Problem Set #58

Jayden Li

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Problem 1

$$\begin{aligned}
A &= \int_0^{2\pi r} y\,\mathrm{d}x = \int_0^{2\pi r} r(1-\cos\theta)\frac{\mathrm{d}}{\mathrm{d}\theta}\left[r(\theta-\sin\theta)\right]\,\mathrm{d}\theta = \int_0^{2\pi r} r(1-\cos\theta)r\left(1-\cos\theta\right)\,\mathrm{d}\theta \\
&= r^2\int_0^{2\pi r}\left(1-2\cos\theta+\cos^2\theta\right)\,\mathrm{d}\theta = r^2\left[\theta-2\sin\theta+\frac{\theta}{2}+\frac{\sin(2\theta)}{4}\right]_0^{2\pi r} \\
&= r^2\left(2\pi r-2\sin(2\pi r)+\frac{2\pi r}{2}+\frac{\sin(4\pi r)}{4}\right) - \cancel{r^2\left(0-2\sin 0+\frac{0}{2}+\frac{\sin 0}{4}\right)} \\
&= \boxed{r^2\left(2\pi r-2\sin(2\pi r)+\pi r+\frac{\sin(4\pi r)}{4}\right)}
\end{aligned}$$

Problem 2

$x=0$ when $t=0$ or $t=2$. The bounds of the integral are $t=0$ and $t=2$.

$$\begin{aligned}
A &= \int_0^2 x\,\mathrm{d}y = \int_0^2\left(t^2-2t\right)\frac{1}{2\sqrt{t}}\,\mathrm{d}t = \int_0^2\left(\frac{1}{2}t^{3/2}-t^{1/2}\right)\mathrm{d}t = \left[\frac{1}{2}\frac{2}{5}t^{5/2}-\frac{2}{3}t^{3/2}\right]_0^2 \\
&= \frac{1}{5}2^{2+1/2}-\frac{2}{3}2^{1+1/2}-0+0 = \frac{4}{5}\sqrt{2}-\frac{4}{3}\sqrt{2}
\end{aligned}$$

Problem 3

Bounds of integration are when $y=0$, which are $t=0$ and $t=1$.

$$\begin{aligned}
A &= \int_0^1 y\,\mathrm{d}x = \int_0^1\left(t-t^2\right)e^t\mathrm{d}t = \cancel{\left[e^t\left(t-t^2\right)\right]_0^1} - \int_0^1 e^t(1-2t)\,\mathrm{d}t \\
&= -\left(\left[(1-2t)e^t\right]_0^1 - \int_0^1\left(-2e^t\right)\mathrm{d}t\right) = -\left(-e-1+2\left[e^t\right]_0^1\right) \\
&= -\left(-e-1+2e-2\right) = -\left(e-3\right) = \boxed{3-e}
\end{aligned}$$

Problem 5

(a) $$\begin{aligned}
L &= \int_1^2\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_1^2\sqrt{(1-2t)^2+\left(2\sqrt{t}\right)^2}\,\mathrm{d}t = \int_1^2\sqrt{1-4t+4t^2+4t}\,\mathrm{d}t \\
&= \int_1^2\sqrt{1+4t^2}\,\mathrm{d}t = \int_1^2 2\sqrt{\frac{1}{4}+t^2}\,\mathrm{d}t \approx \boxed{3.168}
\end{aligned}$$

