Problem Set #61

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Problem 5

We begin by proving two lemmas (lemmata?) which will simply this answer.

Lemma 1. *If* $n \in \mathbb{N}_0$*,* $\lim_{h \to 0} h^n \sin\left(\frac{1}{h}\right)$ *exists if and only if* $n \geq 1$*. Additionally, the limit equals* 0*.*

Proof. First, suppose that $n = 0$, then $\lim_{h \to 0} h^0 \sin\left(\frac{1}{h}\right) = \lim_{h \to 0} \sin\left(\frac{1}{h}\right)$ which does not exist.

We then prove that the limit does exist for $n \geq 2$.

$$-1 \leq \sin\left(\frac{1}{h}\right) \leq 1$$

$$-h^n \leq h^n \sin\left(\frac{1}{h}\right) \leq h^n$$

$$\lim_{h \to 0} [-h^n] \leq \lim_{h \to 0} h^n \sin\left(\frac{1}{h}\right) \leq \lim_{h \to 0} h^n$$

Since $n - 1 \geq 1$:

$$0 \leq \lim_{h \to 0} h^n \sin\left(\frac{1}{h}\right) \leq 0$$

By the squeeze theorem, $\lim_{h \to 0} h^n \sin\left(\frac{1}{h}\right) = 0$. ☺

Lemma 2. *If* $n \in \mathbb{N}_0$*,* $\lim_{h \to 0} h^n \cos\left(\frac{1}{h}\right)$ *exists if and only if* $n \geq 1$*. Additionally, the limit equals* 0*.*

Proof. First, suppose that $n = 1$, then $\lim_{h \to 0} h^0 \cos\left(\frac{1}{h}\right) = \lim_{h \to 0} \cos\left(\frac{1}{h}\right)$ which does not exist.

We then prove that the limit does exist for $n \geq 2$.

$$-1 \leq \cos\left(\frac{1}{h}\right) \leq 1$$

$$-h^n \leq h^n \cos\left(\frac{1}{h}\right) \leq h^n$$

$$\lim_{h \to 0} [-h^n] \leq \lim_{h \to 0} h^n \cos\left(\frac{1}{h}\right) \leq \lim_{h \to 0} h^n$$

Since $n-1 \geq 1$:

$$0 \leq \lim_{h\to 0} h^n \cos\left(\frac{1}{h}\right) \leq 0$$

By the squeeze theorem, $\lim_{h\to 0} h^n \cos\left(\frac{1}{h}\right) = 0$. ☺

For some $n \in \mathbb{N}_1$, we define the function $f : \mathbb{R} \to \mathbb{R}$:

$$f(x) = \begin{cases} x^n \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Then we calculate the derivative f'. We will first calculate $f'(0)$:

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{h^n \sin\left(\frac{1}{h}\right)}{h} \\ &= \lim_{h\to 0} h^{n-1} \sin\left(\frac{1}{h}\right) \end{aligned}$$

By Lemma 1, $n \geq 2$ and the above limit evaluates to 0. Thus, $f'(0) = 0$. Next, we will find $f'(x)$ for $x \neq 0$.

$$\begin{aligned} f'(x) &= nx^{n-1} \sin\left(\frac{1}{x}\right) + x^n \cos\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) \\ &= nx^{n-1} \sin\left(\frac{1}{x}\right) - x^{n-2} \cos\left(\frac{1}{x}\right) \end{aligned}$$

Incorporating our result for $f'(0)$, we have:

$$f'(x) = \begin{cases} nx^{n-1} \sin\left(\frac{1}{x}\right) - x^{n-2} \cos\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Now we can calculate $f''(0)$.

$$\begin{aligned} f''(0) &= \lim_{h\to 0} \frac{f'(0+h) - f'(0)}{h} \\ &= \lim_{h\to 0} \frac{nh^{n-1} \sin\left(\frac{1}{h}\right) - h^{n-2} \cos\left(\frac{1}{h}\right) - 0}{h} \\ &= \lim_{h\to 0} \left[nh^{n-2} \sin\left(\frac{1}{h}\right) - h^{n-3} \cos\left(\frac{1}{h}\right)\right] \\ &= n \lim_{h\to 0} h^{n-2} \sin\left(\frac{1}{h}\right) - \lim_{h\to 0} h^{n-3} \cos\left(\frac{1}{h}\right) \end{aligned}$$

By Lemma 1, the first limit exists if and only if $n-2 \geq 1 \iff n \geq 3$. By Lemma 2, the second limit exists if and only if $n-3 \geq 1 \iff n \geq 4$. Therefore, the second derivative f'' exists at $x = 0$ if and only if $n \geq 4$.

We can also calculate $f''(x)$ for $x \neq 0$.

