Problem Set #33

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Problem 5

(a)
$$\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}$$
$$=\lim_{\theta\to 0}-\frac{-(1-\cos\theta)}{\theta}$$
$$=\lim_{\theta\to 0}-\frac{\cos\theta-1}{\theta}$$
$$=\lim_{\theta\to 0}-1\cdot\lim_{\theta\to 0}\frac{\cos\theta-1}{\theta}$$
$$=-1\cdot 1$$
$$=\boxed{-1}$$

(b)
$$\lim_{x\to 1}\sin\left(\frac{x^2-1}{x-1}\right)$$
$$=\lim_{x\to 1}\sin\frac{(x+1)(x-1)}{x-1}$$
$$=\lim_{x\to 1}\sin\frac{x+1}{1}$$
$$=\sin\frac{1+1}{1}$$
$$=\boxed{\sin 2}$$

(c)
$$\lim_{x\to 0}\frac{\tan x}{4x}$$
$$=\lim_{x\to 0}\frac{\sin x}{4x\cos x}$$
$$=\lim_{x\to 0}\left(\frac{\sin x}{4x}\cdot\frac{1}{\cos x}\right)$$
$$=\lim_{x\to 0}\frac{\sin x}{4x}\cdot\lim_{x\to 0}\frac{1}{\cos x}$$
$$=\frac{1}{4}\cdot\frac{1}{\cos 0}$$
$$=\boxed{\frac{1}{4}}$$

(d)
$$\lim_{x\to\pi}\frac{\sin(\pi-x)}{\pi(\pi-x)}$$
Let $y=\pi-x$
$$=\lim_{y\to 0}\frac{\sin y}{\pi\cdot y}\qquad\text{as } x\to\pi,\ \pi-x\to 0$$
$$=\lim_{y\to 0}\frac{\sin y}{y}\cdot\lim_{y\to 0}\frac{1}{\pi}$$
$$=\boxed{\frac{1}{\pi}}$$

(e)
$$\lim_{x\to 0}\frac{\cos x}{\pi-x}$$
$$=\frac{\cos 0}{\pi-0}$$
$$=\boxed{\frac{1}{\pi}}$$

(f)
$$\lim_{x\to 0}\frac{\cos 2x-1}{\cos x-1}$$
$$=\lim_{x\to 0}\frac{2\cos^2 x-1-1}{\cos x-1}$$
$$=2\lim_{x\to 0}\frac{(\cos x-1)(\cos x+1)}{\cos x-1}$$
$$=2\lim_{x\to 0}\frac{\cos x+1}{1}$$
$$=2\cdot(\cos 0+1)$$
$$=\boxed{4}$$

(h) $\displaystyle\lim_{x\to 0} x\sec x$

$$= \lim_{x\to 0}\left(\frac{x}{\cos x}\right)$$
$$= \frac{0}{\cos 0}$$
$$= \boxed{0}$$

(j) $\displaystyle\lim_{x\to 0}\frac{\sin ax + bx}{ax + \sin bx}$

$$= \lim_{x\to 0}\frac{\sin ax + bx}{ax + \sin bx}$$
$$= \lim_{x\to 0}\left(\frac{\sin ax}{ax + \sin bx} + \frac{bx}{ax + \sin bx}\right)$$
$$= \lim_{x\to 0}\left(\left(\frac{ax + \sin bx}{\sin ax}\right)^{-1} + \left(\frac{ax + \sin bx}{bx}\right)^{-1}\right)$$
$$= \lim_{x\to 0}\left(\left(\frac{ax}{\sin ax} + \frac{\sin bx}{\sin ax}\right)^{-1} + \left(\frac{ax}{bx} + \frac{\sin bx}{bx}\right)^{-1}\right)$$
$$= \lim_{x\to 0}\left(\left(1 + \frac{b}{a}\right)^{-1} + \left(\frac{a}{b} + 1\right)^{-1}\right)$$
$$= \lim_{x\to 0}\left(\left(\frac{a+b}{a}\right)^{-1} + \left(\frac{a+b}{b}\right)^{-1}\right)$$
$$= \lim_{x\to 0}\left(\frac{a}{a+b} + \frac{b}{a+b}\right)$$
$$= \lim_{x\to 0}\frac{a+b}{a+b}$$
$$= \boxed{1}$$

(i) $\displaystyle\lim_{x\to 0}(\csc x - \cot x)$

$$= \lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right)$$
$$= \lim_{x\to 0}\frac{1 - \cos x}{\sin x}$$
$$= \lim_{x\to 0}\left(\frac{1 - \cos x}{\sin x}\cdot\frac{1 + \cos x}{1 + \cos x}\right)$$
$$= \lim_{x\to 0}\frac{1 - \cos^2 x}{\sin(x)(1 + \cos x)}$$
$$= \lim_{x\to 0}\frac{\sin^2 x}{\sin(x)(1 + \cos x)}$$
$$= \lim_{x\to 0}\frac{\sin x}{1 + \cos x}$$
$$= \frac{\sin 0}{1 + \cos 0}$$
$$= \boxed{0}$$

