Problem Set #3

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Problem 4

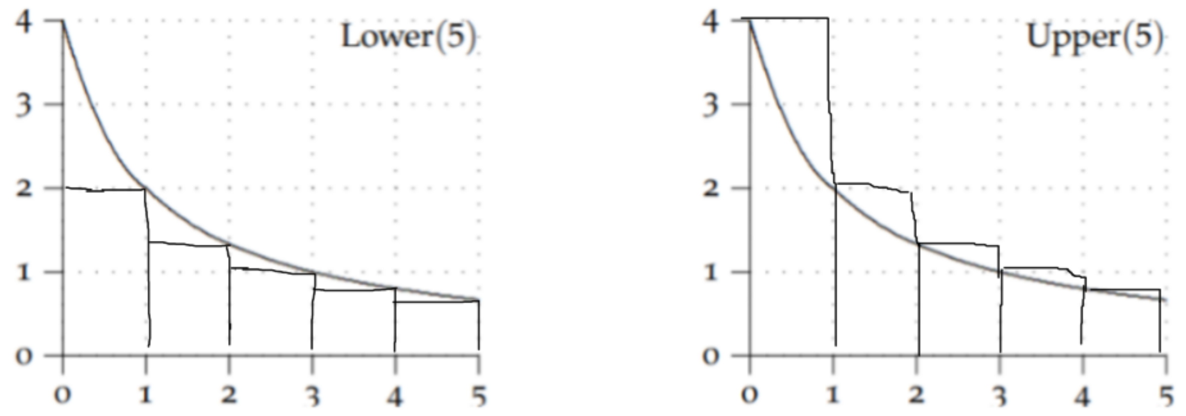

Figure 4.

(a)

$$\text{Lower}(5) \approx 2 + 1.3 + 1 + 0.8 + 0.6 = 5.7$$

(b)

$$\text{Upper}(5) \approx 4 + 2 + 1.3 + 1 + 0.8 = 9.1$$

Problem 5

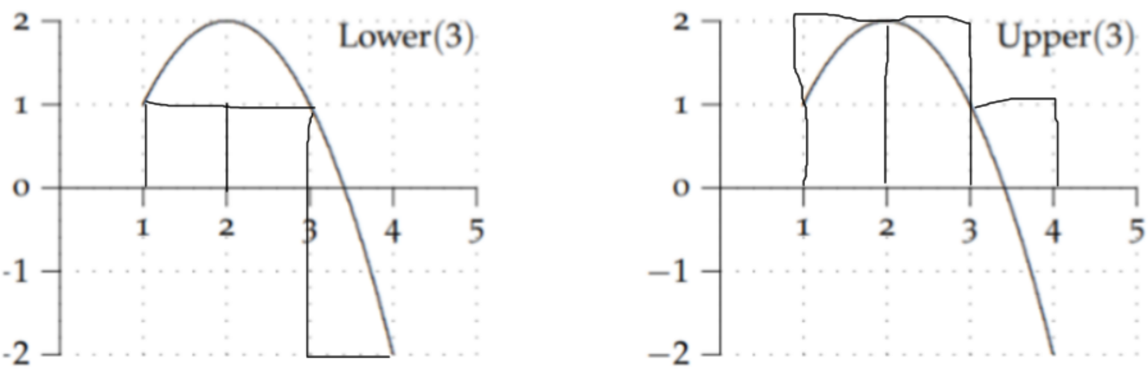

Figure 5.

(a)

$$\text{Lower}(3) = 1 + 1 + (-2) = 0$$

(b)

$$\text{Upper}(3) = 2 + 2 + 1 = 5$$

Problem 6

$$\begin{aligned} \Delta x &= \frac{2}{n} \\ x_i &= 1 + \frac{2i}{n} \\ f(x_i) &= \frac{8i}{n} + \frac{8i^2}{n^2} \\ \sum_{i=1}^{n} f(x_i)\Delta x &= \sum_{i=1}^{n} \left(\frac{16i}{n^2} + \frac{16i^2}{n^3} \right) \end{aligned}$$

$$\begin{aligned} \Delta x &= \frac{4}{n} \\ x_i &= -1 + \frac{4i}{n} \\ f(x_i) &= \frac{64i^3}{n^3} \\ \sum_{i=1}^{n} f(x_i)\Delta x &= \sum_{i=1}^{n} \frac{256i^3}{n^4} \end{aligned}$$

