Integral of $\sec x$ and $\sec^3 x$

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1 Integral of secant

$$\int \sec(x)\,\mathrm{d}x = \boxed{\ln|\sec x + \tan x| + C} \tag{1}$$

1.1 Using black magic

$$\begin{aligned}\int \sec(x)\,\mathrm{d}x &= \int \frac{\sec(x)(\sec x + \tan x)}{\sec x + \tan x}\,\mathrm{d}x = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,\mathrm{d}x \\ &= \begin{bmatrix} u = \sec x + \tan x \\ \mathrm{d}u = (\sec x \tan x + \sec^2 x)\,\mathrm{d}x \end{bmatrix} = \int \frac{1}{u}\,\mathrm{d}u = \ln|u| = \boxed{\ln|\sec x + \tan x| + C}\end{aligned}$$

1.2 Using partial fractions

$$\int \sec(x)\,\mathrm{d}x = \int \frac{\cos x}{\cos^2 x}\mathrm{d}x = \int \frac{\cos x}{1-\sin^2 x}\,\mathrm{d}x = \begin{bmatrix} u = \sin x \\ \mathrm{d}u = \cos(x)\,\mathrm{d}x \end{bmatrix} \int \frac{1}{1-u^2}\,\mathrm{d}u = \int \frac{1}{(1+u)(1-u)}\,\mathrm{d}u$$

$$\frac{1}{(1+u)(1-u)} = \frac{A}{1+u} + \frac{B}{1-u} = \frac{A - Au + B + Bu}{(1+u)(1-u)} = \frac{(A+B)+(B-A)u}{(1+u)(1-u)}$$

But we know there is no u in the numerator, so $B - A = 0 \implies B = A$. Also we know $A + B = 1$, so $A = 1/2, B = 1/2$.

$$\begin{aligned}&= \int \left(\frac{1/2}{1+u} + \frac{1/2}{1-u}\right)\mathrm{d}u = \frac{1}{2}\left(\int \frac{1}{1+u}\,\mathrm{d}u + \int \frac{1}{1-u}\,\mathrm{d}u\right) = \frac{1}{2}\left(\ln|1+u| - \ln|1-u|\right) \\ &= \boxed{\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C} = \frac{1}{2}\ln\left|\frac{1+2\sin x+\sin^2 x}{1-\sin^2 x}\right| = \frac{1}{2}\ln\left|\frac{1+2\sin x+\sin^2 x}{\cos^2 x}\right| \\ &= \frac{1}{2}\ln\left|\sec^2 x + 2\tan(x)\sec(x) + \tan^2 x\right| = \frac{1}{2}\ln\left((\sec x + \tan x)^2\right) = \boxed{\ln|\sec x + \tan x| + C}\end{aligned}$$

2 Integral of secant cubed

$$\int \sec^3(x)\,\mathrm{d}x = \frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\int \sec(x)\,\mathrm{d}x = \boxed{\frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec x + \tan x| + C} \tag{2}$$

2.1 Using integration by parts

$$\begin{aligned}\int \sec^3(x)\,\mathrm{d}x &= \begin{bmatrix} u = \sec x & \mathrm{d}u = \sec(x)\tan(x)\,\mathrm{d}x \\ \mathrm{d}v = \sec^2(x)\,\mathrm{d}x & v = \tan x \end{bmatrix} \sec(x)\tan(x) - \int \sec(x)\tan^2(x)\,\mathrm{d}x \\ &= \sec(x)\tan(x) - \int \sec(x)\left(\sec^2 x - 1\right)\,\mathrm{d}x = \sec(x)\tan(x) - \int \sec^3(x)\,\mathrm{d}x + \int \sec(x)\,\mathrm{d}x\end{aligned}$$

$$2\int \sec^3(x)\,\mathrm{d}x = \sec(x)\tan(x) + \ln|\sec x + \tan x|$$

$$\int \sec^3(x)\,\mathrm{d}x = \boxed{\frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec x + \tan x| + C}$$

2.2 Using a very bad technique

$$\begin{aligned}\int \sec^3 x\,\mathrm{d}x &= \int \frac{1}{\cos^3 x}\,\mathrm{d}x = \int \frac{\cos x}{\cos^4 x}\,\mathrm{d}x = \int \frac{\cos x}{\left(1-\sin^2 x\right)^2}\,\mathrm{d}x = \begin{bmatrix} u = \sin x \\ \mathrm{d}u = \cos(x)\,\mathrm{d}x \end{bmatrix} \int \frac{1}{(1-u^2)^2}\,\mathrm{d}u \\ &= \int \left(\frac{1}{(1-u)(1+u)}\right)^2\mathrm{d}u = \int \left(\frac{1}{2}\left(\frac{1}{1+u} + \frac{1}{1-u}\right)\right)^2\mathrm{d}u \\ &= \frac{1}{4}\int \frac{1}{(1+u)^2}\,\mathrm{d}u + \frac{1}{2}\underbrace{\int \frac{1}{(1+u)(1-u)}\,\mathrm{d}u}_{\text{integral of secant}} + \frac{1}{4}\int \frac{1}{(1-u)^2}\,\mathrm{d}u \\ &= -\frac{1}{4(1+u)} + \frac{1}{4(1-u)} + \frac{1}{2}\ln|\sec x + \tan x| \\ &= \frac{1}{4}\left(\frac{1}{1-\sin x} - \frac{1}{1+\sin x}\right) + \frac{1}{2}\ln|\sec x + \tan x| \\ &= \frac{1}{4}\cdot\frac{1+\sin x - (1-\sin x)}{(1-\sin x)(1+\sin x)} + \frac{1}{2}\ln|\sec x + \tan x| = \frac{2\sin x}{4\left(1-\sin^2 x\right)} + \frac{1}{2}\ln|\sec x + \tan x| \\ &= \frac{1}{2}\cdot\frac{\sin x}{\cos^2 x} + \frac{1}{2}\ln|\sec x + \tan x| = \boxed{\frac{1}{2}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec x + \tan x| + C}\end{aligned}$$