Problem Set #53

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Problem 7

(b)

$$
\begin{aligned}
|3-\sqrt{3}i| &= \sqrt{9+3} = \sqrt{12} \\
\arg(3-\sqrt{3}i) &= \arctan\left(\frac{-\sqrt{3}}{3}\right) = -\frac{\pi}{6}
\end{aligned}
$$

$$
\begin{aligned}
(3-\sqrt{3})^5 &= \left(\sqrt{12}e^{-\frac{i\pi}{6}}\right)^5 \\
&= (\sqrt{12})^5\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)^5 \\
&= (\sqrt{12})^4(\sqrt{12})\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right) \\
&= 144(2\sqrt{3})\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) \\
&= -\frac{144\cdot 2\cdot 3}{2} - \frac{288\sqrt{3}}{2}i \\
&= \boxed{-432-144\sqrt{3}i}
\end{aligned}
$$

Problem 10

(a) Let $z = a+bi$. Then $\overline{z} = a-bi$, $\operatorname{Re}(z) = a$ and $\operatorname{Im}(z) = b$.

$$
\begin{aligned}
\frac{z+\overline{z}}{2} &= \frac{a+bi+a-bi}{2} = \frac{2a}{2} = \operatorname{Re}(z) \\
\frac{z-\overline{z}}{2i} &= \frac{a+bi-a+bi}{2i} = \frac{2bi}{2i} = b = \operatorname{Im}(z)
\end{aligned}
$$

$\square$

(b)

$$
\begin{aligned}
e^{i\theta} &= \cos\theta + i\sin\theta && (1) \\
e^{-i\theta} &= \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta && (2)
\end{aligned}
$$

(1) + (2):

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$
$$\frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) = \cos\theta$$

(1) − (2):

$$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$$
$$\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) = \sin\theta$$

$\square$

(c)

$$\begin{aligned}
\sin(i) &= \frac{1}{2i}\left(e^{i\cdot i} - e^{-i\cdot i}\right) \\
&= \frac{1(-i)}{2i(-i)}\left(e^{-1} - e^{1}\right) \\
&= \frac{-i\left(e^{-1} - e^{1}\right)}{2} \\
&= i\left(\frac{e^{1} - e^{-1}}{2}\right) \\
\boxed{\sin(i) = i\sinh 1} \\
&= \frac{i}{2}\left(\frac{e^2}{e} - \frac{1}{e}\right) \\
\boxed{\sin(i) = \frac{ie^2 - i}{2e}}
\end{aligned}$$

$$\begin{aligned}
\cos(i) &= \frac{1}{2}\left(e^{i\cdot i} + e^{-i\cdot i}\right) \\
&= \frac{e^{-1} + e^{1}}{2} \\
\boxed{\cos(i) = \cosh 1} \\
&= \frac{1}{2}\left(\frac{1}{e} + \frac{e^2}{e}\right) \\
\boxed{\cos(i) = \frac{1 + e^2}{2e}}
\end{aligned}$$

(d) Let $\theta = ix$. Then $x = -i\theta$.

$$\begin{aligned}
\cos(ix) &= \frac{e^{i\cdot ix} + e^{-i\cdot ix}}{2} \\
\cos\theta &= \frac{e^{-x} + e^{x}}{2} \\
&= \cosh x \\
&= \cosh(-i\theta) \\
\boxed{\cos\theta = \cosh(i\theta)}
\end{aligned}$$

$$\begin{aligned}
\sin(ix) &= \frac{e^{i\cdot ix} - e^{-i\cdot ix}}{2i} \\
\sin\theta &= \frac{e^{-x} - e^{x}}{2} \cdot \frac{1}{i} \\
&= \frac{e^{x} - e^{-x}}{2} \cdot \frac{1}{-i}\left(\frac{i}{i}\right) \\
&= i\sinh x \\
&= i\sinh(-i\theta) \\
\boxed{\sin\theta = -i\sinh(i\theta)}
\end{aligned}$$

(e)

$$\sin(i) = i\sinh 1$$
$$\cos(i) = \cosh 1$$