Problem Set #68

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Problem 1

(a)

$$y = \lim_{x\to0^+} (1+\sin 4x)^{\cot x}$$
$$\ln y = \lim_{x\to0^+} \cot(x)\ln(1+\sin 4x) = \lim_{x\to0^+} \frac{\cos(x)\ln(1+\sin 4x)}{\sin x}$$

$$\boxed{\begin{array}{r} \lim_{x\to0^+} \cos(x)\ln(1+\sin 4x) = 0 \\ \lim_{x\to0^+} \sin x = 0 \end{array}}$$

$$\overset{\left[\frac{0}{0}\right]}{\underset{\text{L'H}}{=}} \lim_{x\to0^+} \frac{-\sin(x)\ln(1+\sin 4x) + \frac{4\cos(x)\cos(4x)}{1+\sin 4x}}{\cos x} = \frac{0+\frac{4}{1}}{1} = 4 \implies y = \boxed{e^4}$$

(b)

$$y = \lim_{x\to0^+} x^x \implies \ln y = \lim_{x\to0^+} x\ln x = \lim_{x\to0^+} \frac{\ln x}{\frac{1}{x}}$$

$$\boxed{\begin{array}{c} \lim_{x\to0^+} \ln x = -\infty \\ \lim_{x\to0^+} \frac{1}{x} = \infty \end{array}}$$

$$\overset{\left[\frac{\infty}{\infty}\right]}{\underset{\text{L'H}}{=}} \lim_{x\to0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = -\lim_{x\to0^+} x = 0 \implies y = e^0 = \boxed{1}$$

Problem 2

(a)

$$y = \lim_{x\to0^+} x^{x^2}$$
$$\ln y = \lim_{x\to0^+} x^2\ln x = \lim_{x\to0^+} \frac{\ln x}{\frac{1}{x^2}}$$

$$\boxed{\begin{array}{c} \lim_{x\to0^+} \ln x = -\infty \\ \lim_{x\to0^+} \frac{1}{x^2} = \infty \end{array}}$$

$$\overset{\left[\frac{\infty}{\infty}\right]}{\underset{\text{L'H}}{=}} \lim_{x\to0^+} \frac{\frac{1}{x}}{-\frac{1}{x^3}} = -\lim_{x\to0^+} x^2 = 0$$
$$y = e^0 = \boxed{1}$$

(b)

$$y = \lim_{x\to0} (1-2x)^{\frac{1}{x}}$$
$$\ln y = \lim_{x\to0} \left[\frac{1}{x}\ln(1-2x)\right] = \lim_{x\to0} \frac{\ln(1-2x)}{x}$$

$$\boxed{\begin{array}{r} \lim_{x\to0} \ln(1-2x) = 0 \\ \lim_{x\to0} x = 0 \end{array}}$$

$$\overset{\left[\frac{0}{0}\right]}{\underset{\text{L'H}}{=}} \lim_{x\to0} \frac{\frac{1}{1-2x}\cdot(-2)}{1} = \frac{1}{1-0}\cdot(-2) = -2$$
$$y = e^{-2} = \boxed{-\frac{1}{e^2}}$$

(c)

$$y = \lim_{x\to\infty} \left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x$$
$$\ln y = \lim_{x\to\infty} x\ln\left(1+\frac{3}{x}+\frac{5}{x^2}\right)$$

Let $k = 1/x$. Then $x = 1/k$, $3/x = 3k$ and $5/x^2 = 5k^2$. As $x\to\infty$, $k\to0$.

$$= \lim_{k\to0} \frac{\ln\left(1+3k+5k^2\right)}{k}$$

$$\boxed{\begin{array}{r} \lim_{k\to0} \ln\left(1+3k+5k^2\right) = 0 \\ \lim_{k\to0} k = 0 \end{array}}$$

$$\overset{\left[\frac{0}{0}\right]}{\underset{\text{L'H}}{=}} \lim_{k\to0} \frac{\frac{3+10k}{1+3k+5k^2}}{1} = \frac{3+0}{1+0+0} = 3$$
$$y = \boxed{e^3}$$

