Problem Set #50

Jayden Li

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Problem 9

(a) I think Cartesian is easier.

Cartesian: $\boxed{y = \frac{\pi}{6}x}$

Polar: $r\sin\theta = \frac{\pi}{6}(r\cos\theta)$

$$\tan\theta = \frac{\pi}{6}$$

$$\boxed{\theta = \arctan\frac{\pi}{6} + \pi n}$$

(b) I think Cartesian is easier.

Cartesian: $\boxed{x = 3}$

Polar: $r\cos\theta = 3$

$$\boxed{r = 3\sec\theta}$$

Problem 10

$$\begin{aligned} r_1 = 3 + \cos\theta = 2 &= r^2 \\ \cos\theta &= -1 \\ \theta &\in \left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\} \end{aligned}$$

Let A be the point with $\theta = \frac{2\pi}{3}$ and B be the point with $\theta = \frac{4\pi}{3}$. Any line from A to the origin must satisfy $\theta = \frac{2\pi}{3}$ or $\theta = \frac{2\pi}{3} + \pi$ (because a line rotated 180° is the same line). When $\theta = \frac{2\pi}{3}$, $r_1 = 3 + 2\cos\frac{2\pi}{3} = 3 + (-1) = 2$, which is A. When $\theta = \frac{2\pi}{3} + \pi$, $r_1 = 3 + 2\cos\left(\frac{5\pi}{3}\right) = 3 + 1 = 4$.

Therefore C has rectangular coordinates $(r_1\cos\theta, r_2\sin\theta) = \left(4\cos\frac{5\pi}{3}, 4\sin\frac{5\pi}{3}\right) = (2, -2\sqrt{3})$.

B has polar coordinates $\left(2, \frac{4\pi}{3}\right)$ and rectangular coordinates $\left(2\cos\frac{4\pi}{3}, 2\sin\frac{4\pi}{3}\right) = (-1, -\sqrt{3})$.

The equation of BC is:

$$\begin{aligned} y - y_0 &= \frac{y_1 - y_0}{x_1 - x_0}(x - x_0) \\ y + 2\sqrt{3} &= \frac{-2\sqrt{3} + \sqrt{3}}{2 + 1}(x - 2) \\ y &= -\frac{\sqrt{3}}{3}x + \frac{2\sqrt{3}}{3} - \frac{6\sqrt{3}}{3} \\ y &= -\frac{\sqrt{3}}{3}x - \frac{4\sqrt{3}}{3} \end{aligned}$$

Substituting into r_2:

$$\begin{aligned}
r_2 &= 2 \\
x^2 + y^2 &= r^2 \\
x^2 + \left(-\frac{\sqrt{3}}{3}x - \frac{4\sqrt{3}}{3}\right)^2 &= 4 \\
x^2 + \frac{3}{9}x^2 + 2\left(\frac{\sqrt{3}}{3}x\right)\left(\frac{4\sqrt{3}}{3}\right) + \frac{48}{9} - 4 &= 0 \\
x^2 + \frac{1}{3}x^2 + \frac{8}{3}x + \frac{4}{3} &= 0 \\
4x^2 + 8x + 4 &= 0 \\
x^2 + 2x + 1 &= 0 \\
(x+1)^2 &= 0 \\
x &= -1
\end{aligned}$$

There is only one intersection between the line BC and the circle given by r_2, and the intersection has $x = -1$, which is the x-coordinate of B. Therefore BC is a tangent line of r_2 at point B.

$\square$