Graph Theory

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1 Graph Basics

Definition 1.1 (Graph). *A graph G is a triple containing the vertex set $V(G)$, the edge set $E(G)$ and a relationship between vertices and edges called the endpoints.*

The endpoints are often defined by means of an image.

Definition 1.2 (Loop). *A loop is an edge whose endpoints are equal.*

Definition 1.3 (Multiple edges). *Multiple edges are edges with the same endpoints.*

Definition 1.4 (Simple graph). *A simple graph is a graph without loops or multiple edges.*

The edges of a simple graph can be specified with only the vertices: an edge $e = uv$ where $u, v \in V(G)$.

Definition 1.5 (Adjacency). *If there exists an edge $e \in E(G)$ with endpoints u and v, then u and v are adjacent.*

Definition 1.6 (Null graph). *A null graph G is a graph such that $|V(G)| = |E(G)| = 0$.*

Definition 1.7 (Complement of a graph). *The complement of a simple graph G, denoted $\overline{G}$, is a simple graph such that $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.*

Definition 1.8 (Clique). *A clique is a set of vertices such that each vertex is adjacent to all other vertices.*

Definition 1.9 (Independent set). *An independent set is a set of vertices such that each vertex is not adjacent to all other vertices.*

Definition 1.10 (Bipartite graph). *A bipartite graph is a graph that can be divided into two disjoint and independent sets U, V called partites, so that each edge in $E(G)$ are between a vertex in U and a vertex in V.*

A graph is bipartite if and only if there are no odd cycles.

Definition 1.11 (Path). *A path is a simple graph whose edges join a sequence of vertices, and so that the vertices can be ordered in a list.*

Definition 1.12 (Cycle). *A cycle is a graph with an equal number of edges and vertices, so that the vertices can be placed into a circle.*

Definition 1.13 (Subgraph). *A graph H is a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, and is denoted $H \subseteq G$.*

Definition 1.14 (Connectedness). *A graph G is connected if for all vertices $u, v \in V(G)$, there exists a path between u and v. A graph G is not connected if there exists two vertices $u, v \in V(G)$ such that no path exists between u and v.*

Definition 1.15 (Incidence and Degree)**.** *If $u \in V(G)$ is an endpoint of an edge $e \in E(G)$, then u and e are incident. The degree of a vertex is the number of incident edges.*

Definition 1.16. *P_n denotes a path with n vertices and $n-1$ edges. C_n denotes a cycle with n vertices and n edges.*

Definition 1.17 (Completeness)**.** *A graph is complete if every vertex is adjacent to every other vertex. A complete graph with n vertices is denoted K_n.*

Definition 1.18 (Complete bipartite graph)**.** *A complete bipartite graph is a bipartite graph such that two vertices are adjacent if and only if they are in different partite sets, and is denoted $K_{r,s}$ where r, s are the size of each bipartite set.*

Definition 1.19 (Regularity)**.** *A k-regular graph is one where every vertex is of degree k.*

2 Petersen graph

Definition 2.1 (Petersen graph)**.** *The Peterson graph is a simple graph whose vertices are 2-element subsets of a 5-element set and whose edges are the pairs of disjoint 2-element subsets.*

Assign each point a pair of numbers. There are $\binom{5}{2} = 10$ vertices, and connect two points with an edge if they do not share any number. Every vertex is degree 3 so the Petersen graph so it is 3-regular.

Proposition 2.1. *If two vertices are nonadjacent in the Petersen graph then they have exactly one common neighbor.*

Definition 2.2 (Girth)**.** *The girth of a graph with a cycle is the length of its shortest cycle. A graph with no cycle has infinite girth.*

Corollary 2.2.1. *The Petersen graph has girth* 5*.*

There cannot be a 3-cycle including points ab and cd, because another point connected to both ab and cd must not share any elements in the 5-element set. But there is only one element in the set that is not a, b, c, d, so no such point can exist.

There cannot be a 4-cycle including points ab and bc (share b because they are not connected), because there cannot exist two other points that share one element in the set in order to not be connected.

3 Matrix Representations

Definition 3.1 (Adjacency matrix)**.** *The adjacency matrix $A(G)$ of a graph with n vertices $\{v_1, v_2, \ldots, v_n\}$ is an $n \times n$ matrix in which the entry a_{ij} is the number of edges in G with endpoints v_i, v_j.*

If an edge has endpoints v_i and v_j, it would add to both entries a_{ij} and a_{ji}, so an adjancency matrix must be symmetric.

Definition 3.2 (Incidence matrix)**.** *The incidence matrix $M(G)$ of a graph with n vertices $\{v_1, v_2, \dots v_n\}$ and m edges $\{e_1, e_2, \dots, e_m\}$ is an $n \times m$ matrix in which the entry m_{ij} is 1 if v_i is an endpoint of edge e_j, and 0 if not.*

4 Isomorphism

An equivalence relation is a relation between two sets that is reflexive, symmetric and transitive.

Definition 4.1 (Isomorphism)**.** *An isomorphism from a simple graph G to a simple graph H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. Isomorphism is denoted as $G \cong H$.*

A graph G is isomorphic to a graph H if and only if we can permute the columns and rows of the adjacency matrix of G to create the adjacency matrix of H.

An isomorphism class is an equivalence class:

- Reflexivity: A graph G is isomorphic to itself.
- Symmetry: If G is isomorphic to H, then H is isomorphic to G.
- Transitivity: If G is isomorphic to H, and H is isomorphic to K, then G is isomorphic to K.

Disproving bipartiteness: find an odd cycle. If one exists, one point in the cycle will be connected to both set of points.

To prove an isomorphism, we identify a bijective mapping that preserves all adjacencies, and is often done by establishing a labelling.

To prove graphs are not isomorphic, demonstrate they contradict each other with features.

Theorem 4.1. *$G \cong H$ if and only if $\overline{G} \cong \overline{H}$.*

Proof of Theorem 4.1. Suppose G and H are isomorphic, and let f be a bijection such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. Thus $uv \notin E(G) \iff f(u)f(v) \notin E(H)$.

By the definition of the graph complement, $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$, and $uv \in E(\overline{H})$ if and only if $uv \notin E(H)$. So:

$$uv \notin E(G) \iff f(u)f(v) \notin E(H) \iff (uv \in E(\overline{G}) \iff f(u)f(v) \in E(\overline{H}))$$

Thus, $\overline{G}$ is isomorphic to $\overline{H}$.

Now we prove the only if case: if $\overline{G} \cong \overline{H}$, then $G \cong H$.

$$\overline{G} \cong \overline{H} \implies G = \overline{\overline{G}} \cong \overline{\overline{H}} = H$$

So G is isomorphic to H. $\square$

Definition 4.2 (Self-complementary)**.** *A graph is self-complementary if it is isomorphic to its complement.*

The graph P_4 is isomorphic to its complement $\overline{P_4}$.

5 Decomposition

Definition 5.1 (Decomposition)**.** *A decomposition of a graph is a list of subgraphs such that each edge appears in exactly one subgraph in the list.*

Theorem 5.1. *An n-vertex graph H is self-complementary if and only if K_n has a decomposition consisting of two copies of H.*

For example, K_5 can be decomposed into two 5-cycles. If we placed the vertices in a pentagon and drew the edges, there are two 5-cycles: the outer pentagon and the inner 5-pointed star. So we know the 5-vertex cycle C_5 is self-complementary.

To prove a graph can be decomposed, propose the subgraphs.