Task 1

a)

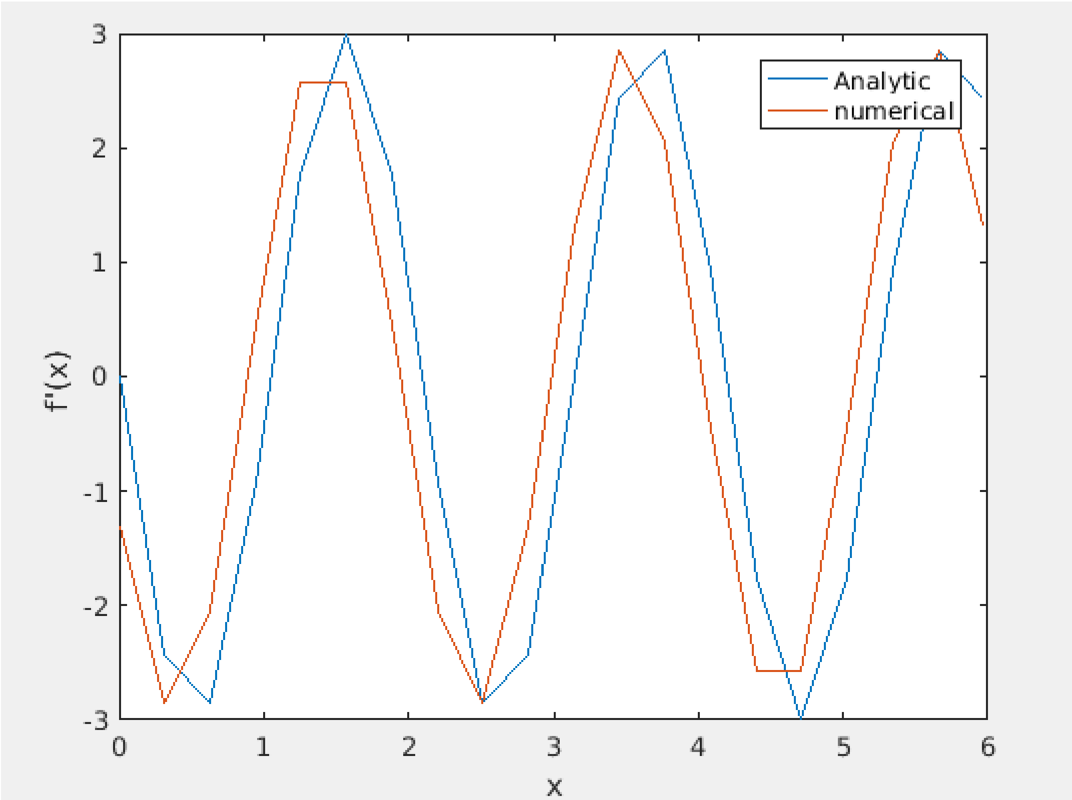
$$\underline{f}'_{num} = \begin{bmatrix} -1 & 1 & & & & \\ & -1 & 1 & & 0 & \\ & & -1 & 1 & & \\ & 0 & & \ddots & \ddots & \\ & & & & & 1 \\ 1 & & & & & -1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{bmatrix}$$

b) $f(x) = e^{ikx}$

$$\underline{f}'_{num} = \frac{e^{ik(x+\Delta x)} - e^{ikx}}{\Delta x} = \frac{\cancel{e^{ikx}}}{\Delta x}\left(e^{ik\Delta x} - 1\right) = \tilde{k}\cancel{e^{ikx}}$$

$\cancel{\tilde{k}e^{ik}}$ $\boxed{\tilde{k}\Delta x = e^{ik\Delta x} - 1}$

c)



d) mu =

```
  -1.3121 + 2.5752i
  -1.3121 + 2.5752i
  -1.3121 + 2.5752i
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  -1.3121 + 2.5752i
```

From the output we can see that ikf = Df for the current case. Analytically this makes sense as well!

k_mod =

```
  -1.3121 + 2.5752i
```

Task 2

Q1 a) FTBS

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\,\frac{u_j^n - u_{j-1}^n}{\Delta x} = 0$$

$$u_j^n = \hat{u}_k^n e^{ikx_j}$$

$$u_j^{n+1} = \hat{u}_k^{n+1} e^{ikx_j}$$

$$u_{j-1}^n = \hat{u}_k^n e^{ikx_j} e^{-ik\Delta x}$$

Sub:

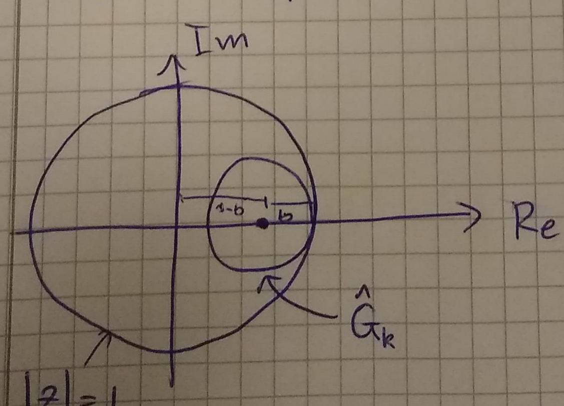
$$\frac{(\hat{u}_k^{n+1} - \hat{u}_k^n)e^{ikx_j}}{\Delta t} + \frac{c e^{ikx_j}(\hat{u}_k^n - \hat{u}_k^n e^{-ik\Delta x})}{\Delta x} = 0$$

$$\hat{u}_k^{n+1} = \hat{u}_k^n \underbrace{\left(1 - \underbrace{\frac{c\Delta t}{\Delta x}}_{b}\left(1 - e^{-ik\Delta x}\right)\right)}_{\hat{G}_k}$$

$$\hat{G}_k = 1 - b(1 - e^{-ik\Delta x})$$

$$\boxed{\hat{G}_k = \underbrace{1 - b + b\cos k\Delta x}_{Re = 1 - b + b\cos k\Delta x} - \underbrace{b\sin k\Delta x}_{Im = -b\sin k\Delta x}}$$

stable if $|\hat{G}_k| \leq 1$



stable if:

