Problem Set #55

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Problem 2

(a) Yes.

Proof. Let $f(x) = \dfrac{|x|}{x}$ and $g(x) = -\dfrac{|x|}{x}$. Then $f(x) + g(x) = \dfrac{|x|}{x} + \left(-\dfrac{|x|}{x}\right) = 0$ for all $x \neq 0$.

$$\lim_{x\to 0^+} f(x) = 1,\ \lim_{x\to 0^-} f(x) = -1 \implies \lim_{x\to 0} f(x)\,\text{DNE}$$

$$\lim_{x\to 0^+} g(x) = -1,\ \lim_{x\to 0^+} g(x) = 1 \implies \lim_{x\to 0} g(x)\,\text{DNE}$$

$$\lim_{x\to 0} [f(x) + g(x)] = \lim_{x\to 0} 0 = 0$$

☺

(b) Yes.

Proof. Let $L = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a} f(x)$. Because $\lim_{x\to a} [f(x) + g(x)]$ exists:

$$\lim_{x\to a^+} [f(x) + g(x)] = \lim_{x\to a^-} [f(x) + g(x)] = \lim_{x\to a} [f(x) + g(x)]$$

$$\lim_{x\to a^+} f(x) + \lim_{x\to a^+} g(x) = \lim_{x\to a^-} f(x) + \lim_{x\to a^-} g(x) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$$

$$L + \lim_{x\to a^+} g(x) = L + \lim_{x\to a^-} g(x) = L + \lim_{x\to a} g(x)$$

$$\lim_{x\to a^+} g(x) = \lim_{x\to a^-} g(x) = \lim_{x\to a} g(x)$$

Therefore $\lim_{x\to a} g(x)$ must exist. ☺

(c) No.

Proof. Let $L = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \lim_{x\to a} f(x)$. Because $\lim_{x\to a} g(x)$ DNE:

$$\lim_{x\to a^+} g(x) \neq \lim_{x\to a^-} g(x)$$

$$L + \lim_{x\to a^+} g(x) \neq L + \lim_{x\to a^-} g(x)$$

$$\lim_{x\to a^+} f(x) + \lim_{x\to a^+} g(x) \neq \lim_{x\to a^-} f(x) + \lim_{x\to a^-} g(x)$$

$$\lim_{x\to a^+} [f(x) + g(x)] \neq \lim_{x\to a^-} [f(x) + g(x)]$$

Therefore $\lim_{x\to a}[f(x)+g(x)]$ DNE. ☺

Problem 3

(b)
$$\begin{aligned}
&\lim_{x\to 0}\frac{\tan^2 x+2x}{x+x^2}\\
&=\lim_{x\to 0}\frac{\sin^2 x+2x\cos^2 x}{x^2\left(\frac{1}{x}+1\right)\cos^2 x}\\
&=\lim_{x\to 0}\left[\frac{\sin^2 x}{x^2\left(\frac{1}{x}+1\right)\cos^2 x}+\frac{2\cancel{x}\cancel{\cos^2 x}}{x^2\left(\frac{1}{x}+1\right)\cancel{\cos^2 x}}\right]\\
&=\lim_{x\to 0}\left[\frac{\sin^2 x}{x^2}\cdot\frac{1}{\left(\frac{1}{x}+1\right)\cos^2 x}\right]+\lim_{x\to 0}\frac{2}{1+x}\\
&=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)^2\lim_{x\to 0}\frac{1}{\frac{(1+x)\cos^2 x}{x}}\lim_{x\to 0}\frac{1}{\cos^2 x}+2\\
&=1^2\cdot\lim_{x\to 0}\left[\frac{x}{(1+x)\cos^2 x}\right]\cdot\frac{1}{1^2}+2\\
&=\boxed{2}
\end{aligned}$$

