Problem Set #38

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Problem 1

In class we found that $f(x) = x^2 - 3x + 2$. For an arbitrary function g, $f(x)$ must equal 0 to cancel the $g(x)$ term.

$$\begin{aligned} f(x) &= 0 \\ (x-2)(x-1) &= 0 \\ x = 2,\ x &= 1 \end{aligned}$$

$$\begin{aligned} f(x)g(x) + a_n x + b_n &= x^{n+1} \\ f(2)g(2) + 2a_n + b_n &= 2^{n+1} \\ f(1)g(1) + 1a_n + b_n &= 1^{n+1} \end{aligned}$$

$$\begin{aligned} 2a_n + b_n &= 2^{n+1} && (1) \\ a_n + b_n &= 1 && (2) \\ 2a_n + 2b_n &= 2 && (3) \end{aligned}$$

$$\begin{aligned} (1) - (2): &\ \boxed{a_n = 2^{n-1} - 1} \\ (3) - (1): &\ \boxed{b_n = 2 - 2^{n+1}} \end{aligned}$$

Problem 2

Given that $a = 10$ and $b = 5$, the equation for the ellipse is $\dfrac{x^2}{100} + \dfrac{y^2}{25} = 1$.

$$\begin{aligned} a_1 + a_2 + a_3 &= 255 \\ a_1 + d + a_1 + d + a_1 + 2d &= 255 \\ 3a_1 + 3d &= 255 \\ a_2 &= 85 \end{aligned}$$

$$\begin{aligned} a_1 &= |OP_1|^2 \\ a_1 &= (\sqrt{10^2 + 0^2})^2 \\ a_1 &= 100 \end{aligned}$$

$$\begin{aligned} d &= a_2 - a_1 \\ d &= 85 - 100 \\ d &= -15 \end{aligned}$$

$$\begin{aligned} a_3 &= a_2 + d \\ a_3 &= 70 \end{aligned}$$

$$\begin{aligned} \frac{x_3^2}{100} + \frac{y_3^2}{25} &= 1 \\ x_3^2 + y_3^2 &= 70 \end{aligned}$$

$$\begin{aligned} x_3^2 + 4y_3^2 &= 100 \\ x_3^2 + y_3^2 &= 70 \end{aligned}$$

$$\begin{aligned} 3y_3^2 &= 30 \\ y_3 &= \pm\sqrt{10} \end{aligned}$$

$$\begin{aligned} x_3^2 + 10 &= 70 \\ x_3 &= \pm\sqrt{60} \end{aligned}$$

$$\boxed{\begin{array}{c} (\sqrt{60}, \sqrt{10}),\ (\sqrt{60}, -\sqrt{10}), \\ (-\sqrt{60}, \sqrt{10}),\ (-\sqrt{60}, -\sqrt{10}) \end{array}}$$

Problem 4

(a)

$$f(x) = \sqrt{x^2 - 4},\ x \leq -1$$

Range of f: $f(x) \geq 0$

$$\begin{aligned} x &= \sqrt{f^{-1}(x)^2 - 4},\ f^{-1}(x) \leq -2 \\ x^2 &= f^{-1}(x)^2 - 4 \\ f^{-1}(x) &= \pm\sqrt{x^2 + 4} \end{aligned}$$

If the $\pm$ were plus, then $f^{-1}(x) \geq 2$. If the $\pm$ were minus, then $f^{-1}(x) \leq 2$. In addition, the range of f is the domain of f^{-1}. Therefore:

$$\boxed{f^{-1}(x) = -\sqrt{x^2 + 4},\ x \geq 0}$$

(b)

$$\begin{aligned} a_n &= -f^{-1}(a_{n-1}) \\ a_n &= \sqrt{a_{n-1}^2 + 4} \\ a_n^2 &= a_{n-1}^2 + 4 \end{aligned}$$

a_n^2 is an arithmetic sequence with $d = 4$

$$\begin{aligned} a_n^2 &= a_1 + 4(n-1) \\ a_n^2 &= 4n - 3 \\ a_n &= \pm\sqrt{4n - 3} \end{aligned}$$