(d)

$$y = \lim_{x\to\infty} x^{\frac{1}{x}}$$
$$\ln y = \lim_{x\to\infty} \frac{1}{x}\ln x = \lim_{x\to\infty} \frac{\ln x}{x} = 0$$

(This miraculous fact is proven in Problem 9.)

$$y = e^0 = \boxed{1}$$

(e)

$$y = \lim_{x\to0^+} (4x+1)^{\cot x}$$
$$\ln y = \lim_{x\to0^+} \cot(x)\ln(4x+1) = \lim_{x\to0^+} \frac{\cos(x)\ln(4x+1)}{\sin x}$$

$$\boxed{\begin{array}{r} \lim_{x\to0^+} \cos(x)\ln(4x+1) = 0 \\ \lim_{x\to0^+} \sin x = 0 \end{array}}$$

$$\overset{\left[\frac{0}{0}\right]}{\underset{\text{L'H}}{=}} \lim_{x\to0^+} \frac{-\cos(x)\ln(4x+1) + \frac{4\cos x}{4x+1}}{\cos x} = \frac{-1\cdot0+\frac{4}{1}}{1} = 4 \implies y = \boxed{e^4}$$

(f)

$$y = \lim_{x\to0^+} (\cos x)^{\frac{1}{x^2}}$$
$$\ln y = \lim_{x\to0^+} \frac{1}{x^2}\ln(\cos x) = \lim_{x\to0^+} \frac{\ln(\cos x)}{x^2}$$

$$\boxed{\begin{array}{r} \lim_{x\to0^+} \ln(\cos x) = 0 \\ \lim_{x\to0^+} x^2 = 0 \\ \lim_{x\to0^+} \sin x = 0 \\ \lim_{x\to0^+} 2x\cos x = 0 \end{array}}$$

$$\overset{\left[\frac{0}{0}\right]}{\underset{\text{L'H}}{=}} \lim_{x\to0^+} \frac{\frac{-\sin x}{\cos x}}{2x} = -\lim_{x\to0^+} \frac{\sin x}{2x\cos x}$$
$$\overset{\left[\frac{0}{0}\right]}{\underset{\text{L'H}}{=}} -\lim_{x\to0^+} \frac{\cos x}{2\cos x - 2x\sin x} = -\frac{1}{2-0} = -\frac{1}{2}$$
$$y = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}} = \boxed{\frac{\sqrt{e}}{e}}$$

Problem 4

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$$

$$\boxed{\begin{array}{c} \lim_{x\to\infty} x = \infty \\ \lim_{x\to\infty} \sqrt{x^2+1} = \infty \end{array}}$$

$$\overset{\left[\frac{\infty}{\infty}\right]}{\underset{\text{L'H}}{=}} \lim_{x\to\infty} \frac{1}{\frac{2x}{2\sqrt{x^2+1}}} = \lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$$

Which is the reciprocal of what we started with. If we keep applying LHR it will just go back to the original function.

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x}{|x|\sqrt{1+\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{\operatorname{sgn}(x)}{\sqrt{1+\frac{1}{x^2}}} = \boxed{1}$$

Problem 6

First, we change the variable E to x because it is too confusing.

$$\lim_{x\to0^+} P(x) = \lim_{x\to0^+} \left[\frac{e^x+e^{-x}}{e^x-e^{-x}} - \frac{1}{x}\right] = \lim_{x\to0^+} \left[\coth x - \frac{1}{x}\right] = \lim_{x\to0^+} \left[\frac{\cosh x}{\sinh x} - \frac{1}{x}\right] = \lim_{x\to0^+} \frac{x\cosh x - \sinh x}{x\sinh x}$$

$$\boxed{\begin{array}{r} \lim_{x\to0^+} [x\cosh x - \sinh x] = 0 \\ \lim_{x\to0^+} x\sinh x = 0 \end{array}}$$

$$\overset{\left[\frac{0}{0}\right]}{\underset{\text{L'H}}{=}} \lim_{x\to0^+} \frac{\cosh x + x\sinh x - \cosh x}{\sinh x + x\cosh x} = \frac{\cosh 0 + 0\sinh 0 - \cosh 0}{\sinh 0 + 0\cosh 0} = \frac{0}{1} = 0$$

