Problem Set #32

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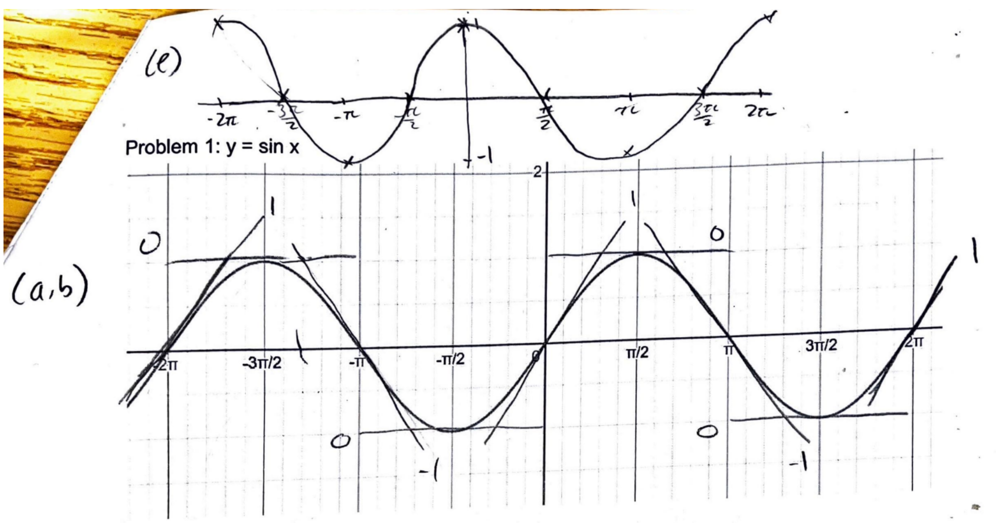
Problem 1

(a, b, d) (e) should be corrected to (d)



(c)

$$
\begin{aligned}
f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\
&\approx \frac{f(0+0.0001) - f(0)}{0.0001} \\
&= \frac{\sin(0.0001) - \sin 0}{0.0001} \\
&= 0.9999999983 \approx 1
\end{aligned}
$$

Because $\sin x$ has a period of 2π, its derivative will also have a period of 2π. This suggests that $f'(2\pi) = f'(-2\pi) = 1$.

(e) $y = \cos x$.

Problem 3

(a)

$$
\begin{aligned}
h(t) &= 3\cos t - 4\sin t \\
h'(t) &= \frac{\mathrm{d}}{\mathrm{d}x} 3\cos t - \frac{\mathrm{d}}{\mathrm{d}x} 4\sin t \\
&\boxed{h'(t) = -3\sin t - 4\cos t}
\end{aligned}
$$

(b)

$$
\begin{aligned}
y = f(t) &= 2x + \frac{\sin x}{2} \\
f'(t) &= \frac{\mathrm{d}}{\mathrm{d}t} 2t + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\sin t\right) \\
&= 2 + \frac{\cos t}{2}
\end{aligned}
$$

$$
\begin{aligned}
f'\left(\frac{\pi}{6}\right) &= 2 + \frac{\cos\frac{\pi}{6}}{2} \\
&= 2 + \frac{\sqrt{3}}{2}/2 \\
&= \boxed{2 + \frac{\sqrt{3}}{4}}
\end{aligned}
$$

(c)

$$\begin{aligned}
y = g(x) &= x^2 + 2\cos x \\
g'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}x^2 + \frac{\mathrm{d}}{\mathrm{d}x}2\cos x \\
&= 2x + (-2\sin x) \\
&= 2x - 2\sin x
\end{aligned}$$

$$\begin{aligned}
g'\left(\frac{\pi}{2}\right) &= 2 \cdot \frac{\pi}{2} - 2\sin\frac{\pi}{2} \\
&= \pi - 2
\end{aligned}$$

$$\begin{aligned}
y &= mx + c \\
g\left(\frac{\pi}{2}\right) &= (\pi - 2) \cdot \frac{\pi}{2} + c \\
\left(\frac{\pi}{2}\right)^2 + 2\cos\frac{\pi}{2} &= \frac{\pi^2 - 2\pi}{2} + c \\
\frac{\pi^2}{4} + 2 \cdot 0 &= \frac{\pi^2 - 2\pi}{2} + c \\
\pi^2 &= 2\left(\pi^2 - 2\pi\right) + 4c \\
4c &= \pi^2 - 2\pi^2 + 4\pi \\
c &= \frac{-\pi^2 + 4\pi}{4} \\
c &= -\frac{\pi^2}{4} + \pi \\
&\boxed{y = (\pi - 2)\,x - \frac{\pi^2}{4} + \pi}
\end{aligned}$$

