A Day at the Blair Academy

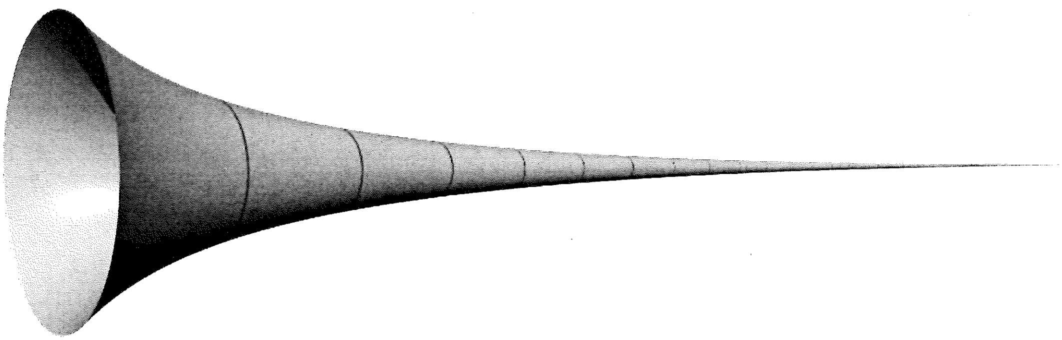
~~AP Calculus BC~~ Calculus, 2025-26[7]

After a winter term trapped inside the robotics lab, Jayden Li wants to connect with nature more. To do so, he has enrolled at the Blair Academy for the spring term.

After a humiliating defeat at the hands of the Peddie School at the 2024 "Peddie Day" event, the students of Blair are finding innovative ways to win the 2025 "Peddie Day," where they seek to snatch the "Potter-Kelly Cup" from its rightful owners.

They discussed many ways to do so. Someone proposed to simply win more matches at "Peddie Day," but this idea was dismissed as "unrealistic." Instead, they are attempting to find innovative devices and techniques to distract Peddie atheletes, allowing the Blair Academy to snatch the win right under Peddie's noses.

They have come up with the following object to use as a Vuvuzela[8]. This is known as a *Gabriel's Horn*, and is obtained by revolving the function $f(x) = 1/x$ on the interval $[1, \infty)$ around the x-axis.



https://en.wikipedia.org/wiki/Gabriel%27s_horn

(a) The Blair Academy's version of Mr. Tackett wants to know how much material the ungraciously-unprofessional object will use (which equals its surface area).

(b) What is the volume of the Vuvuzela?

(c) Before they can use this object at the 2025 "Peddie Day," they must paint the device in navy blue. Is this possible? Why or why not?

Jayden's classmates at the Blair Academy are unable to answer the above questions, despite being seniors who will graduate in a few weeks. Save the day by solving it for them![9]

Note: these problems include *improper integrals* in the form $\int_a^\infty f(x)\,\mathrm{d}x$. If F is the antiderivative of f, then:

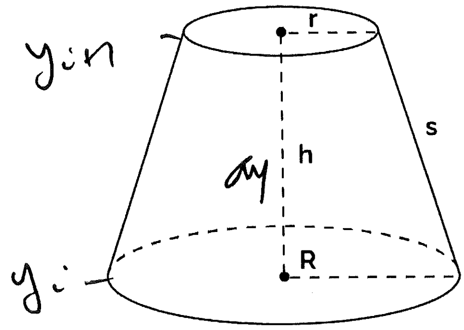
$$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{t\to\infty}\int_a^t f(x)\,\mathrm{d}x = \lim_{t\to\infty}\left[F(x)\right]_a^t = \lim_{t\to\infty} F(t) - F(a)$$

If $\lim_{t\to\infty} F(t)$ diverges, then the integral also diverges.

[7] Apparently, the Blair Academy does not offer AP courses.
[8] https://www.youtube.com/watch?v=OXViRxTvOkc
[9] Inspired by https://www.reddit.com/r/UofT/comments/1ijkea6/uwaterloo_math_138_question_yall_are_dumb_asl_lmao/

(b) 1. Let's consider the surface area of the vase as a Riemann sum. Let there be n subdivisions, each of height Δy.

Each subdivision is a **truncated cone**:



https://www.omnicalculator.com/math/truncated-cone

2. Let's consider the ith subdivision. We call the bottom y-value y_i, and the top y_{i+1}. In the above picture, we would have $r =$ $f(y_{i+1})$ and $R =$ $f(y_i)$ (hint: the radius r and R is the distance between the y-axis and the function $f(y)$).

3. The height of the truncated cone is $y_{i+1} - yi =$ Δy.

4. We need to calculate the surface area of the truncated cone, excluding the top and bottom faces. This is because in the Riemann sum, the top and bottom faces are directly against another subdivision and do not count towards the area of the surface of revolution. This is exactly the area of the slant, which is $\pi s(R + r)$.

5. We know R and r, but need to find s. Notice we can construct a right triangle with s as the hypotenuse. One of the legs is the height of the truncated cone: Δy.

6. The other leg is: R − r = $f(y_i) - f(y_{i+1})$.

7. By the Pythagorean theorem, s equals $\sqrt{(\Delta y)^2 + (f(y_{i+1}) - f(y_i))^2}$

8. Using the formula Slant area $= \pi s(R + r)$, write down the area of the slant below.

$$\text{Slant area} = \pi\sqrt{(\Delta y)^2 + (f(y_{i+1}) - f(y_i))^2}\,(f(y_i) + f(y_{i+1}))$$

You should have $\pi\sqrt{(\Delta y)^2 + (f(y_{i+1}) - f(y_i))^2}\,(f(y_i) + f(y_{i+1}))$, or something equivalent.

9. Now write down the sum of the slant areas of n subdivisions, as n approaches infinity.

$$SA = \lim_{n\to\infty} \sum_{i=1}^{n} \pi\sqrt{(\Delta y)^2 + (f(y_{i+1}) - f(y_i))^2}\,(f(y_i) + f(y_{i+1}))$$