Problem Set #36

Jayden Li

December 21, 2023

Problem 1

We did this in class on Wednesday, after we finished PS35.

Problem 2

(a)

$$\frac{\sin A}{1+\cos A}+\frac{1+\cos A}{\sin A}\stackrel{?}{=}\frac{2}{\sin A}$$

Proof. LHS:
$$\begin{aligned}\frac{\sin A}{1+\cos A}+\frac{1+\cos A}{\sin A}&=\frac{\sin^2 A+(1+\cos A)^2}{(1+\cos A)\sin A}\\&=\frac{\sin^2 A+1+2\cos A+\cos^2 A}{(1+\cos A)\sin A}\\&=\frac{1+1+2\cos A}{(1+\cos A)\sin A}\\&=\frac{2\left(1+\cos A\right)}{(1+\cos A)\sin A}\\&=\frac{2}{\sin A}\end{aligned}$$

□

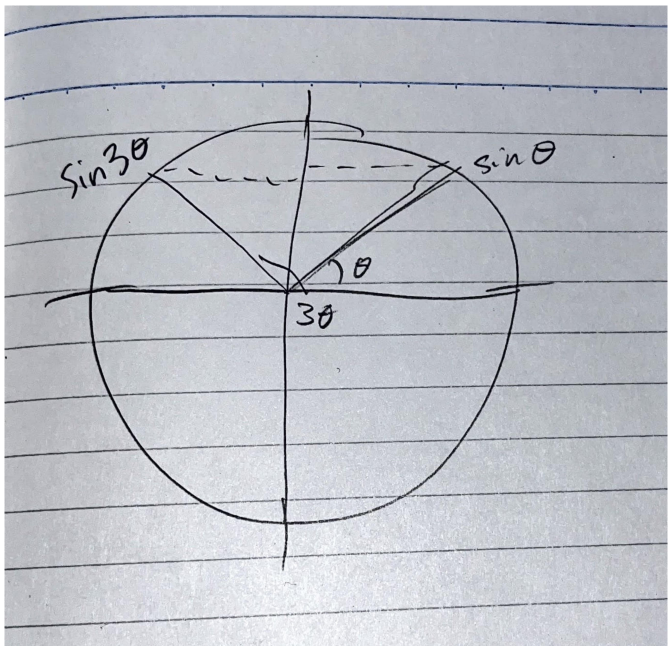
(b)

$$\begin{aligned}\frac{\sin x}{1+\cos x}+\frac{1+\cos x}{\sin x}&=1+3\sin x\\1+3\sin x&=\frac{2}{\sin x}\\\sin x+3\sin^2 x&=2\\3\left(\sin^2 x+\frac{1}{3}\sin x-\frac{2}{3}\right)&=0\\3\left(\left(\sin x+\frac{1}{6}\right)^2-\frac{1}{36}-\frac{2}{3}\right)&=0\\\left(\sin x+\frac{1}{6}\right)^2-\frac{25}{36}&=0\\\sin x+\frac{1}{6}&=\pm\frac{5}{6}\end{aligned}$$

$$\text{C1: } \sin x + \frac{1}{6} = \frac{5}{6}$$
$$\sin x = \frac{2}{3}$$
$$\boxed{x = (-1)^n \arcsin \frac{2}{3} + \pi n}$$

$$\text{C2: } \sin x + \frac{1}{6} = -\frac{5}{6}$$
$$\sin x = -1$$
$$x = (-1)^n \arcsin(-1) + \pi n$$
$$\boxed{x = -(-1)^n \cdot \frac{\pi}{2} + \pi n}$$

Problem 3



$$\begin{aligned}
\sin 3\theta &= \sin(2\theta + \theta) \\
&= \sin 2\theta \cos\theta + \cos 2\theta \sin\theta \\
&= (2\sin\theta\cos\theta)\cos\theta + \left(1 - 2\sin^2\theta\right)\sin\theta \\
&= 2\sin\theta\cos^2\theta + \sin\theta - 2\sin^3\theta \\
&= \sin(\theta)\left(2\cos^2\theta + 1 - 2\sin^2\theta\right) \\
&= \sin(\theta)\left(2 - 2\sin^2\theta + 1 - 2\sin^2\theta\right) \\
&= \sin(\theta)\left(3 - 4\sin^2\theta\right)
\end{aligned}$$

$$\sin\theta > \sin 3\theta$$
$$\sin\theta > \sin(\theta)\left(3 - 4\sin^2\theta\right)$$

