$$f(y) = \cos(y-1) + 2$$

(a) 1. First, we are revolving $f(y)$ around the y-axis. Since the function f is in terms of y, we integrate along the __y__ axis (with respect to __y__), which is parallel to the axis of rotation. Consider the cross section of the vase along the axis of integration. This means we can use (disc / washer / cylindrical shells).

2. The base of the vase is at $y =$ __0__, and since the vase's height is 6 meters, the top of the vase is at $y =$ __6__. The limits of integration is [__0__, __6__].

3. The radius of each disc is __$f(y) = \cos(y-1)+2$__ .

4. Recall the disc method formula: $V = \pi \int_a^b (\text{Radius})^2\, dy$. As an integral, the volume of the solid is:

$$V = \pi \int_0^6 (f(y))^2 dy = \pi \int_0^6 (\cos(y-1)+2)^2 dy$$

5. Evaluate the integral below. **The vase should store more than 80 cubic meters of water.**

$$u = y-1 \qquad du = dy$$

$$V = \pi \int_{-1}^{5} (\cos u + 2)^2 du = \pi \int_{-1}^{5} (\cos^2 u + 4\cos u + 4) du$$

$$= \pi \int_{-1}^{5} \frac{1+\cos 2u}{2} du + \pi \int_{-1}^{5} (4\cos u + 4) du$$

$$= \frac{\pi}{2}\left[u + \frac{\sin 2u}{2}\right]_{-1}^{5} + \pi \left[4\sin u + 4u\right]_{-1}^{5}$$

$$= \frac{\pi}{2}\left(5 + \frac{\sin 10}{2} - \left(-1 + \frac{\sin(-2)}{2}\right)\right) + \pi\left(4\sin 5 + 20 - (4\sin(-1) - 4)\right)$$

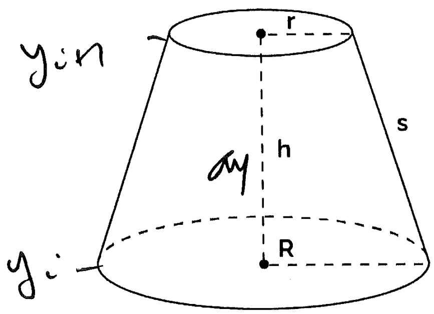
$$= \frac{\pi}{2}\left(6 + \frac{\sin 10}{2} - \frac{\sin(-2)}{2}\right) + \pi\left(24 + 4\sin 5 - 4\sin(-1)\right)$$

$$= \ldots \text{ [more simplification]}$$

$$\approx 83.634 \text{ m}^3 \geq 80 \text{ m}^3$$

(b) 1. Let's consider the surface area of the vase as a Riemann sum. Let there be n subdivisions, each of height Δy.

Each subdivision is a **truncated cone**:



https://www.omnicalculator.com/math/truncated-cone

2. Let's consider the ith subdivision. We call the bottom y-value y_i, and the top y_{i+1}. In the above picture, we would have $r =$ $f(y_{i+1})$ and $R =$ $f(y_i)$ (hint: the radius r and R is the distance between the y-axis and the function $f(y)$).

3. The height of the truncated cone is $y_{i+1} - yi =$ Δy.

4. We need to calculate the surface area of the truncated cone, excluding the top and bottom faces. This is because in the Riemann sum, the top and bottom faces are directly against another subdivision and do not count towards the area of the surface of revolution. This is exactly the area of the slant, which is $\pi s(R + r)$.

5. We know R and r, but need to find s. Notice we can construct a right triangle with s as the hypotenuse. One of the legs is the height of the truncated cone: Δy.

6. The other leg is: R − r = $f(y_i) - f(y_{i+1})$.

7. By the Pythagorean theorem, s equals $\sqrt{(\Delta y)^2 + (f(y_{i+1}) - f(y_i))^2}$

8. Using the formula Slant area $= \pi s(R + r)$, write down the area of the slant below.

Slant area $= \pi \sqrt{(\Delta y)^2 + (f(y_{i+1}) - f(y_i))^2}\,(f(y_i) + f(y_{i+1}))$

You should have $\pi\sqrt{(\Delta y)^2 + (f(y_{i+1}) - f(y_i))^2}\,(f(y_i) + f(y_{i+1}))$, or something equivalent.

9. Now write down the sum of the slant areas of n subdivisions, as n approaches infinity.

$$SA = \lim_{n\to\infty} \sum_{i=1}^{n} \pi \sqrt{(\Delta y)^2 + (f(y_{i+1}) - f(y_i))^2}\,(f(y_i) + f(y_{i+1}))$$

10. See if you can factor out the $(\Delta y)^2$ from the square root. The root should read $\sqrt{1 + \frac{\rule{4cm}{0.4pt}}{(\Delta y)^2}}$. Also, $y_{i+1} = y_i +$ $\underline{\Delta y}$. Make this substitution in the two appearances of y_{i+1}.

$$SA = \lim_{n\to\infty} \sum_{i=1}^{n} \pi \sqrt{(\Delta y)^2} \sqrt{1 + \frac{(f(y_{i+1}) - f(y_i))^2}{(\Delta y)^2}} \left(f(y_i) + f(y_i + \Delta y)\right)$$

11. As $n \to \infty$, $\Delta y \to$ $\underline{0}$. So $y_{i+1} = y_i + \Delta y \to$ $\underline{y_i}$. In which of the two appearances of $f(y_i + \Delta y)$ in the above equation can we actually make this substitution? (hint: notice $(\Delta y)^2$ in the denominator, and remember the limit definition of the derivative)

not in root b/c Δy in denominator.

$$SA = \lim_{n\to\infty} \sum_{i=1}^{n} \pi \Delta y \sqrt{1 + \left(\frac{f(y_{i+1}) - f(y_i)}{\Delta y}\right)^2} \left(f(y_i) + f(y_i)\right)$$

You should have $\lim_{n\to\infty} \sum_{i=1}^{n} \pi \Delta y \sqrt{1 + \frac{(f(y_i + \Delta y) - f(y_i))^2}{(\Delta y)^2}}\,(f(y_i) + f(y_i))$, or something similar.

12. By the limit definition, $f'(y_i) = \lim_{\Delta y \to 0} \frac{f(y_i + \Delta y) - f(y_i)}{\Delta y}$

Do you see this in your answer to 11? If yes, rewrite it in terms of $f'(y_i)$. Make any trivial simplifications.

$$SA = \lim_{n\to\infty} \sum_{i=1}^{n} \pi \Delta y (2f(y_i)) \sqrt{1 + (f'(y_i))^2}$$

You should have $\lim_{n\to\infty} \sum_{i=1}^{n} 2\pi f(y_i) \Delta y \sqrt{1 + (f'(y))^2}$, or something similar.

recall bounds of integ. $[0, 6]$.

13. Is this in the form of a Riemann sum? Write the above sum as an integral, and evaluate it by calculator.

$$SA = \int_a^b 2\pi f(y) \sqrt{1 + (f'(y))^2}\, dy = \int_0^6 2\pi f(y) \sqrt{1 + (f'(y))^2}\, dy$$

$$\approx 90.216 \ m^2$$