Progress on Problem Set #39

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Problem 2

Let $\{a_n\}$ be the geometric sequence.

(a)

$$\begin{aligned}
a_1 - a_4 &= 5(a_2 - a_3) \\
a_1 - a_1 \cdot r^{4-1} &= 5(a_1 \cdot r^{2-1} - a_1 \cdot r^{3-1}) \\
a_1 - a_1 r^3 &= 5a_1 r - 5a_1 r^2 \\
1 - r^3 &= 5r - 5r^2
\end{aligned}$$

$$\boxed{r^3 - 5r^2 + 5r - 1 = 0}$$

$$\begin{aligned}
(1)^3 - 5(1)^2 + 5(1) - 1 &= 0 \\
(-1)^3 - 5(-1)^2 + 5(-1) - 1 &\neq 0
\end{aligned}$$

$$\begin{aligned}
\frac{r^3 - 5r^2 + 5r - 1}{r - 1} &= 0 \\
r^2 - 4r + 1 &= 0 \\
r &= \frac{4 \pm \sqrt{16 - 4}}{2} \\
r &= \frac{4 \pm \sqrt{4 \cdot 3}}{2} \\
r &= 2 \pm \sqrt{3}
\end{aligned}$$

$$\boxed{r = 1, 2 + \sqrt{3}, 2 - \sqrt{3}}$$

(b)

$$S_\infty = \frac{a_1}{1 + r}$$

Case 1. $r = 1$
$r \geq 1$ so S_∞ diverges.

Case 2. $r = 2 + \sqrt{3}$
$r \geq 1$ so S_∞ diverges.

Case 3. $r = 2 - \sqrt{3}$
S_∞ converges because $|2 - \sqrt{3}| < 1$.

$$\begin{aligned}
\sqrt{2} + \sqrt{6} &= \frac{a_1}{1 - (2 - \sqrt{3})} \\
\sqrt{2} + \sqrt{6} &= \frac{a_1}{\sqrt{3} - 1} \\
a_1 &= \sqrt{6} - \sqrt{2} + \sqrt{18} - \sqrt{6} \\
a_1 &= -\sqrt{2} + 3\sqrt{2}
\end{aligned}$$

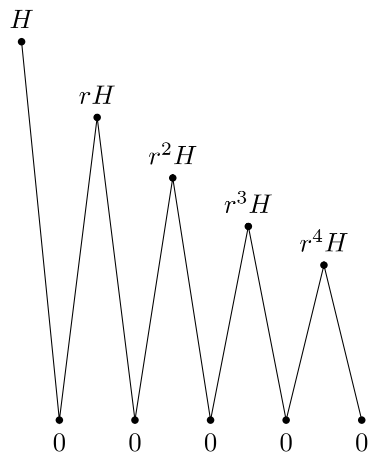
$$\boxed{a_1 = 2\sqrt{2}}$$

Problem 3

$$\begin{aligned}
b_{n+2} &= 5b_{n+1} - 6b_n \\
b_n r^2 &= 5b_n r - 6b_n \\
r^2 &= 5r - 6 \\
r^2 - 5r + 6 &= 0 \\
(r - 2)(r - 3) &= 0
\end{aligned}$$

$$\boxed{r = 2, r = 3}$$

Problem 5



(Of course the ball would not actually move to the right like in this diagram - it would only travel the vertical distance indicated)
Let b_n be the height of the ball before the nth bounce. It is a geometric sequence with common ratio d and first term $b_1 = H$. Let D be the total distance traveled by the ball.

$$D = b_1 + b_2 + b_2 + b_3 + b_3 + \dots$$
$$D = \sum_{k=1}^{\infty} b_k + \sum_{k=2}^{\infty} b_k$$
$$D = 2\sum_{k=1}^{\infty} b_k - b_1$$
$$D = 2\left(\frac{H}{1-r}\right) - H$$
$$\frac{D+H}{2} = \frac{H}{1-r}$$
$$(1-r)\left(\frac{D+H}{2}\right) = H$$
$$1 - r = \frac{2H}{D+H}$$
$$r = \frac{D+H}{D+H} - \frac{2H}{D+H}$$
$$\boxed{r = \frac{D-H}{D+H}}$$

Problem 6

$$r = \frac{\sqrt{3}\tan\theta}{\sqrt{2}\sin\theta} = \frac{\sqrt{2}\sin\theta}{\cos\theta}$$
$$\frac{\sqrt{3}\tan\theta}{\sqrt{2}\sin\theta}\cdot\cos\theta = \frac{\sqrt{2}\sin\theta}{\cos\theta}\cdot\cos\theta$$
$$\frac{\sqrt{3}\sin\theta}{\sqrt{2}\sin\theta} = \sqrt{2}\sin\theta$$
$$\frac{\sqrt{3}}{\sqrt{2}} = \sqrt{2}\sin\theta$$
$$2\sin\theta = \sqrt{3}$$
$$\sin\theta = \frac{\sqrt{3}}{2}$$
$$\theta = \frac{\pi}{3}$$

$$r = \frac{\sqrt{2}\sin\frac{\pi}{3}}{\cos\frac{\pi}{3}}$$
$$r = \frac{\sqrt{2}\cdot\frac{\sqrt{3}}{2}}{\frac{1}{2}}$$
$$r = \sqrt{2}\cdot\frac{\sqrt{3}}{2}\cdot 2$$
$$r = \sqrt{6}$$

$$a_1 = \frac{\cos\frac{\pi}{3}}{\sqrt{6}}$$
$$a_1 = \frac{\frac{1}{2}}{\sqrt{6}}$$
$$a_1 = \frac{1}{2\sqrt{6}}$$

$$S_6 = a_1\left(\frac{1-r^6}{1-r}\right)$$
$$S_6 = \frac{1}{2\sqrt{6}}\left(\frac{1-(\sqrt{6})^6}{1-\sqrt{6}}\right)$$
$$S_6 = \frac{1-6^3}{2\sqrt{6}-12}$$
$$S_6 = \frac{-215}{2\sqrt{6}-12}\cdot\frac{2\sqrt{6}+12}{2\sqrt{6}+12}$$
$$S_6 = \frac{-430\sqrt{6}-2580}{24-144}$$
$$S_6 = \frac{430\sqrt{6}+2580}{144-24}$$
$$S_6 = \frac{43\sqrt{6}+258}{12}$$
$$\boxed{S_6 = \frac{43(6+\sqrt{6})}{12}}$$