Problem Set #49

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Problem 2

The x-coordinate of every point on the parametric equation must satisfy $x \in [1, 4]$ and the y-coordinate must satisfy $y \in [2, 3]$. The points on the equation must be within this closed region.

Problem 3

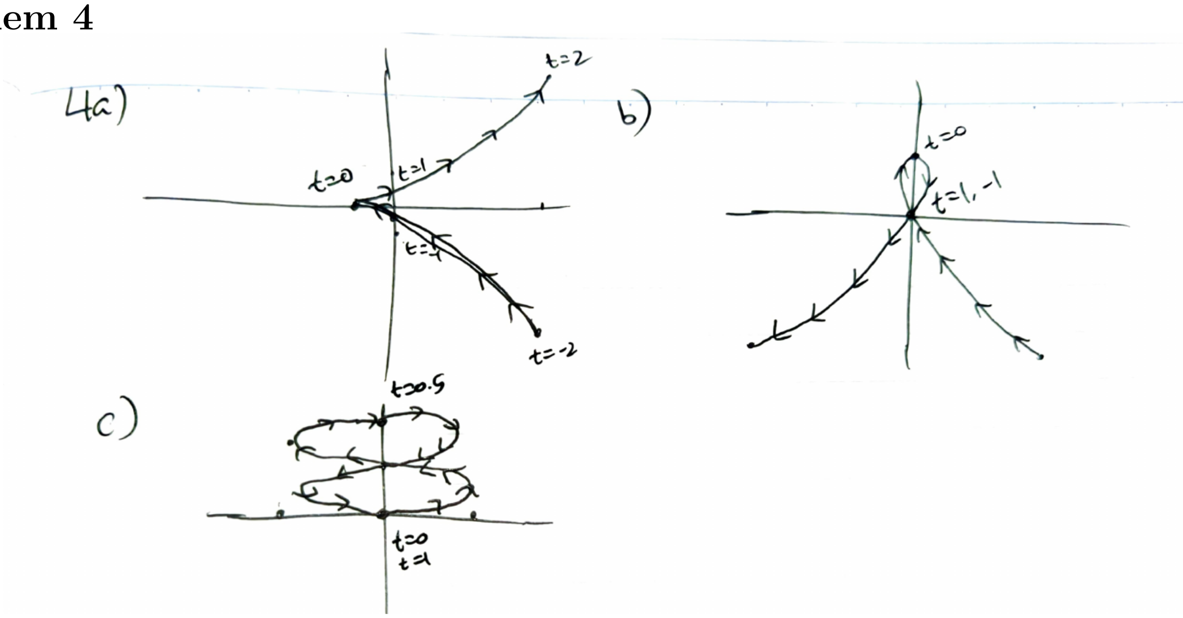
(a) III. The range of x is $[1, 2]$ and III is the only graph where every coordinate has x-coordinate inside this interval.

(d) I. This is the only graph that passes through $(2, 2)$, $(2, -2)$, $(-2, 2)$ and $(-2, -2)$.

(c) IV. The graph for y shows that $y \geq 0$. IV is the only graph without a negative y-coordinate.

(d) II. This is the only curve with the same x and y intercepts as shown in the graph.

Problem 4



Problem 7

(a) Let $t = x$. $-3 \leq x \leq 2 \implies -3 \leq t \leq 2$, $y = x^2 \implies y = t^2$.

$$\boxed{x(t) = t, y(t) = t^2, t \in [-3, 2]}$$

(b) Let $t = f^{-1}(x) \implies f(t) = x$. $y = f^{-1}(x) \implies y = t$.

$$\boxed{x(t) = t^5 + 2t + 1, y(t) = t}$$

(c)

$$\begin{aligned} x &= x_0 + (x_1 - x_0)t \\ y &= y_0 + (y_1 - y_0)t \\ 0 &\le t \le 1 \end{aligned} \implies \begin{aligned} x &= 2 + (1-2)t \\ y &= -3 + (5-(-3))t \\ 0 &\le t \le 1 \end{aligned} \implies \boxed{\begin{aligned} x(t) &= 2 - t \\ y(t) &= -3 + 8t \\ 0 &\le t \le 1 \end{aligned}}$$

