Problem Set #58

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Problem 4

It is known from Problem 3 that $f'(x) = 2x - 8$. The slope at $(3, -6)$ is $f'(3) = 6 - 8 = -2$. The tangent line is $\boxed{y + 6 = -2(x - 3)}$.

Problem 5

Proof. If f is continuous, then $\lim_{x \to a} f(x) = f(a) \iff \lim_{x \to a} f(x) - f(a) = 0 \iff \lim_{x \to a} [f(x) - f(a)] = 0$.

$$\lim_{x \to a} [f(x) - f(a)] = \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \cdot (x - a) \right] = \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \right] \cdot \lim_{x \to a} [x - a] = f'(x) \cdot 0 = 0$$

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Problem 7

$$\begin{aligned}
\text{velocity} &= \lim_{h \to 0} \frac{40(2+h) - 16(2+h)^2 - (40(2) - 16(2)^2)}{h} \\
&= \lim_{h \to 0} \frac{80 + 40h - 16\left(4 + 4h + h^2\right) - 80 + 64}{h} \\
&= \lim_{h \to 0} \frac{80 + 40h - 64 - 64h - 16h^2 - 80 + 64}{h} \\
&= \lim_{h \to 0} \frac{-24h - 16h^2}{h} \\
&= \lim_{h \to 0} [-24 - 16h] \\
&= \boxed{-24}
\end{aligned}$$

Problem 8

$$\begin{aligned}
\frac{\mathrm{d}s}{\mathrm{d}t} &= \lim_{h \to 0} \frac{\frac{1}{(t+h)^2} - \frac{1}{t^2}}{h} \\
&= \lim_{h \to 0} \frac{t^2 - (t+h)^2}{ht^2(t+h)^2}
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{h\to 0} \frac{t^2 - 2th - h^2 - t^2}{ht^2(t+h)^2} \\
&= \lim_{h\to 0} \frac{-2t-h}{t^2(t+h)^2} \\
&= \frac{\lim_{h\to 0}\,[-2t-h]}{\lim_{h\to 0}\left[t^2(t+h)^2\right]} \\
&= \frac{-2t}{t^4} \\
&= -\frac{2}{t^3}
\end{aligned}
$$

evaluated at $t = a$: $-2/t^2$
evaluated at $t = 1$: -2
evaluated at $t = 2$: $-1/4$
evaluated at $t = 3$: $-2/27$

Problem 10

(b)

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{\frac{1}{\sqrt{a+h+2}} - \frac{1}{\sqrt{a+2}}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{a+2} - \sqrt{a+h+2}}{h\sqrt{a+h+2}\sqrt{a+2}} \\
&= \lim_{h\to 0} \frac{a+2-a-h-2}{h\sqrt{a+h+2}\sqrt{a+2}\left(\sqrt{a+2}+\sqrt{a+h+2}\right)} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{a+h+2}\sqrt{a+2}\left(\sqrt{a+2}+\sqrt{a+h+2}\right)} \\
&= \frac{-1}{(a+2)\left(\sqrt{a+2}+\sqrt{a+2}\right)} \\
&= \frac{-1}{2(a+2)\sqrt{a+2}} \cdot \frac{\sqrt{a+2}}{\sqrt{a+2}} \\
&= \boxed{\frac{-\sqrt{a+2}}{2(a+2)^2}}
\end{aligned}
$$

Problem 12

(a) The rate of change of the price of producing x ounces of gold. Units are dollars per ounce per ounce or dollars per square ounce ($/oz^2)

(b) The rate of change at 800 ounces is 17 dollars per square ounces.

(c) It depends.

If there is a large amount of gold, then $f'(x)$ will decrease over the long term as the mine can utilize economies of scale.

If there is not a large amount of gold, then $f'(x)$ will increase over the long term since the cost of producing more gold as the gold runs out is higher.

Problem 13

$T'(10)$ is the rate at which the temperature changes in the neighborhood of 100 Fahrenheit. I estimate $T'(10) = 9$ because the average of $T(10) - T(9)$ and $T(11) - T(10)$ is 9.

Problem 14

(a) The rate of change of solubility in the neighborhood around a certain temperature. Units are milligrams per liter per degree Celsius (mg/L/°C)

(b) Approximately $-1/3$? In the neighborhood around 16°C, the rate of change of solubility is $-1/3\,\text{mg/L/°C}$.

Problem 16

Proof.

$$\begin{aligned} f'(-x) &= \lim_{h\to 0} \frac{f(-x+h) - f(-x)}{h} \\ &= \lim_{h\to 0} \frac{f(-(x-h)) - f(-x)}{h} \\ &= \lim_{h\to 0} \frac{f(x-h) - f(x)}{h} \\ &= -\lim_{h\to 0} \frac{f(x) - f(x-h)}{h} \\ &= -f'(x) \end{aligned}$$

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