Problem 7

(a) Let $S(n)$ be the left Riemann sum with n subdivisions.
We have $\Delta x = 2/n$, $x_k = 1 + k\Delta x = 1 + 2k/n$.

$$\begin{aligned} S(n) &= \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n} \frac{2}{n} \left(27 - \left(1 + \frac{2k}{n}\right)^3 \right) = \frac{2}{n} \sum_{k=1}^{n} 27 - \frac{2}{n} \sum_{k=1}^{n} \left(1 + \frac{2k}{n}\right)^3 \\ &= 54 - \frac{2}{n} \sum_{k=1}^{n} \left(1 + \frac{2k}{n}\right)^3 = 54 - \frac{2}{n} \sum_{k=1}^{n} \left[1 + \frac{6k}{n} + \frac{12k^2}{n^2} + \frac{8k^3}{n^3}\right] \\ &= 54 - 2 - \frac{12}{n^2} \sum_{k=1}^{n} k - \frac{24}{n^3} \sum_{k=1}^{n} k^2 - \frac{16}{n^4} \sum_{k=1}^{n} k^3 \\ &= 52 - \frac{12n(n+1)}{2n^2} - \frac{24n(n+1)(2n+1)}{6n^3} - \frac{16n^2(n+1)^2}{4n^4} \\ &= 52 - \frac{6n+1}{n} - \frac{8n^2+12n+4}{n^2} - \frac{4n^2+8n+4}{n^2} \\ \lim_{n\to\infty} S(n) &= \lim_{n\to\infty} \left[52 - \frac{6n+1}{n} - \frac{8n^2+12n+4}{n^2} - \frac{4n^2+8n+4}{n^2} \right] \\ &= \lim_{n\to\infty} [52 - 6 - 8 - 4] \qquad \text{(Trivial and left as exercise to the reader.)} \\ &= \boxed{34} \end{aligned}$$

(b) Let $S(n)$ be the left Riemann sum with n subdivisions.
We have $\Delta x = 2/n$, $x_k = -1 + k\Delta x = -1 + 2k/n$.

$$\begin{aligned} S(n) &= \sum_{k=1}^{n} f(x_k)\Delta x = \sum_{k=1}^{n} \frac{2}{n} \left(\left(-1 + \frac{2k}{n}\right)^2 - \left(-1 + \frac{2k}{n}\right)^3 \right) \\ &= \frac{2}{n} \sum_{k=1}^{n} \left(1 - \frac{4k}{n} + \frac{4k^2}{n^2} - \left(-1 + \frac{6k}{n} - \frac{12k^2}{n^2} + \frac{8k^3}{n^3} \right) \right) \\ &= \frac{2}{n} \sum_{k=1}^{n} \left(1 - \frac{4k}{n} + \frac{4k^2}{n^2} + 1 - \frac{6k}{n} + \frac{12k^2}{n^2} - \frac{8k^3}{n^3} \right) \\ &= \frac{2}{n} \sum_{k=1}^{n} \left(2 - \frac{10k}{n} + \frac{16k^2}{n^2} - \frac{8k^3}{n^3} \right) = 4 - \frac{20}{n^2} \sum_{k=1}^{n} k + \frac{32}{n^3} \sum_{k=1}^{n} k^2 - \frac{16}{n^4} \sum_{k=1}^{n} k^3 \\ &= 4 - \frac{10n(n+1)}{n^2} + \frac{16n(n+1)(2n+1)}{3n^3} - \frac{4n^2(n+1)^2}{n^4} \\ \lim_{n\to\infty} S(n) &= \lim_{n\to\infty} \left[4 - \frac{10n(n+1)}{n^2} + \frac{16n(n+1)(2n+1)}{3n^3} - \frac{4n^2(n+1)^2}{n^4} \right] \\ &= \lim_{n\to\infty} \left[4 - 10 + \frac{32}{3} - 4 \right] \qquad \text{(This statement does not warrant a proof.)} \\ &= \boxed{\frac{2}{3}} \end{aligned}$$