Eigenvalue Algorithms

Numerical Strategies Providing for the Time and Resource Efficient Computation of Eigenvalues for Any General Square Matrix

Jayden Li

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The Problem

How can we find the eigenvalues of any $n \times n$ matrix A?

$$A\mathbf{x} = \lambda\mathbf{x}$$

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Power Method

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Reference: [Koc25].

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- Now, if $|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n|$, then $\lambda_i/\lambda_1 < 1$ for all $i \neq 1$.

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$$A^k v = \lim_{k\to\infty} \lambda_1^k \left(\cancel{\left(\frac{\lambda_1}{\lambda_1}\right)^k} v_{0_1} p_1 + \cancel{\left(\frac{\lambda_2}{\lambda_1}\right)^k} v_{0_2} p_2 + \cancel{\sum_{i=3}^{n} \left(\frac{\lambda_i}{\lambda_1}\right)^k v_{0_i} p_i} \right)$$

$$A^k v = \lim_{k\to\infty} \lambda_1^k \left(v_{0_1} p_1 + v_{0_2} p_2 \right)$$

is not an eigenvector of A if $\lambda_1 \neq \lambda_2$.

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- Cannot calculate complex eigenvalues.
- Can use shifts (calculate eigenvalues of $A - cI$), but that is complicated.
- Even worse: only calculates dominant eigenvalue.
- Speed of convergence depends on choice of starting vector.

Section 2

Basic QR Algorithm

Basic QR Algorithm

Schur form

Let A be an $n \times n$ real matrix. The **Schur form** of A is:

$$A = QUQ^T = QUQ^{-1}$$

where Q is orthogonal and U is upper triangular. A and U are similar so they share the same eigenvalues, which are the diagonal entries of U.

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Reference: [Arb16, p. 64].

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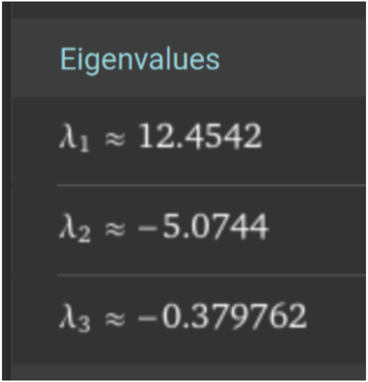
Proof is hard and complicated. We will "prove" by example.

Example of the Basic QR Algorithm (1)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 1 \end{bmatrix}$$



Example of the Basic QR Algorithm (2)

$$A_1 = \begin{bmatrix} 1.00 & 2.00 & 3.00 \\ 4.00 & 5.00 & 6.00 \\ 7.00 & 8.00 & 1.00 \end{bmatrix}$$

Example of the Basic QR Algorithm (2)

$$A_1 = \begin{bmatrix} 1.00 & 2.00 & 3.00 \\ 4.00 & 5.00 & 6.00 \\ 7.00 & 8.00 & 1.00 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} -0.12 & 0.90 & 0.41 \\ -0.49 & 0.30 & -0.82 \\ -0.86 & -0.30 & 0.41 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} -8.12 & -9.60 & -4.19 \\ 0.00 & 0.90 & 4.22 \\ 0.00 & 0.00 & -3.27 \end{bmatrix}$$

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$$A_2 = \begin{bmatrix} 9.33 & -8.98 & 2.81 \\ -4.08 & -1.00 & 0.98 \\ 2.81 & 0.98 & -1.33 \end{bmatrix}$$

Example of the Basic QR Algorithm (3)

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$$Q_2 = \begin{bmatrix} -0.88 & -0.47 & 0.02 \\ 0.39 & -0.70 & 0.60 \\ -0.27 & 0.54 & 0.80 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} -10.57 & 7.28 & -1.75 \\ 0.00 & 5.44 & -2.73 \\ 0.00 & 0.00 & -0.42 \end{bmatrix}$$