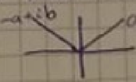
$$\boxed{b = \frac{c\Delta t}{\Delta x} \leq 1}$$

Task 2 b) cont..

Q1

wont $|\hat{G}_k| \le 1$

Since the $\pm$ is only on the $Re(\hat{G}_k)$ the modulus will be the same: $z = \pm a + ib$



$Re(\hat{G}_k) = \pm\sqrt{1-b^2\sin^2 k\Delta x}$ $\quad$ $Im(\hat{G}_k) = -ib\sin k\Delta x$

$$|\hat{G}_k| = \sqrt{(1-b^2\sin^2 k\Delta x) + b^2\sin^2 k\Delta x}$$

$$= 1$$

Task 2

Q2 a general solution over characteristic line $x = x_0 - ct$

$$u(x(t), t) = u_0 e^{ikx(t)} = u_0 e^{ik(x_0 - ct)}$$

$$u(x(t+\Delta t), t) = u_0 e^{ik(x_0 - ct - c\Delta t)}$$

$$\hat{G}_k = \frac{u_0 e^{ik(x_0 - ct - c\Delta t)}}{u_0 e^{ik(x_0 - ct)}} = \frac{\cancel{e^{ik(x_0 - ct)}}\, e^{-ikc\Delta t}}{\cancel{e^{ik(x_0 - ct)}}}$$

$$\boxed{\hat{G}_k = e^{-ikc\Delta t}} = e^{-ikc\Delta t \times \frac{\Delta x}{\Delta x}} = \boxed{e^{-ib\phi}}$$

Analytic

now $|\hat{G}_k| = 1$

$$\tan(\boxed{\tilde{\Phi}}) = -\frac{\sin(-b\phi)}{\cos(-b\phi)} = \boxed{\tan(b\phi)}$$

First Order

$$|\hat{G}_k| = (1 - b + b\cos k\Delta x)^2 + b^2 \sin^2 k\Delta x$$

$$= (1-b)^2 + 2b(1-b)\cos k\Delta x + b^2\cos^2 k\Delta x + b^2 \sin^2 k\Delta x$$

$$= (1-b)^2 + 2b(1-b)\cos k\Delta x + b^2$$

$$= 1 - 2b + b^2 + b^2 + 2b(1-b)\cos k\Delta x$$

$$= 1 - 2b + 2b^2 + 2b(1-b)\cos k\Delta x$$

$$= 1 - 2b(1-b) + 2b(1-b)\cos k\Delta x$$

$$= 1 - 2b(1-b)(1 - \cos k\Delta x)$$

$$\boxed{\varepsilon_A = \frac{1 - 2b(1-b)(1-\cos k\Delta x)}{1}}$$

Task 2
Q2

First Order cont...

$$\tan(\Phi) = + \frac{b \sin k\Delta x}{1 - b + b\cos k\Delta x}$$

ϕ

$$\boxed{\varepsilon_p = \tan^{-1}\left(\frac{b \sin k\Delta x}{1 - b + b\cos k\Delta x}\right) \times \frac{1}{b\phi}}$$

Leapfrog:

$|\hat{G}_k| = 1$ (from question 1)

$$\boxed{\varepsilon_A = 1}$$

$$\tan(\Phi) = - \frac{-b\sin \cancel{k\Delta x}\,\phi}{\pm\sqrt{1 - b^2\sin^2\phi}}$$

$$\tan(\Phi) = \pm \frac{b\sin\phi}{\sqrt{1 - b^2\sin^2\phi}}$$

$$\boxed{\varepsilon_p = \tan^{-1}\left(\pm \frac{b\sin\phi}{\sqrt{1 - b^2\sin^2\phi}}\right) \times \frac{1}{b\phi}}$$

Task 2
Q2

First Order cont...

$$\tan(\Phi) = + \frac{b \sin k\Delta x}{1 - b + b\cos k\Delta x}$$

$$\boxed{\varepsilon_P = \tan^{-1}\left(\frac{b \sin k\Delta x}{1 - b + b\cos k\Delta x}\right) \times \frac{1}{b\phi}}$$

ϕ

Leapfrog:

$|\hat{G}_k| = 1$ (from question 1)

$$\boxed{\varepsilon_A = 1}$$

$$\tan(\Phi) = -\frac{-b\sin\phi}{\pm\sqrt{1 - b^2\sin^2\phi}}$$

$$\tan(\Phi) = \pm\frac{b\sin\phi}{\sqrt{1 - b^2\sin^2\phi}}$$

$$\boxed{\varepsilon_P = \tan^{-1}\left(\pm\frac{b\sin\phi}{\sqrt{1 - b^2\sin^2\phi}}\right) \times \frac{1}{b\phi}}$$