Problem Set #67

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May 9, 2024

Problem 1

(a)

$$\lim_{x\to\infty}\frac{e^x}{x^2}$$

$$\boxed{\begin{gathered}\lim_{x\to\infty} e^x = \infty\\ \lim_{x\to\infty} x^2 = \infty\\ \lim_{x\to\infty} 2x = \infty\end{gathered}}$$

$$\overset{[\frac{\infty}{\infty}]}{\underset{\text{L'H}}{=}} \lim_{x\to\infty}\frac{e^x}{2x} \overset{[\frac{\infty}{\infty}]}{\underset{\text{L'H}}{=}} \lim_{x\to\infty}\frac{e^x}{2} = \boxed{\infty}$$

(d)

$$\lim_{x\to 0}\frac{\tan x - x}{x^3}$$

$$\boxed{\begin{gathered}\lim_{x\to 0}[\tan x - x] = 0\\ \lim_{x\to 0} x^3 = 0\end{gathered}}$$

$$\overset{[\frac{0}{0}]}{\underset{\text{L'H}}{=}} \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2}$$

$$\boxed{\begin{gathered}\lim_{x\to 0}\left[\sec^2 x - 1\right] = 0\\ \lim_{x\to 0} 3x^2 = 0\end{gathered}}$$

$$\overset{[\frac{0}{0}]}{\underset{\text{L'H}}{=}} \lim_{x\to 0}\frac{\frac{2\sin x}{\cos^3 x}}{6x}$$

$$\boxed{\begin{gathered}\lim_{x\to 0}\left[\sec^2 x - 1\right] = 0\\ \lim_{x\to 0} 3x^2 = 0\end{gathered}}$$

$$\overset{[\frac{0}{0}]}{\underset{\text{L'H}}{=}} \lim_{x\to 0}\frac{\frac{2\cos^2 x + 6\sin^2 x}{\cos^4 x}}{6} = \boxed{\frac{1}{3}}$$

Problem 4

$$\begin{aligned} f(x) &= xe^x\\ f'(x) &= e^x + xe^x = e^x(1+x)\\ f''(x) &= e^x + e^x + xe^x = e^x(2+x)\end{aligned}$$

$$\begin{aligned} f(x) &= 0\\ xe^x &= 0\\ x &= 0\end{aligned}$$

$-\infty \quad - \quad 0 \quad + \quad \infty$

$$\begin{aligned} f'(x) &= 0\\ e^x(1+x) &= 0\\ x &= -1\end{aligned}$$

$-\infty \quad - \quad -1 \quad + \quad \infty$

$$\begin{aligned} f''(x) &= 0\\ e^x(2+x) &= 0\\ x &= -2\end{aligned}$$

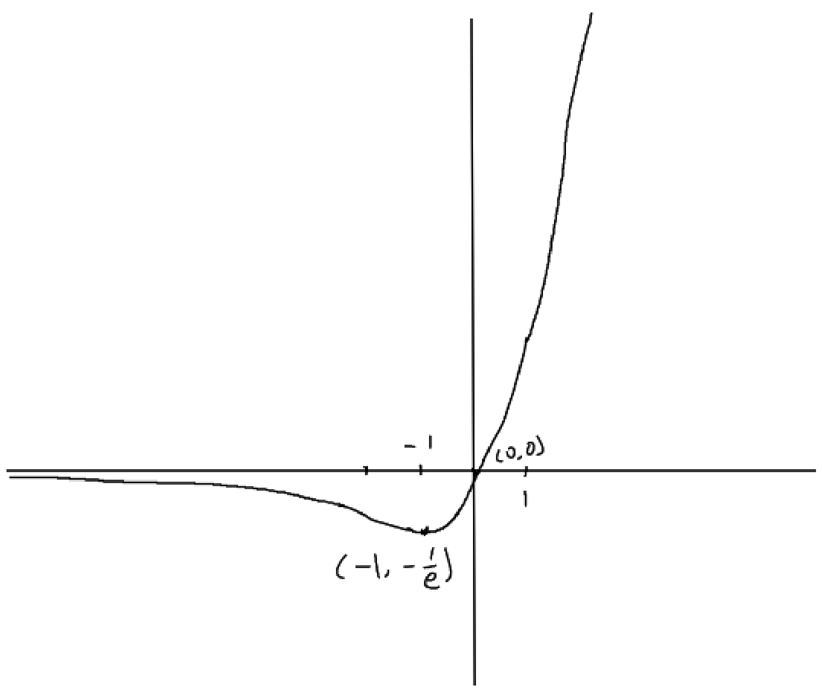
$-\infty \quad - \quad -2 \quad + \quad \infty$

- The domain of f is $\mathbb{R}$.
- f is increasing on $(-1, \infty)$.
- f is decreasing on $(-\infty, -1)$.
- f has an absolute minimum at $x = -1$ (because $f'(c) < 0$ for all $c \in (-\infty, -1)$.).
- f is concave up on $(-2, \infty)$.
- f is concave down on $(-\infty, -2)$.
- f has an inflection point at $x = 2$.
- $\lim_{x\to\infty} f(x) = \infty$.

