Problem Set #57

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Problem 1

$$\begin{aligned}\lim_{x\to 3}\left[2f(x)-g(x)\right]&=4\\ 2\lim_{x\to 3}f(x)-\lim_{x\to 3}g(x)&=4\\ 2f(3)-g(3)&=4\\ 10-g(3)&=4\\ \boxed{g(3)=6}\end{aligned}$$

Problem 2

$$\begin{aligned}\lim_{x\to a}f(x)&=\lim_{x\to -1}\left(x+2x^3\right)^4\\ &=\left(\lim_{x\to -1}\left[x+2x^3\right]\right)^4\\ &=\left(\lim_{x\to -1}x+\lim_{x\to -1}2x^3\right)^4\\ &=\left(-1+2\lim_{x\to -1}x^3\right)^4\\ &=\left(-1+2\left(\lim_{x\to -1}x\right)^3\right)^4\\ &=\left(-1+2\left(-1\right)^3\right)^4\\ &=f(a)\end{aligned}$$

Because $\lim_{x\to a}f(x)=f(a)$, f is continuous at a by the definition of continuity.

Problem 3

Let $a\in(2,\infty)$.

$$\lim_{x\to a}f(x)=\lim_{x\to a}\frac{2x+3}{x-2}$$

$$= \frac{\lim_{x\to a}[2x+3]}{\lim_{x\to a}[x-2]}$$

(this is allowed because $x-2 \neq 0$ since $a \neq 2$.)

$$= \frac{\lim_{x\to a} 2x + \lim_{x\to a} 3}{\lim_{x\to a} x - \lim_{x\to a} 2}$$

$$= \frac{2a+3}{a-2}$$

$$= f(a)$$

Because $\lim_{x\to a} f(x) = f(a)$, f is continuous at a by the definition of continuity for all $a \in (2, \infty)$.

Problem 4

(a) f is not defined at 1 so the function is discontinuous at $x = 1$.

(b)

$$\lim_{x\to 1^+} f(x) = \frac{1}{1} = 1$$

$$\lim_{x\to 1^-} f(x) = 1 - (1)^2 = 0$$

$\lim_{x\to 1} f(x)$ DNE so the function is discontinuous at $x = 1$.

(c)

$$\lim_{x\to 0^+} f(x) = 1 - (0^2) = 1$$

$$\lim_{x\to 0^-} f(x) = \cos 0 = 1$$

$$f(0) = 0$$

$\lim_{x\to 0} f(x) = 1 \neq 0 = f(0)$ so the function is discontinuous at $x = 0$.

Problem 5

(a)

$$x^2 + 5x + 6 \neq 0$$

$$(x+2)(x+3) \neq 0$$

$$x \notin \{-2, -3\}$$

Domain is $(-\infty, -3) \cup (-3, -2) \cup (-2, \infty)$. F is a rational function so it is continuous on its domain.

(b)

$$2x - 1 \geq 0$$

$$2x \geq 1$$

$$x \geq \frac{1}{2}$$

Domain is $\left[\frac{1}{2}, \infty\right)$. x^2 is a polynomial and continuous. $2x$ is a polynomial and continuous. 1 is a constant and continuous. $2x-1$ is the difference of two continuous function and is continuous. The root function is continuous on all points on its domain so it is continuous on the image of $2x-1$. $x^2 + \sqrt{2x-1}$ is the sum of two continuous functions and is continuous on all points on its domain.

(c) Domain is $\mathbb{R}$. Cosine is continuous on $\mathbb{R}$ and the image of $1 - x^2$ is a subset of $\mathbb{R}$.

(d) Domain is $(0, \infty]$. $\sqrt{x}$ is a root function and continuous on its domain. Sine is continuous on $\mathbb{R}$. F is the product of two continuous function and is therefore continuous.

Problem 6

This function is continuous on its domain. Let $f(x) = 1/x$ and $g(x) = 1 + \sin x$, then $y = f(g(x))$. f is a rational function and continuous on its domain, which is $x \neq 0$. g is continuous on $\mathbb{R}$. So the graph is continuous on all $g(x) \neq 0$.

$$\begin{aligned} 1 + \sin x &\neq 0 \\ \sin x &\neq -1 \\ x &\neq -\frac{\pi}{2} + \pi n \end{aligned}$$

Discontinuities are at all points $-\pi/2 + \pi n$ where $n \in \mathbb{Z}$.

Problem 7

(a)

$$\lim_{x \to 4} \frac{5 + \sqrt{x}}{\sqrt{5 + x}}$$

Denominator not equal to 0: $\sqrt{5+4} = 3 \neq 0$

$$= \frac{\lim_{x \to 4} \left[5 + \sqrt{x}\right]}{\lim_{x \to 4} \sqrt{5 + x}}$$

$$= \frac{\lim_{x \to 4} 5 + \sqrt{\lim_{x \to 4} x}}{\sqrt{\lim_{x \to 4} 5 + \lim_{x \to 4} x}}$$

x is continuous.

$$= \frac{5 + 2}{\sqrt{5 + 4}}$$

$$= \boxed{\frac{7}{9}}$$

(b)

$$\lim_{x \to \pi/4} x \cos^2 x$$

$$= \lim_{x \to \pi/4} x \cdot \left(\lim_{x \to \pi/4} \cos x\right)^2$$

It is known that cosine and x are continuous.

$$= \frac{\pi}{4} \left(\frac{\sqrt{2}}{2}\right)^2$$

$$= \boxed{\frac{\pi}{8}}$$

Problem 8

Problem 9

Problem 10

Problem 11

Problem 12

Problem 13

Problem 14