Problem Set #56

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Problem 1

- $x = 1$ because $f(1)$ is not defined.
- $x = 3$ because $\lim_{x\to 3^-} f(x) = -1 \neq 3 = \lim_{x\to 3^+} f(x)$ so $\lim_{x\to 3} f(x)$ DNE.
- $x = 5$ because $\lim_{x\to 5^-} f(x) = \lim_{x\to 5^+} f(x) = \lim_{x\to 5} f(x) = 1 \neq 3 = f(5)$

Problem 2

(d) **Case 1.** $a \in \mathbb{R} \cap \mathbb{Z}'$

$$\begin{aligned} \lim_{x\to a^+} f(x) &= \lfloor a \rfloor \\ \lim_{x\to a^-} f(x) &= \lfloor a \rfloor \\ \implies \lim_{x\to a} f(x) &= \lfloor a \rfloor \\ f(a) &= \lfloor a \rfloor \end{aligned}$$

Therefore f is continuous on a.

Case 2. $a \in \mathbb{Z}$

$$\begin{aligned} \lim_{x\to a^+} f(x) &= \lfloor a \rfloor = a \\ \lim_{x\to a^-} f(x) &= \lfloor a \rfloor = a - 1 \\ \implies \lim_{x\to a} f(x) &\text{ DNE} \end{aligned}$$

Therefore f is discontinuous on a.

$\boxed{f \text{ is discontinuous for all } a \in \mathbb{Z}}$

Problem 3

Proof. Let a be an integer. We have:

$$\begin{aligned} f(a) &= \lfloor a \rfloor = a \\ \lim_{x\to a^+} f(x) &= \lim_{x\to a^+} \lfloor x \rfloor = a \\ \lim_{x\to a^-} f(x) &= \lim_{x\to a^-} \lfloor x \rfloor = a - 1 \end{aligned}$$

Therefore $\lim_{x\to a^+} f(x) = f(a)$, so f is continuous from the right at any integer. $\lim_{x\to a^-} f(x) \neq f(a)$ so f is discontinuous from the left at any integer. ☺

Problem 5

If f and g are continuous at a, then $\lim_{x\to a} f(x) = f(a)$ and $\lim_{x\to a} g(x) = g(a)$.
Limit rules from PS#54 are in parentheses at the end of the line.

1.
$$\begin{aligned} \lim_{x\to a}[f(x)+g(x)] &\stackrel{?}{=} f(a)+g(a) \\ \lim_{x\to a}f(x)+\lim_{x\to a}g(x) &\stackrel{?}{=} f(a)+g(a) \\ f(a)+g(a) &= f(a)+g(a) \end{aligned} \tag{1}$$

2.
$$\begin{aligned} \lim_{x\to a}[f(x)-g(x)] &\stackrel{?}{=} f(a)-g(a) \\ \lim_{x\to a}f(x)-\lim_{x\to a}g(x) &\stackrel{?}{=} f(a)-g(a) \\ f(a)-g(a) &= f(a)-g(a) \end{aligned} \tag{1}$$

3.
$$\begin{aligned} \lim_{x\to a}cf(x) &\stackrel{?}{=} cf(a) \\ c\lim_{x\to a}f(x) &\stackrel{?}{=} cf(a) \\ cf(a) &= cf(a) \end{aligned} \tag{2}$$

4.
$$\begin{aligned} \lim_{x\to a}f(x)g(x) &\stackrel{?}{=} f(a)g(a) \\ \lim_{x\to a}f(x)\cdot\lim_{x\to a}g(x) &\stackrel{?}{=} f(a)g(a) \\ f(a)g(a) &= f(a)g(a) \end{aligned} \tag{3}$$

5.
$$\begin{aligned} \lim_{x\to a}\frac{f(x)}{g(x)} &\stackrel{?}{=} \frac{f(a)}{g(a)} \\ \frac{\lim_{x\to a}f(x)}{\lim_{x\to a}g(x)} &\stackrel{?}{=} \frac{f(a)}{g(a)} \\ \frac{f(a)}{g(a)} &= \frac{f(a)}{g(a)} \end{aligned} \tag{4}$$

Problem 6

Lemma 1. *For any $k \in \mathbb{N}_0$, cx^k is continuous for all $c, x \in \mathbb{R}$.*

