Problem Set #21

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Problem 5

(a) area of 1 quadrant $= \displaystyle\int_0^a \sqrt{b^2 - \frac{b^2x^2}{a^2}}\,\mathrm{d}x = \int_0^a b\sqrt{1-\frac{x^2}{a^2}}\,\mathrm{d}x$

$$= \begin{bmatrix} x = a\sin\theta \\ \mathrm{d}x = a\cos\theta\,\mathrm{d}\theta \end{bmatrix} \int_0^{\pi/2} b\sqrt{1-\frac{a^2\sin^2\theta}{a^2}}\cdot a\cos\theta\,\mathrm{d}\theta = ab\int_0^{\pi/2}\cos^2\theta\,\mathrm{d}\theta$$

$$= ab\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}\,\mathrm{d}\theta = ab\cdot\frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \frac{ab}{2}\left(\frac{\pi}{2}+0-0-0\right) = \frac{\pi ab}{4}$$

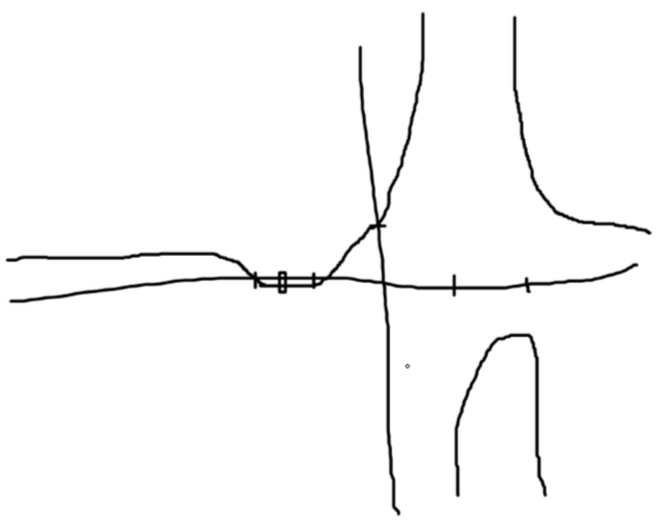
$$\text{area} = 4\cdot\text{area of 1 quadrant} = 4\cdot\frac{\pi ab}{4} = \boxed{\pi ab}$$

(b) Circle is larger than ellipse.

Area of an ellipse with semi-axes a and $a-b$ is $\pi a(a-b)$.

$A_{\text{between circle and ellipse}} = A_{\text{circle}} - A_{\text{ellipse}} = \pi a^2 - \pi ab = \pi a(a-b) = S_{\text{other ellipse}}$

Problem 6



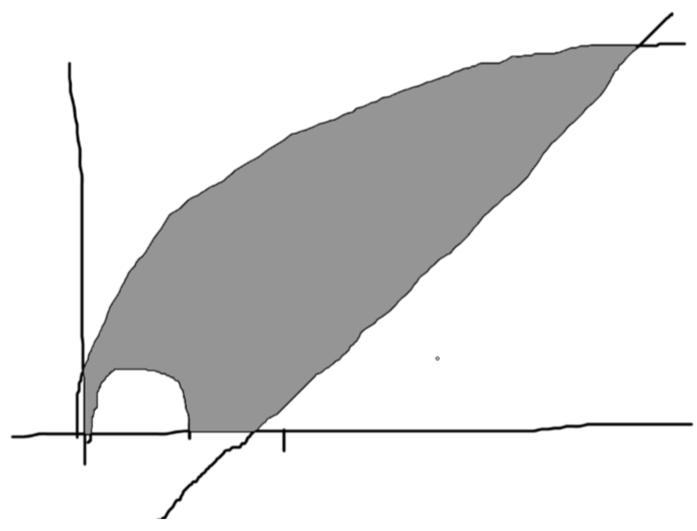
$$\text{signed area} = \int_{-2}^{-1}\frac{x^2+3x+2}{x^2-3x+2}\,\mathrm{d}x = \int_{-2}^{-1}\left(1+\frac{6x}{(x-1)(x-2)}\right)\mathrm{d}x = \int_{-2}^{-1}\left(1+\frac{A}{x-1}+\frac{B}{x-2}\right)\mathrm{d}x$$

