Problem Set #50

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Problem 1

(a) Center of convergence is $x = 0$.

$$L < 1 \iff \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{|x|^{n+1}}{\sqrt{n+1}}\frac{\sqrt{n}}{|x|^n}\right| = \lim_{n\to\infty}\frac{|x|\sqrt{n}}{\sqrt{n+1}} = \lim_{n\to\infty}\frac{|x|\sqrt{n}}{\sqrt{n}\sqrt{1+\frac{1}{n}}}$$
$$= \lim_{n\to\infty}\frac{|x|}{\sqrt{1+\cancelto{0}{1/n}}} = |x| < 1$$

$x = 1 : \sum a_n = \sum \frac{1}{n^{1/2}}$ diverges by the p-series test: $p = 1/2 \le 1$.

$x = -1 : \sum a_n = \sum \frac{(-1)^n}{n^{1/2}}$ converges by the AST since $\lim\limits_{n\to\infty} a_n = 0$ and $|a_n|$ is decreasing.

Radius of convergence: 1, Interval of convergence: $[0 + (-1), 0 + 1) = \boxed{[-1, 1)}$.

(b) Center of convergence is $x = 0$.

$$L < 1 \iff \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{(n+1)^3}\frac{n^3}{(-1)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)xn^3}{(n+1)^3}\right|$$
$$= \lim_{n\to\infty}|x|\frac{n^3}{n^3+3n^2+3n+1} = \lim_{n\to\infty}|x|\frac{n^3}{n^3\left(1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}\right)} = |x| < 1$$

$x = 1 : \sum a_n = \sum \frac{(-1)^n}{n^3}$ converges by the absolute convergence test.

$x = -1 : \sum a_n = \sum \frac{(-1)^n(-1)^n}{n^3} = \sum \frac{1}{n^3}$ converges by the p-series test.

Radius of convergence: 1, Interval of convergence: $\boxed{[-1, 1]}$.

(c) Center of convergence is $x = 0$.

$$L < 1 \iff \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0 < 1$$

Interval of convergence: $\boxed{(-\infty, \infty)}$.

(d) Center of convergence is $x = 0$.

$$L < 1 \iff \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)^2x^{n+1}}{2^{n+1}}\frac{2^n}{(-1)^n n^2 x^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)(n+1)^2x}{2n^2}\right|$$
$$= \lim_{n\to\infty}\frac{|x|(n+1)^2}{2n^2} = \frac{|x|}{2} < 1 \iff |x| < 2$$

$x = 2 : \sum a_n = \sum (-1)^n n^2$ diverges by the nth term divergence test.

$x = -1 : \sum a_n = \sum \frac{(-1)^n n^2(-2)^n}{2^n} = \sum \frac{n^2 2^n}{2^n}$ diverges by the nth term divergence test.

Radius of convergence: 2, Interval of convergence: $\boxed{(-2, 2)}$.

(e) Center of convergence is $x = 0$.

$$L < 1 \iff \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}x^{n+1}}{\sqrt[4]{n+1}}\frac{\sqrt[4]{n}}{(-2)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{(-2)x\sqrt[4]{n}}{\sqrt[4]{n+1}}\right|$$
$$= 2\lim_{n\to\infty}|x|\frac{n^{1/4}}{(n+1)^{1/4}} = 2|x| < 1 \iff x < \frac{1}{2}$$

$x = \frac{1}{2} : \sum a_n = \sum \frac{(-2)^n\left(\frac{1}{2}\right)^n}{\sqrt[4]{n}} = \sum \frac{(-1)^n}{\sqrt[4]{n}}$ converges by the alternating series test since $\lim_{n\to\infty} a_n = 0$ and $|a_n$ is decreasing.

$x = -\frac{1}{2} : \sum a_n = \sum \frac{(-2)^n\left(-\frac{1}{2}\right)^n}{\sqrt[4]{n}} = \sum \frac{1}{\sqrt[4]{n}}$ diverges by the p-series test: $p = 1/4 \le 1$.

Radius of convergence: 1/2, Interval of convergence: $\boxed{(-1/2, 1/2]}$.

(f) Center of convergence: $x = 0$.

$$L < 1 \iff \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{4^{n+1}\ln(n+1)}\frac{4^n\ln n}{(-1)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)x\ln n}{4\ln(n+1)}\right|$$
$$= \frac{1}{4}\lim_{n\to\infty}|x|\frac{\ln n}{\ln(n+1)} = \frac{|x|}{4} < 1 \iff |x| < 4$$

$x = 4 : \sum \frac{(-1)^n 4^n}{4^n \ln n} = \sum \frac{(-1)^n}{\ln n}$ converges by the alternating series test since $\lim_{n\to\infty} a_n = 0$ and $|a_n|$ is decreasing when $x = 4$.

