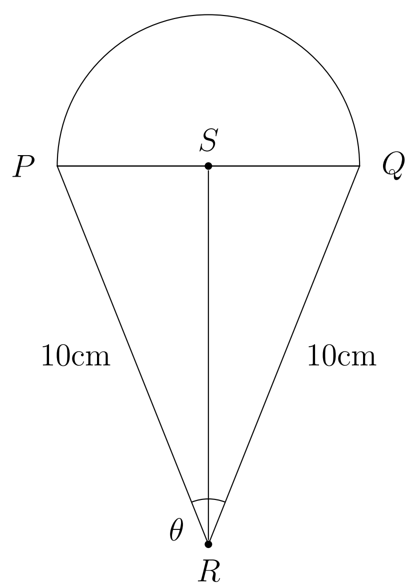
Problem Set #33, Part 3

Jayden Li

December 11, 2023

Problem 6



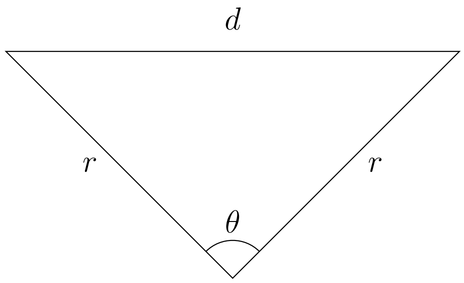
$$\angle QPR = \angle PQR = \frac{180^\circ - \theta}{2}$$
$$\overline{SR} = 10\sin\left(\frac{180^\circ - \theta}{2}\right)$$
$$\overline{PS} = 10\cos\left(\frac{180^\circ - \theta}{2}\right)$$
$$A(\theta) = \frac{1}{2}\cdot\left(\overline{SR}\right)^2\cdot\pi$$
$$A(\theta) = \frac{1}{2}\pi\left(10\cos\left(\frac{180^\circ - \theta}{2}\right)\right)^2$$
$$A(\theta) = 50\pi\left(\sqrt{\frac{1+\cos(180^\circ - \theta)}{2}}\right)^2$$
$$A(\theta) = 25\pi\cdot(1-\cos\theta)$$
$$B(\theta) = 2\cdot\frac{1}{2}\cdot 10\sin\left(\frac{180^\circ - \theta}{2}\right)\cdot 10\cos\left(\frac{180^\circ - \theta}{2}\right)$$
$$B(\theta) = 50\cdot 2\sin\left(\frac{180^\circ - \theta}{2}\right)\cos\left(\frac{180^\circ - \theta}{2}\right)$$
$$B(\theta) = 50\sin(180^\circ - \theta)$$
$$B(\theta) = 50\sin\theta$$

$$\begin{aligned}\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} &= \lim_{\theta\to 0^+}\frac{25\pi\cdot(1-\cos\theta)}{50\sin\theta}\\ &= \frac{\pi}{2}\cdot\lim_{\theta\to 0^+}\left(\frac{1-\cos\theta}{\sin\theta}\cdot\frac{1+\cos\theta}{1+\cos\theta}\right)\\ &= \frac{\pi}{2}\cdot\lim_{\theta\to 0^+}\frac{1-\cos^2\theta}{\sin(\theta)(1+\cos\theta)}\\ &= \frac{\pi}{2}\cdot\lim_{\theta\to 0^+}\frac{\sin\theta}{1+\cos\theta}\end{aligned}$$

$$\boxed{\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} = 0}$$

Problem 7

Let r be the radius of the circle.



$$d = 2 \cdot \frac{1}{2} \cdot r\cos\frac{\pi-\theta}{2} \cdot r\sin\frac{\pi-\theta}{2}$$
$$d = \frac{1}{2}r^2 \cdot 2\cos\frac{\pi-\theta}{2}\sin\frac{\pi-\theta}{2}$$
$$d = \frac{1}{2}r^2 \cdot \sin(\pi-\theta)$$
$$d = \frac{r^2\sin\theta}{2}$$

$$s = r^2\pi \cdot \frac{\theta}{2\pi}$$
$$s = \frac{r^2\theta}{2}$$

$$\begin{aligned}\lim_{\theta\to 0^+}\frac{s}{d} &= \lim_{\theta\to 0^+}\frac{r^2\theta}{r^2\sin\theta}\\ &= \lim_{\theta\to 0^+}\left(\frac{\sin\theta}{\theta}\right)^{-1}\\ &= 1^{-1}\end{aligned}$$
$$\boxed{\lim_{\theta\to 0^+}\frac{s}{d} = 1}$$