Problem 6

(b) $$\begin{aligned}
L &= \int_0^2\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_0^2\sqrt{\left(\frac{1+t-t}{(1+t)^2}\right)^2+\left(\frac{1}{1+t}\right)^2}\,\mathrm{d}t \\
&= \int_0^2\frac{\sqrt{1+(1+t)^2}}{(1+t)^2}\,\mathrm{d}t = \begin{bmatrix} t=\tan\theta-1 \\ \mathrm{d}t=\sec^2\theta\,\mathrm{d}\theta\end{bmatrix}\int_{\arctan(0+1)}^{\arctan(2+1)}\frac{\sqrt{1+(1+\tan\theta-1)^2}}{(1+\tan\theta-1)^2}\sec^2\theta\,\mathrm{d}\theta \\
&= \int_{\pi/4}^{\arctan 3}\frac{\sqrt{1+\tan^2\theta}}{\tan^2\theta}\sec^2\theta\,\mathrm{d}\theta = \int_{\pi/4}^{\arctan 3}\frac{\sec^3\theta}{\tan^2\theta}\,\mathrm{d}\theta = \int_{\pi/4}^{\arctan 3}\frac{1}{\sin^2\theta\cos\theta}\,\mathrm{d}\theta \\
&= \int_{\pi/4}^{\arctan 3}\frac{\mathrm{d}\theta}{(1+\cos\theta)(1-\cos\theta)\cos\theta} = \int_{\pi/4}^{\arctan 3}\left(\frac{A}{1+\cos\theta}+\frac{B}{1-\cos\theta}+\frac{C}{\cos\theta}\right)\mathrm{d}\theta
\end{aligned}$$

$$\boxed{\begin{aligned}
& A(1-\cos\theta)\cos\theta+B(1+\cos\theta)\cos\theta+C(1+\cos\theta)(1-\cos\theta)=1 \\
\Longrightarrow\ & A\cos\theta-A\cos^2\theta+B\cos\theta+B\cos^2\theta+C-C\cos^2\theta=1 \\
\Longrightarrow\ & \begin{cases} A+B=0 \\ -A+B-C=0 \\ C=1\end{cases} \Longrightarrow \begin{cases} A+B=0 \\ -A+B=1\end{cases} \Longrightarrow \begin{cases} A=-1/2 \\ B=1/2\end{cases}
\end{aligned}}$$

$$\begin{aligned}
&= \int_{\pi/4}^{\arctan 3}\left(\frac{-\frac{1}{2}}{1+\cos\theta}+\frac{\frac{1}{2}}{1-\cos\theta}+\frac{1}{\cos\theta}\right)\mathrm{d}\theta \\
&= \int_{\pi/4}^{\arctan 3}\left(-\frac{1}{2}I_1+\frac{1}{2}I_2+I_3\right)\mathrm{d}\theta
\end{aligned}$$

$$\boxed{\begin{aligned}
I_1 &= \int\frac{\mathrm{d}\theta}{1+\cos\theta} = \int\frac{1-\cos\theta}{\sin^2\theta}\,\mathrm{d}\theta = \int\left(\csc^2\theta-\cot\theta\csc\theta\right)\mathrm{d}\theta = -\cot\theta+\csc\theta+C \\
I_2 &= \int\frac{\mathrm{d}\theta}{1-\cos\theta} = \int\frac{1+\cos\theta}{\sin^2\theta}\,\mathrm{d}\theta = \int\left(\csc^2\theta+\cot\theta\csc\theta\right)\mathrm{d}\theta = -\cot\theta-\csc\theta+C \\
I_3 &= \int\frac{\mathrm{d}\theta}{\cos\theta} = \int\sec\theta\,\mathrm{d}\theta = \ln\left|\sec\theta+\tan\theta\right|+C
\end{aligned}}$$

$$= \left[\frac{\cot\theta}{2}-\frac{\csc\theta}{2}-\frac{\cot\theta}{2}-\frac{\csc\theta}{2}+\ln\left|\sec\theta+\tan\theta\right|\right]_{\pi/4}^{\arctan 3} = \Big[-\csc\theta+\ln\left|\sec\theta+\tan\theta\right|\Big]_{\pi/4}^{\arctan 3}$$

It is known, by drawing triangles, that if $\theta=\arctan 3$, then $\sec\theta=\sqrt{10}$ and $\csc\theta=\sqrt{10}/3$.

$$\begin{aligned}
&= \left(-\frac{\sqrt{10}}{3}+\ln\left(\sqrt{10}+3\right)\right)-\left(-\sqrt{2}+\ln\left(\sqrt{2}+1\right)\right) \\
&= \boxed{-\frac{\sqrt{10}}{3}+\ln\left(\sqrt{10}+3\right)+\sqrt{2}-\ln\left(\sqrt{2}+1\right)}
\end{aligned}$$

