Problem Set #43

Jayden Li

January 19, 2024

Problem 2

$$
\begin{aligned}
(x+y)^5 + (x-y)^5 &= \sum_{k=0}^{5} \binom{5}{k} x^{5-k} y^k + \sum_{k=0}^{5} \binom{5}{k} x^{5-k} (-y)^k \\
&= \sum_{k=0}^{5} \left(\binom{5}{k} x^{5-k} y^k + \binom{5}{k} x^{5-k} (-1)^k y^k \right) \\
&= \sum_{k=0}^{5} \binom{5}{k} x^{5-k} y^k \left(1 + (-1)^k\right) \\
&= \sum_{1 \le k \le 5, k \text{ odd}} \binom{5}{k} x^{5-k} y^k \left(1 + (-1)^k\right) + \sum_{0 \le k \le 4, k \text{ even}} \binom{5}{k} x^{5-k} y^k \left(1 + (-1)^k\right) \\
&= \sum_{1 \le k \le 5, k \text{ odd}} \binom{5}{k} x^{5-k} y^k (0) + \sum_{0 \le k \le 4, k \text{ even}} \binom{5}{k} x^{5-k} y^k (2) \\
&= 2 \sum_{0 \le k \le 4, k \text{ even}} \binom{5}{k} x^{5-k} y^k \\
&= 2 \left(\binom{5}{0} x^{5-0} y^0 + \binom{5}{2} x^{5-2} y^2 + \binom{5}{4} x^{5-4} y^4 \right) \\
&= 2 \left(x^5 + 10x^3y^2 + 5xy^4 \right) \\
&= 2x^5 + 20x^3y^2 + 10xy^4
\end{aligned}
$$

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\begin{aligned}
\left(\sqrt{2}+1\right)^5 + \left(\sqrt{2}-1\right)^5 &= 2\left(\sqrt{2}\right)^5 + 20\left(\sqrt{2}\right)^3 (1)^2 + 10\left(\sqrt{2}\right)(1)^4 \\
&= 2\left(\sqrt{2}\right)^4 \sqrt{2} + 20\left(\sqrt{2}\right)^2 \sqrt{2} + 10\sqrt{2} \\
&= 8\sqrt{2} + 40\sqrt{2} + 10\sqrt{2} \\
&= \boxed{58\sqrt{2}}
\end{aligned}
$$

Problem 3

Let $a = \sqrt{x+1}, b = \sqrt{x-1}$.

$$
\begin{aligned}
&\left(\sqrt{x+1}+\sqrt{x-1}\right)^6 + \left(\sqrt{x+1}-\sqrt{x-1}\right)^6,\ x \geq 1 \\
=& (a+b)^6 + (a-b)^6 \\
=& \sum_{k=0}^{6} \left(\binom{6}{k} a^{6-k} b^k + \binom{6}{k} a^{6-k} (-1)^k b^k\right) \\
=& \sum_{k=0}^{6} \binom{6}{k} a^{6-k} b^k \left(1+(-1)^k\right) \\
=& \sum_{1 \leq k \leq 5, k \text{ odd}} \binom{6}{k} a^{6-k} b^k \left(1+(-1)^k\right) + \sum_{0 \leq k \leq 6, k \text{ even}} \binom{6}{k} a^{6-k} b^k \left(1+(-1)^k\right) \\
=& \sum_{1 \leq k \leq 5, k \text{ odd}} \binom{6}{k} a^{6-k} b^k (0) + \sum_{0 \leq k \leq 6, k \text{ even}} \binom{6}{k} a^{6-k} b^k (2) \\
=& 2 \sum_{0 \leq k \leq 6, k \text{ even}} \binom{6}{k} a^{6-k} b^k \\
=& 2\left(\binom{6}{0} a^{6-0} b^0 + \binom{6}{2} a^{6-2} b^2 + \binom{6}{4} a^{6-4} b^4 + \binom{6}{6} a^{6-6} b^6\right) \\
=& 2\left(a^6 + 15a^4b^2 + 15a^2b^4 + b^6\right) \\
=& 2a^6 + 30a^4b^2 + 30a^2b^4 + 2b^6 \\
=& 2\left(\sqrt{x+1}\right)^6 + 30\left(\sqrt{x+1}\right)^4\left(\sqrt{x-1}\right)^2 + 30\left(\sqrt{x+1}\right)^2\left(\sqrt{x-1}\right)^4 + 2\left(\sqrt{x-1}\right)^6 \\
=& 2(x+1)^3 + 30(x+1)^2(x-1) + 30(x+1)(x-1)^2 + 2(x-1)^3 \\
=& 2(x+1)^3 + 30(x+1)(x+1)(x-1) + 30(x+1)(x-1)(x-1) + 2(x-1)^3 \\
=& 2\left((x+1)^3 + (x-1)^3\right) + 30(x+1)(x-1)((x+1)+(x-1)) \\
=& 2\left(x^3 + 3x^2 + 3x + 1 + x^3 - 3x^2 + 3x - 1\right) + 30(x^2-1)(2x) \\
=& 2\left(2x^3 + 6x\right) + 60x(x^2-1) \\
=& 4x^3 + 12x + 60x^3 - 60x \\
=& \boxed{64x^3 - 48x,\ x \geq 1}
\end{aligned}
$$