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$$A_3 = \begin{bmatrix} 12.61 & -1.09 & 2.74 \\ 2.83 & -5.28 & 1.08 \\ 0.11 & -0.22 & -0.33 \end{bmatrix}$$

Example of the Basic QR Algorithm (4)

$$A_4 = \begin{bmatrix} 12.15 & -4.87 & -3.01 \\ -1.07 & -4.77 & -0.65 \\ 0.00 & 0.02 & -0.38 \end{bmatrix}$$

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$$A_5 = \begin{bmatrix} 12.54 & -3.34 & 2.96 \\ 0.46 & -5.16 & 0.93 \\ 0.00 & -0.00 & -0.38 \end{bmatrix}$$

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Example of the Basic QR Algorithm (5)

$$A_{11} = \begin{bmatrix} \boxed{12.45} & -3.79 & 2.98 \\ 0.00 & \boxed{-5.07} & 0.85 \\ 0.00 & -0.00 & \boxed{-0.38} \end{bmatrix}$$

Eigenvalues

$\lambda_1 \approx 12.4542$

$\lambda_2 \approx -5.0744$

$\lambda_3 \approx -0.379762$

We found EVERY eigenvalue!

Is it good enough?

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- Similar convergence problems to Power Method: only guaranteed to work if $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$
- Needs modifications to find complex eigenvalues.
- Could be faster: runs in $\mathcal{O}(n^3)$ time for $n \times n$ matrix.

Bad Example (1)

$$A = \begin{bmatrix} 1 & 5 \\ -1 & -3 \end{bmatrix}$$

Eigenvalues

$\lambda_1 = -1 + i$

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$$|\lambda_1| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$
$$|\lambda_2| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

Bad Example (2)

$$A_1 = \begin{bmatrix} 1.00 & 5.00 \\ -1.00 & -3.00 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -3.00 & -5.00 \\ 1.00 & 1.00 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -1.40 & 5.80 \\ -0.20 & -0.60 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} -0.60 & -5.80 \\ 0.20 & -1.40 \end{bmatrix}$$

Bad Example (2)

$$A_1 = \begin{bmatrix} 1.00 & 5.00 \\ -1.00 & -3.00 \end{bmatrix} \qquad A_5 = \begin{bmatrix} 1.00 & 5.00 \\ -1.00 & -3.00 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -3.00 & -5.00 \\ 1.00 & 1.00 \end{bmatrix} \qquad A_6 = \begin{bmatrix} -3.00 & -5.00 \\ 1.00 & 1.00 \end{bmatrix}$$

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$\lim_{k\to\infty} A_k$ will not converge onto a triangular matrix.

Another Bad Example

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

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$$\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 1, \lambda_4 = -1$$

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$$A_1 = A = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{Q_1} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{R_1} = A_1 I$$

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This will never converge to a triangular matrix, since $A_i = A$ for all i.

Section 3

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Used by the Linear Algebra PACKage (LAPACK), which is used by NumPy and SciPy behind the scenes.

LAPACK Algorithm

Improved QR Algorithm

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LAPACK Algorithm

1. Reduce A to an *upper Hessenberg* form by calculating $A = QHQ^T$, where Q is orthogonal and H is an upper Hessenberg matrix. H and A are similar and share the same eigenvalues.

Improved QR Algorithm

Used by the Linear Algebra PACKage (LAPACK), which is used by NumPy and SciPy behind the scenes.

LAPACK Algorithm

1. Reduce A to an *upper Hessenberg* form by calculating $A = QHQ^T$, where Q is orthogonal and H is an upper Hessenberg matrix. H and A are similar and share the same eigenvalues.
2. Calculate the Schur form of H.

Reference: [Bla99].

Section 4

Conclusion

References

[Arb16] Peter Arbenz. *Lecture Notes on Solving Large Scale Eigenvalue Problems*. 2016. URL: https://people.inf.ethz.ch/arbenz/ewp/Lnotes/lsevp.pdf.

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