Problem Set #66

Jayden Li

April 27, 2024

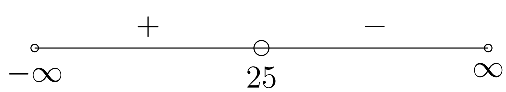
Problem 2

Let h, w be the height and width of the rectangle, respectively. Let $f(w)$ be the area of a rectangle.

$$\begin{aligned} 2(h+w) &= 100 \\ h &= 50 - w \end{aligned}$$

$$\begin{aligned} f(w) &= hw \\ f(w) &= w(50-w) \end{aligned}$$

$$\begin{aligned} f'(w) = \frac{\mathrm{d}}{\mathrm{d}w}\left[50w - w^2\right] 0 &= 0 \\ 50 - 2w &= 0 \\ w &= 25 \end{aligned}$$



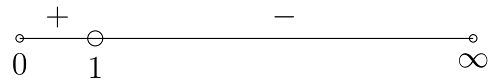
Because $f'(c) > 0$ for all $c \in (-\infty, 25)$ and $f'(c) < 0$ for all $c \in (25, \infty)$, the absolute maximum of f must be at $w = 25$.

$$f(25) = 25(50-25) = \boxed{625}$$

Problem 4

$$\begin{aligned} \frac{\mathrm{d}Y}{\mathrm{d}N} = \frac{\mathrm{d}}{\mathrm{d}N}\left[\frac{kN}{1+N^2}\right] &= 0 \\ \frac{k\left(1+N^2\right) - kN\left(2N\right)}{\left(1+N^2\right)^2} &= 0 \\ k + kN^2 - 2kN^2 &= 0 \\ kN^2 &= k \\ N^2 &= 1 \\ N &= \pm 1 \end{aligned}$$

Nitrogen level is positive so we discard the negative case.



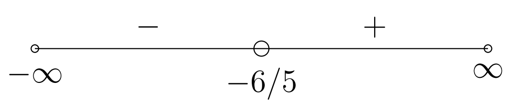
Because $f'(c) > 0$ for all $c \in (0, 1)$ and $f'(c) < 0$ for all $c \in (1, \infty)$, the absolute maximum of f must be at $\boxed{N = 1}$.

Problem 6

Let $f(x)$ be the distance of the point on the line with a given x coordinate to the origin.

$$\begin{aligned} f(x) &= \sqrt{x^2 + y^2} \\ &= \sqrt{x^2 + (2x+3)^2} \\ &= \sqrt{x^2 + 4x^2 + 12x + 9} \\ &= \sqrt{5x^2 + 12x + 9} \end{aligned}$$

$$\begin{aligned} f'(x) = \frac{1}{2\sqrt{5x^2 + 12x + 9}} \cdot (10x + 12) &= 0 \\ 10x + 12 &= 0 \\ x &= -\frac{6}{5} \end{aligned}$$



Because $f'(c) < 0$ for all $c \in (-\infty, -6/5)$ and $f'(c) > 0$ for all $c \in (-6/5, \infty)$, the absolute minimum of f must be at $x = -6/5$.

$$f\left(-\frac{6}{5}\right) = \sqrt{\frac{36}{5} - \frac{72}{5} + 9} = \sqrt{\frac{36 - 72 + 45}{5}} = \sqrt{\frac{9}{5}} = \frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \boxed{\frac{3\sqrt{5}}{5}}$$

Problem 8

Let w, h be the width and height of the rectangle, respectively. As side lengths, w and h must be positive ($w, h > 0$). We see that the bases of the rectangle and the radius of the circle form a right triangle with legs $w/2$ and $h/2$ and hypotenuse r.

$$\begin{aligned} \left(\frac{w}{2}\right)^2 + \left(\frac{h}{2}\right)^2 &= r^2 \\ \frac{w^2}{4} + \frac{h^2}{4} &= r \\ h^2 &= 4r - w^2 \\ h &= \pm\sqrt{4r - w^2} \end{aligned}$$

h must be positive.

$$h = \sqrt{4r - w^2}$$

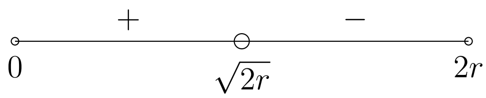
Let $f(w)$ be the area of the rectangle in terms of width.

$$\begin{aligned} f(w) &= wh \\ &= w\sqrt{4r - w^2} \end{aligned}$$

$$\begin{aligned} f'(w) = \sqrt{4r - w^2} + w \cdot \frac{1}{2\sqrt{4r-w^2}} \cdot (-2w) &= 0 \\ \sqrt{4r-w^2} &= \frac{\cancel{2}w^2}{\cancel{2}\sqrt{4r-w^2}} \\ 4r - w^2 &= w^2 \\ 2w^2 &= 4r \\ w &= \pm\sqrt{2r} \end{aligned}$$

w must be positive.

