Problem Set #52

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Problem 5

$$
\begin{aligned}
z_1 &= 0 & |z_1| &= 0 \\
z_2 &= 0^2 + i = i & |z_2| &= 1 \\
z_3 &= (-i)^2 + i = -1 + i & |z_3| &= \sqrt{2} \\
z_4 &= (-1 + i)^2 + i = -i & |z_4| &= 1 \\
z_5 &= (-i)^2 + i = -1 + i & |z_5| &= \sqrt{2} \\
z_6 &= (-i + 1)^2 + i = -i & |z_6| &= 1
\end{aligned}
$$

I claim that:

$$
z_n = \begin{cases} 0 & n = 1 \\ 1 & n \text{ even} \\ \sqrt{2} & \text{else} \end{cases}
$$

Therefore $\boxed{z_{111} = \sqrt{2}}$

Problem 6

Let $F = a + bi$ where $a, b \in \mathbb{R}$, $a > 1$ and $0 < b < 1$.

$$
\begin{aligned}
\frac{1}{F} &= \frac{1}{a + bi} \cdot \frac{a - bi}{a - bi} \\
&= \frac{a - bi}{a + b} \\
&= \frac{a}{a + b} + \left(-\frac{b}{a + b}\right) i
\end{aligned}
$$

Because $a, b < a + b$ and $a, b, (a + b) \in \mathbb{R}^+$, $0 < \dfrac{a}{a + b} < 1$ and $-1 < -\dfrac{b}{a + b} < 0$. $\boxed{C}$ is the reciprocal of F.

Problem 7

(a) *Proof.* Let $z = a + bi$.

$$
z\overline{z} = (a + bi)(a - bi)
$$

$$
\begin{aligned}
&= a^2 + b^2 \\
&= \left(\sqrt{a^2+b^2}\right) \\
&= |z|^2
\end{aligned}
$$

$\square$

(b) Let $z = a + bi$ and $w = c + di$.

$$
\begin{aligned}
|z| &= |w| = 3 \\
\sqrt{a^2+b^2} &= \sqrt{c^2+d^2} = 3 \\
a^2 + b^2 &= c^2 + d^2 = 9
\end{aligned}
$$

$$
\begin{aligned}
|z+w| &= 2 \\
|a+bi+c+di| &= 2 \\
\sqrt{(a+c)^2+(b+d)^2} &= 2 \\
(a+c)^2 + (b+d)^2 &= 4 \\
a^2 + 2ac + c^2 + b^2 + 2bd + d^2 &= 4 \\
9 + 2ac + 9 + 2bd &= 4 \\
2ac + 2bd &= -14 \\
-2ac - 2bd &= 14
\end{aligned}
$$

$$
\begin{aligned}
|z-w| &= |a+bi-c-di| \\
&= \sqrt{(a-c)^2+(b-d)^2} \\
&= \sqrt{a^2-2ac+c^2+b^2-2bd+d^2} \\
&= \sqrt{9+9+14} \\
&= \sqrt{32} \\
&= \boxed{4\sqrt{2}}
\end{aligned}
$$

Problem 8

Let $z = a + bi$.

(a)

$$
\begin{aligned}
|z-3| &= |z+2i| \\
|(a-3)+bi| &= |a+(b+2)i| \\
\sqrt{(a-3)^2+b^2} &= \sqrt{a^2+(b+2)^2} \\
\cancel{a^2} - 6a + 9 + \cancel{b^2} &= \cancel{a^2} + \cancel{b^2} + 4b + 4 \\
6a + 4b &= 5 \\
b &= \frac{5-6a}{4}
\end{aligned}
$$

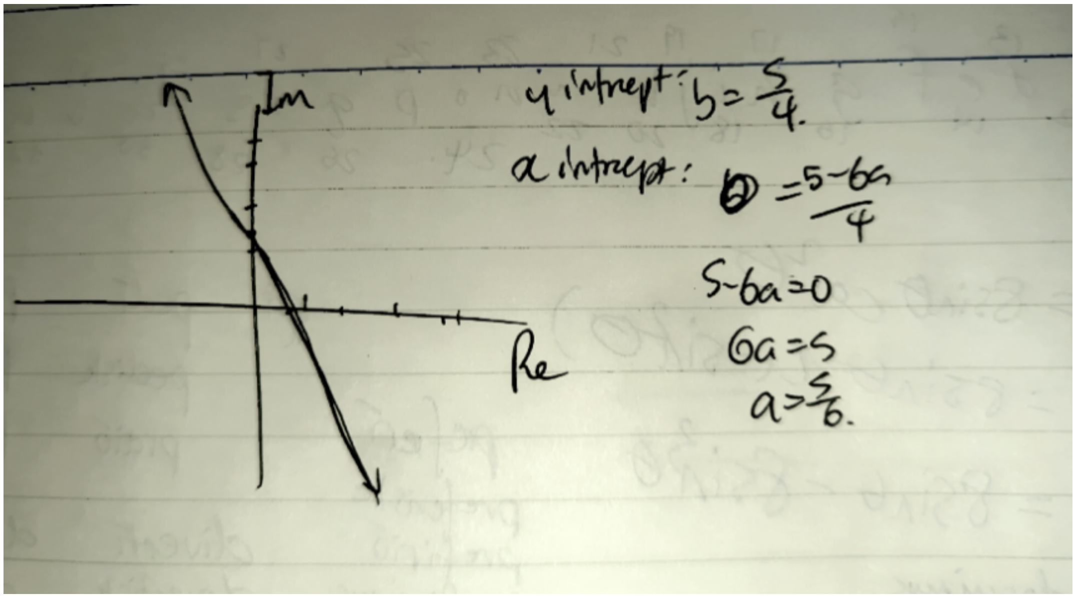
(b)

$$
\begin{aligned}
z + |z| &= 2 + 8i \\
a + bi + \sqrt{a^2+b^2} &= 2 + 8i \\
\left(a + \sqrt{a^2+b^2}\right) + bi &= 2 + 8i
\end{aligned}
$$

$$\operatorname{Re}\left(\left(a+\sqrt{a^2+b^2}\right)+bi\right)=\operatorname{Re}(2+8i)$$
$$a+\sqrt{a^2+8^2}=2$$
$$(a-2)^2=a^2+64$$
$$\cancel{a^2}-4a+4=\cancel{a^2}+64$$
$$4a=-60$$
$$a=-15$$

$$\operatorname{Im}\left(\left(a+\sqrt{a^2+b^2}\right)+bi\right)=\operatorname{Im}(2+8i)$$
$$b=8$$

$$|z|=|-15+8i|$$
$$|z|=\sqrt{225+64}$$
$$|z|=\sqrt{289}$$
$$\boxed{|z|=17}$$



Problem 9

Let $z=a+bi$.

$$|z-4+5i|=2\sqrt{3}$$
$$|(a-4)+(b+5)i|=2\sqrt{3}$$
$$(a-4)^2+(b+5)^2=\left(2\sqrt{3}\right)^2$$

Turn this from the complex plane to the real Cartesian plane by applying $a=x, b=y$.

$$(x-4)^2+(y+5)^2=\left(2\sqrt{3}\right)^2$$

This is the equation of a circle with radius $2\sqrt{3}$. The radius of this circle is:

$$\pi\left(2\sqrt{3}\right)^2=\boxed{12\pi}$$