Homework 1.1 (3)

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3. Square matrix made up of four square block matrices, as follows:

$$\begin{array}{c} \vdots \\ m \text{ entries} \\ \vdots \\ \hline \vdots \\ n \text{ entries} \\ \vdots \end{array} \left[\begin{array}{ccc|ccc} 0 & \dots & 0 & 1 & \dots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \dots & 1 \\ \hline 1 & \dots & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \dots & 1 & 0 & \dots & 0 \end{array}\right]$$

12. The Petersen graph is girth 5 so there is a 5-cycle, so it is not bipartite.

    The largest independent set is size 4. There cannot exist an independent set of size 5. The vertex 12 is connected to three vertices (34, 35 and 45) and not connected to six vertices: $S = \{13, 14, 15, 23, 24, 25\}$. If an independent set of size 5 exists, then we can add four of these vertices into an independent set with 12. But for any subset of size 4 in S, at least one of the vertices is adjacent to another vertex. We know this because we can find a cycle in S: $13, 24, 15, 23, 14, 25$. We cannot choose 4 vertices that are not adjacent to each other because the largest independent set in the cycle is $6/2 = 3$.

14. Done in class.

15. Done in class.

21. Done in class.

22. G_1, G_2, G_5 are isomorphic: $G_1 \cong G_2 \cong G_5 \cong C_7$.

    G_3, G_4 are isomorphic but are not isomorphic to C_7: $G_3 \cong G_4$.

24.

25.

26. Done in class.

29. Let each person be a vertex. An "acquaintance" of a person is any vertex it is adjacent to. A "stranger" is any vertex it is not adjacent to. We are to prove any graph G with six vertices contains either an independent set of size 3 or a clique of size 3. This occurs if and only if $\overline{G}$ has an independent set of size 3 or a clique of size 3.

    Let $u \in V(G)$. There might be 3 or more vertices adjacent to u in G. If not, and there are less than 3 vertices adjacent to u in G, then there are less than 3 vertices not adjacent to u in $\overline{G}$, so the degree of u in $\overline{G}$ is greater than or equal to 3. In either case, u will have degree greater than or equal to 3 in either G or $\overline{G}$.

Let us take the first case (G). The vertex u is adjacent to at least three vertices x, y, z. If (x, y), (x, z) or (y, z) are adjacent, then those two vertices and x form a clique. If not, then x, y, z form an independent set.

This is also true in $\overline{G}$.

There is either an independent set or clique of size 3 in G or in $\overline{G}$. If either exists in $\overline{G}$, it must also exist in G.

$\square$