Progress on Problem Set #40

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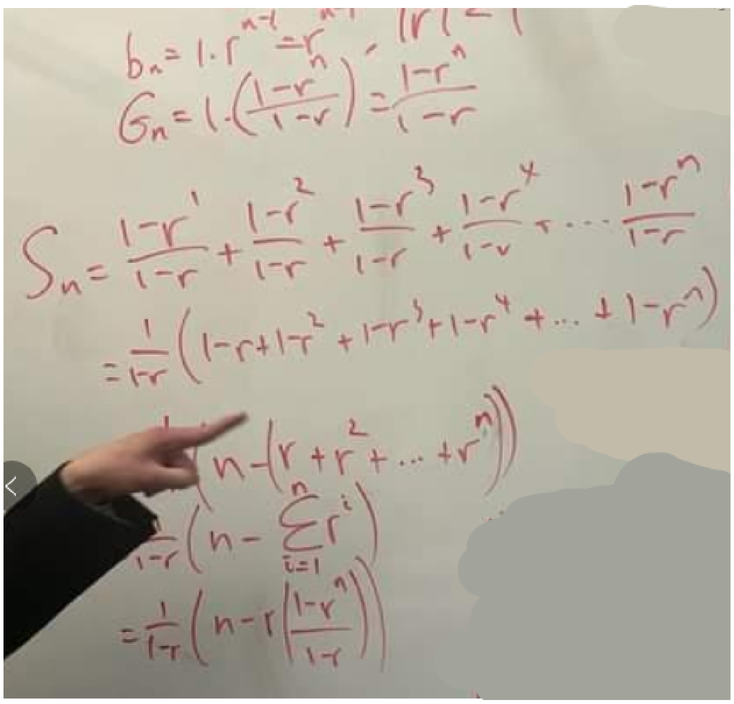
Problem 1



$$
\begin{aligned}
a_n &= a_1 + d(n-1) \\
&= 1 + d(n-1) \\
A_n &= \sum_{i=1}^{n} a_i \\
&= \frac{(1+1+d(n-1))n}{2} \\
&= \frac{(2+dn-d)n}{2}
\end{aligned}
$$

$$
\begin{aligned}
S_n &= \frac{1}{1-r}\left(n - r\left(\frac{1-r^n}{1-r}\right)\right) \\
&= \frac{n}{1-r} - \frac{r(1-r^n)}{(1-r)^2} \\
&= \frac{n(1-r)}{(1-r)(1-r)} - \frac{r(1-r^n)}{(1-r)(1-r)} \\
&= \frac{n-nr-r(1-r^n)}{(1-r)^2}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty}\left(\frac{A_n}{n} - S_n\right) &= 1 \\
\lim_{n\to\infty}\left(\frac{(2+dn-d)n}{2n} - \frac{1}{1-r}\left(n - r\left(\frac{1-r^n}{1-r}\right)\right)\right) &= 1 \\
\lim_{n\to\infty}\left(\frac{2+dn-d}{2} - \frac{n-nr-r(1-\cancelto{0}{r^n})}{(1-r)^2}\right) &= 1 \\
\lim_{n\to\infty}\frac{2+dn-d}{2} - \lim_{n\to\infty}\frac{n-nr-r}{(1-r)^2} &= 1 \\
\lim_{n\to\infty}\frac{2+dn-d}{2} - 1 &= \lim_{n\to\infty}\frac{n-nr-r}{(1-r)^2} \\
2(1-r)^2 \cdot \lim_{n\to\infty}\frac{dn-d}{2} &= 2(1-r)^2 \cdot \lim_{n\to\infty}\frac{n-nr-r}{(1-r)^2} \\
\lim_{n\to\infty}\left((dn-d)(1-r)^2\right) &= \lim_{n\to\infty} 2\left(n-nr-r\right)
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty}\left((dn-d)(1-2r+r^2)\right) &= \lim_{n\to\infty} 2\left(n-nr-r+r^{n+1}\right)\\
\lim_{n\to\infty}\left(dn-2rdn+r^2dn-d+2rd-dr^2\right) &= \lim_{n\to\infty}\left(2n-2nr-2r+2r^{n+1}\right)\\
\lim_{n\to\infty}\left(dn-2rdn+r^2dn-2n+2nr\right) &= \lim_{n\to\infty}\left(-2r+\cancelto{0}{2r^{n+1}}+d-2rd+dr^2\right)\\
\lim_{n\to\infty} n\left(d-2rd+r^2d-2+2r\right) &= -2r+d-2rd+dr^2
\end{aligned}
$$

