Problem Set #33, Part 2

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December 8, 2023

Problem 2

(b) *Proof.*

$$\lim_{h\to 0}\frac{\cos(h)-1}{h}$$
$$=\lim_{h\to 0}\frac{\cos\left(2\cdot\frac{h}{h}\right)-1}{2}$$
$$=\lim_{h\to 0}\frac{2\cos^2 h-1-1}{h}$$
$$=2\lim_{h\to 0}\frac{\cos^2 h-1}{h}$$
$$=2\lim_{h\to 0}\frac{-\sin^2 h}{h}$$
$$=2\lim_{h\to 0}\left(\frac{-\sin h}{h}\cdot\sin h\right)$$
$$=-2\sin(0)\cdot\lim_{h\to 0}\frac{\sin h}{h}$$
$$=-2\cdot 0\cdot 1$$
$$=\boxed{0}$$

□

Problem 3

(a) $\lim_{h\to\infty}\frac{\sin h}{h}=0$

$\lim_{h\to-\infty}\frac{\sin h}{h}=0$

In both limits, $\sin h$ stays between -1 and 1. As $h\to\infty$, the quotient of a number between -1 and 1 approaches 0. Even though $\sin h$ is alternating, the quotient still approaches 0 as the denonimator grows.

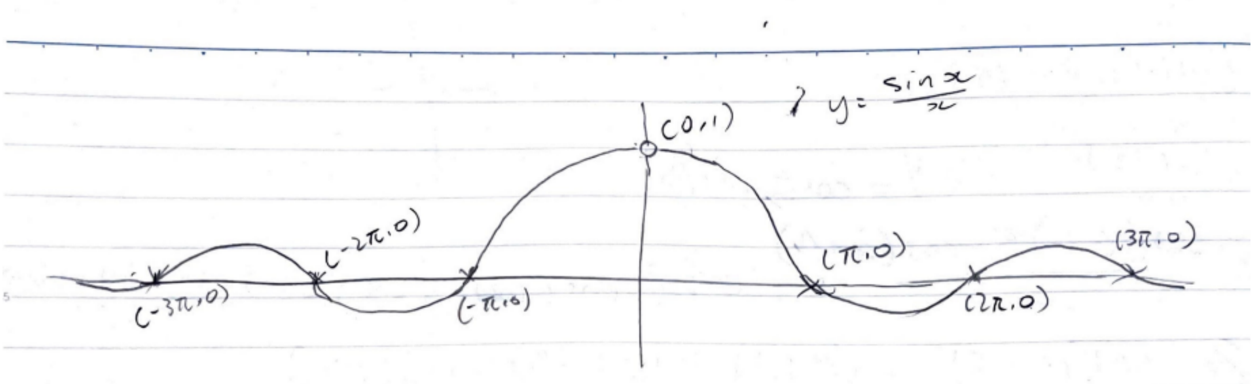
(b) $f(x)=\frac{\sin x}{x}$

$\frac{\sin x}{x}=0$

$\sin x=0,\ x\neq 0$

$\boxed{x=\pi n,\ x\neq 0,\ n\in\mathbb{Z}}$

$x=\ldots,\ -2\pi,\ -\pi,\ \pi,\ 2\pi,\ \ldots$



Problem Set #34

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Problem 5

(a) True. The domain of $\arcsin y$ is $-1 \leq y \leq 1$, and by the definition of the $\arcsin y$ as the inverse of $\sin y$, $\sin(\arcsin y)$ must equal y inside the domain of $\arcsin y$.

(b) False. $\arcsin(\sin 5\pi) = \arcsin 0 = 0$, $0 \neq 5\pi$.

(c) False. $\arccos(\cos 5\pi) = \arccos(-1) = \pi$, $\pi \neq 5\pi$.

(d) True. $\cos(\arccos y) = y$ within the domain of $\arccos y$, and $-1 \leq y \leq 1$ is the domain of $\arccos y$.

(e) True.

