Problem Set #47

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Problem 1

(a) Vertical:

$$\begin{aligned}
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} &= 1 \\
b^2(x-h)^2 + a^2(y-k)^2 &= a^2b^2 \\
b^2\left(x^2 - 2hx + h^2\right) + a^2\left(y^2 - 2ky + k^2\right) &= a^2b^2 \\
b^2x^2 - 2b^2hx + b^2h^2 + a^2y^2 - 2a^2ky + a^2k^2 - a^2b^2 &= 0 \\
B^2 - 4AC &= (0)^2 - 4\left(b^2\right)\left(a^2\right) \\
&= \boxed{-4a^2b^2}
\end{aligned}$$

Horizontal:

$$\begin{aligned}
\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} &= 1 \\
b^2(y-k)^2 + a^2(x-h)^2 &= a^2b^2 \\
b^2\left(y^2 - 2ky + k^2\right) + a^2\left(x^2 - 2hx + h^2\right) &= a^2b^2 \\
b^2y^2 - 2b^2ky + b^2k^2 + a^2x^2 - 2a^2hx + a^2h^2 - a^2b^2 &= 0 \\
B^2 - 4AC &= (0)^2 - 4\left(a^2\right)\left(b^2\right) \\
&= \boxed{-4a^2b^2}
\end{aligned}$$

(b) Vertical:

$$\begin{aligned}
(x-h)^2 &= 4p(y-k) \\
x^2 - 2hx + h^2 &= 4py - 4pk \\
x^2 - 2hx + (-4p)y + \left(h^2 + 4pk\right) &= 0 \\
B^2 - 4AC &= 0 - 4(1)(0) \\
&= \boxed{0}
\end{aligned}$$

Horizontal:

$$\begin{aligned}
(y-k)^2 &= 4p(x-h) \\
y^2 - 2ky + k^2 &= 4px - 4ph \\
y^2 - 2ky + (-4p)x + \left(k^2 + 4ph\right) &= 0 \\
B^2 - 4AC &= 0 - 4(0)(1) \\
&= \boxed{0}
\end{aligned}$$

(c) Vertical:

$$\begin{aligned}
\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2} &= 1 \\
b^2(y-k)^2-a^2(x-h)^2 &= a^2b^2 \\
b^2\left(y^2-2ky+k^2\right)-a^2\left(x^2-2hx+h^2\right)-a^2b^2 &= 0 \\
b^2y^2-2b^2ky+b^2k^2-a^2x^2+2a^2hx-a^2h^2-a^2b^2 &= 0 \\
B^2-4AC &= 0-4\left(-a^2\right)\left(b^2\right) \\
&= \boxed{4a^2b^2}
\end{aligned}$$

Horizontal:

$$\begin{aligned}
\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2} &= 1 \\
b^2(x-h)^2-a^2(y-k)^2 &= a^2b^2 \\
b^2\left(x^2-2hx+h^2\right)-a^2\left(y^2-2ky+k^2\right)-a^2b^2 &= 0 \\
b^2x^2-2b^2hx+b^2h^2-a^2y^2+2a^2ky-a^2k^2-a^2b^2 &= 0 \\
B^2-4AC &= 0-4\left(b^2\right)\left(-a^2\right) \\
&= \boxed{4a^2b^2}
\end{aligned}$$

Because $a, b \in \mathbb{R}^+$, I claim that the discriminant of an ellipse is negative, the discriminant of a parabola is 0, and the discriminant of a hyperbola is positive.

Problem 2

(a)

$$\begin{aligned}
C_1 : y^2+2y+12x+25 &= 0 \\
B^2-4AC &= 0-4(0)(1) \\
&= 0 \ \boxed{\text{Parabola}}
\end{aligned}$$

$$\begin{aligned}
C_2 : x^2+2y^2-6x+4y+7 &= 0 \\
B^2-4AC &= 0-4(1)(2) \\
&= -8 \ \boxed{\text{Ellipse}}
\end{aligned}$$

$$\begin{aligned}
C_3 : 2y^2-3x^2-4y+12x+8 &= 0 \\
B^2-4AC &= 0-4(-3)(2) \\
&= 12 \ \boxed{\text{Hyperbola}}
\end{aligned}$$

(b)

