Problem Set #44

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Problem 1

(a)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2 + f(x)\right] = \boxed{2x + f'(x)}$$

(b)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2 f(x)\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[x^2\right] f(x) + x^2 \frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)\right] = \boxed{2xf(x) + x^2 f'(x)}$$

(c)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[c + x + f(x)\right] = \cancel{\frac{\mathrm{d}}{\mathrm{d}x}c} + \frac{\mathrm{d}}{\mathrm{d}x}x + \frac{\mathrm{d}}{\mathrm{d}x}f(x) = \boxed{1 + f'(x)}$$

(d)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x^2)\right] = f'(x^2) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[x^2\right] = \boxed{2xf'(x^2)}$$

(e)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left[xf(x) + f(cx) + cf(x)\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[xf(x)\right] + \frac{\mathrm{d}}{\mathrm{d}x}\left[f(cx)\right] + \frac{\mathrm{d}}{\mathrm{d}x}\left[cf(x)\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[x\right] f(x) + x\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)\right] + f'(cx)\frac{\mathrm{d}}{\mathrm{d}x}\left[cx\right] + c\frac{\mathrm{d}}{\mathrm{d}x}\left[f(x)\right] \\
&= \boxed{f(x) + xf'(x) + cf'(cx) + cf'(x)}
\end{aligned}$$

Problem 2

(a)

$$\begin{aligned}
x^2 + y^2 &= 1 \\
x^2 + (f(x))^2 &= 1 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2 + (f(x))^2\right] &= \frac{\mathrm{d}}{\mathrm{d}x}1 \\
2x + 2f(x)f'(x) &= 0 \\
f'(x) &= \frac{-2x}{2f(x)} \\
\boxed{f'(x) = -\frac{x}{f(x)}}
\end{aligned}$$

The tangent line to the circle defined by $x^2 + y^2 = 1$ at the point (x, y) has slope $-\frac{x}{y}$.

(b) Let $y = f(x)$. The equation $x^2 + y^2 = 1$ is equal to $x^2 + (f(x))^2 = 1$. We have shown in (a) that $f'(x) = -\dfrac{x}{f(x)}$, and substituting $f(x)$ for y gives $\boxed{\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{x}{y}}$.

The derivative at $\left(\dfrac{\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$ is $-\dfrac{\sqrt{2}/2}{\sqrt{2}/2} = \boxed{1}$

(c)

$$\boxed{y - \frac{\sqrt{2}}{2} = -\left(x - \frac{\sqrt{2}}{2}\right)}$$

(d) The center of the circle is $O(0, 0)$. The slope of the radius OP is" $\dfrac{y-0}{x-0} = \dfrac{y}{x}$. The slope of any line perpendicular to OP is $-\dfrac{x}{y}$. The tangent line of the circle at (x, y) is $-\dfrac{x}{y}$. $\square$

Problem 3

(a) Because y is not a function as it fails the vertical line test.

(b)

$$\begin{aligned}
x &= y^5 - 5y^3 + 4y \\
\frac{\mathrm{d}}{\mathrm{d}x}x &= \frac{\mathrm{d}}{\mathrm{d}x}\left[y^5 - 5y^3 + 4y\right] \\
1 &= 5y^4\frac{\mathrm{d}y}{\mathrm{d}x} - 15y^2\frac{\mathrm{d}y}{\mathrm{d}x} + 4\frac{\mathrm{d}y}{\mathrm{d}x} \\
1 &= \frac{\mathrm{d}y}{\mathrm{d}x}\left(5y^4 - 15y^2 + 4\right) \\
&\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{5y^4 - 15y^2 + 4}}
\end{aligned}$$

(c)

$$\begin{aligned}
y - y_0 &= \frac{1}{5y_0^4 - 15y_0^2 + 4}(x - x_0) \\
y - 1 &= \frac{1}{5(1)^4 - 15(1)^2 + 4}(x - 0) \\
&\boxed{y - 1 = -\frac{1}{6}x}
\end{aligned}$$