Problem 7

$$A = \lim_{n\to\infty} \left[A_0\left(1+\frac{r}{n}\right)^{nt}\right]$$
$$\ln A = \ln(A_0)\lim_{n\to\infty} \left[nt\ln\left(1+\frac{r}{n}\right)\right]$$

Let $k = 1/n$. Then $n = 1/k$ and $r/n = rk$. As $n\to\infty$, $k\to0$.

$$= \ln(A_0)\lim_{k\to0} \frac{t\ln(1+rk)}{k}$$

$$\boxed{\begin{array}{r} \lim_{k\to0} [t\ln(1+rk)] = 0 \\ \lim_{k\to0} k = 0 \end{array}}$$

$$\overset{\left[\frac{0}{0}\right]}{\underset{\text{L'H}}{}} \ln(A_0)\lim_{k\to0} \frac{\frac{rt}{1+rk}}{1} = rt\ln(A_0) \implies A = \exp\left(rt\ln(A_0)\right) = \exp\left(\ln(A_0)\right)\cdot e^{rt} = A_0e^{rt}$$

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Problem 8

$$\lim_{x\to0} \frac{f(2+3x)+f(2+5x)}{x} = \lim_{x\to0} \frac{f(2+3x)-f(2)+f(2+5x)-f(2)}{x}$$
$$= \lim_{x\to0} \frac{f(2+3x)-f(2)}{x} + \lim_{x\to0} \frac{f(2+5x)-f(2)}{x}$$

Let $h_1 = 3x$ and $h_2 = 5x$. Then $x = h_1/3 = h_2/5$. As $x\to0$, $h_1, h_2\to0$.

$$= \lim_{h_1\to0} \frac{f(2+h_1)-f(2)}{\frac{h_1}{3}} + \lim_{h_2\to0} \frac{f(2+h_2)-f(2)}{\frac{h_2}{5}}$$
$$= 3\lim_{h_1\to0} \frac{f(2+h_1)-f(2)}{h_1} + 5\lim_{h_2\to0} \frac{f(2+h_2)-f(2)}{h_2}$$
$$= 3f'(2) + 5f'(2) = 3\cdot7 + 5\cdot7 = \boxed{56}$$

Problem 9

$$\boxed{\begin{array}{c} \lim_{x\to\infty} \ln x = \infty \\ \lim_{x\to\infty} x^p = \infty \end{array}}$$

$$\lim_{x\to\infty} \frac{\ln x}{x^p} \overset{\left[\frac{\infty}{\infty}\right]}{\underset{\text{L'H}}{=}} \lim_{x\to\infty} \frac{\frac{1}{x}}{px^{p-1}} = \lim_{x\to\infty} \frac{1}{px^p} = 0$$

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Problem 10

$$\boxed{\begin{array}{c} \lim_{x\to0^+} \ln x = -\infty \\ \lim_{x\to0^+} x^{-a} = \lim_{x\to0^+} \frac{1}{x^a} = \pm\infty \end{array}}$$

$$\lim_{x\to0^+} x^a\ln x = \lim_{x\to0^+} \frac{\ln x}{x^{-a}} \overset{\left[\frac{\infty}{\infty}\right]}{\underset{\text{L'H}}{=}} \lim_{x\to0^+} \frac{\frac{1}{x}}{-ax^{-a-1}} = -\frac{1}{a}\lim_{x\to0^+} \frac{1}{x^{-a}} = -\frac{1}{a}\lim_{x\to0^+} x^a = \frac{1}{a}\cdot0 = 0$$

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