Problem Set #45

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Problem 2

2(a) Parabola, focus $F(1,1)$, directrix $x=7$ $\Rightarrow$ horizontal.

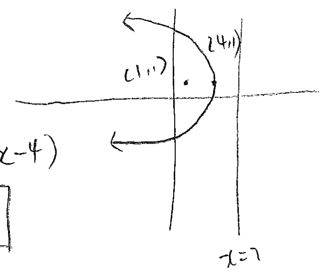
vertex: $(4,1)$

$\Rightarrow h=4,\ k=1$

$(y-k)^2 = 4p(x-h)$

$(y-1)^2 = 4(1-4)(x-4)$

$\boxed{(y-1)^2 = -12(x-4)}$



(b) Ellipse, center $(-1,4)$, vertex $(-1,1)$, focus $(-1,6)$ $\Rightarrow$ vertical.

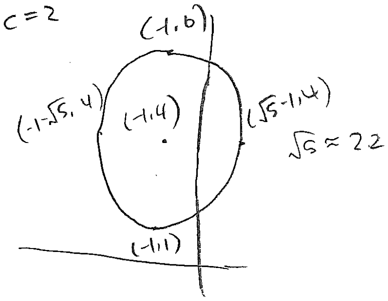
$\Rightarrow h=-1,\ k=4$ $\quad\Rightarrow a=3$ $\quad\Rightarrow c=2$

$a^2-b^2=c^2$

$9-b^2=4$

$b^2=5$

$\boxed{\dfrac{(x+1)^2}{5} + \dfrac{(y-4)^2}{9} = 1}$



(c) Hyperbola. vertices $(-1,2)$, $(7,2)$, foci $(-2,2)$, $(8,2)$ $\Rightarrow$ horizontal

$\Rightarrow$ center: $(3,2)$ $\quad\Rightarrow c=5$

$\Rightarrow a=4$ $\quad a^2+b^2=c^2$

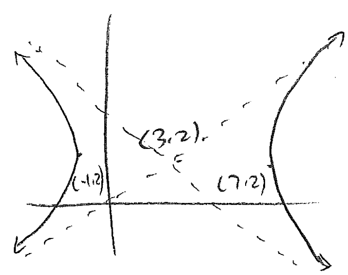
$\Rightarrow h=3,\ k=2$ $\quad 16+b^2=25$

$b^2=9$

$b=3$

$\boxed{\dfrac{(x-3)^2}{16} - \dfrac{(y-2)^2}{9} = 1}$

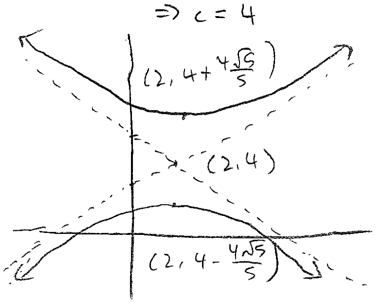
Asymptotes: $y-2 = \pm\frac{4}{5}(x-3)$



(d) Hyperbola. foci $(2,0)$, $(2,8)$. asymptotes: $y=3+\frac{x}{2}$, $y=5-\frac{x}{2}$

$\Rightarrow$ vertical, $h=2,\ k=4$ $\quad\Rightarrow y-4=\frac{1}{2}(x-2)$, $y-4=-\frac{1}{2}(x-2)$

$\Rightarrow c=4$



$\begin{cases} a^2+b^2=16 \\ \frac{a}{b}=\frac{1}{2} \end{cases}$

$\frac{a^2}{b^2}=\frac{1}{4}$

$4a^2=b^2$

$a^2+4a^2=16$

$5a^2=16$

$a^2=\frac{16}{5}$

$b^2=4\left(\frac{16}{5}\right)=\frac{64}{5}$

$a=\frac{4}{\sqrt{5}}=\frac{4\sqrt{5}}{5}$

$\boxed{\dfrac{(y-4)^2}{\frac{16}{5}} - \dfrac{(x-2)^2}{\frac{64}{5}} = 1}$

Problem 3

(a) **Ellipse**. If $k > 16$, then $k > 0$ and $k - 16 > 0$.

(b) **Hyperbola**. If $0 < k < 16$, then $k > 0$ and $k - 16 < 0$.

(c) **No real values**. If $k < 0$, then $k < 0$ and $k - 16 < 0$. $x^2 > 0$ and $y^2 > 0$ for $x, y \in \mathbb{R}$. Therefore, $\frac{x^2}{k}$ and $\frac{y^2}{k-16}$ are negative and the sum of two negative numbers cannot be 1.

