Question 1

Given: $R = 5\,km$ $\qquad P_e = 300\,kr/kg$

$v = 2.5\,km/h$ $\qquad P_{C_0} = 10\,kr/kg$

$T = 1$ hour

$P_f = P_e = 300\,kr/kg$ at time $t = T$

Also given:

"price decrease linearly with x" $\rightarrow \frac{\partial P}{\partial x} = f(t) = at + b$

"price change linearly with t" $\frac{\partial P}{\partial t} = g(x) = cx + d$

ALSO given: (BC's)

$$\left.\frac{\partial P}{\partial t}\right|_{x=0} = 0 \quad \forall t \Rightarrow d = 0$$

$$\left.\frac{\partial P}{\partial t}\right|_{x=R} = \frac{P_e - P_{C_0}}{T} \Rightarrow c = \frac{P_e - P_{C_0}}{RT}$$

$$\Rightarrow \boxed{\frac{\partial P}{\partial t} = \frac{P_e - P_{C_0}}{RT}x}$$

$$\left.\frac{\partial P}{\partial x}\right|_{t=0} = -\frac{(P_e - P_{C_0})}{R} \Rightarrow b = -\frac{(P_e - P_{C_0})}{R}$$

$$\left.\frac{\partial P}{\partial x}\right|_{t=T} = 0 \Rightarrow a = \frac{P_e - P_{C_0}}{RT}$$

$$\Rightarrow \boxed{\frac{\partial P}{\partial x} = \frac{P_e - P_{C_0}}{RT}(t - T)}$$

Question 1 cont...

using the notion of "material derivative" with our little man walk ~~&~~ having trajectory $x = vt$ $\left(\begin{array}{l}\text{at } t=0,\ x=0 \\ \text{constant velocity}\end{array}\right)$

$\frac{\partial x}{\partial t} = v$

we have $\frac{dP}{dt} = \frac{\partial P}{\partial t} + \frac{\partial x}{\partial t} \times \frac{\partial P}{\partial x}$

$$= \frac{\partial P}{\partial t} + v\frac{\partial P}{\partial x}$$

$$= \frac{P_e - P_{co}}{RT}x + v\left(\frac{P_e - P_{co}}{RT}\right)(t - T)$$

$$= \frac{P_e - P_{co}}{RT}vt + v\left(\frac{P_e - P_{co}}{RT}\right)(t - T)$$

$$\boxed{\frac{dP}{dt} = \frac{v(P_e - P_{co})}{RT}(2t - T)}$$

max/min @ $\frac{dP}{dt} = 0 \Rightarrow t = T/2$

we can also see by ~~subbing in~~ $\frac{d^2P}{dt^2} = \frac{2v(P_e - P_{co})}{RT} > 0 \Rightarrow$ min

$\boxed{\text{cheapest price @ } t = T/2,\ x = \frac{vT}{2}}$

$$P = \int \frac{v(P_e - P_{co})}{RT}(2t - T)\,dt = \frac{v(P_e - P_{co})}{RT}\left[t^2 - Tt\right] + C$$

@ $t = 0$, $P = P_e$

$$\rightarrow P = \frac{v(P_e - P_{co})}{RT}\left[t^2 - Tt\right] + P_e$$

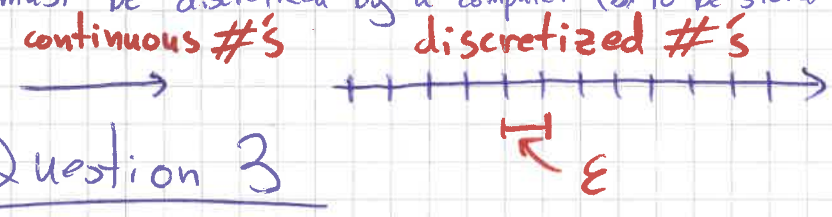
@ $t = T/2$ and subbing in values $P =$ ~~[illegible]~~ 263.75 kr/kg