$$\boxed{\begin{aligned}&\frac{A}{x-1}+\frac{B}{x-2} = \frac{6x}{x^2-3x+2} \implies \frac{Ax-2A+Bx-B}{x^2-3x+2} = \frac{6x}{x^2-3x+2} = \begin{cases} A+B=6 \\ -2A-B=0\end{cases} \\ &\implies -A = 6 \implies A=-6 \implies B=12\end{aligned}}$$

$$\begin{aligned}
&= \int_{-2}^{-1}\left(1-\frac{6}{x-1}+\frac{12}{x-2}\right)\mathrm{d}x = \Big[x - 6\ln|x-1| + 12\ln|x-2|\Big]_{-2}^{-1} \\
&= -1-6\ln 2+12\ln 3-(-2-6\ln 3+12\ln 4) = -1-6\ln 2+12\ln 3+2+6\ln 3-12\ln 4 \\
&= 1-6\ln 2+18\ln 3-24\ln 2 = 1-30\ln 2+18\ln 3 = 1-6\,(5\ln 2-3\ln 3) \\
&= 1-6\left(\ln\left(2^5\right)-\ln\left(3^3\right)\right) = 1-6\ln\left(\frac{2^5}{3^3}\right) = 1-6\ln\left(\frac{32}{27}\right)
\end{aligned}$$

$$\text{area} = |\text{signed area}| = \left|1-6\ln\left(\frac{32}{27}\right)\right| = \boxed{6\ln\left(\frac{32}{27}\right)-1}$$

Problem 7



We integrate with respect to y because it is easier. We need to find the upper bound of integration.

$$\frac{y^2}{4} = y+2 \implies y^2 = 4y+8 \implies y^2-4y-8=0 \implies y = \frac{4\pm\sqrt{16+32}}{2} \implies y = \frac{4\pm 2\sqrt{12}}{2}$$
$$\implies y = 2\pm\sqrt{12}$$

We only care care about first quadrant, so we keep $y = 2+\sqrt{12}$.

Equation of the circle is $x^2+y^2=2x \implies (x-1)^2+y^2=1$. Radius of circle is 1.

$S_{\text{semicircle}} = \dfrac{1}{2}\cdot\pi(1)^2 = \dfrac{\pi}{2}$

$$S_{\text{other}} = \int_0^{2+\sqrt{12}}\left(y+2-\frac{y^2}{4}\right)\mathrm{d}y = \int_0^{2+2\sqrt{3}}\left(y+2-\frac{y^2}{4}\right)\mathrm{d}y = \left[\frac{y^2}{2}+2y-\frac{y^3}{12}\right]_0^{2+2\sqrt{3}}$$

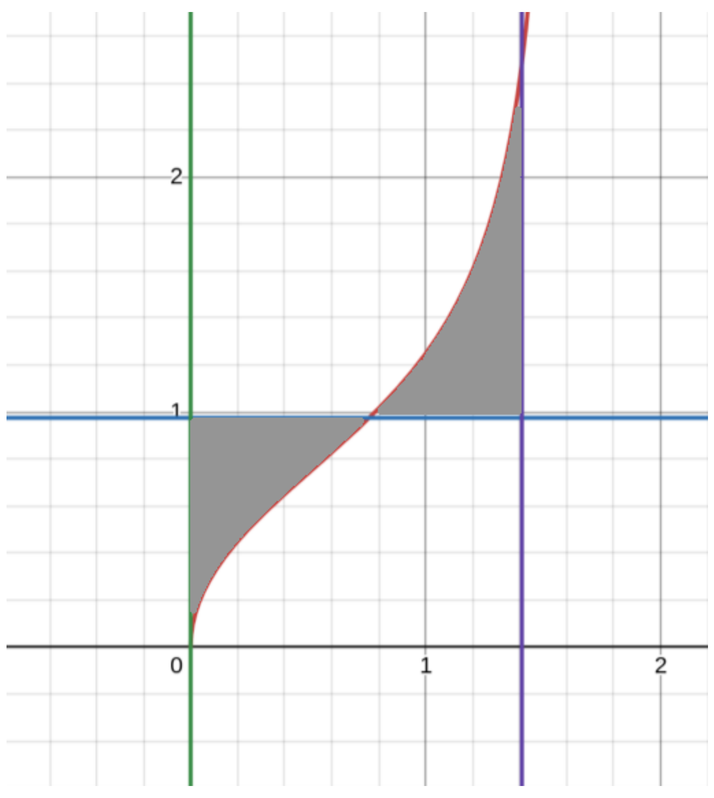
$$= \frac{\left(2+2\sqrt{3}\right)^2}{2} + 2\left(2+2\sqrt{3}\right) - \frac{\left(2+2\sqrt{3}\right)^3}{12} = 2\left(1+\sqrt{3}\right)^2 + 4\left(1+\sqrt{3}\right) - \frac{2\left(1+\sqrt{3}\right)^3}{3}$$

