# Problem Set #35

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## Problem 3

(a) $$\frac{\mathrm{d}y}{\mathrm{d}t} = ky\left(1 - \frac{y}{K}\right) \implies y(0) = \frac{K}{1 + Ae^{-kt}} = \frac{8 \times 10^7}{1 + \frac{8\times10^7 - 2\times10^7}{2\times10^7}e^{-0.71t}} = \frac{8 \times 10^7}{1 + 3e^{-0.71t}}$$
$$\implies y(1) = \frac{8 \times 10^7}{1 + 3e^{-0.71}}$$

Biomass the following year is approximately $3.2 \times 10^7$ kg.

(b) $$\frac{8 \times 10^7}{1 + 3e^{-0.71t}} = 4 \times 10^7 \implies 1 + 3e^{-0.71t} = 2 \implies -0.71t = \ln\frac{1}{3} \implies t = 1.547$$

It will take approximately 1.547 years for the biomass to reach $4 \times 10^7$ kg.

## Problem 4

(a,b) *I'll derive everything again, because why not*

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ky(1-y) \implies \frac{1}{ky(1-y)}\frac{\mathrm{d}y}{\mathrm{d}t} = 1 \implies \int \frac{1}{ky(1-y)}\,\mathrm{d}y \implies \int \left(\frac{A}{ky} + \frac{B}{1-y}\right)\mathrm{d}y = t + C$$

$$\boxed{\begin{aligned} &\frac{A}{ky} + \frac{B}{1-y} = \frac{1}{ky(1-y)} \implies A - Ay + Bky = 1 \implies \begin{cases} A = 1 \\ -A + Bk = 0 \end{cases} \implies A = 1 \\ &\implies -1 + Bk = 0 \implies B = 1/k \end{aligned}}$$

$$\implies \int \left(\frac{1}{ky} + \frac{\frac{1}{k}}{1-y}\right)\mathrm{d}y = \frac{1}{k}\int\left(\frac{1}{y} + \frac{1}{1-y}\right)\mathrm{d}y = \frac{1}{k}\left(\ln|y| - \ln|1-y|\right)$$
$$\implies \exp(\ln|y| - \ln|1-y|) = \exp(kt + C) \implies \frac{|y|}{|1-y|} = Ce^{kt}$$
$$\implies y = Ce^{kt} - Cye^{kt} \implies y\left(1 + Ce^{kt}\right) = Ce^{kt} \implies y = \frac{Ce^{kt}}{1 + Ce^{kt}}$$
$$\implies y = \frac{\cancel{Ce^{kt}}}{\cancel{Ce^{kt}}\left(Ce^{-kt} + 1\right)} \implies \boxed{y = \frac{1}{1 + Ce^{-kt}}}$$

where $C$ is a constant determined by the size of the population and the size of the original rumor.

(c) Let $t$ denote the number of hours since 8am. Then the time at noon is $t = 4$.

$$\frac{1}{1 + Ce^{-k\cdot 0}} = \frac{80}{1000} \implies \frac{1}{1 + C} = \frac{2}{25} \implies 2 + 2C = 25 \implies 2C = 23 \implies C = \frac{23}{2}$$
$$\frac{1}{1 + Ce^{-k\cdot 4}} = \frac{1}{2} \implies \frac{1}{1 + \frac{23}{2}e^{-4k}} = \frac{1}{2} \implies 2 + 23e^{-4k} = 4 \implies e^{-4k} = \frac{2}{23}$$
$$\implies -4k = \ln\frac{2}{23} \implies k = -\frac{1}{4}\ln\frac{2}{23}$$

$$\frac{1}{1 + \frac{23}{2}\exp\left(\frac{1}{4}t\ln\frac{2}{23}\right)} = \frac{9}{10} \implies 9 + \frac{207}{2}\exp\left(\frac{t}{4}\ln\frac{2}{23}\right) = 10 \implies \exp\left(\ln\frac{2}{23}\right)^{t/4} = \frac{2}{207}$$
$$\implies \left(\frac{2}{23}\right)^{t/4} = \frac{2}{207} \implies \frac{t}{4} = \log_{2/23}\left(\frac{2}{207}\right) \implies t \approx 7.599$$

After 7.599 hours, or 7 hours and 36 minutes, 90% of the town would have heard the rumor.

