Problem Set #65

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Problem 2

(a) At $x = 4$, where the f goes from decreasing to increasing. By the first derivative test it is a local minimum.

(b) We need to compare whether the function decreases more where the derivative is a semicircle, or whether it increases more. We can find out by calculating the area under or above the graph, which represents the quantity of change accumulated by f'.

The area above the semicircle is $2^2 \cdot \pi/2 = 2\pi \approx 6.28$, so the absolute difference between $f(0)$ and $f(4)$ is approximately 6.28. Because the function is decreasing, we see that $f(0)$ is approximately 6.28 higher than $f(4)$.

The area below the line is the area of a triangle with legs of length 2 and 6. Its area is $2 \cdot 6/2 = 6$. Therefore $f(10)$ is 6 higher than $f(4)$.

It is known that $f(10) = 2$. Therefore, we see that $f(4) = 2 - 6 = -4$.

We found earlier that $f(0) - 6.28 \approx f(4)$, so $f(0) \approx -4 + 6.28 = 2.28$.

Because $2.28 > 2$, f reaches an absolute maximum at $x = 0$.

Problem 3

R has an absolute maximum at $t = 2.29$ and $R(2.29) = 3.95$. Therefore the fastest rate in which water fills into the rank is 3.95 gallons per hour.

Problem 5

(a) If $(1, -2)$ were a critical point, then the derivative evaluated at $x = 1, y = -2$ is zero or not differentiable at that point (definition of a critical point). $2(1) + (-2) = 0$, so it is a critical point.

(b) We use the second derivative test. We first find the second derivative $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left[2x + y\right] = 2 + \frac{\mathrm{d}y}{\mathrm{d}x} = 2 + 2x + y$$

The second derivative evaluated at $(1, -2)$ is $2 + 2(1) + (-2) = 2 > 0$. Thus, by the second derivative test the point $(1, -2)$ is a local minimum.

Problem 6

(a) An inflection point is where the function f changes from concave down to concave up, or from concave up to concave down (definition of inflection point). A point is concave up if the second derivative f'' is greater than 0 at that point, and concave down if f'' is less than 0 (definition of concavity). So we can say that an inflection point is where the second derivative f'' changes sign – in other words, where the first derivative f' changes from increasing to decreasing, or from decreasing to increasing. On the graph of f', we see that this happens at $x = 2$ and $x = 6$, which are the inflection points of f.

(b)

$$g(x) = f(x) - x$$
$$g'(x) = f'(x) - 1$$

Decreasing on $(0, 5)$, which is where the graph of f' translated down 1 unit is negative.

(c) Suppose that $c \in [0, 7]$ is a critical point of g. Then g is either not differentiable at c, or $g'(c) = 0$. f is differentiable on $[0, 7]$. x is differentiable for all real numbers. Therefore, g is differentiable on $[0, 7]$, so there is no point c where g is not differentiable. So all critical points c are such that $g'(c) = 0$.

$$g'(c) = 0$$
$$f'(x) - 1 = 0$$
$$f'(c) = 1$$

By looking at the graph of f', c must equal 5. So we have a critical point at $x = 5$.

By the closed interval method, the absolute minimum is either at the endpoints (which are $x = 0$ and $x = 7$), or at critical points ($x = 5$). If we were to draw the graph of g', it is obvious that the area above the semicircle connected to a line segment is greater than the area under the triangle. So I claim that the absolute minimum occurs at $x = 4$.

It is known that $f(4) = 3$, so $g(4) = f(4) - 4 = 3 - 4 = -1$. The absolute maximum of g on the interval $[0, 7]$ is -1.

Problem 7

(a)

$$H'(6) \approx \frac{H(7) - H(5)}{7 - 5} = \frac{11 - 6}{2} = \frac{5}{2}$$

The rate of growth of the tree 6 years after being grown, in meters per year.

(b) Consider the interval $[3, 5]$. The average rate of change of H over this interval is given by:

$$\frac{H(5) - H(3)}{5 - 3} = \frac{6 - 2}{2} = 2$$

H is differentiable on $(3, 5)$ (given by the problem), so it must be continuous on $[3, 5]$. Then there must exist some $c \in (3, 5)$ such that $H'(c) = 2$ (by the mean value theorem).

Problem 8

(a) Assuming that r' has a constant rate of change on $[7, 10]$, $r''(8.5)$ will equal the average rate of change on $[7, 10]$.

$$r''(8.5) = \frac{r'(10) - r'(7)}{10 - 7} = \frac{-3.8 - (4.4)}{3} = \frac{0.6}{3} = 0.2$$

(b) Yes by the intermediate value theorem.

Because r is twice-differentiable, r' must be differentiable and therefore continuous on $[0, 3]$.

Let $c = -6$. We have that $c \in (r'(0), r'(3))$. By the intermediate value theorem, $r'(t) = c$ for some $t \in (0, 3)$.

Problem 9

(a) f is increasing on $[-6, -2) \cup (2, 5)$ because that is where f' is positive.

(b) Critical points are $x = -2$ and $x = 2$. Endpoints are $x = -6$ and $x = 5$. f' exists for all values on $[-6, 5]$ so f must also be continuous on $[-6, 5]$ so we can use the closed interval method. Because I don't want to think of an easier way, we use integration to calculate $f(-6)$, $f(2)$ and $f(5)$.

$$f(-2) = 7$$

$$\int_{-6}^{-2} f'(x)\,\mathrm{d}x = f(-2) - f(-6) \implies 4 = 7 - f(-6) \implies f(-6) = 3$$

$$\int_{-2}^{2} f'(x)\,\mathrm{d}x = f(2) - f(-2) \implies -2\pi = f(2) - 7 \implies f(2) = 7 - 2\pi$$

$$\int_{-2}^{5} f'(x)\,\mathrm{d}x = f(5) - f(-2) \implies -2\pi + 3 = f(5) - 7 \implies f(5) = 10 - 2\pi$$

The absolute minimum value is $7 - 2\pi$ because it is less than all the other points.

(c) $f''(3)$ doesn't exist because f' has a sharp corner at $x = 3$ and sharp corners are not differentiable.

$f''(5)$ doesn't exist. Since f' is continuous on the closed interval $[-6, 5]$, f' is differentiable on the open interval $(-6, 5) \not\ni 5$.