Problem 1

> *Prove or disprove: If $x, y \in \mathbb{R}$ with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.*

Observe that the two regions are bounded by hyperbolas. We will calculate the standard forms for the two hyperbolas and parameterize them. Define $R_0 : y(y+1) \leq (x+1)^2$ and $R_1 : y(y-1) \leq x^2$. Notice that if R_0 were a subset of R_1, then any point on R_0 must automatically be on R_1.

$$
\begin{aligned}
y(y+1) &\leq (x+1)^2 \\
y^2 + y &\leq (x+1)^2 \\
\left(y + \frac{1}{2}\right)^2 - \frac{1}{4} - (x+1)^2 &\leq 0 \\
4\left(y + \frac{1}{2}\right)^2 - 4(x+1)^2 &\leq 1 \\
\frac{\left(y + \frac{1}{2}\right)^2}{(1/2)^2} - \frac{(x+1)^2}{(1/2)^2} &\leq 1
\end{aligned}
$$

$$x_0(t) = -1 + \frac{1}{2}\tan t \tag{1}$$

$$y_0(t) = -\frac{1}{2} + \frac{1}{2}\sec t \tag{2}$$

$$
\begin{aligned}
y(y-1) &\leq x^2 \\
y^2 - y &\leq x^2 \\
\left(y - \frac{1}{2}\right)^2 - \frac{1}{4} - x^2 &\leq 0 \\
4\left(y - \frac{1}{2}\right)^2 - 4x^2 &\leq 1 \\
\frac{\left(y - \frac{1}{2}\right)^2}{(1/2)^2} - \frac{x^2}{(1/2)^2} &\leq 1
\end{aligned}
$$

$$x_1(t) = \frac{1}{2}\tan t \tag{3}$$

$$y_1(t) = \frac{1}{2} + \frac{1}{2}\sec t \tag{4}$$

Next, we calculate the interval for y which satisfies each of the two inequalities at some arbitrary value x.

Observe that x_0, x_1 maps the parameter t to its corresponding x coordinate. We need to calculate x_0^{-1}, x_1^{-1} which will map the x coordinate to the parameter. We can then evaluate y_0, y_1 at the value t calculated by these inverse functions to obtain the desired interval for y

However, x_0^{-1} and y_0^{-1} are actually multivalued, because there are two values t for which a the x-coordinate of a point on a hyperbola equals some constant. Define the upper branch to be x_{00}^{-1} and x_{10}^{-1}, and define the lower branch to be x_{01}^{-1} and x_{11}^{-1}. It follows from this definition that $x_{00}^{-1}(x) > x_{01}^{-1}(x)$ and $x_{10}^{-1}(x) > x_{11}^{-1}(x)$.

$$
\begin{aligned}
&x_0(t) = -1 + \frac{1}{2}\tan t \implies 2x_0(t) + 2 = \tan t \implies x_{00}^{-1}(x) = \arctan(2x+2) + \pi,\ x_{01}^{-1}(x) = \arctan(2x+2) \\
&x_1(t) = \frac{1}{2}\tan t \implies 2x_1(t) = \tan t \implies x_{10}^{-1}(x) = \arctan(2x) + \pi,\ x_{11}^{-1}(x) = \arctan(2x)
\end{aligned}
$$

It is known that:

$$\sec(\arctan\theta) = \sqrt{1+\theta^2} \tag{5}$$

$$\sec(\arctan\theta + \pi) = \frac{1}{\cos(\arctan\theta + \pi)} = \frac{1}{-\cos(\arctan\theta)} = -\sec(\arctan\theta) = -\sqrt{1+\theta^2} \tag{6}$$

The square root of any positive number is positive, and $1 + \theta^2$ must be positive for any real value θ. Therefore:

$$\sec(\arctan\theta) \geq \sec(\arctan\theta + \pi) \leq -1 \tag{7}$$

Therefore, $y_0(\arctan\theta) > y_0(\arctan\theta + \pi)$ for all θ, since y_0 is the secant function scaled by a positive number added to a constant. Likewise, $y_1(\arctan\theta) > y_1(\arctan\theta + \pi)$.

Because $x_{00}^{-1}(x) = x_{01}^{-1}(x) + \pi$, from the above inequality we see that $y_0(x_{01}^{-1}(x)) > y_0(x_{00}^{-1}(x))$. Likewise, by similar logic $y_1(x_{11}^{-1}(x)) > y_1(x_{10}^{-1}(x))$.

