Harmonic Series is Divergent

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January 12, 2025

Proof. Let $a_n = \dfrac{1}{n}$ and $s_n = \sum_{k=1}^{n} a_n = \sum_{k=1}^{n} \dfrac{1}{k}$.

Define the sequence $\{b_n\}$ where the nth term is the reciprocal of the smallest power of two that is larger than n. For all integer $m \geq 1$, the kth term $b_k = 1/2^m$ if and only if $2^{m-1} + 1 \leq k \leq 2^m$. There are exactly $2^m - 2^{m-1} - 1 + 1 = 2^m - 2^{m-1} = 2^{m-1}(2-1) = 2^m/2$ values of k satisfying this inequality.

Note that the reciprocal of every term b_k is larger than or equal to k.

$$(b_k)^{-1} \geq k \implies b_k \leq \frac{1}{k} = a_k$$

Let $\{u_n\}$ be the nth partial sum of $\{b_n\}$. Then for all k, $u_k \leq s_k$ because every term of the sum u_k is less than equal to the corresponding term in s_k.

Let us calculate the value of u_k.

$$u_n = 1 + \sum_{k=1}^{\log_2 n} \frac{1}{2^k} \cdot \frac{2^k}{2} = 1 + \sum_{k=1}^{\log_2 n} \frac{1}{2} = 1 + \frac{\log_2 n}{2}$$

(this diverges.)

Formal definition for the divergence of a series is that it grows arbitrarily:

$$\lim_{n\to\infty} s_n = \infty \iff (\forall M > 0)(\exists c > 0) : \mathbb{N} \ni n > c \implies s_n > M \tag{1}$$

Because we know $u_n \geq s_n$:

$$\lim_{n\to\infty} s_n = \infty \iff (\forall M > 0)(\exists c > 0) : \mathbb{N} \ni n > c \implies s_n \geq u_n > M$$

If $u_n > M$, it is necessarily true that $s_n > M$. Therefore, the following implies statement (1):

$$\begin{aligned}
&\impliedby (\forall M > 0)(\exists c > 0) : \mathbb{N} \ni n > c \implies u_n > M \\
&\iff (\forall M > 0)(\exists c > 0) : \mathbb{N} \ni n > c \implies 1 + \frac{\log_2 n}{2} > M \\
&\iff (\forall M > 0)(\exists c > 0) : \mathbb{N} \ni n > c \implies 2 + \log_2 n > 2M \\
&\iff (\forall M > 0)(\exists c > 0) : \mathbb{N} \ni n > c \implies \log_2 n > 2M - 2 \\
&\iff (\forall M > 0)(\exists c > 0) : \mathbb{N} \ni n > c \implies n > 2^{2M-2}
\end{aligned}$$

Let $c = 2^{2M-2}$. Then, for all integer $M > 0$, c must also be an integer. If some integer $n > c$:

$$\begin{aligned}
n > c &\implies n > 2^{2M-2} \\
&\implies \log_2 n > 2M - 2 && \text{(logarithm of all bases is increasing)} \\
&\implies 2 + \log_2 n > 2M \\
&\implies 1 + \frac{\log_2 n}{2} > M
\end{aligned}$$

Therefore, by definition (1), the sequence $\left\{1 + \dfrac{\log_2 n}{2}\right\} = \{u_n\}$ diverges. The divergence of $\{u_n\}$ implies the divergence of $\{s_n\}$, so the harmonic series diverges. □