Problem 7

(a) $$\begin{aligned}
L &= \int_0^\pi\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_0^\pi\sqrt{\left(e^t\cos t-e^t\sin t\right)^2+\left(e^t\sin t+e^t\cos t\right)^2}\,\mathrm{d}t \\
&= \int_0^\pi\sqrt{e^{2t}\cos^2 t-2e^{2t}\cos t\sin t+e^{2t}\sin^2 t+e^{2t}\sin^2 t+2e^{2t}\sin t\cos t+e^{2t}\cos^2 t}\,\mathrm{d}t \\
&= \int_0^\pi\sqrt{2e^{2t}\cos^2 t+2e^{2t}\sin^2 t}\,\mathrm{d}t = \int_0^\pi\sqrt{2}e^t\sqrt{\cos^2 t+\sin^2 t}\,\mathrm{d}t = \sqrt{2}\int_0^\pi e^t\,\mathrm{d}t \\
&= \sqrt{2}\left[e^t\right]_0^\pi = \boxed{\sqrt{2}\left(e^\pi-1\right)}
\end{aligned}$$

(b) $$\begin{aligned}
L &= \int_{-8}^3\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t = \int_{-8}^3\sqrt{\left(e^t-1\right)^2+\left(2e^{t/2}\right)^2}\,\mathrm{d}t = \int_{-8}^3\sqrt{e^{2t}-2e^t+1+4e^t}\,\mathrm{d}t \\
&= \int_{-8}^3\sqrt{e^{2t}+2e^t+1}\,\mathrm{d}t = \int_{-8}^3\sqrt{\left(e^t+1\right)^2}\,\mathrm{d}t = \int_{-8}^3\left(e^t+1\right)\mathrm{d}t = \left[e^t+x\right]_{-8}^3 \\
&= e^3+3-e^{-8}-(-8) = \boxed{e^3-\frac{1}{e^8}+11}
\end{aligned}$$

Problem 9

$$\begin{cases} x(\theta)=R\cos\theta+\theta R\sin\theta \\ y(\theta)=R\sin\theta-\theta R\cos\theta\end{cases} \Longrightarrow x'(\theta)=-R\sin\theta+R\sin\theta+\theta R\cos\theta=\theta R\sin\theta$$

The left half of the involute is a semicircle containing the point:

$$(x(\pi),y(\pi))=(R\cos\pi+\pi R\sin\pi, R\sin\pi-\pi R\cos\pi)=(-R,-R\pi)$$

The semicircle is centered at $(-R,0)$. Thus the radius of the semicircle is $R\pi$ and its area $A_1=\pi\left(R\pi\right)^2/2=R^2\pi^3/2$.

The other areas are $A_2=A_3$, where:

$$\begin{aligned}
A_2 &= \int_0^\pi\left(R\sin\theta-\theta R\cos\theta\right)\theta R\cos\theta\,\mathrm{d}\theta = \int_0^\pi\theta R^2\sin\theta\cos\theta\,\mathrm{d}\theta-\int_0^\pi\theta^2R^2\cos^2\theta\,\mathrm{d}\theta \\
&= R^2\left(\int_0^\pi\theta\sin\theta\cos\theta\,\mathrm{d}\theta-\int_0^\pi\theta^2\cos^2\theta\,\mathrm{d}\theta\right)
\end{aligned}$$

$$\boxed{\begin{aligned}
I &= \int\theta\sin\theta\cos\theta\,\mathrm{d}\theta = \begin{bmatrix} u=\theta\sin\theta & \mathrm{d}v=\cos\theta\,\mathrm{d}\theta \\ \mathrm{d}u=(\sin\theta+\theta\cos\theta)\,\mathrm{d}\theta & v=\sin\theta\end{bmatrix} \\
&= \theta\sin^2\theta-\int\sin(\theta)\left(\sin\theta+\theta\cos\theta\right)\mathrm{d}\theta = \theta\sin^2\theta-\int\sin^2\theta\,\mathrm{d}\theta-\int\theta\sin\theta\cos\theta\,\mathrm{d}\theta \\
&= \theta\sin^2\theta-\int\frac{1-\cos 2\theta}{2}\,\mathrm{d}\theta-I = \theta\sin^2\theta-\frac{1}{2}\left(\theta-\frac{\sin 2\theta}{2}\right)-I \\
&= \theta\sin^2\theta-\frac{\theta}{2}+\frac{\sin 2\theta}{4}-I \Longrightarrow I=\frac{\theta\sin^2\theta}{2}-\frac{\theta}{4}+\frac{\sin 2\theta}{8}
\end{aligned}}$$

