Problem Set #54

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Problem 4

(c)

$$\lim_{x\to 0^-}\left(\frac{1}{x}-\frac{1}{|x|}\right)=\lim_{x\to 0^-}\left(\frac{1}{x}-\frac{1}{-x}\right)=\lim_{x\to 0^-}\left(\frac{1}{x}+\frac{1}{x}\right)=\lim_{x\to 0^-}\left(\frac{2}{x}\right)=\boxed{-\infty}$$

Problem 5

(a)

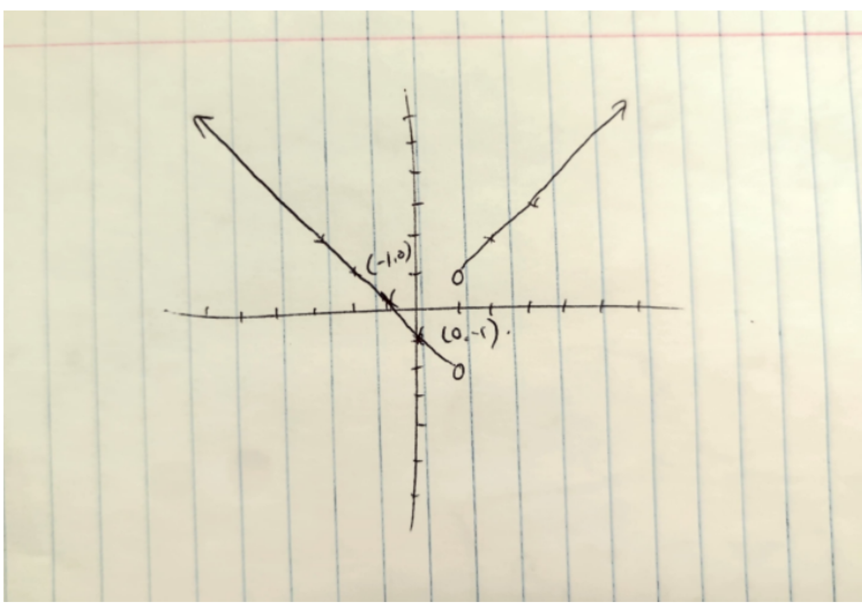
$$\lim_{x\to 1^+}F(x)=\lim_{x\to 1^+}\frac{x^2-1}{|x-1|}=\lim_{x\to 1^+}\frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}=\lim_{x\to 1^+}(x+1)=\boxed{2}$$

$$\lim_{x\to 1^-}F(x)=\lim_{x\to 1^-}\frac{x^2-1}{|x-1|}=\lim_{x\to 1^-}\frac{(x+1)\cancel{(x-1)}}{-\cancel{(x-1)}}=\lim_{x\to 1^-}\frac{x+1}{-1}=-\lim_{x\to 1^-}(x+1)=\boxed{-2}$$

(b) No because $\lim_{x\to 1^+}F(x)\neq\lim_{x\to 1^-}F(x)$.

(c)

$$F(x)=\frac{x^2-1}{|x-1|}=\begin{cases}\dfrac{(x+1)(x-1)}{x-1} & x-1>0\\ \dfrac{(x+1)(x-1)}{-(x-1)} & x-1<0\end{cases}=\begin{cases}x+1 & x>1\\ -x-1 & x<1\end{cases}$$



Problem 6

$$\lim_{x\to 3+} \lfloor x \rfloor = 3 \quad (x \text{ is greater than 3 but less than 4 since } x \text{ approaches 3 from the right.})$$
$$\lim_{x\to 3-} \lfloor x \rfloor = 2 \quad (x \text{ is less than 3 but greater than 2 since } x \text{ approaches 3 from the left.})$$

$\lim_{x\to 3^+} \lfloor x \rfloor \neq \lim_{x\to 3^-} \lfloor x \rfloor$ so the limit does not exist.

