Problem Set #58, Problem 9

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Problem 9

$$\begin{cases} x(\theta) = R\cos\theta + \theta R\sin\theta \\ y(\theta) = R\sin\theta - \theta R\cos\theta \end{cases} \implies x'(\theta) = -R\sin\theta + R\sin\theta + \theta R\cos\theta = \theta R\sin\theta$$

The left half of the involute is a semicircle containing the point:

$$(x(\pi), y(\pi)) = (R\cos\pi + \pi R\sin\pi, R\sin\pi - \pi R\cos\pi) = (-R, -R\pi)$$

The semicircle is centered at $(-R, 0)$. Thus the radius of the semicircle is $R\pi$ and its area $A_1 = \pi\left(R\pi\right)^2/2 = R^2\pi^3/2$.

The other areas are $A_2 = A_3$, where:

$$\begin{aligned} A_2 &= \int_0^\pi (R\sin\theta - \theta R\cos\theta)\,\theta R\cos\theta\,\mathrm{d}\theta = \int_0^\pi \theta R^2\sin\theta\cos\theta\,\mathrm{d}\theta - \int_0^\pi \theta^2 R^2\cos^2\theta\,\mathrm{d}\theta \\ &= R^2\left(\int_0^\pi \theta\sin\theta\cos\theta\,\mathrm{d}\theta - \int_0^\pi \theta^2\cos^2\theta\,\mathrm{d}\theta\right) \end{aligned}$$

$$\begin{aligned} I &= \int \theta\sin\theta\cos\theta\,\mathrm{d}\theta = \begin{bmatrix} u = \theta\sin\theta & \mathrm{d}v = \cos\theta\,\mathrm{d}\theta \\ \mathrm{d}u = (\sin\theta + \theta\cos\theta)\,\mathrm{d}\theta & v = \sin\theta \end{bmatrix} \\ &= \theta\sin^2\theta - \int \sin(\theta)\left(\sin\theta + \theta\cos\theta\right)\,\mathrm{d}\theta = \theta\sin^2\theta - \int \sin^2\theta\,\mathrm{d}\theta - \int \theta\sin\theta\cos\theta\,\mathrm{d}\theta \\ &= \theta\sin^2\theta - \int \frac{1-\cos 2\theta}{2}\,\mathrm{d}\theta - I = \theta\sin^2\theta - \frac{1}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) - I \\ &= \theta\sin^2\theta - \frac{\theta}{2} + \frac{\sin 2\theta}{4} - I \implies I = \frac{\theta\sin^2\theta}{2} - \frac{\theta}{4} + \frac{\sin 2\theta}{8} \end{aligned}$$

$$\begin{aligned} \int \theta^2\cos^2\theta\,\mathrm{d}\theta &= \int \theta^2\frac{1+\cos 2\theta}{2}\,\mathrm{d}\theta = \frac{1}{2}\int\left(\theta^2 + \theta^2\cos 2\theta\right)\mathrm{d}\theta = \frac{1}{2}\left(\frac{\theta^3}{3} + \int\theta^2\cos 2\theta\,\mathrm{d}\theta\right) \\ &= \begin{bmatrix} u = \theta^2 & \mathrm{d}v = \cos 2\theta\,\mathrm{d}\theta \\ \mathrm{d}u = 2\theta\,\mathrm{d}\theta & v = \frac{1}{2}\sin 2\theta \end{bmatrix} \frac{\theta^3}{6} + \frac{1}{2}\left(\frac{1}{2}\theta^2\sin 2\theta - \int\theta\sin 2\theta\,\mathrm{d}\theta\right) \\ &= \begin{bmatrix} u = \theta & \mathrm{d}v = \sin 2\theta\,\mathrm{d}\theta \\ \mathrm{d}u = \mathrm{d}\theta & v = -\frac{1}{2}\cos 2\theta \end{bmatrix} \frac{\theta^3}{6} + \frac{1}{2}\left(\frac{1}{2}\theta^2\sin 2\theta - \left(-\frac{1}{2}\theta\cos 2\theta + \int\frac{1}{2}\cos 2\theta\mathrm{d}\theta\right)\right) \\ &= \frac{\theta^3}{6} + \frac{1}{2}\left(\frac{1}{2}\theta^2\sin 2\theta - \left(-\frac{1}{2}\theta\cos 2\theta + \frac{1}{4}\sin 2\theta\right)\right) + C \\ &= \frac{\theta^3}{6} + \frac{1}{2}\left(\frac{1}{2}\theta^2\sin 2\theta + \frac{1}{2}\theta\cos 2\theta - \frac{1}{4}\sin 2\theta\right) + C \\ &= \frac{\theta^3}{6} + \frac{\theta^2\sin 2\theta}{4} + \frac{\theta\cos 2\theta}{4} - \frac{\sin 2\theta}{8} + C \end{aligned}$$

$$\begin{aligned} &= R^2\left(\left[\frac{\theta\sin^2\theta}{2} - \frac{\theta}{4} + \frac{\sin 2\theta}{8}\right]_0^\pi - \left[\frac{\theta^3}{6} + \frac{\theta^2\sin 2\theta}{4} + \frac{\theta\cos 2\theta}{4} - \frac{\sin 2\theta}{8}\right]_0^\pi\right) \\ &= R^2\left(0 - \frac{\pi}{4}\right) - R^2\left(\frac{\pi^3}{6} + \frac{\pi}{4}\right) = -\frac{\pi R^2}{4} - \frac{R^2\pi^3}{6} - \frac{\pi R^2}{4} = -\frac{R^2\pi^3}{6} - \frac{\pi R^2}{2} \end{aligned}$$

Then we need to subtract the area of the fence, which is a circle of radius R. Its area is $A_4 = \pi R^2$.

The total area is:

$$\begin{aligned} A &= |A_1| + |A_2| + |A_3| - |A_4| = \frac{R^2\pi^3}{2} + 2\left|-\frac{R^2\pi^3}{6} - \frac{\pi R^2}{2}\right| - \pi R^2 = \frac{R^2\pi^3}{2} + \frac{R^2\pi^3}{3} + \pi R^2 - \pi R^2 \\ &= \frac{3R^2\pi^3}{6} + \frac{2R^2\pi^3}{6} = \boxed{\frac{5R^2\pi^3}{6}} \end{aligned}$$