(d)
$$\begin{aligned}
&\lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\
&=\lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\
&=\lim_{h\to 0}\frac{\sin(x)(\cos h-1)+\cos x\sin h}{h}\\
&=\lim_{h\to 0}\frac{\sin(x)(\cos h-1)}{h}+\lim_{h\to 0}\frac{\cos x\sin h}{h}\\
&=\lim_{h\to 0}\frac{\cos h-1}{h}\lim_{h\to 0}\sin x+\lim_{h\to 0}\frac{\sin h}{h}\lim_{h\to 0}\cos x\\
&=0\cdot\sin x+1\cdot\cos x\\
&=\boxed{\cos x}
\end{aligned}$$

Problem 4

(b)
$$\begin{aligned}
&\lim_{x\to\infty}x\sin\left(\frac{1}{x}\right)\\
&=\lim_{x\to\infty}\left[\frac{x\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}\cdot\frac{1}{x}\right]\\
&=\lim_{x\to\infty}\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}\lim_{x\to\infty}\left[x\cdot\frac{1}{x}\right]
\end{aligned}$$

Let $y=\frac{1}{x}$. As $x\to\infty$, $y\to 0$.

$$\begin{aligned}
&=\lim_{y\to 0}\frac{\sin(y)}{y}\lim_{x\to\infty}1\\
&=\boxed{1}
\end{aligned}$$

(d)
$$\begin{aligned}
&\lim_{x\to\infty}\frac{x\sin x}{x^2+5}\\
&=\lim_{x\to\infty}\frac{\cancel{x^2}\left(\frac{\sin x}{x^2}\right)}{\cancel{x^2}\left(1+\frac{5}{x^2}\right)}\\
&=\frac{\lim_{x\to\infty}\frac{\sin x}{x}\lim_{x\to\infty}\frac{1}{x}}{\lim_{x\to\infty}1+\lim_{x\to\infty}\frac{5}{x^2}}\\
&=\frac{0\cdot 0}{1+0}\\
&=\boxed{0}
\end{aligned}$$

Problem 5

(b) $\lim_{x\to\infty} x(1+\sin^2 x)$

$= \lim_{x\to\infty} x + \lim_{x\to\infty} x\sin^2 x$

$\lim_{x\to\infty} x\sin^2 x$ oscillates on the interval $[0,\infty)$.

$= \boxed{\infty}$

(d) $\lim_{x\to\infty} x^2 \sin\left(\frac{1}{x}\right)$

$= \lim_{x\to\infty} \left[\frac{x^2\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \cdot \frac{1}{x}\right]$

$= \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} \lim_{x\to\infty} \left[x^2 \cdot \frac{1}{x}\right]$

Let $y = \frac{1}{x}$. As $x \to \infty$, $y \to 0$.

$= \lim_{y\to 0} \frac{\sin(y)}{y} \lim_{x\to\infty} x$

$= \boxed{\infty}$

(e) $\lim_{x\to\infty} \left[\sqrt{x^2+2x} - x\right]$

$= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+2x}-x\right)\left(\sqrt{x^2+2x}+x\right)}{\sqrt{x^2+2x}+x}$

$= \lim_{x\to\infty} \frac{x^2+2x-x^2}{\sqrt{x^2}\sqrt{1+\frac{2}{x}}+x}$

$= \lim_{x\to\infty} \frac{2x}{|x|\sqrt{1+\frac{2}{x}}+x}$

Because $x\to\infty$, x is positive.

$= \lim_{x\to\infty} \frac{2\cancel{x}}{\cancel{x}\sqrt{1+\frac{2}{x}}+\cancel{x}^{1}}$

$= \lim_{x\to\infty} \frac{2}{\sqrt{1+0}+1}$

$= \boxed{1}$

(f) $\lim_{x\to\infty} \left[x(\sqrt{x+2}-\sqrt{x})\right]$

$= \lim_{x\to\infty} \frac{x(\sqrt{x+2}-\sqrt{x})(\sqrt{x+2}+\sqrt{x})}{\sqrt{x+2}+\sqrt{x}}$

$= \lim_{x\to\infty} \frac{x(x+2-x)}{\sqrt{x+2}+\sqrt{x}}$

$= \lim_{x\to\infty} \frac{2x}{\sqrt{x+2}+\sqrt{x}}$

$= \lim_{x\to\infty} \frac{2x}{\sqrt{x^2\left(\frac{1}{x}+\frac{2}{x^2}\right)}+\sqrt{x^2\left(\frac{1}{x}\right)}}$

$= \lim_{x\to\infty} \frac{2x}{|x|\left(\sqrt{\frac{1}{x}+\frac{2}{x^2}}+\sqrt{\frac{1}{x}}\right)}$

x is positive. As $x\to\infty$, $\frac{1}{x}$ and $\frac{2}{x^2}$ are positive and approach 0 but do not equal 0.