Problem 4

(a)

$$
\begin{aligned}
x^2y^2 + x\sin y &= 4 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2y^2 + x\sin y\right] &= \frac{\mathrm{d}}{\mathrm{d}x}4 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2\right]y^2 + x^2\frac{\mathrm{d}}{\mathrm{d}x}\left[y^2\right] + \frac{\mathrm{d}}{\mathrm{d}x}[x]\sin y + x\frac{\mathrm{d}}{\mathrm{d}x}[\sin y] &= 0 \\
2xy^2 + 2x^2y\frac{\mathrm{d}y}{\mathrm{d}x} + \sin y + x\cos(y)\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\frac{\mathrm{d}y}{\mathrm{d}x}\left(2x^2y + x\cos y\right) &= -2xy^2 - \sin y \\
\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2xy^2 + \sin y}{2x^2y + x\cos y}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
4\cos x\sin y &= 1 \\
\frac{\mathrm{d}}{\mathrm{d}x}[4\cos x\sin y] &= \frac{\mathrm{d}}{\mathrm{d}x}1 \\
4\left(\frac{\mathrm{d}}{\mathrm{d}x}[\cos(x)]\sin(y) + \cos(x)\frac{\mathrm{d}}{\mathrm{d}x}[\sin(y)]\right) &= 0 \\
-4\sin x\sin y + 4\cos x\cos y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
4\cos x\cos y\frac{\mathrm{d}y}{\mathrm{d}x} &= 4\sin x\sin y \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4\sin x\sin y}{4\cos x\cos y} \\
\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \tan x\tan y}
\end{aligned}
$$

(c)

$$
\begin{aligned}
x\ln y + y^3 &= 3\ln x \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[x\ln y + y^3\right] &= \frac{\mathrm{d}}{\mathrm{d}x}[3\ln x] \\
\ln y + x\left(\frac{1}{y}\right)\frac{\mathrm{d}y}{\mathrm{d}x} + 3y^2\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{3}{x} \\
\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{x}{y} + 3y^2\right) &= \frac{3}{x} - \ln y \\
\frac{\mathrm{d}y}{\mathrm{d}x}\left(x^2 + 3xy^3\right) &= 3y - xy\ln y \\
\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3y - xy\ln y}{x^2 + 3xy^3}}
\end{aligned}
$$

(d)

$$\begin{aligned}
\tan(x-y) &= \frac{y}{1+x^2} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[\tan(x-y)\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{y}{1+x^2}\right] \\
\sec^2(x-y)\frac{\mathrm{d}}{\mathrm{d}x}\left[x-y\right] &= \frac{\frac{\mathrm{d}}{\mathrm{d}x}[y]\left(1+x^2\right) - y\frac{\mathrm{d}}{\mathrm{d}x}\left[1+x^2\right]}{\left(1+x^2\right)^2} \\
\sec^2(x-y)\left(1-\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= \frac{\frac{\mathrm{d}y}{\mathrm{d}x}\left(1+x^2\right) - 2xy}{1+2x^2+x^4} \\
\left(\sec^2(x-y) - \sec^2(x-y)\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(1+2x^2+x^4\right) &= \frac{\mathrm{d}y}{\mathrm{d}x}\left(1+x^2\right) - 2xy \\
\sec^2(x-y)\left(1+2x^2+x^4\right) - \frac{\mathrm{d}y}{\mathrm{d}x}\sec^2(x-y)\left(1+2x^2+x^4\right) &= \frac{\mathrm{d}y}{\mathrm{d}x}\left(1+x^2\right) - 2xy \\
\sec^2(x-y)\left(1+2x^2+x^4\right) + 2xy &= \frac{\mathrm{d}y}{\mathrm{d}x}\left(1+x^2\right) + \frac{\mathrm{d}y}{\mathrm{d}x}\sec^2(x-y)\left(1+2x^2+x^4\right) \\
\sec^2(x-y)\left(1+2x^2+x^4\right) + 2xy &= \frac{\mathrm{d}y}{\mathrm{d}x}\left(1+x^2+\sec^2(x-y)\left(1+2x^2+x^4\right)\right) \\
&\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sec^2(x-y)\left(1+2x^2+x^4\right)+2xy}{1+x^2+\sec^2(x-y)\left(1+2x^2+x^4\right)}}
\end{aligned}$$

