Brownian motion

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Wiener process: Definition

The stochastic process $\{W(t); t \geq 0\}$ is a (standard) Wiener process (standard Brownian motion) if:

- $W(0) = 0$
- The process has independent increments

$$\big(W(t_4) - W(t_3)\big) \perp \big(W(t_2) - W(t_1)\big);$$

$$\forall t_1, t_2, t_3, t_4: \;\; t_4 \geq t_3 \geq t_2 \geq t_1$$

- $W(t) - W(s) \sim N\big(0, \sqrt{t - s}\big), \quad \forall\, t, s: \;\; t > s \geq 0$
- The sample trajectories are continuous (almost surely).

With probability one
(i.e. excluding negligible events, which have probability 0)

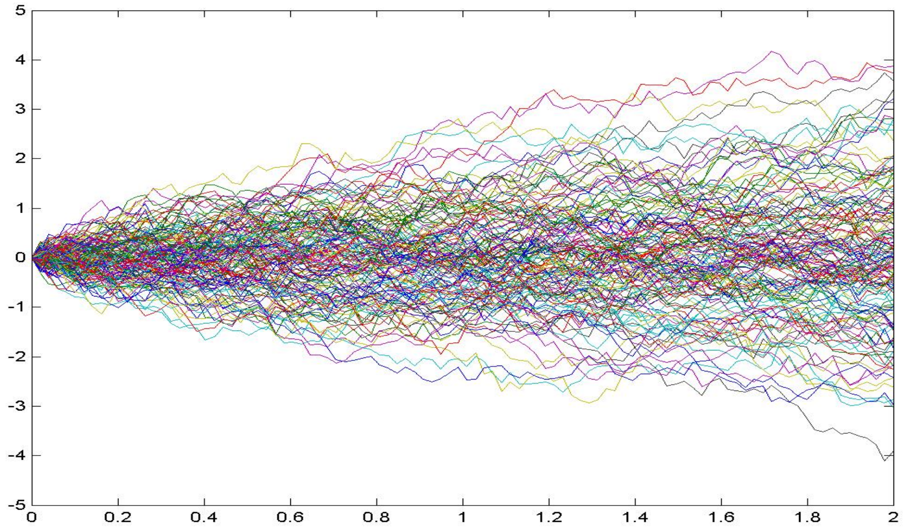
Wiener process: Sample trajectories

- A Wiener process is a stochastic process defined by the following equations:

Δt arbitrary

$$W(0) = 0$$

$$W(t + \Delta t) = W(t) + \sqrt{\Delta t}\, Z; \quad Z \sim N(0,1); \quad Z \perp W(t).$$



Wiener process: Properties

- $\mathbb{E}[W(t)] = 0$
- $Var[W(t)] = t$
- $Cov[W(t), W(s)] = \min(t, s)$
- Sample trajectories are continuous

$$\lim_{\Delta t \to 0^+} \mathbb{E}\left[|W(t+\Delta t) - W(t)|^2\right] = \lim_{\Delta t \to 0^+} \Delta t = 0$$

- Sample trajectories are nowhere differentiable

$$\frac{W(t+\Delta t) - W(t)}{\Delta t} \sim N\left(0, \frac{1}{\sqrt{\Delta t}}\right), \quad \text{limit distribution not defined for } \Delta t \to 0^+$$

Wiener process: simulation

$$W(t+\Delta t) = W(t) + \sqrt{\Delta t}\, Z; \quad Z \sim N(0,1)$$

- Initial condition: $W(0) = 0$
- Integration interval $[0\,, T]$
- Grid of points for integration: $t_0 < t_1 < \cdots < t_N = T$;
- Solution: $M \to \infty$ trajectories

$$W_{n+1}^{(m)} = W_n^{(m)} + \sqrt{\Delta t_n}\, Z_n^{(m)}; \qquad n = 0,1,\ldots,N-1$$

$$m = 1,\ldots,M$$

$W_n^{(m)} \stackrel{\text{def}}{=} W^{(m)}(t_n)$; $\quad \Delta t_n \stackrel{\text{def}}{=} t_{n+1} - t_n$; $\quad \left\{Z_n^{(m)} \sim N(0,1)\right\}$ iidrv's

