Problem Set #13

Jayden Li

September 30, 2024

Problem 1

(c)

$$\int \frac{(\ln x)^2}{x}\,\mathrm{d}x = \begin{bmatrix} u = \ln x \\ \mathrm{d}u = \dfrac{\mathrm{d}x}{x} \end{bmatrix} \int u^2\,\mathrm{d}u = \frac{u^3}{3} = \boxed{\frac{(\ln x)^3}{3} + C}$$

(d)

$$\int \frac{\sin(\ln x)}{x}\,\mathrm{d}x = \begin{bmatrix} u = \ln x \\ \mathrm{d}u = \dfrac{\mathrm{d}x}{x} \end{bmatrix} \int \sin(u)\,\mathrm{d}u = -\cos(u) = \boxed{-\cos(\ln x) + C}$$

Problem 2

(b)

$$\int \frac{x}{36 + 25x^4}\,\mathrm{d}x = \frac{1}{36}\int \frac{x}{1 + \frac{25}{36}x^4}\,\mathrm{d}x = \begin{bmatrix} u = \dfrac{5}{6}x^2 \\ \mathrm{d}u = \dfrac{5}{3}x\,\mathrm{d}x \end{bmatrix} \frac{1}{36}\cdot\frac{3}{5}\int \frac{1}{1+u^2}\,\mathrm{d}u$$

$$= \frac{1}{60}\arctan(u) = \boxed{\frac{1}{60}\arctan\left(\frac{5}{6}x^2\right) + C}$$

(f)

$$\int \frac{\sec^2(4x)}{9 + \tan^2(4x)}\,\mathrm{d}x = \frac{1}{9}\int \frac{\sec^2(4x)}{1 + \left(\frac{\tan(4x)}{3}\right)^2}\,\mathrm{d}x = \begin{bmatrix} u = \dfrac{\tan(4x)}{3} \\ \mathrm{d}u = \dfrac{4}{3}\sec^2(4x)\,\mathrm{d}x \end{bmatrix} \frac{1}{9}\cdot\frac{3}{4}\int \frac{1}{1+u^2}\,\mathrm{d}u$$

$$= \frac{1}{12}\arctan(u) = \boxed{\frac{1}{12}\arctan\left(\frac{\tan(4x)}{3}\right) + C}$$

Problem 3

(b)

$$\int x^3\sqrt[3]{x+1}\,\mathrm{d}x = \begin{bmatrix} u = x+1 \\ \mathrm{d}u = \mathrm{d}x \end{bmatrix} \int (u-1)^3\sqrt[3]{u}\,\mathrm{d}u = \int \left(u^{3+1/3} - 3u^{2+1/3} + 3u^{1+1/3} - 1\cdot u^{1/3}\right)\mathrm{d}u$$

$$= \frac{u^{13/3}}{13/3} - \frac{3u^{10/3}}{10/3} + \frac{3u^{7/3}}{7/3} - \frac{u^{4/3}}{4/3} = \frac{3u^4\sqrt[3]{u}}{13} - \frac{9u^3\sqrt[3]{u}}{10} + \frac{9u^2\sqrt[3]{u}}{7} - \frac{3u\sqrt[3]{u}}{4}$$

$$= \boxed{\frac{3(x+1)^4\sqrt[3]{x+1}}{13} - \frac{9(x+1)^3\sqrt[3]{x+1}}{10} + \frac{9(x+1)^2\sqrt[3]{x+1}}{7} - \frac{3(x+1)\sqrt[3]{x+1}}{4} + C}$$

Problem 4

(b)

$$\int_0^{1/4} \sin^2(\pi x)\,\mathrm{d}x = \frac{1}{2}\int_0^{1/4}(1 - \cos(2\pi x))\,\mathrm{d}x = \begin{bmatrix} u = 2\pi x \\ \mathrm{d}u = 2\pi\,\mathrm{d}x \end{bmatrix} \frac{1}{2}\int_0^{1/4} 1\,\mathrm{d}x - \frac{1}{2\cdot 2\pi}\int_0^{\pi/2}\cos(u)\,\mathrm{d}u$$

$$= \frac{1}{2}[x]_0^{1/4} - \frac{1}{4\pi}[\sin u]_0^{\pi/2} = \boxed{\frac{1}{8} - \frac{1}{4\pi}}$$