Problem Set #35

Jayden Li

December 11, 2023

Problem 4

(a)

$$
\begin{aligned}
y &= 2\cot^2 x - 4\cot x + 5 \\
y &= 2\left(\cot^2 x - 2\cot x + \frac{5}{2}\right) \\
y &= 2\left((\cot x - 1)^2 - 1 + \frac{5}{2}\right) \\
y &= 2(\cot x - 1)^2 + 3
\end{aligned}
$$

$$
\begin{aligned}
\cot x &\in (-\infty, \infty) \\
\cot x - 1 &\in (-\infty, \infty) \\
(\cot x - 1)^2 &\in [0, \infty) \\
2(\cot x - 1)^2 &\in [0, \infty) \\
2(\cot x - 1)^2 + 3 &\in [3, \infty)
\end{aligned}
$$

$$\boxed{\text{Range: } [3, \infty)}$$

(b)

$$
\begin{aligned}
y &= \frac{2\tan x}{1 + \tan^2 x} \\
y &= \frac{2 \cdot \frac{\sin x}{\cos x}}{1 + \frac{\sin^2 x}{\cos^2 x}} \cdot \frac{\cos^2 x}{\cos^2 x} \\
y &= \frac{2\cos x \sin x}{\cos^2 x + \sin^2 x} \\
y &= \sin 2x
\end{aligned}
$$

$$\boxed{\text{Range: } [-1, 1]}$$

(c)

$$y = \arccos\left(2x - x^2\right)$$

Because $\arccos x$ is decreasing, its range can be calculated from its input.

$$\begin{aligned}2x - x^2 \text{ vertex x: } & \frac{-2}{-2} \\ =& 1 \\ 2x - x^2 \text{ vertex y: } & 2\left(1\right) - 1^2 \\ =& 1\end{aligned}$$

The leading coefficient of $2x - x^2$ is negative, so its vertex must be the maximum. Therefore, its range is $(-\infty, 1]$. This is fully within the domain of $\arccos x$, $[-1, 1]$, so the range of y is $\boxed{[0, \pi]}$.

(d)

$$y = \arctan\left(1 - 2|x|\right)$$

Because $\arctan x$ is increasing, its range can be calculated from its input.

$$\begin{aligned}1 - 2|x| \text{ min: } & -\infty \\ 1 - 2|x| \text{ max: } & 1 \\ \lim_{x \to -\infty} \arctan x =& -\frac{\pi}{2} \\ \arctan 1 =& \frac{\pi}{4}\end{aligned}$$

$$\boxed{\text{Range: } \left(-\frac{\pi}{2}, \frac{\pi}{4}\right]}$$

(d)

$$\begin{aligned}y &= \arctan\frac{x^2 + \sqrt{3}}{x^2 + 1} \\ \tan y &= \frac{x^2 + \sqrt{3}}{x^2 + 1}, \, -\frac{\pi}{2} < y < \frac{\pi}{2} \\ x^2 \tan y + \tan y &= x^2 + \sqrt{3}, \, -\frac{\pi}{2} < y < \frac{\pi}{2} \\ x^2\left(\tan y - 1\right) &= \sqrt{3} - \tan y, \, -\frac{\pi}{2} < y < \frac{\pi}{2} \\ x^2 &= \frac{\sqrt{3} - \tan y}{\tan y - 1}, \, -\frac{\pi}{2} < y < \frac{\pi}{2} \\ x &= \pm\sqrt{\frac{\sqrt{3}\cos y - \sin y}{\sin y - \cos y}}, \, -\frac{\pi}{2} < y < \frac{\pi}{2}\end{aligned}$$

Find inverse.

$$y = \pm\sqrt{\frac{\sqrt{3}\cos x - \sin x}{\sin x - \cos x}}, \, -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Rewriting the expressions as a single trigonometric function:

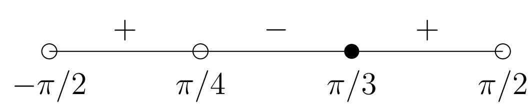
$$
\begin{aligned}
&\sqrt{3}\cos x-\sin x\\
=&r\cos\left(x-a\right)\\
=&r\cos a\cos x+r\sin a\sin x\\
r\cos a=&\sqrt{3}\\
r\sin a=&-1\\
r^2\cos^2 a=&3\\
r^2\sin^2 a=&1\\
r^2-r^2\cos^2 a=&1\\
r=&\pm 2
\end{aligned}
$$

$$
\begin{aligned}
4\cos^2 a=&3\\
4-4\cos^2 a=&1\\
8\cos^2 a-4=&2\\
\cos^2 a=&\frac{3}{4}\\
\cos a=&-\frac{\sqrt{3}}{2}\\
a=&-\frac{\pi}{6}\\
\sqrt{3}\cos x-\sin x=&2\cos\left(x+\frac{\pi}{6}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\sin x-\cos x\\
=&\frac{2}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}\sin x-\frac{\sqrt{2}}{2}\cos x\right)\\
=&\sqrt{2}\left(\sin\frac{\pi}{4}\sin x-\cos\frac{\pi}{4}\cos x\right)\\
=&-\sqrt{2}\left(\cos\frac{\pi}{4}\cos x-\sin\frac{\pi}{4}\sin x\right)\\
=&-\sqrt{2}\cos\left(\frac{\pi}{4}+x\right)
\end{aligned}
$$