C1: $\sin\theta > 0$

$$\begin{aligned} 1 &> 3 - 4\sin^2\theta \\ 4\sin^2\theta &> 2 \\ \sin^2\theta &> \frac{1}{2} \\ \sin\theta &> \frac{\sqrt{2}}{2} \end{aligned}$$

$$\theta \in \left(\frac{\pi}{4} + 2\pi n, \frac{3\pi}{4} + 2\pi n\right)$$

C2: $\sin\theta < 0$

$$\begin{aligned} 1 &< 3 - 4\sin^2\theta \\ 4\sin^2\theta &< 2 \\ \sin^2\theta &< \frac{1}{2} \end{aligned}$$

Only negative case since $\sin\theta$ is restricted to be negative in this case. Flip inequality sign as a result.

$$\sin\theta > -\frac{\sqrt{2}}{2}$$

$$\theta \in \left(-\frac{\pi}{4} + 2\pi n, \frac{5\pi}{4} + 2\pi n\right) \cap (2\pi n - \pi, 2\pi n)$$

$$\theta \in \left(-\frac{\pi}{4} + 2\pi n, 2\pi n\right) \cup \left(\pi + 2\pi n, \frac{5\pi}{4} + 2\pi n\right)$$

$$\boxed{\begin{array}{c} \theta \in \left(\frac{\pi}{4} + 2\pi n, \frac{3\pi}{4} + 2\pi n\right) \cup \\ \left(-\frac{\pi}{4} + 2\pi n, 2\pi n\right) \cup \\ \left(\pi + 2\pi n, \frac{5\pi}{4} + 2\pi n\right) \end{array}}$$

Problem 4

(a)

$$\begin{aligned} y &= \sin x + \cos x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \cos x - \sin x \\ 0 &= \cos x - \sin x \\ \sin x &= \cos x \end{aligned}$$

$$\boxed{x = \frac{\pi}{4},\ x = \frac{5\pi}{4}}$$

I personally think the Calculus method is more elegant. There are less steps and the intuition of slope being zero is simplier. There are also less opportunities for error.

(b)

$$\begin{aligned} y &= \sin x + \cos x \\ y &= \frac{2}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}\cos x + \frac{\sqrt{2}}{2}\sin x\right) \\ y &= \sqrt{2}\left(\sin\frac{\pi}{4}\cos x + \cos\frac{\pi}{4}\sin x\right) \\ y &= \sqrt{2}\sin\left(\frac{\pi}{4} + x\right) \end{aligned}$$

The turning points of $\sin\theta$ are when $\sin\theta = 1$. $\theta = \pi/2$, $\theta = 3\pi/2$

$$\begin{aligned} \frac{\pi}{4} + x &= \frac{\pi}{2} \\ x &= \frac{2\pi}{4} - \frac{\pi}{4} \\ x &= \frac{\pi}{4} \end{aligned}$$

$$\begin{aligned} \frac{\pi}{4} + x &= \frac{3\pi}{2} \\ x &= \frac{6\pi}{4} - \frac{\pi}{4} \\ x &= \frac{5\pi}{4} \end{aligned}$$

$$\boxed{x = \frac{\pi}{4},\ x = \frac{5\pi}{4}}$$

Problem 5

(b)

$$2\sin x\cos 5x - \cos 5x = 0,\ x \in [0, \pi]$$
$$\cos(5x)(2\sin x - 1) = 0$$

$$\begin{aligned} \cos(5x) &= 0 \\ 5x &= \pm\arccos(0) + 2\pi n \\ 5x &= \pm\frac{\pi}{2} + 2\pi n \\ x &= \pm\frac{\pi}{10} + \frac{2\pi n}{5} \\ x &= \frac{\pm\pi + 4\pi n}{10} \end{aligned}$$

$$x = \frac{\pi}{10}, \frac{\pi}{2}, \frac{3\pi}{10}, \frac{9\pi}{10}, \frac{7\pi}{10}$$

$$\begin{aligned} 2\sin x - 1 &= 0 \\ \sin x &= \frac{1}{2} \\ x &= (-1)^n \cdot \arcsin\frac{1}{2} + \pi n \\ x &= (-1)^n \cdot \frac{\pi}{6} + \pi n \end{aligned}$$

$$n = 0,\ n = 1$$
$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\boxed{x = \frac{\pi}{10}, \frac{\pi}{2}, \frac{3\pi}{10}, \frac{9\pi}{10}, \frac{7\pi}{10}, \frac{\pi}{6}, \frac{5\pi}{6}}$$

