Problem Set #46

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Problem 1

(a)

$$
\begin{aligned}
PF_1 - PF_2 &= \pm 2a \\
\sqrt{(x-(-c))^2+(y-0)^2} - \sqrt{(x-c)^2+(y-0)^2} &= \pm 2a \\
\left(\sqrt{x^2+2xc+c^2+y^2} - \sqrt{x^2-2xc+c^2+y^2}\right)^2 &= (\pm 2a)^2 \\
\begin{array}{r} x^2+2xc+c^2+y^2+x^2-2xc+c^2+y^2 \\ -2\sqrt{\left(x^2+2xc+c^2+y^2\right)\left(x^2-2xc+c^2+y^2\right)} \end{array} &= 4a^2 \\
2\sqrt{\left(x^2+2xc+c^2+y^2\right)\left(x^2-2xc+c^2+y^2\right)} &= 2x^2+2c^2+2y^2-4a^2 \\
\left(x^2+2xc+c^2+y^2\right)\left(x^2-2xc+c^2+y^2\right) &= \left(x^2+c^2+y^2-2a^2\right)^2 \\
\begin{array}{r} \cancel{x^4} - \cancel{2x^3c} + \cancel{x^2c^2} + \cancel{x^2y^2} + \\ \cancel{2x^3c} - 4x^2c^2 + \cancel{2xc^3} + \cancel{2xcy^2} + \\ \cancel{x^2c^2} - \cancel{2xc^3} + \cancel{c^4} + \cancel{y^2c^2} + \\ \cancel{x^2y^2} - \cancel{2xcy^2} + \cancel{y^2c^2} + \cancel{y^4} \end{array} &= \begin{array}{l} \cancel{x^4} + \cancel{x^2c^2} + \cancel{x^2y^2} - 2x^2a^2 + \\ \cancel{x^2c^2} + \cancel{c^4} + \cancel{y^2c^2} - 2c^2a^2 + \\ \cancel{x^2y^2} + \cancel{y^2c^2} + \cancel{y^4} - 2y^2a^2 - \\ 2x^2a^2 - 2c^2a^2 - 2y^2a^2 + 4a^4 \end{array} \\
2x^2a^2+2c^2a^2+2y^2a^2+2x^2a^2+2c^2a^2+2y^2a^2-4a^4 &= 4x^2c^2 \\
2x^2a^2+2\left(a^2+b^2\right)a^2+2y^2a^2-2a^4 &= 2x^2\left(a^2+b^2\right) \\
\cancel{x^2a^2}+\cancel{a^4}+a^2b^2+y^2a^2-\cancel{a^4} &= \cancel{x^2a^2}+x^2b^2 \\
x^2b^2-y^2a^2 &= a^2b^2
\end{aligned}
$$

$$\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1} \tag{1}$$

(b) The hyperbola given by equation (1) is centered at the origin $(0,0)$. A hyperbola centered at a point (h,k) is given by applying to (1) a horizontal translation of h units and a vertical translation of k units. Let (x_1, y_1) be members of the solution set of the new hyperbola. Because the new hyperbola is given by a translation to (1), we have $x_1 = x+h, y_1 = y+k \implies x = x_1-h, y = y_1-k$. Substituting this to (1) gives:

$$\frac{(x_1-h)^2}{a^2}-\frac{(y_1-k)^2}{b^2}=1$$

Thus, the equation of the new hyperbola is:

$$\boxed{\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1} \tag{2}$$

(c) First, we will find the equation of a hyperbola centered at the origin whose transverse axis is vertical. This hyperbola is a reflection of the hyperbola given by (1) across the line $y = x$. This is equivalent to the inverse of (1), which is given by switching the x and y variables.

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

To find the equation centered at (h, k) we apply a horizontal translation of h units and a vertical translation of k units.

$$\boxed{\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1} \tag{3}$$

Problem 2

(a)

$$\begin{aligned}
\frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 \\
x^2b^2 - y^2a^2 &= a^2b^2 \\
y^2a^2 &= x^2b^2 - a^2b^2 \\
y^2 &= \frac{x^2b^2}{a^2} - b^2 \\
y &= \pm\sqrt{\frac{b^2}{a^2}(x^2 - a^2)} \\
&\boxed{y = \pm\frac{b}{a}\sqrt{x^2 - a^2}}
\end{aligned}$$

Explicit formula for y from (2):

$$\begin{aligned}
\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} &= 1 \\
(x-h)^2b^2 - (y-k)^2a^2 &= a^2b^2 \\
(y-k)^2a^2 &= (x-h)^2b^2 - a^2b^2 \\
(y-k)^2 &= \frac{(x-h)^2b^2}{a^2} - b^2 \\
y - k &= \pm\sqrt{\frac{b^2}{a^2}((x-h)^2 - a^2)} \\
&\boxed{y = \pm\frac{b}{a}\sqrt{(x-h)^2 - a^2} + k}
\end{aligned}$$

$\square$

(b) i.

$$\begin{aligned}
y &\to \lim_{x\to\infty} \pm\frac{b}{a}\sqrt{x^2 - a^2} \\
y &\to \pm\lim_{x\to\infty} \frac{b}{a}\sqrt{x^2\left(1 - \cancel{\frac{a^2}{x^2}}\right)} \\
y &\to \pm\lim_{x\to\infty} \frac{b|x|}{a} \\
&\boxed{y \to \pm\infty}
\end{aligned}$$

ii.

$$\begin{aligned}
y &\to \lim_{x\to\infty} \left(\pm\frac{b}{a}\sqrt{(x-h)^2 - a^2} + k\right) \\
y &\to \pm\lim_{x\to\infty} \frac{b}{a}\sqrt{(x-h)^2\left(1 - \cancel{\frac{a^2}{(x-h)^2}}\right)} + k \\
y &\to \pm\lim_{x\to\infty} \frac{b|x-h|}{a} + k \\
&\boxed{y \to \pm\infty}
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} &= 1 \\
(y-k)^2b^2 - (x-h)^2a^2 &= a^2b^2 \\
(y-k)^2 &= \frac{a^2b^2 + (x-h)^2a^2}{b^2} \\
y - k &= \pm\sqrt{\frac{a^2}{b^2}(b^2 + (x-h)^2)} \\
y &= \pm\frac{a}{b}\sqrt{b^2 + (x-h)^2} + k
\end{aligned}$$

$$\begin{aligned}
y &\to \lim_{x\to\infty} \left(\pm\frac{a}{b}\sqrt{b^2 + (x-h)^2} + k\right) \\
y &\to \pm\lim_{x\to\infty} \frac{a}{b}\sqrt{(x-h)^2\left(\cancel{\frac{b^2}{(x-h)^2}} + 1\right)} + k \\
y &\to \pm\lim_{x\to\infty} \frac{a|x-h|}{b} + k \\
&\boxed{y \to \pm\infty}
\end{aligned}$$

The behavior of y as $x \to \infty$ does not change.

(d)

$$e = \frac{c}{a}$$
$$e^2 = \frac{a^2 + b^2}{a^2}$$
$$e^2 = 1 + \frac{b^2}{a^2}$$
$$e = \pm\sqrt{1 + \frac{b^2}{a^2}}$$

Eccentricity must be positive.

$$\boxed{e = \sqrt{1 + \frac{b^2}{a^2}}}$$