Problem Set #48

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Problem 1

(a)

$$a_n = \frac{(-1)^n}{\sqrt[3]{x}^2}$$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{x^{2/3}} \text{ diverges by the } p\text{-series test as } p = 2/3 \leq 1.$$

$$\lim_{n\to\infty} a_n = 0$$

$$\sqrt[3]{x+1} > \sqrt[3]{x} \implies \sqrt[3]{x+1}^2 > \sqrt[3]{x}^2 \implies \frac{1}{\sqrt[3]{x+1}^2} = a_{n+1} < a_n = \frac{1}{\sqrt[3]{x}^2}$$

The series is not absolutely convergent since $\sum |a_n|$ diverges, but $\sum a_n$ converges by the alternating series test. So $\sum a_n$ is conditionally convergent.

(b)

$$a_n = \frac{\cos n}{n^2}$$

$$\forall n \in \mathbb{Z}^+ : |\cos n| \leq 1 \implies \frac{|\cos n|}{n^2} \leq \frac{1}{n^2} \implies \left|\frac{\cos n}{n^2}\right| \leq \frac{1}{n^2}$$

$\sum 1/n^2$ converges by the p-series test since $p = 2 > 1$. Then $\sum \cos |n/n^2|$ converges by the direct comparison test, so $\sum a_n$ is absolutely convergent.

(c)

$$a_n = \frac{(-1)^n}{\sqrt{n^2-1}}$$

$$\sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2-1}} \text{ diverges by limit comparison to } \sum \frac{1}{n}.$$

$$\lim_{n\to\infty} \frac{1}{\sqrt{n^2-1}} = 0$$

$$\begin{aligned}
\forall \mathbb{N} \ni n \geq 2 : 2n \geq -1 &\implies n^2 + 2n \geq n^2 - 1 \implies (n+1)^2 - 1 \geq n^2 - 1 \\
&\implies \sqrt{(n+1)^2 - 1} \geq \sqrt{n^2-1} \\
&\implies \frac{1}{\sqrt{(n+1)^2-1}} = a_{n+1} \leq a_n = \frac{1}{\sqrt{n^2-1}}
\end{aligned}$$

$\sum a_n$ converges by the alternating series test so the series conditionally converges.

Problem 2

$$a_n = \frac{b_n^n \cos(\pi n)}{n}, |a_n| = \frac{b_n^n}{n}$$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\frac{b_n^n}{n}} = \lim_{n\to\infty} \frac{b_n}{\sqrt[n]{n}} = \frac{1}{2} \lim_{n\to\infty} \frac{1}{n^{1/n}} = \frac{1}{2} \lim_{n\to\infty} n^{-1/n} = \frac{1}{2}$$

$$\boxed{\ln \lim_{n\to\infty} n^{-1/n} = \lim_{n\to\infty} \left[-\frac{1}{n} \ln n\right] = -\lim_{n\to\infty} \frac{\ln n}{n} = 0 \implies \lim_{n\to\infty} n^{-1/n} = e^0 = 1}$$

The series is absolutely convergent by the root test.