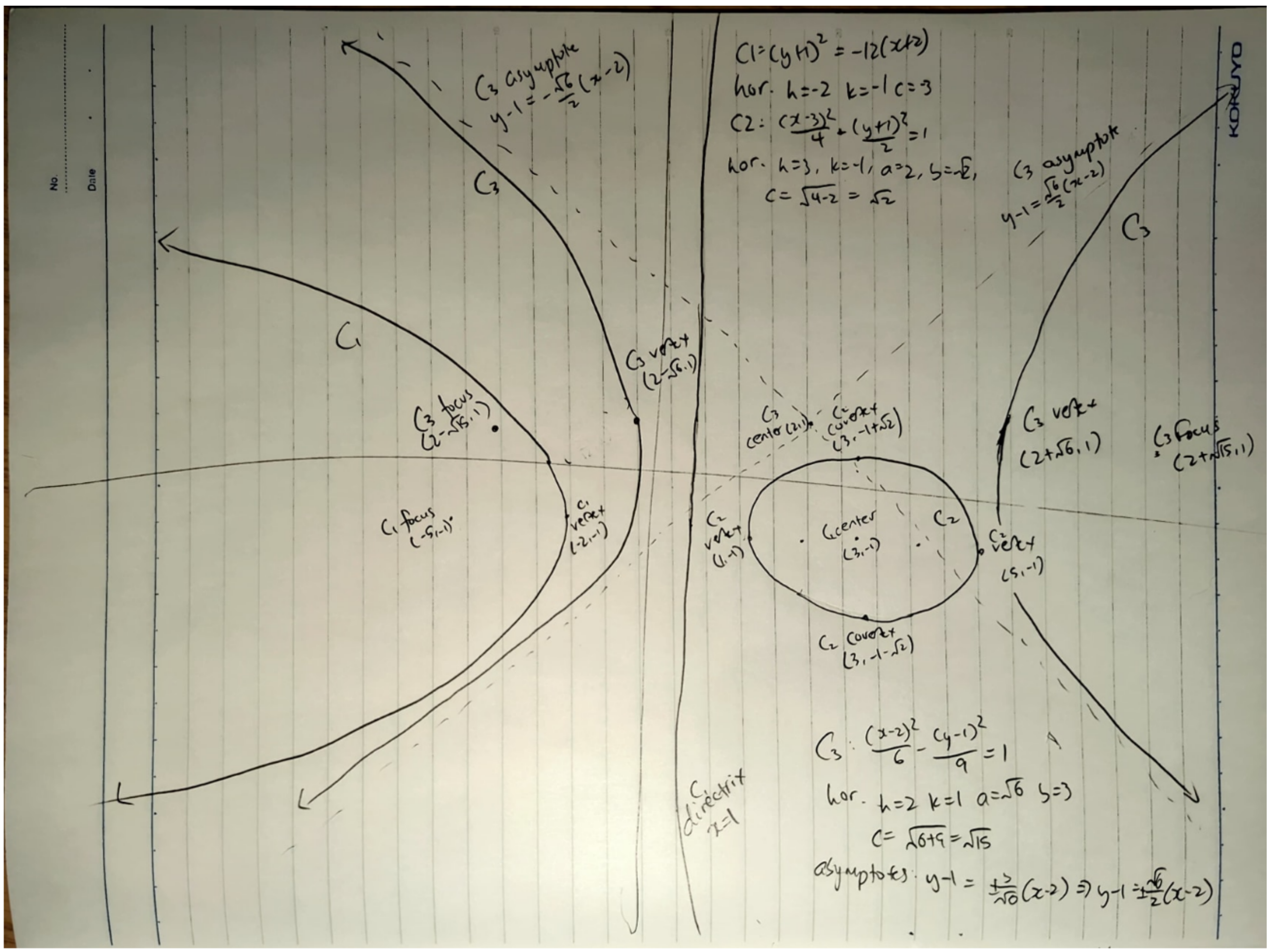
$$\begin{aligned}
C_1 : y^2+2y+12x+25 &= 0 \\
(y+1)^2-1+12x+25 &= 0 \\
(y+1)^2 &= -24-12x \\
\Aboxed{(y+1)^2 &= -12(x+2)}
\end{aligned}$$

$$C_2 : x^2+2y^2-6x+4y+7 = 0$$

$$(x-3)^2 - 9 + 2\left((y+1)^2 - 1\right) + 7 = 0$$
$$(x-3)^2 + 2(y+1)^2 = 4$$
$$\boxed{\frac{(x-3)^2}{4} + \frac{(y+1)^2}{2} = 1}$$

$$C_3 : 2y^2 - 3x^2 - 4y + 12x + 8 = 0$$
$$2\left((y-1)^2 - 1\right) - 3\left((x-2)^2 - 4\right) + 8 = 0$$
$$2(y-1)^2 - 3(x-2)^2 = -18$$
$$3(x-2)^2 - 2(y-1)^2 = 18$$
$$\boxed{\frac{(x-2)^2}{6} - \frac{(y-1)^2}{9} = 1}$$