$$w = \sqrt{2r}$$



Because $f'(c) > 0$ for all $c \in (0, \sqrt{2r})$ and $f'(c) < 0$ for all $c \in (\sqrt{2r}, 2r)$, the absolute maximum of f must be at $w = \sqrt{2r}$.

$$f\left(\sqrt{2r}\right) = \sqrt{2r} \cdot \sqrt{4r - \left(\sqrt{2r}\right)^2} = \sqrt{2r}\sqrt{4r-2r} = \boxed{2r}$$

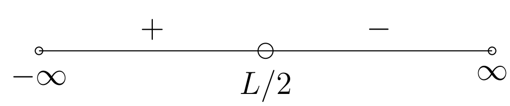
Problem 9

Let h, w be the height and width of the rectangle, respectively, and let $f(w)$ be the area of the rectangle.

$$\begin{aligned} \frac{\sqrt{L^2 - \left(\frac{L}{2}\right)^2} - h}{\sqrt{L^2 - \left(\frac{L}{2}\right)^2}} &= \frac{w/2}{L/2} \\ \frac{\sqrt{\frac{3L^2}{4}} - h}{\sqrt{\frac{3L^2}{4}}} &= \frac{w}{L} \\ \cancel{L}\left(\frac{L\sqrt{3}}{2} - h\right) &= w \cdot \frac{\cancel{L}\sqrt{3}}{2} \\ \frac{L\sqrt{3}}{2} - h &= \frac{w\sqrt{3}}{2} \\ h &= \frac{L\sqrt{3}}{2} - \frac{w\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} f(w) &= hw \\ f(w) &= \frac{Lw\sqrt{3}}{2} - \frac{w^2\sqrt{3}}{2} \end{aligned}$$

$$\begin{aligned} f'(w) = \frac{L\sqrt{3}}{2} - w\sqrt{3} &= 0 \\ w &= \frac{L}{2} \end{aligned}$$



Because $f'(c) > 0$ for all $c \in (-\infty, L/2)$ and $f'(c) < 0$ for all $c \in (L/2, \infty)$, the absolute maximum of f must be at $w = L/2$.

$$f\left(\frac{L}{2}\right) = \frac{L^2\sqrt{3}}{4} - \frac{L^2\sqrt{3}}{8} = \frac{2L^2\sqrt{3}}{8} - \frac{L^2\sqrt{3}}{8} = \boxed{\frac{L^2\sqrt{3}}{8}}$$

Problem 10

Let w be the base length of the isosceles triangle. We see that its height, h, can be expressed as:

$$h = r + \sqrt{r^2 - \left(\frac{w}{2}\right)^2} = r + \sqrt{r^2 - \frac{w^2}{4}}$$

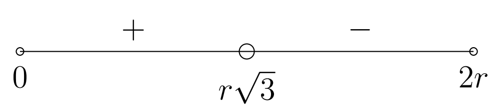
Let $f(w)$ be the area of the triangle in terms of width. w and h must be positive $(h, w > 0)$.

$$\begin{aligned} f(w) &= \frac{hw}{2} \\ &= \frac{1}{2}w\left(r + \sqrt{r^2 - \frac{w^2}{4}}\right) \end{aligned}$$

$$\begin{aligned} f'(w) = \frac{1}{2} \cdot \frac{\mathrm{d}}{\mathrm{d}w}\left[w\left(r + \sqrt{r^2 - \frac{w^2}{4}}\right)\right] &= 0 \\ \frac{1}{2}\left(r + \sqrt{r^2 - \frac{w^2}{4}} + w\left(\frac{1}{2\sqrt{r^2 - \frac{w^2}{4}}}\right)\left(-\frac{w}{2}\right)\right) &= 0 \\ r + \sqrt{r^2 - \frac{w^2}{4}} &= \frac{w^2}{4\sqrt{r^2 - \frac{w^2}{4}}} \\ r\sqrt{r^2 - \frac{w^2}{4}} + r^2 - \frac{w^2}{4} &= \frac{w^2}{4} \\ r\sqrt{r^2 - \frac{w^2}{4}} &= \frac{w^2}{2} - r^2 \\ r^2\left(r^2 - \frac{w^2}{4}\right) &= \frac{w^4}{4} - r^2w^2 + r^4 \\ \cancel{r^4} - \frac{r^2w^2}{4} &= \frac{w^4}{4} - r^2w^2 + \cancel{r^4} \\ \frac{3r^2w^2}{4} &= \frac{w^4}{4} \\ 3r^2 &= w^2 \\ w &= \pm r\sqrt{3} \end{aligned}$$

Discard negative case since $w > 0$.