$x = -4 : \sum \frac{(-1)^n(-4)^n}{4^n \ln n} = \frac{1}{\ln n}$ diverges by comparison to $\frac{1}{n}$.

Radius of convergence: 4, Interval of convergence: $\boxed{(-4, 4]}$.

(g) Center of convergence: $x = 2$.

$$L < 1 \iff \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^2+1}\frac{n^2+1}{(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)\left(n^2+1\right)}{(n+1)^2+1}\right|$$
$$= \lim_{n\to\infty}|x-2|\frac{n^2+1}{n^2+2n+2} = |x-2| < 1$$

$x = 3 : \sum \frac{1}{n^2+1}$ converges by comparison to $\frac{1}{n^2}$.

$x = 1 : \sum \frac{(-1)^n}{n^2+1}$ converges by the absolute convergence test.

Radius of convergence: 1, Interval of convergence: $\boxed{[1, 3]}$.

(h) Center of convergence: $x = -4$.

$$L < 1 \iff \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}(x+4)^{n+1}}{\sqrt{n+1}}\frac{\sqrt{n}}{3^n(x+4)^n}\right| = \lim_{n\to\infty}\frac{|3(x+4)|\sqrt{n}}{\sqrt{n+1}}$$
$$= |3(x+4)| < 1 \iff |x+4| < \frac{1}{3}$$

$x = -\frac{11}{3} : \sum \frac{3^n\left(\frac{1}{3}\right)^n}{\sqrt{n}} = \sum \frac{1}{n^{1/2}}$ diverges by the p-series test.

$x = -\frac{13}{3} : \sum \frac{3^n\left(-\frac{1}{3}\right)^n}{\sqrt{n}} = \sum \frac{(-1)^n}{\sqrt{n}}$ converges by the alternating series test.

Radius of convergence: 1/2, Interval of convergence: $\boxed{[-13/3, -11/3)}$.

(i) Center of convergence: $x = 2$.

$$L < 1 \iff \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(n+1)^{n+1}}\frac{n^n}{(x-2)^n}\right| = \lim_{n\to\infty}|x-2|\frac{n^n}{(n+1)^n}\frac{1}{(n+1)^n}$$
$$= \lim_{n\to\infty}|x-2|\frac{1}{(n+1)^n} = 0 < 1$$

Interval of convergence: $\boxed{(-\infty, \infty)}$.

(j) Center of convergence: $x = a$.

$$L < 1 \iff \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-a)^{n+1}}{b^{n+1}}\frac{b^n}{n(x-a)^n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-a)}{bn}\right|$$
$$= \lim_{n\to\infty}\left|\frac{n+1}{n}\frac{x-a}{b}\right| = \left|\frac{x-a}{b}\right| < 1 \iff |x-a| < b$$

$x = a + b : \sum \frac{n}{b^n}(b+a-a)^n = \sum \frac{n}{b^n}b^n = \sum n$ diverges by the nth term divergence test.

$x = a - b : \sum \frac{n}{b^n}(a-b+a)^n = \sum (-1)^n n$ diverges by the nth term divergence test.

Radius of convergence: b, Interval of convergence: $\boxed{(a-b, a+b)}$.

(k) Center of convergence: $x = 1/2$.

$$L < 1 \iff \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!(2x-1)^{n+1}}{n!(2x-1)^n}\right| = \lim_{n\to\infty}(n+1)\,|2x-1| = \infty < 1$$

This is clearly not possible.

$x = \frac{1}{2} : \sum n!0^n = \sum 0$ is convergent.

Radius of convergence: 0, Interval of convergence: $\boxed{[1/2, 1/2]}$.

(l) Center of convergence: $x = -1/4$.

$$L < 1 \iff \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(4x+1)^{n+1}}{(n+1)^2}\frac{n^2}{(4x+1)^n}\right| = \lim_{n\to\infty}|4x+1|\frac{n^2}{(n+1)^2}$$
$$= |4x+1| < 1$$

$x = 0 : \sum \frac{1}{n^2}$ is convergent by the p-series test.

$x = -1/2 : \sum \frac{(-1)^n}{n^2}$ is convergent by the absolute convergence test.