(d) (a) $\frac{x^2}{k} + \frac{x^2}{x-16} = 1 \implies a = \sqrt{k}, b = \sqrt{k-16} \implies c = \sqrt{a^2 - b^2} = \sqrt{k - (k-16)} = 4$. The focus of the ellipse is at $(4, 0)$ and $(-4, 0)$ for all $k > 16$.

(b) $\frac{x^2}{k} + \frac{x^2}{x-16} = 1 \implies \frac{x^2}{k} - \frac{x^2}{16-x} = 1 \implies a = \sqrt{k}, b = \sqrt{16-k} \implies c = \sqrt{a^2 + b^2} = \sqrt{k + (16-k)} = 4$. The focus of the hyperbola is at $(4, 0)$ and $(-4, 0)$ for all $0 < k < 16$.

Problem 6

(a)

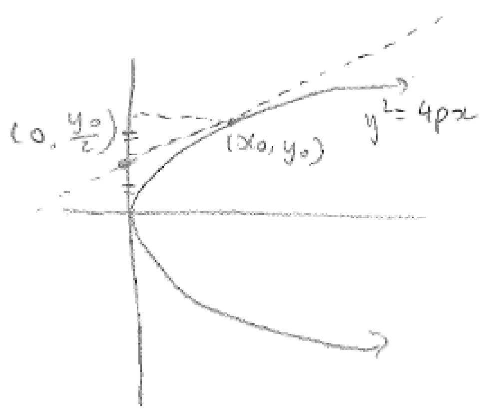
$$\begin{aligned} y^2 &= 4px \\ \frac{\mathrm{d}}{\mathrm{d}x} y^2 &= \frac{\mathrm{d}}{\mathrm{d}x} 4px \\ 2y \frac{\mathrm{d}y}{\mathrm{d}x} &= 4p \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2p}{y} \end{aligned}$$

$$\begin{aligned} y - y_0 &= \frac{2p}{y_0}(x - x_0) \\ y_0 y - y_0^2 &= 2p(x - x_0) \\ y_0 y &= 2px - 2px_0 + 4px_0 \\ y_0 y &= 2px + 2px_0 \\ & \boxed{y_0 y = 2p(x + x_0)} \end{aligned}$$

(b)

$$\begin{aligned} y_0 y &= 2p(x + x_0) \\ y &= \frac{2px}{y_0} + \frac{4px_0}{2y_0} \\ y &= \frac{2px}{y_0} + \frac{y_0^{\cancel{2}}}{2\cancel{y_0}} \\ & \boxed{y = \frac{2px}{y_0} + \frac{y_0}{2}} \end{aligned}$$

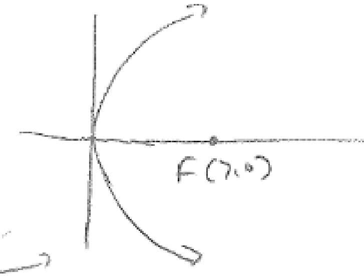
$$y = \frac{2px}{y_0} + \frac{y_0}{2}$$



Problem 7

7. (a): $y^2 = 28x$

$x = \frac{y^2}{28}$

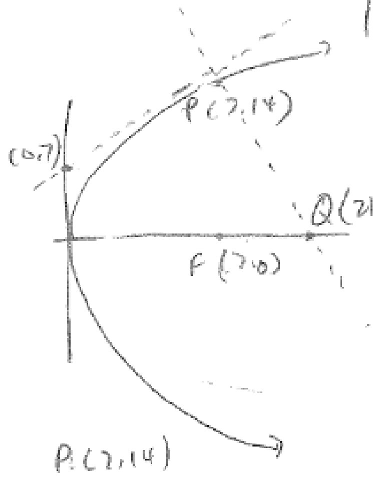


vertex: $(0,0)$

$p = \frac{28}{4} = 7$

focus: $(0+7, 0) = (7,0)$

(b)



Tangent line = $y = \frac{2\left(\frac{28}{4}\right)x}{14} + \frac{14}{2}$

$y = x + 7$

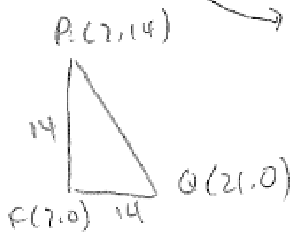
Nomal line $(y-14) = -\frac{1}{1}(x-7)$

$\boxed{y = -x + 21}$

$0 = -x + 21$

$x = 21$

(c)



$S_{\Delta PQR} = \frac{1}{2}(14-0)(21-7)$

$= \frac{1}{2}(196)$

$\boxed{= 98}$