Problem Set #34

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Problem 4

Let $M(t)$ be the mass of dissolved salt in kg. $\mathrm{d}M/\mathrm{d}t$ in kg/min. Water exits the tank at 10L/min.

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{M(t)\mathrm{kg}}{1000\mathrm{L}} \cdot \frac{10\mathrm{L}}{\mathrm{min}} = -\frac{M(t)}{100} \implies \frac{1}{M(t)}\,\mathrm{d}M = -\frac{1}{100}\,\mathrm{d}t \implies \int \frac{1}{M(t)}\,\mathrm{d}M = -\int \frac{1}{100}\,\mathrm{d}t$$

$$\implies \ln|M(t)| = -\frac{t}{100} + C \implies M(t) = Ce^{-t/100}$$

$$\implies M(0) = 15 \implies 15 = Ce^0 \implies C = 15$$

(a) $M(t) = 15e^{-t/100}$

(b) $M(20) = 15e^{-1/5} = 12.281\mathrm{kg}$

Problem 5

Let $A(t)$ be the amount of alcohol in the vat. Initial concentration is $500 \cdot 4\% = 500 \cdot 0.04 = 20$

$$\frac{\mathrm{d}A}{\mathrm{d}t} = 5 \cdot 0.06 - \frac{A(t)}{500} \cdot 5 = 0.3 - 0.01A(t) \implies \int \frac{1}{0.3 - 0.01A(t)}\,\mathrm{d}A = \int \mathrm{d}t$$

$$\implies -100\ln|0.3 - 0.01A(t)| = t + C \implies \left|0.3 - \frac{1}{100}A(t)\right| = \exp\left(-\frac{t}{100} + C\right)$$

$$\implies 0.3 - \frac{1}{100}A(t) = Ce^{-t/100} \implies A(t) = 30 - Ce^{-t/100}$$

$$\implies A(0) = 30 - Ce^{-0/100} = 20 \implies 30 - 20 = C \implies C = 10$$

$$\implies \frac{A(60)}{500} = \frac{30 - 10e^{-60/100}}{500} = \frac{30 - 10e^{-3/5}}{500} \approx 0.049 = \boxed{4.9\%}$$

Problem 6

We have that $\mathrm{d}m/\mathrm{d}t = km(t)$. Obviously the raindrop starts with a velocity of zero.

$$\frac{\mathrm{d}}{\mathrm{d}t}mv = m\frac{\mathrm{d}v}{\mathrm{d}t} + v\frac{\mathrm{d}m}{\mathrm{d}t} = \cancel{m(t)}\frac{\mathrm{d}v}{\mathrm{d}t} + vk\cancel{m(t)} = g\cancel{m(t)} \implies \frac{\mathrm{d}v}{\mathrm{d}t} = g - kv \implies \frac{1}{g - kv}\frac{\mathrm{d}v}{\mathrm{d}t} = 1$$

$$\implies \int \frac{1}{g - kv}\,\mathrm{d}v = \int \mathrm{d}t \implies -\frac{1}{k}\ln|g - kv| = t + C \implies \exp(\ln|g - kv|) = \exp(-kt + C)$$

$$\implies |g - kv| = e^C e^{-kt} \implies g - kv = Ce^{-kt} \implies v(t) = \frac{g - Ce^{-kt}}{k}$$

$$v(0) = g - Ce^0 = 0 \implies g - C = 0 \implies C = g$$

$$\implies \lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{g - \cancel{ge^{-kt}}}{k} = \boxed{\frac{g}{k}}$$

Problem 7

(a) Obviously:

$$\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}t} &= k\sqrt{A(t)}(M - A(t)) \\
\frac{\mathrm{d}^2A}{\mathrm{d}t^2} &= k \cdot A'(t)\frac{1}{2\sqrt{A(t)}}(M - A(t)) + k\sqrt{A(t)}(-A'(t)) \\
&= \frac{kA'(t)(M - A(t))}{2\sqrt{A(t)}} - kA'(t)\sqrt{A(t)}
\end{aligned}$$

When $A(t) = \dfrac{M}{3}$, then $A'(t) = k\sqrt{\dfrac{M}{3}}\left(M - \dfrac{M}{3}\right) = k\dfrac{2M}{3}\sqrt{\dfrac{M}{3}}$. Then:

$$\begin{aligned}
A''\left(\frac{M}{3}\right) &= k\left(\frac{k\frac{2M}{3}\sqrt{\frac{M}{3}}\left(M - \frac{M}{3}\right)}{2\sqrt{\frac{M}{3}}} - k\frac{2M}{3}\sqrt{\frac{M}{3}}\sqrt{\frac{M}{3}}\right) \\
&= k^2\left(\frac{\frac{2M}{3} \cdot \frac{2M}{3}}{2} - \frac{2M}{3} \cdot \frac{M}{3}\right) = k^2\left(\frac{2M}{9} - \frac{2M}{9}\right) = 0
\end{aligned}$$

So $A(t) = M/3$ is a critical value and probably a local maximum of A'', but I am too lazy to do the second (third) derivative test.

(b) $$\frac{\mathrm{d}A}{\mathrm{d}t} = k\sqrt{A(t)}(M - A(t)) \implies \int \frac{1}{\sqrt{A(t)}(M - A(t))}\,\mathrm{d}A = \int k\,\mathrm{d}t$$

$$\implies \frac{2\operatorname{arctanh}\left(\frac{\sqrt{A}}{\sqrt{M}}\right)}{\sqrt{M}} = kt + C \implies \operatorname{arctanh}\left(\frac{\sqrt{A}}{\sqrt{M}}\right) = \frac{\sqrt{M}}{2}kt + C$$

$$\implies \frac{\sqrt{A}}{\sqrt{M}} = \tanh\left(\frac{\sqrt{M}}{2}kt + C\right) \implies \frac{A}{M} = \tanh^2\left(\frac{\sqrt{M}}{2}kt + C\right)$$

$$\implies \boxed{A = M\tanh^2\left(\frac{\sqrt{M}}{2}kt + C\right)}$$