Problem 5

(a)

$$\begin{aligned}
\left(ax^2 + \frac{1}{bx}\right)^{11} &= \left(ax^2 + \frac{1}{b}x^{-1}\right)^{11} \\
&= \sum_{k=0}^{11} \binom{11}{k} \left(ax^2\right)^{11-k} \left(\frac{1}{b}x^{-1}\right)^k
\end{aligned}$$

Let $\deg(P)$ be the degree of a polynomial P. We will find k s.t. $\deg\left(\binom{11}{k}\left(ax^2\right)^{11-k}\left(\frac{1}{b}x^{-1}\right)^k\right) = 7$.

$$\begin{aligned}
\deg\left(\binom{11}{k}\left(ax^2\right)^{11-k}\left(\frac{1}{b}x^{-1}\right)^k\right) &= 7 \\
\deg\left(\left(x^2\right)^{11-k}\left(x^{-1}\right)^k\right) &= 7 \\
\deg\left(x^{22-2k} \cdot x^{-k}\right) &= 7 \\
\deg\left(x^{22-2k+(-k)}\right) &= 7 \\
22 - 3k &= 7 \\
3k &= 15 \\
k &= 5
\end{aligned}$$

The term with $k = 5$ is given by:

$$\begin{aligned}
\binom{11}{5} a^{11-5} \left(\frac{1}{b}\right)^5 &= \frac{11!}{5! \cdot 6!} \cdot a^6 \cdot \frac{1}{b^5} \\
&= \frac{11 \cdot \cancel{10} \cdot \cancel{9}^{3} \cdot \cancel{8}^{2} \cdot 7}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2}} \cdot \frac{a^6}{b^5} \\
&= 11 \cdot 3 \cdot 2 \cdot 7 \cdot \frac{a^6}{b^5} \\
&= \boxed{\frac{462a^6}{b^5}}
\end{aligned}$$

(b) Let Midterm(P) be the middle term of the polynomial P, and let $P = (a+b)^n$ where $n \in \mathbb{Z}^+$. By the binomial theorem, we have that:

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

For even n, the middle term of the above sum is $\dfrac{n}{2}$. For odd n, there are 2 middle terms: $\dfrac{n-1}{2}$ and $\dfrac{n+1}{2}$. Thus Midterm$\left((a+b)^n\right)$ is multi-valued for odd n and single-valued for even n.

For even n:

$$\begin{aligned}
\text{Midterm}\left((a+b)^n\right) &= \binom{n}{\frac{n}{2}} a^{n-\frac{n}{2}} b^{\frac{n}{2}} \\
&= \frac{n!}{\left(\frac{n}{2}\right)!\left(n-\frac{n}{2}\right)!} \cdot a^{\frac{n}{2}} b^{\frac{n}{2}} \\
&= \frac{n!(ab)^{\frac{n}{2}}}{\left(\left(\frac{n}{2}\right)!\right)^2}
\end{aligned}$$

For odd n:

$$\begin{aligned}
\text{Midterm}\left((a+b)^n\right) &= \left\{ \frac{n!}{\left(\frac{n-1}{2}\right)!\left(n-\frac{n-1}{2}\right)!} \cdot a^{n-\frac{n-1}{2}} b^{\frac{n-1}{2}},\ \frac{n!}{\left(\frac{n+1}{2}\right)!\left(n-\frac{n+1}{2}\right)!} \cdot a^{n-\frac{n+1}{2}} b^{\frac{n+1}{2}} \right\} \\
&= \left\{ \frac{n!}{\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!} \cdot a^{\frac{n+1}{2}} b^{\frac{n-1}{2}},\ \frac{n!}{\left(\frac{n+1}{2}\right)!\left(\frac{n-1}{2}\right)!} \cdot a^{\frac{n-1}{2}} b^{\frac{n+1}{2}} \right\}
\end{aligned}$$

