Task 1:

$$\frac{du}{dt} = \lambda u \qquad u(0) = 1 \qquad \lambda \in \mathbb{C}$$
$$t \in [0, T]$$

$$u = e^{\lambda t} + C$$

but $u(0) = 1 \Rightarrow C = 0$

$$\boxed{u = e^{\lambda t}}$$

(2) (see figure 1)

(3) EE: $u^{n+1} - u^n = \Delta t \lambda u^n$

$$\boxed{u^{n+1} = (1 + \lambda \Delta t) u^n} \rightarrow \boxed{G(z) = 1 + z} \quad \lim_{z \to \infty} G(z) \to \infty$$

IE: $$\boxed{u^{n+1} = \frac{1}{1 - \lambda \Delta t} u^n} \rightarrow \boxed{G(z) = \frac{1}{1 - z}} \quad \lim_{z \to \infty} G(z) \to 0$$

CN: $u^{n+1} - u^n = \frac{1}{2} \Delta t [\lambda u^{n+1} + \lambda u^n]$ ~~$\lim_{z \to 0} G(z) \to$~~

$$u^{n+1} = u^n \left(\frac{1 + \lambda \Delta t \frac{1}{2}}{1 - \lambda \Delta t \cdot \frac{1}{2}} \right)$$

$$\boxed{u^{n+1} = \left(\frac{2 + \lambda \Delta t}{2 - \lambda \Delta t} \right) u^n} \rightarrow \boxed{G(z) = \frac{2 + z}{2 - z}} \quad \lim_{z \to \infty} (G(z)) = \frac{\frac{2}{z} + \frac{z}{z}}{\frac{2}{z} - \frac{z}{z}} \rightarrow 1$$

Analytic: $$\boxed{G(z) = e^{z}}$$

(see figure 2)

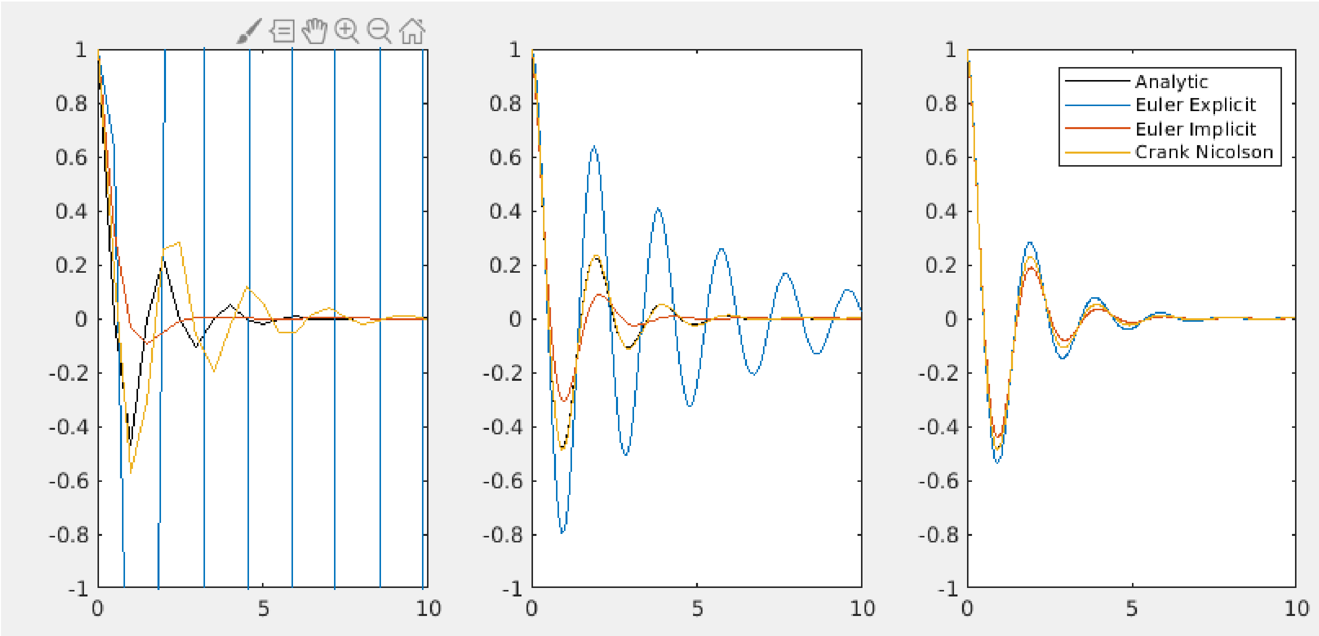


Figure 1) Left to right, N = 20, 100, 500.

