Problem Set #51

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Problem 1

(a) $f(x) = \dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \displaystyle\sum_{n=0}^{\infty}(-x)^n = \boxed{\sum_{n=0}^{\infty}(-1)^n x^n}$

Interval of convergence: $(-1, 1)$.

(c) $f(x) = \dfrac{x}{9+x^2} = \dfrac{x}{9}\dfrac{1}{1-\left(-\frac{x^2}{9}\right)} = \dfrac{x}{9}\displaystyle\sum_{n=1}^{\infty}\left(-\frac{x^2}{9}\right)^n = \frac{x}{9}\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n}}{9^n} = \boxed{\sum_{n=1}^{\infty}(-1)^n\frac{x^{2n+1}}{9^{n+1}}}$

Converges when $\left|-\dfrac{x^2}{9}\right| < 1 \iff x^2 < 9 \iff |x| < 3$.

Interval of convergence: $(-3, 3)$.

Problem 6

(b)

First, we calculate the power series expansion of the arctangent function.

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = \frac{1}{1+x^2} = \frac{1}{1-\underbrace{(-x^2)}_{|-x^2|<1 \iff |x|<1}} = \sum_{n=0}^{\infty}\left(-x^2\right)^n = \sum_{n=0}^{\infty}(-1)^n x^{2n}$$

$$\arctan x = \int \frac{\mathrm{d}}{\mathrm{d}x}\arctan x \,\mathrm{d}x = \int \sum_{n=0}^{\infty}(-1)^n x^{2n}\,\mathrm{d}x = \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} + C$$

$$\arctan 0 = 0 = \sum_{n=1}^{\infty}\frac{(-1)^n 0^{2n+1}}{2n+1} + C = C \implies C = 0$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} = \boxed{\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}}$$

$$\int \frac{x-\arctan x}{x^3}\,\mathrm{d}x = \int\left(\frac{1}{x^2} - \frac{1}{x^3}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1}\right)\mathrm{d}x = -\frac{1}{x} - \int\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n-2}}{2n+1}\,\mathrm{d}x$$

$$= -\frac{1}{x} - \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n-1}}{(2n+1)(2n-1)} + C$$

$$\text{LHS} = \lim_{n\to 0}\left[-\frac{1}{x} - \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n-1}}{(2n+1)(2n-1)} + C\right] = \lim_{x\to 0}\left[-\frac{1}{x} - x^{-1} + C\right] = C$$

$$\text{RHS} = \lim_{x\to 0}\frac{x-\arctan x}{x^3} \overset{\left[\frac{0}{0}\right]}{\underset{\text{L'H}}{=}} \lim_{x\to 0}\frac{1-\frac{1}{1+x^2}}{3x^2}\frac{1+x^2}{1+x^2} = \lim_{x\to 0}\frac{1+x^2-1}{3x^2\left(1+x^2\right)} = \lim_{x\to 0}\frac{x^2}{3x^2+3x^4}$$

$$\overset{\left[\frac{0}{0}\right]}{\underset{\text{L'H}}{=}} \lim_{x\to 0}\frac{2x}{6x+12x^3} \overset{\left[\frac{0}{0}\right]}{\underset{\text{L'H}}{=}} \lim_{x\to 0}\frac{2}{6+36x^2} = \frac{1}{3}$$

Therefore, $C = 1/3$.

$$= \boxed{-\frac{1}{x} - \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n-1}}{(2n+1)(2n-1)} + \frac{1}{3}}$$

Radius of convergence would be the same as that of arctangent power series above, so $R = 1$.

Problem 7

(a) $\displaystyle\int_0^{0.2}\frac{1}{1+x^5}\,\mathrm{d}x = \int_0^{0.2}\frac{1}{1-(-x^5)}\,\mathrm{d}x = \int_0^{0.2}\sum_{n=0}^{\infty}(-1)^n x^{5n}\,\mathrm{d}x = \left[\sum_{n=0}^{\infty}\frac{(-1)^n x^{5n+1}}{5n+1}\right]_0^{0.2}$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(0.2)^{5n+1}}{5n+1} - \cancel{\sum_{n=0}^{\infty}\frac{(-1)^n(0)^{5n+1}}{5n+1}} \approx \boxed{0.199989}$$

Interval of convergence is $(-1, 1)$, both bounds $0, 0.2 \in (-1, 1)$.

(b) $\displaystyle\int_0^{0.1} x\arctan(3x)\,\mathrm{d}x = \int_0^{0.1} x\sum_{n=0}^{\infty}\frac{(-1)^n(3x)^{2n+1}}{2n+1}\,\mathrm{d}x = \int_0^{0.1}\sum_{n=0}^{\infty}\frac{(-1)^n 3^{2n+1}x^{2n+2}}{2n+1}\,\mathrm{d}x$

$$= \left[\sum_{n=0}^{\infty}\frac{(-1)^n 3^{2n+1}x^{2n+3}}{(2n+1)(2n+3)}\right]_0^{0.1} = \sum_{n=0}^{\infty}\frac{(-1)^n 3^{2n+1}(0.1)^{2n+3}}{(2n+1)(2n+3)} \approx \boxed{0.000983}$$

Radius of convergence is one third of that of arctangent, so interval of convergence is $(-1/3, 1/3)$. Bounds are in the interval of convergence.

Problem 8

(a) $f'(x) = \dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=1}^{\infty}\frac{nx^{n-1}}{n(n-1)!} = \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty}\frac{x^n}{n!} = f(x)$

(b)

Let k be a positive integer.

$$\lim_{n\to\infty}\frac{n!}{(n-k)!n^k} = \lim_{n\to\infty}\frac{\overbrace{n(n-1)(n-2)\times\ldots\times}^{k\text{ terms}}\cancel{(n-k)!}}{\underbrace{n\times n\times\ldots\times n}_{k\text{ terms}}\cancel{(n-k)!}} = \lim_{n\to\infty}\frac{n}{n}\frac{n-1}{n}\frac{n-2}{n}\ldots\frac{n-k+1}{n} = 1$$

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{x}{n}\right)^k = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{\cancel{n!}}{k!\cancel{(n-k)!}}\frac{x^k}{\cancel{n^k}} = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{x^k}{k!} = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

Problem 9

We know that:

$$\tan\frac{\pi}{6} = \frac{\sin\frac{\pi}{6}}{\cos\frac{\pi}{6}} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}} \iff \arctan\left(\tan\frac{\pi}{6}\right) = \arctan\left(\frac{1}{\sqrt{3}}\right) \iff \frac{\pi}{6} = \arctan\left(\frac{1}{\sqrt{3}}\right)$$

$$\pi = 6\arctan\left(\frac{1}{\sqrt{3}}\right) = 6\sum_{n=0}^{\infty}\frac{(-1)^n\left(\frac{1}{\sqrt{3}}\right)^{2n+1}}{2n+1} = 6\sum_{n=0}^{\infty}\frac{(-1)^n 3^{-\frac{2n+1}{2}}}{2n+1} = 6\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^{n+1/2}}$$

$$= \frac{6}{3^{1/2}}\frac{\sqrt{3}}{\sqrt{3}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^n} = \frac{6\sqrt{3}}{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^n} = 2\sqrt{3}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)3^n}$$