$= \lim_{x\to\infty} \frac{2}{\sqrt{\frac{1}{x}+\frac{2}{x^2}}+\sqrt{\frac{1}{x}}}$

The denominator is positive and approaches 0. A positive real number divided by a positive number approaching 0 is ∞.

$= \boxed{\infty}$

(g) $\lim_{x\to\infty} \frac{\sqrt{|x|}}{x}$

x is positive.

$= \lim_{x\to\infty} \frac{\sqrt{x}}{|x|}$

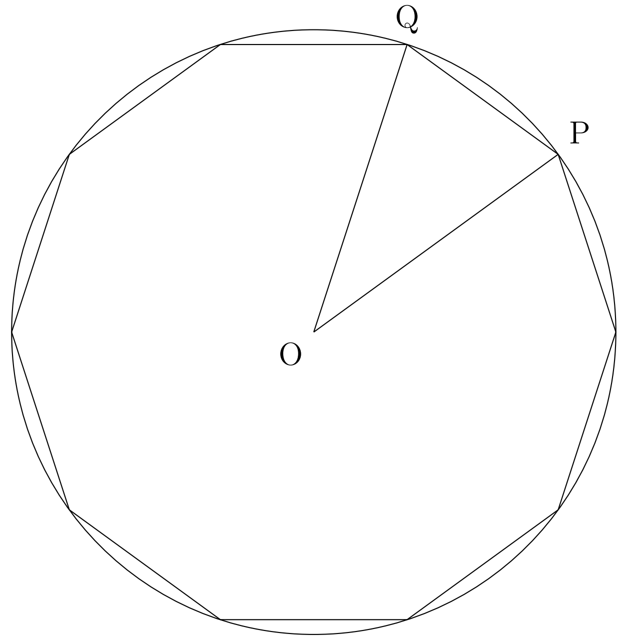
$= \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x^2}}$

$= \lim_{x\to\infty} \sqrt{\frac{1}{x}}$

$= \boxed{0}$

Problem 6

(a)



$$\angle QOP = \frac{2\pi}{n}$$
$$\angle QPO = \angle PQO = \frac{1}{2}\left(180^\circ - \frac{2\pi}{n}\right)$$
$$= \frac{\pi}{2} - \frac{\pi}{n}$$

By the law of sines:

$$\frac{\overline{PQ}}{\sin \angle QOP} = \frac{\overline{OP}}{\sin \angle PQO}$$
$$\overline{PQ} \sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right) = r \sin \frac{2\pi}{n}$$
$$\overline{PQ} \cancel{\cos \frac{\pi}{n}} = r\left(2 \sin \frac{\pi}{n} \cancel{\cos \frac{\pi}{n}}\right)$$
$$\overline{PQ} = 2r \sin \frac{\pi}{n}$$

Perimeter $= \boxed{2nr \sin \frac{\pi}{n}}$

(b)

$$\lim_{n\to\infty}\left[2nr \sin \frac{\pi}{n}\right]$$
$$= \lim_{n\to\infty}\left[\frac{2nr \sin \frac{\pi}{n}}{\frac{\pi}{n}} \cdot \frac{\pi}{n}\right]$$
$$= \lim_{n\to\infty} \frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}} \cdot \lim_{n\to\infty}\left[\frac{\pi}{n} \cdot 2nr\right]$$

Let $m = \frac{\pi}{n}$. As $n \to \infty$, $m \to 0$.

$$= \lim_{m\to 0} \frac{\sin m}{m} \cdot \lim_{x\to\infty} 2\pi r$$
$$= 1 \cdot 2\pi r$$
$$= \boxed{2\pi r}$$

(c) Perimeter of a circle