Problem Set #64

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Problem 1

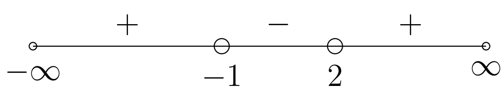
$$f'(x) = 12x^3 - 12x^2 - 24x$$

Increasing:

$$12x^3 - 12x^2 - 24x > 0$$
$$12\left(x^2 - x - 2\right) > 0$$

C1: $x = 0$. Then $12x\left(x^2 - x - 2\right) = 0 \not> 0$.
C2: $x > 0$.

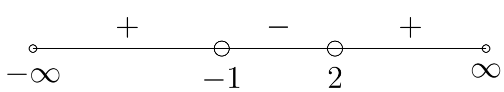
$$x^2 - x - 2 > 0$$
$$(x-2)(x+1) > 0$$


.

$$x \in ((-\infty, -1) \cup (2, \infty)) \cap (0, \infty)$$
$$x \in (2, \infty)$$

C3: $x < 0$.

$$x^2 - x - 2 < 0$$
$$(x-2)(x+1) < 0$$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$$x \in (-1, 2) \cap (-\infty, 0)$$
$$x \in (-1, 0)$$

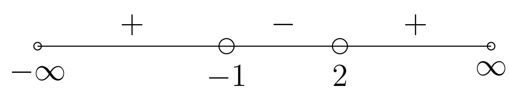
$$\boxed{x \in (-1, 0) \cup (2, \infty)}$$

Decreasing:

$$12x^3 - 12x^2 - 24x < 0$$
$$12\left(x^2 - x - 2\right) < 0$$

C1: $x = 0$. Then $12x\left(x^2 - x - 2\right) = 0 \not< 0$.
C2: $x > 0$.

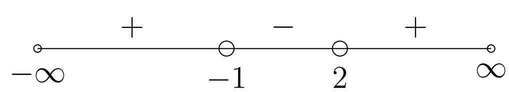
$$x^2 - x - 2 < 0$$
$$(x-2)(x+1) < 0$$


.

$$x \in (-1, 2) \cap (0, \infty)$$
$$x \in (0, 2)$$

C3: $x < 0$.

$$x^2 - x - 2 > 0$$
$$(x-2)(x+1) > 0$$


.

$$x \in ((-\infty, -1) \cup (2, \infty)) \cap (-\infty, 0)$$
$$x \in (-\infty, -1)$$

$$\boxed{x \in (\infty, -1) \cup (0, 2)}$$

Problem 2

(a) Increasing: $(1, 5)$.
Decreasing: $[0, 1) \cup (5, 6]$.

(b) $x \in \{1, 5\}$.

Problem 3

Local maximum: $x = 1$. Local maximum: $x = 5$.

Problem 4

$$\begin{aligned} g(x) &= x + 2\sin x \\ g'(x) &= 1 + 2\cos x \\ g''(x) &= -2\sin x \end{aligned}$$

Critical points:

$$\begin{aligned} g'(x) &= 0 \\ 1 + 2\cos x &= 0 \\ \cos x &= -\frac{1}{2} \\ x &= \pm\frac{2\pi}{3} + 2\pi n \end{aligned}$$

Use second derivative test:

$$\begin{aligned} g''\left(\frac{2\pi}{3} + 2\pi n\right) &= -2\sin\left(\frac{2\pi}{3} + 2\pi n\right) \\ &= -2 \cdot \left(\frac{\sqrt{3}}{2}\right) \\ &= -\sqrt{3} \end{aligned}$$

Local maxima: $\boxed{\frac{2\pi}{3} + 2\pi n}$

$$\begin{aligned} g''\left(-\frac{2\pi}{3} + 2\pi n\right) &= -2\sin\left(-\frac{2\pi}{3} + 2\pi n\right) \\ &= -2 \cdot \left(-\frac{\sqrt{3}}{2}\right) \\ &= \sqrt{3} \end{aligned}$$

Local minima: $\boxed{-\frac{2\pi}{3} + 2\pi n}$

Problem 5

(i)

(ii)

(iii)

Problem 6

(i)

$$
\begin{aligned}
f(x) &= \sin x + \cos x \\
f'(x) &= \cos x - \sin x \\
f''(x) &= -\sin x - \cos x
\end{aligned}
$$

(a) Increasing:

$$
\begin{aligned}
f'(x) &> 0 \\
\cos x - \sin x &> 0 \\
\cos x &> \sin x
\end{aligned}
$$

$$\boxed{x \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{5\pi}{4}, 2\pi\right)}$$

Decreasing:

$$
\begin{aligned}
f'(x) &< 0 \\
\cos x - \sin x &< 0 \\
\cos x &< \sin x
\end{aligned}
$$

$$\boxed{x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)}$$

(I figured out the answers by looking at a unit circle.)