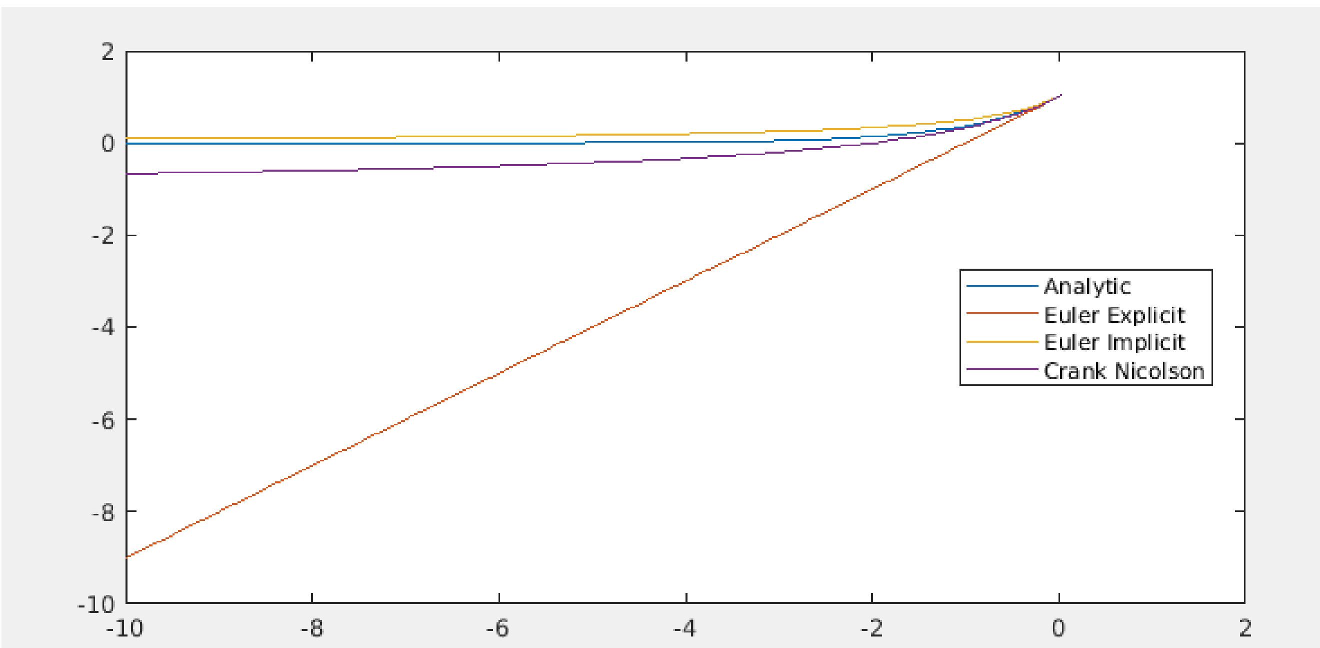
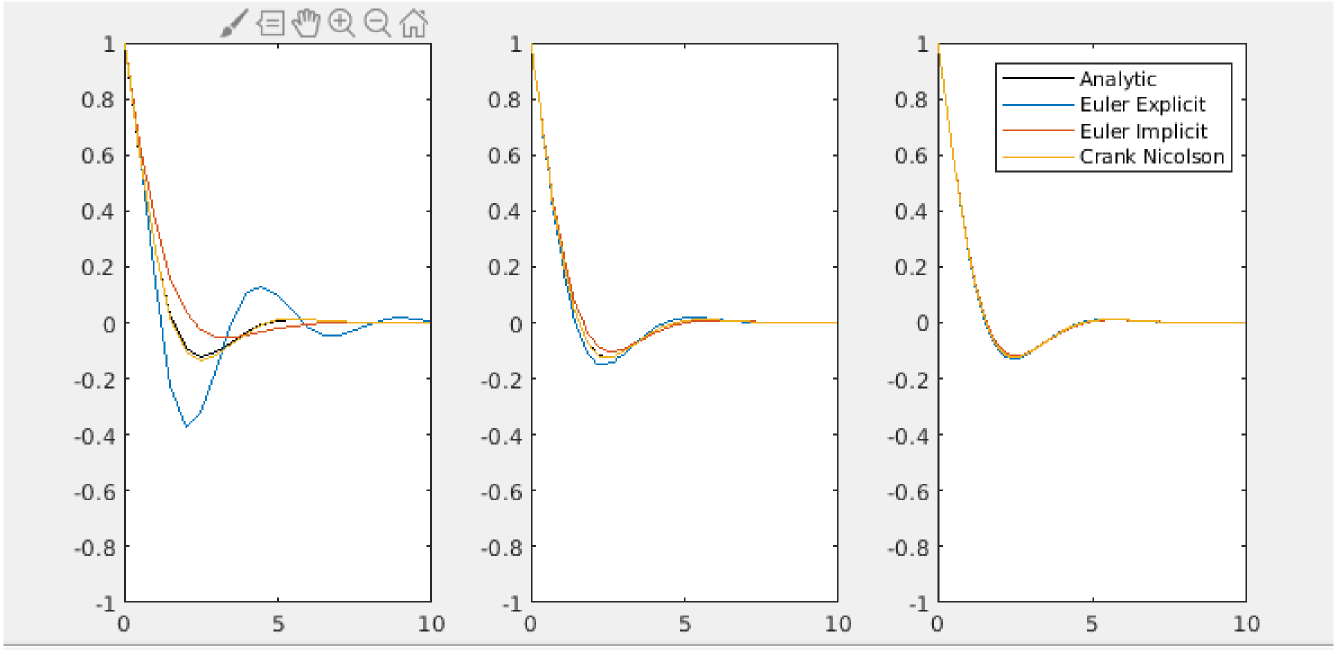