$\boxed{\text{with 100kr little man buys } 0.3791\ldots\text{kg}}$

Question 2

① single : 1.1921×10^{-7}
double : 2.2204×10^{-16}

② I would interperate "Machine accuracy" (based on the Matlab code) to be the smallest number such that the computer can't tell what the next nearest nuber is. Ie, the continuous number line must be discretized by a computer (to be ~~stored~~ described in memory)



Question 3

① In Matlab plot
(with $f'(x) = -\dfrac{1}{(x+2)^2} + 2x$

② we have X_1 with uncertainty ε_1
X_2 ε_2

and $g(X_1, X_2) = X_1 + X_2$

$$\frac{\partial g}{\partial X_1} = 1 \quad , \quad \frac{\partial g}{\partial X_2} = 1$$

$$\varepsilon_p = \sum_{j=1}^{n} \left| \frac{a_j}{g} \frac{\partial g}{\partial a_j} \right| \varepsilon_j$$

$$= \left| \frac{X_1}{g(X_1, X_2)} \times \frac{\partial g}{\partial X_1} \right| \varepsilon_1 + \left| \frac{X_2}{g(X_1, X_2)} \times \frac{\partial g}{\partial X_2} \right| \varepsilon_2$$

$$= \left| \frac{X_1}{X_1 + X_2} \right| \varepsilon_1 + \left| \frac{X_2}{X_1 + X_2} \right| \varepsilon_2$$

$$\boxed{\varepsilon_p = \frac{|X_1|}{|X_1 + X_2|} \varepsilon_1 + \frac{|X_2|}{|X_1 + X_2|} \varepsilon_2}$$

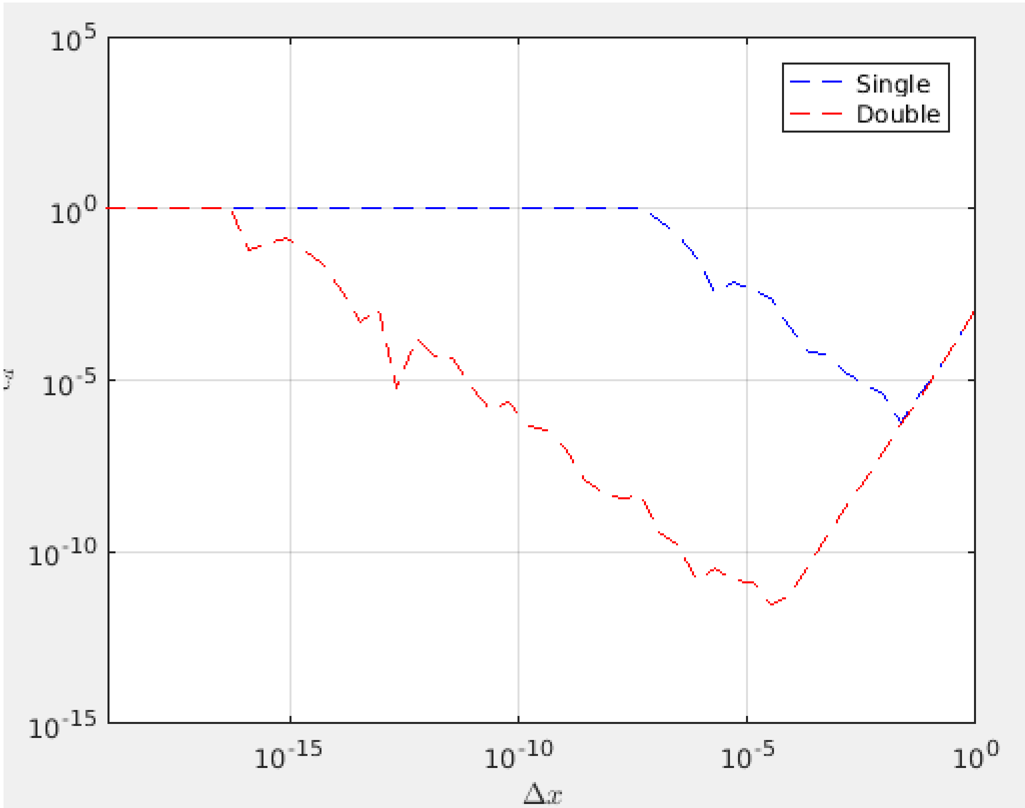


Figure 1: Loglog plot of Δx against relative discretization error for both single and double precision.

