Eigenvalue Algorithms

Numerical Strategies Providing for the Time and Resource Efficient Computation of Eigenvalues for Any General Square Matrix

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The Problem

How can we find the eigenvalues of any $n \times n$ matrix A?

$$A\mathbf{x} = \lambda\mathbf{x}$$

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Power Method

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3. Iterate the above process for m steps. Now, we have have calculated x_m, which is parallel to $A^m x_0$.
4. Calculate the dominant eigenvalue $\lambda_1 = \dfrac{\|AA^m x_0\|}{\|A^m x_0\|} = \dfrac{\|Ax_m\|}{\|x_m\|} = \dfrac{Ax_m \cdot x_m}{x_m \cdot x_m}$.

Reference: [Koc25].

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- Now, if $|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n|$, then $\lambda_i/\lambda_1 < 1$ for all $i \neq 1$.

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&= \lim_{k\to\infty} \lambda_1^k \left(\cancel{\left(\frac{\lambda_1}{\lambda_1}\right)^k} v_{0_1} p_1 + \cancel{\left(\frac{\lambda_2}{\lambda_1}\right)^k} v_{0_2} p_2 + \cancel{\sum_{i=3}^{n} \left(\frac{\lambda_i}{\lambda_1}\right)^k v_{0_i} p_i} \right) \\
&= \lim_{k\to\infty} \lambda_1^k \left(v_{0_1} p_1 + v_{0_2} p_2 \right)
\end{aligned}
$$

is not an eigenvector of A if $\lambda_1 \neq \lambda_2$.

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- Cannot calculate complex eigenvalues.
- Can use shifts (calculate eigenvalues of $A - cI$), but that is complicated.
- Only calculates dominant eigenvalue.
- Speed of convergence depends on choice of starting vector.

Section 2

Basic QR Algorithm

Basic QR Algorithm

Schur form

Let A be an $n \times n$ real matrix. The **Schur form** of A is:

$$A = QUQ^T = QUQ^{-1}$$

where Q is orthogonal and U is upper triangular. A and U are similar so they share the same eigenvalues, which are the diagonal entries of U.

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Reference: [Arb16, p. 64].

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Proof is hard and complicated. We will "prove" by example.

Example of the Basic QR Algorithm (1)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 1 \end{bmatrix}$$

Eigenvalues

$\lambda_1 \approx 12.4542$

$\lambda_2 \approx -5.0744$

$\lambda_3 \approx -0.379762$

Example of the Basic QR Algorithm (2)

$$A_1 = \begin{bmatrix} 1.00 & 2.00 & 3.00 \\ 4.00 & 5.00 & 6.00 \\ 7.00 & 8.00 & 1.00 \end{bmatrix}$$

Example of the Basic QR Algorithm (2)

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$$Q_1 = \begin{bmatrix} -0.12 & 0.90 & 0.41 \\ -0.49 & 0.30 & -0.82 \\ -0.86 & -0.30 & 0.41 \end{bmatrix}$$

$$R_1 = \begin{bmatrix} -8.12 & -9.60 & -4.19 \\ 0.00 & 0.90 & 4.22 \\ 0.00 & 0.00 & -3.27 \end{bmatrix}$$

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$$A_2 = \begin{bmatrix} 9.33 & -8.98 & 2.81 \\ -4.08 & -1.00 & 0.98 \\ 2.81 & 0.98 & -1.33 \end{bmatrix}$$

Example of the Basic QR Algorithm (3)

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$$Q_2 = \begin{bmatrix} -0.88 & -0.47 & 0.02 \\ 0.39 & -0.70 & 0.60 \\ -0.27 & 0.54 & 0.80 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} -10.57 & 7.28 & -1.75 \\ 0.00 & 5.44 & -2.73 \\ 0.00 & 0.00 & -0.42 \end{bmatrix}$$

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$$A_3 = \begin{bmatrix} 12.61 & -1.09 & 2.74 \\ 2.83 & -5.28 & 1.08 \\ 0.11 & -0.22 & -0.33 \end{bmatrix}$$

Example of the Basic QR Algorithm (4)

$$A_4 = \begin{bmatrix} 12.15 & -4.87 & -3.01 \\ -1.07 & -4.77 & -0.65 \\ 0.00 & 0.02 & -0.38 \end{bmatrix}$$

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$$A_5 = \begin{bmatrix} 12.54 & -3.34 & 2.96 \\ 0.46 & -5.16 & 0.93 \\ 0.00 & -0.00 & -0.38 \end{bmatrix}$$

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$$A_{11} = \begin{bmatrix} 12.45 & -3.79 & 2.98 \\ 0.00 & -5.07 & 0.85 \\ 0.00 & -0.00 & -0.38 \end{bmatrix}$$

Example of the Basic QR Algorithm (5)

$$A_{11} = \begin{bmatrix} \boxed{12.45} & -3.79 & 2.98 \\ 0.00 & \boxed{-5.07} & 0.85 \\ 0.00 & -0.00 & \boxed{-0.38} \end{bmatrix}$$

Eigenvalues

$\lambda_1 \approx 12.4542$

$\lambda_2 \approx -5.0744$

$\lambda_3 \approx -0.379762$

We found <u>EVERY</u> eigenvalue!