Wiener process: simulation in a regular grid

$$W(t+\Delta t) = W(t) + \sqrt{\Delta t}\, Z; \quad Z \sim N(0,1)$$

- Initial condition: $W(0) = 0$
- Integration interval $[0\,, T]$
- Regular grid of integration points : $t_n = t_0 + n\Delta T; \quad n = 0, 1, \dots, N$
- Solution: $M \longrightarrow \infty$ trajectories

$$W_{n+1}^{(m)} = W_n^{(m)} + \sqrt{\Delta t}\, Z_n^{(m)}; \qquad n = 0,1, \dots, N-1$$

$$m = 1, \dots, M$$

$$W_n^{(m)} \stackrel{\text{def}}{=} W^{(m)}(t_n); \qquad \Delta t \stackrel{\text{def}}{=} T/N; \qquad \left\{ Z_n^{(m)} \sim N(0,1) \right\}$$ iidrv's

Wiener process: simulation in a regular grid

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Wiener process: simulation in a regular grid

Simulation in the interval $[0, T]$ in N steps of length $\Delta T = T/N$, M trajectories

$$t_n = n\Delta T \qquad W_n^{(m)} \stackrel{\text{def}}{=} W^{(m)}(t_n) \qquad n = 0,1,2, \dots N \qquad m = 0,1,2, \dots , M$$

$$W_0^{(m)} = W_0$$

$$W_{n+1}^{(m)} = W_n^{(m)} + \sqrt{\Delta T} Z_{n+1}^{(m)} \qquad Z_{n+1}^{(m)} \sim N(0,1) \quad n = 0,1,2, \dots N-1 \quad m = 1,2, \dots , M$$

t_0	t_1	t_2	…	t_N
$W_0^{(1)} = W_0$	$W_1^{(1)} = W_0^{(1)} + \sqrt{\Delta T} Z_1^{(1)}$	$W_2^{(1)} = W_1^{(1)} + \sqrt{\Delta T} Z_2^{(1)}$	…	$W_N^{(1)} = W_{N-1}^{(1)} + \sqrt{\Delta T} Z_N^{(1)}$
$W_0^{(2)} = W_0$	$W_1^{(2)} = W_0^{(2)} + \sqrt{\Delta T} Z_1^{(2)}$	$W_2^{(2)} = W_1^{(2)} + \sqrt{\Delta T} Z_2^{(2)}$	…	$W_N^{(2)} = W_{N-1}^{(2)} + \sqrt{\Delta T} Z_N^{(2)}$
	…			
$W_0^{(M)} = W_0$	$W_1^{(M)} = W_0^{(M)} + \sqrt{\Delta T} Z_1^{(M)}$	$W_2^{(M)} = W_1^{(M)} + \sqrt{\Delta T} Z_2^{(M)}$	…	$W_N^{(M)} = W_{N-1}^{(M)} + \sqrt{\Delta T} Z_N^{(M)}$
	Time t_1			

Wiener process: $W(t) \sim N\left(0, \sqrt{t}\right)$

- Assume $W(t) \sim N\left(0, \sqrt{t}\right)$

$$W(t) = \sqrt{t}\, Z; \quad Z \sim N(0,1)$$
$$W(t+\tau) = W(t) + \sqrt{\tau} Z'; \quad Z' \sim N(0,1); \quad Z' \perp Z$$

The sum of two independent Gaussian variables is also Gaussian

$$W(t+\tau) = W(t) + \sqrt{\tau}\, Z' = \sqrt{t}\, Z + \sqrt{\tau}\, Z' = \sqrt{t+\tau}\, Z''; \quad Z'' \sim N(0,1)$$

The process at $t+\tau$ has the assumed form $W(t) \sim N\left(0, \sqrt{t+\tau}\right)$

