Problem Set #28

Jayden Li

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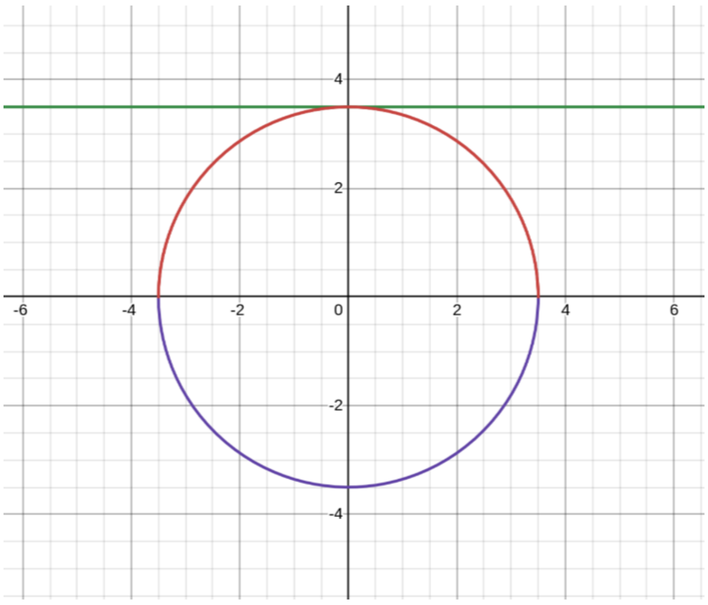
Problem 6

$y = r$ is perpendicular to the y-axis, so we will use $\mathrm{d}x$.



$$x^2 + y^2 = r^2 \implies y = \pm\sqrt{r^2 - x^2} \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{-2x}{2\sqrt{r^2-x^2}} = \mp\frac{x}{\sqrt{r^2-x^2}}$$

$$S_{\text{red}} = \int_{-r}^{r} 2\pi \cdot \text{radius} \cdot \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x$$

Radius is distance between $y = r$ and the circle, which is $r - \sqrt{r^2 - x^2}$.

$$= \int_{-r}^{r} 2\pi\left(r - \sqrt{r^2-x^2}\right)\sqrt{1 + \left(\mp\frac{x}{\sqrt{r^2-x^2}}\right)^2}\,\mathrm{d}x = \boxed{2\pi\int_{-r}^{r}\left(r - \sqrt{r^2-x^2}\right)\sqrt{1+\frac{x^2}{r^2-x^2}}\,\mathrm{d}x}$$

$$= 2\pi\int_{-r}^{r}\left(r\sqrt{1+\frac{x^2}{r^2-x^2}} - \sqrt{r^2-x^2+\frac{x^2}{\cancel{r^2-x^2}}\cdot(\cancel{r^2-x^2})}\right)\mathrm{d}x$$

$$= 2\pi\int_{-r}^{r}\left(r\sqrt{\frac{r^2-x^2}{r^2-x^2}+\frac{x^2}{r^2-x^2}} - \sqrt{r^2-x^2+x^2}\right)\mathrm{d}x = 2\pi\int_{-r}^{r}\left(r\sqrt{\frac{r^2}{r^2-x^2}} - r\right)\mathrm{d}x$$

$$= 2\pi r\int_{-r}^{r}\left(\frac{r}{\sqrt{r^2-x^2}} - 1\right)\mathrm{d}x = 2\pi r\int_{-r}^{r}\left(\frac{\cancel{r}}{\cancel{r}\sqrt{1-\frac{x^2}{r^2}}} - 1\right)\mathrm{d}x$$

$$= \begin{bmatrix} u = x/r \\ \mathrm{d}u = \mathrm{d}x/r \end{bmatrix} 2\pi r^2\int_{-1}^{1}\left(\frac{1}{\sqrt{1-u^2}} - 1\right)\mathrm{d}u = 2\pi r^2\Big[\arcsin u - u\Big]_{-1}^{1}$$

$$= 2\pi r^2\left(\frac{\pi}{2} - 1 - \left(-\frac{\pi}{2} - (-1)\right)\right) = 2\pi r^2\left(\frac{\pi}{2} - 1 + \frac{\pi}{2} - 1\right) = 2\pi r^2(\pi - 2)$$

Notice that the "radius" of revolution for the purple curve (whose equation is $y = -\sqrt{r^2-x^2}$) is $r + |y| = r + \left|-\sqrt{r^2-x^2}\right| = r + \sqrt{r^2-x^2}$.

$$S_{\text{purple}} = \int_{-r}^{r} 2\pi\left(r + \sqrt{r^2-x^2}\right)\sqrt{1+\left(\mp\frac{x}{\sqrt{r^2-x^2}}\right)^2}\,\mathrm{d}x = 2\pi\int_{-r}^{r}\left(r+\sqrt{r^2-x^2}\right)\sqrt{1+\frac{x^2}{r^2-x^2}}\,\mathrm{d}x$$

Notice this differs from the above boxed expression by only a plus sign.

$$= 2\pi\int_{-r}^{r}\left(\frac{r}{\sqrt{r^2-x^2}} + 1\right)\mathrm{d}x = \ldots = 2\pi r^2\Big[\arcsin u + u\Big]_{-1}^{1} = 2\pi r^2\left(\frac{\pi}{2} + 1 - \left(-\frac{\pi}{2} - 1\right)\right)$$

$$= 2\pi r^2\left(\frac{\pi}{2} + 1 + \frac{\pi}{2} + 1\right) = 2\pi r^2(\pi + 2)$$

Total surface area is the sum of the surface areas of the two individual solids.

$$S = S_{\text{red}} + S_{\text{purple}} = 2\pi r^2(\pi - 2) + 2\pi r^2(\pi + 2) = 2\pi^2 r^2 - \cancel{4\pi r^2} + 2\pi^2 r^2 + \cancel{4\pi r^2} = \boxed{4\pi^2 r^2}$$