Linear Algebra 3.1

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2. No; multiplication by scalar 1 is not the identity operation. $1 \cdot (a, b) = (a, 0) \neq (a, b)$.

3. (a) Not closed under scalar multiplication. $-1 \cdot (2) = (-2) \notin \text{V}$.

   (b) $c(\mathbf{x} + \mathbf{y}) = 3(\mathbf{2} + \mathbf{1}) = 3 \cdot \mathbf{2} = 2^3 = 8 \stackrel{?}{=} c\mathbf{x} + c\mathbf{y} = 3 \cdot \mathbf{2} + 3 \cdot \mathbf{1} = \mathbf{8} + \mathbf{1} = 8 \cdot 1 = 8$. It works. "Zero vector" is the scalar value 1.

4. Zero vector is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. $\dfrac{1}{2}A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$. $-A = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$.

   Smallest subspace containing A is $\left\{ a \cdot \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \middle| a \in \mathbb{R} \right\}$.

5. (a) $\left\{ a \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \middle| a \in \mathbb{R} \right\}$.

   (b) Yes. Any vector space containing both A and B must be closed under linear combination.

$$\underbrace{1 \cdot A + (-1) \cdot B}_{\text{linear combination}} = 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

   Because the set is closed, I must be in it.

   (c) $\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

8. (a) $\left\{ \begin{bmatrix} a \\ b \end{bmatrix} \middle| a, b \in \mathbb{Z} \right\}$

   (b) $\left\{ \begin{bmatrix} a \\ b \end{bmatrix} \middle| a \in \mathbb{R}, b \in \mathbb{R} \setminus \{0\} \right\}$
   (product of two nonzero real numbers cannot be zero, but their sum could be)

9. (a) Yes. $r(a_1, a_2, a_3) + s(b_1, b_2, b_3) = (ra_1 + sb_1, ra_1 + sb_1, ra_3 + sb_3)$.

   (b) No. Not closed under addition. $(1, 0, 0) + (1, 0, 0) = (2, 0, 0)$, $2 \neq 1$.

   (c) No. Not closed under addition. $(1, 1, 0) + (0, 1, 1) = (1, 2, 1)$, $(1)(2)(1) = 2 \neq 0$.

   (d) Yes. Span is a vector space.

   (e) Yes. The plane passes through the origin.

   (f) No. Not closed under scalar multiplication. $-1 \cdot (1, 2, 3) = (-1, -2, -3)$, $-1 \not\leq -2$.

10. (a) $\left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \middle| a, b \in \mathbb{R} \right\}$

(b) $\left\{ \begin{bmatrix} a & a \\ 0 & 0 \end{bmatrix} \middle| \, a \in \mathbb{R} \right\}$

(c) $\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| \, a, b \in \mathbb{R} \right\}$

11. $(1, 1, -1) \in P \ (1 + 1 + (-2)(-1) = 1 + 1 + 2 = 4)$

$(4, 0, 0) \in P \ (4 + 0 + 0 = 4)$

$(1, 1, -1) + (4, 0, 0) = (5, 1, -1) \notin P \ (5 + 1 + (-2)(-1) = 5 + 1 + 2 = 8 \neq 4)$

12. $\boxed{\{(x, y, z) | \, x + y - 2z = 0\}}$

$(1, 1, 1) \in P \ (1 + 1 + (-2)(1) = 1 + 1 - 2 = 0)$

$(4, -4, 0) \in P \ (4 + (-4) + 0 = 0)$

$(1, 1, 1) + (4, -4, 0) = (5, -3, 1) \notin P \ (5 + (-3) + (-2)(1) = 5 - 3 - 2 = 0)$

14. (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ which is singular and not invertible.

(b) $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which is nonsingular.

15. (a) True. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 4 \\ 2 & 4 & 9 \\ 4 & 9 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 8 & 14 \\ 7 & 14 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 7 \\ 4 & 8 & 14 \\ 7 & 14 & 7 \end{bmatrix}^T$

(b) True. $\begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 5 \\ -3 & -5 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 4 \\ -2 & 0 & 9 \\ -4 & -9 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 7 \\ -4 & 0 & 14 \\ -7 & -14 & 0 \end{bmatrix} = -\left(\begin{bmatrix} 0 & 4 & 7 \\ -4 & 0 & 14 \\ -7 & -14 & 0 \end{bmatrix} \right)^T$

(c) True. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 5 & 8 & 11 \\ 7 & 10 & 13 \end{bmatrix}$ which is not symmetric.