- $\mathbb{E}\left[\sqrt{t}\, Z + \sqrt{\tau}\, Z'\right] = \sqrt{t}\, \mathbb{E}[Z] + \sqrt{\tau}\, \mathbb{E}[Z'] = 0$
- $\mathrm{Var}\left[\sqrt{t}\, Z + \sqrt{\tau}\, Z'\right] = \mathrm{Var}\left[\sqrt{t}\, Z\right] + \mathrm{Var}\left[\sqrt{\tau}\, Z'\right] + 2\, \mathrm{Cov}\left[\sqrt{t}\, Z, \sqrt{\tau}\, Z'\right]$
  $= t\, \mathrm{Var}[Z] + \tau\, \mathrm{Var}[Z'] + 2\sqrt{t\tau}\, \mathrm{Cov}[Z, Z'] = t + \tau$

$\mathrm{Var}[Z] = 1$

$\mathrm{Var}[Z'] = 1$

$\mathrm{Cov}[Z, Z'] = 0$

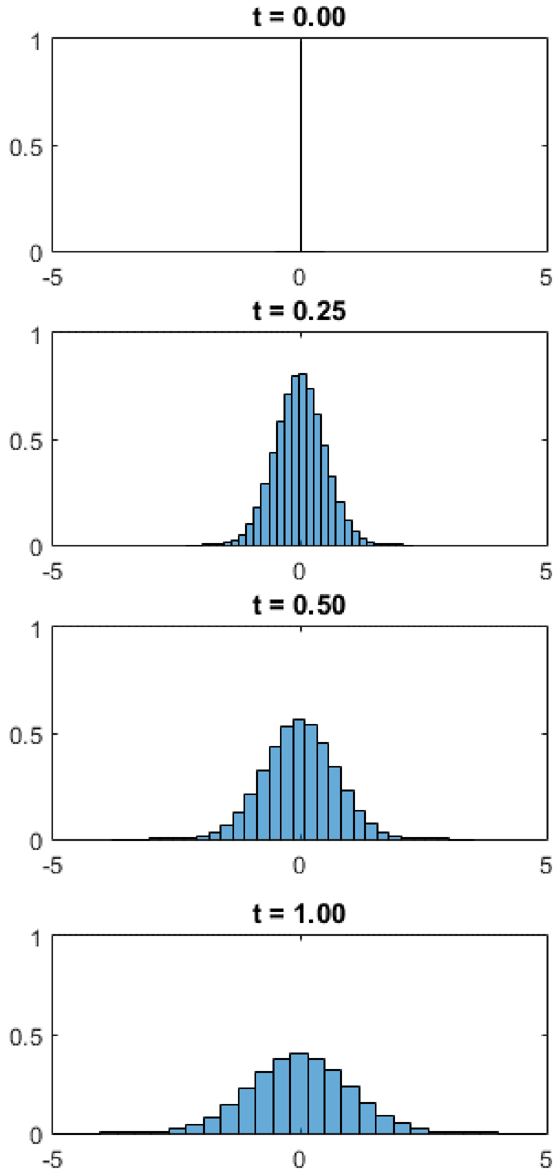
Wiener process: Density

$$p(W(t) = x, t) = \frac{1}{\sqrt{2\pi t}} exp\left\{-\frac{x^2}{2t}\right\};$$

- $\lim_{t \to 0^+} p(W(t) = x, t) = \delta(x)$

Initially, all the probability is concentrated at $x = 0$

- $\mathbb{E}[W(t)] = 0$
- $\mathrm{Var}[W(t)] = t$
- $\mathrm{Cov}[W(t), W(t')] = \min(t, t')$



