Problem Set #39

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Problem 1

For all $k \geq 1$:

$$\begin{aligned} a_k > a_{k+1} &\iff \frac{k+1}{k} > \frac{k+2}{k+1} \\ &\iff k^2 + 2k + 1 > k^2 + 2k \qquad \text{(multiplying positive } k(k+1)) \\ &\iff k^2 + 1 > k^2 \end{aligned}$$

Clearly, the above must be true. Thus, $\{a_n\}$ is an decreasing sequence. □

Problem 2

(a) Done in class

(b) Done in class (yes because sine function is between -1 and 1)

(c) $a_n = \left\{\sqrt{2a_{n-1}}\right\}, a_1 = \sqrt{2}$.

Proof increasing

Base case. When $n = 1$, $a_n = \sqrt{2}$ and $a_{n+1} = \sqrt{2\sqrt{2}}$. Because square function is increasing for positive values, $a_1 < a_2 \iff \sqrt{2} < \sqrt{2\sqrt{2}} \iff 2 < 2\sqrt{2}$. Clearly true because $\sqrt{2} > 1$.

Hypothesis. Suppose $a_k < a_{k+1} \iff a_k < \sqrt{2a_k}$ when $k \geq 1$.

Inductive Step. Want to show the statement holds for $k + 1$, that is, $a_{k+1} < a_{k+2}$.

$$\begin{aligned} a_{k+1} < a_{k+2} &\iff \sqrt{2a_k} < \sqrt{2\sqrt{2a_k}} \\ &\iff 2a_k < 2\sqrt{2a_k} \qquad \text{(square function increasing for positive values)} \\ &\iff a_k < \sqrt{2a_k} \end{aligned}$$

By the principle of mathematical induction, the statement $a_k < a_{k+1}$ is true for all $k \geq 1$. □

Because the sequence is increasing, it is clearly bounded at the lowest possible value, which is $a_1 = \sqrt{2}$. So $\{a_n\}$ is bounded below.

Proof bounded above

I claim that the sequence is bounded above at 2. That is, $a_k < 2$ for all $k \geq 1$.

Base case. When $n = 1$, $a_n = \sqrt{2} < 2$.

Hypothesis. Suppose $a_k < 2$ when $k \geq 1$.

Inductive Step. Want to show the statement holds for $k + 1$, that is, $a_{k+1} < 2$.

$$a_{k+1} < 2 \iff \sqrt{2a_k} < 2 \iff 2a_k < 4 \iff a_k < 2$$

True from hypothesis.

By the principle of mathematical induction, the statement $a_k < \sqrt{2}$ is true for all $k \geq 1$. □

Thus, because $\{a_n\}$ is bounded above and below, it is bounded. □

Problem 3

(a) Done in class

(b) Alternating sequence, so not monotonic. Clearly not bounded as a_∞ tends to infinity.

(c) Let $f(x) = \dfrac{x}{x^2+1}$. Then $f(n) = a_n$ for all $n \in \mathbb{N}$.

$$\begin{aligned} f'(x) < 0 &\iff \frac{(x^2+1) - x(2x)}{(x^2+1)^2} < 0 \\ &\iff x^2 + 1 - 2x^2 < 0 \qquad \text{(denominator always positive)} \\ &\iff x^2 > 1 \iff x \in (-\infty, -1) \cup (1, \infty) \end{aligned}$$

For all $x > 1$, $f'(x) < 0$, so f is decreasing for all $x > 1$. So $\{a_n\}$ is decreasing for all $n > 1$. But $a_1 = 1/2 > 2/5 = a_2$, so it is decreasing for all $n \geq 1$. Therefore $\{a_n\}$ is decreasing and monotonic.

Clearly it is also bounded because the range of f on the positive real numbers is $(0, \infty)$ (too lazy).

Problem 4

(a) Done in class

(b) Same proof as 2c, but with 6 instead of 2. Or:

$a_n = \left\{\sqrt{6a_{n-1}}\right\}, a_1 = \sqrt{6}$.