$$w = r\sqrt{3}$$



Because $f'(c) > 0$ for all $c \in (0, r\sqrt{3})$ and $f'(c) < 0$ for all $c \in (r\sqrt{3}, 2r)$, the absolute maximum of f must be at $w = r\sqrt{3}$.

$$f\left(r\sqrt{3}\right) = \frac{1}{2}r\sqrt{3}\left(r + \sqrt{r^2 - \frac{\left(r\sqrt{3}\right)^2}{4}}\right) = \frac{1}{2}r\sqrt{3}\left(r + \sqrt{\frac{4r^2}{4} - \frac{3r^2}{4}}\right) = \frac{1}{2}r\sqrt{3}\left(r + \left|\frac{r}{2}\right|\right)$$

Because r is positive, its absolute value equals itself.

$$= \frac{r\sqrt{3}}{2} \cdot \frac{2r+r}{2} = \boxed{\frac{3r^2\sqrt{3}}{4}}$$

Problem 11

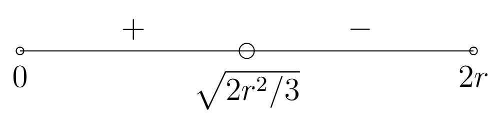
Let x be the base radius of the cylinder and let h be the height of the cylinder. Notice that x and h must be positive. h can be expressed in terms of x and r:

$$h = 2\sqrt{r^2 - x^2}$$

Let $f(x)$ be the volume of the cylinder.

$$\begin{aligned} f(x) &= \pi x^2 h \\ &= 2\pi x^2 \sqrt{r^2 - x^2} \end{aligned}$$

$$\begin{aligned} f'(x) = 2\pi \left(2x\sqrt{r^2 - x^2} + x^2 \cdot \frac{1}{2\sqrt{r^2 - x^2}} \cdot (-2x) \right) &= 0 \\ 2x\sqrt{r^2 - x^2} - \frac{2x^3}{2\sqrt{r^2 - x^2}} &= 0 \\ 2x\left(r^2 - x^2\right) &= \frac{2x^3}{2} \\ 2r^2x - 2x^3 &= x^3 \\ 3x^2 &= 2r^2 \\ x &= \sqrt{\frac{2r^2}{3}} \end{aligned}$$



Because $f'(c) > 0$ for all $c \in (0, \sqrt{2r^2/3})$ and $f'(c) < 0$ for all $c \in (\sqrt{2r^2/3}, 2r)$, the absolute maximum of f must be at $x = \sqrt{2r^2/3}$.

$$\begin{aligned} f\left(\sqrt{\frac{2r^2}{3}}\right) &= 2\pi \left(\sqrt{\frac{2r^2}{3}}\right)^2 \sqrt{r^2 - \left(\sqrt{\frac{2r^2}{3}}\right)^2} = 2\pi \left(\frac{2r^2}{3}\right) \sqrt{r^2 - \frac{2r^2}{3}} = \frac{4\pi r^2}{3} \sqrt{\frac{r^2}{3}} \\ &= \frac{4\pi r^2}{3} \cdot \frac{r}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \boxed{\frac{4\pi\sqrt{3}r^3}{9}} \end{aligned}$$

Problem 12

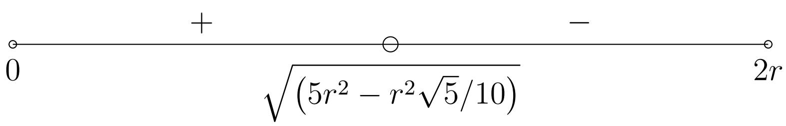
Use the same variables and notation as before. The only difference is a different definition of f.

$$\begin{aligned} f(x) &= 2\pi x^2 + 2\pi x h \\ &= 2\pi x^2 + 4\pi x \sqrt{r^2 - x^2} \end{aligned}$$

$$f'(x) = 4\pi x + 4\pi\sqrt{r^2 - x^2} + 4\pi x \cdot \frac{1}{2\sqrt{r^2 - x^2}} \cdot (-2x) = 0$$

$$
\begin{aligned}
4\pi x + 4\pi\sqrt{r^2 - x^2} - \frac{8\pi x^2}{2\sqrt{r^2 - x^2}} &= 0 \\
\frac{r^2 - x^2}{\sqrt{r^2 - x^2}} - \frac{x^2}{\sqrt{r^2 - x^2}} &= -x \\
r^2 - 2x^2 &= -x\sqrt{r^2 - x^2} \\
r^4 - 4r^2x^2 + 4x^4 &= x^2\left(r^2 - x^2\right) \\
r^4 - 4r^2x^2 + 4x^4 &= x^2r^2 - x^4 \\
5x^4 - 5r^2x^2 + r^4 &= 0 \\
x^2 &= \frac{5r^2 \pm \sqrt{25r^4 - 20r^4}}{10} \\
x^2 &= \frac{5r^2 \pm |r^2|\sqrt{5}}{10} \\
x &= \pm\sqrt{\frac{5r^2 \pm r^2\sqrt{5}}{10}} \\
x &= \sqrt{\frac{5r^2 \pm r^2\sqrt{5}}{10}}
\end{aligned}
$$

According to my calculator, $x = \sqrt{\dfrac{5r^2 + r^2\sqrt{5}}{10}}$ is valid solution but $x = \sqrt{\dfrac{5r^2 - r^2\sqrt{5}}{10}}$ is not.