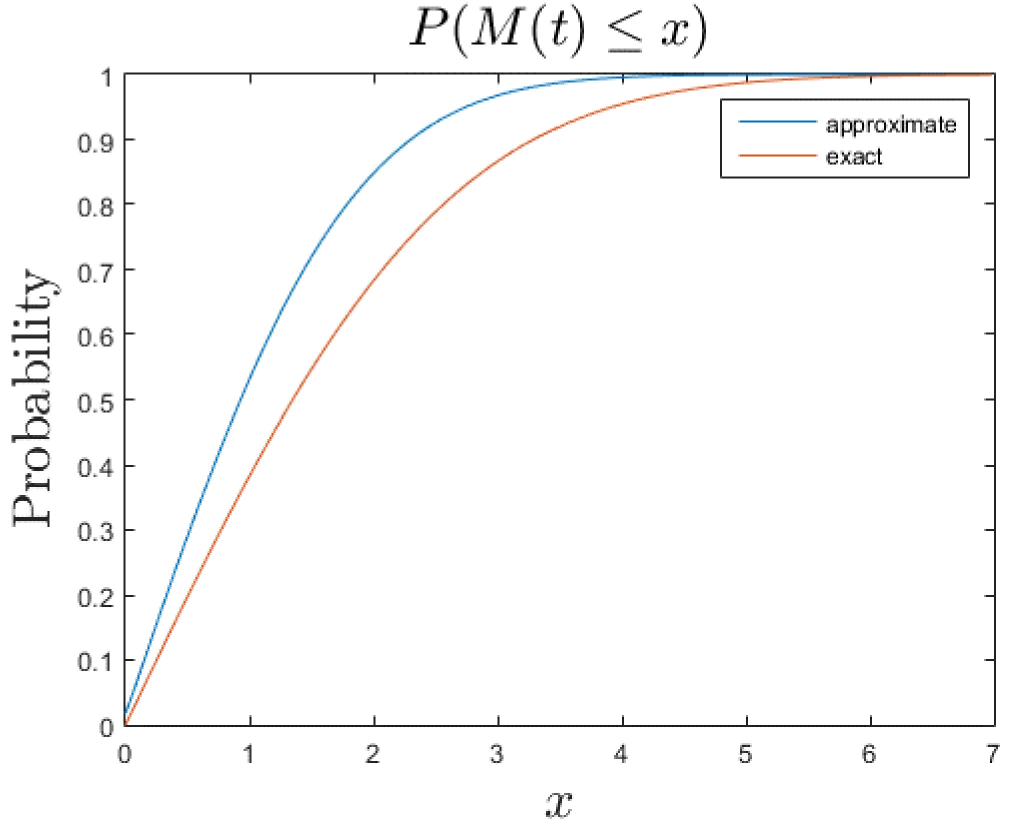
Wiener process: Distribution of maxima

- Define the random variable $M(\mathrm{t}) = \max\limits_{0 \le \tau \le t} W(\tau)$

$$p(M(\mathrm{t}) = x) = \sqrt{\frac{2}{\pi t}} exp\left\{-\frac{x^2}{2t}\right\};$$

$$\mathbb{P}(M(\mathrm{t}) \le x) = \sqrt{\frac{2}{\pi t}} \int_0^x exp\left\{-\frac{y^2}{2t}\right\} dy;$$

$$x \ge 0$$



Wiener process: First passage time

- Define the random variable $T(x) = \inf\{t \geq 0:\ W(t) = x\}$

$$p\big(T(x) = t\big) = \frac{|x|}{\sqrt{2\pi t^3}} exp\left\{-\frac{x^2}{2t}\right\}; \qquad t \geq 0$$

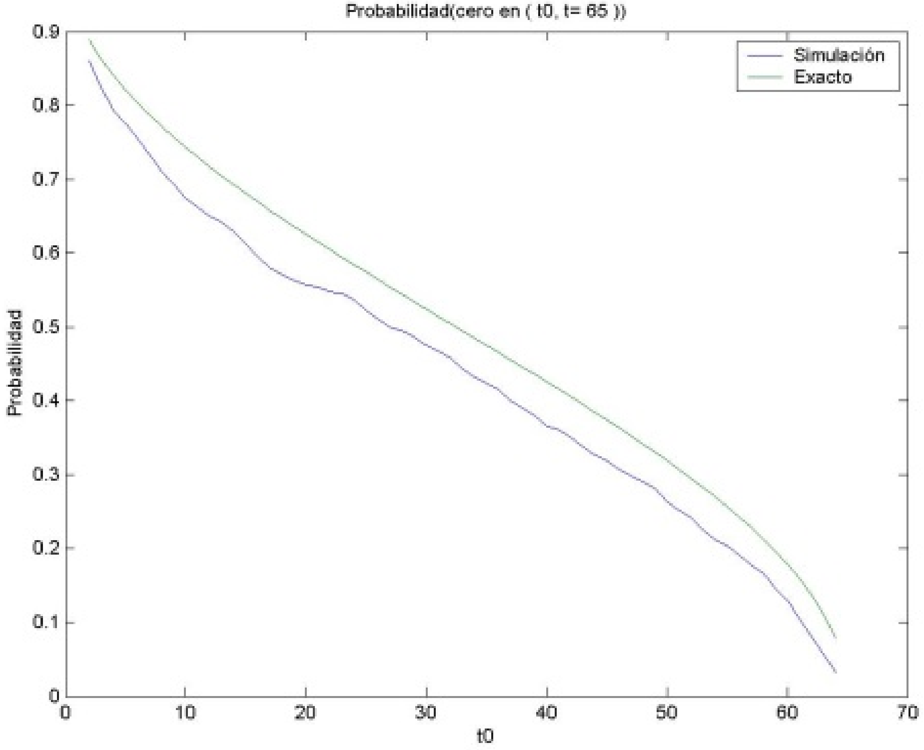
$$\mathbb{P}(T(x) \leq t) = \sqrt{\frac{2}{\pi t}} \int_x^\infty exp\left\{-\frac{y^2}{2t}\right\} dy\,; \qquad t \geq 0$$

