Problem Set #69

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Problem 2

Let s be the side length of the square.

$$\begin{aligned} V &= s^2 \\ \frac{\mathrm{d}V}{\mathrm{d}t} &= 2s\frac{\mathrm{d}s}{\mathrm{d}t} = 2 \cdot 4 \cdot 6 = 48 \end{aligned}$$

The area of the square is increasing at a rate of $48\,cm^2/s$.

Problem 4

$$\begin{aligned} y &= x^3 + 2x \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= 3x^2\frac{\mathrm{d}x}{\mathrm{d}t} + 2\frac{\mathrm{d}x}{\mathrm{d}t} = 3 \cdot 2^2 \cdot 5 + 2 \cdot 5 = \boxed{70} \end{aligned}$$

Problem 6

Let d_1 be the ground distance between the plane and the radio station and let d_2 be the actual distance.

$$\begin{aligned} d_2^2 &= 1^2 + d_1^2 \\ 2d_2\frac{\mathrm{d}d_2}{\mathrm{d}t} &= 2d_1\frac{\mathrm{d}d_1}{\mathrm{d}t} \\ \sqrt{1+2^2}\frac{\mathrm{d}d_2}{\mathrm{d}t} &= 2 \cdot 500 \\ \frac{\mathrm{d}d_2}{\mathrm{d}t} &= \frac{1000}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} \\ \frac{\mathrm{d}d_2}{\mathrm{d}t} &= \boxed{200\sqrt{5}\,mi/h} \end{aligned}$$

This is not a very well worded question – is the actual distance (accounting for altitude) between the plane and the radio station or the ground distance 2 miles? I took it as the latter, so I had $d_1 = 2$ instead of $d_2 = 2$.

Problem 8

Let s be the distance traveled by the first car and w the distance traveled by the second.

$$\begin{aligned} D^2 &= s^2 + w^2 \\ 2D\frac{\mathrm{d}D}{\mathrm{d}t} &= 2s\frac{\mathrm{d}s}{\mathrm{d}t} + 2w\frac{\mathrm{d}w}{\mathrm{d}t} \\ \sqrt{(2 \cdot 60^2) + (2 \cdot 25)^2}\frac{\mathrm{d}D}{\mathrm{d}t} &= (2 \cdot 60)(60) + (2 \cdot 25)(25) \\ 130\frac{\mathrm{d}D}{\mathrm{d}t} &= 8450 \\ \frac{\mathrm{d}D}{\mathrm{d}t} &= 65 \end{aligned}$$

The cars are moving apart at a rate of $65\,mi/h$.

Problem 10

Let h be the altitude of the triangle and b be the base.

$$\begin{aligned} A &= \frac{1}{2}hb \\ \frac{\mathrm{d}A}{\mathrm{d}t} &= \frac{1}{2}\frac{\mathrm{d}h}{\mathrm{d}t}b + \frac{1}{2}h\frac{\mathrm{d}b}{\mathrm{d}t} \\ 2 &= \frac{1}{2}20 + \frac{1}{2}10\frac{\mathrm{d}b}{\mathrm{d}t} \\ 5\frac{\mathrm{d}b}{\mathrm{d}t} &= 2 - 10 \\ \frac{\mathrm{d}b}{\mathrm{d}t} &= -\frac{8}{5} \end{aligned}$$

The base of the triangle is decreasing at a rate of $1.6\,cm/min$.

Problem 12

Let h be the height of the water. The water in the tank has the shape of a cone with height h, and by similar triangles we see that the radius of the cone must be $h/3$.
Let V be the volume of the cone representing the water.

$$\begin{aligned} V &= \frac{1}{3}\pi\left(\frac{h}{3}\right)^2 h \\ &= \frac{\pi h^3}{27} \\ \frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{3\pi h^2}{27}\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{\pi(200)^2}{9}(20) \approx 279252.680 \end{aligned}$$

Let x be the rate at which water is being pumped into the tank.

$$x - 10000 = 279252.680 \implies x = 289252.680$$

Water is being pumped into the tank at a rate of $289252.680\,cm^3/min$.

