Problem Set #60

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Problem 1

Proof.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{c-c}{h} \\
&= \lim_{h\to 0} \frac{0}{h} \\
&= 0
\end{aligned}
$$

☺

Problem 2

Proof. Let $f_n(x) = x^n$, where $n \in \mathbb{Z}^+$.

Base case. $n = 1$.

$$\frac{\mathrm{d}}{\mathrm{d}x} x = 1 = 1 \cdot 1 = 1x^{1-1}$$

Hypothesis. Suppose that $f_n'(x) = nx^{n-1}$. NTS $f_{n+1}'(x) = (n+1)x^n$.

Inductive step.

$$
\begin{aligned}
f_{n+1}'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[x^{n+1}\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[x \cdot x^n\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[x\right] \cdot x^n + x \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[x^n\right] \\
&= x^n + x \cdot \left(nx^{n-1}\right) \\
&= x^n + nx^n \\
&= (n+1)x^n
\end{aligned}
$$

☺

Problem 3

(a)

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\sin x + \sin\left(x^2\right)\right] \\ &= \boxed{\cos x + 2x\cos\left(x^2\right)} \end{aligned}$$

(b)

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\sin(\sin x) \\ &= \boxed{\cos(\sin x)\cos x} \end{aligned}$$

(c)

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\sin(\cos x)}{x}\right] \\ &= \frac{x\cos(\cos x)(-\sin x) - \sin(\cos x)}{x^2} \\ &= \boxed{\frac{-x\cos(\cos x)\sin x - \sin(\cos x)}{x^2}} \end{aligned}$$

(d)

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\sin(\cos(\sin x))\right] \\ &= \cos(\cos(\sin x)) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[\cos(\sin x)\right] \\ &= \cos(\cos(\sin x)) \cdot (-\sin(\sin x) \cdot \cos x) \\ &= \boxed{-\cos(\cos(\sin x))\sin(\sin x)\cos x} \end{aligned}$$

Problem 4

(a)

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\sin^3\left(x^2 + \sin x\right)\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\left(\sin\left(x^2 + \sin x\right)\right)^3\right] \\ &= 3\left(\sin\left(x^2 + \sin x\right)\right)^2 \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[\sin\left(x^2 + \sin x\right)\right] \\ &= \boxed{3\left(\sin\left(x^2 + \sin x\right)\right)^2 \cdot \cos\left(x^2 + \sin x\right) \cdot (2x + \cos x)} \end{aligned}$$

(b)

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\sin\left(\frac{x^3}{\cos\left(x^3\right)}\right)\right] \\ &= \cos\left(\frac{x^3}{\cos\left(x^3\right)}\right) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^3}{\cos\left(x^3\right)}\right] \\ &= \cos\left(\frac{x^3}{\cos\left(x^3\right)}\right) \cdot \frac{3x^2\cos\left(x^3\right) - x^3\left(-\sin\left(x^3\right) \cdot 3x^2\right)}{\cos^2\left(x^3\right)} \\ &= \boxed{\cos\left(\frac{x^3}{\cos\left(x^3\right)}\right) \cdot \frac{3x^2\cos\left(x^3\right) + 3x^5\sin\left(x^3\right)}{\cos^2\left(x^3\right)}} \end{aligned}$$

(c)

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[(\cos x)^{31^2}\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[(\cos x)^{961}\right] \end{aligned}$$

$$= \boxed{-961\cos^{960}(x)\sin(x)}$$

Problem 5

(a)

$$\begin{aligned} f'(x) &= \cos x \\ f'(f(x)) &= \boxed{\cos(\sin x)} \end{aligned}$$

(b)

$$\begin{aligned} f'(x) &= 0 \\ f'(f(x)) &= \boxed{0} \end{aligned}$$

Problem 6

(a)

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[g(xg(a))\right] \\ &= g'(xg(a)) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[xg(a)\right] \\ &= \boxed{g'(xg(a))\,(g(a) + xg'(a))} \end{aligned}$$

(b)

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[g(x)(x-a)\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[xg(x)\right] - \frac{\mathrm{d}}{\mathrm{d}x}\left[ag(x)\right] \\ &= \boxed{g(x) + xg'(x) - ag'(x)} \end{aligned}$$

(c) Let $y = x + 3$.

$$\begin{aligned} f(y) &= g\left((y-3)^2\right) \\ f'(y) &= g'\left((y-3)^2\right) \cdot 2(y-3) \cdot 1 \\ &\boxed{f'(y) = 2g'\left((y-3)^2\right)(y-3)} \end{aligned}$$

Problem 7

Let $f(x)$ and $g(x)$ be the radii of the larger and smaller circle, respectively, then:

$$\begin{aligned} \pi(f(x))^2 - \pi(g(x))^2 &= 9\pi \\ \pi(g(x))^2 &= \pi(f(x))^2 - 9\pi \\ (g(x))^2 &= (f(x))^2 - 9 \\ \frac{\mathrm{d}}{\mathrm{d}x}\left[(g(x))^2\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[(f(x))^2 - 9\right] \\ 2g(x)g'(x) &= 2f(x)f'(x) \\ g'(x) &= \frac{f(x)f'(x)}{g(x)} \end{aligned} \tag{1}$$