(k)

$$\lim_{x\to 0}\frac{1-\cos 4x}{x^2}$$

$$=\lim_{x\to 0}\frac{1-\left(2\cos^2 2x-1\right)}{x^2}$$

$$=\lim_{x\to 0}\frac{1-\left(2\left(2\cos^2 x-1\right)^2-1\right)}{x^2}$$

$$=\lim_{x\to 0}\frac{1-\left(2\left(4\cos^4 x-4\cos^2 x+1\right)-1\right)}{x^2}$$

$$=\lim_{x\to 0}\frac{1-\left(8\cos^4 x-8\cos^2 x+2-1\right)}{x^2}$$

$$=\lim_{x\to 0}\frac{1-8\cos^4 x+8\cos^2 x-1}{x^2}$$

$$=8\lim_{x\to 0}\frac{\cos^2 x-\cos^4 x}{x^2}$$

$$=8\lim_{x\to 0}\frac{\cos^2(x)\left(1-\cos^2 x\right)}{x^2}$$

$$=8\cos^2 0\cdot\lim_{x\to 0}\frac{\sin^2 x}{x^2}$$

$$=8\cdot\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2$$

$$=\boxed{8}$$

(l)

$$\lim_{x\to \pi}\frac{\sin 3x-3\sin x}{(\pi-x)^3}$$

Let $y=\pi-x$

$\therefore x=\pi-y$

$$=\lim_{y\to 0}\frac{\sin 3(\pi-y)-3\sin(\pi-y)}{y^3}$$

$$=\lim_{y\to 0}\frac{\sin(2y+y)-3\sin y}{y^3}$$

$$=\lim_{y\to 0}\frac{\sin 2y\cos y+\cos 2y\sin y-3\sin y}{y^3}$$

$$=\lim_{y\to 0}\frac{2\sin y\cos y\cos y+\left(2\cos^2 y-1\right)\sin y-3\sin y}{y^3}$$

$$=\lim_{y\to 0}\frac{2\sin y\cos^2 y+2\sin y\cos^2 y-\sin y-3\sin y}{y^3}$$

$$=\lim_{y\to 0}\frac{4\sin y\cos^2 y-4\sin y}{y^3}$$

$$=\lim_{y\to 0}\frac{4\sin(y)\left(\cos^2 y-1\right)}{y^3}$$

$$=4\lim_{y\to 0}\left(\frac{\sin y}{y}\cdot\frac{-\sin^2 y}{y^2}\right)$$

$$=4\cdot 1\cdot(-1)^2$$

$$=\boxed{4}$$

(m)

$$\lim_{x\to 0}\frac{\tan 2x-x}{2x-\sin x}$$

$$=\lim_{x\to 0}\left(\frac{\tan 2x}{2x-\sin x}-\frac{x}{2x-\sin x}\right)$$

$$=\lim_{x\to 0}\left(\left(\frac{2x-\sin x}{\tan 2x}\right)^{-1}-\left(\frac{2x-\sin x}{x}\right)^{-1}\right)$$

$$=\lim_{x\to 0}\left(\left(\frac{2x}{\tan 2x}-\frac{\sin x}{\tan 2x}\right)^{-1}-\left(\frac{2x}{x}-\frac{\sin x}{x}\right)^{-1}\right)$$

$$=\lim_{x\to 0}\left(\left(\left(\frac{\tan 2x}{2x}\right)^{-1}-\sin(x)\cdot\frac{\cos 2x}{\sin 2x}\right)^{-1}-(2-1)^{-}\right.$$

$$=\lim_{x\to 0}\left(\left(1-\frac{\sin x\cos 2x}{2\sin x\cos x}\right)^{-1}-1\right)$$

$$=\lim_{x\to 0}\left(\left(1-\frac{\cos 2x}{2\cos x}\right)^{-1}-1\right)$$

(m) (cont.) $$= \left(1 - \frac{\cos 0}{2\cos 0}\right)^{-1} - 1$$
$$= \left(\frac{1}{2}\right)^{-1} - 1$$
$$= \boxed{1}$$

(n) $$\lim_{x\to\frac{\pi}{4}} \frac{\sin x - \cos x}{x - \frac{\pi}{4}}$$
Let $y = x - \dfrac{\pi}{4}$

$\therefore x = y + \dfrac{\pi}{4}$

$$= \lim_{y\to 0} \frac{\sin\left(y + \frac{\pi}{4}\right) - \cos\left(x + \frac{\pi}{4}\right)}{y}$$
$$= \lim_{y\to 0} \frac{1}{y}\left(\sin y \cdot \frac{\sqrt{2}}{2} + \cos y \cdot \frac{\sqrt{2}}{2} - \cos y \cdot \frac{\sqrt{2}}{2} + \sin y \cdot \frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{2}}{2} \cdot \lim_{y\to 0} \frac{2\sin y}{y}$$
$$= \sqrt{2} \cdot \lim_{y\to 0} \frac{\sin y}{y}$$
$$= \boxed{\sqrt{2}}$$