RHS is a real (and therefore bounded) value so LHS must also converge. This means that:

$$
d-2rd+r^2d-2+2r=0=-2r+d-2rd+dr^2
$$

$$
\begin{gathered}
\cancel{d-2rd+dr^2}-2+2r=-2r+\cancel{d-2rd+dr^2}\\
2r-2=-2r\\
4r=2\\
\boxed{r=\frac{1}{2}}
\end{gathered}
$$

$$
\begin{aligned}
\lim_{n\to\infty}\left(\frac{(2+dn-d)n}{2n}-\frac{1}{1-r}\left(n-r\left(\frac{1-r^n}{1-r}\right)\right)\right) &= 1\\
\lim_{n\to\infty}\left(\frac{2+dn-d}{2}-\frac{1}{1-\frac{1}{2}}\left(n-\left(\frac{1}{2}\right)\left(\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right)\right)\right) &= 1\\
\lim_{n\to\infty}\left(\frac{2+dn-d}{2}-2\left(n-\left(\frac{1}{2}\right)\left(\frac{1-0}{\frac{1}{2}}\right)\right)\right) &= 1\\
\lim_{n\to\infty}\left(\frac{2+dn-d}{2}-2\left(n-1\right)\right) &= 1\\
2\lim_{n\to\infty}\left(\frac{2+dn-d}{2}-2n+2\right) &= 2\\
\lim_{n\to\infty}(2+dn-d-4n+4) &= 2\\
\lim_{n\to\infty}(dn-d-4n) &= -4\\
\lim_{n\to\infty} n(d-4) &= d-4\\
d-4=d-4 &= 0\\
\boxed{d=4}
\end{aligned}
$$

Problem 2

$$\begin{cases} 2\displaystyle\sum_{i=0}^{\infty}(\log_2 p)^i = \sum_{k=1}^{\infty}(1+q)^{-k} & (1) \\ \displaystyle\sum_{k=1}^{1}(1+q)^{-k} - \sum_{i=0}^{1}(\log_2 p)^i = \frac{7}{5} & (2) \end{cases}$$

$$\begin{aligned}
\sum_{i=0}^{1}(\log_2 p)^i + \frac{7}{5} &= \sum_{k=1}^{1}(1+q)^{-k} \\
(\log_2 p)^0 + (\log_2 p)^1 + \frac{7}{5} &= (1+q)^{-1} \\
\frac{5}{5} + \log_2 p + \frac{7}{5} &= \frac{1}{1+q} \\
\log_2 p + \frac{12}{5} &= \frac{1}{1+q} \\
2^{\log_2 p + \frac{12}{5}} &= 2^{\frac{1}{1+q}} \\
2^{\log_2 p} \cdot 2^{\frac{12}{5}} &= 2^{\frac{1}{1+q}} \\
p2^{\frac{12}{5}} &= 2^{\frac{1}{1+q}} \\
p &= \frac{2^{\frac{1}{1+q}}}{2^{\frac{12}{5}}} \\
p &= 2^{\frac{1}{1+q} - \frac{12}{5}}
\end{aligned}$$

