Problem Set #37

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Problem 1

(b)
$$\begin{aligned}
a_n &= (-1)^n \cdot \frac{n+1}{3^n} \\
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \left((-1)^n \cdot \frac{n+1}{3^n} \right) \\
&= \lim_{n\to\infty} (-1)^n \cdot \lim_{n\to\infty} \frac{n+1}{3^n} \\
&= \lim_{n\to\infty} (-1)^n \cdot \lim_{n\to\infty} \left(\frac{n}{3^n} + \frac{1}{3^n} \right) \\
&= \lim_{n\to\infty} (-1)^n \cdot \lim_{n\to\infty} \left(\frac{3^{\log_3 n}}{3^n} \right) \\
&= \lim_{n\to\infty} (-1)^n \cdot \lim_{n\to\infty} 3^{\log_3(n)-n}
\end{aligned}$$

$\log_3 n$ is always less than n for positive numbers, hence $\log_3 n - n \to -\infty$ as $n \to \infty$.

$$\begin{aligned}
&= \lim_{n\to\infty} (-1)^n \cdot \lim_{n\to\infty} 3^{-n} \\
&= \lim_{n\to\infty} (-1)^n \cdot 0
\end{aligned}$$

$(-1)^n$ oscillates between -1 and 1. Any real number multiplied by 0 equals 0.

$$= \boxed{0}$$

(f)
$$\begin{aligned}
a_n &= (n+2)! \\
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} (n+2)! \\
&= \boxed{\infty}
\end{aligned}$$

(a)
$$\begin{aligned}
a_n &= \frac{n}{n+1} \\
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{n}{n+1} \\
&= \lim_{n\to\infty} \frac{n}{n\left(1+\frac{1}{n}\right)} \\
&= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} \\
&= \frac{1}{1+0} \\
&= \boxed{1}
\end{aligned}$$

(c)
$$\begin{aligned}
a_n &= \sqrt{n-1} \\
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \sqrt{n-1} \\
&= \boxed{\infty}
\end{aligned}$$

(d)
$$\begin{aligned}
a_n &= \frac{\sqrt{n}}{2} \\
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \frac{\sqrt{n}}{2} \\
&= \boxed{\infty}
\end{aligned}$$

(e)
$$\begin{aligned}
a_n &= -7n+9 \\
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} (-7n+9) \\
&= \boxed{-\infty}
\end{aligned}$$

Problem 4

$$\begin{aligned}
S_n - 2S_{n+1} + S_{n+2} &= S_n - 2(S_n + u_{n+1}) + S_n + u_{n+1} + u_{n+2} \\
&= S_n - 2S_n - 2u_{n+1} + S_n + u_{n+1} + u_{n+1} + d \\
&= -2u_{n+1} + u_{n+1} + u_{n+1} + d \\
&= \boxed{d}
\end{aligned}$$

Problem 7

$$\begin{aligned}
S_n &= \frac{(2a+d(n-1))n}{2} = \frac{2an+dn^2-dn}{2} = m \\
S_m &= \frac{(2a+d(m-1))m}{2} = \frac{2am+dm^2-dm}{2} = n
\end{aligned}$$

$$\begin{aligned}
\frac{m}{n} &= \frac{2an+dn^2-dn}{2n} \\
&= \frac{2a+dn-d}{2} \\
\frac{n}{m} &= \frac{2am+dm^2-dm}{2m} \\
&= \frac{2a+dm-d}{2}
\end{aligned}$$

$$\begin{aligned}
\frac{m}{n}-\frac{n}{m} &= \frac{dn-dm}{2} \\
\frac{m^2-n^2}{mn} &= \frac{d(n-m)}{2} \\
\frac{(m+n)\cancel{(m-n)}}{mn} &= \frac{-d\cancel{(m-n)}}{2} \\
\frac{m+n}{mn} &= -\frac{d}{2} \\
d &= -2\left(\frac{m+n}{mn}\right)
\end{aligned}$$

$$\begin{aligned}
S_{m+n} &= \frac{(2a+d(m+n-1))(m+n)}{2} \\
&= \frac{2am+2an+dm(m+n-1)+dn(m+n-17)}{2} \\
&= \frac{2am+2an+dm^2+dmn-dm+dmn+dn^2-dn}{2} \\
&= \frac{\overbrace{\boxed{2am+dm^2-dm}}^{=2S_m=2n}+\overbrace{\boxed{2an+dn^2-dn}}^{=2S_n=2m}+2dmn}{2} \\
&= \frac{2n+2m+2dmn}{2} \\
&= n+m+dmn \\
&= n+m-2\left(\frac{m+n}{mn}\right)mn \\
&= n+m-2n-2m \\
&= -m-n
\end{aligned}$$

$\square$

Problem 8

$$\begin{aligned}
f(x) &= \log_2 x - \log_x 2,\ 0<x<1 \\
f(2^{u_n}) &= 2n \\
\log_2 2^{u_n} - \log_{2^{u_n}} 2 &= 2n \\
u_n - \frac{1}{u_n}\log_2 2 &= 2n \\
u_n^2 - 1 &= 2nu_n^2 \\
u_n^2 - 2nu_n - 1 &= 0 \\
u_n &= \frac{2n \pm \sqrt{4n^2+4}}{2} \\
u_n &= n \pm \sqrt{n^2+1}
\end{aligned}$$

C1: $+$

$$\begin{aligned}
u_n &= n+\sqrt{n^2+1} \\
2^{u_n} &= 2^{n+\sqrt{n^2+1}}
\end{aligned}$$

Range of $n+\sqrt{n^2+1}$ for $n \in \mathbb{N}$ is $[1+\sqrt{2},\infty)$. Therefore $2^{n+\sqrt{n^2+1}}$ is outside the domain of f.

C2: $-$

$$\begin{aligned}
u_n &= n-\sqrt{n^2+1} \\
2^{u_n} &= 2^{n+\sqrt{n^2-1}}
\end{aligned}$$

Range of $n-\sqrt{n^2+1}$ for $n \in \mathbb{N}$ is $(0, 1-\sqrt{2}]$. Therefore $2^{n-\sqrt{n^2+1}}$ is within the domain of f.

$$\boxed{u_n = n-\sqrt{n^2+1}}$$