Problem Set #60

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Problem 1

(a) $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\pi/6} = \left.\dfrac{2\cos\theta\sin\theta + 2\sin\theta\cos\theta}{2\cos^2\theta - 2\sin^2\theta}\right|_{\theta=\pi/6} = \dfrac{\frac{\sqrt{3}}{2}\frac{1}{2}\cdot 2}{\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^2} = \dfrac{\frac{\sqrt{3}}{2}}{\frac{3}{4}-\frac{1}{4}} = \sqrt{3}$

$$x_0 = r\cos\frac{\pi}{6} = 2\sin\frac{\pi}{6}\cos\frac{\pi}{6} = 2\frac{\sqrt{3}}{2}\frac{1}{2} = \frac{\sqrt{3}}{2}$$
$$y_0 = r\sin\frac{\pi}{6} = 2\sin\frac{\pi}{6}\sin\frac{\pi}{6} = 2\frac{1}{2}\frac{1}{2} = \frac{1}{2}$$
$$\boxed{y - \frac{1}{2} = \sqrt{3}\left(x - \frac{\sqrt{3}}{2}\right)}$$

(b) $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\pi} = \left.\dfrac{-\frac{1}{\theta^2}\sin\theta + \frac{1}{\theta}\cos\theta}{-\frac{1}{\theta^2}\cos\theta - \frac{1}{\theta}\sin\theta}\right|_{\theta=\pi} = \dfrac{-\frac{1}{\pi^2}\cdot 0 + \frac{1}{\pi}(-1)}{-\frac{1}{\pi^2}(-1) - \frac{1}{\pi}\cdot 0} = \dfrac{-\frac{1}{\pi}}{\frac{1}{\pi^2}}\cdot\dfrac{\pi^2}{\pi^2} = -\pi$

$$x_0 = r\cos\pi = \frac{1}{\pi}\cos\pi = \frac{1}{\pi}(-1) = -\frac{1}{\pi}$$
$$y_0 = r\sin\pi = \frac{1}{\pi}\cdot 0 = 0$$
$$\boxed{y = -\pi\left(x + \frac{1}{\pi}\right)}$$

(c) $\left.\dfrac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\pi/4} = \left.\dfrac{-2\sin 2\theta\sin\theta + \cos 2\theta\cos\theta}{-2\sin 2\theta\cos\theta - \cos 2\theta\sin\theta}\right|_{\theta=\pi/4} = \dfrac{-2\cdot 1\cdot\frac{\sqrt{2}}{2} + 0}{-2\cdot 1\cdot\frac{\sqrt{2}}{2} - 0} = 1$

$$x_0 = r\cos\frac{\pi}{4} = \cos\frac{\pi}{2}\cos\frac{\pi}{4} = 0$$
$$y_0 = r\sin\frac{\pi}{4} = \cos\frac{\pi}{2}\sin\frac{\pi}{4} = 0$$
$$\boxed{y = x}$$

Problem 2

For some polar graph $r = f(\theta)$, we have:

$$x(\theta) = r\cos(\theta) = f(\theta)\cos(\theta)$$
$$y(\theta) = r\sin(\theta) = f(\theta)\sin(\theta)$$

From here, we can calculate $\mathrm{d}y/\mathrm{d}x$ and $\mathrm{d}x/\mathrm{d}y$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\sin\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$
$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\mathrm{d}x/\mathrm{d}\theta}{\mathrm{d}y/\mathrm{d}\theta} = \frac{f'(\theta)\cos\theta - f(\theta)\sin\theta}{f'(\theta)\sin\theta + f(\theta)\sin\theta} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1}$$

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-3\sin\theta\sin\theta + 3\cos\theta\cos\theta}{-3\sin\theta\cos\theta - 3\cos\theta\sin\theta} = \dfrac{-\sin^2\theta + \cos^2\theta}{-\sin\theta\cos\theta - \cos\theta\sin\theta} = \dfrac{\sin^2\theta - \cos^2\theta}{2\sin\theta\cos\theta} = 0$

$$\implies \sin^2\theta - \cos^2\theta = 0 \implies \sin^2\theta = \cos^2\theta \implies \cos\theta = \sin\theta = \pm\frac{\sqrt{2}}{2}$$
$$\implies x = r\cos\theta \implies x = 3\cos\theta\cos\theta = 3\cos^2\theta = 3\frac{1}{2} = \frac{3}{2}$$
$$\implies y = r\sin\theta \implies y = 3\cos\theta\sin\theta = \pm\frac{3}{2}$$

Horizontal tangent line: $\left(\dfrac{3}{2}, \pm\dfrac{3}{2}\right)$.

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{2\sin\theta\cos\theta}{\sin^2\theta - \cos^2\theta} = 0 \implies \theta = \frac{n\pi}{2}, n \in \mathbb{Z}$$

Notice: $\cos\theta, \sin\theta \in \{0, 1\}$.

$$\implies x = r\cos\theta = 3\cos^2\theta \in \{0, 1\}$$
$$\implies y = r\sin\theta = 3\cos\theta\sin\theta = 0$$

Vertical tangent line: $(0, 0), (1, 0)$.

(b)

(c)

Problem 3

$$r = a\sin\theta + b\cos\theta$$
$$r^2 = ar\sin\theta + br\cos\theta$$
$$x^2 + y^2 = ay + bx$$
$$x^2 - bx + y^2 - ay = 0$$
$$\left(x - \frac{b}{2}\right)^2 - \frac{b^2}{4} + \left(y - \frac{a}{2}\right)^2 - \frac{a^2}{4} = 0$$
$$\left(x - \frac{b}{2}\right)^2 + \left(y - \frac{a}{2}\right)^2 = \frac{b^2}{4} + \frac{a^2}{4}$$

Center: $\left(\dfrac{b}{2}, \dfrac{a}{2}\right)$. Radius: $\dfrac{1}{2}\sqrt{a^2 + b^2}$.

Problem 4