(c)

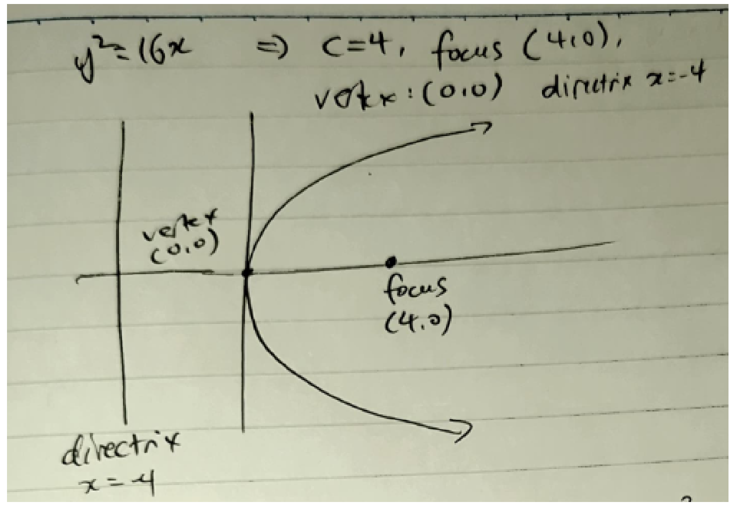


Problem 3

(a)

$$
\begin{aligned}
y^2 &= 16x \\
2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 16 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{8}{y} \\
y - 8a &= \frac{\cancel{8}}{\cancel{8}a}\left(x - 4a^2\right) \\
ay - 8a^2 &= x - 4a^2 \\
ay &= x + 4a^2 \\
&\boxed{y = \frac{x}{a} + 4a}
\end{aligned}
$$

(b)



□

(c) The center is $(0,0)$ and $c = 16/4 = 4$. The directrix is $x = 0 - c = -4$ and the focus is $F(4,0)$. In part (a) we showed that the tangent line to the parabola at some point $P\left(4a^2, 8a\right)$ is $y = \dfrac{x}{a} + 4a$. The point $Q\left(-4, \dfrac{42}{5}\right)$ is also on this tangent line.

$$
\begin{aligned}
\frac{42}{5} &= \frac{-4}{a} + 4a \\
42a &= -20 + 20a^2 \\
0 &= 10a^2 - 21a - 10 \\
a &= \frac{21 \pm \sqrt{441 + 400}}{20}
\end{aligned}
$$

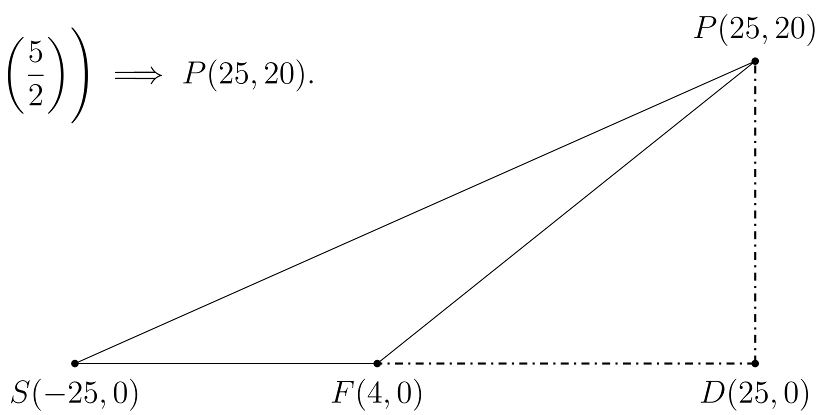
We only consider the positive case because $a \geq 0$.

$$
\begin{aligned}
a &= \frac{21 + \sqrt{841}}{20} \\
a &= \frac{21 + 29}{20} \\
a &= \frac{5}{2}
\end{aligned}
$$

$$
\begin{aligned}
y &= \frac{x}{\frac{5}{2}} + 4\left(\frac{5}{2}\right) \\
y &= \frac{2x}{5} + 10 \\
x\text{-intercept} : 0 &= \frac{2x}{5} + 10 \\
2x &= -50 \\
x &= -25 \implies S(-25, 0)
\end{aligned}
$$

