Problem Set $\#55\frac{1}{2}$

Jayden Li

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Problem 2

$$
\begin{aligned}
& |\tan x - 1| < 0.2 \\
\implies & -0.2 < \tan x - 1 < 0.2 \\
\implies & 0.8 < \tan x < 1.2 \\
\implies & \arctan(0.8) < x < \arctan(1.2) \\
& \left|x - \frac{\pi}{4}\right| < \delta \\
\implies & -\delta < x - \frac{\pi}{4} < \delta \\
\implies & \frac{\pi}{4} - \delta < x < \frac{\pi}{4} + \delta
\end{aligned}
$$

$$
\begin{aligned}
\delta &= \min\left(\frac{\pi}{4} + \arctan(0.8), \arctan(1.2) - \frac{\pi}{4}\right) \\
&= \boxed{\arctan(1.2) - \frac{\pi}{4}}
\end{aligned}
$$

Problem 3

(a)

$$r^2\pi = 1000 \implies \boxed{r = \sqrt{\frac{1000}{\pi}}\text{cm}}$$

(b)

$$
\begin{gathered}
995 < r^2\pi < 1005 \\
\frac{995}{\pi} < r^2 < \frac{1005}{\pi} \\
\sqrt{\frac{995}{\pi}} < r < \sqrt{\frac{1005}{\pi}} \\
\sqrt{\frac{1000}{\pi}} - \delta < r < \sqrt{\frac{1000}{\pi}} + \delta \\
\delta = \min\left(\sqrt{\frac{1000}{\pi}} - \sqrt{\frac{995}{\pi}}, \sqrt{\frac{1005}{\pi}} - \sqrt{\frac{1000}{\pi}}\right)
\end{gathered}
$$

$$\delta \approx \boxed{0.044547\text{cm}}$$

(c) x is the area. f is the function that calculates the radius needed for a circle with radius x. a is the desired radius that x tends to. As x tends to a, L tends to $f(x)$. $\varepsilon = 5$cm. $\delta \approx 0.044547$cm.

Problem 4

$$\begin{aligned}|4x-8| &< \varepsilon\\ 4|x-2| &< \varepsilon\\ |x-2| &< \frac{\varepsilon}{4}\end{aligned}$$

Obviously/clearly/trivially $\delta = \dfrac{\varepsilon}{4}$.

Proof. Let $\delta = \dfrac{\varepsilon}{4}$. If $|x-2| < \delta$, $4|x-2| < 4 \cdot \dfrac{\varepsilon}{4}$. Therefore $|4x-8| < \varepsilon$. ☺

(a) $\delta = \dfrac{\varepsilon}{4} = \dfrac{0.1}{4} = \boxed{0.025}$

(b) $\delta = \dfrac{\varepsilon}{4} = \dfrac{0.01}{4} = \boxed{0.0025}$

Problem 5

(b)

$$\begin{aligned}\left|\frac{x^2+x-6}{x-2} - 5\right| &< \varepsilon\\ \left|\frac{(x+3)\cancel{(x-2)}-5}{\cancel{x-2}}\right| &< \varepsilon\\ |x+3-5| &< \varepsilon\\ |x-2| &< \varepsilon\\ |x-2| &< \delta\end{aligned}$$

I claim that $\delta = \varepsilon$.

Proof. Let $\delta = \varepsilon$.

$$|x-2| < \delta \implies |x+3-5| < \varepsilon \implies \left|\frac{(x+3)(x-2)}{x-2} - 5\right| < \varepsilon \implies \left|\frac{x^2+x-6}{x-2} - 5\right| < \varepsilon$$

☺

(d) Because x^2 and $\sqrt{x}$ are always positive:

$$\left|x^2 - 0\right| < \varepsilon \implies |x^2| < \varepsilon \implies |x-0| < \sqrt{\varepsilon}$$

I claim that $\delta = \sqrt{\varepsilon}$.

Proof. Let $\delta = \sqrt{\varepsilon}$.

$$|x-0| < \delta \implies |x| < \sqrt{\varepsilon} \implies (|x|)^2 < \varepsilon \implies \left|x^2 - 0\right| < \varepsilon$$

☺

(e)

$$||x| - 0| < \varepsilon \implies |x| < \varepsilon \implies |x - 0| < \varepsilon$$

I claim that $\delta = \varepsilon$.

Proof. Let $\delta = \varepsilon$.

$$|x-0| < \delta \implies ||x-0|| < \varepsilon \implies ||x| - 0| < \varepsilon$$

☺

(f)

$$\left|x^2 - 4x + 5 - 1\right| < \varepsilon \implies \left|x^2 - 4x + 4\right| < \varepsilon \implies \left|(x-2)^2\right| < \varepsilon \implies |x-2| < \sqrt{\varepsilon}$$

I claim that $\delta = \sqrt{\varepsilon}$.

Proof. Let $\delta = \sqrt{\varepsilon}$.

$$|x-2| < \delta \implies \left|(x-2)^2\right| < \delta^2 \implies \left|x^2 - 4x + 4\right| < \varepsilon \implies \left|x^2 - 4x + 5 - 1\right| < \varepsilon$$

☺

(g)

$$\left|x^2 - 1 - 3\right| < \varepsilon \implies |x+2|\,|x-2| < \varepsilon \tag{1}$$

Suppose $\delta = 1$. By definition:

$$|x+2| < \delta \implies -1 < x+2 < 1 \implies -5 < x-2 < -3 \implies |x-2| < 5 \tag{2}$$

Substitute (2) into (1):

$$5\,|x+2| < \varepsilon \implies |x+2| < \frac{\varepsilon}{5}$$

I claim that $\delta = \min(1, \varepsilon/5)$.

Proof. Let $\delta = \min(1, \varepsilon/5)$. If $|x+2| < \delta$ then all of the following are true:

$$|x+2| < 1 \tag{3}$$

$$|x+2| < \frac{\varepsilon}{5} \tag{4}$$

From (3) we have:

$$|x+2| < 1 \implies -1 < x+2 < 1 \implies -5 < x-2 < -3 \implies |x-2| < 5 \tag{5}$$

Multiple (3) and (5):

$$|x-2|\,|x+2| = \left|x^2 - 16\right| < 5 \cdot \frac{\varepsilon}{5} = \varepsilon$$

Thus by the ε-δ definition $\lim_{x \to -2} \left[x^2 - 1\right] = 3$. ☺

Problem 7

Lemma. Density of rational and irrational numbers: there exists both a rational and irrational number on the open interval (x, y) where $x < y$. (by obvious intuition)

Proof. Suppose not. Let $L = \lim_{x \to 0} f(x)$ and choose $\varepsilon = 0.1$. By the ε-δ definition there must exist some $\delta > 0$ such that $|x| < \delta$. On the open interval $(-\delta, \delta)$, there exists both a rational number p and an irrational number q (by the Lemma).

On the interval $(-\delta, \delta)$, we have $|f(p) - f(q)| = 1 \not< \varepsilon = 0.1$. We have a contradiction and the limit $\lim_{x \to 0} f(x)$ does not exist. ☺

Problem 8

$$
\begin{aligned}
\frac{1}{(x+3)^4} &> 10000 \\
1 &> 10000(x+3)^4 \\
\sqrt[4]{1} &> \sqrt[4]{10000(x+3)^4} \\
1 &> 10(x+3) \\
-29 &> 10x \\
\end{aligned}
$$

$$\boxed{x < -\frac{29}{10}}$$