Proof. Notice that because k is a positive integer:

$$x^k = \underbrace{x\cdot x\cdot\ldots\cdot x}_{k\text{ times}} = \prod_{m=1}^{k} x$$

Let $a \in \mathbb{R}$. We will prove that x^k is continuous at $x = a$.

$$\begin{aligned} \lim_{x\to a}cx^k &\stackrel{?}{=} ca^k \\ \lim_{x\to a}c\prod_{m=1}^{k} x &\stackrel{?}{=} ca^k \end{aligned}$$

By limit rule #2 and #3 from PS#54:

$$c\prod_{m=1}^{k}\lim_{x\to a}x \stackrel{?}{=} ca^k$$

By limit rule #7 from PS#54:

$$\begin{aligned} c\prod_{m=1}^{k} a &\stackrel{?}{=} ca^k \\ c\cdot\underbrace{a\cdot a\cdot\ldots\cdot a}_{k\text{ times}} &\stackrel{?}{=} ca^k \\ ca^k &= ca^k \end{aligned}$$

☺

Proof. Any n-degree polynomial can be written in the following form:

$$f(x) = \sum_{k=0}^{n} a_k x^k$$

Where $a_k \in \mathbb{R}$. Then, by Lemma 1, each term of the polynomial is continuous on $\mathbb{R}$. By Theorem 1, f must be continuous on $\mathbb{R}$. ☺

Problem 7

Let $f(x) = \dfrac{x^3 + 2x^2 - 1}{5 - 3x}$. Observe that f is a rational function so f is continuous on its domain by Theorem 2. The domain of f is $5 - 3x \neq 0 \implies x \neq 5/3$.

Because $-2 \neq 5/3$, -2 is within the domain of f. By the definition of continuity, we have:

$$\lim_{x \to -2} f(x) = f(-2)$$

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)}$$

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{-8 + 8 - 1}{5 - (-6)}$$

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \boxed{-\frac{1}{11}}$$

Problem 8

(a) f is a polynomial, so it is continuous on $\boxed{\mathbb{R}}$.

(b) g is a rational function, so it is continuous on its domain. The denominator of g must not equal 0, so $x^2 - 1 \neq 0 \implies x \neq 1, x \neq -1$. Therefore g is continuous on $\boxed{(-\infty, -1) \cup (-1, 1) \cup (1, \infty)}$.

(c)

$$h(x) = \sqrt{x} + \frac{x+1}{x-1} + \frac{x+1}{x^2+1}$$

$$h(x) = \frac{\sqrt{x}(x-1)\left(x^2+1\right)}{(x-1)\left(x^2+1\right)} + \frac{(x+1)\left(x^2+1\right)}{(x-1)\left(x^2+1\right)} + \frac{(x+1)(x-1)}{(x-1)\left(x^2+1\right)}$$

$$h(x) = \frac{\sqrt{x}(x-1)\left(x^2+1\right) + (x+1)\left(x^2+1\right) + (x+1)(x-1)}{(x-1)\left(x^2+1\right)}$$

So h is a rational function. Its domain is $x \neq 1, x \geq 0$.

Therefore, h is continuous on $\boxed{[0, 1) \cup (1, \infty)}$.

Problem 9

Let $f(x) = \dfrac{\sin x}{2 + \cos x}$. Observe that f is a rational function so f is continuous on its domain by Theorem 2. The domain of f is $2 + \cos x \neq 0 \implies \cos x = -2$. $\cos x$ does not equal -2 for real values of x, so we can say that the domain of f is $\mathbb{R}$. It is also continuous on $\mathbb{R}$.

Clearly $\pi \in \mathbb{R}$. By the definition of continuity:

$$\lim_{x\to\pi} \frac{\sin x}{2+\cos x} = \frac{\sin \pi}{2+\cos\pi}$$
$$\lim_{x\to\pi} \frac{\sin x}{2+\cos x} = \frac{0}{2+(-1)}$$
$$\lim_{x\to\pi} \frac{\sin x}{2+\cos x} = \boxed{0}$$

Problem 10

Proof. 1. Let $y = g(x)$.

