Finish Problem Set #40

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Problem 4

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{1}{2^{m+n}} = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{1}{2^m}\frac{1}{2^n}\right)$$

$$= \sum_{n=0}^{\infty}\frac{1}{2^n}\sum_{m=0}^{n}\frac{1}{2^m}$$

$$= \sum_{n=0}^{\infty}\frac{1}{2^n}\left(\frac{1}{2^0}\left(\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}\right)\right)$$

$$= \sum_{n=0}^{\infty}\frac{2}{2^n}\left(1-\frac{1}{2^{n+1}}\right)$$

$$= \sum_{n=0}^{\infty}\left(\frac{2}{2^n}-\frac{2}{2^n 2^{n+1}}\right)$$

$$= 2\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n - 2\sum_{n=0}^{\infty}\frac{1}{2^{2n+1}}$$

$$= 2\left(\frac{1}{1-\frac{1}{2}}\right) - 2\sum_{n=0}^{\infty}\left(\left(\frac{1}{2}\right)^2\right)^{n+\frac{1}{2}}$$

$$= 4 - 2\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^{n+\frac{1}{2}}$$

$$= 4 - 2\left(\frac{\left(\frac{1}{4}\right)^{\frac{1}{2}}}{1-\frac{1}{4}}\right)$$

$$= 4 - 2\left(\frac{1}{2}\cdot\frac{4}{3}\right)$$

$$= \frac{12}{3} - \frac{4}{3}$$

$$= \boxed{\frac{8}{3}}$$

Problem 5

$$4\sum_{k=2}^{\infty}(1+n)^{-k} = 27n$$

$$\implies -1 < 1+n < 1$$
$$\implies -2 < n < 0$$

$$4\sum_{k=2}^{\infty}\left(\frac{1}{1+n}\right)^k = 27n$$

$$4\left(\frac{\left(\frac{1}{1+n}\right)^2}{1-\frac{1}{1+n}}\right) = 27n$$

$$4\left(\frac{\frac{1}{(1+n)^2}}{\frac{(1+n)^2}{(1+n)^2}-\frac{1+n}{(1+n)^2}}\right) = 27n$$

$$4\left(\frac{1}{(1+n)^2-1-n}\right) = 27n$$

$$\frac{4}{1+2n+n^2-1-n} = 27n$$

$$\frac{4}{n^2+n} = 27n$$

$$27n^3 + 27n^2 - 4 = 0$$

Rational roots: $\pm 1, 2, 4, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{1}{27}, \frac{2}{27}, \frac{4}{27}$

$$27\left(\frac{1}{3}\right)^3 + 27\left(\frac{1}{3}\right)^2 - 4 = 0$$
$$27\left(-\frac{2}{3}\right)^3 + 27\left(-\frac{2}{3}\right)^2 - 4 = 0$$

The restriction on n so that the geometric series converges is $-2 < n < 0$. $\frac{1}{3} \geq 0$, so the only solution is $n = -\frac{2}{3}$. □

Progress on Problem Set #41

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Problem 1

$$
\begin{aligned}
\frac{12x+11}{(x-2)(2x+3)} &= \frac{A}{x-2} + \frac{B}{2x+3} \\
\frac{12x+11}{(x-2)(2x+3)} &= \frac{A(2x+3)+B(x-2)}{(x-2)(2x+3)} \\
\frac{12x+11}{(x-2)(2x+3)} &= \frac{2Ax+Bx+3A-2B}{(x-2)(2x+3)} \\
12x+11 &= (2A+B)x + (3A-2B)
\end{aligned}
$$

$$
\begin{cases} 12 = 2A + B \\ 11 = 3A - 2B \end{cases}
$$

$$
\begin{cases} 24 = 4A + 2B & (1) \\ 11 = 3A - 2B & (2) \end{cases}
$$

$$
\begin{aligned}
24 + 11 &= 4A + 2B + 3A - 2B \\
7A &= 35 \\
A &= 5
\end{aligned}
$$

$$
\begin{aligned}
11 &= 3(5) - 2B \\
2B &= 4 \\
B &= 2
\end{aligned}
$$

$$\boxed{A = 5, B = 2}$$

Problem 2

(a) $$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{n}\right)$$

$$= \left(\frac{2}{\cancel{3}}\right)\left(\frac{\cancel{3}}{\cancel{4}}\right)\left(\frac{\cancel{4}}{\cancel{5}}\right)\cdots\left(\frac{\cancel{n-1}}{n}\right)$$

$$= \boxed{\frac{2}{n}}$$

(b) $$\sum_{n=1}^{99}\left(\frac{1}{n} - \frac{1}{n+1}\right)$$

$$= \frac{1}{1} \cancel{- \frac{1}{2} + \frac{1}{2}} \cancel{- \frac{1}{3} + \cdots + \frac{1}{99}} - \frac{1}{100}$$

$$= \frac{100}{100} - \frac{1}{100}$$

$$= \boxed{\frac{99}{100}}$$

(c) $$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{99\cdot 100}$$

$$=\sum_{n=1}^{99}\frac{1}{n(n+1)}$$

$$=\sum_{n=1}^{99}\left(\frac{A}{n}-\frac{B}{n+1}\right)$$

$$A(n+1)-Bn=1$$

There are infinitely many solutions for A and B. To ensure telescoping works, add another condition:

$$\begin{aligned} A&=B\\ An+A-An&=1\\ A,B&=1 \end{aligned}$$

$$=\sum_{n=1}^{99}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots+\frac{1}{99}-\frac{1}{100}$$

$$=\frac{100}{100}-\frac{1}{100}$$

$$=\boxed{\frac{99}{100}}$$