(d)

$$\begin{aligned}
p(z) &= z^4 + 4^z + 4\cos z - \sin\frac{\pi}{2} \\
p'(z) &= \frac{\mathrm{d}}{\mathrm{d}z}z^4 + \frac{\mathrm{d}}{\mathrm{d}z}4^z + \frac{\mathrm{d}}{\mathrm{d}z}4\cos z - \frac{\mathrm{d}}{\mathrm{d}z}\sin\frac{\pi}{2} \\
&= 4z^3 + 4^z \cdot \ln 4 + (-\sin z) - 0
\end{aligned}$$

$$\boxed{p'(z) = 4z^3 + 4^z \cdot \ln 4 - 4\sin z}$$

(e)

$$\begin{aligned}
P(t) &= 24 + 8\sin t \\
P'(t) &= 8 \cdot \frac{\mathrm{d}}{\mathrm{d}t}\sin t \\
P'(t) &= 8\cos t
\end{aligned}$$

2 decades have passed between January 1, 2010 and January 1, 2030.

$$\begin{aligned} P'(2) &= 8\cos 2 \\ &\approx -3.32917 \ \frac{\text{hundred animals}}{\text{decade}} \end{aligned}$$

as $P(t)$ is in hundreds of animals, t is in decades. On January 1, 2030, the population of animals is decreasing at a rate of 3.32917 hundred animals per decade.

Problem 5

(c)

$$\begin{aligned} f(x) &= 4x^{25}\cos x \\ f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(4x^{25}\right)\cdot\cos x + 4x^{25}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) \\ f'(x) &= (4\cdot 25)\,x^{24}\cdot\cos x + 4x^{25}\cdot(-\sin x) \\ &\boxed{f'(x) = 100x^{24}\cos x - 4x^{25}\sin x} \end{aligned}$$

(d)

$$\begin{aligned} g(x) &= \cos(2021x)\cos(2022x) + \sin(2021x)\sin(2022x) \\ &= \cos(2021x - 2022x) \\ &= \cos(-x) \\ &= \cos x \\ g'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\cos x \\ &\boxed{g'(x) = -\sin x} \end{aligned}$$

(e)

$$\begin{aligned} h(x) &= \sin(2022x)\cos(2021x) - \sin(2021x)\cos(2022x) \\ &= \sin(2022x - 2021x) \\ &= \sin x \\ h'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\sin x \\ &\boxed{h'(x) = \cos x} \end{aligned}$$

Problem 6

$$\begin{aligned}
y &= \frac{1}{\sin x + \cos x} \\
\text{Let } f(x) &= \frac{1}{\sin x + \cos x} \\
&= (\sin x + \cos x)^{-1}
\end{aligned}$$

$$\begin{aligned}
f'(x) &= \frac{-1}{(\sin x + \cos x)^2} \cdot \frac{\mathrm{d}}{\mathrm{d}x} (\sin x + \cos x) \\
&= \frac{-(\cos x - \sin x)}{(\sin x + \cos x)^2} \\
&= \frac{\sin x - \cos x}{(\sin x + \cos x)^2}
\end{aligned}$$

$$\begin{aligned}
f'(0) &= \frac{\sin 0 - \cos 0}{(\sin 0 + \cos 0)^2} \\
&= \frac{0 - 1}{(0 + 1)^2} \\
&= -1
\end{aligned}$$

$$\begin{aligned}
y &= mx + c \\
f(0) &= f'(0)x + c \\
\frac{1}{\sin 0 + \cos 0} &= -1 \cdot 0 + c \\
\frac{1}{0 + 1} &= c \\
c &= 1 \\
&\boxed{y = -x + 1}
\end{aligned}$$

Problem 9

$$
\begin{aligned}
y &= \sec x - 2\cos x \\
\text{Let } f(x) &= \sec x - 2\cos x \\
&= (\cos x)^{-1} - 2\cos x \\
f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(\cos x)^{-1} - \frac{\mathrm{d}}{\mathrm{d}x}(2\cos x) \\
&= \frac{-1}{(\cos x)^2} \cdot (-\sin x) - 2(-\sin x) \\
&= \frac{\sin x}{\cos^2 x} + 2\sin x
\end{aligned}
$$

$$
\begin{aligned}
f'\left(\frac{\pi}{3}\right) &= \frac{\sin\frac{\pi}{3}}{\cos^2\frac{\pi}{3}} + 2\sin\frac{\pi}{3} \\
&= \frac{\sqrt{3}}{2} \Big/ \left(\frac{1}{2}\right)^2 + 2 \cdot \frac{\sqrt{3}}{2} \\
&= \frac{\sqrt{3}}{2} \cdot \frac{4}{1} + \sqrt{3} \\
&= 2\sqrt{3} + \sqrt{3} \\
&= 3\sqrt{3}
\end{aligned}
$$

