Problem Set #19

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Problem 2

$$\boxed{\begin{aligned}\int \csc x \,\mathrm{d}x &= \int \csc(x)\cdot\frac{\csc x+\cot x}{\csc x+\cot x}\,\mathrm{d}x = \int \frac{\csc^2 x+\csc x\cot x}{\csc x+\cot x}\,\mathrm{d}x \\ &= \begin{bmatrix} u = \csc x+\cot x \\ \mathrm{d}u = \left(-\csc x\cot x-\csc^2 x\right)\mathrm{d}x \\ = -\left(\csc^2 x+\csc x\cot x\right)\mathrm{d}x\end{bmatrix}\int \frac{-1}{u}\,\mathrm{d}u = \boxed{-\ln|\csc x+\cot x|+C}\end{aligned}}$$

(b)
$$\begin{aligned}\int \frac{1}{\sqrt{3}\sin x+\cos x}\,\mathrm{d}x &= \int\left(2\left(\frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\cos x\right)\right)^{-1}\mathrm{d}x = \frac{1}{2}\int\left(\cos\frac{\pi}{6}\sin x+\sin\frac{\pi}{6}\cos x\right)^{-1}\mathrm{d}x \\ &= \frac{1}{2}\int\left(\sin\left(x+\frac{\pi}{6}\right)\right)^{-1}\mathrm{d}x = \frac{1}{2}\int\csc\left(x+\frac{\pi}{6}\right)\,\mathrm{d}x \\ &= \boxed{-\frac{1}{2}\ln\left|\csc\left(x+\frac{\pi}{6}\right)+\cot\left(x+\frac{\pi}{6}\right)\right|+C}\end{aligned}$$

(e) $$\int \frac{1}{x}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,\mathrm{d}x = \begin{bmatrix} u = \sqrt{x} \\ \mathrm{d}u = \dfrac{\mathrm{d}x}{2\sqrt{x}} \implies \mathrm{d}x = 2u\,\mathrm{d}u \end{bmatrix} \int \frac{1}{u^2}\sqrt{\frac{1-u}{1+u}} \cdot 2u\,\mathrm{d}u = 2\int \frac{\sqrt{1-u}}{u\sqrt{1+u}}\,\mathrm{d}u$$

$$= 2\int \frac{\sqrt{(1-u)(1+u)}}{u(1+u)}\,\mathrm{d}u = 2\int \frac{\sqrt{1-u^2}}{u(1+u)}\,\mathrm{d}u$$

$$= \begin{bmatrix} u = \sin\theta \\ \mathrm{d}u = \cos\theta\,\mathrm{d}\theta \end{bmatrix} 2\int \frac{\sqrt{1-\sin^2\theta}}{\sin(\theta)(1+\sin\theta)} \cdot \cos\theta\,\mathrm{d}\theta = 2\int \frac{\cos^2\theta}{\sin(\theta)(1+\sin\theta)}\,\mathrm{d}\theta$$

$$= 2\int \frac{1-\sin^2\theta}{\sin(\theta)(1+\sin\theta)}\,\mathrm{d}\theta = 2\int \frac{1}{\sin(\theta)(1+\sin\theta)}\,\mathrm{d}\theta - 2\int \frac{\sin^2\theta}{\sin(\theta)(1+\sin\theta)}\,\mathrm{d}\theta$$

$$= 2\int \left(\frac{1}{\sin\theta} - \frac{1}{1+\sin\theta}\right)\,\mathrm{d}\theta - 2\int \frac{\sin\theta}{1+\sin\theta}\,\mathrm{d}\theta$$

$$= 2\int \csc\theta\,\mathrm{d}\theta - 2\int \frac{1-\sin\theta}{\cos^2\theta}\,\mathrm{d}\theta - 2\int \frac{\sin\theta - \sin^2\theta}{\cos^2\theta}\,\mathrm{d}\theta$$