Problem 5

$$\begin{aligned}
\sqrt{x}+\sqrt{y} &= \sqrt{c} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{x}+\sqrt{y}\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\sqrt{c} \\
\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{y}}} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{\sqrt{y}}{\sqrt{x}}
\end{aligned}$$

The tangent line at any point (x_0, y_0) on the curve is $y - y_0 = -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x-x_0)$.

$$
\begin{aligned}
x\text{-intercepts: } 0 - y_0 &= -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0) \\
-y_0 &= -\frac{x\sqrt{y_0}}{\sqrt{x_0}} + \frac{x_0\sqrt{y_0}}{\sqrt{x_0}} \\
\frac{x\sqrt{y_0}}{\sqrt{x_0}} &= y_0 + \sqrt{\frac{x_0^{\cancel{2}} y_0}{\cancel{x_0}}} \\
x &= \frac{\sqrt{x_0}}{\sqrt{y_0}}\left(y_0 + \sqrt{x_0 y_0}\right) \\
x &= \frac{y_0\sqrt{x_0}}{\sqrt{y_0}} + \frac{\sqrt{x_0 y_0}\sqrt{x_0}}{\sqrt{y_0}} \\
x &= \sqrt{\frac{x_0 y_0^2}{y_0}} + \frac{x_0\cancel{\sqrt{y_0}}}{\cancel{\sqrt{y_0}}} \\
x &= \sqrt{x_0 y_0} + x_0
\end{aligned}
$$

$$
\begin{aligned}
y\text{-intercepts: } y - y_0 &= -\frac{\sqrt{y_0}}{\sqrt{x_0}}(0 - x_0) \\
y - y_0 &= \sqrt{\frac{y_0}{x_0}}\sqrt{x_0^2} \\
y &= \sqrt{x_0 y_0} + y_0
\end{aligned}
$$

$$
\begin{aligned}
\text{Sum of intercepts: } &\sqrt{x_0 y_0} + x_0 + \sqrt{x_0 y_0} + y_0 \\
&= \left(\sqrt{x_0}\right)^2 + 2\sqrt{x_0 y_0} + \left(\sqrt{y_0}\right)^2 \\
&= \left(\sqrt{x_0} + \sqrt{y_0}\right)^2 \\
&= \left(\sqrt{c}\right)^2 \\
&= c
\end{aligned}
$$

$\square$

Problem 6

$$
\begin{aligned}
x^2y^2 + xy &= 2 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[x^2y^2 + xy\right] &= \frac{\mathrm{d}}{\mathrm{d}x}2 \\
2xy^2 + 2x^2y\frac{\mathrm{d}y}{\mathrm{d}x} + y + x\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\frac{\mathrm{d}y}{\mathrm{d}x}\left(2x^2y + x\right) &= -\left(2xy^2 + y\right) \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{2xy^2 + y}{2x^2y + x}
\end{aligned}
$$

$$
\begin{aligned}
-\frac{2xy^2 + y}{2x^2y + x} &= -1 \\
2xy^2 + y &= 2x^2y + x \\
2xy^2 - 2x^2y &= x - y \\
2xy(y - x) + (y - x) &= 0 \\
(2xy + 1)(y - x) &= 0
\end{aligned}
$$

$$x^2y^2 + xy = 2$$

$$
\begin{aligned}
2xy + 1 &= 0 \\
y &= -\frac{1}{2x} \\
x^2\left(-\frac{1}{2x}\right)^2 + x\left(-\frac{1}{2x}\right) &= 2 \\
\frac{x^2}{4x^2} - \frac{x}{2x} &= 2 \\
\frac{1}{4} - \frac{2}{4} &= 2
\end{aligned}
$$