$$
\begin{aligned}
y &= mx + c \\
f\left(\frac{\pi}{3}\right) &= 3\sqrt{3} \cdot \frac{\pi}{3} + c \\
\left(\cos\frac{\pi}{3}\right)^{-1} - 2\cos\frac{\pi}{3} &= \sqrt{3}\pi + c \\
\left(\frac{1}{2}\right)^{-1} - 2 \cdot \frac{1}{2} &= \sqrt{3}\pi + c \\
2 - 1 &= \sqrt{3}\pi + c \\
c &= 1 - \sqrt{3}\pi
\end{aligned}
$$

$$
\boxed{y = 3\sqrt{3}x + 1 - \sqrt{3}\pi}
$$

Problem 10

$$
\begin{aligned}
f\left(\frac{\pi}{3}\right) &= 4 \\
f'\left(\frac{\pi}{3}\right) &= -2 \\
g(x) &= f(x) \cdot \sin x \\
h(x) &= \frac{\cos x}{f(x)}
\end{aligned}
$$

(a)

$$
\begin{aligned}
g'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(f(x) \cdot \sin x\right) \\
&= f'(x) \cdot \sin x + f(x) \cdot \frac{\mathrm{d}}{\mathrm{d}x}(\sin x) \\
&= f'(x) \cdot \sin x + f(x) \cdot \cos x \\
g'\left(\frac{\pi}{3}\right) &= f'\left(\frac{\pi}{3}\right) \cdot \sin\frac{\pi}{3} + f\left(\frac{\pi}{3}\right) \cdot \cos\frac{\pi}{3} \\
&= -2 \cdot \frac{\sqrt{3}}{2} + 4 \cdot \frac{1}{2} \\
&= -\sqrt{3} + 2 \\
&= \boxed{2 - \sqrt{3}}
\end{aligned}
$$

(b)

$$
\begin{aligned}
h'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cos x}{f(x)}\right) \\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}x}(\cos x)\cdot f(x) - \cos(x)\cdot \frac{\mathrm{d}f}{\mathrm{d}x}}{\left(f(x)\right)^2} \\
&= \frac{-\sin x\cdot f(x) - \cos x\cdot f'(x)}{\left(f(x)\right)^2} \\
h'\left(\frac{\pi}{3}\right) &= \frac{-\sin\frac{\pi}{3}\cdot f\left(\frac{\pi}{3}\right) - \cos\frac{\pi}{3}\cdot f'\left(\frac{\pi}{3}\right)}{\left(f\left(\frac{\pi}{3}\right)\right)^2} \\
&= \frac{-\frac{\sqrt{3}}{2}\cdot 4 - \frac{1}{2}(-2)}{4^2} \\
&= \frac{-2\sqrt{3}+1}{16} \\
&= \boxed{\frac{1-2\sqrt{3}}{16}}
\end{aligned}
$$

Problem 11

(a)

$$
\begin{aligned}
y &= 5\cos(10x) \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 5\cdot\frac{\mathrm{d}}{\mathrm{d}x}\cos(10x) \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= 5\cdot(-\sin(10x))\cdot 10 \\
&\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = -50\sin(10x)}
\end{aligned}
$$

(b)

$$
\begin{aligned}
f(x) &= \tan(\cos x) \\
f'(x) &= \sec^2(\cos x)\cdot(-\sin x) \\
&\boxed{f'(x) = -sec^2(\cos x)\sin x}
\end{aligned}
$$

(c)

$$
\begin{aligned}
f(x) &= e^{x\cos x}\\
f'(x) &= e^{x\cos x}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(x\cos x)\\
f'(x) &= e^{x\cos x}\cdot(1\cdot\cos x + x\cdot(-\sin x))\\
&\boxed{f'(x) = e^{x\cos x}\cdot(cosx - x\sin x)}
\end{aligned}
$$

(d)