Thus we have $P\left(4\left(\frac{5}{2}\right)^2, 8\left(\frac{5}{2}\right)\right) \implies P(25, 20)$.

$$
\begin{aligned}
S_{FSP} &= S_{SPD} - S_{FPD} \\
S_{FSP} &= \frac{50 \cdot 20}{2} - \frac{21 \cdot 20}{2} \\
S_{FSP} &= 500 - 210 \\
&\boxed{S_{FSP} = 290}
\end{aligned}
$$

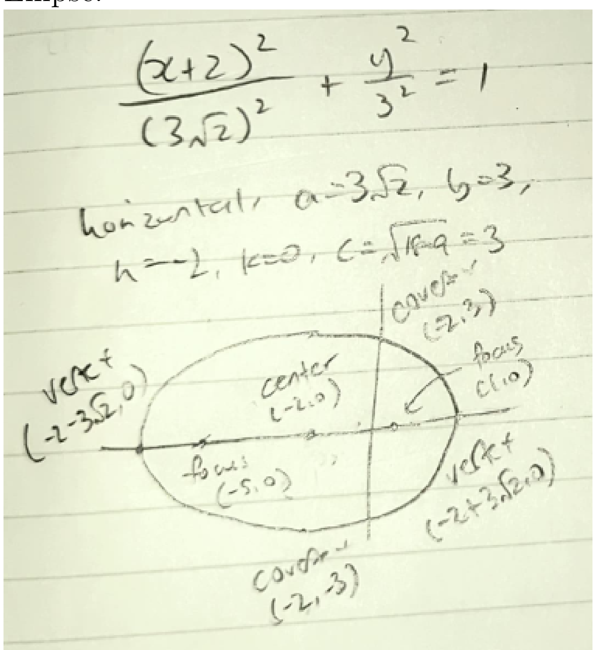


Problem 4

(a)

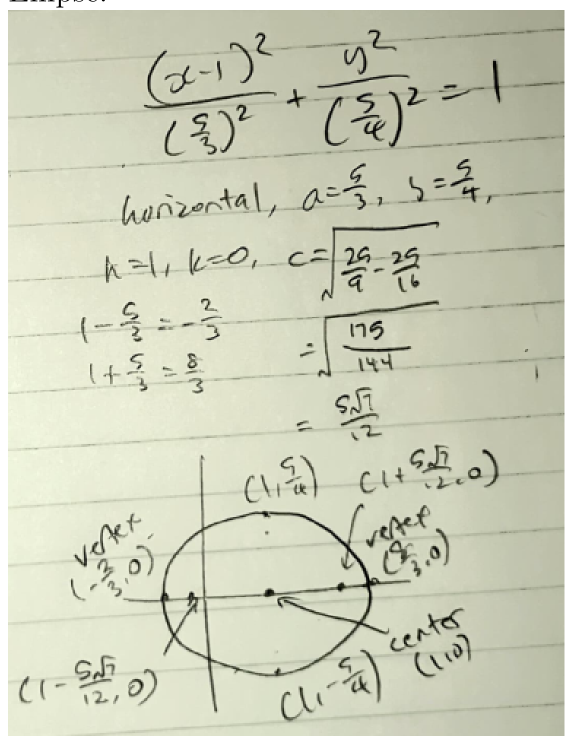
$$
\begin{aligned}
x^2+4x+2y^2-14&=0\\
(x+2)^2-4+2y^2-14&=0\\
(x+2)^2+2y^2&=18\\
\frac{(x+2)^2}{18}+\frac{y^2}{9}&=1\\
\frac{(x+2)^2}{\left(3\sqrt{2}\right)^2}+\frac{y^2}{3^2}&=1
\end{aligned}
$$

Ellipse.



$$
\begin{aligned}
9x^2-18x+16y^2-16&=0\\
9\left((x-1)^2-1\right)+16y^2-16&=0\\
9(x-1)^2+16y^2&=25\\
\frac{(x-1)^2}{\frac{25}{9}}+\frac{y^2}{\frac{25}{16}}&=1\\
\frac{(x-1)^2}{\left(\frac{5}{3}\right)^2}+\frac{y^2}{\left(\frac{5}{4}\right)^2}&=1
\end{aligned}
$$

Ellipse.