Radius of convergence: 1/4, Interval of convergence: $\boxed{[-1/2, 0]}$.

(m) Center of convergence: $x = 0$.

$$L < 1 \iff \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)(2n+1)}\frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{x^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{x}{2n+1}\right| = 0 < 1$$

Interval of convergence: $\boxed{(-\infty, \infty)}$.

Problem 2

By the ratio test, if $\sum\limits_{n=0}^{\infty} c_n 4^n$ is convergent and $L = \lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \le 1$. (*If $L < 1$, then the series must converge, and if $L > 1$, the series must diverge. But if $L = 1$ then the test is inconclusive, so L could be 1 even when series is convergent.*) Therefore:

$$L = \lim_{n\to\infty}\left|\frac{c_{n+1}4^{n+1}}{c_n 4^n}\right| = \lim_{n\to\infty} 4\left|\frac{c_{n+1}}{c_n}\right| \le 1 \implies \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| \le \frac{1}{4}$$

(a)

$$L = \lim_{n\to\infty}\left|\frac{c_{n+1}(-2)^{n+1}}{c_n(-2)^n}\right| = 2\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| \le 2\cdot\frac{1}{4} = \frac{1}{2}$$

Therefore the series converges by the ratio test as $L = 1/2 < 1$.

(b)

$$L = \lim_{n\to\infty}\left|\frac{c_{n+1}(-4)^{n+1}}{c_n(-4)^n}\right| = 4\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| \le 4\cdot\frac{1}{4} = 1$$

The ratio test is inconclusive since $L = 1$, so the series could either converge or diverge. It does not follow that the series must be convergent.

Problem 3

Center of convergence: $x = 0$.

$$L < 1 \iff \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{((n+1)!)^k x^{n+1}}{(k(n+1))!}\frac{(kn)!}{(n!)^k x^n}\right| = \lim_{n\to\infty}\left|\frac{(n!)^k(n+1)^k x^n x(kn)!}{(kn+k)!(n!)^k x^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(n+1)^k x(kn)!}{(kn+k)!}\right| = \lim_{n\to\infty}|x|\frac{(n+1)^k(kn)!}{(kn)!(kn+1)(kn+2)\ldots(kn+k)}$$
$$= |x|\lim_{n\to\infty}\frac{(n+1)^k}{\underbrace{(kn+1)(kn+2)\ldots(kn+k)}_{\text{product with } k \text{ terms}}}$$
$$= |x|\lim_{n\to\infty}\frac{n^k + \overbrace{\cdots\cdots\cdots}^{\text{terms of lower power}}}{k^k n^k + \underbrace{\cdots\cdots\cdots}_{\text{terms of lower power}}} = |x|\lim_{n\to\infty}\frac{1}{k^k} = \frac{|x|}{k^k} < 1 \iff |x| < k^k$$

Radius of convergence: $\boxed{k^k}$.

Problem 4

No. Suppose we are able to find such a series with radius R and center of convergence c. Then, we must have that $c = R$ so that the distance to 0 is R. However, c cannot be a finite distance R from infinity, so this series cannot exist.

Problem 5

$$f(x) = 1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + \ldots = \left(1 + x + x^2 + x^3 + x^4 + x^5\right) + \left(x + x^3 + x^5 + \ldots\right)$$
$$= \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} x^{2n+1} = \sum_{n=0}^{\infty} x^n + x\sum_{n=0}^{\infty}\left(x^2\right)^n = \boxed{\frac{1}{1-x} + \frac{x}{1-x^2}}$$

Notice that $f(x)$ equals the sum of two geometric series with common ratios x and x^2. By the geometric series test, we must have:

$$\begin{cases} |x| < 1 \\ \left|x^2\right| < 1 \iff |x| < 1 \end{cases} \iff x \in (-1, 1)$$

Therefore, the interval of convergence is $\boxed{(-1, 1)}$.

Problem 6

By the ratio test, $\sum a_n$ converges if $L = \lim\limits_{n\to\infty}\sqrt[n]{|a_n|} < 1$. We also have $c = \lim\limits_{n\to\infty}\sqrt[n]{|c_n|} \ne 0$.

$$L < 1 \iff \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{c_n x^n} = \lim_{n\to\infty}|x|\sqrt[n]{c_n} = c\,|x| < 1 \iff |x| < \frac{1}{c}$$

So we have the radius of convergence is $1/c$. ☺

Problem 7

Radius of convergens is $\min(2, 3) = 2$.