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x} f'(x)$$

$$
\begin{aligned}
&= \frac{\mathrm{d}}{\mathrm{d}x}\left[nx^{n-1}\sin\left(\frac{1}{x}\right)\right] - \frac{\mathrm{d}}{\mathrm{d}x}\left[x^{n-2}\cos\left(\frac{1}{x}\right)\right] \\
&= n(n-1)x^{n-2}\sin\left(\frac{1}{x}\right) + nx^{n-1}\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) - (n-2)x^{n-3}\cos\left(\frac{1}{x}\right) + x^{n-2}\sin\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) \\
&= n(n-1)x^{n-2}\sin\left(\frac{1}{x}\right) - nx^{n-3}\cos\left(\frac{1}{x}\right) - (n-2)x^{n-3}\cos\left(\frac{1}{x}\right) - x^{n-4}\sin\left(\frac{1}{x}\right)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{x\to 0} f''(x) &= \lim_{x\to 0}\left[n(n-1)x^{n-2}\sin\left(\frac{1}{x}\right) - nx^{n-3}\cos\left(\frac{1}{x}\right) - (n-2)x^{n-3}\cos\left(\frac{1}{x}\right) - x^{n-4}\sin\left(\frac{1}{x}\right)\right] \\
&= n(n-1)\lim_{x\to 0} x^{n-2}\sin\left(\frac{1}{x}\right) - n\lim_{x\to 0} x^{n-3}\cos\left(\frac{1}{x}\right) - (n-2)\lim_{x\to 0} x^{n-3}\cos\left(\frac{1}{x}\right) - \lim_{x\to 0} x^{n-4}\sin\left(\frac{1}{x}\right)
\end{aligned}
$$

By Lemma 1 and 2, all limits evaluate to 0 if $n-2 \geq 1$, $n-3 \geq 1$ and $n-4 \geq 1$. If $n \geq 5$, $\lim_{x\to 0} f''(x) = 0 = f(0)$, and f'' is continuous at $x = 0$. If $n \ngeq 5$, then $\lim_{x\to 0} f''(x)$ DNE and f'' is not continuous at $x = 0$.

Therefore, the second derivative of f at 0 exists iff $x \geq 4$, and is continuous at $x = 0$ iff $x \geq 5$.

Problem 6

$$
\begin{aligned}
f'(0) &= \lim_{h\to 0}\frac{f'(0+h) - f'(0)}{h} \\
&= \lim_{h\to 0}\frac{g(h)\sin\left(\frac{1}{h}\right) - 0}{h} \\
&= \lim_{h\to 0}\left[\frac{g(h)}{h}\right]\lim_{h\to 0}\left[\sin\left(\frac{1}{h}\right)\right] \\
&= \lim_{h\to 0}\left[\frac{g(0+h) - g(0)}{h}\right]\lim_{h\to 0}\left[\sin\left(\frac{1}{h}\right)\right] \\
&= g'(0)\cdot\lim_{h\to 0}\left[\sin\left(\frac{1}{h}\right)\right] \\
&= 0\cdot\lim_{h\to 0}\left[\sin\left(\frac{1}{h}\right)\right] \\
&= \lim_{h\to 0}\left[0\cdot\sin\left(\frac{1}{h}\right)\right] \\
&= \boxed{0}
\end{aligned}
$$

$$
\begin{gathered}
-1 \leq \sin\left(\frac{1}{h}\right) \leq 1 \\
0\cdot -1 \leq 0\cdot\sin\left(\frac{1}{h}\right) \leq 0\cdot 1 \\
\lim_{h\to 0} 0 \leq \lim_{h\to 0}\left[0\cdot\sin\left(\frac{1}{h}\right)\right] \leq \lim_{h\to 0} 0
\end{gathered}
$$

By the squeeze theorem, $\lim_{h\to 0}\left[0\cdot\sin\left(\frac{1}{h}\right)\right] = 0$.

Problem 7

$$
\begin{aligned}
f'(0) &= \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} \\
&= \lim_{h\to 0}\frac{hg(h)}{h} \\
&= \lim_{h\to 0} g(h) \\
&= g(0)
\end{aligned}
$$

Because the difference quotient limit exists, f must be differentiable at $x = 0$.

Problem 8

Let $g(x) = f(x)/x$.

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} \frac{f(x)}{x}$$

It is known that $f(0) = 0$.

$$\begin{aligned} g(0) &= \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \\ g(0) &= f'(0) \end{aligned}$$

By taking $g(x) = f(x)/x$, we show that f is differentiable at 0. Thus, multiplying both sides by x gives $f(x) = xg(x)$. ☺

Problem 9

(a)

$$\begin{aligned} g(x) &= \frac{a_n}{n+1}x^{n+1} + \frac{a_{n-1}}{n}x^n + \frac{a_{n-2}}{n-1}x^{n-1} + \ldots + a_0 x + C \\ &= \sum_{k=0}^{n} \frac{a_k}{k+1}x^{k+1} + C \end{aligned}$$

Where C is a constant.

