Problem Set #44

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Problem 7

$$\int\left(\frac{1}{\sqrt{x^2+4}}-\frac{C}{x+2}\right)\mathrm{d}x=\begin{bmatrix}x=2\tan\theta\\ \mathrm{d}x=2\sec^2\theta\,\mathrm{d}\theta\end{bmatrix}\int\frac{2\sec^2\theta}{\sqrt{4\tan^2\theta+4}}\,\mathrm{d}\theta-\int\frac{C}{x+2}\,\mathrm{d}x$$

$$=\int\frac{\cancel{2}\sec^2\theta}{\cancel{\sqrt{4}}\sqrt{\tan^2\theta+1}}\,\mathrm{d}\theta-C\ln|x+2|=\int\frac{\sec\theta\cancel{\tan\theta}}{\cancel{\tan\theta}}\,\mathrm{d}\theta-C\ln|x+2|$$

$$=\ln|\sec\theta+\tan\theta|-C\ln|x+2|+D$$

$$=\ln\left|\frac{x}{2}+\sqrt{1+\frac{x^2}{4}}\right|-C\ln|x+2|+D$$

$$=\ln\left|\frac{x}{2}+\sqrt{1+\frac{x^2}{4}}\right|-\ln\left(|x+2|^C\right)+D$$

$$=\ln\left|\frac{\frac{x}{2}+\sqrt{1+\frac{x^2}{4}}}{(x+2)^C}\right|+D$$

$$\int_0^\infty\left(\frac{1}{\sqrt{x^2+4}}-\frac{C}{x+2}\right)\mathrm{d}x=\lim_{t\to\infty}\left[\ln\left|\frac{\frac{x}{2}+\sqrt{1+\frac{x^2}{4}}}{(x+2)^C}\right|\right]_0^t=\lim_{t\to\infty}\ln\left|\frac{\frac{t}{2}+\sqrt{1+\frac{t^2}{4}}}{(t+2)^C}\right|-\ln\left|\frac{0+\sqrt{1+0}}{2^C}\right|$$

$$=\lim_{t\to\infty}\ln\left|\frac{\frac{t}{2}+\sqrt{1+\frac{t^2}{4}}}{(t+2)^C}\right|-\ln\left(\left(\frac{1}{2}\right)^C\right)=\lim_{x\to\infty}\ln\left|\frac{\frac{x}{2}+\sqrt{1+\frac{x^2}{4}}}{(x+2)^C}\right|+C\ln 2$$

If you go a change of variable $t=x/2$, we arrive at the below. Since $t\to\infty$, also $x\to\infty$, so the limits are the same.

$$\exp\lim_{t\to\infty}\ln\left|\frac{t+\sqrt{t^2+4}}{2(t+2)^C}\right|=\lim_{t\to\infty}\left|\frac{t+\sqrt{t^2+4}}{2(t+2)^C}\right|=\lim_{t\to\infty}\frac{t+\sqrt{t^2+4}}{2(t+2)^C}\overset{\left[\frac{\infty}{\infty}\right]}{\underset{\text{LH}}{=}}\lim_{t\to\infty}\frac{1+\frac{2t}{2\sqrt{t^2+4}}}{2C(t+2)^{C-1}}$$

$$=\lim_{t\to\infty}\frac{1+\frac{t}{t\sqrt{1+\frac{4}{t^2}}}}{2C(t+2)^{C-1}}=\lim_{t\to\infty}\frac{1+1}{2C(t+2)^{C-1}}=\lim_{t\to\infty}\frac{1}{C(t+2)^{C-1}}$$

When $C-1<0$, the denominator tends to 0, meaning the limit does not exist. Hence, the above limit equals a real number when $C-1\ge 0\iff C\ge 1$.

Case 1. $C=1$.

$$\lim_{t\to\infty}\frac{1}{C(t+2)^{C-1}}=\lim_{t\to\infty}\frac{1}{1(t+2)^0}=1$$

Case 2. $C>1$; then $C-1>0$ so $(t+2)^{C-1}$ tends to ∞.

$$\lim_{t\to\infty}\frac{1}{C(t+2)^{C-1}}=0$$

Therefore:

$$\lim_{t\to\infty}\frac{1}{C(t+2)^{C-1}}=\begin{cases}1 & C=1\\ 0 & C>1\end{cases}$$

$$=\begin{cases}1 & C=1\\ 0 & C>1\end{cases}$$

$$\lim_{t\to\infty}\ln\left|\frac{t+\sqrt{t^2+4}}{2(t+2)^C}\right|=\ln\left(\begin{cases}1 & C=1\\ 0 & C>1\end{cases}\right)=0\text{ when }C=1$$

$$=0+C\ln 2=\boxed{\ln 2}$$

When $C=1$, the integral equals $\ln 2$. Otherwise, the integral does not exist.