Problem Set #37

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Problem 3

(a) At any time, the distance traveled by the dog is equal to the distance traveled by the rabbit. Also note that because the dog always runs in a straight line to the rabbit, the tangent line to $f(x)$ passes through the rabbit.

The distance traveled by the dog and rabbit can be calculated using the arc length formula. This is the y-coordinate of the rabbit. The y-coordinate of the dog equals $f(x)$. The x-difference between the animals is always x. Therefore:

$$f'(x) = \frac{1}{x}\left(f(x) - \int_x^L \sqrt{1+(f'(t))^2}\,\mathrm{d}t\right) \implies xf'(x) = f(x) - \int_x^L \sqrt{1+(f'(t))^2}\,\mathrm{d}t$$

$$\implies \frac{\mathrm{d}}{\mathrm{d}x}\left[f(x) - xf'(x)\right] = \frac{\mathrm{d}}{\mathrm{d}x}\int_x^L \sqrt{1+(f'(t))^2}\,\mathrm{d}t \implies f'(x) - f'(x) - xf''(x) = -\sqrt{1+(f'(x))^2}$$

$$\implies xf''(x) = \sqrt{1+(f'(x))^2} \implies \boxed{x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}}$$

(b) $$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \implies \begin{bmatrix} z = \dfrac{\mathrm{d}y}{\mathrm{d}x} \\ \dfrac{\mathrm{d}z}{\mathrm{d}x} = \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}\end{bmatrix} x\frac{\mathrm{d}z}{\mathrm{d}x} = \sqrt{1+z^2} \implies \frac{1}{\sqrt{1+z^2}}\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{1}{x}$$

$$\implies \int \frac{1}{\sqrt{1+z^2}}\,\mathrm{d}z = \int \frac{1}{x}\,\mathrm{d}x = \ln|x|$$

$$\boxed{\begin{aligned}\int \frac{1}{\sqrt{1+z^2}}\,\mathrm{d}z &= \begin{bmatrix} z = \tan\theta \\ \mathrm{d}z = \sec^2\theta\end{bmatrix}\int \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\,\mathrm{d}\theta = \int \sec\theta\,\mathrm{d}\theta = \ln|\sec\theta + \tan\theta| + C_0 \\ &= \ln\left|\sqrt{z^2+1}+z\right| + C_0\end{aligned}}$$

$$\implies \ln\left|\sqrt{z^2+1}+z\right| + C_0 = \ln|x| \implies C_0\sqrt{z^2+1} + C_0z = x$$

$$\implies C_0^2\left(z^2+1\right) = x^2 - 2C_0xz + C_0^2z^2 \implies C_0^2 = x^2 - 2C_0xz$$

$$\implies 2C_0xz = x^2 - C_0^2 \implies z = \frac{x^2 - C_0^2}{2C_0x}$$

$$z(L) = 0 \implies \frac{L^2 - C_0^2}{2C_0L} = 0 \implies L^2 - C_0^2 = 0 \implies C_0^2 = L^2 \implies C_0 = \pm L$$

$$z = \frac{x^2 - L^2}{2Lx} \implies y = \int \frac{x^2-L^2}{2Lx}\,\mathrm{d}x = \frac{1}{2}\int\left(\frac{x}{L} - \frac{L}{x}\right)\mathrm{d}x = \frac{1}{2}\left(\frac{x^2}{2L} - L\ln|x|\right) + C_1$$

$$y(0) = 0 \implies \frac{1}{2}\left(\frac{0}{2L} - L\right) + C_1 = -\frac{L}{2} + C_1 = 0 \implies C_1 = \frac{L}{2}$$

$$\implies y = \frac{1}{2}\left(\frac{x^2}{2L} - L\ln|x|\right) + \frac{L}{2} = \boxed{\frac{x^2}{4L} - \frac{L\ln|x|}{2} + \frac{L}{2}}$$

(c) No. The above function is not defined at $x = 0$, and because the x-component of the rabbit's position always equals 0, the dog will never reach it.