Problem Set # 60

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December 18, 2024

Problem 2

(a)

$$\boxed{I(x) = \exp \int 1 \,\mathrm{d}x = \exp(x)}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} + y = 1 \implies \frac{\mathrm{d}y}{\mathrm{d}x} + 1 \cdot y = 1 \implies e^x \frac{\mathrm{d}y}{\mathrm{d}x} + e^x y = e^x \implies \frac{\mathrm{d}}{\mathrm{d}x} e^x y = e^x \implies e^x y = e^x + C$$
$$\implies \boxed{y = 1 + \frac{C}{e^x}}$$

(b) $$\frac{\mathrm{d}y}{\mathrm{d}x} = x - y \implies \frac{\mathrm{d}y}{\mathrm{d}x} + y = x \implies e^x \frac{\mathrm{d}y}{\mathrm{d}x} + e^x y = xe^x \implies \int \frac{\mathrm{d}}{\mathrm{d}x} e^x y \,\mathrm{d}x = \int xe^x \,\mathrm{d}x$$
$$\implies e^x y = xe^x - e^x + C \implies \boxed{y = x - 1 + \frac{C}{e^x}}$$

(e) $$(1+t)\frac{\mathrm{d}u}{\mathrm{d}t} + u = 1 + t \implies \frac{\mathrm{d}u}{\mathrm{d}t} + \frac{1}{1+t} u = 1$$

$$\boxed{I(t) = \exp \int \frac{1}{1+t} \,\mathrm{d}t = \exp(\ln|1+t| + C) = C(1+t)}$$

$$\implies (1+t)\frac{\mathrm{d}u}{\mathrm{d}t} + u = 1 + t \implies \int \frac{\mathrm{d}}{\mathrm{d}t}(1+t)u \,\mathrm{d}t = \int (1+t) \,\mathrm{d}t$$
$$\implies (1+t)u = t + \frac{t^2}{2} + C \implies u = \frac{t}{1+t} + \frac{t^2}{2(1+t)} + \frac{C}{1+t}$$

Problem 4

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x)y^n \implies \begin{bmatrix} u = y^{1-n} \\ y = u^{\frac{1}{1-n}} \\ \mathrm{d}y = \dfrac{1}{1-n} u^{\frac{n}{1-n}} \,\mathrm{d}u \end{bmatrix} \frac{1}{1-n} \frac{\mathrm{d}u}{\mathrm{d}x} u^{\frac{n}{1-n}} + P(x) u^{\frac{1}{1-n}} = Q(x) u^{\frac{n}{1-n}}$$
$$\implies u^{\frac{-n}{1-n}} \left(\frac{\mathrm{d}u}{\mathrm{d}x} u^{\frac{n}{1-n}} + (1-n)P(x) u^{\frac{1}{1-n}} \right) = (1-n)Q(x) u^{\frac{n}{1-n}} u^{\frac{-n}{1-n}}$$
$$\implies \frac{\mathrm{d}u}{\mathrm{d}x} + (1-n)P(x) u^{\frac{1-n}{1-n}} = (1-n)Q(x)$$
$$\implies \frac{\mathrm{d}u}{\mathrm{d}x} + (1-n)P(x) u = (1-n)Q(x)$$

Problem 5

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{2}{x} y = \frac{y^3}{x^2} \implies \begin{bmatrix} u = y^{-2} \\ y = u^{-1/2} \\ \dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{2} u^{-3/2} \dfrac{\mathrm{d}u}{\mathrm{d}x} \end{bmatrix} - \frac{1}{2} u^{-3/2} \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{2}{x} u^{-1/2} = \frac{\left(u^{-1/2}\right)^3}{x^2}$$
$$\implies u^{3/2} \left(-\frac{1}{2} u^{-3/2} \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{2}{x} u^{-1/2} \right) = u^{3/2} \cdot \frac{u^{-3/2}}{x^2} \implies -2 \left(-\frac{1}{2} \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{2}{x} u \right) = \frac{1}{x^2} \cdot (-2)$$
$$\implies \frac{\mathrm{d}u}{\mathrm{d}x} - \frac{4}{x} u = -\frac{2}{x^2}$$

$$\boxed{I(x) = \exp \int \left(-\frac{4}{x} \right) \mathrm{d}x = (\exp \ln|x|)^{-4} = |x|^{-4} = x^{-4}}$$

$$\implies x^{-4} \frac{\mathrm{d}u}{\mathrm{d}x} - 4x^{-5} u = -2x^{-6} \implies \int \frac{\mathrm{d}}{\mathrm{d}x} \left[x^{-4} u \right] \,\mathrm{d}x = \int \left(-2x^{-6} \right) \mathrm{d}x$$
$$\implies x^4 x^{-4} u = \left(\frac{-2}{-5} x^{-5} + C \right) x^4 \implies \boxed{\frac{1}{y^2} = \frac{2}{5x} + Cx^4}$$