<u>Note:</u> $\mathbb{P}\big(T(x) < t\big) = \mathbb{P}\big(M(\mathrm{t}) \geq x\big)$

Wiener process: Return to zero

- The probability that the Wiener process goes through 0 in the Interval (t_1, t_2)

$$\mathbb{P}(W(t) = 0 : t_1 \leq t \leq t_2) = \frac{2}{\pi} \operatorname{argcos} \sqrt{\frac{t_1}{t_2}}$$



Generalized Wiener process: Arithmetic Brownian motion (BM)

- Brownian motion is a stochastic process defined by the following equations:

$$B(t_0) = B_0$$

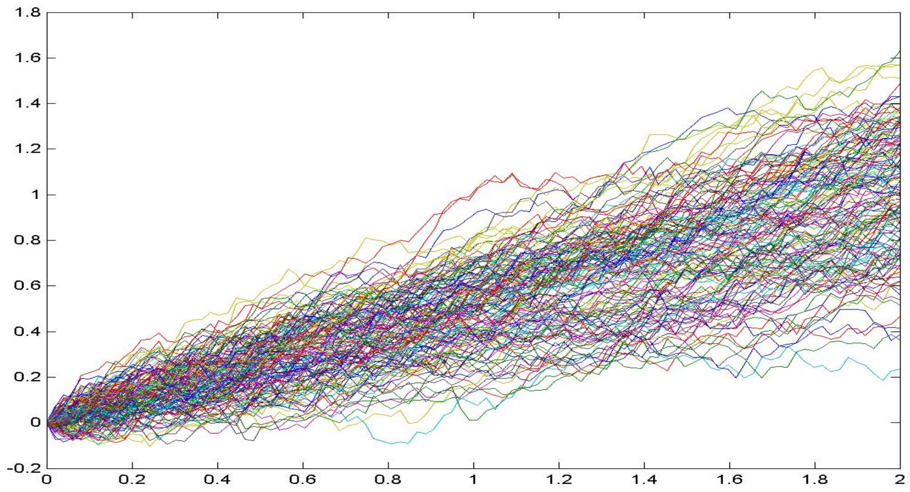
$$dB(t) = \mu dt + \sigma dW(t);$$

dt infinitesimal

$dW(t) = \sqrt{dt}\, Z \sim N\left(0, \sqrt{dt}\right)$

$Z \sim N(0,1);\ \ Z \perp B(t)$

$dB(t) = B(t + dt) - B(t)$



BM: simulation in a regular grid

$$B(t+\Delta t) = B(t) + \mu\Delta t + \sigma\sqrt{\Delta t}\, Z; \qquad Z \sim N(0,1); \;\; Z \perp B(t)$$