$-$ will not work because a_1 would equal $-\sqrt{4-3} = -1 \neq 1$

$$\boxed{a_n = \sqrt{4n - 3}}$$

(c)

$$\begin{aligned} b_n &= \frac{1}{a_n + a_{n+1}} \\ &= \frac{1}{\sqrt{4n-3} + \sqrt{4n+1}} \cdot \frac{\sqrt{4n-3} - \sqrt{4n+1}}{\sqrt{4n-3} - \sqrt{4n+1}} \\ &= \frac{\sqrt{4n-3} - \sqrt{4n+1}}{4n - 3 - (4n+1)} \\ &= \frac{\sqrt{4n-3} - \sqrt{4n+1}}{-4} \\ &= \frac{a_n - a_{n+1}}{-4} \end{aligned}$$

Let S_n be the nth partial sum of b_n.

$$\begin{aligned} S_n &= b_1 + b_2 + \cdots + b_n \\ &= \frac{a_1 - a_2}{-4} + \frac{a_2 - a_3}{-4} + \cdots + \frac{a_{n-1} - a_n}{-4} + \frac{a_n - a_{n+1}}{-4} \\ &= -\frac{1}{4}(a_1 \cancel{- a_2} \cancel{+ a_2} \cancel{- a_3} + \ldots \cancel{+ a_{n-1}} \cancel{- a_n} \cancel{+ a_n} - a_{n+1}) \\ &= -\frac{1}{4}(a_1 - a_{n+1}) \\ &= \boxed{-\frac{1 - \sqrt{4n+1}}{4}} \end{aligned}$$

Problem 6

(a)

$$\begin{aligned} a_3a_4 &= 117 \\ a_2 + a_5 &= 22 \end{aligned}$$

$$\begin{aligned} (a_1+2d)(a_1+3d) &= 117 \\ (a_1+d)+(a_1+4d) &= 22 \end{aligned}$$

$$\begin{aligned} a_1^2 + 5a_1d + 6d^2 &= 117 \\ 2a_1 + 5d &= 22 \\ a_1 &= 11 - \frac{5d}{2} \end{aligned}$$

$$\begin{aligned} \left(11 - \frac{5d}{2}\right)^2 + 5d\left(11 - \frac{5d}{2}\right) + 6d^2 &= 117 \\ 121 \cancel{- 55d} + \frac{25d^2}{4} \cancel{+ 55d} - \frac{25d^2}{2} + 6d^2 &= 117 \\ \frac{25d^2}{4} - \frac{50d^2}{4} + \frac{24d^2}{4} &= -4 \\ 25d^2 - 50d^2 + 24d^2 &= -16 \\ -d^2 &= -16 \\ d &= 4 \end{aligned}$$

$$\begin{aligned} a_1 &= 11 - \frac{5 \cdot 4}{2} = 1 \\ a_n &= 1 + 4(n-1) \\ &\boxed{a_n = 4n - 3} \end{aligned}$$

(b)

$$\begin{aligned} b_n &= \frac{\frac{1}{2}(1+4n-3)n}{n+p} \\ &= \frac{n(2n-1)}{n+p} \qquad (1) \\ &= \frac{2n(n-\frac{1}{2})}{n+p} \qquad (2) \end{aligned}$$

For b_n to be an arithmetic sequence, it can be written as a in the form $b_n = b_1 + d(n-1)$. This implies that in the above formula for b_n, $\frac{n}{2}\left(n - \frac{1}{2}\right)$ is divisble by $n + p$. This leaves two possibilities:

Case 1. Equation (1): $n = n + p$

$p = 0$

However p is a nonzero constant.

Case 2. Equation (2): $n - \dfrac{1}{2} = n + p$

$$\boxed{p = -\frac{1}{2}}$$