Problem 5

(b)

$$\int \frac{1}{\sqrt{1+\sqrt{1+x}}}\,\mathrm{d}x = \int \frac{\sqrt{1+\sqrt{1+x}}}{1+\sqrt{1+x}}\,\mathrm{d}x = \int \frac{\sqrt{1+\sqrt{1+x}}}{2\sqrt{1+x}\left(\frac{1}{2\sqrt{1+x}} + \frac{1}{2}\right)}\,\mathrm{d}x$$

$$= \begin{bmatrix} u = \sqrt{1+x} \\ \mathrm{d}u = \dfrac{\mathrm{d}x}{2\sqrt{1+x}} \end{bmatrix} \int \frac{\sqrt{1+u}}{\frac{1}{2u} + \frac{1}{2}}\,\mathrm{d}u = \int \frac{\sqrt{1+u}}{\frac{1+u}{2u}}\,\mathrm{d}u = \int \frac{2u\sqrt{1+u}}{1+u}\,\mathrm{d}u$$

$$= 2\int \frac{u}{\sqrt{1+u}}\,\mathrm{d}u = \begin{bmatrix} v = 1+u \\ \mathrm{d}v = \mathrm{d}u \end{bmatrix} 2\int \frac{v-1}{\sqrt{v}}\,\mathrm{d}v = 2\left(\int v^{1/2}\,\mathrm{d}v - \int v^{-1/2}\,\mathrm{d}v\right)$$

$$= \frac{2v^{3/2}}{3/2} - \frac{2v^{1/2}}{1/2} = \frac{4\sqrt{(1+u)^3}}{3} - 4\sqrt{1+u}$$

$$= \boxed{\frac{4\left(1+\sqrt{1+x}\right)\sqrt{1+\sqrt{1+x}}}{3} - 4\sqrt{1+\sqrt{1+x}} + C}$$

Problem 6

(a)

$$\int_0^3 \frac{1}{\sqrt{4-x}}\,\mathrm{d}x = \begin{bmatrix} u = 4-x \\ \mathrm{d}u = -\mathrm{d}x \end{bmatrix} -\int_4^1 u^{-1/2}\,\mathrm{d}u = \left[\frac{u^{1/2}}{1/2}\right]_4^1 = 2\sqrt{4} - 2\sqrt{1} = \boxed{2}$$

(b)

$$\int_0^1 \frac{1}{\sqrt{4-x^2}}\,\mathrm{d}x = \frac{1}{2}\int_0^1 \frac{1}{\sqrt{1 - \frac{x^2}{4}}}\,\mathrm{d}x = \begin{bmatrix} u = \dfrac{x}{2} \\ \mathrm{d}u = \dfrac{\mathrm{d}x}{2} \end{bmatrix} \frac{1}{2}\cdot 2\int_0^{1/2} \frac{1}{\sqrt{1-u^2}}\,\mathrm{d}u = [\arcsin u]_0^{1/2} = \boxed{\frac{\pi}{6}}$$

(c)

$$\int_0^{\sqrt{3}} \frac{x}{9+x^4}\,\mathrm{d}x = \frac{1}{9}\int_0^{\sqrt{3}} \frac{x}{1+\frac{x^4}{9}}\,\mathrm{d}x = \begin{bmatrix} u = \dfrac{x^2}{3} \\ \mathrm{d}u = \dfrac{2x}{3}\,\mathrm{d}x \end{bmatrix} \frac{1}{9}\cdot\frac{3}{2}\int_0^1 \frac{1}{1+u^2}\,\mathrm{d}u$$

$$= \frac{1}{6}[\arctan u]_0^1 = \frac{1}{6}\left(\frac{\pi}{4} - 0\right) = \boxed{\frac{\pi}{24}}$$

(d)

$$\int_0^1 \frac{x^9}{1+x^{20}}\,\mathrm{d}x = \begin{bmatrix} u = x^{10} \\ \mathrm{d}u = 10x^9\,\mathrm{d}x \end{bmatrix} \frac{1}{10}\int_0^1 \frac{1}{1+u^2}\,\mathrm{d}u = \frac{1}{10}[\arctan u]_0^1 = \frac{1}{10}\left(\frac{\pi}{4} - 0\right) = \boxed{\frac{\pi}{40}}$$

(e)

$$\int_0^2 \frac{2x}{9+x^2}\,\mathrm{d}x = \begin{bmatrix} u = 9+x^2 \\ \mathrm{d}u = 2x\,\mathrm{d}x \end{bmatrix} \int_9^{13} \frac{1}{u}\,\mathrm{d}u = [\ln u]_9^{13} = \boxed{\ln 13 - \ln 9}$$

(f)

$$\int \frac{\arctan 2x}{1+4x^2}\,\mathrm{d}x = \begin{bmatrix} u = \arctan 2x \\ \mathrm{d}u = \dfrac{2}{1+4x^2}\,\mathrm{d}x \end{bmatrix} \frac{1}{2}\int u\,\mathrm{d}u = \frac{u^2}{4} = \boxed{\frac{(\arctan 2x)^2}{4} + C}$$

(g) The function is not continuous because of a discontinuity at $x = \pi^2/4$, so the integral is $\boxed{\text{undefined}}$.