(o) $$\lim_{x\to 0} \frac{\sin 3x - \tan 4x}{x^2}$$
$$= \lim_{x\to 0} \left(\frac{1}{x} \cdot \frac{\sin 3x - \tan 4x}{x}\right)$$
$$= \lim_{x\to 0} \left(\frac{1}{x} \cdot \left(\frac{\sin 3x}{x} - \frac{\tan 4x}{x}\right)\right)$$
$$= \lim_{x\to 0} \frac{1}{x} \cdot (3 - 4)$$
$$\to \lim_{x\to 0^+} \frac{1}{x} = +\infty, \lim_{x\to 0^-} \frac{1}{x} = -\infty$$
$\boxed{\text{DNE}}$

(p) $$\lim_{x\to 0} \frac{\tan 2x - \sin 2x}{x^3}$$
$$= \lim_{x\to 0} \left(\frac{1}{x^3} \cdot \left(\frac{\sin 2x}{\cos 2x} - \frac{\sin 2x \cos 2x}{\cos 2x}\right)\right)$$
$$= \lim_{x\to 0} \left(\frac{1}{\cos 2x} \cdot \left(\frac{\sin(2x)(1 - \cos 2x)}{x^3}\right)\right)$$
$$= \frac{1}{\cos 0} \cdot \lim_{x\to 0} \frac{\sin(2x)(1 - \cos 2x)}{x^3}$$
$$= \lim_{x\to 0} \frac{\sin 2x}{x} \cdot \lim_{x\to 0} \frac{1 - \cos 2x}{x^2}$$
$$= 2 \cdot \lim_{x\to 0} \frac{1 - 2\cos^2 x + 1}{x^2}$$
$$= 2 \cdot \lim_{x\to 0} \frac{2\sin^2 x}{x^2}$$
$$= 4 \cdot \lim_{x\to 0} \left(\frac{\sin x}{x}\right)^2$$
$$= 4 \cdot (1)^2$$
$$= \boxed{4}$$

(q) $$\lim_{x\to 0} \frac{1 - \cos 2x + \tan^2 x}{x \sin x}$$
$$= \lim_{x\to 0} \left(\frac{1 - 2\cos^2 x + 1}{x \sin x} + \frac{\tan^2 x}{x \sin x}\right)$$
$$= \lim_{x\to 0} \left(\frac{2 - 2\cos^2 x}{x \sin x} + \frac{\sin^2 x}{\cos^2 x \cdot x \sin x}\right)$$
$$= \lim_{x\to 0} \left(\frac{2\sin^2 x}{x \sin x} + \frac{\sin x}{x \cos^2 x}\right)$$
$$= \lim_{x\to 0} \left(\frac{2\sin x}{x} + \frac{\sin x}{x} \cdot \frac{1}{\cos^2 x}\right)$$
$$= \lim_{x\to 0} \frac{2\sin x}{x} + \lim_{x\to 0} \frac{\sin x}{x} \cdot \lim_{x\to 0} \frac{1}{\cos^2 x}$$
$$= 2 + 1 \cdot \frac{1}{\cos^2 0}$$
$$= \boxed{3}$$

(r) $$\lim_{x\to 0}\frac{\sqrt{7}-\sqrt{6+\cos x}}{x\sin x}$$

$$=\lim_{x\to 0}\left(\frac{\sqrt{7}-\sqrt{6+\cos x}}{x\sin x}\cdot\frac{\sqrt{7}+\sqrt{6+\cos x}}{\sqrt{7}+\sqrt{6+\cos x}}\right)$$

$$=\lim_{x\to 0}\left(\frac{7-6-\cos x}{x\sin x}\cdot\frac{1}{\sqrt{7}+\sqrt{6+\cos x}}\right)$$

$$=\lim_{x\to 0}\frac{1-\cos x}{x\sin x}\cdot\lim_{x\to 0}\frac{1}{\sqrt{7}+\sqrt{6+\cos x}}$$

$$=\lim_{x\to 0}\left(\frac{1-\cos x}{x\sin x}\cdot\frac{1+\cos x}{1+\cos x}\right)\cdot\frac{1}{\sqrt{7}+\sqrt{6+\cos 0}}$$

$$=\lim_{x\to 0}\frac{1-\cos^2 x}{x\sin(x)(1+\cos x)}\cdot\frac{1}{2\sqrt{7}}$$

$$=\lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{1}{1+\cos x}\right)\cdot\frac{1}{2\sqrt{7}}$$

$$=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{1+\cos 0}\cdot\frac{1}{2\sqrt{7}}$$

$$=1\cdot\frac{1}{2}\cdot\frac{1}{2\sqrt{7}}$$

$$=\boxed{\frac{1}{4\sqrt{7}}}$$