Problem Set #18

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Problem 1

(b) $$\int \frac{6x-2}{x^2-2x-3}\,\mathrm{d}x = \int \frac{6x-2}{(x-3)(x+1)}\,\mathrm{d}x = \int \left(\frac{A}{x-3} + \frac{B}{x+1}\right)\mathrm{d}x$$

$$\boxed{\begin{aligned} \frac{A}{x-3} + \frac{B}{x+1} = \frac{6x-2}{x^2-2x-3} \implies Ax + A + Bx - 3B = 6x - 2 \implies \begin{cases} A + B = 6 \\ A - 3B = -2 \end{cases} \\ \implies 4B = 8 \implies B = 2 \implies A = 4 \end{aligned}}$$

$$= \int \left(\frac{4}{x-3} + \frac{2}{x+1}\right)\mathrm{d}x = \boxed{4\ln|x-3| + 2\ln|x+1| + C}$$

(c) $$\int \frac{2x+5}{x^2+2x-8}\,\mathrm{d}x = \int \frac{2x+5}{(x+4)(x-2)}\,\mathrm{d}x = \int \left(\frac{A}{x+4} + \frac{B}{x-2}\right)\mathrm{d}x$$

$$\boxed{\begin{aligned} \frac{A}{x+4} + \frac{B}{x-2} = \frac{2x+5}{x^2+2x-8} \implies Ax - 2A + Bx + 4B = 2x + 5 \implies \begin{cases} A + B = 2 \\ -2A + 4B = 5 \end{cases} \\ \implies 6B = 9 \implies B = \frac{3}{2} \implies A = \frac{1}{2} \end{aligned}}$$

$$= \int \left(\frac{1/2}{x+4} + \frac{3/2}{x-2}\right)\mathrm{d}x = \boxed{\frac{1}{2}\ln|x+4| + \frac{3}{2}\ln|x-2| + C}$$

Problem 2

(b) $$\int \frac{x^2+4x-1}{x^3-x}\,\mathrm{d}x = \int \frac{x^2+4x-1}{x(x+1)(x-1)}\,\mathrm{d}x = \int \left(\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-1}\right)\mathrm{d}x$$

$$\boxed{\begin{aligned}
&\frac{A}{x}+\frac{B}{x+1}+\frac{C}{x-1} = \frac{x^2+4x-1}{x^3-x}\\
\implies &A(x+1)(x-1)+Bx(x-1)+Cx(x+1) = x^2+4x-1\\
\implies &Ax^2-A+Bx^2-Bx+Cx^2+Cx = x^2+4x-1 \implies \begin{cases} A+B+C=1\\ \quad -B+C=4\\ \qquad\; -A=-1\end{cases}\\
\implies &A=1 \implies \begin{cases} B+C=0\\ -B+C=4\end{cases} \implies 2C=4 \implies C=2 \implies B=-2
\end{aligned}}$$

$$= \int \left(\frac{1}{x}-\frac{2}{x+1}+\frac{2}{x-1}\right)\mathrm{d}x = \boxed{\ln|x|-2\ln|x+1|+2\ln|x-1|+C}$$

(c) $$\int \frac{4x+28}{(x+1)(x^2-4x+3)}\,\mathrm{d}x = \int \frac{4x+28}{(x+1)(x-1)(x-3)}\,\mathrm{d}x = \int \left(\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{x-3}\right)\mathrm{d}x$$

$$\boxed{\begin{aligned}
&\frac{A}{x+1}+\frac{B}{x-1}+\frac{C}{x-3} = \frac{4x+28}{(x+1)(x-1)(x-3)}\\
\implies &A(x-1)(x-3)+B(x+1)(x-3)+C(x+1)(x-1) = 4x+28\\
\implies &A\left(x^2-4x+3\right)+B\left(x^2-2x-3\right)+C\left(x^2-1\right) = 4x+28\\
\implies &Ax^2-4Ax+3A+Bx^2-2Bx-3B+Cx^2-C = 4x+28 \implies \begin{cases} A+B+C=0\\ -4A-2B=4\\ 3A-3B-C=28\end{cases}\\
\implies &\begin{cases} 4A-2B=28\\ -4A-2B=4\end{cases} \implies -4B=32 \implies B=-8 \implies A=3 \implies C=5
\end{aligned}}$$

$$\begin{aligned}
&= \int \left(\frac{3}{x+1}-\frac{8}{x-1}+\frac{5}{x-3}\right)\mathrm{d}x\\
&= \boxed{3\ln|x+1|-8\ln|x-1|+5\ln|x-3|+C}
\end{aligned}$$