(d)

$$\begin{aligned} x^2 + 2x + y^2 - 4y &= 4 \\ (x+1)^2 - 1 + (y-2)^2 - 4 &= 4 \\ (x+1)^2 + (y-2)^2 &= 9 \end{aligned}$$

$9 = 9\cos^2 t + 9\sin^2 t$. Let $9\cos^2 t = (x+1)^2, 9\sin^2 t = (y-2)^2$.

$$\begin{aligned} 9\cos^2 t &= (x+1)^2 \\ 9\sin^2 t &= (y-2)^2 \\ 0 &\le t \le 2\pi \end{aligned} \implies \begin{aligned} 3\cos t &= x + 1 \\ 3\sin t &= y - 2 \\ 0 &\le t \le 2\pi \end{aligned} \implies \boxed{\begin{aligned} x(t) &= 3\cos t - 1 \\ y(t) &= 3\sin t + 2 \\ 0 &\le t \le 2\pi \end{aligned}}$$

(e) Let $\cos^2 t = \dfrac{x^2}{4}, \sin^2 t = \dfrac{y^2}{9} \implies \cos t = \dfrac{x}{2}, \sin t = \dfrac{y}{3}$. All points on the left half of the ellipse satisfy $x \le 0 \implies \cos t \le 0 \implies \dfrac{\pi}{2} \le t \le \dfrac{3\pi}{2}$.

$$\boxed{x(t) = 2\cos t, y(t) = 3\sin t, \frac{\pi}{2} \le t \le \frac{3\pi}{2}}$$

Problem 8

(a) Let $t = x \implies y = t^2$. $-1 \le x \le 2 \implies -1 \le t \le 2$. Let $t_1 = t + 1 \implies t = t_1 - 1, x = t_1 - 1, y = (t_1 - 1)^2, 0 \le t_1 \le 3$.

$$\boxed{x(t) = t - 1, y(t) = (t-1)^2, 0 \le t \le 3}$$

(b) First path: $x = 0 + (3-0)t = 3t, y = 0 + (4-0)t = 4t, 0 \le t \le 1$.
Second path: We need to increase t by 1 so that it can connect to the first path.
$x = 3 + (5-3)(t-1) = 3 + 2t - 2 = 1 + 2t, y = 4 + (0-4)(t-1) = 4 - 4t + 4 = 8 - 4t, 1 < t \le 2$.

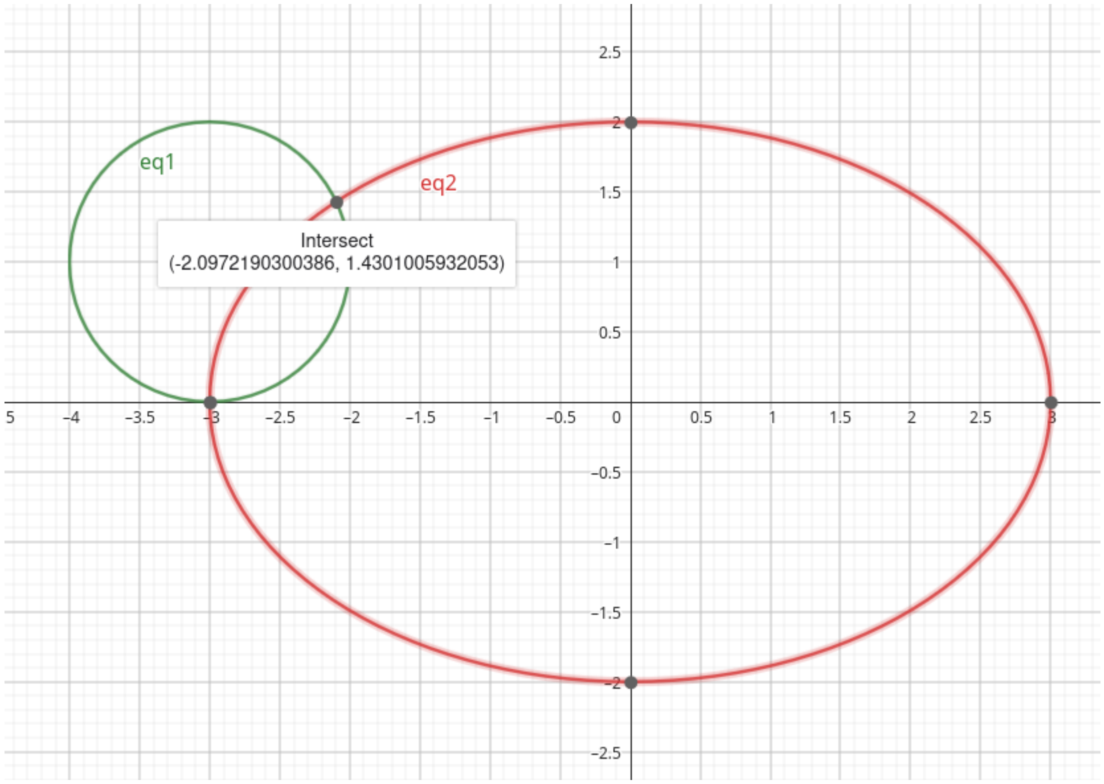
$$\boxed{x(t) = \begin{cases} 3t & 0 \le t \le 1 \\ 1 + 2t & 1 < t \le 2 \end{cases}, y(t) = \begin{cases} 4t & 0 \le t \le 1 \\ 8 - 4t & 1 < t \le 2 \end{cases}}$$