(This case is impossible.)

$$
\begin{aligned}
y - x &= 0 \\
y &= x \\
x^2x^2 + xx &= 2 \\
x^4 + x^2 - 2 &= 0 \\
(x^2 + 2)(x^2 - 1) &= 0
\end{aligned}
$$

$$x = 1, y = 1 \text{ or } x = -1, y = -1$$

$$\boxed{(1, 1), (-1, -1)}$$

Problem 7

If the point $(-5,0)$ is on the edge of the shadow, then the line drawn from $(-5,0)$ to the lamp must be a tangent line of the ellipse $x^2+4y^2=5$ and must intercept it at one point only. Let k be the y-coordinate of the lamp. Then the coordinates of the lamp is $(3,k)$.

$$\begin{aligned}
y-0 &= \frac{k-0}{3-(-5)}(x-(-5)) \\
y &= \frac{k}{8}(x+5) \\
y &= \frac{kx+5k}{8}
\end{aligned}$$

$$\begin{aligned}
x^2+4y^2 &= 5 \\
x^2+4\left(\frac{kx+5k}{8}\right)^2 &= 5 \\
x^2+4\left(\frac{k^2x^2+10k^2x+25k^2}{64}\right) &= 5 \\
x^2+\frac{k^2x^2}{16}+\frac{5k^2x}{8}+\frac{25k^2}{16} &= 5 \\
\left(1+\frac{k^2}{16}\right)x^2+\frac{5k^2x}{8}+\left(\frac{25k^2}{16}-5\right) &= 0 \\
\frac{-\frac{5k^2}{8}\pm\sqrt{\left(\frac{5k^2}{8}\right)^2-4\left(1+\frac{k^2}{16}\right)\left(\frac{25k^2}{16}-5\right)}}{2\left(1+\frac{k^2}{16}\right)} &= x
\end{aligned}$$

Because there is only one intercept, there must only be one solution for x, and the discriminant must be 0.

$$\begin{aligned}
\left(\frac{5k^2}{8}\right)^2-4\left(1+\frac{k^2}{16}\right)\left(\frac{25k^2}{16}-5\right) &= 0 \\
\frac{25k^4}{64}-4\left(\frac{25k^2}{16}-5+\frac{25k^4}{256}-\frac{5k^2}{16}\right) &= 0 \\
\cancel{\frac{25k^4}{64}}-\frac{25k^2}{4}+20-\cancel{\frac{25k^4}{64}}+\frac{5k^2}{4} &= 0 \\
-25k^2+80+5k^2 &= 0 \\
20k^2 &= 80 \\
k &= \pm 2
\end{aligned}$$

The lamp cannot be underneath the ground, so the only solution for k is 2. Hence the lamp is located $\boxed{2}$ units above the x-axis.

Problem 8

$$
\begin{aligned}
x^m y^n &= (x+y)^{m+n} \\
\ln\left(x^m y^n\right) &= \ln\left((x+y)^{m+n}\right) \\
m\ln x + n\ln y &= (m+n)\ln(x+y) \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[m\ln x + n\ln y\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[(m+n)\ln(x+y)\right] \\
\frac{m}{x} + \frac{n}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= (m+n)\left(\frac{1}{x+y}\right)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right) \\
\frac{m}{x} + \frac{n}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) &= \frac{m+n}{x+y} + \frac{m+n}{x+y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \\
\frac{m}{x} - \frac{m+n}{x+y} &= \left(\frac{m+n}{x+y} - \frac{n}{y}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \\
\frac{m(x+y) - x(m+n)}{x(x+y)} &= \left(\frac{y(m+n) - n(x+y)}{y(x+y)}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \\
\frac{\cancel{mx} + my - \cancel{mx} - nx}{x\cancel{(x+y)}} &= \left(\frac{my + \cancel{ny} - nx - \cancel{ny}}{y\cancel{(x+y)}}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \\
\frac{\cancel{my - nx}}{x} &= \left(\frac{\cancel{my - nx}}{y}\right)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \\
\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x}}
\end{aligned}
$$

