Problem Set #53

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Problem 3

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{\pi}{4}\right)^{2n+1} = \sin\left(\frac{\pi}{4}\right) = \boxed{\frac{\sqrt{2}}{2}}$

Problem 4

(a)

$$\int e^{-x^2}\,\mathrm{d}x = \int \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\,\mathrm{d}x = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}\,\mathrm{d}x = \boxed{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!} + C}$$

(b) Let $S = \displaystyle\int_0^1 e^{-x^2}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)n!}\Bigg|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!}$.

Let S_k be the kth partial sum of S. Then $S_k = \displaystyle\sum_{n=0}^{k} \frac{(-1)^n}{(2n+1)n!}$.

Let a_n be the nth term of the sum, then let $b_n = |a_n| = \dfrac{1}{(2n+1)n!}$.

By the alternating series estimation theorem, we have:

$$\text{Error} = |S - S_k| \le \underbrace{b_{k+1} = \frac{1}{(2k+3)(k+1)!} \le 0.001}$$

The braced equation always implies the error is less than 0.001. By calculator, that equation is true when $k \ge 4$. Therefore, $S \approx S_4$ within the acceptable error.

$$S_4 = \sum_{n=0}^{4} \frac{(-1)^n}{(2n+1)n!} = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx \boxed{0.747}$$

Problem 5

$$\lim_{x\to 0} \frac{e^x - 1 - x}{x^2} = \lim_{n\to\infty} \frac{1 + x + \sum_{n=2}^{\infty} \frac{x^n}{n!} - 1 - x}{x^2} = \lim_{x\to 0} \sum_{n=2}^{\infty} \frac{x^n}{n!x^2} = \lim_{x\to 0}\left[\frac{1}{2!} + \sum_{n=3}^{\infty} \frac{x^{n-2}}{n!}\right] = \boxed{\frac{1}{2}}$$

Problem 6

d) $f(x) = e^{-x^2}\cos x = \left(\displaystyle\sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}\right) \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$

$= \left(1 - \dfrac{x^2}{1} + \dfrac{x^4}{2} + \dots\right)\left(1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} + \dots\right)$

$= 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \dfrac{x^2}{1} + \dfrac{x^4}{2} - \cancel{\dfrac{x^6}{24}} + \dfrac{x^4}{2} - \cancel{\dfrac{x^6}{4}} + \cancel{\dfrac{x^8}{48}} + \dots$

$= 1 - \dfrac{3x^2}{2} + \dfrac{x^4}{24} + \dfrac{24x^4}{24x^4} + \dots$

$\boxed{= 1 - \dfrac{3x^2}{2} + \dfrac{25x^4}{24} + \dots}$

e) $f(x) = y = \dfrac{x}{\sin x} = \dfrac{x}{\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}$

$= \dfrac{x}{x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots} = \dfrac{x}{x - \frac{x^3}{6} + \frac{x^5}{120} + \dots}$

$1 + \dfrac{x^2}{6} + \dfrac{7x^4}{360}$

$x - \dfrac{x^3}{6} + \dfrac{x^5}{120} \,\Big)\; x$

$-\quad x - \dfrac{x^3}{6} + \dfrac{x^5}{120}$

$0 + \dfrac{x^3}{6} - \dfrac{x^5}{120}$

$-\quad \dfrac{x^3}{6} - \dfrac{x^5}{36} + \dfrac{x^7}{720}$

$0 + \dfrac{7x^5}{360} + \dots$

$-\dfrac{x^5}{120} + \dfrac{x^5}{36}$

$= \dfrac{-3x^5}{360} + \dfrac{10x^5}{360}$

$= \dfrac{7x^5}{360}$

$\boxed{= 1 + \dfrac{x^2}{6} + \dfrac{7x^4}{360} + \dots}$

f) $f(x) = e^x \ln(1+x)$

$= \left(\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$

$= \left(1 + x + \dfrac{x^2}{2} + \dots\right)\left(x - \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dots\right)$

$= x - \dfrac{x^2}{2} + \dfrac{x^3}{3} + x^2 - \cancel{\dfrac{x^3}{2}} + \cancel{\dfrac{x^4}{3}} + \cancel{\dfrac{x^3}{2}} - \cancel{\dfrac{x^4}{4}} + \cancel{\dfrac{x^3}{6}} + \dots$

$\boxed{= x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dots}$