Finish Problem Set #41

Jayden Li

January 13, 2024

Problem 3

(d)
$$\sum_{k=1}^{7} \frac{1}{\sqrt[3]{k^2} + \sqrt[3]{k(k+1)} + \sqrt[3]{(k+1)^2}}$$
$$= \sum_{k=1}^{7} \frac{1}{\left(\sqrt[3]{k}\right)^2 + \sqrt[3]{k}\sqrt[3]{k+1} + \left(\sqrt[3]{k+1}\right)^2}$$

Let $a = \sqrt[3]{k}$, $b = \sqrt[3]{k+1}$

$$= \left(\sum_{k=1}^{7} \frac{1}{a^2 + ab + b^2} \cdot \frac{a-b}{a-b}\right)$$
$$= \sum_{k=1}^{7} \frac{a-b}{a^3 - b^3}$$
$$= \sum_{k=1}^{7} \frac{\sqrt[3]{k}}{\left(\sqrt[3]{k}\right)^3 - \left(\sqrt[3]{k+1}\right)^3} - \sum_{k=1}^{7} \frac{\sqrt[3]{k+1}}{\left(\sqrt[3]{k}\right)^3 - \left(\sqrt[3]{k+1}\right)^3}$$
$$= \sum_{k=1}^{7} \frac{\sqrt[3]{k}}{k-k-1} - \sum_{k=1}^{7} \frac{\sqrt[3]{k+1}}{k-k-1}$$
$$= -\sum_{k=1}^{7} \sqrt[3]{k} - \left(-\sum_{k=1}^{7} \sqrt[3]{k+1}\right)$$
$$= \sum_{k=1}^{7} \sqrt[3]{k+1} - \sum_{k=1}^{7} \sqrt[3]{k}$$
$$= \cancel{\sqrt[3]{2} + \cdots + \sqrt[3]{7}} + \sqrt[3]{8} - \sqrt[3]{1} - \cancel{\sqrt[3]{2} - \cdots - \sqrt[3]{7}}$$
$$= \sqrt[3]{8} - \sqrt[3]{1}$$
$$= \boxed{1}$$

Problem 4

Case 1. n is even

$$\sum_{k=1}^{n} (-1)^{k+1} k$$
$$= 1 - 2 + 3 - 4 + \cdots - (n-2) + (n-1) - n$$
$$= 1 - (2-3) - \cdots - ((n-2) - (n-1)) - n$$
$$= 1 \underbrace{-(-1) - (-1) - \cdots - (-1)}_{\frac{n}{2}-1 \text{ terms}} - n$$
$$= 1 + \frac{n}{2} - 1 - n$$
$$= \frac{n}{2} - \frac{2n}{2}$$
$$= -\frac{n}{2}$$

Case 2. n is odd

$$\sum_{k=1}^{n} (-1)^{k+1} k$$
$$= 1 - 2 + 3 - 4 + \cdots - (n-1) + n$$
$$= 1 - (2-3) - \cdots - ((n-1) - n)$$
$$= 1 \underbrace{-(-1) - (-1) - \cdots - (-1)}_{\frac{n+1}{2} \text{ terms}}$$
$$= 1 + \frac{n+1}{2} - 1$$
$$= \frac{n+1}{2}$$

(a)
$$\sum_{k=1}^{200} (-1)^{k+1} k = -\frac{200}{2} = \boxed{-100}$$

(b)
$$\sum_{k=1}^{403} (-1)^{k+1} k = \frac{403+1}{2} = \boxed{202}$$

Problem 5

(a) $$\sum_{n=1}^{100} \frac{1}{n(n+2)}$$

$$= \sum_{n=1}^{100} \left(\frac{A}{n} + \frac{B}{n+2} \right)$$

$$A(n+2) + Bn = 1$$
$$n(A+B) + 2A = 1$$

$$2A = 1 \quad A + B = 0$$
$$A = \frac{1}{2} \qquad B = -\frac{1}{2}$$

$$= \sum_{n=1}^{100} \left(\frac{1}{2n} - \frac{1}{2n+4} \right)$$

$$= \frac{1}{2} \sum_{n=1}^{100} \left(\frac{1}{n} - \frac{1}{n+2} \right)$$

$$= \frac{1}{2} \left(\sum_{n=1}^{100} \frac{1}{n} - \sum_{n=1}^{100} \frac{1}{n+2} \right)$$