Proof increasing

Base case. When $n = 1$, $a_n = \sqrt{6}$ and $a_{n+1} = \sqrt{6\sqrt{6}}$. Because square function is increasing for positive values, $a_1 < a_6 \iff \sqrt{6} < \sqrt{6\sqrt{6}} \iff 6 < 6\sqrt{6}$. Clearly true because $\sqrt{6} > 1$.

Hypothesis. Suppose $a_k < a_{k+1} \iff a_k < \sqrt{6a_k}$ when $k \geq 1$.

Inductive Step. Want to show the statement holds for $k + 1$, that is, $a_{k+1} < a_{k+2}$.

$$\begin{aligned} a_{k+1} < a_{k+2} &\iff \sqrt{6a_k} < \sqrt{6\sqrt{6a_k}} \\ &\iff 6a_k < 6\sqrt{6a_k} \qquad \text{(square function increasing for positive values)} \\ &\iff a_k < \sqrt{6a_k} \end{aligned}$$

By the principle of mathematical induction, the statement $a_k < a_{k+1}$ is true for all $k \geq 1$. □

Because the sequence is increasing, it is clearly bounded at the lowest possible value, which is $a_1 = \sqrt{6}$. So $\{a_n\}$ is bounded below.

Proof bounded above

I claim that the sequence is bounded above at 6. That is, $a_k < 6$ for all $k \geq 1$.

Base case. When $n = 1$, $a_n = \sqrt{6} < 6$.

Hypothesis. Suppose $a_k < 6$ when $k \geq 1$.

Inductive Step. Want to show the statement holds for $k + 1$, that is, $a_{k+1} < 6$.

$$a_{k+1} < 6 \iff \sqrt{6a_k} < 6 \iff 6a_k < 36 \iff a_k < 6$$

True from hypothesis.

By the principle of mathematical induction, the statement $a_k < \sqrt{6}$ is true for all $k \geq 1$. □

Thus, because $\{a_n\}$ is bounded above and below, it is bounded. □

Problem 5

Done in class

Problem 6

(a) At the start, there is one pair. That pair takes 2 months to produce another pair, so $f_1 = f_2 = 1$. At any time $n \in \mathbb{N}, n > 2$, no rabbits die, so every rabbit at the previous time of $n - 1$ are also at time n. There are f_{n-1} such rabbits. Rabbits born two months ago at time $n - 2$ now become productive and produce new pairs; there are f_{n-2} of these. So the number of rabbits at time n is $f_n = f_{n-1} + f_{n-2}$.

(b)

$$\begin{aligned} a_{n-1} = 1 + \frac{1}{a_{n-2}} &\iff \frac{f_n}{f_{n-1}} = 1 + \left(\frac{f_{n-1}}{f_{n-2}}\right)^{-1} \iff \frac{f_n}{f_{n-1}} = \frac{f_{n-1}}{f_{n-1}} + \frac{f_{n-2}}{f_{n-1}} \iff \frac{f_n}{f_{n-1}} = \frac{f_n}{f_{n-1}} \\ &\iff a_n = 1 + \frac{1}{a_{n-1}} \end{aligned}$$

Let $L = \lim\limits_{n\to\infty} a_n$, then $L = \lim_{n\to\infty} a_{n-1}$. From above, we have:

$$\begin{aligned} a_n = 1 + \frac{1}{a_{n-1}} &\implies \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left[1 + \frac{1}{a_{n-1}}\right] \implies L = 1 + \frac{1}{L} \implies L^2 - L - 1 = 0 \\ &\implies \left(L - \frac{1}{2}\right)^2 - \frac{1}{4} - \frac{4}{4} = 0 \implies L - \frac{1}{2} = \pm\sqrt{\frac{5}{4}} \implies \boxed{L = \frac{1 \pm \sqrt{5}}{2}} \end{aligned}$$

The golden ratio φ is defined as the positive case.