Problem 7

(a)

$$\lim_{x\to -2^+} \lfloor x \rfloor = \boxed{-2}$$
$$\lim_{x\to -2^-} \lfloor x \rfloor = -3 \implies \lim_{x\to -2^-} \lfloor x \rfloor \neq \lim_{x\to -2^+} \lfloor x \rfloor \implies \boxed{\lim_{x\to -2} \lfloor x \rfloor \text{ DNE}}$$
$$\lim_{x\to -2.4^+} \lfloor x \rfloor = \lim_{x\to -2.4^-} \lfloor x \rfloor = -3 \implies \lim_{x\to -2.4} \lfloor x \rfloor = \boxed{-3}$$

(b)

$$\lim_{x\to n^-} \lfloor x \rfloor = \boxed{n-1}$$
$$\lim_{x\to n^+} \lfloor x \rfloor = \boxed{n}$$

Problem 8

$$f(x) = \lfloor x \rfloor + \lfloor -x \rfloor$$
$$f(2) = \lfloor 2 \rfloor + \lfloor -2 \rfloor = 2 + (-2) = 0$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \lfloor x \rfloor + \lim_{x\to 2^+} \lfloor -x \rfloor = \lim_{x\to 2^+} \lfloor x \rfloor + \lim_{x\to -2^-} \lfloor x \rfloor = 2 + (-3) = -1$$
$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \lfloor x \rfloor + \lim_{x\to 2^-} \lfloor -x \rfloor = \lim_{x\to 2^-} \lfloor x \rfloor + \lim_{x\to -2^+} \lfloor x \rfloor = 2 + (-3) = -1$$
$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = -1 \implies \lim_{x\to 2} f(x) = -1$$

The limit exists and equals -1, which is not equal to $f(x) = 0$.

Problem 9

p is a polynomial and therefore continuous on $\mathbb{R}$. Therefore $\lim_{x\to a} p(x) = p(a)$ on the interval of continuity (all $a \in \mathbb{R}$) (definition of continuous).

Problem 10

If $\lim_{x\to 1} \dfrac{f(x)-8}{x-1} = 10$, then $f(x) - 8$ must be divisible by $x - 1$ and the quotient is 10 when $x = 1$. It follows that when $x = 1$, $10(x-1) = f(x) - 8 \iff f(x) = 10x - 2$. By direct substitution,

$\lim_{x\to 1} f(x) = \lim_{x\to 1}(10x-2) = 10(1)-2 = \boxed{8}$.

Problem 11

Let $f(x) = \frac{|x|}{x}$, $g(x) = -\frac{|x|}{x} = -f(x)$ and $a = 0$.

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x}{x} = 1 \neq -1 = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} f(x)$$

$$\lim_{x\to 0^+} g(x) = -\lim_{x\to 0^+} \frac{x}{x} = -1 \neq 1 = -\lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} g(x)$$

The one-sided limits of f and g are not equal so $\lim_{x\to 0} f(x)$ and $\lim_{x\to 0} g(x)$ DNE.

However, $\lim_{x\to 0} f(x)g(x)$ exists.

$$2\lim_{x\to 0} f(x)g(x) = \lim_{x\to 0}\left(-\frac{|x|\cdot|x|}{x\cdot x}\right) = -\lim_{x\to 0}\frac{|x^2|}{x^2} = -\lim_{x\to 0}\frac{x^2}{x^2} = -1$$

☺

Problem 12

$$\lim_{x\to -2}\frac{3x^2+ax+a+3}{x^2+x-2} = \lim_{x\to -2}\frac{3x^2+ax+a+3}{(x+2)(x-1)}$$

$x+2$ must be a factor of the numerator $3x^2+ax+a+3$ for the discontinuity at $x=-2$ to be removable. By the factor theorem the numerator must equal 0 when $x=-2$.

$$\begin{aligned} 3x^2+ax+a+3 &= 0 \\ 3(-2)^2+a(-2)+a+3 &= 0 \\ 12-2a+a+3 &= 0 \\ -a &= -15 \\ \boxed{a = 15} \end{aligned}$$

$$\lim_{x\to -2}\frac{3x^2+15x+15+3}{(x+2)(x-1)} = \lim_{x\to -2}\frac{3x^2+9x+6x+18}{(x+2)(x-1)} = \lim_{x\to -2}\frac{3\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x-1)} = \frac{3(-2+3)}{-2-1} = \boxed{-1}$$