Finish Problem Set #42

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Problem 6

(b)

$$1^3 + 2^3 + 3^3 + \cdots + n^3 \stackrel{?}{=} (1 + 2 + 3 + \cdots + n)^2$$
$$\sum_{k=1}^{n} k^3 \stackrel{?}{=} \left(\sum_{k=1}^{n} k \right)^2$$

Proof. By induction.

Base case. $n = 1$

LHS: $\displaystyle\sum_{k=1}^{1} k^3 = 1$ RHS: $\displaystyle\left(\sum_{k=1}^{1} k \right)^2 = 1$

Induction Hypothesis. Suppose $\displaystyle\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k \right)^2$. Show $\displaystyle\sum_{k=1}^{n+1} k^3 = \left(\sum_{k=1}^{n+1} k \right)^2$

Inductive Step.
$$\begin{aligned}
\left(\sum_{k=1}^{n+1} k \right)^2 &= \left(\sum_{k=1}^{n} k + (n+1) \right)^2 \\
&= \left(\sum_{k=1}^{n} k \right)^2 + 2 \left(\sum_{k=1}^{n} k \right) (n+1) + (n+1)^2 \\
&= \sum_{k=1}^{n} k^3 + 2 \left(\frac{n(n+1)(n+1)}{2} \right) + (n+1)^2 \\
&= \frac{n^2(n+1)^2}{4} + \frac{4n(n+1)^2}{4} + \frac{4(n+1)^2}{n} \\
&= \frac{(n+1)^2 \left(n^2 + 4n + 4\right)}{4} \\
&= \frac{(n+1)^2(n+2)^2}{4} \\
&= \sum_{k=1}^{n+1} k^3
\end{aligned}$$

$\square$

Problem 7

$$\begin{aligned}
\sum_{i=1}^{4}\sum_{j=1}^{4}(i-j)^2 &= \sum_{i=1}^{4}\sum_{j=1}^{4}\left(i^2-2ij+j^2\right) \\
&= \sum_{i=1}^{4}\left(\sum_{j=1}^{4} i^2 - 2\sum_{j=1}^{4} ij + \sum_{j=1}^{4} j^2\right) \\
&= \sum_{i=1}^{4}\left(4i^2 - 2i\sum_{j=1}^{4} j + \sum_{j=1}^{4} j^2\right) \\
&= \sum_{i=1}^{4} 4i^2 - \sum_{i=1}^{4}\left(2i\left(\frac{4(4+1)}{2}\right) - \frac{4(4+1)(2(4)+1)}{6}\right) \\
&= 4\left(\frac{4(4+1)(2(4)+1)}{6}\right) - \sum_{i=1}^{4}(20i-30) \\
&= 4\cdot 30 - 20\sum_{i=1}^{4} i + \sum_{i=1}^{4} 30 \\
&= 120 - 20\left(\frac{4(4+1)}{2}\right) + 4\cdot 30 \\
&= 240 - 200 \\
&= \boxed{40}
\end{aligned}$$