$$\lim_{x\to-\infty} xe^x = \lim_{x\to-\infty}\frac{x}{\frac{1}{e^x}}$$

$$\boxed{\begin{gathered}\lim_{x\to-\infty} x = -\infty\\ \lim_{x\to-\infty}\frac{1}{e^x} = \infty\end{gathered}}$$

$$\overset{[\frac{0}{0}]}{\underset{\text{L'H}}{=}} \lim_{x\to-\infty}\frac{1}{-\frac{1}{(e^x)^2}\cdot e^x} = -\lim_{x\to-\infty}\frac{e^{2x}}{e^x} = \lim_{x\to-\infty} e^x = 0$$



Problem 5

(b)

$$\boxed{\begin{gathered}\lim_{x\to 1}\left[x^9 - 1\right] = 0\\ \lim_{x\to 1}\left[x^5 - 1\right] = 0\end{gathered}}$$

$$\lim_{x\to 1}\frac{x^9-1}{x^5-1} \overset{[\frac{0}{0}]}{\underset{\text{L'H}}{=}} \lim_{x\to 0}\frac{9x^8}{5x^4} = \lim_{x\to 1}\frac{9x^4}{5} = \boxed{\frac{9}{5}}$$

(d)

$$\boxed{\begin{gathered}\lim_{t\to 0}\left[e^t - 1\right] = 0\\ \lim_{t\to 0} t^3 = 0\end{gathered}}$$

$$\lim_{t\to 0}\frac{e^t - 1}{t^3} \overset{[\frac{0}{0}]}{\underset{\text{L'H}}{=}} \lim_{t\to 0}\frac{e^t}{t^2} = \boxed{\infty}$$

(e)

$$\lim_{x\to 0}\frac{\cancel{\tan}(px)}{\cancel{\tan}(qx)} = \lim_{x\to 0}\frac{p\cancel{x}}{q\cancel{x}} = \boxed{\frac{p}{q}}$$

(f)

$$\boxed{\begin{gathered}\lim_{x\to\infty}\ln x = \infty\\ \lim_{x\to\infty}\sqrt{x} = \infty\end{gathered}}$$

$$\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}} \overset{[\frac{\infty}{\infty}]}{\underset{\text{L'H}}{=}} \lim_{x\to 1}\frac{\frac{1}{x}}{\frac{1}{2\sqrt{x}}} = 2\lim_{x\to\infty}\frac{\sqrt{x}}{x}$$

$$= 2\lim_{x\to\infty}\frac{\sqrt{x}}{\sqrt{x}\cdot\sqrt{x}} = 2\lim_{x\to\infty}\frac{1}{\sqrt{x}} = \boxed{0}$$

(j)

$$\boxed{\begin{gathered}\lim_{t\to 0}\left[5^t - 3^t\right] = 1 - 1 = 0\\ \lim_{t\to 0} t = 0\end{gathered}}$$

$$\lim_{t\to 0}\frac{5^t - 3^t}{t} \overset{[\frac{0}{0}]}{\underset{\text{L'H}}{=}} \lim_{t\to 0}\frac{5^t\ln 5 - 3^t\ln 3}{1} = \boxed{\ln 5 - \ln 3}$$

(l)

$$\boxed{\begin{gathered}\lim_{x\to 0}[1 - \cos x] = 0\\ \lim_{x\to 0} x^2 = 0\end{gathered}}$$

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} \overset{[\frac{0}{0}]}{\underset{\text{L'H}}{=}} \lim_{x\to 0}\frac{\sin x}{2x} = \frac{1}{2}\lim_{x\to 0}\frac{\sin x}{x} = \boxed{\frac{1}{2}}$$

(n)

$$\boxed{\begin{gathered}\lim_{x\to 1}[1 - x + \ln x] = 0\\ \lim_{x\to 1}[1 + \cos\pi x] = 0 \lim_{x\to 1}\left[-1 + \frac{1}{x}\right] \quad = 0\\ \lim_{x\to 1}[-\pi\sin\pi x] = 0\end{gathered}}$$

$$\lim_{x\to 1}\frac{1 - x + \ln x}{1 + \cos\pi x} \overset{[\frac{0}{0}]}{\underset{\text{L'H}}{=}} \lim_{x\to 1}\frac{-1 + \frac{1}{x}}{-\pi\sin\pi x}$$

$$\overset{[\frac{0}{0}]}{\underset{\text{L'H}}{=}} \lim_{x\to 1}\frac{-\frac{1}{x^2}}{-\pi^2\cos\pi x} = \frac{-1}{-\pi^2\cos\pi} = \boxed{-\frac{1}{\pi^2}}$$

(p) Let $y = 1/x$. Then $x = 1/y$ and $\pi/x = \pi y$. As $x \to \infty$, $y \to 0$.