$$= -2\ln|\csc\theta + \cot\theta| - 2\int \left(\sec^2\theta - \tan\theta\sec\theta\right)\mathrm{d}\theta - 2\int \left(\tan\theta\sec\theta - \tan^2\theta\right)\mathrm{d}\theta$$

$$= -2\ln|\csc\theta + \cot\theta| - 2\tan\theta + 2\sec\theta - 2\sec\theta + 2\tan\theta - 2\theta$$

$$= -2\ln|\csc\theta + \cot\theta| - 2\theta = -2\ln\left|\frac{1}{u} + \frac{\sqrt{1-u^2}}{u}\right| - 2\arcsin u$$

$$= \boxed{-2\ln\left|\frac{1+\sqrt{1-x}}{\sqrt{x}}\right| - 2\arcsin\sqrt{x} + C}$$

Problem 3

$$\int \frac{1}{1+\sin x}\,\mathrm{d}x = \tan\left(\frac{x}{2}+a\right) + b$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int \frac{1}{1+\sin x}\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}x}\left[\tan\left(\frac{x}{2}+a\right) + b\right]$$

$$\frac{1}{1+\sin x} = \sec^2\left(\frac{x}{2}+a\right) \cdot \frac{1}{2} = \frac{1}{\left(\cos\frac{x}{2}\cos a - \sin\frac{x}{2}\sin a\right)^2} \cdot \frac{1}{2}$$

$$= \left(\cos^2\left(\frac{x}{2}\right)\cos^2(a) - 2\cos\left(\frac{x}{2}\right)\cos(a)\sin\left(\frac{x}{2}\right)\sin(a) + \sin^2\left(\frac{x}{2}\right)\sin^2(a)\right)^{-1} \cdot \frac{1}{2}$$

$$1+\sin x = 2\cos^2\left(\frac{x}{2}\right)\cos^2(a) + 2\sin^2\left(\frac{x}{2}\right)\sin^2(a) - 2\sin(x)\cos(a)\sin(a)$$

$$
\begin{aligned}
2\cos^2\left(\frac{x}{2}\right)\cos^2(a) + 2\sin^2\left(\frac{x}{2}\right)\sin^2(a) &= 1\\
\cos^2\left(\frac{x}{2}\right)\left(1-\sin^2 a\right) + \left(1-\cos^2\left(\frac{x}{2}\right)\right)\sin^2 a &= \frac{1}{2}\\
\cos^2\left(\frac{x}{2}\right) - 2\cos^2\left(\frac{x}{2}\right)\sin^2 a + \sin^2 a &= \frac{1}{2}\\
\cos^2\left(\frac{x}{2}\right)\left(1-2\sin^2 a\right) + \sin^2 a &= \frac{1}{2}
\end{aligned}
$$

LHS is constant and always equals 1/2.

$$
\begin{cases} 1-2\sin^2 a = 0 \\ \sin^2 a = \frac{1}{2} \end{cases} \implies \begin{cases} \sin^2 a = \frac{1}{2} \\ \sin^2 a = \frac{1}{2} \end{cases}
$$

$$
\sin^2 a = \frac{1}{2} \implies \sin a = \pm\frac{\sqrt{2}}{2}
$$

$$
\begin{aligned}
-2\sin(x)\cos(a)\sin(a) &= \sin x\\
\cos(a)\sin(a) &= -\frac{1}{2}\\
\pm\frac{\sqrt{2}}{2}\cos a &= -\frac{1}{2}\\
\cos a &= \mp\frac{\sqrt{2}}{2}
\end{aligned}
$$

So a is in Quadrant 2 or 4, and b is the constant of integration.

$$
\boxed{a = \frac{3\pi}{4} + \pi n, b \in \mathbb{R}}
$$

Problem 4

(a) $\displaystyle\int_0^a f(a-x)\,\mathrm{d}x = \begin{bmatrix} u = a - x \\ \mathrm{d}u = -\mathrm{d}x \end{bmatrix} \int_a^0 (-f(u))\,\mathrm{d}u = \int_0^a f(u)\,\mathrm{d}u = \int_0^x f(x)\,\mathrm{d}x$