$$= \frac{1}{2} \left(\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{99} + \frac{1}{100} \right) - \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{101} + \frac{1}{102} \right) \right)$$

$$= \frac{1}{2} \left(\frac{1}{1} + \frac{1}{2} + \cancel{\frac{1}{3}} + \cancel{\cdots} + \cancel{\frac{1}{99}} + \cancel{\frac{1}{100}} - \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} - \cancel{\cdots} - \cancel{\frac{1}{100}} - \frac{1}{101} - \frac{1}{102} \right)$$

$$= \frac{1}{2} \left(\frac{3}{2} - \frac{1}{101} - \frac{1}{102} \right)$$

$$= \frac{1}{2} \left(\frac{15453}{10302} - \frac{102}{10302} - \frac{101}{10302} \right)$$

$$= \frac{15453 - 102 - 101}{20604}$$

$$= \frac{15250}{20604}$$

$$= \boxed{\frac{7625}{10302}}$$

(b) $$\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$$

$$= \frac{1}{2} \left(\sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n+2} \right)$$

$$= \frac{1}{2} \left(\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \right) - \left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots \right) \right)$$

$$= \frac{1}{2} \left(\frac{1}{1} + \frac{1}{2} + \cancel{\frac{1}{3}} + \cancel{\frac{1}{4}} + \cancel{\cdots} - \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} - \cancel{\frac{1}{6}} - \cancel{\cdots} \right)$$

$$= \boxed{\frac{3}{4}}$$

Because this is an infinite series, when we break the sum into two sums they will cancel each other out, except for the first two terms.

Problem 6

(a)

$$\begin{aligned}
F_{n+2} &= F_{n+2} \\
F_{n+2} &= F_{n+1} + F_{n+2} - F_{n+1} \\
1 &= \frac{F_{n+1} + F_{n+2} - F_{n+1}}{F_{n+2}} \\
\frac{1}{F_{n+1}(F_{n+1} + F_{n+2})} &= \frac{F_{n+1} + F_{n+2} - F_{n+1}}{F_{n+2}} \cdot \frac{1}{F_{n+1}(F_{n+1} + F_{n+2})} \\
\frac{1}{F_{n+1}(F_{n+1} + F_{n+2})} &= \frac{F_{n+1} + F_{n+2} - F_{n+1}}{F_{n+2}F_{n+1}(F_{n+1} + F_{n+2})} \\
\frac{1}{F_{n+1}F_{n+3}} &= \frac{\cancel{F_{n+1} + F_{n+2}}}{F_{n+2}F_{n+1}\cancel{(F_{n+1} + F_{n+2})}} - \frac{\cancel{F_{n+1}}}{F_{n+2}\cancel{F_{n+1}}(F_{n+1} + F_{n+2})} \\
\frac{1}{F_{n+1}F_{n+3}} &= \frac{1}{F_{n+1}F_{n+2}} - \frac{1}{F_{n+2}F_{n+3}}
\end{aligned}$$

$\square$

(b)

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{F_{n+1}F_{n+3}} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{F_{n+1}F_{n+2}} - \frac{1}{F_{n+2}F_{n+3}} \right) \\
&= \frac{1}{F_1F_2} - \cancel{\frac{1}{F_2F_3}} + \cancel{\frac{1}{F_2F_3}} - \cancel{\frac{1}{F_4F_5}} + \cancel{\frac{1}{F_4F_5}} - \cancel{\frac{1}{F_5F_6}} + \cancel{\frac{1}{F_5F_6}} - \cdots \\
&= \frac{1}{1 \cdot 1} \\
&= \boxed{1}
\end{aligned}$$