Δt arbitrary

- Initial condition: $B(t_0) = B_0$
- Integration interval $[t_0, t_0 + T]$
- Regular grid of integration points : $t_n = t_0 + n\Delta T; \quad n = 0, 1, \dots, N$
- Solution: $M \longrightarrow \infty$ trajectories

$$B_{n+1}^{(m)} = B_n^{(m)} + \mu\Delta t + \sigma\sqrt{\Delta t}\, Z_n^{(m)}; \qquad n = 0,1,\dots,N-1$$

$$m = 1, \dots, M$$

$$B_n^{(m)} \stackrel{\text{def}}{=} B^{(m)}(t_n); \qquad \Delta t \stackrel{\text{def}}{=} T/N; \qquad \left\{Z_n^{(m)} \sim N(0,1)\right\} \text{ iidrv's}$$

Brownian motion: Simulation

Simulation in the interval $[0, T]$ in N steps of length $\Delta T = T/N$, M trajectories

$t_n = t_0 + n\Delta T \qquad B_n^{(m)} \stackrel{\text{def}}{=} B^{(m)}(t_n) \qquad n = 0,1,2, \dots N \qquad m = 0,1,2, \dots, M$

$B_0^{(m)} = B_0$

$B_{n+1}^{(m)} = B_n^{(m)} + d_{n+1}^{(m)} \qquad n = 0,1,2, \dots N-1$

$m = 1,2, \dots, M$

$$d_{n+1}^{(m)} = \mu\Delta T + \sigma\sqrt{\Delta T} Z_{n+1}^{(m)}, \qquad Z_{n+1}^{(m)} \sim N(0,1)$$

	$B_0^{(1)} = B_0$	$B_1^{(1)} = B_0^{(1)} + d_1^{(1)}$	$B_2^{(1)} = B_1^{(1)} + d_2^{(1)}$	…	$B_N^{(1)} = B_{N-1}^{(1)} + d_N^{(1)}$
Trajectory 2	$B_0^{(2)} = B_0$	$B_1^{(2)} = B_0^{(2)} + d_1^{(2)}$	$B_2^{(2)} = B_1^{(2)} + d_2^{(2)}$	…	$B_N^{(2)} = B_{N-1}^{(2)} + d_N^{(2)}$
				…	
	$B_0^{(M)} = B_0$	$B_1^{(M)} = B_0^{(M)} + d_1^{(M)}$	$B_2^{(M)} = B_1^{(M)} + d_2^{(M)}$	…	$B_N^{(M)} = B_{N-1}^{(M)} + d_N^{(M)}$
		Time t_1			

Brownian motion: $B(t) \sim N(B_0 + \mu(t - t_0), \sigma\sqrt{t - t_0})$

- Assume $B(t) \sim N(B_0 + \mu(t - t_0), \sigma\sqrt{t - t_0})$

$$B(t) = B_0 + \mu(t - t_0) + \sigma\sqrt{t - t_0}\, Z; \quad Z \sim N(0,1)$$

$$B(t + \tau) = B(t) + \mu\tau + \sigma\sqrt{\tau}\, Z'; \quad Z' \sim N(0,1); \quad Z' \perp Z$$

$$B(t + \tau) = \left(B_0 + \mu(t - t_0) + \sigma\sqrt{t - t_0} Z\right) + \mu\tau + \sigma\sqrt{\tau} Z'$$

$$= B_0 + \mu\,(t + \tau - t_0) + \sigma\left(\sqrt{t - t_0}\, Z + \sqrt{\tau} Z'\right)$$

$\sqrt{t - t_0}\, Z + \sqrt{\tau}\, Z'$
$\sim N\left(0, \sqrt{t + \tau - t_0}\right)$

$$= B_0 + \mu\,(t + \tau - t_0) + \sigma\sqrt{t + \tau - t_0}\, Z''; \quad Z'' \sim N(0,1)$$

The process at $t + \tau$ has the assumed form
$B(t + \tau) \sim N\left(B_0 + \mu(t + \tau - t_0), \sigma\sqrt{t + \tau - t_0}\right)$