Is it good enough?

- Similar convergence problems to Power Method: only guaranteed to work if $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$

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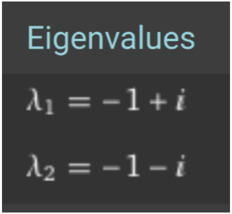
- Similar convergence problems to Power Method: only guaranteed to work if $|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$
- Needs modifications to find complex eigenvalues.
- Could be faster: runs in $\mathcal{O}(n^3)$ time for $n \times n$ matrix.

Bad Example (1)

$$A = \begin{bmatrix} 1 & 5 \\ -1 & -3 \end{bmatrix}$$



Bad Example (1)

$$A = \begin{bmatrix} 1 & 5 \\ -1 & -3 \end{bmatrix}$$

Eigenvalues

$$\lambda_1 = -1 + i$$

$$\lambda_2 = -1 - i$$

$$|\lambda_1| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$
$$|\lambda_2| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$$

Bad Example (2)

$$A_1 = \begin{bmatrix} 1.00 & 5.00 \\ -1.00 & -3.00 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -3.00 & -5.00 \\ 1.00 & 1.00 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -1.40 & 5.80 \\ -0.20 & -0.60 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} -0.60 & -5.80 \\ 0.20 & -1.40 \end{bmatrix}$$

Bad Example (2)

$$A_1 = \begin{bmatrix} 1.00 & 5.00 \\ -1.00 & -3.00 \end{bmatrix} \qquad A_5 = \begin{bmatrix} 1.00 & 5.00 \\ -1.00 & -3.00 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -3.00 & -5.00 \\ 1.00 & 1.00 \end{bmatrix} \qquad A_6 = \begin{bmatrix} -3.00 & -5.00 \\ 1.00 & 1.00 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} -1.40 & 5.80 \\ -0.20 & -0.60 \end{bmatrix} \qquad A_7 = \begin{bmatrix} -1.40 & 5.80 \\ -0.20 & -0.60 \end{bmatrix}$$

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$\lim_{k\to\infty} A_k$ will not converge to a triangular matrix.

Another Bad Example

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

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$$\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 1, \lambda_4 = -1$$

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$$A_1 = A = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{Q_1} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{R_1} = A_1 I$$

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This will never converge to a triangular matrix, since $A_i = A$ for all i.

Section 3

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Hessenberg matrix

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A Hessenberg matrix is a matrix with nonzero entries only in the diagonal just above or just below its main diagonal. A **lower** Hessenberg matrix H satisfies $H_{ij} = 0$ for all $j > i + 1$, and an **upper** Hessenberg matrix H satisfies $H_{ij} = 0$ for all $i > j + 1$.

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$$H_l = \begin{bmatrix} \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & 0 \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix} \qquad H_u = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \\ 0 & 0 & 0 & \times & \times \end{bmatrix}$$

Improved QR Algorithm

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LAPACK Algorithm

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2. Calculate the Schur form of H using the Francis double-step algorithm.

Reference: [Bla99].

Hessenberg Example

$$A = \begin{bmatrix} 5 & 4 & 2 & 0 & 59 \\ 12 & 5 & 3 & 12 & 6 \\ 96 & 4 & 696 & 12 & 3 \\ 23 & 4 & 1 & 2 & 2 \\ 66 & 7 & 8 & 22 & 1 \end{bmatrix}$$

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$$A = \begin{bmatrix} 5 & 4 & 2 & 0 & 59 \\ 12 & 5 & 3 & 12 & 6 \\ 96 & 4 & 696 & 12 & 3 \\ 23 & 4 & 1 & 2 & 2 \\ 66 & 7 & 8 & 22 & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 5 & -34.64 & 43.54 & -4.41 & 19.65 \\ -119.35 & 461.78 & 325.31 & 0.33 & 22.18 \\ 0 & 324.49 & 245.37 & 8.50 & -9.67 \\ 0 & 0 & 10.68 & 2.38 & -6.14 \\ 0 & 0 & 0 & -1.57 & -5.52 \end{bmatrix}$$