Because $f'(c) > 0$ for all $c \in \left(0, \sqrt{\left(5r^2 - r^2\sqrt{5}/10\right)}\right)$ and $f'(c) < 0$ for all $c \in \left(\sqrt{\left(5r^2 - r^2\sqrt{5}/10\right)}, 2r\right)$, the absolute maximum of f must be at $x = \sqrt{\left(5r^2 - r^2\sqrt{5}/10\right)}$.

$$
\begin{aligned}
f\left(\sqrt{\frac{5r^2 + r^2\sqrt{5}}{10}}\right) &= 2\pi\left(\sqrt{\frac{5r^2 + r^2\sqrt{5}}{10}}\right)^2 + 4\pi\left(\sqrt{\frac{5r^2 + r^2\sqrt{5}}{10}}\right)\sqrt{r^2 - \left(\sqrt{\frac{5r^2 + r^2\sqrt{5}}{10}}\right)^2} \\
&= 2\pi\cdot\frac{5r^2 + r^2\sqrt{5}}{10} + 4\pi\left(\sqrt{\frac{5r^2 + r^2\sqrt{5}}{10}}\right)\sqrt{r^2 - \frac{5r^2 + r^2\sqrt{5}}{10}} \\
&= 2\pi\left(\frac{5r^2 + r^2\sqrt{5}}{10} + 2\sqrt{\frac{5r^2 + r^2\sqrt{5}}{10}}\sqrt{\frac{5r^2 - r^2\sqrt{5}}{10}}\right) \\
&= 2\pi\left(\frac{5r^2 + r^2\sqrt{5}}{10} + 2\sqrt{\frac{25r^4 - 5r^4}{100}}\right) \\
&= 2\pi\left(\frac{5r^2 + r^2\sqrt{5}}{10} + \frac{2\sqrt{20r^4}}{10}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= 2\pi\left(\frac{5r^2 + r^2\sqrt{5} + 4\left|r^2\right|\sqrt{5}}{10}\right) \\
&= 2\pi\left(\frac{5r^2 + 5r^2\sqrt{5}}{10}\right) \\
&= 10\pi r^2\left(\frac{1+\sqrt{5}}{10}\right) \\
&= \boxed{\pi r^2\left(1+\sqrt{5}\right)} \\
&= \pi r^2\left(\frac{1+\sqrt{5}}{2}\right)\cdot 2 \\
&= \boxed{2\pi r^2\phi}
\end{aligned}
$$

(Golden ratio appears somehow – nice!)

Problem 13

(a) Let s, t be the side lengths of the square and triangle, respectively.

$$
\begin{aligned}
4s + 3t &= 10 \\
t &= \frac{10-4s}{3}
\end{aligned}
$$

Let $f(s)$ be the combined area of the square and triangle.

$$
\begin{aligned}
f(s) &= s^2 + \frac{\sqrt{3}}{4}\cdot t^2 \\
&= s^2 + \frac{\sqrt{3}}{4}\cdot\left(\frac{10-4s}{3}\right)^2
\end{aligned}
$$

$$
\begin{aligned}
f'(s) = 2s + \frac{\sqrt{3}}{4}\cdot\frac{\mathrm{d}}{\mathrm{d}s}\left[\left(\frac{10-4s}{3}\right)^2\right] &= 0 \\
2s + \frac{\sqrt{3}}{4}\left(2\left(\frac{10-4s}{3}\right)\left(-\frac{4}{3}\right)\right) &= 0 \\
2s - \frac{4\sqrt{3}\,(10-4s)}{12} &= 0 \\
24s &= 40\sqrt{3} - 16\sqrt{3}s \\
\left(12 + 8\sqrt{3}\right)s &= 20\sqrt{3} \\
s &= \frac{20\sqrt{3}}{12+8\sqrt{3}} \\
s &= \frac{5\sqrt{3}}{3+2\sqrt{3}}\cdot\frac{3-2\sqrt{3}}{3-2\sqrt{3}} \\
s &= \frac{15\sqrt{5}-30}{9-12}
\end{aligned}
$$

$$s = -\frac{3\left(5\sqrt{3} - 10\right)}{3}$$
$$s = -\left(5\sqrt{3} - 10\right)$$
$$s = 10 - 5\sqrt{3}$$

(b)