Progress on Problem Set #42

Jayden Li

January 13, 2024

Problem 1

(a)
$$(k+1)^2 - k^2$$
$$= k^2 + 2k + 1 - k^2$$
$$= \boxed{2k+1}$$

(b)
$$\sum_{k=1}^{n} \left((k+1)^2 - k^2\right)$$
$$= \sum_{k=1}^{n} (2k+1)$$
$$= \sum_{k=1}^{n} 2k + \sum_{k=1}^{n} 1$$
$$= \boxed{2\sum_{k=1}^{n} k + n}$$

(c) Let $S_n = \sum_{k=1}^{n} \left((k+1)^2 - k^2\right)$.

$$S_1 = \underset{+5}{3} \qquad S_4 = \overset{+9}{\underset{+11}{24}}$$
$$S_2 = \underset{+7}{8} \qquad S_5 = \underset{+13}{35}$$
$$S_3 = 15 \qquad S_5 = 48$$

Let $D_n = S_{n+1} - S_n = 2n + 3$.

$$S_{n+1} = D_n + S_n$$
$$\begin{aligned}
S_{n+1} &= 2n + 3 + S_n \\
&= 2n + 3 + 2(n-1) + 3 + S_{n-1} \\
&= 2n + 3 + 2(n-1) + 3 + 2(n-2) + 3 + \cdots + 2(n-(n-1)) + 3 + 2(n-n) + 3 \\
&= 2n + 3 + 2n - 2 + 3 + 2n - 4 + 3 + \cdots + 2n - 2n + 2 + 3 + 2n - 2n + 3 \\
&= \underbrace{3 + 3 + \cdots + 3}_{n \text{ terms}} + \underbrace{2n + 2n + \cdots + 2n}_{n \text{ terms}} - 2 - 4 - \cdots - 2n + 2 - 2n \\
&= 3n + 2n^2 - (2 + 4 + \cdots + 2n - 2 + 2n) \\
&= 3n + 2n^2 - (2 + 2n + 4 + 2n - 2 + \cdots) \\
&= 3n + 2n^2 - \underbrace{((2+2n) + (2+2n) + \cdots)}_{\frac{n}{2} \text{ terms}} \\
&= 3n + 2n^2 - \frac{n}{2}(2+2n) \\
&= 3n + 2n^2 - n - n^2 \\
&= \boxed{n^2 + 2n}
\end{aligned}$$

(d)
$$\sum_{k=1}^{n} \left((k+1)^2 - k^2\right) = n^2 + 2n$$
$$2\sum_{k=1}^{n} k + n = n^2 + 2n$$
$$2\sum_{k=1}^{n} k = n^2 + n$$
$$\boxed{\sum_{k=1}^{n} k = \frac{n^2+n}{2}}$$

Problem 2

(a) $$\sum_{k=1}^{n}\big((k+1)^3-k^3\big)$$

$$= \sum_{k=1}^{n}(k+1)^3 - \sum_{k=1}^{n}k^3$$

$$= \sum_{k=2}^{n+1}k^3 - \sum_{k=1}^{n}k^3$$

$$= (n+1)^3 + \sum_{k=2}^{n}k^3 - \sum_{k=2}^{n}k^3 - 1$$

$$= (n+1)^3 - 1$$

$$\sum_{k=1}^{n}\big((k+1)^3-k^3\big) = \sum_{k=1}^{n}\big(k^3+3k^2+3k+1-k^3\big)$$