□

Problem 9

(a)

$$
\begin{aligned}
(fg)'' &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}}{\mathrm{d}x}(fg)\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[f'g + fg'\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[f'g\right] + \frac{\mathrm{d}}{\mathrm{d}x}\left[fg'\right] \\
&= f''g + f'g' + f'g' + fg'' \\
\boxed{(fg)''} & \boxed{= f''g + 2f'g' + fg''}
\end{aligned}
$$

□

(b) *Proof.* By induction.
Base case. $n = 1$

$$(fg)' = f'g + fg'$$

$$\sum_{k=0}^{1}\binom{1}{k} f^{(k)} g^{(n-k)} = \binom{1}{0} f^{(0)} g^{(1-0)} + \binom{1}{1} f^{(1)} g^{(1-0)} = fg' + f'g$$

Induction Hypothesis. Suppose that:

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

We will show that:

$$(fg)^{(n+1)} = \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)} g^{(n-k+1)}$$

Inductive Step.

$$\begin{aligned}
(fg)^{(n+1)} &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}\right] \\
&= \sum_{k=0}^{n} \binom{n}{k} \frac{\mathrm{d}}{\mathrm{d}x}\left[f^{(k)} g^{(n-k)}\right] \\
&= \sum_{k=0}^{n} \binom{n}{k} \left(f^{(k+1)} g^{(n-k)} + f^{(k)} g^{(n-k+1)}\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} f^{(k+1)} g^{(n-k)} + \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k+1)} \\
&= \sum_{k=1}^{n+1} \binom{n}{k-1} f^{(k)} g^{(n-k+1)} + \sum_{k=1}^{n+1} \binom{n}{k} f^{(k)} g^{(n-k+1)} + \binom{n}{0} f^{(0)} g^{(n+1)} - \binom{n}{n+1} f^{(n+1)} g^{(n-(n+1)+1)} \\
&= \sum_{k=1}^{n+1} \left(\binom{n}{k-1} + \binom{n}{k}\right) f^{(k)} g^{(n-k+1)} + f^{(0)} g^{(n+1)} \\
&= \sum_{k=1}^{n+1} \left(\frac{n!}{(n-(k-1))!(k-1)!} + \frac{n!}{k!(n-k)!}\right) f^{(k)} g^{(n-k+1)} + f^{(0)} g^{(n+1)} \\
&= \sum_{k=1}^{n+1} \left(\frac{n!}{(n-k+1)(n-k)!(k-1)!} + \frac{n!}{k(k-1)!(n-k)!}\right) f^{(k)} g^{(n-k+1)} + f^{(0)} g^{(n+1)} \\
&= \sum_{k=1}^{n+1} \left(\frac{(k)(n!) + (n-k+1)(n!)}{k(n-k+1)(n-k)!(k-1)!}\right) f^{(k)} g^{(n-k+1)} + f^{(0)} g^{(n+1)} \\
&= \sum_{k=1}^{n+1} \left(\frac{(n-\cancel{k}+1+\cancel{k})(n!)}{((n-k+1)(n-k)!)(k(k-1)!)}\right) f^{(k)} g^{(n-k+1)} + \binom{n+1}{0} f^{(0)} g^{(n-0+1)} \\
&= \sum_{k=1}^{n+1} \left(\frac{(n+1)!}{((n+1)-k)!(k!)}\right) f^{(k)} g^{(n-k+1)} + \binom{n+1}{0} f^{(0)} g^{(n-0+1)} \\
&= \sum_{k=1}^{n+1} \binom{n+1}{k} f^{(k)} g^{(n-k+1)} + \binom{n+1}{0} f^{(0)} g^{(n-0+1)} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(k)} g^{(n-k+1)}
\end{aligned}$$

□