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i.

$$\begin{aligned}
\text{Midterm}\left(\left(1-\frac{x^2}{2}\right)^{14}\right) &= \frac{14!\left(1\left(-\frac{x^2}{2}\right)\right)^{\frac{14}{2}}}{\left(\left(\frac{14}{2}\right)!\right)^2} \\
&= \frac{14!\left(-\frac{x^2}{2}\right)^7}{(7!)^2} \\
&= -\frac{\cancelto{2}{14} \cdot 13 \cdot \cancelto{2}{12} \cdot 11 \cdot \cancelto{2}{10} \cdot \cancelto{3}{9} \cdot \cancelto{2}{8}}{\cancel{7} \cdot \cancel{6} \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot 1} \cdot \frac{x^{14}}{\cancel{2} \cdot \cancel{2} \cdot \cancel{2} \cdot 2^4} \\
&= -\frac{13 \cdot 11 \cdot 3 \cdot x^{14}}{16} \\
&= \boxed{-\frac{429x^{14}}{16}}
\end{aligned}$$

ii.

$$
\begin{aligned}
\text{Midterm}\left(3a-\frac{a^3}{6}\right)^9 &= \left\{\frac{9!}{\left(\frac{9-1}{2}\right)!\left(\frac{9+1}{2}\right)!}\cdot(3a)^{\frac{9+1}{2}}\left(-\frac{a^3}{6}\right)^{\frac{9-1}{2}},\ \frac{9!}{\left(\frac{9+1}{2}\right)!\left(\frac{9-1}{2}\right)!}\cdot(3a)^{\frac{9-1}{2}}\left(-\frac{a^3}{6}\right)^{\frac{9+1}{2}}\right\} \\
&= \left\{\frac{9!}{4!\cdot 5!}\cdot(3a)^5\left(-\frac{a^3}{6}\right)^4,\ \frac{9!}{5!\cdot 4!}\cdot(3a)^4\left(-\frac{a^3}{6}\right)^5\right\} \\
&= \left\{9\cdot 7\cdot 2\cdot(3a)^5\left(\frac{a^{12}}{6^4}\right),\ 9\cdot 7\cdot 2\cdot(3a)^4\left(-\frac{a^{15}}{6^5}\right)\right\} \\
&= \left\{9\cdot 7\cdot 2\cdot\cancel{3}\cdot\cancel{3}\cdot\cancel{3}\cdot\cancel{3}\cdot 3\cdot\frac{a^{17}}{\cancelto{2}{3}\cdot\cancelto{2}{3}\cdot\cancelto{2}{3}\cdot\cancelto{2}{3}},\ -9\cdot 7\cdot 2\cdot\cancel{3}\cdot\cancel{3}\cdot\cancel{3}\cdot\cancel{3}\cdot\frac{a^{19}}{\cancelto{2}{3}\cdot\cancelto{2}{3}\cdot\cancelto{2}{3}\cdot\cancelto{2}{3}\cdot 6}\right\} \\
&= \left\{63\cdot 3\cdot 2\cdot\frac{a^{17}}{2\cdot 8},\ -63\cdot 2\cdot\frac{a^{19}}{2\cdot 48}\right\} \\
&= \left\{\frac{189a^{17}}{8},\ -\frac{63a^{19}}{48}\right\} \\
&= \boxed{\left\{\frac{189a^{17}}{8},\ -\frac{21a^{19}}{16}\right\}}
\end{aligned}
$$