Proof.

$$
\begin{aligned}
\text{Let } y &= \arcsin x \\
\sin y &= \sin(\arcsin x), \ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, \ \text{as} -\frac{\pi}{2} \leq \arcsin x \leq \frac{\pi}{2} \\
\sin y &= x, \ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \\
-\sin y &= -x, \ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \\
\sin(-y) &= -x, \ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \\
\arcsin(\sin(-y)) &= \arcsin(-x), \ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \\
-y &= \arcsin(-x) \\
\arcsin x &= -\arcsin(-x)
\end{aligned}
$$

$\square$

(f) False. $\arccos(-1) = \pi$, $\arccos(1) = 0$, $\pi \neq 0$.

Problem 8

(e)

$$y = \arccos\left(\frac{x-3}{2x-1}\right)$$

$$\begin{aligned}
\frac{x-3}{2x-1} &\geq -1 \\
\frac{x-3}{2x-1} + \frac{2x-1}{2x-1} &\geq 0 \\
\frac{3x-4}{2x-1} &\geq 0
\end{aligned}$$

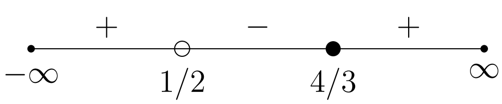
$$\begin{aligned}
2x-1 &= 0 \\
x &= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
3x-4 &= 0 \\
x &= \frac{4}{3}
\end{aligned}$$

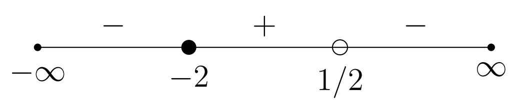
$$\begin{aligned}
\frac{x-3}{2x-1} &\leq 1 \\
\frac{x-3}{2x-1} - \frac{2x-1}{2x-1} &\leq 0 \\
\frac{-x-2}{2x-1} &\leq 0
\end{aligned}$$

$$\begin{aligned}
2x-1 &= 0 \\
x &= \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
-x-2 &= 0 \\
x &= -2
\end{aligned}$$



$$x \in \left(-\infty, \frac{1}{2}\right) \cup \left[\frac{4}{3}, \infty\right)$$



$$x \in (-\infty, -2] \cup \left(\frac{1}{2}, \infty\right)$$

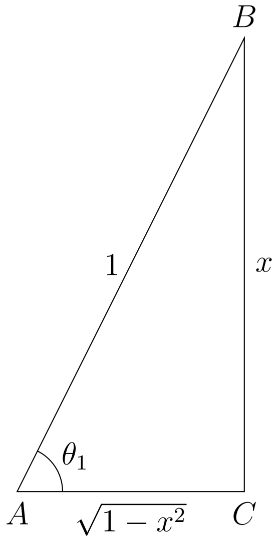
$$x \in \left(\left(-\infty, \frac{1}{2}\right) \cup \left[\frac{4}{3}, \infty\right)\right) \cap \left((-\infty, -2] \cup \left(\frac{1}{2}, \infty\right)\right)$$

$$\boxed{x \in (-\infty, -2] \cup \left[\frac{4}{3}, \infty\right)}$$

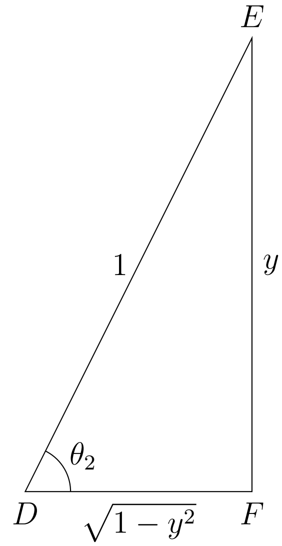
Problem 9

Theorem. $\arcsin x - \arcsin y = \arcsin\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$

Proof.



