Problem Set #49

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Problem 1

(a) $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cos(n+1)}{(n+1)^2}\frac{n^2}{\cos n}\right| = \lim_{n\to\infty}\frac{n^2}{n^2+2n+1}\frac{\cos n\cos 1 - \sin n \sin 1}{\cos n}$

$= \lim_{n\to\infty}\frac{n^2}{n^2\left(1+\frac{2}{n}+\frac{1}{n^2}\right)}(\cos 1 - \tan n \sin 1) = \cos 1 - \sin(1)\lim_{n\to\infty}\tan(n)$ does not exist.

Not sure how would ratio test, but series is clearly absolutely convergent by direct comparison to $1/n^2$ as $|\cos n| < 1$.

(b) $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{n+1}{5^{n+1}}\frac{5^n}{n}\right| = \lim_{n\to\infty}\frac{n\left(1+\frac{1}{n}\right)}{5n} = \frac{1}{5} < 1$

We have $L < 1$ so the series is absolutely convergent by the ratio test.

(d) $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(k+1)\left(\frac{2}{3}\right)^{k+1}}{k\left(\frac{2}{3}\right)^k}\right| = \lim_{n\to\infty}\frac{k+1}{k}\frac{2}{3} = \frac{2}{3} < 1$

The series is absolutely convergent by the ratio test.

(e) $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(1.1)^{n+1}}{(n+1)^4}\frac{n^4}{(-1)^n(1.1)^n}\right| = \lim_{n\to\infty}\frac{n^4}{(n+1)^4}|(-1)(1.1)| = 1.1 > 1$

The series is divergent by the ratio test.

(h) $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{\ln(n+1)}\frac{\ln n}{(-1)^n}\right| = \lim_{n\to\infty}\frac{\ln n}{\ln(n+1)} \overset{\left[\frac{\infty}{\infty}\right]}{\underset{\text{L'H}}{=}} \lim_{n\to\infty}\frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty}\frac{n+1}{n} = 1$

Ratio test is inconclusive.

However notice that $\sum 1/\ln n$ is divergent (trivial) but also $1/\ln n = |a_n|$ is decreasing so by the absolute convergence test the series is conditionally convergent.

(i) $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\cos\left(\frac{\pi n}{3}+\frac{\pi}{3}\right)}{(n+1)!}\frac{n!}{\cos(\pi n/3)}\right| = \lim_{n\to\infty}\left|\frac{\cos\frac{\pi n}{3}\cos\frac{\pi}{3} - \sin\frac{\pi n}{3}\sin\frac{\pi}{3}}{(n+1)\cos\frac{\pi n}{3}}\right| = 0 < 1$

The series is absolutely convergent by the ratio test.

(j) $L = \lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\sqrt[n]{\left(\frac{n^2+1}{2n^2+1}\right)^n} = \lim_{n\to\infty}\frac{n^2+1}{2n^2+1} = \frac{1}{2} < 1$

The series is absolutely convergent by the root test.

(l) $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{100}\cdot 100^{n+1}}{(n+1)!}\frac{n!}{n^{100}\cdot 100^n}\right| = \lim_{n\to\infty}\frac{(n+1)^{100}}{n^{100}}\frac{100}{n+1} = 1\cdot 0 = 0 < 1$

The series is absolutely convergent by the ratio test.

Problem 2

(a) Convergent with $L = 0$.

(b) Convergent with $L = 2/3$.

Problem 3

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{5n+1}{4n+3}a_n\cdot\frac{1}{a_n}\right| = \lim_{n\to\infty}\frac{5n+1}{4n+3} = \lim_{n\to\infty}\frac{5n\left(1+\frac{1}{5n}\right)}{4n\left(1+\frac{3}{4n}\right)} = \frac{5}{4} > 1$$

By the ratio test the series is divergent.

Problem 4

Absolutely convergent (see PS48 last question).

Problem 5

(a)

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1}{(n+1)^3}\frac{n^3}{1}\right| \lim_{n\to\infty}\frac{n^3}{(n+1)^3} = 1$$

Ratio test is inconclusive.

(b)

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^n}{\sqrt{n+1}}\frac{\sqrt{n}}{(-3)^{n-1}}\right| = \lim_{n\to\infty}\frac{3\sqrt{n}}{\sqrt{n+1}} = 3 > 1$$

The series is divergent by the ratio test.

Problem 6

(a)

$$L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)!}\frac{n!}{x^n}\right| = \lim_{n\to\infty}\frac{x}{n+1} = 0 < 1$$

Therefore for all $x \in \mathbb{R}$ the series is convergent by the ratio test.

(b) Yeah did that.

Problem 7

(a) Written on the messy previous submission.

(b) A lot.