Problem Set #44

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Problem 6

$$\int x^p \ln x \, \mathrm{d}x = \begin{bmatrix} u = \ln x & \mathrm{d}u = \dfrac{\mathrm{d}x}{x} \\ \mathrm{d}v = x^p \, \mathrm{d}x & v = \dfrac{x^{p+1}}{p+1} \end{bmatrix} \frac{x^{p+1} \ln x}{p+1} - \int \frac{1}{x} \frac{x^{p+1}}{p+1} \, \mathrm{d}x = \frac{x^{p+1} \ln x}{p+1} - \frac{1}{p+1} \int x^p \, \mathrm{d}x$$

$$= \frac{x^{p+1} \ln x}{p+1} - \frac{x^{p+1}}{(p+1)^2} + C$$

$$\int_0^1 x^p \ln x \, \mathrm{d}x = \lim_{t \to 0^+} \int_t^1 x^p \ln x \, \mathrm{d}x = \lim_{t \to 0^+} \left[\frac{x^{p+1} \ln x}{p+1} - \frac{x^{p+1}}{(p+1)^2} \right]_t^1$$

$$= 0 - \frac{1}{(p+1)^2} - \lim_{t \to 0^+} \left[\frac{t^{p+1} \ln t}{p+1} + \frac{t^{p+1}}{(p+1)^2} \right] = -\frac{1}{(p+1)^2} - \lim_{t \to 0^+} \frac{t^{p+1} \ln t}{p+1} + \frac{0}{(p+1)^2} \tag{1}$$

$$\boxed{\lim_{t \to 0^+} t^{p+1} \ln t = \lim_{t \to 0^+} \frac{\ln t}{t^{-p-1}} \overset{\left[\frac{0}{0}\right]}{\underset{\text{LH}}{=}} \lim_{t \to 0^+} \left[\frac{\frac{1}{t}}{(-p-1)t^{-p-2}} \cdot \frac{t}{t} \right] = \lim_{t \to 0^+} \frac{1}{-(p+1)t^{-p-1}} = \lim_{t \to 0^+} \frac{-t^{p+1}}{p+1} = 0}$$

$$= -\frac{1}{(p+1)^2} - 0 + 0 = \boxed{-\frac{1}{(p+1)^2}}$$

Notice: when $p = -1$, the above is obviously undefined. When $p < -1$, t^{p+1} is not real when $t = 1$, so we are not able to evaluate it. So the integral only converges to $-1/(p+1)^2$ when $p > -1$.