2. For all $\varepsilon > 0$, there exists some $\delta > 0$ such that $|x-a| < \delta \implies |y-b| < \varepsilon$.
3. Because f is continuous at b, by the definition of continuity $\lim\limits_{y\to b} f(y) = f(b) \iff$ for all $\varepsilon' > 0$, there exists some $\delta' > 0$ such that $|y-b| < \delta' \implies |f(y) - f(b)| < \varepsilon'$.
4. Let $\varepsilon = \delta'$. Then, $|x-a| < \delta \implies |y-b| < \delta' \implies |f(y)-f(b)| < \varepsilon'$.
5. Recall that $y = g(x)$. Therefore $|x-a| < \delta \implies |f(g(x)) - f(b)| < \varepsilon'$. In other words $\lim\limits_{x\to a} f(g(x)) = f(b)$.

☺

Problem 11

(a) Let $f(x) = \sin x$ and $g(x) = x^2$. Then $h(x) = \sin(x^2) = f(g(x)) = (f \circ g)(x)$. Notice that the domain of both f and g are $\mathbb{R}$, and the range of g is a subset of $\mathbb{R}$.

Let a be some real number. g is continuous at a because g is a polynomial and is continuous on $\mathbb{R}$. Likewise, f is a trigonometric function and is continuous on its domain. f is defined on $\mathbb{R}$ so it must be continuous on $\mathbb{R}$. Because $g(a)$ must be real, f is continuous at $g(a)$.

Because g is continuous at a and f is continuous at $g(a)$, by Theorem 5 we have that $f \circ g$ is continuous at a. a is any real number. Therefore h is continuous on $\mathbb{R}$.

(b) Let $p(x) = \dfrac{1}{x}$ and $q(x) = \sqrt{x^2+7} - 4$. Then $F(x) = (p \circ q)(x)$.

Let a be some real number. By Theorem 5, $p \circ q$ is continuous at a if p is continuous at $q(a)$ and g is continuous at a.

Let $r(x) = \sqrt{x^2+7}$ and $s(x) = 4$. Then $q(x) = (r - s)(x)$. r is continuous on $\mathbb{R}$ because it is a root function and root functions are continuous on its domain by Theorem 3, and the domain of r is $\mathbb{R}$ because $x^2 + 7 \not< 0$ for all $x \in \mathbb{R}$. s is obvious continuous everywhere. Therefore, by Theorem 1 q is continuous on $\mathbb{R}$.

p is a rational function and by Theorem 3 is continuous everyone on its domain. So p is continuous at $q(a)$ if and only if it is defined at $q(a)$. p is defined at q(a) if $q(a) \neq 0$.

$$\sqrt{a^2+7} - 4 \neq 0$$
$$a^2 + 7 \neq 16$$
$$a^2 \neq 9$$

$$a \neq \pm 3$$

Thus q is defined for all real numbers except for 3 and -3.

Therefore, F is continuous on $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$

Problem 12

(a)
1. Let $f(x) = 4x^3 - 6x^2 + 3x - 2$.
2. Let $N = 0$.
3. f is continuous on $\mathbb{R}$ because f is a polynomial and polynomials are continuous on $\mathbb{R}$ by Theorem 2.
4. $f(1) = 4 - 6 + 3 - 2 = -1$ and $f(2) = 32 - 24 + 6 - 2 = 12$. Observe that $f(1) < N < f(2)$.
5. Therefore, by the IVT, there exists some $c \in (1, 2)$ such that $f(c) = 0$.
6. Thus the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one real root.

(b)
1. Let $g(x) = \cos x - x$.
2. Let $N = 0$.
3. g is continuous on $\mathbb{R}$. cos is continuous on its domain ($\mathbb{R}$) as it is a trigonometric function, while x is a polynomial and continuous on $\mathbb{R}$.
4. $g(0) = 1 - 0 = 1$ and $g(\pi/2) = -\pi/2$. Observe that $g(\pi/2) < N < g(0)$.
5. Therefore, by the IVT, there exists some $c \in (0, \pi/2)$ such that $g(c) = 0$ and the equation $g(x) = 0$ has at least one real root.
6. Thus $\cos x - x = 0$ has at least one real root, so $\cos x = x$ has at least one real root.

AP Corner

13. A
14. f is continuous (given). $f(-\pi) \approx 0.14$ and $f(-\pi/2) \approx 0.45$. $f(-\pi) < 0.240 < f(-\pi/2)$. Therefore there exists some $c(-\pi, -\pi/2)$ such that $f(c) = 0.240$ by the IVT. $c \approx -2.09$.
15. A