Collection of Calculus Problems

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Problem 1

$$\begin{aligned} x^2+xy+y^2 &= 12 \\ y^2+xy+x^2-12 &= 0 \\ \frac{-x \pm \sqrt{x^2-4x^2+48}}{2} &= y \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \frac{1}{2}\left(-1 \pm \frac{2x-8x}{2\sqrt{x^2-4x^2+48}}\right) &= 0 \\ \pm\frac{-6x}{2\sqrt{-3x^2+48}} &= 1 \\ \pm(-6x) &= 2\sqrt{-3x^2+48} \\ 36x^2 &= 4\left(-3x^2+48\right) \\ 48x^2 &= 4 \cdot 48 \\ x^2 &= 4 \\ x &= \pm 2 \end{aligned}$$

$$\begin{aligned} x^2-4x^2+48 &\geq 0 \\ 3x^2 &\leq 48 \\ |x| &\leq 4 \\ x &\in [-4,4] \end{aligned}$$

CPs: $x=\pm 2$. Endpoints: $x=\pm 4$.

r	$v(r)$
-4	2
-2	$-2, 4$
2	$-4, 2$
4	-2

Lowest point: $(2,-4)$.
Highest point: $(-2,4)$.

Problem 2

Let Sgn be a function defined as follows:

$$\operatorname{Sgn}(x) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

It follows that the derivative of $|x|$ is $\operatorname{Sgn}(x)$.

$$f(x) = \frac{1}{1+|x|} + \frac{1}{1+|x-2|}$$

$1+|x| \neq 0$ and $1+|x-2| \neq 0$, so the domain of f is $\mathbb{R}$.

$$\begin{aligned} f'(x) &= 0 \\ \frac{\operatorname{Sgn}(x)}{(1+|x|)^2} + \frac{\operatorname{Sgn}(x-2)}{(1+|x-2|)^2} &= 0 \\ \frac{(1+|x-2|)^2\operatorname{Sgn}(x) + (1+|x|)^2\operatorname{Sgn}(x-2)}{(1+|x|)^2(1+|x-2|)^2} &= 0 \end{aligned}$$

Because $\operatorname{Sgn}(0)$ is undefined, f is not differentiable at $\boxed{x=0}$ and $\boxed{x=2}$. The denominator is never 0.

$$(1+|x-2|)^2\operatorname{Sgn}(x) + (1+|x|)^2\operatorname{Sgn}(x-2) = 0$$

Case 1. $x > 2$.

$$\begin{aligned} (1+x-2)^2 + (1+x)^2 &= 0 \\ x^2-2x+1+x^2+2x+1 &= 0 \\ 2x^2 &= -2 \end{aligned}$$

No real solutions on $(2,\infty)$.

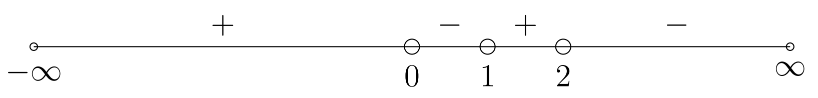
Case 2. $0 < x < 2$.

$$\begin{aligned} (1-(x-2))^2 - (1+x)^2 &= 0 \\ (3-x)^2 - (x+1)^2 &= 0 \\ 9-6x+x^2-x^2-2x-1 &= 0 \\ -8x+8 &= 0 \\ \boxed{x=1} \end{aligned}$$

Case 3. $x < 0$.

$$\begin{aligned} -(1-(x-2))^2 - (1-x)^2 &= 0 \\ -(3-x)^2 - (1-x)^2 &= 0 \\ -9+6x-x^2-1+2x-x^2 &= 0 \\ -2x^2+8x-10 &= 0 \\ x^2-4x+5 &= 0 \\ (x-2)^2-4+5 &= 0 \\ (x-2)^2 &= -1 \end{aligned}$$

No real solutions on $(-\infty,0)$.
CPs: $x=0, x=1, x=2$.



$$f(0) = f(2) = 1 + \frac{1}{3} = \frac{4}{3}$$

- f is increasing on $(\infty, 0)$. So $f(0) = 4/3 > f(c)$ for all $c \in (-\infty, 0)$.
- f is decreasing on $(2,\infty)$. So $f(2) = 4/3 > f(c)$ for all $c \in (2,\infty)$.
- f is decreasing on $(0,1)$. So $f(0) = 4/3 > f(c)$ for all $c \in (0,1)$.
- f is increasing on $(1,2)$. So $f(2) = 4/3 > f(c)$ for all $c \in (1,2)$.

$4/3 > f(c)$ for all $c \in \mathbb{R} \setminus \{0,2\}$. Therefore, $\boxed{4/3}$ is the absolute maximum.

Problem 4

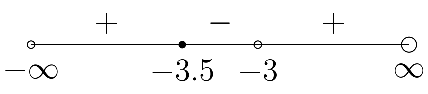
$$\begin{aligned} f(x) &= \left(a^2+a-6\right)\cos 2x + (a-2)x + \cos 1 \\ f'(x) &= (a+3)(a-2)(-2\sin 2x) + (a-2) \\ f'(x) &= (a-2)\left(-2(a+3)\sin 2x + 1\right) \end{aligned}$$

f is differentiable on $\mathbb{R}$. Therefore, some point c is a critical point of f if and only if $f'(c) = 0$.

$$\begin{aligned} f'(c) &= 0 \\ (a-2)\left(-2(a+3)\sin 2c + 1\right) &= 0 \\ 2(a+3)\sin 2c &= 1 \\ \sin 2c &= \frac{1}{2(a+3)} \end{aligned}$$

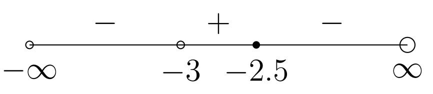
Notice there is a solution iff $1/2(a+3) \in [-1,1]$. Also, if $a=2$, then $f'(x)=0$.

$$\begin{aligned} \frac{1}{2(a+3)} &\geq -1 \\ \frac{1}{2(a+3)} + \frac{2(a+3)}{2(a+3)} &\geq 0 \\ \frac{2a+7}{2a+6} &\geq 0 \end{aligned}$$



$$x \in \left(-\infty, -\frac{7}{2}\right] \cup (-3,\infty)$$

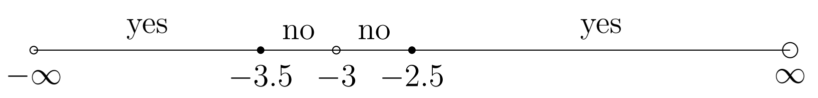
$$\begin{aligned} \frac{1}{2(a+3)} &\leq 1 \\ \frac{1}{2(a+3)} - \frac{2(a+3)}{2(a+3)} &\leq 0 \\ \frac{-2a-5}{2a+6} &\geq 0 \end{aligned}$$



$$a \in (-\infty,-3) \cup \left[-\frac{5}{2},\infty\right)$$

Notice that if $a=-3$, then $1/2(a+3)$ would be undefined, which means $\sin 2c$ must be equal to it and thus f would have no critical points.

$$a \in \left(\left(-\infty,-\frac{7}{2}\right] \cup (-3,\infty)\right) \cap \left((-\infty,-3) \cup \left[-\frac{5}{2},\infty\right)\right) \cup \{3\}$$



But, we want the values of a that do not satisfy the inequality, so we take the intervals marked "no."

$$a \in \left[-\frac{7}{2},-3\right) \cup \left(-3,-\frac{5}{2}\right] \cup \{3\}$$

$$\boxed{a \in \left[-\frac{7}{2},-\frac{5}{2}\right]}$$

This set does not contain 2, which is good.