Thus, for any value of c, the intersection between the set of all points on R_0 and the set of all points contained on the vertical line $x = c$ is precisely the interval $I_0 = [y_0(x_{00}^{-1}(c)), y_0(x_{01}^{-1}(c))]$. Similarly, the intersection between the line $x = c$ and the region R_1 is the interval $I_1 = [y_1(x_{10}^{-1}(c)), y_1(x_{11}^{-1}(c))]$.

Notice that $\sec(x_{00}^{-1}(c)), \sec(x_{10}^{-1}(c)) < -1$, which means that $y_0(x_{00}^{-1}(c)), y_1(x_{10}^{-1}(c)) < 0$. However, $y \geq 0$ so the intervals should actually be defined as $I_0 = [0, y_0(x_{01}^{-1}(c))]$ and $I_1 = [0, y_1(x_{11}^{-1}(c))]$.

Notice that because y_0 and y_1 differ by a constant, $y_0(x_{01}^{-1}(c)) < y_1(x_{11}^{-1}(c))$. Thus, the right end point of I_0 must be less than the right end point of I_1. This implies that $I_0 \subset I_1$, which implies that $R_0 \subset R_1$.

Since R_0 is a subset of R_1, it follows that every point $(x, y) \in R_0$ must also be on R_1. ☺

Problem 2

Find all polynomial P such that $P(x^2+1) = (P(x))^2 + 1$ and $P(0) = 0$

We start by writing out some values of $P(x)$.

- $P(0) = 0$.
- $P(1) = P(0^2+1) = (P(0))^2 + 1 = 1$.
- $P(2) = P(1^2+1) = (P(1))^2 + 1 = 2$.
- $P(5) = P(2^2+1) = (P(2))^2 + 1 = 5$.

...and so on. We define a sequence $\{a_n\}$ where $a_{n+1} = a_n^2 + 1$ and $a_0 = 0$. I claim that for any $n \in \mathbb{N}_0$, $P(a_n) = a_n$.

Proof. We will use induction.

Base case. $n = 0$. It is known that $P(a_0) = P(0) = 0 = a_0$.

Hypothesis. Suppose that $P(a_n) = a_n$. Need to show that $P(a_{n+1}) = a_{n+1}$.

Inductive step. $P(a_{n+1}) = P\left(a_n^2+1\right) = (P(a_n))^2 + 1 = (a_n)^2 + 1 = a_{n+1}$. ☺

Lemma 1. *$m^2 \geq m$ for all $m \in \mathbb{N}_0$.*

Proof. If $m \geq 1$, we can divide both sides by m to obtain $m \geq 1$, which is true. If $m = 0$, then $0^2 = 0 \geq 0$. ☺

I also claim that a diverges and tends to ∞, which can be shown with the lemma.

Proof. First, we show that $a_n \geq n$ by induction.

Base case. $a_0 = 0 \geq 0$.

Hypothesis. Suppose that $a_n \geq n$. Need to show that $a_{n+1} \geq n+1$.

Inductive step. $a_n \geq n \implies a_n^2 \geq n \implies a_n^2 + 1 \geq n+1 \implies a_{n+1} \geq n+1$.

Therefore, if n was arbitrarily large, a_n would always be larger than or equal to n. Thus, a diverges and tends to ∞. ☺

I will now use some nice calculus theorems we just learned in PCH. P is a polynomial, so it must be differentiable and continuous on $\mathbb{R}$. By the Mean Value theorem, for all $m \in \mathbb{N}_0$, the following must be true for some $c \in (a_m, a_{m+1})$:

$$P'(c) = \frac{P(a_{m+1}) - P(a_m)}{a_{m+1} - a_m} = \frac{a_{m+1} - a_m}{a_{m+1} - a_m} = 1$$

However, because there are infinitely many natural numbers, $P'(x) = 1$ must have infinitely many solutions. Because P is a polynomial, P' is also a polynomial. Let $f(x) = P'(x) - 1$, so f has a root at any and all points where $P'(x) = 1$. $P'(x)$ cannot equal 1 at infinitely many points because f cannot have infinitely many roots. Therefore, P' must be the constant function with $P'(x) = 1$.

$$P(x) = \int 1 \, \mathrm{d}x = x + C$$

$P(0) = 0$ so the constant is 0. Thus, $\boxed{P(x) = x}$.