(b)

$$
\begin{aligned}
f'(x) &= 0 \\
\cos x - \sin x &= 0 \\
\cos x &= \sin x \\
x &\in \left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\} \\
f''\left(\frac{\pi}{4}\right) &= -\sin\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{4}\right) \\
&= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \\
&= -\sqrt{2} < 0 \\
f''\left(\frac{5\pi}{4}\right) &= -\sin\left(\frac{5\pi}{4}\right) - \cos\left(\frac{5\pi}{4}\right) \\
&= \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right) \\
&= \sqrt{2} > 0
\end{aligned}
$$

Local maximum: $\boxed{x = \frac{5\pi}{4}}$. Local minimum: $\boxed{x = \frac{\pi}{4}}$

(c)

Concave up:

$$
\begin{aligned}
f''(x) &> 0 \\
-\sin x - \cos x &> 0 \\
-\sin x &> \cos x
\end{aligned}
$$

$$\boxed{x \in \left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)}$$

Concave down:

$$
\begin{aligned}
f''(x) &< 0 \\
-\sin x - \cos x &< 0 \\
-\sin x &< \cos x
\end{aligned}
$$

$$\boxed{x \in \left(0, \frac{3\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)}$$

Inflection points:

$$
\begin{aligned}
f''(x) &= 0 \\
-\sin x - \cos x &= 0 \\
-\sin x &= \cos x
\end{aligned}
$$

$$\boxed{x \in \left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}}$$

(ii)

$$
\begin{aligned}
f(x) &= e^{2x} + e^{-x} \\
f'(x) &= 2e^{2x} - e^{-x} \\
f''(x) &= 4e^{2x} + e^{-x}
\end{aligned}
$$

(a) Increasing:

$$
\begin{aligned}
f'(x) &> 0 \\
2e^{2x} - e^{-x} &> 0 \\
e^{x}\left(2e^{2x} - \frac{1}{e^{x}}\right) &> 0
\end{aligned}
$$

(e^x is always positive.)

$$
\begin{aligned}
2e^{3x} &> 1 \\
e^{3x} &> \frac{1}{2} \\
3x &> \ln\left(\frac{1}{2}\right)
\end{aligned}
$$

(natural logarithm is an increasing function.)

$$x > \frac{\ln(0.5)}{3}$$

$$\boxed{x \in \left(\frac{\ln(0.5)}{3}, \infty\right)}$$

Decreasing:

$$
\begin{aligned}
f'(x) &< 0 \\
2e^{2x} - e^{-x} &< 0 \\
e^{x}\left(2e^{2x} - \frac{1}{e^{x}}\right) &< 0 \\
2e^{3x} &< 1 \\
e^{3x} &< \frac{1}{2} \\
3x &< \ln\left(\frac{1}{2}\right) \\
x &< \frac{\ln(0.5)}{3}
\end{aligned}
$$

$$\boxed{x \in \left(-\infty, \frac{\ln(0.5)}{3}\right)}$$

(b)

$$
\begin{aligned}
f'(x) &= 0 \\
2e^{2x} - e^{-x} &= 0 \\
2e^{3x} &= 1 \\
e^{3x} &= \frac{1}{2}
\end{aligned}
$$

$$x = \frac{\ln(0.5)}{3}$$
$$f''\left(\frac{\ln(0.5)}{3}\right) = 4\exp\left(\frac{2\ln(0.5)}{3}\right) + \exp\left(-\frac{\ln(0.5)}{3}\right) > 0$$

(since the exponential function is always positive, the sum of two exp's must be positive.)