Problem 14

Let x be the diameter and height of the pile of gravel. Then $x/2$ is the radius of the cone.

$$\begin{aligned} V &= \frac{1}{3}\pi\left(\frac{x}{2}\right)^2 x \\ &= \frac{\pi x^3}{12} \\ \frac{\mathrm{d}V}{\mathrm{d}t} &= \frac{3\pi x^2}{12}\frac{\mathrm{d}x}{\mathrm{d}t} \\ 30 &= \frac{\pi 10^2}{4}\frac{\mathrm{d}x}{\mathrm{d}t} \\ \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{6}{5\pi} \end{aligned}$$

The height of the pile is increasing at a rate of $6/5\pi\,cm/min$.

Problem 18

Let $\sin_d$ and $\cos_d$ be sine and cosine functions where the argument is in degrees. We use $\sin_r$ and $\cos_r$ to denote standard sine and cosine functions. We start by finding the derivatives of $\sin_d$ and $\cos_d$.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\sin_d x &= \frac{\mathrm{d}}{\mathrm{d}x}\sin_r\left(\frac{\pi}{180}x\right) = \frac{\pi}{180}\cos_r\left(\frac{\pi}{180}x\right) = \frac{\pi}{180}\cos_d x \\ \frac{\mathrm{d}}{\mathrm{d}x}\cos_d x &= \frac{\mathrm{d}}{\mathrm{d}x}\cos_r\left(\frac{\pi}{180}x\right) = -\frac{\pi}{180}\sin_r\left(\frac{\pi}{180}x\right) = -\frac{\pi}{180}\sin_d x \end{aligned}$$

Let x be the third side of the triangle. By the law of cosines:

$$\begin{aligned} x^2 &= 12^2 + 15^2 - 2(12)(15)\cos_d\theta \\ &= 369 - 360\cos_d\theta \\ 2x\frac{\mathrm{d}x}{\mathrm{d}t} &= -360\left(-\frac{\pi}{180}\sin_d\theta\right)\frac{\mathrm{d}\theta}{\mathrm{d}t} \\ \sqrt{369 - 360\cos_d(60)}\frac{\mathrm{d}x}{\mathrm{d}t} &= \pi\sin_d(60)(2) \\ \frac{\mathrm{d}x}{\mathrm{d}t} &\approx 0.396 \end{aligned}$$

The length of the third side is increasing at a rate of $0.396\,cm/min$.

Problem 19

Let h be the height of the rocket, let θ be the rotation of the camera and let d be the distance from the camera to the rocket.

(a)

$$\begin{aligned} d^2 &= 4000^2 + h^2 \\ 2d\frac{\mathrm{d}d}{\mathrm{d}t} &= 2h\frac{\mathrm{d}h}{\mathrm{d}t} \\ \sqrt{4000^2 + 3000^2}\frac{\mathrm{d}d}{\mathrm{d}t} &= 3000 \cdot 600 \\ \frac{\mathrm{d}d}{\mathrm{d}t} &= 360 \end{aligned}$$

The distance between the rocket and the camera is increasing at a rate of $360\,ft/s$.

(b)

$$\begin{aligned} \tan\theta &= \frac{h}{4000} \implies \theta = \arctan\frac{h}{4000} \\ \sec^2(\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{1}{4000}\frac{\mathrm{d}h}{\mathrm{d}t} \\ \sec^2\left(\arctan\frac{3000}{4000}\right)\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \frac{600}{4000} \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= 0.096 \end{aligned}$$

The angle of the camera is changing at a rate of $0.096\,rad/s$.

Problem 20

Let θ be the angle of elevation of the telescope and let d be the distance between the plane and telescope.

$$\begin{aligned} \tan\theta &= \frac{5}{d} \implies \frac{1}{d^2} = \left(\frac{\tan\theta}{5}\right)^2 \\ \sec^2(\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t} &= -\frac{5}{d^2}\frac{\mathrm{d}d}{\mathrm{d}t} \\ \sec^2\left(\frac{\pi}{3}\right)\left(-\frac{\pi}{6}\right) &= -5\left(\frac{\tan\frac{\pi}{3}}{5}\right)^2\frac{\mathrm{d}d}{\mathrm{d}t} \\ \frac{4\pi}{6} &= \frac{3}{5}\frac{\mathrm{d}d}{\mathrm{d}t} \\ \frac{\mathrm{d}d}{\mathrm{d}t} &= \frac{10\pi}{9} \end{aligned}$$

The plane is traveling at a speed of $10\pi/9\,km/min$.

Problem 21

Problem 22