Problem 3

(a) $$\int \frac{3x^2-7x+2}{x^3-2x^2+x}\,\mathrm{d}x = \int \frac{3x^2-7x+2}{x(x-1)(x-1)}\,\mathrm{d}x = \int \left(\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2}\right)\mathrm{d}x$$

$$\boxed{\begin{aligned}
&\frac{A}{x}+\frac{B}{x-1}+\frac{C}{(x-1)^2} = \frac{3x^2-7x+2}{x(x-1)(x-1)}\\
\Longrightarrow\ &\frac{A(x-1)(x^2-2x+1)+Bx(x^2-2x+1)+Cx(x-1)}{x(x-1)(x-1)^2} = \frac{3x^2-7x+2}{x(x-1)(x-1)}\\
\Longrightarrow\ &A\left(x^3-3x^2+3x-1\right)+B\left(x^3-2x^2+x\right)+C\left(x^2-x\right) = (x-1)\left(3x^2-7x+2\right)\\
\Longrightarrow\ &Ax^3-3Ax^2+3Ax-A+Bx^3-2Bx^2+Bx+Cx^2-Cx = 3x^3-10x^2+9x-2\\
\Longrightarrow\ &\begin{cases} A+B=3\\ -3A-2B+C=-10\\ 3A+B-C=9\\ -A=-2\end{cases} \Longrightarrow A=2 \Longrightarrow B=1 \Longrightarrow C=-2
\end{aligned}}$$

$$= \int \left(\frac{2}{x}+\frac{1}{x-1}-\frac{2}{(x-1)^2}\right)\mathrm{d}x = \boxed{2\ln|x|+\ln|x-1|+\frac{2}{x-1}+C}$$

(b) $$\int \frac{3x^2-2x-3}{x^3-x^2}\,\mathrm{d}x = \int \frac{3x^2-2x-3}{x^2(x-1)}\,\mathrm{d}x = \int \left(\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1}\right)\mathrm{d}x$$

$$\boxed{\begin{aligned}
&\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x-1} = \frac{3x^2-2x-3}{x^2(x-1)} \Longrightarrow \frac{Ax^2(x-1)+Bx(x-1)+Cx^3}{x^3(x-1)} = \frac{3x^2-2x-3}{x^2(x-1)}\\
\Longrightarrow\ &Ax^3-Ax^2+Bx^2-Bx+Cx^3 = 3x^3-2x^2-3x \Longrightarrow \begin{cases} A+C=3\\ -A+B=-2\\ -B=-3\end{cases}\\
\Longrightarrow\ &B=3 \Longrightarrow A=5 \Longrightarrow C=-2
\end{aligned}}$$

$$= \int \left(\frac{5}{x}+\frac{3}{x^2}-\frac{2}{x-1}\right)\mathrm{d}x = \boxed{5\ln|x|-\frac{3}{x}-2\ln|x-1|+C}$$

(c) $$\int \frac{x^2}{(x+1)^3}\,\mathrm{d}x \left[\begin{matrix} t=x+1\\ \mathrm{d}t=\mathrm{d}x\end{matrix}\right] \int \frac{(t-1)^2}{t^3}\,\mathrm{d}x = \int \frac{t^2-2t+1}{t^3}\,\mathrm{d}t = \int \left(\frac{1}{t}-\frac{2}{t^2}+\frac{1}{t^3}\right)\mathrm{d}t$$

$$= \ln|t|+\frac{2}{t}+\frac{-1/2}{t^2} = \boxed{\ln|x+1|+\frac{2}{x+1}-\frac{1}{2(x+1)^2}+C}$$

