Problem Set #54

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Problem 1

(a) $f(x) = \sec x = \dfrac{1}{\cos x} = \dfrac{1}{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}}$

$$P_2(x) = \frac{1}{\sum_{n=0}^{1} \frac{(-1)^n x^{2n}}{(2n)!}} = \frac{1}{1 - \frac{x^2}{2!}} = \boxed{1 + \frac{x^2}{2}} \cancel{+ \dots}$$

Problem 2

(a)

$$f(x) = \frac{2}{x} = 2x^{-1}$$

n	$f^{(n)}(x)$	$f^{(n)}(1)$	c_n
0	$2x^{-1}$	2	2
1	$-2x^{-2}$	-2	-2
2	$4x^{-3}$	4	$4/2! = 2$
3	$-12x^{-4}$	-12	$-12/3! = -2$

$$\boxed{P_3(x) = 2 - 2(x-1) + 2(x-a)^2 - 2(x-1)^3}$$

(b)

$$f(x) = \sqrt[3]{x} = x^{1/3}$$

n	$f^{(n)}(x)$	$f^{(n)}(8)$	c_n
0	$x^{1/3}$	2	2
1	$(1/3)x^{-2/3}$	$1/12$	$1/12$
2	$(-2/9)x^{-5/3}$	$-2/(9 \cdot 32)$	$-1/288$
3	$(10/27)x^{-8/3}$	$10/(27 \cdot 256)$	$5/20736$

$$\boxed{P_3(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2 + \frac{5}{20736}(x-8)^3}$$

Problem 4

(b) Let $f(x) = \arcsin x$. We start by calculating the 5th degree Maclaurin polynomial of f.

n	$\arcsin^{(n)}(x)$	$\arcsin^{(n)}(0)$	c_n
0	$\arcsin x$	0	0
1	$\dfrac{1}{\sqrt{1-x^2}}$	1	1
2	$\dfrac{-\frac{-2x}{2\sqrt{1-x^2}}}{1-x^2} = \dfrac{x}{\left(1-x^2\right)^{3/2}}$	0	0
3	$\begin{aligned} \frac{\left(1-x^2\right)^{3/2} - x\frac{3}{2}\sqrt{1-x^2}(-2x)}{\left(1-x^2\right)^3} &= \frac{\left(1-x^2\right)^{3/2} + 3x^2\left(1-x^2\right)^{1/2}}{\left(1-x^2\right)^3} \\ &= \left(1-x^2\right)^{-3/2} + 3x^2\left(1-x^2\right)^{-5/2} \end{aligned}$	1	$\dfrac{1}{3!}$
4	$\begin{aligned} &-2x\frac{-3}{2}\left(1-x^2\right)^{-5/2} + 6x\left(1-x^2\right)^{-5/2} + 3x^2\frac{-5}{2}(-2x)\left(1-x^2\right)^{-7/2} \\ &= 3x\left(1-x^2\right)^{-5/2} + 6x\left(1-x^2\right)^{-5/2} + 15x^3\left(1-x^2\right)^{-7/2} \\ &= 9x\left(1-x^2\right)^{-5/2} + 15x^3\left(1-x^2\right)^{-7/2} \end{aligned}$	0	0
5	$\begin{aligned} &\left[\begin{matrix} 9\left(1-x^2\right)^{-5/2} + 9x\frac{-5}{2}(-2x)\left(1-x^2\right)^{-7/2} + \\ 45x^2\left(1-x^2\right)^{-7/2} + 15x^3\frac{-7}{2}(-2x)\left(1-x^2\right)^{-9/2} \end{matrix}\right] \\ &= \left[\begin{matrix} 9\left(1-x^2\right)^{-5/2} + 45x^2\left(1-x^2\right)^{-7/2} + 45x^2\left(1-x^2\right)^{-7/2} + \\ 105x^4\left(1-x^2\right)^{-9/2} \end{matrix}\right] \\ &= 9\left(1-x^2\right)^{-5/2} + 90x^2\left(1-x^2\right)^{-7/2} + 105x^4\left(1-x^2\right)^{-9/2} \end{aligned}$	9	$\dfrac{9}{5!}$