(c) We will reverse the direction of the parametric equation by using $-t$ instead of t, and apply a parameter shift of $-\dfrac{\pi}{2}$ to move the starting point to $(0, -1)$.

$$\begin{aligned} x &= \cos\left(-\left(t + \frac{\pi}{2}\right)\right) \\ y &= \sin\left(-\left(t + \frac{\pi}{2}\right)\right) \\ 0 &\le t \le 2\pi \end{aligned} \implies \begin{aligned} x &= \cos t \cos\frac{\pi}{2} - \sin t \sin\frac{\pi}{2} \\ y &= -\sin t \cos\frac{\pi}{2} - \cos t \sin\frac{\pi}{2} \\ 0 &\le t \le 2\pi \end{aligned} \implies \boxed{\begin{aligned} x(t) &= -\sin t \\ y(t) &= -\cos t \\ 0 &\le t \le 2\pi \end{aligned}}$$

Problem 9

(a)



Intersections: $(-2.097, 1.430), (-3, 0)$

(b) A collision point must satisfy $x_1 = y_1, x_2 = y_2$ at the same value t.

$$\begin{aligned} 3\sin t &= -3 + \cos t \\ 2\cos t &= 1 + \sin t \end{aligned}$$

$$\begin{aligned} 9 - 9\cos^2 t &= \cos^2 t - 6\cos t + 9 \\ 4 - 4\sin^2 t &= \sin^2 t + 2\sin t + 1 \end{aligned}$$

$$\begin{aligned} 0 &= 10\cos^2 t - 6\cos t \\ 0 &= 5\sin^2 t + 2\sin t - 3 \end{aligned}$$

$$\begin{aligned} 0 &= 2\cos(t)(5\cos t - 3) \\ t &\in \left\{\frac{\pi}{2}, \frac{3\pi}{2}, \arccos\frac{3}{5}, 2\pi - \arccos\frac{3}{5}\right\} \end{aligned}$$

$$\begin{aligned} 0 &= 5\sin^2 t + 5\sin t - 3\sin t - 3 \\ 0 &= 5\sin(t)(\sin t + 1) - 3(\sin t + 1) \\ 0 &= (5\sin t - 3)(\sin t + 1) \\ t &\in \left\{\arcsin\frac{3}{5}, \pi - \arcsin\frac{3}{5}, \frac{3\pi}{2}\right\} \end{aligned}$$

A collision occurs at $t = \dfrac{3\pi}{2}$. Thus the one collision point is $\boxed{(-3, 0)}$.