Problem 4

$$\int \ln\left(x^2-x+2\right)\mathrm{d}x = \begin{bmatrix} u=\ln\left(x^2-x+2\right) & \mathrm{d}u=\dfrac{(2x-1)\,\mathrm{d}x}{x^2-x+2} \\ \mathrm{d}v=\mathrm{d}x & v=x \end{bmatrix} x\ln\left(x^2-x+2\right) - \int \frac{2x^2-x}{x^2-x+2}\,\mathrm{d}x$$

$$= x\ln\left(x^2-x+2\right) - \int\left(2+\frac{x-4}{x^2-x+2}\right)\mathrm{d}x$$

$$= x\ln\left(x^2-x+2\right) - 2x - \frac{1}{2}\int\left(\frac{2x-1}{x^2-x+2}-\frac{7}{x^2-x+2}\right)\mathrm{d}x$$

$$= \begin{bmatrix} t=x^2-x+2 \\ \mathrm{d}t=(2x-1)\,\mathrm{d}x \end{bmatrix} x\ln\left(x^2-x+2\right) - 2x - \frac{1}{2}\int\frac{1}{t}\,\mathrm{d}t + \frac{1}{2}\int\frac{7}{x^2-x+2}\,\mathrm{d}x$$

$$= x\ln\left(x^2-x+2\right) - 2x - \frac{1}{2}\ln\left|x^2-x+2\right| + \frac{1}{2}\int\frac{7}{x^2-x+2}\,\mathrm{d}x$$

$$\int\frac{7}{x^2-x+2}\,\mathrm{d}x = \int\frac{7}{\left(x-\frac{1}{2}\right)^2-\frac{1}{4}+2}\,\mathrm{d}x = \begin{bmatrix} t=x-\dfrac{1}{2} \\ \mathrm{d}t=\mathrm{d}x \end{bmatrix} 7\int\frac{1}{t^2+\frac{7}{4}}\,\mathrm{d}t = 7\int\frac{1}{\frac{7}{4}\left(\frac{4}{7}t^2+1\right)}\,\mathrm{d}t$$

$$= 7\cdot\frac{4}{7}\int\frac{1}{\frac{4}{7}t^2+1}\,\mathrm{d}t = \begin{bmatrix} s=\dfrac{2}{\sqrt{7}}t \\ \mathrm{d}s=\dfrac{2}{\sqrt{7}}\,\mathrm{d}t \end{bmatrix} 4\cdot\frac{\sqrt{7}}{2}\int\frac{1}{s^2+1}\,\mathrm{d}s = 2\sqrt{7}\arctan s$$

$$= 2\sqrt{7}\arctan\left(\frac{2}{\sqrt{7}}t\right) = 2\sqrt{7}\arctan\left(\frac{2}{\sqrt{7}}\left(x-\frac{1}{2}\right)\right)+C$$

$$= x\ln\left(x^2-x+2\right) - 2x - \frac{1}{2}\ln\left|x^2-x+2\right| + \frac{1}{2}\left(2\sqrt{7}\arctan\left(\frac{2}{\sqrt{7}}\left(x-\frac{1}{2}\right)\right)\right)$$

$$= \boxed{x\ln\left(x^2-x+2\right) - 2x - \frac{1}{2}\ln\left|x^2-x+2\right| + \sqrt{7}\arctan\left(\frac{2}{\sqrt{7}}\left(x-\frac{1}{2}\right)\right)+C}$$

Problem 5

$$\cos\left(2\cdot\frac{\theta}{2}\right) = 2\cos^2\left(\frac{\theta}{2}\right)-1$$

$$\cos\theta+1 = 2\cos^2\left(\frac{\theta}{2}\right)$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{\cos\theta+1}{2}}$$

$\cos(\theta/2)\geq 0$ if $\theta\in[-\pi,\pi]$

$$\cos\frac{\theta}{2} = \boxed{\sqrt{\frac{\cos\theta+1}{2}}}$$

$$\cos\left(2\cdot\frac{\theta}{2}\right) = 1-2\sin^2\left(\frac{\theta}{2}\right)$$

$$2\sin^2\left(\frac{\theta}{2}\right) = 1-\cos\theta$$

$$\sin\frac{\theta}{2} = \boxed{\pm\sqrt{\frac{1-\cos\theta}{2}}}$$

$$\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}\Big/\sqrt{\frac{1+\cos\theta}{2}}$$

$$= \frac{\sqrt{1-\cos\theta}\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}\sqrt{1-\cos\theta}}$$

$$= \frac{1-\cos\theta}{\sqrt{1-\cos^2\theta}} = \boxed{\frac{1-\cos\theta}{\sin\theta}}$$