$$(n+1)^3-1 = \sum_{k=1}^{n}3k^2 + \sum_{k=1}^{n}3k + \sum_{k=1}^{n}1$$

$$(n+1)^3-1 = 3\sum_{k=1}^{n}k^2 + 3\cdot\frac{n(n+1)}{2} + n$$

$$3\sum_{k=1}^{n}k^2 = n^3+3n^2+3n+1-1-\frac{3n(n+1)}{2} - n$$

$$\sum_{k=1}^{n}k^2 = \frac{n^3+3n^2+2n-\frac{3n^2+3n}{2}}{3}$$

$$\sum_{k=1}^{n}k^2 = \frac{2n^3+6n^2+4n-3n^2-3n}{6}$$

$$\boxed{\sum_{k=1}^{n}k^2 = \frac{2n^3+3n^2+n}{6}}$$

Problem 3

$$1\cdot 2 + 3\cdot 4 + \cdots + (2n-1)2n$$

$$= \sum_{k=1}^{n}(2k-1)2k$$

$$= \sum_{k=1}^{n}4k^2 - \sum_{k=1}^{n}2k$$

$$= 4\cdot\frac{2n^3+3n^2+n}{6} - 2\cdot\frac{n(n+1)}{2}$$

$$= \frac{4n^3+6n^2+2n}{3} - \frac{3n(n+1)}{3}$$

$$= \frac{4n^3+6n^2+2n-3n^2-3}{3}$$

$$= \boxed{\frac{4n^3+3n^2+2n-3}{3}}$$

Problem 4

(a)
$$\begin{aligned}
&\sum_{n=1}^{20} f(n) \\
&= \sum_{n=1}^{20} \left(3x^2 - 7x + 2\right) \\
&= \sum_{n=1}^{20} 3x^2 - \sum_{n=1}^{20} 7x + \sum_{n=1}^{20} 2 \\
&= \frac{3\left(2(20)^3 + 3(20)^2 + 20\right)}{6} - \frac{7 \cdot 20(20+1)}{2} + 40 \\
&= \frac{2(8000) + 3(400) + 20}{2} - \frac{7 \cdot 420}{2} + 40 \\
&= \frac{16000 + 1200 + 20}{2} - 1470 + 40 \\
&= 8000 + 600 + 10 - 1430 \\
&= \boxed{7180}
\end{aligned}$$

(b)
$$\begin{aligned}
\sum_{k=1}^{n} f(n) &= 0 \\
\sum_{k=1}^{n} \left(k^2 - 11k - 28\right) &= 0 \\
\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} 11k - \sum_{k=1}^{n} 28 &= 0 \\
\frac{2n^3 + 3n^2 + n}{6} - 11 \cdot \frac{n(n+1)}{2} - 28n &= 0 \\
2n^3 + 3n^2 + n - 33n(n+1) - 168n &= 0 \\
2n^3 + 3n^2 + n - 33n^2 - 33n - 168n &= 0 \\
2n^3 - 30n^2 - 200n &= 0 \\
n\left(n^2 - 15n - 100\right) &= 0 \\
n(n-20)(n+5) &= 0 \\
\cancel{n = 0}, n = 20, \cancel{n = -5} \;\Rightarrow\; &\boxed{x = 20}
\end{aligned}$$

Problem 5

$$\begin{aligned}
&1 \cdot 4 + 3 \cdot 6 + \cdots + (2n-1)(2n+2) \\
&= \sum_{k=1}^{n} (2k-1)(2k+2) \\
&= \sum_{k=1}^{n} \left(4k^2 + 2k - 2\right) \\
&= 4\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 2 \\
&= 4 \cdot \frac{2n^3 + 3n^2 + n}{6} + 2 \cdot \frac{n(n+1)}{2} - 2n \\
&= \frac{4n^3 + 6n^2 + 2n}{3} + \frac{3n(n+1)}{3} - \frac{6n}{3} \\
&= \frac{4n^3 + 6n^2 + 2n + 3n^2 + 3 - 6n}{3} \\
&= \boxed{\frac{4n^3 + 9n^2 - 4n + 3}{3}}
\end{aligned}$$