Problem 6

(b)

$$\sin 4x - \cos 2x = 0$$
$$2\sin 2x\cos 2x - \cos 2x = 0$$
$$\cos(2x)(2\sin 2x - 1) = 0$$

$$\begin{aligned} \cos 2x &= 0 \\ 2x &= \pm\arccos(0) + 2\pi n \\ x &= \frac{1}{2}\left(\pm\frac{\pi}{2} + 2\pi n\right) \\ x &= \pm\frac{\pi}{4} + \pi n \end{aligned}$$

$$\begin{aligned} 2\sin 2x - 1 &= 0 \\ \sin 2x &= \frac{1}{2} \\ 2x &= (-1)^n \cdot \arcsin\frac{1}{2} + \pi n \\ x &= (-1)^n \cdot \frac{\pi}{12} + \frac{\pi n}{2} \end{aligned}$$

$$\boxed{x = \pm\frac{\pi}{4} + \pi n,\ x = (-1)^n \cdot \frac{\pi}{12} + \frac{\pi n}{2}}$$

(c)

$$\begin{aligned} \cos^2 3x &= \frac{3}{4} \\ \cos 3x &= \pm\frac{\sqrt{3}}{2} \\ 3x &= \pm\arccos\left(\pm\frac{\sqrt{3}}{2}\right) + 2\pi n \end{aligned}$$

$$3x = \pm \arccos\left(\frac{\sqrt{3}}{2}\right) + 2\pi n$$
$$3x = \pm\frac{\pi}{6} + 2\pi n$$
$$x = \pm\frac{\pi}{18} + \frac{2\pi n}{3}$$

$$3x = \pm \arccos\left(-\frac{\sqrt{3}}{2}\right) + 2\pi n$$
$$3x = \pm\frac{5\pi}{6} + 2\pi n$$
$$x = \pm\frac{5\pi}{18} + \frac{2\pi n}{3}$$

$$\boxed{x = \pm\frac{\pi}{18} + \frac{2\pi n}{3},\ x = \pm\frac{5\pi}{18} + \frac{2\pi n}{3}}$$

(d)

$$\begin{aligned}
\sin 17x &= \sin 7x \\
17x &= (-1)^n \cdot \arcsin(\sin 7x) + \pi n \\
17x &= (-1)^n \cdot 7x + \pi n \\
17x - (-1)^n \cdot 7x &= \pi n \\
x\left(17 - (-1)^n \cdot 7\right) &= \pi n
\end{aligned}$$
$$\boxed{x = \frac{\pi n}{17 - 7(-1)^n}}$$

(e)

$$\begin{aligned}
\cos^2 x + \sin^2 3x &= 1 \\
\cos^2 x + \sin^2 3x - \left(\cos^2 x + \sin^2 x\right) &= 1 + 1 \\
\cos^2 x + \sin^2 3x - \left(\cos^2 x + \sin^2 x\right) &= 1 - \left(\cos^2 x + \sin^2 x\right) \\
\sin^2 3x - \sin^2 x &= 0 \\
\left(\sin(x)\left(3 - 4\sin^2 x\right)\right)^2 - \sin^2 x &= 0 \\
\sin^2(x)\left(3 - 4\sin^2 x\right)^2 - \sin^2 x &= 0 \\
\sin^2(x)\left(\left(3 - 4\sin^2 x\right)^2 - 1\right) &= 0
\end{aligned}$$

$$\begin{aligned}
\sin^2(x) = 0 \sin x &= 0 \\
x &= \pi n
\end{aligned}$$

$$\begin{aligned}
\left(3 - 4\sin^2 x\right)^2 - 1 &= 0 \\
3 - 4\sin^2 x &= \pm 1 \\
4\sin^2 x &= 3 \pm 1 \\
\sin x &= \pm\frac{\sqrt{3 \pm 1}}{2}
\end{aligned}$$

$$\sin x = 1,\ \sin x = -1$$
$$x = \frac{\pi}{2} + 2\pi n,\ \frac{3\pi}{2} + 2\pi n$$
$$x = \frac{\pi}{2} + \pi n$$

$$\sin x = \frac{\sqrt{2}}{2}$$
$$x = (-1)^n \cdot \frac{\pi}{4} + \pi n$$

$$\sin x = -\frac{\sqrt{2}}{2}$$
$$x = (-1)^n \cdot \left(-\frac{\pi}{4}\right) + \pi n$$

$$\boxed{x = \pi n,\ x = \frac{\pi}{2} + \pi n,\ x = (-1)^n \cdot \frac{\pi}{4} + \pi n,\ x = -(-1)^n \cdot \frac{\pi}{4} + \pi n}$$