$$\begin{aligned}
\text{Let } \theta_1 &= \arcsin x \\
\sin\theta_1 &= \sin(\arcsin x) \\
\frac{\text{opp}}{\text{hyp}} &= x \\
\frac{BC}{AB} &= x \\
\text{Let } BC = x,\ AB &= 1 \\
AC &= \sqrt{1-x^2} \\
\\
\sin\theta_1 &= x \\
\cos\theta_1 &= \sqrt{1-x^2}
\end{aligned}$$



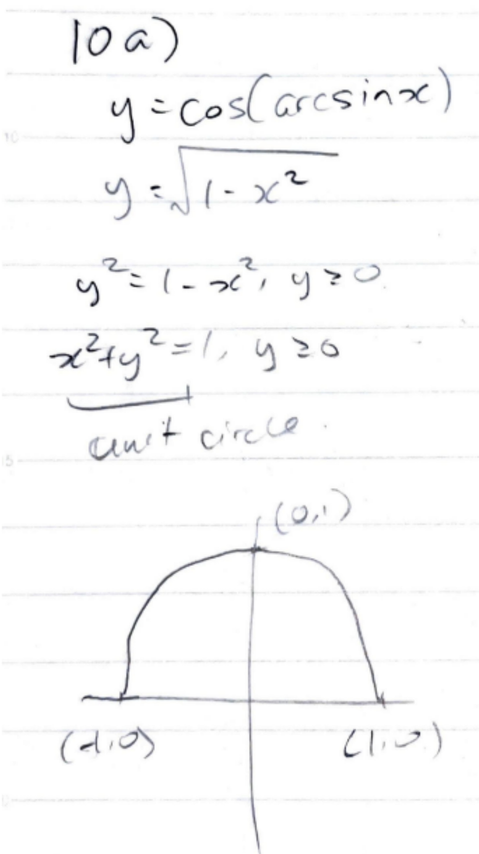
$$\begin{aligned}
\text{Let } \theta_2 &= \arcsin y \\
\sin\theta_2 &= \sin(\arcsin y) \\
\frac{\text{opp}}{\text{hyp}} &= y \\
\frac{EF}{DE} &= y \\
\text{Let } EF = y,\ DE &= 1 \\
DF &= \sqrt{1-y^2} \\
\\
\sin\theta_2 &= y \\
\cos\theta_2 &= \sqrt{1-y^2}
\end{aligned}$$

$$\begin{aligned}
\text{RHS: } &\arcsin\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right) \\
&= \arcsin(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1) \\
&= \arcsin(\sin(\theta_1 - \theta_2)) \\
&= \theta_1 - \theta_2 \\
&= \arcsin x - \arcsin y
\end{aligned}$$

□

Problem 10

(a)



(b) $\arccos(\cos x)$ is periodic with period 2π. This means that the graph on $[-\pi, \pi]$ repeats. Because $\arccos x$ is defined as the invrse of $\cos x$ on $[0, \pi]$, $\arccos(\cos x) = x$.

Let $\theta \in [-\pi, 0]$ and $\varphi = \theta + \pi$, so $\varphi \in [0, \pi]$ and $\theta = \pi - \varphi$.
$\arccos(\cos(\pi - \varphi)) = \arccos(\cos\theta)$. By angle addition identites,
$\arccos(-\cos\varphi) = \arccos(\cos\theta)$. Now, we will prove that $\arccos(-\cos\varphi) = \pi - \varphi$.

Proof.

$$-\cos\varphi = -\cos\varphi$$
$$-\cos\varphi = \cos\pi\cos\varphi + \sin\pi\sin\varphi$$
$$\cos(\arccos(-\cos\varphi)) = \cos(\pi - \varphi)$$
$$(\text{This is fine because } \varphi, \pi - \varphi \in [0, 2\pi])$$
$$\arccos(-\cos\varphi) = \pi - \varphi$$

□

From this identity, $\pi - \varphi = \arccos(\cos\theta)$. Substituting $\varphi = \theta + \pi$ gives $-\theta = \arccos(\cos\theta)$. Now, changing variable θ to x, we see that $y = -x$ on $[-\pi, 0]$.