + \theta) \\ &= \frac{\tan \theta + \tan \theta}{1 - \tan \theta \tan \theta} \\ &= \boxed{\frac{2 \tan \theta}{1 - \tan^2 \theta}} \end{aligned}$$

□

## 2.6 Sine Half Angle

$$\begin{aligned} \text{Proof. } \cos \theta &= \cos \left( 2 \cdot \frac{\theta}{2} \right) \\ \cos \theta &= 1 - 2 \sin^2 \frac{\theta}{2} \\ 2 \sin^2 \frac{\theta}{2} &= 1 - \cos \theta \\ \boxed{\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}} \end{aligned}$$

□

## 2.7 Cosine Half Angle

$$\begin{aligned} \text{Proof. } \cos \theta &= \cos \left( 2 \cdot \frac{\theta}{2} \right) \\ \cos \theta &= 2 \cos^2 \frac{\theta}{2} - 1 \\ 2 \cos^2 \frac{\theta}{2} &= 1 + \cos \theta \\ \boxed{\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}} \end{aligned}$$

□

## 2.8 Tangent Half Angle

$$\begin{aligned}
 \text{Proof. } \tan \frac{\theta}{2} &= \sin \frac{\theta}{2} / \cos \frac{\theta}{2} \\
 &= \sqrt{\frac{1 - \cos \theta}{2}} / \sqrt{\frac{1 + \cos \theta}{2}} \\
 &= \sqrt{\frac{1 - \cos \theta}{2}} / \frac{1 + \cos \theta}{2} \\
 &= \boxed{\pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}} \\
 &= \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta} \cdot \frac{1 - \cos \theta}{1 - \cos \theta}} \\
 &= \sqrt{\frac{(1 - \cos \theta)^2}{1 - \cos^2 \theta}} \\
 &= \sqrt{\frac{(1 - \cos \theta)^2}{\sin^2 \theta}} \\
 &= \boxed{\frac{1 - \cos \theta}{\sin \theta}} \\
 &= \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta} \cdot \frac{1 + \cos \theta}{1 + \cos \theta}} \\
 &= \sqrt{\frac{1 - \cos^2 \theta}{(1 + \cos \theta)^2}} \\
 &= \sqrt{\frac{\sin^2 \theta}{(1 + \cos \theta)^2}} \\
 &= \boxed{\frac{\sin \theta}{1 + \cos \theta}}
 \end{aligned}$$

□

## 2.9 Product-to-Sum Identities

*Proof.*

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \quad (1)$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \quad (2)$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \quad (3)$$

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta \quad (4)$$

$$(1) + (2)$$

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta + \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta$$

$$\boxed{\sin \alpha \cos \beta = \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{2}}$$

$$(1) - (2)$$

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta - \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\sin(\alpha + \beta) - \sin(\alpha - \beta) = 2 \cos \alpha \sin \beta$$

$$\boxed{\cos \alpha \sin \beta = \frac{\sin(\alpha + \beta) - \sin(\alpha - \beta)}{2}}$$

$$(3) + (4)$$

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta + \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos \alpha \cos \beta$$

$$\boxed{\cos \alpha \cos \beta = \frac{\cos(\alpha - \beta) + \cos(\alpha + \beta)}{2}}$$

$$(3) - (4)$$

$$\cos(\alpha + \beta) - \cos(\alpha - \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta - \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos(\alpha + \beta) - \cos(\alpha - \beta) = -2 \sin \alpha \sin \beta$$

$$\sin \alpha \sin \beta = -\frac{\cos(\alpha + \beta) - \cos(\alpha - \beta)}{2}$$

$$\boxed{\sin \alpha \sin \beta = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2}}$$

□

## 2.10 Sum-to-Product Identities

*Proof.*

$$\sin \alpha \cos \beta = \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{2}$$

$$\cos \alpha \sin \beta = \frac{\sin(\alpha + \beta) - \sin(\alpha - \beta)}{2}$$

$$\cos \alpha \cos \beta = \frac{\cos(\alpha - \beta) + \cos(\alpha + \beta)}{2}$$

$$\sin \alpha \sin \beta = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2}$$



$$\text{Let } \theta = \alpha + \beta, \varphi = \alpha - \beta$$

$$\alpha = \frac{\theta + \varphi}{2}, \beta = \frac{\theta - \varphi}{2}$$

$$\sin \frac{\theta + \varphi}{2} \cos \frac{\theta - \varphi}{2} = \frac{\sin \theta + \sin \varphi}{2}$$

$$\cos \frac{\theta + \varphi}{2} \sin \frac{\theta - \varphi}{2} = \frac{\sin \theta - \sin \varphi}{2}$$

$$\cos \frac{\theta + \varphi}{2} \cos \frac{\theta - \varphi}{2} = \frac{\cos \varphi + \cos \theta}{2}$$

$$\sin \frac{\theta + \varphi}{2} \sin \frac{\theta - \varphi}{2} = \frac{\cos \varphi - \cos \theta}{2}$$

$$\sin \theta + \sin \varphi = 2 \sin \frac{\theta + \varphi}{2} \cos \frac{\theta - \varphi}{2}$$

$$\sin \theta - \sin \varphi = 2 \cos \frac{\theta + \varphi}{2} \sin \frac{\theta - \varphi}{2}$$

$$\cos \varphi + \cos \theta = 2 \cos \frac{\theta + \varphi}{2} \cos \frac{\theta - \varphi}{2}$$

$$\cos \varphi - \cos \theta = 2 \sin \frac{\theta + \varphi}{2} \sin \frac{\theta - \varphi}{2}$$

$$\cos \theta + \cos \varphi = 2 \cos \frac{\theta + \varphi}{2} \cos \frac{\theta - \varphi}{2}$$

$$\cos \theta - \cos \varphi = -2 \sin \frac{\theta + \varphi}{2} \sin \frac{\theta - \varphi}{2}$$

□