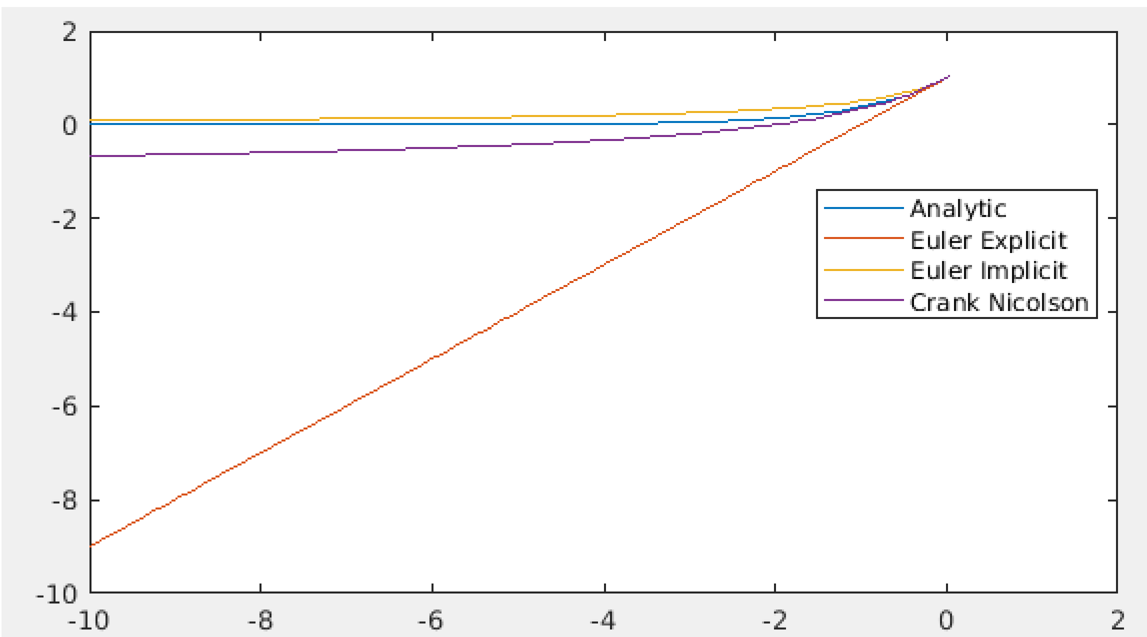