(b)

$$
\begin{cases}
(x+2)^2+2y^2=18\\
9(x-1)^2+16y^2=25
\end{cases}
\implies
\begin{cases}
16y^2=144-8(x+2)^2 & (1)\\
16y^2=25-9(x-1)^2 & (2)
\end{cases}
$$

$$
\begin{aligned}
(1)-(2):0&=119-8(x+2)^2+9(x-1)^2\\
0&=119-8x^2-32x-32+9x^2-18x+9\\
0&=x^2-50x+96\\
0&=(x-2)(x-48)\\
x&=2,x=-48
\end{aligned}
$$

$$
\begin{aligned}
(2+2)^2+2y^2&=18\\
2y^2&=2\\
y&=\pm1
\end{aligned}
$$

$$
\begin{aligned}
(-48+2)^2+2y^2&=18\\
2y^2&=18-46^2
\end{aligned}
$$

No real solution for y.

Intersections: $\boxed{(2,1),(2,-1)}$

(c)

$$\begin{aligned}
(x-c)^2 + y^2 &= r^2 \\
(2-c)^2 + 1^2 &= r^2 \\
4 - 4c + c^2 + 1 &= r^2 \\
c^2 - 4c + 5 &= r^2
\end{aligned}$$

$$\begin{aligned}
(x-c)^2 + y^2 &= c^2 - 4c5 \\
x^2 - 2xc + \cancel{c^2} + y^2 &= \cancel{c^2} - 4c + 5 \\
x^2 - 2xc + y^2 &= 5 - 4c
\end{aligned}$$

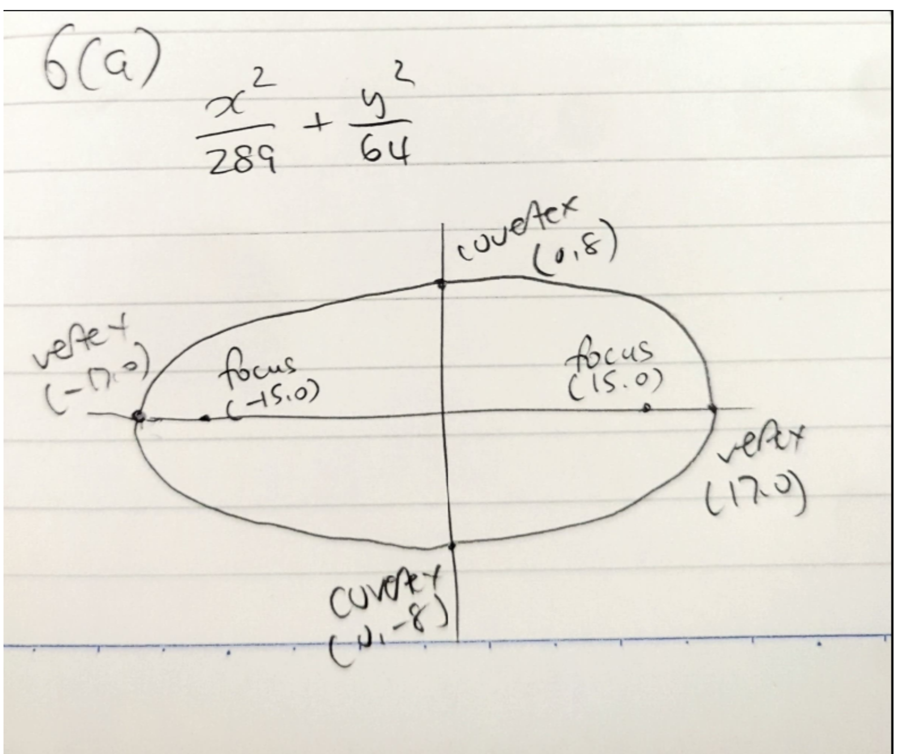
□

Problem 6

(a)

$$\begin{aligned}
\frac{x^2}{289} + \frac{y^2}{64} &= 1 \\
\frac{x^2}{17^2} + \frac{y^2}{8^2} &= 1
\end{aligned}$$

Horizontal, $h = 0, k = 0, a = 17, b = 8, c = \sqrt{289 - 64} = \sqrt{225} = 15, e = \frac{c}{a} = \frac{15}{17}$, $\boxed{\text{foci } (-15, 0), (15, 0)}$, $\boxed{\text{directrices } x = \pm\frac{a}{e} = \pm\frac{17}{\frac{15}{17}} = \pm\frac{289}{15}}$



