Problem Set #27

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Problem 4

(d) $$L = \int_0^{1/2} \sqrt{1 + \left(\frac{\mathrm{d}}{\mathrm{d}x} \ln\left(1 - x^2\right)\right)^2}\,\mathrm{d}x = \int_0^{1/2} \sqrt{1 + \left(\frac{-2x}{1-x^2}\right)^2}\,\mathrm{d}x$$

$$= \int_0^{1/2} \sqrt{\frac{\left(1-x^2\right)^2}{\left(1-x^2\right)^2} + \frac{4x^2}{\left(1-x^2\right)^2}}\,\mathrm{d}x = \int_0^{1/2} \sqrt{\frac{1 - 2x^2 + x^4 + 4x^2}{\left(1-x^2\right)^2}}\,\mathrm{d}x$$

$$= \int_0^{1/2} \sqrt{\frac{1 + 2x^2 + x^4}{\left(1-x^2\right)^2}}\,\mathrm{d}x = \int_0^{1/2} \sqrt{\frac{\left(1+x^2\right)^2}{\left(1-x^2\right)^2}}\,\mathrm{d}x = \int_0^{1/2} \frac{1+x^2}{1-x^2}\,\mathrm{d}x$$

$$= \int_0^{1/2} \left(-1 + \frac{2}{1-x^2}\right)\mathrm{d}x = \int_0^{1/2} \left(-1 + \frac{A}{1+x} + \frac{B}{1-x}\right)\mathrm{d}x$$

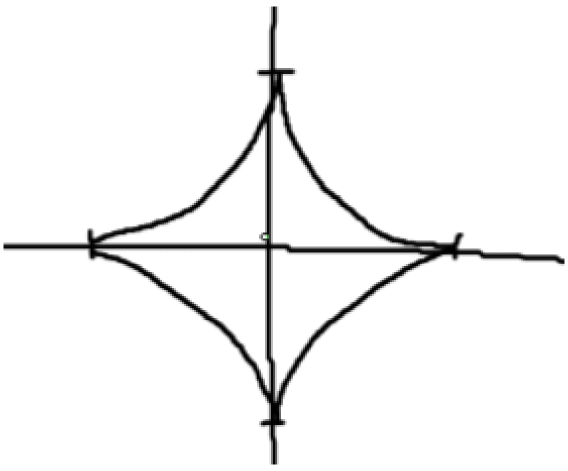
$$\boxed{\frac{A}{1+x} + \frac{B}{1-x} = \frac{2}{1-x^2} \implies A - Ax + B + Bx = 2 \implies \begin{cases} A + B = 2 \\ -A + B = 0 \end{cases} \implies A = 1, B = 1}$$

$$= \int_0^{1/2} \left(-1 + \frac{1}{1+x} + \frac{1}{1-x}\right)\mathrm{d}x = \Big[-x + \ln|1+x| - \ln|1-x| \Big]_0^{1/2}$$

$$= -\frac{1}{2} + \ln\left(1 + \frac{1}{2}\right) - \ln\left(1 - \frac{1}{2}\right) + 0 - \ln 1 + \ln 1 = -\frac{1}{2} + \ln\frac{3}{2} - \ln\frac{1}{2}$$

$$= -\frac{1}{2} + \ln 3 - \ln 2 - \ln 1 + \ln 2 = \boxed{\ln 3 - \frac{1}{2}}$$

Problem 5



$$x^{2/3} + y^{2/3} = 1 \implies y^{2/3} = 1 - x^{2/3} \implies y = \left(1 - x^{2/3}\right)^{3/2}$$

$$\implies \frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = 0 \implies y^{-1/3} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = -x^{-1/3} \implies \frac{1}{\sqrt[3]{y}}\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sqrt[3]{x}}$$

$$\implies \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\sqrt[3]{y}}{\sqrt[3]{x}} = -\frac{\sqrt[3]{\left(1 - x^{2/3}\right)^{3/2}}}{\sqrt[3]{x}} = -\frac{\left(1 - x^{2/3}\right)^{1/2}}{\sqrt[3]{x}} = -\frac{\sqrt{1 - x^{2/3}}}{\sqrt[3]{x}}$$

$$L = 4\int_0^1 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = 4\int_0^1 \sqrt{1 + \left(-\frac{\sqrt{1 - x^{2/3}}}{\sqrt[3]{x}}\right)^2}\,\mathrm{d}x = 4\int_0^1 \sqrt{\frac{\left(\sqrt[3]{x}\right)^2}{\left(\sqrt[3]{x}\right)^2} + \frac{1 - \left(\sqrt[3]{x}\right)^2}{\left(\sqrt[3]{x}\right)^2}}\,\mathrm{d}x$$

$$= 4\int_0^1 \sqrt{\frac{1}{\left(\sqrt[3]{x}\right)^2}}\,\mathrm{d}x = 4\int_0^1 \frac{1}{\sqrt[3]{x}}\,\mathrm{d}x = 4\int_0^1 x^{-1/3}\,\mathrm{d}x = 4\left[\frac{x^{2/3}}{2/3}\right]_0^1 = 4 \cdot \frac{3}{2}(1 - 0) = \boxed{6}$$

Problem 6

$$180 - \frac{x^2}{45} = 0 \implies \frac{x^2}{45} = 180 \implies x^2 = 8100 \implies x = \pm 90 \implies x = 90 \quad \text{(keep only positive root)}$$

$$D = \int_0^{90} \sqrt{1 + \left(\frac{\mathrm{d}}{\mathrm{d}x}\left[180 - \frac{x^2}{45}\right]\right)^2}\,\mathrm{d}x = \int_0^{90} \sqrt{1 + \left(-\frac{2x}{45}\right)^2}\,\mathrm{d}x = \int_0^{90} \sqrt{1 + \frac{4x^2}{45^2}}\,\mathrm{d}x \approx \boxed{209.1\text{m}}$$

Problem 7

$$L = \int_1^4 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_1^4 \sqrt{1 + \left(\frac{\mathrm{d}}{\mathrm{d}x}\int_1^x \sqrt{t^3 - 1}\,\mathrm{d}t\right)^2}\,\mathrm{d}x = \int_1^4 \sqrt{1 + \left(\sqrt{x^3 - 1}\right)^2}\,\mathrm{d}x$$

$$= \int_1^4 \sqrt{1 + x^3 - 1}\,\mathrm{d}x = \int_1^4 x^{3/2}\,\mathrm{d}x = \left[\frac{x^{5/2}}{5/2}\right]_1^4 = \frac{2}{5}(32 - 1) = \frac{2 \cdot 31}{5} = \boxed{\frac{62}{5}}$$