Linear Algebra 3.2

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1. If $A\mathbf{x} = \mathbf{0}$, then $EA\mathbf{x} = E(A\mathbf{x}) = E\mathbf{0} = \mathbf{0}$.

   If $EA\mathbf{x} = \mathbf{0}$, then $E^{-1}EA\mathbf{x} = E^{-1}\mathbf{0} = IA\mathbf{x} = \mathbf{0}$ (E^{-1} must exist because E is invertible)

   So $EA\mathbf{x} = \mathbf{0} \iff A\mathbf{x} = \mathbf{0}$, which means that $N(A) = N(EA)$.

2. $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 3 \\ 4 & 8 & c \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & c-4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & c-4 \end{bmatrix}$

   If $c = 4$, then $R = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$ is rank 1; column 1 is the pivot and columns 2 and 3 are free.

   The special solutions are $\begin{bmatrix} -2 & 1 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T$.

   If $c \neq 4$, then $R = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & c-4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}$ is rank 2; columns 1 and 3 are the pivots.

   The special solution is $\begin{bmatrix} -2 & 1 & 0 \end{bmatrix}^T$.

   If $c \neq 0$, $B = \begin{bmatrix} c & c \\ c & c \end{bmatrix} \to \begin{bmatrix} c & c \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \end{bmatrix}$ is rank 1; column 1 is the pivot and column 2 is free.

   The special solution is $\begin{bmatrix} -1 & 1 \end{bmatrix}^T$.

   If $c = 0$, then $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is rank 0; both columns are free.

   The special solutions are $\begin{bmatrix} 0 & 1 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & 0 \end{bmatrix}^T$.

3. $$S = \begin{bmatrix} s_1 & s_2 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 1 & 0 \\ 0 & -6 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ 0 & -6 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = P^T \begin{bmatrix} -F \\ I \end{bmatrix}$$

   $$R = \begin{bmatrix} I & F \end{bmatrix} P = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 6 \end{bmatrix}$$

   $$N(A) = \{ cs_1 + ds_2 | \, c, d \in \mathbb{R} \} = \left\{ c \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \end{bmatrix} \middle| \, c, d \in \mathbb{R} \right\}$$

4. $A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$

   $A = CR = \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$

   $B = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 0 & 8 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

   $B = CR = \begin{bmatrix} 2 & 4 \\ 0 & 4 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$

5. Columns 1 and 3 (x_1, x_3) are pivots, columns 2, 4 and 5 (x_2, x_4, x_5) are free.

   When $x_2 = 1, x_4 = 0, x_5 = 0$, the special solution is $\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \end{bmatrix}^T$.

   When $x_2 = 0, x_4 = 1, x_5 = 0$, the special solution is $\begin{bmatrix} 0 & 0 & -2 & 1 & 0 \end{bmatrix}^T$.

   When $x_2 = 0, x_4 = 0, x_5 = 1$, the special solution is $\begin{bmatrix} 0 & 0 & -3 & 0 & 1 \end{bmatrix}^T$.

6. (a) False. In $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, every variable is free.

   (b) True. An invertible matrix must be square and nonsingular. For a nonsingular square matrix A, then the only solution to $A\mathbf{x} = 0$ is $\mathbf{x} = \mathbf{0}$, so there are no free variables.

   (c) True. Rank of matrix is at most $\min(m, n)$. In any case, rank $\leq \min(m, n) \leq n$, so rank $\leq n$.

   (d) True. Rank of matrix is at most $\min(m, n)$. In any case, rank $\leq \min(m, n) \leq m$, so rank $\leq m$.