We need to calculate g' at $x = t$. It is known that the circumference of the smaller circle is 16π, so $2g(t)\pi = 16\pi$, thus $g(t) = 8$. Also, the rate of change of the area of the larger circle is 10π.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left[\pi(f(x))^2\right] &= 10\pi \\ \pi \cdot 2f(x)f'(x) &= 10\pi \\ f(x)f'(x) &= 5 \\ f(t)f'(t) &= 5 \end{aligned}$$

Substituting into (1), we have:

$$\begin{aligned} g'(x) &= \frac{f(x)f'(x)}{g(x)} \qquad (1) \\ g'(t) &= \frac{f(t)f'(t)}{g(t)} \\ &= \boxed{\frac{5}{8}} \end{aligned}$$

Problem 8

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[x^2 \sin\left(\frac{1}{x}\right)\right] \\ &= 2x\sin\left(\frac{1}{x}\right) + x^2\cos\left(\frac{1}{x}\right)\cdot\left(-\cdot\frac{1}{x^2}\right) \\ &= 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \end{aligned}$$

f and f' are both undefined at $x = 0$.

(a)

$$\begin{aligned} (f \circ h)'(0) &= f'(h(0)) \cdot h'(0) \\ &= f'(0) \cdot \sin^2(\sin(1)) \end{aligned}$$

$\boxed{\text{Undefined.}}$

(b)

$$(k \circ f)'(0) = k'(f(0)) \cdot f'(0)$$

$\boxed{\text{Undefined.}}$

Problem 9

(a)

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1-x^2} \\ &= \frac{1}{2\sqrt{1-x^2}}\cdot(0-2x) \\ &= \boxed{-\frac{x}{\sqrt{1-x^2}}} \end{aligned}$$

(b) *Proof.*

$$\begin{aligned} y - \sqrt{1-a^2} &= -\frac{a}{\sqrt{1-a^2}}(x-a) \\ y &= -\frac{a}{\sqrt{1-a^2}}(x-a) + \sqrt{1-a^2} \\ \sqrt{1-x^2} &= \frac{-ax}{\sqrt{1-a^2}} + \frac{a^2}{\sqrt{1-a^2}} + \frac{1-a^2}{\sqrt{1-a^2}} \end{aligned}$$

$$
\begin{aligned}
\sqrt{1-x^2} &= \frac{1-ax}{\sqrt{1-a^2}} \\
1-x^2 &= \frac{1-2ax+a^2x^2}{1-a^2} \\
1-a^2-x^2+a^2x^2 &= 1-2ax+a^2x^2 \\
-a^2-x^2 &= -2ax \\
x^2-2ax+a^2 &= 0 \\
x &= \frac{2a \pm \sqrt{4a^2-4a^2}}{2} \\
x &= \frac{2a}{2} \\
x &= a
\end{aligned}
$$

The line intersects the curve at only one point, so it is a tangent line. ☺

Problem 10

(a)

$$g' = 2f \cdot f'$$

(b)

$$g' = 2f' \cdot f''$$

(c)

$$
\begin{aligned}
(f')^2 &= f + \frac{1}{f^3} \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[(f')^2\right] &= \frac{\mathrm{d}}{\mathrm{d}x}[f] + \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{f^3}\right] \\
2f' \cdot f'' &= f' + \left(\frac{-3}{f^4}\right) \cdot f' \\
&\boxed{f'' = \frac{f' - \frac{3f'}{f^4}}{2f'}}
\end{aligned}
$$

Problem 11

Proof. By induction

Base case. $k = 1$.

$$f'(x) = -nx^{-n-1} = (-1)^1 \cdot n \cdot x^{-n-1} = (-1)^1 \cdot \frac{n \cdot (n-1)!}{(n-1)!} x^{-n-1} = (-1)^1 \frac{(n+1-1)!}{(n-1)!} x^{-n-1}$$

Hypothesis. Suppose that

$$f^{(k)}(x) = (-1)^k \frac{(n+k-1)!}{(n-1)!} x^{-n-k}$$

NTS:

$$
\begin{aligned}
f^{(k+1)}(x) &= (-1)^{k+1}\frac{(n+k+1-1)!}{(n-1)!}x^{-n-(k+1)} \\
&= -(-1)^k\frac{(n+k)!}{(n-1)!}x^{-n-k-1}
\end{aligned}
$$

Inductive step.

$$
\begin{aligned}
f^{(k+1)}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[f^{(k)}(x)\right] \\
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[(-1)^k\frac{(n+k-1)!}{(n-1)!}x^{-n-k}\right] \\
&= (-1)^k\frac{(n+k-1)!}{(n-1)!}\cdot\frac{\mathrm{d}}{\mathrm{d}x}x^{-n-k} \\
&= (-1)^k\frac{(n+k-1)!}{(n-1)!}\cdot(-n-k)\cdot x^{-n-k-1} \\
&= -(-1)^k\frac{(n+k)\cdot(n+k-1)!}{(n-1)!}\cdot x^{-n-k-1} \\
&= -(-1)^k\frac{(n+k)!}{(n-1)!}x^{-n-k-1}
\end{aligned}
$$

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