It can be seen that ε = 1 for very small Δx (indicating Matlab has evaluated is f'_n(x) = 0). It is clear that the notion of "very small Δx" is dependent on whether single or double precision is used.
As Δx increases, ε decreases. This trend continues until a certain minimum at which point the relative discretization error increases again.

Question 3 cont...

③ $f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + \frac{\Delta x^3}{3!} f'''(x) + O(\Delta x^4)$

$f(x-\Delta x) = f(x) - \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) - \frac{\Delta x^3}{3!} f'''(x) + O(\Delta x^4)$

$$\frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} = \frac{2\Delta x f'(x) + 2\frac{\Delta x^3}{3!} f'''(x) + \dots}{2\Delta x}$$

$$= f'(x) + \frac{\Delta x^2}{3!} f'''(x) + O(\Delta x^4)$$

$$\varepsilon_d = \left| \frac{f'(x) - f_n'(x)}{f'(x)} \right|$$

$$= \left| \frac{\cancel{f'(x)} - \cancel{f'(x)} - \frac{\Delta x^2}{3!} f'''(x)}{f'(x)} \right|$$

$$= \left| \frac{-\frac{\Delta x^2}{3!} f'''(x)}{f'(x)} \right| = \boxed{\frac{\Delta x^2}{6} \frac{|f'''(x)|}{|f'(x)|}}$$

this minus is included in X_2

$$f_n'(x) = \underbrace{\frac{f(x+\Delta x)}{2\Delta x}}_{X_1} - \underbrace{\frac{f(x-\Delta x)}{2\Delta x}}_{X_2}$$

$$\mathcal{E}_p = \frac{|X_1|}{|X_1+X_2|} \varepsilon_1 + \frac{|X_2|}{|X_1+X_2|} \varepsilon_2$$

but each has the same machine accuracy.

$\Rightarrow \varepsilon_1 = \varepsilon_2 = \varepsilon$

Question 3 cont...

Note: $|X_1 + X_2| = \frac{1}{2\Delta x}\left|\left[2\Delta x f'(x) + \frac{2\Delta x^3}{3!} f'''(x) + \dots\right]\right|$ as before

$$|X_1| + |X_2| = \frac{1}{2\Delta x}\Big[\left|f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + \dots\right|$$
$$+ \left|f(x) - \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + \dots\right|\Big]$$

since Δx is small we only take leading terms

$$|X_1 + X_2| \approx \frac{1}{2\Delta x}|2\Delta x f'(x)| = |f'(x)|$$

$$|X_1| + |X_2| \approx \frac{1}{2\Delta x}\left[|f(x)| + |f(x)|\right] = \frac{|f(x)|}{\Delta x}$$

now $\boxed{\mathcal{L}_p} = \frac{\varepsilon\left[|X_1| + |X_2|\right]}{|X_1 + X_2|}$ (as $\varepsilon_1 = \varepsilon_2 = \varepsilon$)

$$\approx \frac{\varepsilon\left[\frac{|f(x)|}{\Delta x}\right]}{|f'(x)|} = \boxed{\frac{|f(x)|\varepsilon}{|f'(x)|\Delta x}}$$

Question 3 cont.

$$\mathcal{E}_g = \mathcal{E}_d + \mathcal{E}_p = \frac{\Delta x^2 |f'''(x)|}{6|f'(x)|} + \frac{|f(x)|\varepsilon}{|f'(x)|\Delta x}$$