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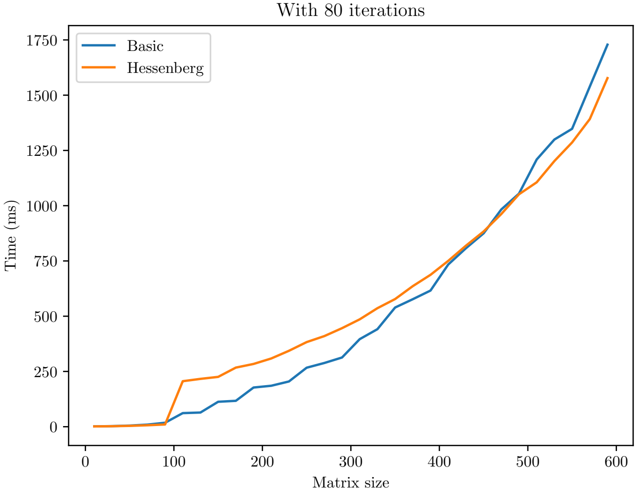
- Hessenberg reduction reduces each QR step to $\mathcal{O}(n^2)$.
- Very easy to recover complex eigenvalues.

Basic vs Improved QR Algorithm



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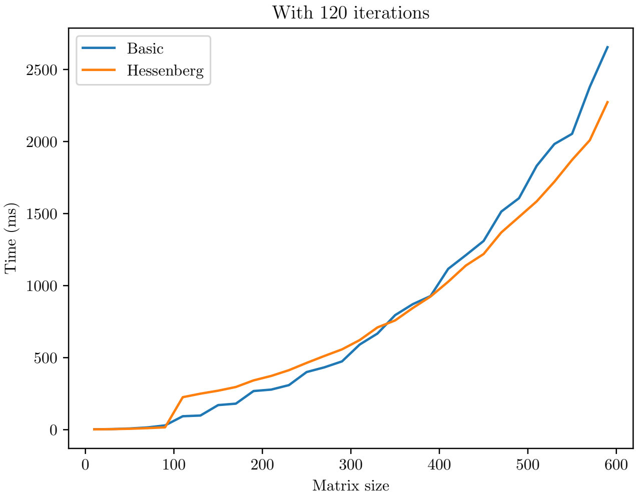
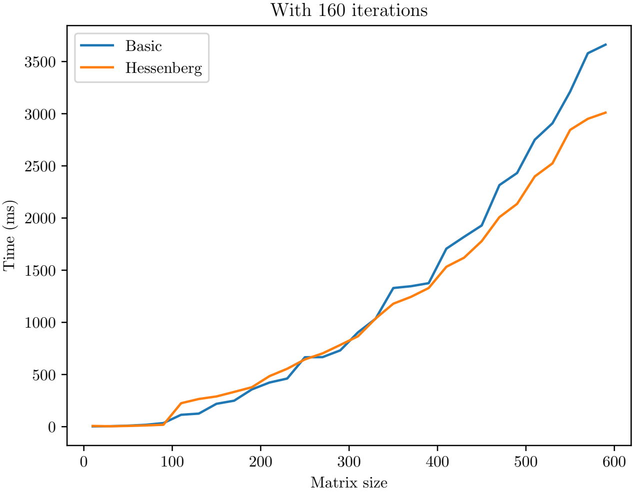


Basic vs Improved QR Algorithm



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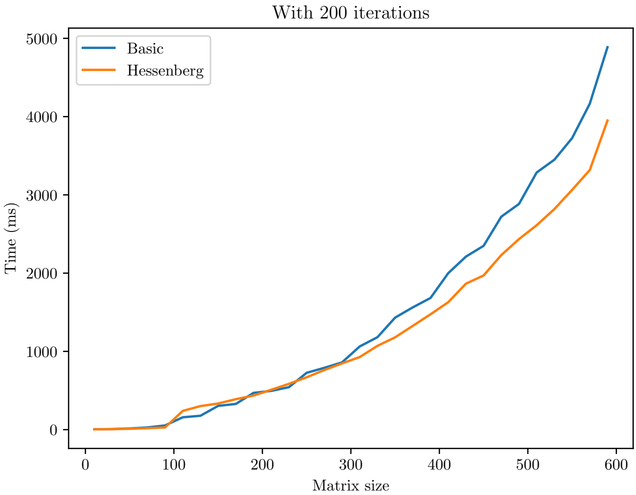

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