Local minimum: $\boxed{x = \frac{\ln(0.5)}{3}}$

(c) Concave up:

$$\begin{aligned} f''(x) &> 0 \\ 4e^{2x} + e^{-x} &> 0 \\ 4e^{3x} &> -1 \end{aligned}$$

$\boxed{x \in \mathbb{R}}$

Concave down:

$$\begin{aligned} f''(x) &< 0 \\ 4e^{2x} + e^{-x} &< 0 \\ 4e^{3x} &< -1 \end{aligned}$$

Which is never true for real values of x.

Inflection points:

$$\begin{aligned} f''(x) &= 0 \\ 4e^{2x} + e^{-x} &= 0 \\ 4e^{3x} &= -1 \end{aligned}$$

Never true $\implies$ no inflection points.

(iii)

$$f(x) = \frac{\ln x}{\sqrt{x}}$$
$$f'(x) = \frac{\frac{1}{x}\cdot\sqrt{x} - \ln(x)\cdot\frac{1}{2\sqrt{x}}}{(\sqrt{x})^2} = \frac{\frac{\sqrt{x}}{x} - \frac{\sqrt{x}\ln x}{2x}}{x} = \frac{2\sqrt{x} - \sqrt{x}\ln x}{2x^2}$$
$$f''(x) = \frac{\left(\frac{2}{2\sqrt{x}} - \frac{\ln x}{2\sqrt{x}} - \frac{\sqrt{x}}{x}\right)2x^2 - (2\sqrt{x} - \sqrt{x}\ln x)\,4x}{4x^4} = \frac{\frac{2x^2}{\sqrt{x}} - \frac{2x^2\ln x}{2\sqrt{x}} - \frac{2x^2}{\sqrt{x}} - \frac{8x^2}{\sqrt{x}} + \frac{4x^2\ln x}{\sqrt{x}}}{4x^4}$$
$$= \frac{4x^2 - 2x^2\ln x - 4x^2 - 16x^2 + 8x^2\ln x}{8x^4\sqrt{x}} = \frac{-\ln x - 8 + 4\ln x}{4x^2\sqrt{x}} = \frac{3\ln x - 8}{4x^2\sqrt{x}}$$

(a) Increasing:

$$\begin{aligned} f'(x) &> 0 \\ \frac{2\sqrt{x} - \sqrt{x}\ln x}{2x^2} &> 0 \\ 2\sqrt{x} &> \sqrt{x}\ln x \\ 2 &> \ln x \end{aligned}$$
$$\boxed{x \in (0, e^2)}$$

(($2x^2$ and $\sqrt{x}$ must be positive and natural logarithm is increasing, and domain is $(0,\infty)$)

Decreasing:

$$\begin{aligned} f'(x) &< 0 \\ \frac{2\sqrt{x} - \sqrt{x}\ln x}{2x^2} &< 0 \\ 2\sqrt{x} &< \sqrt{x}\ln x \\ 2 &< \ln x \end{aligned}$$
$$\boxed{x \in (e^2, \infty)}$$

(b)

$$\begin{aligned}
f'(x) &= 0 \\
\frac{2\sqrt{x} - \sqrt{x}\ln x}{2x^2} &= 0 \\
2\sqrt{x} &= \sqrt{x}\ln x \\
2 &= \ln x \\
x &= e^2 \\
f''(e^2) &= \frac{3\ln e^2 - 8}{4\left(e^2\right)^2\sqrt{e^2}} \\
&= \frac{3\cdot 2 - 8}{4e^4 \cdot e} \\
&= -\frac{1}{2e^5} < 0
\end{aligned}$$

Local maximum: $\boxed{x = e^2}$

(c) Concave up:

$$\begin{aligned}
f''(x) &> 0 \\
\frac{3\ln x - 8}{4x^2\sqrt{x}} &> 0 \\
3\ln x &> 8
\end{aligned}$$

(denominator is always positive)

$$x > e^{8/3}$$

$$\boxed{x \in \left(e^{8/3}, \infty\right)}$$

Concave down:

$$\begin{aligned}
f''(x) &< 0 \\
\frac{3\ln x - 8}{4x^2\sqrt{x}} &< 0 \\
3\ln x &< 8 \\
x &< e^{8/3}
\end{aligned}$$

$$\boxed{x \in \left(0, e^{8/3}\right)}$$

Inflection points:

$$\begin{aligned}
f''(x) &= 0 \\
\frac{3\ln x - 8}{4x^2\sqrt{x}} &= 0 \\
3\ln x &= 8
\end{aligned}$$

$$\boxed{x = e^{8/3}}$$

Problem 7

(a)

