Figure 1: youo

1 2D lid-driven cavity

1.1 Case A

1.2 Case B

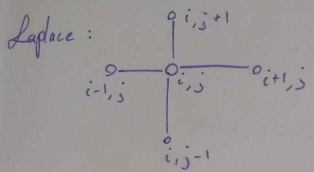
1.3 Case C

1.4 Question 1

Q1

Start with $P_{m\times n}$.

Laplace:



$$\frac{\partial^2 P_{i,j}}{\partial x^2} \doteq \frac{P_{i-1,j}-2P_{i,j}+P_{i+1,j}}{\Delta x^2} + \frac{P_{i,j-1}-2P_{i,j}+P_{i,j+1}}{\Delta y^2}$$

In Matrix form:

$$\underbrace{\frac{1}{\Delta x^2}\begin{bmatrix}\ddots & \ddots & \\ 1 & -2 & 1 \\ & \ddots & \ddots\end{bmatrix}}_{L_x}\underbrace{\begin{bmatrix}u_{11} & u_{12} & \cdots \\ u_{21} & \ddots & \\ \vdots & & u_{mn}\end{bmatrix}}_{P} + \underbrace{\begin{bmatrix}u_{11} & u_{12} & \cdots \\ u_{21} & \ddots & \\ \vdots & & u_{mn}\end{bmatrix}}_{P}\underbrace{\frac{1}{\Delta y^2}\begin{bmatrix}\ddots & \ddots & \\ 1 & -2 & 1 \\ & \ddots & \ddots\end{bmatrix}}_{L_y}$$

$$L_x P + P L_y$$

multiply by I

$$= L_x P I + I P L_y$$

$(B^T \otimes A)\,\mathrm{vec}(X) = \mathrm{vec}(AXB)$

$$= (I^T \otimes L_x)\,\mathrm{vec}(P) + (L_y \otimes I)\,\mathrm{vec}(P)$$

$$= \left[I \otimes L_x + L_y \otimes I\right]\mathrm{vec}(P)$$

1.5 Question 2

The two stability conditions for a general 2D advection diffusion problem are stated in equations (1) and (2) and the combined condition includes equations (2) and (3).
Note: *These equations were derived in lectures and part of the exam so the derivation has been omitted from this report.*

$$\text{Advection (CFL): } \sigma_x + \sigma_y \leq 1 \tag{1}$$

$$\text{Diffusion: } \beta_x + \beta_y \leq \frac{1}{2} \tag{2}$$

$$\text{Combined: } \frac{\sigma_x^2}{\beta_x} + \frac{\sigma_y^2}{\beta_y} \leq 2 \tag{3}$$

In the case of the non-linear Navier Stokes equations we take $a_x = \max|u|$ and $a_y = \max|v|$. Furthermore, by assuming $\Delta x = \Delta y = h$, equations (1) - (3) can be further simplified to equations (4) - (6). A final conservative assumption to equations (4) and (6) is that $\max|u| \approx \max|v| < U_{lid} = 1$.

$$\text{CFL: } \Delta t \leq \frac{h}{\max(|u|) + \max(|v|)} \approx \frac{h}{2} \tag{4}$$

$$\text{Viscous limit: } \Delta t \leq \frac{h^2}{4}\text{Re} \tag{5}$$

$$\text{Combined: } \Delta t \leq \frac{2}{\text{Re}\left(\max(|u|)^2 + \max(|v|)^2\right)} \approx \frac{1}{Re} \tag{6}$$

These conditions have been tabulated for cases A - C below:

	Re	h	max($\lvert u\rvert$)	max($\lvert v\rvert$)	CFL	Viscous	Combined
Case A	20	$\frac{1}{30}$	•	•	0.0167	0.0056	0.0500
Case B	200	$\frac{1}{50}$	•	•	0.0100	0.0200	0.0050
Case C	4000	$\frac{1}{100}$	•	•	0.0050	0.1000	0.0003

It can be seen that the energy equation takes a similar form to the momentum equation with the Peclet number replacing the Reynolds number. Hence we can assert the following analogous conditions to equations (5) and (6):

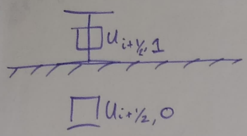
$$\text{Viscous: } \Delta t_{energy} \leq \frac{2}{\text{Pe}\left(\max(|u|)^2 + \max(|v|)^2\right)} \approx \frac{1}{Pe} = \frac{1}{Pr}\Delta t_{momentum} \tag{7}$$

$$\text{Combined: } \Delta t_{energy} \leq \frac{h^2}{4}\text{Pe} = Pr\Delta t_{momentum} \tag{8}$$

In the case that $Pr = 0.71 < 1$ then in the viscous limit (equation (8))$\Delta t_{energy} < \Delta t_{momentum}$ is a stricter condition.

1.6 Question 3

BC's for u:



Assume at the boundary we want $u = u_{BC}$.

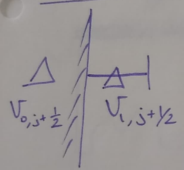
$u_{BC} = 1$ top (lid driven)

$u_{BC} = 0$ bottom (wall)

Take the average (linear interpolation): $\frac{u_{i+\frac{1}{2},1} + u_{i+\frac{1}{2},0}}{2} = u_{BC}$.

$$\Rightarrow \boxed{u_{i+\frac{1}{2},0} = 2u_{BC} - u_{i+\frac{1}{2},1}}$$

BC's for v:



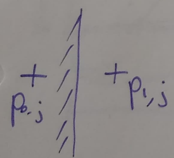
At boundary we wish to have $v = v_{BC}$.

$v_{BC} = 0$ (both walls)

$$\frac{v_{0,j+\frac{1}{2}} + v_{1,j+\frac{1}{2}}}{2} = v_{BC}$$

$$\Rightarrow \boxed{v_{0,j+\frac{1}{2}} = 2v_{BC} - v_{1,j+\frac{1}{2}}}$$

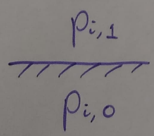
BC's for pressure:



We wish to have $\frac{\partial p}{\partial n} = 0$ on ~~BCs~~ Boundaries

Using finite difference:

$$\frac{\partial p}{\partial n} \approx \frac{p_{1,j} - p_{0,j}}{\Delta x} = 0 \Rightarrow \boxed{p_{0,j} = p_{1,j}}$$



$$\boxed{p_{i,0} = p_{i,1}}$$

Laplacian: (in 1D).

$$\frac{\partial^2 p}{\partial x^2} \doteq \frac{p_{i-1} - 2p_i + p_{i+1}}{\Delta x^2}$$

at $i = 1$ (boundary) $\rightarrow \dfrac{p_0 - 2p_1 + p_2}{\Delta x^2} = \dfrac{p_2 - p_1}{\Delta x^2}$ $(p_0 = p_1)$

1.7 Question 4

1.8 Question 5