$$\boxed{\begin{aligned}
\int\theta^2\cos^2\theta\,\mathrm{d}\theta &= \int\theta^2\frac{1+\cos 2\theta}{2}\,\mathrm{d}\theta = \frac{1}{2}\int\left(\theta^2+\theta^2\cos 2\theta\right)\mathrm{d}\theta = \frac{1}{2}\left(\frac{\theta^3}{3}+\int\theta^2\cos 2\theta\,\mathrm{d}\theta\right) \\
&= \begin{bmatrix} u=\theta^2 & \mathrm{d}v=\cos 2\theta\,\mathrm{d}\theta \\ \mathrm{d}u=2\theta\,\mathrm{d}\theta & v=\frac{1}{2}\sin 2\theta\end{bmatrix}\frac{\theta^3}{6}+\frac{1}{2}\left(\frac{1}{2}\theta^2\sin 2\theta-\int\theta\sin 2\theta\,\mathrm{d}\theta\right) \\
&= \begin{bmatrix} u=\theta & \mathrm{d}v=\sin 2\theta\,\mathrm{d}\theta \\ \mathrm{d}u=\mathrm{d}\theta & v=-\frac{1}{2}\cos 2\theta\end{bmatrix}\frac{\theta^3}{6}+\frac{1}{2}\left(\frac{1}{2}\theta^2\sin 2\theta-\left(-\frac{1}{2}\theta\cos 2\theta+\int\frac{1}{2}\cos 2\theta\mathrm{d}\theta\right)\right) \\
&= \frac{\theta^3}{6}+\frac{1}{2}\left(\frac{1}{2}\theta^2\sin 2\theta-\left(-\frac{1}{2}\theta\cos 2\theta+\frac{1}{4}\sin 2\theta\right)\right)+C \\
&= \frac{\theta^3}{6}+\frac{1}{2}\left(\frac{1}{2}\theta^2\sin 2\theta+\frac{1}{2}\theta\cos 2\theta-\frac{1}{4}\sin 2\theta\right)+C \\
&= \frac{\theta^3}{6}+\frac{\theta^2\sin 2\theta}{4}+\frac{\theta\cos 2\theta}{4}-\frac{\sin 2\theta}{8}+C
\end{aligned}}$$

$$\begin{aligned}
&= R^2\left(\left[\frac{\theta\sin^2\theta}{2}-\frac{\theta}{4}+\frac{\sin 2\theta}{8}\right]_0^\pi-\left[\frac{\theta^3}{6}+\frac{\theta^2\sin 2\theta}{4}+\frac{\theta\cos 2\theta}{4}-\frac{\sin 2\theta}{8}\right]_0^\pi\right) \\
&= R^2\left(0-\frac{\pi}{4}\right)-R^2\left(\frac{\pi^3}{6}+\frac{\pi}{4}\right) = -\frac{\pi R^2}{4}-\frac{R^2\pi^3}{6}-\frac{\pi R^2}{4} = -\frac{R^2\pi^3}{6}-\frac{\pi R^2}{2}
\end{aligned}$$

Then we need to subtract the area of the fence, which is a circle of radius R. Its area is $A_4=\pi R^2$.

The total area is:

$$\begin{aligned}
A &= |A_1|+|A_2|+|A_3|-|A_4| = \frac{R^2\pi^3}{2}+2\left|-\frac{R^2\pi^3}{6}-\frac{\pi R^2}{2}\right|-\pi R^2 = \frac{R^2\pi^3}{2}+\frac{R^2\pi^3}{3}+\pi R^2-\pi R^2 \\
&= \frac{3R^2\pi^3}{6}+\frac{2R^2\pi^3}{6} = \boxed{\frac{5R^2\pi^3}{6}}
\end{aligned}$$