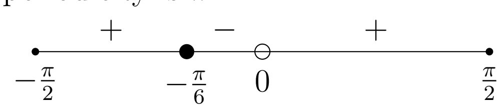
Problem 7

(b)

$$\begin{aligned}
\cot x &\le -\sqrt{3} \\
\frac{\cos x}{\sin x} + \sqrt{3} &\le 0 \\
\frac{\cos x + \sqrt{3}\sin x}{\sin x} &\le 0 \\
\frac{2\left(\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x\right)}{\sin x} &\le 0 \\
\frac{2\left(\sin\frac{\pi}{6}\cos x + \cos\frac{\pi}{6}\sin x\right)}{\sin x} &\le 0 \\
\frac{\sin\left(\frac{\pi}{6} + x\right)}{\sin x} &\le 0
\end{aligned}$$

$$\begin{aligned}
\sin\left(\frac{\pi}{6} + x\right) &= 0 \\
\frac{\pi}{6} + x &= \arcsin 0 \\
x &= -\frac{\pi}{6}
\end{aligned}
\qquad
\begin{aligned}
\sin x &= 0 \\
x &= \arcsin 0 \\
x &= 0
\end{aligned}$$

The range of $\arcsin x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, so the periodicity is π.



$$\boxed{x \in \left[-\frac{\pi}{6} + \pi n, \pi n\right)}$$

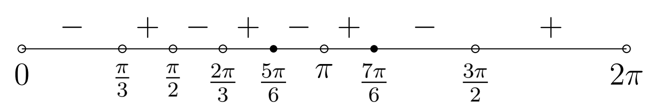
(d)

$$\frac{2\cos x + \sqrt{3}}{\sin(2x)\left(2\sin x - \sqrt{3}\right)} \le 0$$

$$\begin{aligned}
2\cos x + \sqrt{3} &= 0 \\
\cos x &= -\frac{\sqrt{3}}{2} \\
x &= \frac{5\pi}{6}, \frac{7\pi}{6}
\end{aligned}$$

$$\sin(2x)\left(2\sin x - \sqrt{3}\right) = 0$$

$$\begin{aligned}
\sin(2x) &= 0 \\
2x &= 0,\ \pi,\ 2\pi,\ 3\pi \\
x &= 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}
\end{aligned}
\qquad
\begin{aligned}
2\sin x - \sqrt{3} &= 0 \\
\sin x &= \frac{\sqrt{3}}{2} \\
x &= \frac{\pi}{3}, \frac{2\pi}{3}
\end{aligned}$$



$$\boxed{\begin{aligned}
x \in \left(2\pi n, \frac{\pi}{3} + 2\pi n\right) \cup \left(\frac{\pi}{2} + 2\pi n, \frac{2\pi}{3} + 2\pi n\right) \cup \\
\left[\frac{5\pi}{6} + 2\pi n, \pi + 2\pi n\right) \cup \left[\frac{7\pi}{6} + 2\pi n, \frac{3\pi}{2} + 2\pi n\right)
\end{aligned}}$$

(e)

$$\frac{2\sin x - 1}{2\cos x - \sqrt{3}} \ge 0$$

$$\begin{aligned}
2\sin x - 1 &= 0 \\
\sin x &= \frac{1}{2} \\
x &= \frac{\pi}{6}, \frac{5\pi}{6}
\end{aligned}
\qquad
\begin{aligned}
2\cos x - \sqrt{3} &= 0 \\
\cos x &= \frac{\sqrt{3}}{2} \\
x &= \frac{\pi}{6}, \frac{11\pi}{6}
\end{aligned}$$

$$\boxed{x \in \left[\frac{5\pi}{6} + 2\pi n, \frac{11\pi}{6} + 2\pi n\right)}$$