$$
\begin{aligned}
y &= \cot^2(\sin\theta)\\
&= \frac{(\cos(\sin\theta))^2}{(\sin(\sin\theta))^2}\\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(\cos(\sin\theta))^2}{(\sin(\sin\theta))^2}\right)\\
&= \frac{\frac{\mathrm{d}}{\mathrm{d}x}(\cos(\sin\theta))^2\cdot\sin^2(\sin\theta) - \frac{\mathrm{d}}{\mathrm{d}x}(\sin(\sin\theta))^2\cdot\cos^2(\sin\theta)}{(\sin(\sin\theta))^4}\\
&= \frac{2\cos(\sin\theta)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\cos(\sin\theta))\cdot\sin^2(\sin\theta) - 2\sin(\sin\theta)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\sin(\sin\theta))\cdot\cos^2(\sin\theta)}{(\sin(\sin\theta))^4}\\
&= \frac{2\cos(\sin\theta)(-\sin(\sin\theta)\cos(\theta))\cdot\sin^2(\sin\theta) - 2\sin(\sin\theta)(\cos(\sin\theta)\cos(\theta))\cdot\cos^2(\sin\theta)}{(\sin(\sin\theta))^4}\\
&= \frac{-2\cos(\sin\theta)\sin(\sin\theta)\cos(\theta)\sin^2(\sin\theta)}{(\sin(\sin\theta))^4} - \frac{2\sin(\sin\theta)\cos(\sin\theta)\cos(\theta)\cos^2(\sin\theta)}{(\sin(\sin\theta))^4}\\
&= \frac{-2\cos(\sin\theta)\cos(\theta)}{\sin(\sin\theta)} - \frac{2(\cos(\sin\theta))^3\cos(\theta)}{(\sin(\sin\theta))^3}\\
&\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = -2\cot(\sin\theta)\cos\theta - 2\cot^3(\sin\theta)\cos\theta}
\end{aligned}
$$

(e)

$$
\begin{aligned}
\frac{\mathrm{d}\tan}{\mathrm{d}x}(x) &= \sec^2 x\\
\frac{\mathrm{d}\sec}{\mathrm{d}x}(x) &= \frac{\mathrm{d}}{\mathrm{d}x}(\cos x)^{-1}\\
&= \frac{-1}{(-\cos x)^2}\cdot(-\sin x)\\
&= \tan x\sec x
\end{aligned}
$$

$$
\begin{aligned}
y &= \sec(2x) + \tan(2x) \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \tan(2x)\sec(2x)\cdot 2 + \sec^2(2x)\cdot 2
\end{aligned}
$$

$$
\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = 2\sec(2x)\left(\tan 2x + \sec 2x\right)}
$$

(f)

$$
\begin{aligned}
y &= \cos\left(\frac{1-e^{2x}}{1+e^{2x}}\right) \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1-e^{2x}}{1+e^{2x}}\right) \\
&= -\sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right)\cdot\frac{\frac{\mathrm{d}}{\mathrm{d}x}\left(1-e^{2x}\right)\cdot\left(1+e^{2x}\right)-\left(1-e^{2x}\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(1+e^{2x}\right)}{\left(1+e^{2x}\right)^2} \\
&= -\sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right)\cdot\frac{-2e^{2x}\cdot\left(1+e^{2x}\right)-\left(1-e^{2x}\right)\cdot 2e^{2x}}{\left(1+e^{2x}\right)^2} \\
&= -\sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right)\cdot\frac{-2e^{2x}-2e^{2x}\cdot e^{2x}-2e^{2x}+e^{2x}\cdot 2e^{2x}}{\left(1+e^{2x}\right)^2} \\
&= -\sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right)\cdot\frac{-2e^{2x}-2e^{4x}-2e^{2x}+2e^{4x}}{\left(1+e^{2x}\right)^2} \\
&= -\sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right)\cdot\frac{-4e^{2x}}{\left(1+e^{2x}\right)^2}
\end{aligned}
$$

$$
\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right)\cdot\frac{4e^{2x}}{\left(1+e^{2x}\right)^2}}
$$

(g)

$$
\begin{aligned}
y &= \cos\left(\sqrt{\sin(\tan x)}\right) \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\sin\left(\sqrt{\sin(\tan x)}\right)\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{\sin(\tan x)}\right) \\
&= -\sin\left(\sqrt{\sin(\tan x)}\right)\cdot\frac{1}{2\sqrt{\sin(\tan x)}}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin(\tan x)\right) \\
&= -\sin\left(\sqrt{\sin(\tan x)}\right)\cdot\frac{1}{2\sqrt{\sin(\tan x)}}\cdot\cos(\tan x)\cdot\frac{\mathrm{d}}{\mathrm{d}x}(\tan x)
\end{aligned}
$$

$$
\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\sin\sqrt{\sin(\tan x)}}{2\sqrt{\sin(\tan x)}}\cdot\cos(\tan x)\cdot\sec^2 x}
$$

(h)

$$
\begin{aligned}
y &= \sin\left(\sin\left(\sin x\right)\right) \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \cos\left(\sin\left(\sin x\right)\right) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\left(\sin x\right)\right) \\
&= \cos\left(\sin\left(\sin x\right)\right) \cdot \cos\left(\sin x\right) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\sin x
\end{aligned}
$$

$$
\boxed{\frac{\mathrm{d}y}{\mathrm{d}x} = \cos\left(\sin\left(\sin x\right)\right) \cdot \cos\left(\sin x\right) \cdot \cos x}
$$