$$\begin{aligned}
2\sum_{i=0}^{\infty}(\log_2 p)^i &= \sum_{k=1}^{\infty}(1+q)^{-k} \\
2\sum_{i=0}^{\infty}\left(\log_2\left(2^{\frac{1}{1+q}-\frac{12}{5}}\right)\right)^i &= \sum_{k=1}^{\infty}\frac{1}{(1+q)^k} \\
2\sum_{i=0}^{\infty}\left(\frac{1}{1+q}-\frac{12}{5}\right)^i &= \sum_{k=1}^{\infty}\left(\frac{1}{1+q}\right)^k \\
2\cdot\frac{\left(\frac{1}{1+q}-\frac{12}{5}\right)^0}{1-\left(\frac{1}{1+q}-\frac{12}{5}\right)} &= \frac{\left(\frac{1}{1+q}\right)^1}{1-\left(\frac{1}{1+q}\right)} \\
\frac{2}{1-\frac{1}{1+q}+\frac{12}{5}} &= \frac{\frac{1}{1+q}}{1-\frac{1}{1+q}} \\
\frac{2}{\frac{17}{5}-\frac{1}{1+q}} &= \frac{\frac{1}{1+q}}{\frac{1+q}{1+q}-\frac{1}{1+q}} \\
\frac{2}{\frac{17(1+q)}{5(1+q)}-\frac{5}{5(1+q)}} &= \frac{\frac{1}{1+q}}{\frac{q}{1+q}} \\
\frac{2}{\frac{17+17q-5}{5+5q}} &= \frac{1}{q} \\
2\cdot\frac{5+5q}{17+17q-5} &= \frac{1}{q} \\
\frac{10+10q}{17+17q-5} &= \frac{1}{q} \\
10q+10q^2 &= 17+17q-5 \\
10q^2-7q-12 &= 0 \\
10q^2-15q+8q-12 &= 0 \\
5q(2q-3)+4(2q-3) &= 0 \\
(5q+4)(2q-3) &= 0
\end{aligned}$$

$$q = -\frac{4}{5}, \frac{3}{2}$$

Case 1. $q = -\dfrac{4}{5}$

$$\begin{aligned}
p &= 2^{\frac{1}{-\frac{4}{5}}-\frac{12}{5}} \\
&= 2^{-\frac{5}{4}-\frac{12}{5}} \\
&= 2^{-\frac{25}{20}-\frac{48}{20}} \\
&= 2^{-\frac{73}{20}}
\end{aligned}$$

$$\begin{aligned}
&2\sum_{i=0}^{\infty}(\log_2 p)^i \\
&= 2\sum_{i=0}^{\infty}\left(-\frac{73}{20}\right)^i
\end{aligned}$$

This series does not converge as $\left|-\dfrac{73}{20}\right| \geq 1$.

Case 2. $q = \dfrac{3}{2}$

$$\begin{aligned}
p &= 2^{\frac{1}{\frac{3}{2}}-\frac{12}{5}} \\
&= 2^{\frac{2}{3}-\frac{12}{5}} \\
&= 2^{-\frac{10}{15}-\frac{36}{15}} \\
&= 2^{-\frac{26}{15}}
\end{aligned}$$

$$\begin{aligned}
&2\sum_{i=0}^{\infty}(\log_2 p)^i \\
&= 2\sum_{i=0}^{\infty}\left(-\frac{26}{15}\right)^i
\end{aligned}$$

This series does not converge as $\left|-\dfrac{26}{15}\right| \geq 1$.

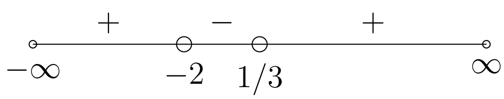
$\boxed{\text{No Solutions.}}$

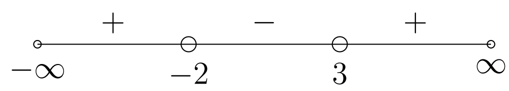
Problem 3

$$\sum_{k=1}^{\infty}\left(\frac{2x-1}{x+2}\right)^k = R$$
$$-1 < \frac{2x-1}{x+2} < 1$$

$$-1 < \frac{2x-1}{x+2}$$
$$\frac{2x-1}{x+2} + \frac{x+2}{x+2} > 0$$
$$\frac{3x-1}{x+2} > 0$$