(b) Let $P(x_0, y_0)$ be a point. Because PF where F is the focus with positive x coordinate, we have $F(15, 0)$ and $x_0 = 15$. By the equation of the ellipse:

$$\begin{aligned}
\frac{x_0^2}{289} + \frac{y_0^2}{64} &= 1 \\
\frac{225}{289} + \frac{y_0^2}{64} &= \frac{289}{289} \\
\frac{y_0^2}{64} &= \frac{289 - 225}{289} \\
y_0^2 &= \frac{64^2}{17^2} \\
y_0 &= \pm\frac{64}{17}
\end{aligned}$$

$$\begin{aligned}
\frac{x^2}{289} + \frac{y^2}{64} &= 1 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^2}{289}\right] + \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{y^2}{64}\right] &= 0 \\
\frac{2x}{289} + \frac{y}{32}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-\frac{2x}{289}}{\frac{y}{32}} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{64x}{289y}
\end{aligned}$$

$$\begin{aligned}
\text{Tangent} : y - \left(\pm\frac{64}{17}\right) &= \left(-\frac{\cancel{64}(15)}{289\left(\pm\frac{\cancel{64}}{17}\right)}\right)(x - 15) \\
x\text{-intercept} : 0 - \left(\pm\frac{64}{17}\right) &= \mp\frac{15}{17}(x - 15) \\
\mp 64 &= \mp 15(x - 15) \\
\mp 64 &= \mp 15x \pm 225 \\
\mp(64 + 225) &= \mp 15x \\
\mp 15x &= \mp 289 \\
x &= \pm\frac{289}{15}
\end{aligned}$$

The x-intercepts of the tangent line at point P is always at the directrix.

$\square$

Problem 7

(a)

$$\begin{aligned}
\frac{x^2}{16} + \frac{y^2}{4} &= 1 \\
\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{x^2}{16}\right] + \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{y^2}{4}\right] &= 0 \\
\frac{x}{8} + \frac{y}{2}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-\frac{x}{8}}{\frac{y}{2}} \\
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{x}{4y}
\end{aligned}$$

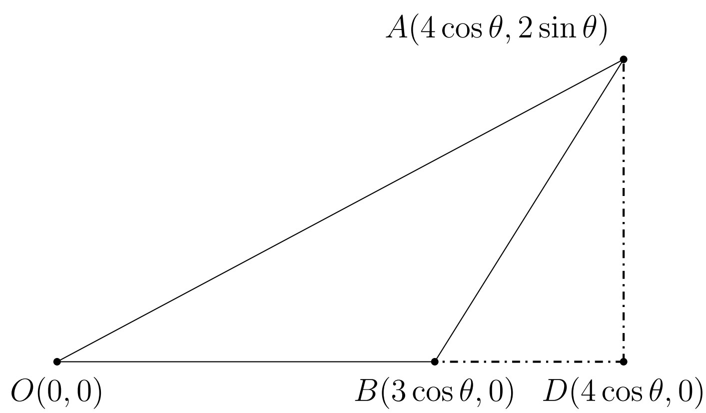
$$\text{Normal line} : y - y_0 = \left(-\left(-\frac{x_0}{4y_0}\right)\right)^{-1}(x - x_0)$$

$$
\begin{aligned}
y - 2\sin\theta &= \frac{\cancel{4}\,(2\sin\theta)}{\cancel{4}\cos\theta}(x - 4\cos\theta) \\
y\cos\theta - 2\sin\theta\cos\theta &= 2\sin(\theta)\,(x - 4\cos\theta) \\
y\cos\theta - 2\sin\theta\cos\theta &= 2x\sin\theta - 8\sin\theta\cos\theta \\
2x\sin\theta - y\cos\theta &= 6\sin\theta\cos\theta
\end{aligned}
$$

$\square$

(b) We will find the x-intercept of the normal line at point A.

$$
\begin{aligned}
2x\sin\theta - \cancel{y\cos\theta} &= 6\sin\theta\cos\theta \\
x &= \frac{6\cancel{\sin\theta}\cos\theta}{2\cancel{\sin\theta}} \\
x &= 3\cos\theta
\end{aligned}
$$



$$
\begin{aligned}
S_{OAB} &= S_{OAD} - S_{ABD} \\
S_{OAB} &= \frac{8\cos(\theta)\sin(\theta)}{2} - \frac{2\cos(\theta)\sin(\theta)}{2} \\
S_{OAB} &= 4\cos\theta\sin\theta - \cos\theta\sin\theta \\
S_{OAB} &= \frac{3}{2}\,(2\cos\theta\sin\theta) \\
S_{OAB} &= \frac{3}{2}\sin 2\theta
\end{aligned}
$$

$$\boxed{\max S_{OAB} = \frac{3}{2}}$$