Figure 2) G(z) The Crank Nicolson scheme is a 2nd order scheme and should provide a better approximation of the exact solution.
The imaginary parts lead to phase shifts/oscillatory behavior as opposed to amplification and are hence not as important to the current stability analysis.

4)

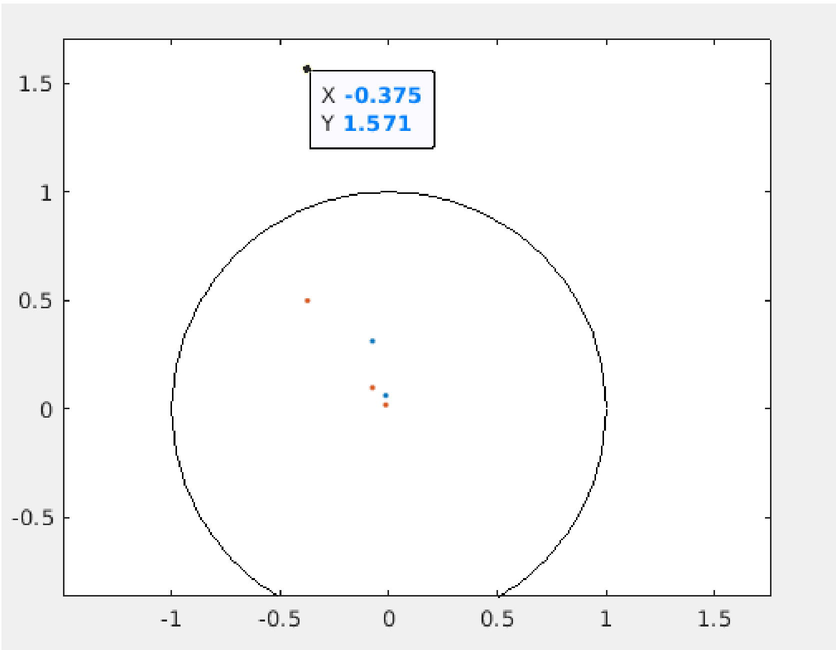

Analytic
Euler Explicit
Euler Implicit
Crank Nicolson


Analytic
Euler Explicit
Euler Implicit
Crank Nicolson

5) Stability:
For question 2 and task 3 we have the following z values and magnitudes:

Question 2:
-0.3750 + 1.5708i
-0.0750 + 0.3142i
-0.0150 + 0.0628i
Question 4:
-0.3750 + 0.5000i
-0.0750 + 0.1000i
-0.0150 + 0.0200i

Euler explicit:
We know that stability is ensured if |1 + z| <= 1
Hence it is only N = 20 for question 2 which is unstable. This can be seen in the following stability diagram:



Euler Implicit:
We know that the Euler implicit method is unconditionally stable (shown in lectures), hence we have stability for all.

Crank-Nicoloson:
instead of drawing the diagram I have just computed the G(z) for all cases:

Question 2	Question 4
0.7937	0.7005
0.9294	0.9279
0.9851	0.9851

They are all below 1.

Accuracy:
For accuracy both Euler methods are equally accurate (first order) and Crank-Nicolson is 2^{nd} order.

Task 2:

let $A, B, C, D \in \mathbb{R}$ constants.

$$Au(x-\Delta x) = Au(x) - A\Delta x\, u'(x) + A\frac{\Delta x^2}{2}u''(x) - A\frac{\Delta x^3}{3!}u'''(x) + A\frac{\Delta x^4}{4!}u''''(x)$$

$$Bu(x) = Bu(x)$$

$$Cu(x+\Delta x) = Cu(x) + C\Delta x\, u'(x) + C\frac{\Delta x^2}{2}u''(x) + C\frac{\Delta x^3}{3!}u'''(x) + C\frac{\Delta x^4}{4!}u''''(x)$$

$$Du(x+2\Delta x) = Du(x) + D(2\Delta x)u'(x) + D\frac{4\Delta x^2}{2}u''(x) + D\frac{8\Delta x^3}{3!}u'''(x) + D\frac{16\Delta x^4}{4!}u''''(x)$$

we want $Au(x-\Delta x) + Bu(x) + Cu(x+\Delta x) + D(x+2\Delta x)$

such that the following cancel: (1) $u(x)=0$, (2) $u'(x)$ remains!
(3) $u''(x)=0$
(4) $u'''(x)=0$

(1) $A+B+C+D=0$

(2) $-A\Delta x + C\Delta x + 2D\Delta x = 1$

(3) $A\frac{\Delta x^2}{2} + C\frac{\Delta x^2}{2} + D\frac{4\Delta x^2}{2} = 0$

(4) $-A\frac{\Delta x^3}{3!} + C\frac{\Delta x^3}{3!} + D\frac{8\Delta x^3}{3!} = 0$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 8 \end{bmatrix}\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\Delta x \\ 0 \\ 0 \end{bmatrix}$$

Matlab $\rightarrow$ $$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \frac{1}{\Delta x}\begin{bmatrix} -1/3 \\ -1/2 \\ 1 \\ -1/6 \end{bmatrix}$$

$$\Rightarrow u'(x) = \frac{1}{6\Delta x}\left(-2u(x-\Delta x) - 3u(x) + 6u(x+\Delta x) - u(x+2\Delta x)\right)$$

Task 2 cont...

b) leading error term:

$$\left(A\frac{\Delta x^4}{4!} + 0 + C\frac{\Delta x^4}{4!} + D\frac{16\Delta x^4}{4!}\right)u''''(x)$$

$$= \frac{\Delta x^4}{6\Delta x \times 4!}\left(-2 + 6 - 16\right)u''''(x)$$

$$= \frac{-12\Delta x^3}{6 \times 4!} = \frac{-\Delta x^3}{12}$$

order 3.