$$\frac{2x-1}{x+2} < 1$$
$$\frac{2x-1}{x+2} - \frac{x+2}{x+2} < 0$$
$$\frac{x-3}{x+2} < 0$$





$$x \in \left(-\frac{1}{3}, 3\right)$$

$$\sum_{k=1}^{\infty}\left(\frac{2x-1}{x+2}\right)^k = \frac{\frac{2x-1}{x+2}}{1-\frac{2x-1}{x+2}} = R$$
$$\frac{\frac{2x-1}{x+2}}{\frac{x+2}{x+2}-\frac{2x-1}{x+2}} = R$$
$$\frac{2x-1}{\cancel{(x+2)}\cdot\frac{x+2-2x+1}{\cancel{x+2}}} = R$$
$$\frac{2x-1}{-x+3} = R$$

The range of R is the range of $\dfrac{2x-1}{-x+3}$ where $x \in \left(-\dfrac{1}{3}, 3\right)$.

Problem 6

Let r be the common ratio.

$$\begin{aligned}
\sin\theta &= \frac{R_{n+1}}{x} \\
\sin\theta &= \frac{R_n}{x + R_n + R_{n+1}}
\end{aligned}$$

$$\begin{aligned}
\sin\theta &= \frac{rR_n}{x} \\
\sin\theta &= \frac{R_n}{x + R_n + rR_n}
\end{aligned}$$

$$\begin{aligned}
\frac{rR_n}{x} &= \frac{R_n}{x + R_n + rR_n} \\
\frac{r}{x} &= \frac{1}{x + R_n + rR_n} \\
rx + rR_n + r^2R_n &= x \\
R_n(r + r^2) &= x - rx \\
R_n &= \frac{x(1-r)}{r(1+r)}
\end{aligned}$$

$$\begin{aligned}
\sin\theta &= \frac{r \cdot \frac{x(1-r)}{r(1+r)}}{x} \\
\sin\theta &= \frac{\frac{x(1-r)}{r(1+r)}}{x + \frac{x(1-r)}{r(1+r)} + r \cdot \frac{x(1-r)}{r(1+r)}}
\end{aligned}$$

$$\begin{aligned}
\sin\theta &= \frac{1-r}{1+r} \\
\sin\theta &= \frac{\frac{x(1-r)}{r(1+r)}}{\frac{xr(1+r)}{r(1+r)} + \frac{x(1-r)}{r(1+r)} + \frac{rx(1-r)}{r(1+r)}} \\
\sin\theta &= \frac{x(1-r)}{xr(1+r) + x(1-r) + rx(1-r)} \\
\sin\theta &= \frac{1-r}{r(1+r) + (1-r) + r(1-r)} \\
\sin\theta &= \frac{1-r}{r + r^2 + 1 - r + r - r^2} \\
\sin\theta &= \frac{1-r}{r+1} \\
r\sin\theta + \sin\theta &= 1 - r \\
r\sin\theta + r &= 1 - \sin\theta \\
r(\sin\theta + 1) &= 1 - \sin\theta \\
r &= \frac{1 - \sin\theta}{\sin\theta + 1}
\end{aligned}$$

$$\begin{aligned}
\sum_{k=1}^{\infty} R_k &= \frac{R_1}{1-r} \\
2\pi &= \frac{\frac{4}{3}}{1 - \frac{1-\sin\theta}{\sin\theta+1}} \\
6\pi &= \frac{4}{\frac{\sin\theta+1}{\sin\theta+1} - \frac{1-\sin\theta}{\sin\theta+1}} \\
6\pi &= \frac{4}{\frac{2\sin\theta}{\sin\theta+1}} \\
6\pi &= 4 \cdot \frac{\sin\theta + 1}{2\sin\theta} \\
12\pi\sin\theta &= 4\sin\theta + 4 \\
3\pi\sin\theta - \sin\theta &= 1 \\
\sin(\theta)(3\pi - 1) &= 1 \\
\sin\theta &= \frac{1}{3\pi - 1} \\
&\boxed{\theta = \arcsin\left(\frac{1}{3\pi - 1}\right)}
\end{aligned}$$