Problem Set #38

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Problem 5

(k) Let $f(x) = x \sin\left(\frac{1}{x}\right)$. Then $a_n = f(n)$.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = \lim_{y\to 0^+} \frac{\sin y}{y} = 1$$

Because the limit $\lim_{x\to\infty} f(x)$ exists, $\{a_n\}$ converges.

(l) Let $f(x) = \left(1 + \frac{2}{x}\right)^x$. Then $a_n = f(n)$.

$$\ln \lim_{x\to\infty} f(x) = \lim_{x\to\infty} x \ln\left(1 + \frac{2}{x}\right) = \lim_{y\to 0^+} \frac{\ln(1+2y)}{y} = \lim_{y\to 0^+} \frac{2}{1+2y} = 2$$
$$\lim_{x\to\infty} f(x) = e^2$$

Because the limit $\lim_{x\to\infty} f(x)$ exists, $\{a_n\}$ converges.

(m) Let $f(x) = \ln\left(2x^2+1\right) - \ln\left(x^2+1\right)$. Then $a_n = f(n)$.

$$\exp \lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2+1}{x^2+1} = \lim_{x\to\infty} \frac{4\cancel{x}}{2\cancel{x}} = 2$$
$$\lim_{x\to\infty} f(x) = \ln 2$$

Because the limit $\lim_{x\to\infty} f(x)$ exists, $\{a_n\}$ converges.

(n) $\{a_n\}$ diverges, pick $0 < \varepsilon < 1$ and epsilon-delta definition will fail.

(o)

$$a_\infty = \lim_{n\to\infty} \frac{n!}{2^n} = \lim_{n\to\infty} \frac{n \times (n-1) \times \ldots \times 3 \times 2 \times 1}{2 \times 2 \times \ldots \times 2 \times 2} = \lim_{n\to\infty} \frac{n}{2}\frac{n-1}{2} \cdots \frac{3}{2}\frac{1}{2}$$

$n/2$ tends to infinity as n tends to infinity. Assuming the rest are finite the product tends to infinity. So $\{a_n\}$ diverges.

Problem 6

(a)

$$a_1 = 1060$$
$$a_2 = 1123.6$$
$$a_3 = 1191.016$$
$$a_4 \approx 1262.477$$
$$a_5 \approx 1338.226$$

(b) Divergent. 1.06^n tends to infinity as n tends to infinity.

Problem 7

(a) -1 create an alternating sequence between -1 and 1 which does not converge, other negative values greater than -1 will still alternate but tend to 0. Values greater than 1 grow without bound. Converges when $-1 < r \le 1$.

(b) $-1 < r < 1$.