$$
y=\pm\sqrt{\frac{2\cos\left(x+\frac{\pi}{6}\right)}{-\sqrt{2}\cos\left(\frac{\pi}{4}+x\right)}},\ -\frac{\pi}{2}<x<\frac{\pi}{2}
$$

$$
\begin{aligned}
&-\sqrt{2}\cdot\frac{\cos\left(x+\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{4}+x\right)}\geq 0,\ -\frac{\pi}{2}<x<\frac{\pi}{2}\\
&\frac{\cos\left(x+\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{4}+x\right)}\leq 0,\ -\frac{\pi}{2}<x<\frac{\pi}{2}\\
&\cos\left(x+\frac{\pi}{6}\right)=0,\ -\frac{\pi}{2}<x<\frac{\pi}{2}\\
&x=\frac{\pi}{3}\\
&\cos\left(\frac{\pi}{4}+x\right)=0,\ -\frac{\pi}{2}<x<\frac{\pi}{2}\\
&x=\frac{\pi}{4}
\end{aligned}
$$



$$x\in\left(\frac{\pi}{4},\frac{\pi}{3}\right]$$

Domain of inverse is range of function

$$\boxed{\text{Range: }\left(\frac{\pi}{4},\frac{\pi}{3}\right]}$$

Problem 5

(a)

$$y = \tan\left(\frac{x}{2}\right) + \cot\left(\frac{x}{2}\right)$$

Period of $\tan x$, $\cot x$ are π. There is a horizontal dilation by a factor of 2, making their respective periods 2π. The LCM of 2π and 2π is 2π, so the period of y is $\boxed{2\pi}$.

(b)

$$\begin{aligned} y &= \frac{\tan\frac{x}{3} + \tan\frac{2x}{3}}{1 - \tan\frac{x}{3}\tan\frac{2x}{3}} \\ &= \tan\left(\frac{x}{3} + \frac{2x}{3}\right) \\ &= \tan x \end{aligned}$$

$$\boxed{\text{Period: } \pi}$$

(c)

$$f(x) = \cos 2x + 3\sin\left(3x - \frac{\pi}{3}\right) - \frac{1}{2}\cot\left(\frac{4x}{5} + 1\right) + 7$$

Period $\cos 2x$

$$\begin{aligned} &= \frac{1}{2} \cdot 2\pi \\ &= \pi \end{aligned}$$

Period $3\sin\left(3x - \frac{\pi}{3}\right)$

$$\begin{aligned} &= \frac{1}{3} \cdot 2\pi \\ &= \frac{2\pi}{3} \end{aligned}$$

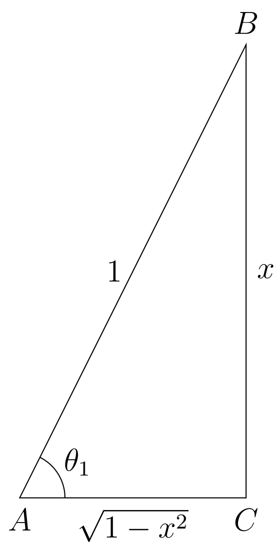
Period $\frac{1}{2}\cot\left(\frac{4x}{5} + 1\right)$

$$\begin{aligned} &= \frac{5}{4} \cdot \pi \\ &= \frac{5\pi}{4} \end{aligned}$$

$$\text{LCM of } \pi, \frac{2\pi}{3}, \frac{5\pi}{4} = 10\pi$$

$$\therefore \boxed{\text{Period: } 10\pi}$$

Problem 6



$$\text{Let } \theta = \arcsin x$$
$$\sin\theta = \sin(\arcsin x)$$
$$\frac{\text{opp}}{\text{hyp}} = x$$
$$\frac{BC}{AB} = x$$
$$\text{Let } BC = x,\ AB = 1$$
$$AC = \sqrt{1-x^2}$$

$$\tan\theta = \frac{x}{\sqrt{1-x^2}}$$

$$\tan\theta = \frac{x}{\sqrt{1-x^2}}$$
$$\tan(\arcsin x) = \frac{x}{\sqrt{1-x^2}}$$

Becuase the domain of $\arctan x$ is $\mathbb{R}$, we can safely take the arctangent of both sides.

$$\arctan(\tan(\arcsin x)) = \arctan\frac{x}{\sqrt{1-x^2}}$$
$$\arcsin x = \arctan\frac{x}{\sqrt{1-x^2}}$$

Proof. $\square$