Task 3

$$\boxed{k_0 = \lambda u^n}$$

$$\boxed{k_1} = \lambda\left(u^n + \frac{\Delta t}{2}(\lambda u^n)\right) = u^n\lambda\left(1 + \frac{\lambda\Delta t}{2}\right) = \boxed{u^n\lambda\left(1 + \frac{z}{2}\right)}$$

$$\boxed{k_2} = \lambda\left(u^n + \frac{\Delta t}{2}\left(u^n\lambda\left(1+\frac{z}{2}\right)\right)\right)$$

$$= \lambda u^n\left(1 + \frac{z}{2}\left(1 + \frac{z}{2}\right)\right) = \boxed{\lambda u^n\left(1 + \frac{z}{2} + \frac{z^2}{4}\right)}$$

$$\boxed{k_3} = \lambda\left(u^n + \Delta t\left(\lambda u^n\left(1 + \frac{z}{2} + \frac{z^2}{4}\right)\right)\right)$$

$$= \lambda u^n\left(1 + z + \frac{z^2}{2} + \frac{z^3}{4}\right)$$

$$u^n = u^n + \frac{\Delta t}{6}\left[u^n\lambda\left(1+\frac{z}{2}\right)\times 2 + \lambda u^n\left(1+\frac{z}{2}+\frac{z^2}{4}\right)\times 2 + \lambda u^n\left(1+z+\frac{z^2}{2}+\frac{z^3}{4}\right) + \lambda u^n\right]$$

$$u^{n+1} = u^n\left[1 + \frac{z}{6}\left(1 + 2 + z + 2 + z + \frac{z^2}{2} + 1 + z + \frac{z^2}{2} + \frac{z^3}{4}\right]\right.$$

$$\boxed{u^{n+1} = u^n\left[1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24}\right]}$$

see figure 3.

Task 3

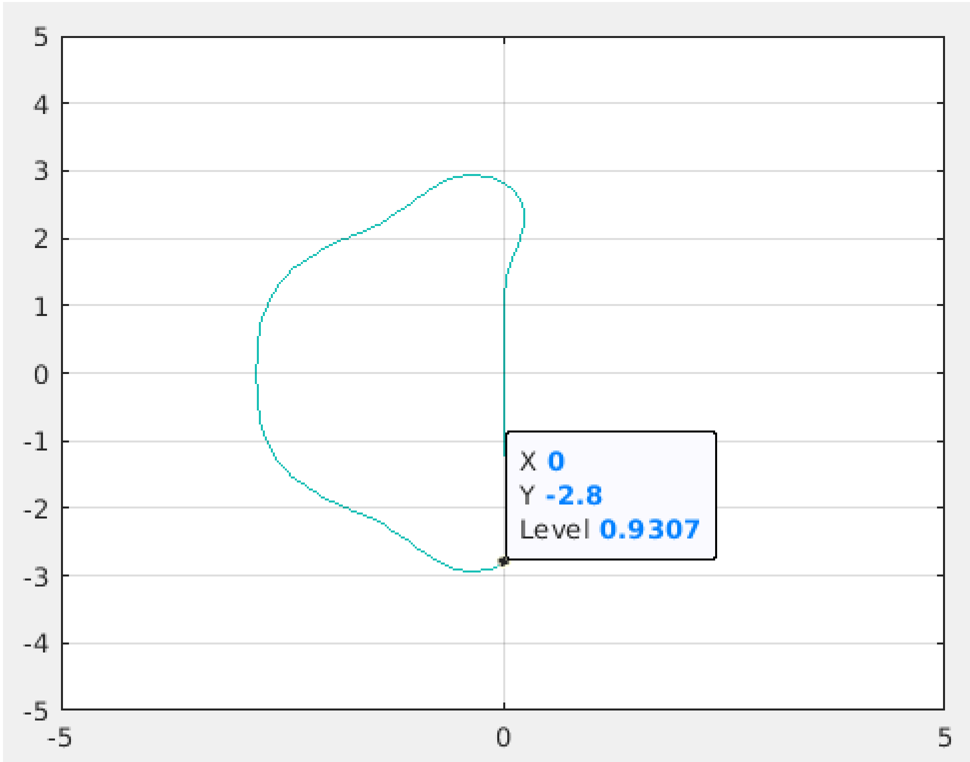


Figure 3) Complex plane showing region of stability (note it should be shaded inside to show stability)