## Problem 5

(a) $$\frac{\mathrm{d}P}{\mathrm{d}t} = kP\left(1 - \frac{P}{K}\right) \implies \frac{\mathrm{d}^2P}{\mathrm{d}t^2} = k\left(1 - \frac{P}{K}\right)\frac{\mathrm{d}P}{\mathrm{d}t} + kP\left(-\frac{1}{K}\frac{\mathrm{d}P}{\mathrm{d}t}\right)$$
$$= kP\left(1 - \frac{P}{K}\right)\left(k\left(1 - \frac{P}{K}\right) - \frac{kP}{K}\right)$$
$$= k^2P\left(1 - \frac{P}{K}\right)\left(1 - \frac{P}{K} - \frac{P}{K}\right) = k^2P\left(1 - \frac{P}{K}\right)\left(1 - \frac{2P}{k}\right)$$

(b) At the time $t$ when population grows the fastest, $P''(t) = 0$.

$$\left.\frac{\mathrm{d}^2P}{\mathrm{d}t^2}\right|_{P=k/2} = k^2 \cdot \frac{k}{2}\left(1 - \frac{\frac{k}{2}}{K}\right)\left(1 - \frac{2\cdot\frac{k}{2}}{k}\right) = \frac{k^3}{2}\left(1 - \frac{k}{2K}\right)\left(1 - \frac{k}{k}\right) = \frac{k^3}{2}\left(1 - \frac{k}{2K}\right)\cdot 0 = 0$$

## Problem 6

(a) $$\frac{\mathrm{d}P}{\mathrm{d}t} = kP - m \implies \int \frac{1}{kP - m}\,\mathrm{d}P = \int \mathrm{d}t \implies \frac{1}{k}\ln|kP - m| = t + C$$
$$\implies \exp(\ln|kP - m|) = \exp(kt + C) \implies kP - m = Ce^{kt} \implies P = \frac{Ce^{kt} + m}{k}$$
$$P(0) = P_0 \implies Ce^{k\cdot 0} + m = kP_0 \implies C = kP_0 - m \implies \boxed{P(t) = \frac{(kP_0 - m)\,e^{kt} + m}{k}}$$

(b) When $kP_0 - m > 0$ or $\boxed{m < kP_0}$.

(c) Constant population when $\boxed{m = kP_0}$ and declining if $\boxed{m > kP_0}$.

(d) $kP_0 = 0.016 \times 8000000 = 128000 < 210000 = m$ so $m > kP_0$ and the population is declining.

## Problem 7

(a) 15 fish leave the population "unnaturally" (possibly by being caught) every week.

(b)

(c)

(d)

(e) $$\frac{\mathrm{d}P}{\mathrm{d}t} = 0.08P\left(1 - \frac{P}{1000}\right) - 15 \implies \int \frac{1}{0.08P\left(1 - \frac{P}{1000}\right) - 15}\,\mathrm{d}P = \int \mathrm{d}t = t + C$$
$$\text{(by Wolfram Alpha)} \implies t + C = 25\ln(250 - P) - 25\ln(750 - P) = 25\ln\left(\frac{250 - P}{750 - P}\right)$$
$$\implies \exp\left(\ln\frac{250 - P}{750 - P}\right) = \exp\left(\frac{t}{25} + C\right) \implies \frac{250 - P}{750 - P} = Ce^{t/25}$$
$$\implies 250 - P = 750Ce^{t/25} - CPe^{t/25}$$
$$\implies P\left(Ce^{t/25} - 1\right) = 750Ce^{t/25} - 250 \implies P = \frac{750Ce^{t/25} - 250}{Ce^{t/25} - 1}$$
$$P(0) = 200 \implies \frac{750Ce^{0/25} - 250}{Ce^{0/25} - 1} = 200 \implies \frac{750C - 250}{C - 1} = 200$$
$$\implies 750C - 250 = 200C - 200 \implies 550C = 50 \implies C = \frac{1}{11}$$
$$\implies P = \frac{\frac{750}{11}e^{t/25} - 250}{\frac{1}{11}e^{t/25} - 1} = \boxed{P = \frac{750e^{t/25} - 2750}{e^{t/25} - 11}}$$
$$P(0) = 300 \implies \frac{750Ce^{0/25} - 250}{Ce^{0/25} - 1} = 300 \implies \frac{750C - 250}{C - 1} = 300$$
$$\implies 750C - 250 = 300C - 300 \implies 450C = -50 \implies C = -\frac{1}{9}$$
$$\implies P = \frac{-\frac{750}{9}e^{t/25} - 250}{-\frac{1}{9}e^{t/25} - 1} = \boxed{P = \frac{750e^{t/25} + 2250}{e^{t/25} + 9}}$$