(a) $$\frac{1}{\sqrt{1+t^2}} = \left(\sqrt{1+\left(\frac{1-\cos\theta}{\sin\theta}\right)^2}\right)^{-1} = \left(\sqrt{\frac{\sin^2\theta}{\sin^2\theta}+\frac{1-2\cos\theta+\cos^2\theta}{\sin^2\theta}}\right)^{-1}$$
$$= \sqrt{\left(\frac{1-2\cos\theta+\cos^2\theta+\sin^2\theta}{\sin^2\theta}\right)^{-1}} = \sqrt{\frac{1-\cos^2\theta}{2-2\cos\theta}} = \sqrt{\frac{1}{2}\cdot\frac{(1+\cos\theta)(1-\cos\theta)}{1-\cos\theta}}$$
$$= \sqrt{\frac{1+\cos\theta}{2}} = \cos\frac{\theta}{2}$$
$$\frac{t}{\sqrt{1+t^2}} = \tan\frac{\theta}{2}\cdot\sqrt{\frac{1+\cos\theta}{2}} = \pm\sqrt{\frac{(1-\cos\theta)^2(1+\cos\theta)}{2\sin^2\theta}} = \pm\sqrt{\frac{(1-\cos\theta)^2(1+\cos\theta)}{2(1+\cos\theta)(1-\cos\theta)}}$$
$$= \pm\sqrt{\frac{1-\cos\theta}{2}} = \sin\frac{\theta}{2}$$

(b) $$\frac{1-t^2}{1+t^2} = \frac{\dfrac{\sin^2 x}{\sin^2 x}-\dfrac{(1-\cos x)^2}{\sin^2 x}}{\dfrac{\sin^2 x}{\sin^2 x}+\dfrac{(1-\cos x)^2}{\sin^2 x}} = \frac{\sin^2 x-(1-2\cos x+\cos^2 x)}{\sin^2 x+(1-2\cos x+\cos^2 x)} = \frac{\sin^2 x-1+2\cos x-\cos^2 x}{\sin^2 x+1-2\cos x+\cos^2 x}$$
$$= \frac{1-\cos^2 x-1+2\cos x-\cos^2 x}{2-2\cos x} = \frac{2\cos x-2\cos^2 x}{2-2\cos x} = \frac{\cos(x)(1-\cos x)}{1-\cos x} = \cos x$$
$$\frac{2t}{1+t^2} = \frac{\dfrac{2-2\cos x}{\sin x}}{\dfrac{\sin^2 x}{\sin^2 x}+\dfrac{(1-\cos x)^2}{\sin^2 x}} = \frac{2-2\cos x}{\sin^2 x+1-2\cos x+\cos^2 x}\cdot\sin x = \frac{2-2\cos x}{2-2\cos x}\cdot\sin x = \sin x$$

(c) $$t=\tan\left(\frac{x}{2}\right) \implies \arctan t = \frac{x}{2} \implies \frac{1}{1+t^2}\,\mathrm{d}t = \frac{1}{2}\,\mathrm{d}x \implies \mathrm{d}x = \frac{2}{1+t^2}\,\mathrm{d}t$$

Problem 6

$$\int_1^2 \frac{x^2+1}{3x-x^2}\,\mathrm{d}x = \int_1^2\left(-1+\frac{3x+1}{x(3-x)}\right)\mathrm{d}x = \int_1^2(-1)\,\mathrm{d}x + \int_1^2\left(\frac{A}{x}+\frac{B}{3-x}\right)\mathrm{d}x$$

$$\boxed{\begin{aligned}&\frac{A}{x}+\frac{B}{3-x}=\frac{3x+1}{x(3-x)} \implies 3A-Ax+Bx=3x+1 \implies \begin{cases}-A+B=3\\ 3A=1\end{cases} \implies A=\frac{1}{3}\\ &\implies -\frac{1}{3}+B=3 \implies B=\frac{10}{3}\end{aligned}}$$

$$= [-x]_1^2 + \int_1^2\left(\frac{1/3}{x}+\frac{10/3}{3-x}\right)\mathrm{d}x = -1+\left[\frac{1}{3}\ln|x|-\frac{10}{3}\ln|3-x|\right]_1^2$$
$$= -1+\frac{1}{3}\ln 2-\cancel{\frac{10}{3}\ln 1}-\cancel{\frac{1}{3}\ln 1}+\frac{10}{3}\ln 2 = \boxed{\frac{11}{3}\ln 2-1}$$