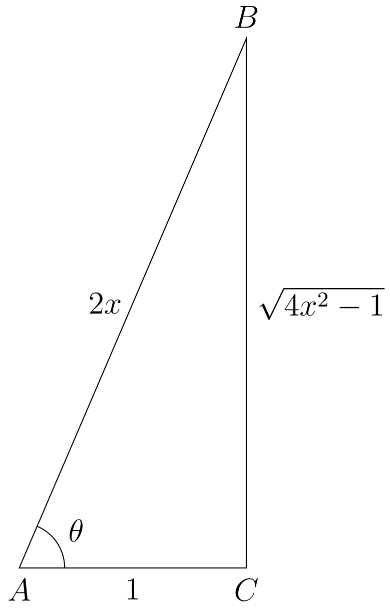
Problem Set #17

Jayden Li

October 9, 2024

Problem 3

(a)

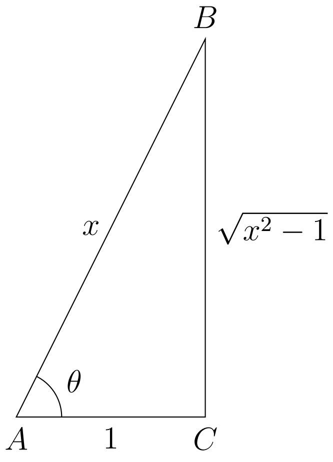


$$\cos\theta = \frac{1}{2x}$$
$$2x\cos\theta = 1$$
$$x = \frac{1}{2}\sec\theta$$
$$\mathrm{d}x = \frac{1}{2}\sec(\theta)\tan(\theta)\,\mathrm{d}\theta$$

$$1 = \frac{1}{2\cos\theta} \implies \theta = \frac{\pi}{3}$$
$$\frac{1}{2} = \frac{1}{2\cos\theta} \implies \theta = 0$$

$$\int_{1/2}^{1} \frac{x^3}{\sqrt{4x^2-1}}\,\mathrm{d}x$$
$$= \int_0^{\pi/3} \frac{\frac{1}{8}\sec^3(\theta)\frac{1}{2}\sec(\theta)\tan(\theta)}{\sqrt{\sec^2 - 1}}\,\mathrm{d}\theta$$
$$= \frac{1}{16}\int_0^{\pi/3} \frac{\sec^4\theta}{\cancel{\tan\theta}}\cdot\cancel{\tan(\theta)}\,\mathrm{d}\theta$$
$$= \frac{1}{16}\int_0^{\pi/3} \sec^2(\theta)(1+\tan^2\theta)\,\mathrm{d}\theta$$
$$= \begin{bmatrix} u = \tan\theta \\ \mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta \end{bmatrix}$$
$$\frac{1}{16}\int_0^{\sqrt{3}} \left(1+u^2\right)\mathrm{d}u$$
$$= \frac{1}{16}\left[u + \frac{u^3}{3}\right]_0^{\sqrt{3}} = \boxed{\frac{\sqrt{3}}{8}}$$

Problem 4



$$\cos\theta = \frac{1}{x}$$
$$x = \sec\theta$$
$$\mathrm{d}x = \sec(\theta)\tan(\theta)\,\mathrm{d}\theta$$

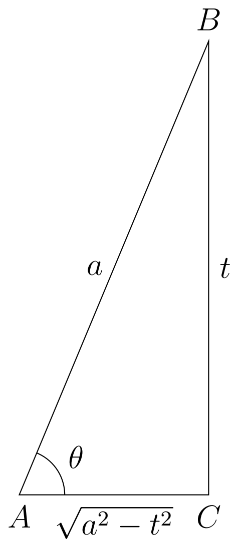
$$1 = \sec\theta$$
$$\implies \theta = \operatorname{arcsec} 1 = 0$$
$$7 = \sec\theta$$
$$\implies \theta = \operatorname{arcsec} 7$$

$$\tan\theta = \sqrt{x^2-1}$$

$$\int_1^7 \frac{\sqrt{x^2-1}}{x}\,\mathrm{d}x$$
$$= \int_0^{\operatorname{arcsec} 7} \frac{\sqrt{\sec^2\theta - 1}\cdot\sec\theta\tan\theta}{\sec\theta}\,\mathrm{d}\theta$$
$$= \int_0^{\operatorname{arcsec} 7} \tan^2\theta\,\mathrm{d}\theta = [\tan\theta - \theta]_0^{\operatorname{arcsec} 7}$$
$$= \sqrt{x^2-1}\Big|_1^7 - \theta\Big|_0^{\operatorname{arcsec} 7} = \sqrt{48} - \operatorname{arcsec} 7$$
$$= 4\sqrt{3} - \operatorname{arcsec} 7$$
$$\text{Average} = \frac{1}{7-1}\int_1^7 \frac{\sqrt{x^2-1}}{x}\,\mathrm{d}x$$
$$= \frac{1}{6}\left(4\sqrt{3} - \operatorname{arcsec} 7\right) = \boxed{\frac{2\sqrt{3}}{3} - \frac{\operatorname{arcsec} 7}{6}}$$

Problem 5

(a)



$$\sin\theta = \frac{t}{a}$$
$$t = a\sin\theta$$
$$\mathrm{d}t = a\cos\theta\,\mathrm{d}\theta$$

$$0 = a\sin\theta$$
$$\implies \theta = 0$$

$$x = a\sin\theta$$
$$\implies \theta = \arcsin\frac{x}{a}$$

$$\int_0^x \sqrt{a^2-t^2}\,\mathrm{d}t = \int_0^{\arcsin\frac{x}{a}} \sqrt{a^2 - a^2\sin^2\theta}\cdot a\cos\theta\,\mathrm{d}\theta$$
$$= a^2\int_0^{\arcsin\frac{x}{a}} \cos^2\theta\,\mathrm{d}\theta = \frac{a^2}{2}\int_0^{\arcsin\frac{x}{a}} (1+\cos 2\theta)\,\mathrm{d}\theta$$
$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\arcsin\frac{x}{a}} = \frac{a^2}{2}\left[\arcsin\frac{x}{a} + \frac{\sin\left(2\arcsin\frac{x}{a}\right)}{2}\right]$$
$$= \frac{a^2}{2}\left[\arcsin\left(\frac{x}{a}\right) + \frac{2\sin\left(\arcsin\frac{x}{a}\right)\cos\left(\arcsin\frac{x}{a}\right)}{2}\right]$$
$$= \frac{a^2}{2}\left[\arcsin\left(\frac{x}{a}\right) + \frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}\right]$$
$$= \boxed{\frac{1}{2}a^2\arcsin\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2-x^2}}$$

(b) The area bound by a circle, the y-axis, and some line between the y-axis and the intersection of the circle with the x-axis equals $\frac{1}{2}a^2\arcsin\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2-x^2}$ if x is the x-value of the line.