$$\lim_{x\to\infty} x\sin\frac{\pi}{x} = \lim_{y\to 0}\frac{1}{y}\sin\pi y = \lim_{y\to 0}\frac{\sin\pi y}{y}$$

$$\boxed{\begin{gathered}\lim_{y\to 0}\sin\pi y = 0\\ \lim_{y\to 0} y = 0\end{gathered}}$$

$$\overset{[\frac{0}{0}]}{\underset{\text{L'H}}{=}} \lim_{y\to 0}\frac{\pi\cos\pi y}{1} = \boxed{\pi}$$

(r)

$$\lim_{x\to\infty} x^3 e^{-x^2} = \lim_{x\to\infty}\frac{x^3}{e^{x^2}}$$

$$\boxed{\begin{gathered}\lim_{x\to\infty} x^3 = \infty\\ \lim_{x\to\infty} e^{x^2} = \infty\\ \lim_{x\to\infty} 3x = \infty\\ \lim_{x\to\infty} 2e^{x^2} = \infty\end{gathered}}$$

$$\overset{[\frac{\infty}{\infty}]}{\underset{\text{L'H}}{=}} \lim_{x\to\infty}\frac{3x^{\cancel{2}}}{2\cancel{x}e^{x^2}} \overset{[\frac{\infty}{\infty}]}{\underset{\text{L'H}}{=}} \lim_{x\to\infty}\frac{3}{2\cdot 2x\cdot e^{x^2}} = \boxed{0}$$

(s)

$$\lim_{x\to 1^+}\ln x\tan\frac{\pi x}{2} = \lim_{x\to 1^+}\frac{\ln x\sin\frac{\pi x}{2}}{\cos\frac{\pi x}{2}}$$

$$\boxed{\begin{gathered}\lim_{x\to 1^+}\ln x\sin\frac{\pi x}{2} = 0\\ \lim_{x\to 1^+}\cos\frac{\pi x}{2} = 0\end{gathered}}$$

$$\overset{[\frac{0}{0}]}{\underset{\text{L'H}}{=}} \lim_{x\to 1^+}\frac{\frac{\sin\frac{\pi x}{2}}{x} - \ln(x)\cdot\frac{\pi}{2}\cdot\cos\frac{\pi x}{2}}{-\frac{\pi}{2}\sin\frac{\pi x}{2}}$$

$$= \frac{\frac{1}{1} - 0\cdot\frac{\pi}{2}\cdot 0}{-\frac{\pi}{2}\cdot 1} = \boxed{-\frac{2}{\pi}}$$

(t)

$$\lim_{x\to 1}\left[\frac{x}{x-1} - \frac{1}{\ln x}\right] = \lim_{x\to 1}\frac{x\ln x - x + 1}{(x-1)\ln x}$$

$$\boxed{\begin{gathered}\lim_{x\to 1}[x\ln x - x + 1] = 0\\ \lim_{x\to 1}(x-1)\ln x = 0\\ \lim_{x\to 1}\ln x = 0\\ \lim_{x\to 1}\left[\ln x + \frac{x-1}{x}\right] = 0\end{gathered}}$$

$$\overset{[\frac{0}{0}]}{\underset{\text{L'H}}{=}} \lim_{x\to 1}\frac{\ln x + x\cdot\frac{1}{x} - 1}{\ln x + \frac{x-1}{x}} = \lim_{x\to 1}\frac{\ln x}{\ln x + \frac{x-1}{x}}$$

$$\overset{[\frac{0}{0}]}{\underset{\text{L'H}}{=}} \lim_{x\to 1}\frac{\frac{1}{x}}{\frac{1}{x} + \frac{x-(x-1)}{x^2}} = \lim_{x\to 1}\frac{1}{x\left(\frac{1}{x} + \frac{1}{x^2}\right)}$$

$$= \frac{1}{1(1+1)} = \boxed{\frac{1}{2}}$$

(u)

$$\begin{aligned}&\lim_{x\to\infty}\left[\sqrt{x^2+x} - x\right]\\ &= \lim_{x\to\infty}\left[\sqrt{x^2}\sqrt{1+\frac{1}{x}} - x\right]\\ &= \lim_{x\to\infty}\left[|x|\sqrt{1+\frac{1}{x}} - x\right]\\ &= \lim_{x\to\infty} x\left(\sqrt{1+\frac{1}{x}} - 1\right)\end{aligned}$$

Let $y = 1/x$. Then $x = 1/y$. As $x \to \infty$, $y \to 0$.

$$= \lim_{y\to 0}\frac{1}{y}\left(\sqrt{1+y} - 1\right) = \lim_{y\to 0}\frac{\sqrt{1+y} - 1}{y}$$

$$\boxed{\begin{gathered}\lim_{y\to 0}\left[\sqrt{1+y} - 1\right] = 0\\ \lim_{y\to 0} y = 0\end{gathered}}$$

$$\overset{[\frac{0}{0}]}{\underset{\text{L'H}}{=}} \lim_{y\to 0}\frac{\frac{1}{2\sqrt{1+y}}}{1} = \frac{1}{2\sqrt{1+0}} = \boxed{\frac{1}{2}}$$