(b)

Problem 8

(a)

(b)

(c)

(d)

(e)

Problem 9

Problem 10

(a)

(b)

(c)

Problem 11

Problem 12

Problem 13

Problem 14

$$\begin{aligned}
y'' &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{(1+x^2)-2x(1+x)}{(1+x^2)^2}\right] &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1-x^2-2x}{(1+x^2)^2}\right] &= 0 \\
\frac{(-2x-2)\left(1+x^2\right)^2-\left(1-x^2-2x\right)\left(2\left(1+x^2\right)\right)(2x)}{\left(1+x^2\right)^4} &= 0 \\
(-2x-2)\left(1+x^2\right)-\left(1-x^2-2x\right)(2)(2x) &= 0 \\
-2x-2x^3-2-2x^2-4x\left(1-x^2-2x\right) &= 0 \\
-2x-2x^3-2-2x^2-4x+4x^3+8x^2 &= 0 \\
2x^3+6x^2-6x-2 &= 0 \\
x^3+3x^2-3x-1 &= 0 \\
(x-1)\left(x^2+4x+1\right) &= 0 \\
x &\in \left\{-1,-2+\sqrt{3},-2-\sqrt{3}\right\}
\end{aligned}$$

Let $x_1 = 1$, $x_2 = -2+\sqrt{3}$, $x_3 = -2-\sqrt{3}$. Then:

$$y_1 = 1$$

$$y_2 = \frac{1+(-2+\sqrt{3})}{1+(-2+\sqrt{3})^2} = \frac{-1+\sqrt{3}}{1+4-4\sqrt{3}+3} = \frac{-1+\sqrt{3}}{8-4\sqrt{3}} = \frac{1}{4}\cdot\frac{-1+\sqrt{3}}{2-\sqrt{3}}\cdot\frac{2+\sqrt{3}}{2+\sqrt{3}}$$

$$= \frac{-2-\sqrt{3}+2\sqrt{3}+3}{4(4-3)} = \frac{1+\sqrt{3}}{4}$$

$$y_3 = \frac{1+(-2-\sqrt{3})}{1+(-2-\sqrt{3})^2} = \frac{-1-\sqrt{3}}{1+4+4\sqrt{3}+3} = \frac{1}{4}\cdot\frac{-1-\sqrt{3}}{2+\sqrt{3}}\cdot\frac{2-\sqrt{3}}{2-\sqrt{3}}$$

$$= \frac{-2+\sqrt{3}-2\sqrt{3}+3}{4(4-3)} = \frac{1-\sqrt{3}}{4}$$

Equation of a line between (x_2, y_2) and (x_3, y_3):

$$y - \left(\frac{1+\sqrt{3}}{4}\right) = \left(\frac{\frac{1-\sqrt{3}}{4} - \frac{1+\sqrt{3}}{4}}{-2+\sqrt{3}-(-2-\sqrt{3})}\right)\left(x - \left(-2+\sqrt{3}\right)\right)$$

$$y - \frac{1+\sqrt{3}}{4} = \left(\frac{\frac{1-\sqrt{3}-1-\sqrt{3}}{4}}{-2+\sqrt{3}+2+\sqrt{3}}\right)\left(x+2-\sqrt{3}\right)$$

$$y - \frac{1+\sqrt{3}}{4} = \left(\frac{-2\sqrt{3}}{8\sqrt{3}}\right)\left(x+2-\sqrt{3}\right)$$

$$y - \frac{1+\sqrt{3}}{4} = -\frac{1}{4}\left(x+2-\sqrt{3}\right)$$

Check if (x_1, y_1) is on the line:

$$1 - \frac{1+\sqrt{3}}{4} \stackrel{?}{=} -\frac{1}{4}\left(1+2-\sqrt{3}\right)$$

$$\frac{4-(1+\sqrt{3})}{4} \stackrel{?}{=} -\frac{3-\sqrt{3}}{4}$$

$$4-1-\sqrt{3} \stackrel{?}{=} 3-\sqrt{3}$$

$$3-\sqrt{3} = 3-\sqrt{3}$$

Therefore the three inflection points lie on a straight line.

Problem 15