(b)

$$\begin{aligned} g(x) &= -\frac{b_2}{x} - \frac{b_3}{2x^2} - \frac{b_4}{3x^3} - \ldots - \frac{b_m}{(m-1)x^{m-1}} + C \\ &= -\sum_{k=2}^{m} \frac{b_k}{(k-1)x^{k-1}} + C \end{aligned}$$

Where C is a constant.

(c) No. We know that $f(x) = \ln|x|$ and logarithms cannot be expressed as a rational function.

Problem 10

(a) *Proof.* **If part.** NTS that if a is a double root of f, then a is a root of f and f'. If a is a double root, then $f(x) = (x-a)^2 g(x)$ where g is a polynomial.

$$\begin{aligned} f'(x) &= 2(x-a)g(x) + (x-a)^2 g'(x) \\ &= (x-a)(2g(x) + (x-a)g'(x)) \end{aligned}$$

g is a polynomial and g' is a polynomial. The product of two polynomials is a polynomial. The sum of two polynomials is a polynomial. Thus, f' is a polynomial with $x - a$ as a factor, so a is a root of f'.

Only if part. NTS that if a is not a double root of f, then it is not a root of either f or f'.

There are three cases, that a is not a root of f (case 1), that a is a single root of f (but not a double root) (case 2), or f is a triple or higher root (case 3).

In case 1, if a is not a root of f, then a is not a root of f (proof is left as exercise to the grader).

In case 2, a is a root of f, but only a single root. In that case, $f(x) = (x-a)h(x)$ where h is a polynomial that is not divisible by $(x-a)$.

$$\begin{aligned}
f(x) &= (x-a)h(x) \\
f'(x) &= (x-a)h'(x) + h(x) \\
&= (x-a)\left(h'(x) + \frac{h(x)}{x-a}\right)
\end{aligned}$$

h is not divisible by $x-a$, therefore the quotient of f' and $x-a$ is not a polynomial, and a is not a root of f'.

In case 3, if a is a triple or higher root of f, then $f(x) = (x-a)^n k(x)$ where $n \in \mathbb{N}$ and $n \geq 3$, and a is not a root of k.

$$\begin{aligned}
f(x) &= (x-a)^n k(x) \\
f'(x) &= n(x-a)^{n-1}k(x) + (x-a)^n k(x) \\
&= (x-a)^2 \left(n(x-a)^{n-1-2}k(x) + (x-a)^{n-2}k(x)\right) \\
&= (x-a)^2 \left(n(x-a)^{n-3}k(x) + (x-a)^{n-2}k(x)\right)
\end{aligned}$$

Because $n \geq 3$, $n-3 \geq 0$ so the above must be a polynomial. Therefore, in this case a must be a double root.

We have shown that if a is a double root of f, then a is a root of f and f'. In addition, if a is not a double root of f, then it is not true that a is a root of both f and f'. Therefore, a is a double root of f if and only if a is a double root of f and f'. ☺

(b) Let n be a double root of f. Then, by the conclusion reached in (a):

$$f(n) = an^2 + bn + c = 0 \tag{1}$$

$$f'(n) = 2an + b = 0 \tag{2}$$

By the quadratic formula, from (1), we have:

$$n = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

After manipulations on (2) and substituting into the above:

$$\begin{aligned}
\frac{-b}{2a} &= \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
-b &= -b \pm \sqrt{b^2-4ac} \\
b^2 - 4ac &= 0
\end{aligned}$$

Therefore, double roots occur at the vertex of a function (because $n = \frac{-b}{2a}$ which is the x-coordinate of the vertex). Geometrically, this means that the graph does not cross to the other side of the x axis at a double root.

Problem 11

(i)

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}x} &= \cos(y) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left[x + x^2\right] \\ &= \boxed{(1+2x)\cos(x+x^2)}\end{aligned}$$

(ii)

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}x} &= \cos(y) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\cos(x) \\ &= \cos(\cos x)(-\sin x) \\ &= \boxed{-\sin(x)\cos(\cos(x))}\end{aligned}$$

(iii)

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}x} &= \cos(u) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\sin x \\ &= \boxed{\cos(\sin(x))\sin(x)}\end{aligned}$$

(iv)

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}x} &= \cos(v)(-\sin(u))(\cos(x)) \\ &= -\cos(\cos(u))\sin(\sin(x))\cos(x) \\ &= \boxed{-\cos(\cos(\sin(x)))\sin(\sin(x))\cos(x)}\end{aligned}$$