Problem Set #14

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Problem 1

(e) $$\int p^5 \ln p \,\mathrm{d}p = \begin{bmatrix} u = \ln p & \mathrm{d}u = \dfrac{\mathrm{d}p}{p} \\ \mathrm{d}v = p^5\,\mathrm{d}p & v = \dfrac{p^6}{6} \end{bmatrix} \frac{p^6 \ln p}{6} - \int \frac{p^6}{6p}\,\mathrm{d}p = \boxed{\frac{p^6\ln p}{6} - \frac{p^6}{36} + C}$$

(f) $$\int_0^1 t\cosh t\,\mathrm{d}t = \begin{bmatrix} u = t & \mathrm{d}u = \mathrm{d}t \\ \mathrm{d}v = \cosh t & v = \sinh t \end{bmatrix} \left[t\sinh t - \int \sinh t\,\mathrm{d}t\right]_0^1 = [t\sinh t - \cosh t]_0^1$$
$$= 1\sinh 1 - \cosh 1 - 0 + \cosh 0 = \frac{e^1 - e^{-1}}{2} - \frac{e^1 + e^{-1}}{2} + \frac{e^0 + e^{-0}}{2}$$
$$= \frac{e^1 - e^{-1} - e^1 - e^{-1}}{2} + 1 = \boxed{-\frac{1}{e} + 1}$$

(g) $$\int_1^2 \frac{\ln x}{x^2}\,\mathrm{d}x = \begin{bmatrix} u = \ln x & \mathrm{d}u = \dfrac{\mathrm{d}x}{x} \\ \mathrm{d}v = \dfrac{\mathrm{d}x}{x^2} & v = -\dfrac{1}{x} \end{bmatrix} \left[-\frac{\ln x}{x}\right]_1^2 + \int_1^2 \frac{1}{x^2}\,\mathrm{d}x = -\frac{\ln 2}{2} + 0 + \left[-\frac{1}{x}\right]_1^2$$
$$= -\frac{\ln 2}{2} + \left(-\frac{1}{2} + 1\right) = \frac{-\ln 2}{2} + \frac{1}{2} = \boxed{\frac{1 - \ln 2}{2}}$$

Problem 2

$\boxed{\text{Average} = \dfrac{81\ln 3 - 26}{18}}$ (obvious.)

Problem 8

(a) $$\int \sec x\,\mathrm{d}x = \int \frac{1}{\cos x}\,\mathrm{d}x = \int \frac{\cos x}{1 - \sin^2 x}\,\mathrm{d}x = \begin{bmatrix} u = \sin x \\ \mathrm{d}u = \cos(x)\,\mathrm{d}x \end{bmatrix} \int \frac{1}{1-u^2}\,\mathrm{d}u = \int \frac{1}{(1+u)(1-u)}\,\mathrm{d}u$$

We can use partial fraction decomposition here. We need to find A, B such that

$$\begin{aligned} \frac{A}{1+u} + \frac{B}{1-u} &= \frac{1}{(1+u)(1-u)} \\ \frac{A - Au + B + Bu}{(1+u)(1-u)} &= \frac{1}{(1+u)(1-u)} \\ \frac{A + B - (A-B)u}{(1+u)(1-u)} &= \frac{1}{(1+u)(1-u)} \end{aligned}$$

We know that $(A-B)u = 0u \implies A = B$. But also $A + B = 1$, so $A, B = 1/2$.

$$= \int \left(\frac{1/2}{1+u} + \frac{1/2}{1-u}\right)\mathrm{d}u = \frac{1}{2}\left(\int \frac{1}{1+u}\,\mathrm{d}u + \int \frac{1}{1-u}\,\mathrm{d}u\right)$$
$$= \frac{1}{2}\left(\ln|1+u| - \ln|1-u|\right) = \boxed{\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C}$$

(I did look up some hints for this one, such as substituting $u = \sin x$ and using partial fractions.)

(b) $$\int \sec^3 x\,\mathrm{d}x = \int \frac{1}{\cos^3 x}\,\mathrm{d}x = \int \frac{\cos x}{\cos^4 x}\,\mathrm{d}x = \int \frac{\cos x}{\left(1 - \sin^2 x\right)^2}\,\mathrm{d}x = \begin{bmatrix} u = \sin x \\ \mathrm{d}u = \cos(x)\,\mathrm{d}x \end{bmatrix} \int \frac{1}{(1-u^2)^2}\,\mathrm{d}u$$
$$= \int \left(\frac{1}{(1-u)(1+u)}\right)^2 \mathrm{d}u = \int \left(\frac{1}{2}\left(\frac{1}{1+u} + \frac{1}{1-u}\right)\right)^2 \mathrm{d}u$$
$$= \frac{1}{4}\int \frac{1}{(1+u)^2}\,\mathrm{d}u + \frac{1}{2}\int \frac{1}{(1+u)(1-u)}\,\mathrm{d}u + \frac{1}{4}\int \frac{1}{(1-u)^2}\,\mathrm{d}u$$
$$= -\frac{1}{4(1+u)^2} + \frac{1}{4(1-u)^2} + \frac{1}{4}\int \left(\frac{1}{1+u} + \frac{1}{1-u}\right)\mathrm{d}u$$
$$= -\frac{1}{4(1+u)^2} + \frac{1}{4(1-u)^2} + \frac{1}{4}(\ln|1+u| - \ln|1-u|)$$
$$= \boxed{-\frac{1}{4(1+\sin x)} + \frac{1}{4(1-\sin x)} + \frac{1}{4}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C}$$