Want to find min:

$$\frac{\partial \mathcal{E}_g}{\partial \Delta x} = \frac{\Delta x |f'''(x)|}{3|f'(x)|} - \frac{|f(x)|\varepsilon}{|f'(x)|\Delta x^2}$$

solve $\frac{\partial \mathcal{E}_g}{\partial \Delta x} = 0$

$$\Rightarrow \Delta x^3 = \frac{|f(x)|\varepsilon \times 3\cancel{|f'(x)|}}{\cancel{|f'(x)|} \times |f'''(x)|}$$

$$\boxed{\Delta x = \sqrt[3]{\frac{|f(x)|}{|f'''(x)|} \times 3\varepsilon}}$$

now $\frac{\partial^2 \mathcal{E}_g}{\partial \Delta x^2} = \frac{|f'''(x)|}{3|f'(x)|} + \frac{2|f(x)|\varepsilon}{|f'(x)|\Delta x^3} > 0 \quad \forall \Delta x, \varepsilon > 0$

$\Rightarrow$ min!

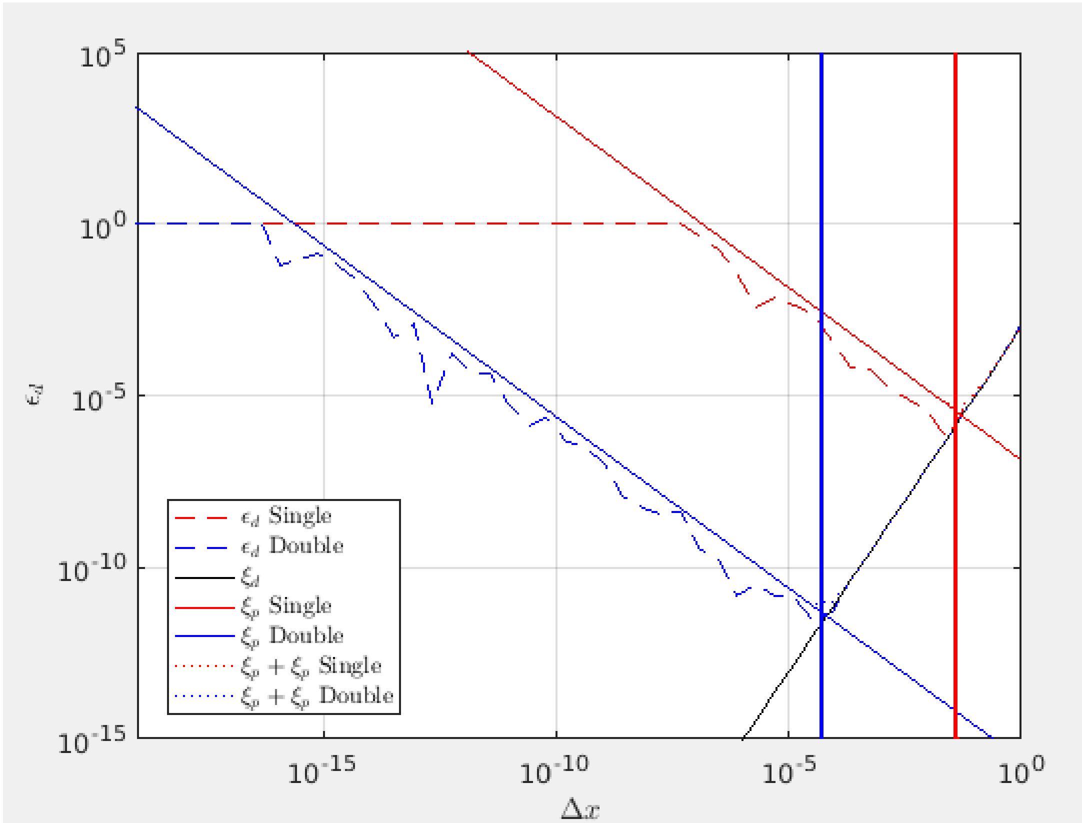


Figure 2: Loglog plot of Δx against relative discretization error for ξd ξp ξg.

The trend in figure 1 is explained in more detail here. A similar trend can be seen for both single and double precision albeit somewhat skewed and shifted.

The decrease in ε (with increasing Δx) initially observed can be explain by the propagation error while the subsequent increase in ε can be explained by the discretization error.

The trend noticed in figure 1 that " ε = 1 for very small Δx" is not apparent in the ξp plots as f'_n(x) is not being explicitly calculated.