Brownian motion: Density

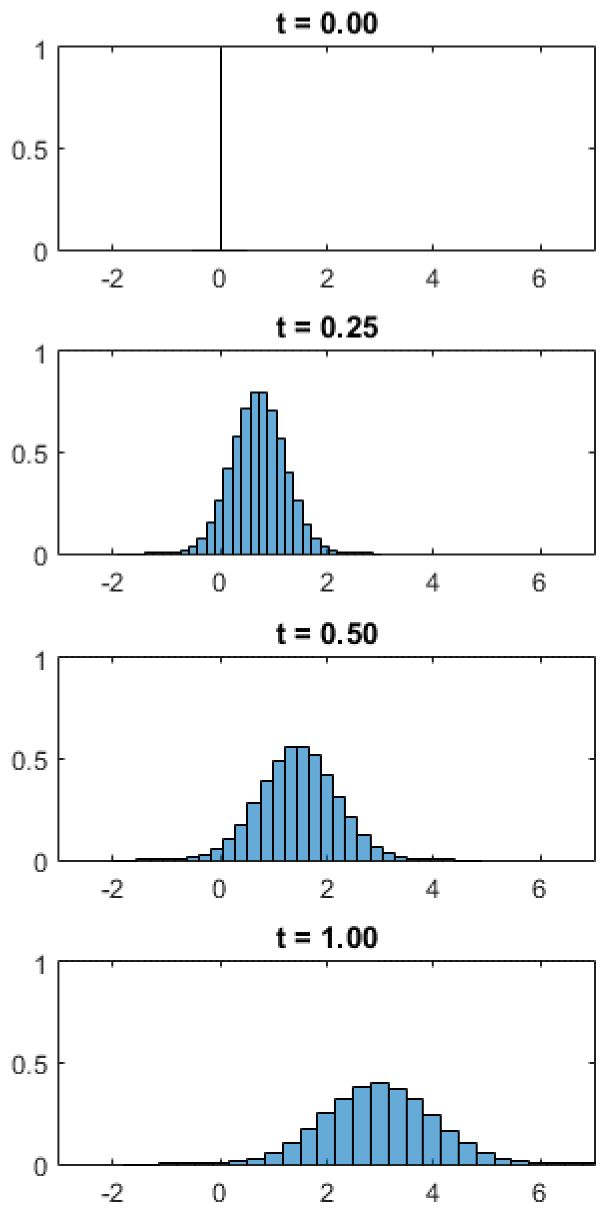
$$p(B(t) = x, t \,|B(t_0) = B_0, t_0)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2(t - t_0)}} exp\left\{-\frac{(x - [B_0 + \mu(t - t_0)])^2}{2\sigma^2(t - t_0)}\right\};$$

- $\lim_{t \to 0^+} p(B(t) = x, t \,|B(t_0) = B_0, t_0) \; = \delta(x - B_0)$

Initially, all the probability is concentrated at $x = B_0$

- $\mathbb{E}[B(t)] = B_0 + \mu(t - t_0)$
- $\mathrm{Var}[B(t)] = \sigma^2(t - t_0)$
- $\mathrm{Cov}[B(t), B(t')] = \sigma^2 \min(t - t_0, t' - t_0)$

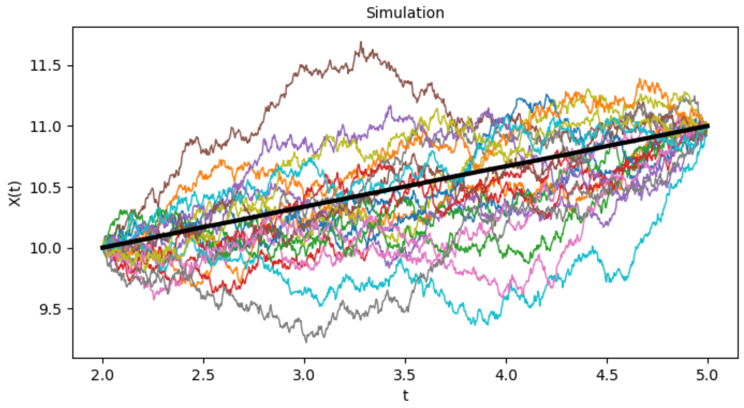


Brownian bridge

- Consider an interval $[t_0, t_1]$
- Simulate a Brownian trajectory such that $BB(t_0) = B_0;\ \ BB(t_1) = B_1$ for B_0, B_1 given.