Problem 6

$$
\begin{aligned}
\binom{n}{1}x^{n-1}a^1 = 240 &\implies \frac{n\cancel{(n-1)!}}{1!\cancel{(n-1)!}}\cdot x^{n-1}a = 240 \implies nx^{n-1}a = 240 \\
\binom{n}{2}x^{n-2}a^2 = 720 &\implies \frac{n(n-1)\cancel{(n-2)!}}{2!\cancel{(n-2)!}}\cdot x^{n-2}a^2 = 720 \implies n(n-1)x^{n-2}a^2 = 1440 \\
\binom{n}{3}x^{n-3}a^3 = 1080 &\implies \frac{n(n-1)(n-2)\cancel{(n-3)!}}{3!\cancel{(n-3)!}}\cdot x^{n-3}a^3 = 1080 \implies n(n-1)(n-2)x^{n-3}a^3 = 6480
\end{aligned}
$$

$$
\begin{aligned}
\frac{\cancel{n}(n-1)x^{n-2}a^{\cancel{2}}}{\cancel{n}x^{n-1}\cancel{a}} &= \frac{1440}{240} \\
\frac{a(n-1)\cancel{x^{n-2}}}{x\cancel{x^{n-2}}} &= 6 \\
\frac{a(n-1)}{x} &= 6 \\
\frac{a(n-1)}{6} &= x
\end{aligned}
$$

$$
\begin{aligned}
\frac{\cancel{n(n-1)}(n-2)x^{n-3}a^{\cancel{3}}}{\cancel{n(n-1)}x^{n-2}\cancel{a^2}} &= \frac{6480}{1440} \\
\frac{(n-2)\cancel{x^{n-3}}a}{x\cancel{x^{n-3}}} &= \frac{9}{2} \\
\frac{(n-2)a}{x} &= \frac{9}{2} \\
\frac{2a(n-2)}{9} &= x
\end{aligned}
$$

$$
\begin{aligned}
\frac{(n-1)a}{6} &= \frac{2a(n-2)}{9} \\
9\cancel{a}(n-1) &= 12\cancel{a}(n-2) \\
9n-9 &= 12n-24 \\
-3n &= -15 \\
\boxed{n=5}
\end{aligned}
$$

$$
\begin{aligned}
nx^{n-1}a = 240 &\implies 5x^4a = 240 \implies 25x^8a^2 = 57600 \\
n(n-1)x^{n-2}a^2 = 1440 &\implies 20x^3a^2 = 1440 \\
n(n-1)(n-2)x^{n-3}a^3 = 6480 &\implies 60x^2a^3 = 6480
\end{aligned}
$$

$$\frac{25x^{\overset{5}{\cancel{8}}}\cancel{a^2}}{20\cancel{x^3}\cancel{a^2}} = \frac{57600}{1440} \qquad 5(2)^4a = 240$$

$$\frac{5}{4}x^5 = 40 \qquad 80a = 240$$

$$x^5 = 32 \qquad \boxed{a = 3}$$

$$\boxed{x = 2}$$

Problem 8

$$\begin{aligned}
\left(\sqrt{2}+1\right)^6 &= \left(1+\sqrt{2}\right)^6 \\
&= \sum_{k=0}^{6} \binom{6}{k} 1^{6-k} \left(\sqrt{2}\right)^k \\
&= \sum_{k=0}^{6} \binom{6}{k} \left(\sqrt{2}\right)^k \\
&= \sum_{0\le k\le 6, k \text{ even}} \binom{6}{k} \left(\sqrt{2}\right)^k + \sum_{1\le k\le 5, k \text{ odd}} \binom{6}{k} \left(\sqrt{2}\right)^k \\
&= \sum_{i=0}^{3} \binom{6}{2i} \left(\sqrt{2}\right)^{2i} + \sum_{i=0}^{2} \binom{6}{2i+1} \left(\sqrt{2}\right)^{2i+1} \\
&= \sum_{i=0}^{3} \binom{6}{2i} 2^i + \sum_{i=0}^{2} \binom{6}{2i+1} 2^i \sqrt{2} \\
&= \binom{6}{0} + \binom{6}{2}2 + \binom{6}{4}4 + \binom{6}{6}8 + \binom{6}{1}\sqrt{2} + \binom{6}{3}2\sqrt{2} + \binom{6}{5}4\sqrt{2} \\
&= 1 + 15\cdot 2 + 15\cdot 4 + 8 + 6\sqrt{2} + 20\cdot 2\sqrt{2} + 6\cdot 4\sqrt{2} \\
&= 99 + 70\sqrt{2} \\
&= 99 + \sqrt{9800}
\end{aligned}$$

$\left\lfloor\sqrt{9800}\right\rfloor$ is the square root of the smallest perfect square under 9800. $99^2 = 9801$ and $98^2 = 9154$. Thus $\lfloor 9800 \rfloor = 98$, so $\left\lfloor 99 + \sqrt{9800}\right\rfloor = 99 + 98 = \boxed{197}$.