$$S = S_{\text{other}} - S_{\text{semicircle}} = \boxed{2\left(1+\sqrt{3}\right)^2 + 4\left(1+\sqrt{3}\right) - \frac{2\left(1+\sqrt{3}\right)^3}{3} - \frac{\pi}{2}}$$

(this is equivalent to the answer given on the set, but I can't figure out how to simplify)

Problem 8

In the picture below, the green line is $x=0$, red is $y=f(x)=\sqrt{\tan x}$, purple is $x=a$, and blue is $y=f(c)=\sqrt{\tan c}$.



$$A = \int_0^c (f(c)-f(x))\,\mathrm{d}x + \int_c^a (f(x)-f(c))\,\mathrm{d}x = \int_0^c f(c)\,\mathrm{d}x - \int_0^c f(x)\,\mathrm{d}x + \int_c^a f(x)\,\mathrm{d}x - \int_c^a f(c)\,\mathrm{d}x$$
$$= cf(c) - \int_0^c f(x)\,\mathrm{d}x + \int_c^a f(x)\,\mathrm{d}x - (a-c)f(c)$$

$$\frac{\mathrm{d}A}{\mathrm{d}c} = \frac{\mathrm{d}}{\mathrm{d}c}\left[cf(c) - \int_0^c f(x)\,\mathrm{d}x + \int_c^a f(x)\,\mathrm{d}x - (a-c)f(c)\right]$$
$$0 = f(c)+cf'(c)-f(c)-f(c)-(-f(c)+(a-c)f'(c)) = cf'(c)-f(c)+f(c)-(a-c)f'(c)$$
$$= f'(c)\,(c-(a-c)) = \frac{\mathrm{d}}{\mathrm{d}c}\left[\sqrt{\tan c}\right](c-a+c) = \frac{\sec^2 c}{2\sqrt{\tan c}}(2c-a)$$

$$0 = \underbrace{\frac{1}{2\cos^2(c)\sqrt{\tan c}}}_{\text{cannot equal 0}}\underbrace{(2c-a)}_{\text{might equal 0}} \implies 2c-a=0 \implies c = \frac{a}{2}$$

We also know that $\left.\dfrac{\mathrm{d}^2A}{\mathrm{d}c^2}\right|_{c=a/2} > 0$ (by computer), which means that $\boxed{c=a/2}$ is a minimum.

Problem 9

The line $y=g(x)=f^{-1}(x)$ is equivalent to $x=f(y)$. The upper bound of integration is $y=f^{-1}(37)=3$ because $f(3)=37$.

$$\text{area} = \int_0^{f^{-1}(37)}(37-f(y))\,\mathrm{d}y = \int_0^3\left(37-y^3-3y-1\right)\mathrm{d}y = \int_0^3\left(36-y^3-3y\right)\mathrm{d}y$$
$$= \left[36y-\frac{y^4}{4}-\frac{3y^2}{2}\right]_0^3 = 108-\frac{81}{4}-\frac{27}{2}-0+0+0 = \frac{432-81-54}{4} = \boxed{\frac{297}{4}}$$