Let R_3 be the remainder of the 3rd degree Maclaurin polynomial of arcsine:

$$\arcsin(x) \approx P_3(x) = x - \frac{x^3}{6}$$

By Taylor's Theorem, we have the following upper bound on $R_3(0.4)$:

$$R_3(0.4) \le \frac{\left|0 - 0.4\right|^{3+1}}{(3+1)!} \max_{0 \le z \le 0.4} \left|f^{(3+1)}(z)\right| = \frac{(0.4)^4}{24} \max_{0 \le z \le 0.4} \left|f^{(4)}(z)\right|$$

Notice that $\dfrac{\mathrm{d}}{\mathrm{d}x} f^{(4)}(x) = f^{(5)}(x) > 0$ for all $0 \le x \le 0.4$, so $f^{(4)}$ is increasing. Thus, $f^{(4)}$ reaches its global maximum on the interval $[0, 0.4]$ at $z = 0.4$. Therefore, we have:

$$R_3(0.4) \le \frac{(0.4)^4}{24} f^{(4)}(0.4) \approx \frac{0.0256}{24} \cdot 7.33402 = \boxed{0.00782}$$

The actual error is:

$$\left|\arcsin(0.4) - \left(0.4 + \frac{(0.4)^3}{6}\right)\right| = 0.000850 < 0.00782 = \text{Lagrange error bound}$$

Problem 5

(b) Let $f(x) = e^x$. Then $f^n(x) = e^x$ for all $n \in \mathbb{N}$.

$$f(0.6) = e^{0.6} \approx P_n(0.6) = \sum_{k=0}^{n} \frac{(0.6)^k}{k!}$$

The Lagrange error bound of the above approximation, in terms of n, is:

$$R_n(0.6) \le \frac{\left|0 - 0.6\right|^{n+1}}{(n+1)!} \max_{0 \le z \le 0.6} \left|f^{(n+1)}(z)\right| = \frac{(0.6)^{n+1}}{(n+1)!} e^{0.6} \le 0.001$$

Because $f^{n+1}(z) = e^z$ is increasing, the global maximum occurs at the highest value $z = 0.6$.

Using a calculator, we see that the above is true when $\boxed{n \ge 5}$.

Problem 6

$$f(x) = \sqrt[3]{x} = x^{1/3}$$

(a)

n	$f^{(n)}(x)$	$f^{(n)}(8)$	c_n
0	$x^{1/3}$	2	2
1	$\dfrac{1}{3}x^{-2/3}$	$\dfrac{1}{12}$	$\dfrac{1}{12}$
2	$\dfrac{1}{3}\left(-\dfrac{2}{3}\right)x^{-5/3} = -\dfrac{2}{9}x^{-5/3}$	$-\dfrac{1}{144}$	$-\dfrac{1}{288}$
3	$-\dfrac{2}{9}\left(-\dfrac{5}{3}\right)x^{-8/3} = \dfrac{10}{27}x^{-8/3}$	$\dots$	$\dots$

$$f(x) \approx \boxed{P_2(x) = 2 + \frac{x-8}{12} - \frac{(x-8)^2}{288}}$$

(b) Let R_2 be the remainder of the 2nd degree polynomial: $R_2(x) = \left|f(x) - P_2(x)\right|$. We calculate $R_2(x)$ where $7 \le x \le 9$.

Let I be an interval defined as follows:

$$I = \begin{cases} [x, 8] & 7 \le x \le 8 \\ [8, x] & 8 < x \le 9 \end{cases}$$

This is the interval which z must be on.

$$R_2(x) \le \frac{\left|0 - x\right|^{2+1}}{(2+1)!} \max_{z \in I} \left|f^{(2+1)}(z)\right| = \frac{x^3}{6} \max_{z \in I} \left|f^{(3)}(z)\right| = \frac{x^3}{6} \max_{z \in I} \left|\frac{10}{27} z^{-8/3}\right|$$

Notice $z^{-8/3} = 1/z^{8/3}$ must be decreasing on $(0, \infty)$ because $z^{8/2}$ is increasing on the same interval. Thus, the maximum value of $z^{-8/3}$ is reached for the lowest value of $z \in I$.

$$\le \frac{x^3}{6} \frac{10}{27} \begin{cases} x & 7 \le x \le 8 \\ 8 & 8 \le x < 9 \end{cases} = \boxed{\frac{5x^3}{81} \begin{cases} x & 7 \le x \le 8 \\ 8 & 8 \le x < 9 \end{cases}}$$