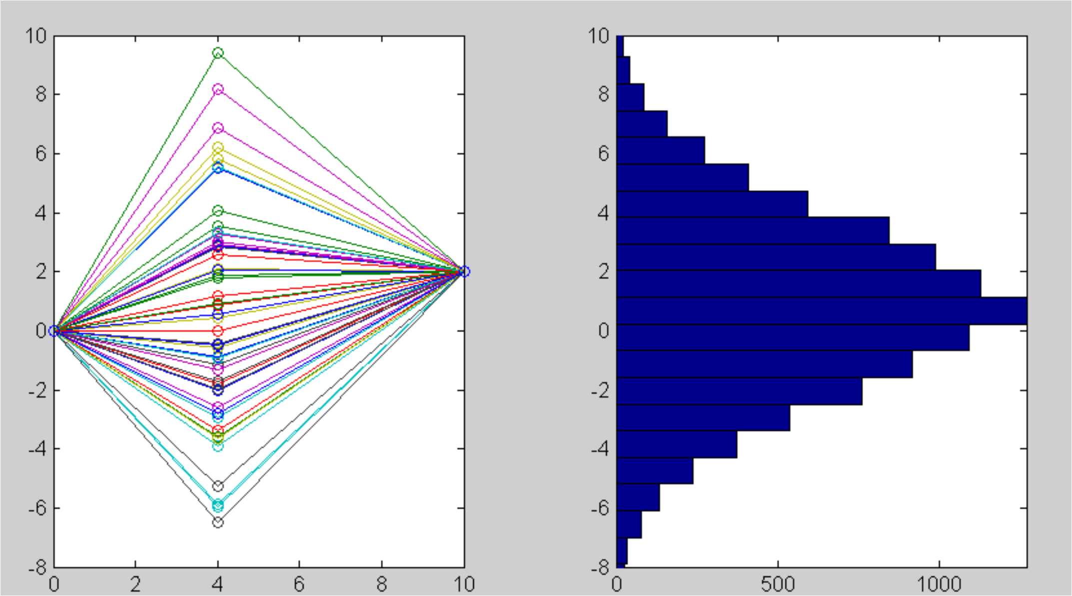
$$BB(t) = B_0 + (B_1 - B_0)\frac{t - t_0}{t_1 - t_0} + \sigma\sqrt{\frac{(t - t_0)(t_1 - t)}{t_1 - t_0}}Z; \quad Z \sim N(0,1)$$

$$t_0 \ \le t \ \le t_1$$



Brownian bridge: distribution

$$BB(t) \sim N\left(B_0 + (B_1 - B_0)\frac{t - t_0}{t_1 - t_0}, \sigma\sqrt{\frac{(t - t_0)(t_1 - t)}{t_1 - t_0}}\right)$$



Brownian bridge: simulation

- Consider an interval $[t_0, t_1]$.
- Let $B(t)$ a Brownian motion process such that $B(t_0) = B_0$
- A Brownian bridge process such that $BB(t_0) = B_0,\ BB(t_1) = B_1$ is:

$$BB(t) = B(t) + \left(B_1 - B(t_1)\right)\frac{t - t_0}{t_1 - t_0}; \qquad t_0 \le t \le t_1$$

- $\mathbb{E}[BB(t)] \ = \mathbb{E}[B(t)] + (B_1 - \mathbb{E}[B(t_1)])\frac{t-t_0}{t_1-t_0} = B_0 + \mu(t - t_0) + \left(B_1 - B_0 - \mu(t_1 - t_0)\right)\frac{t-t_0}{t_1-t_0}$

$$= B_0 + (B_1 - B_0)\frac{t-t_0}{t_1-t_0}$$

- $\mathrm{Var}[BB(t)] \ = \mathrm{Var}[B(t)] + \mathrm{Var}[B(t_1)]\left(\frac{t-t_0}{t_1-t_0}\right)^2 - 2\mathrm{Cov}[B(t), B(t_1)]\frac{t-t_0}{t_1-t_0} = \sigma^2(t - t_0) +$ $\sigma^2(t_1 - t_0)\left(\frac{t-t_0}{t_1-t_0}\right)^2 - 2\sigma^2(t - t_0)\frac{t-t_0}{t_1-t_0} = \sigma^2\,\frac{(t-t_0)(t_1-t)}{t_1-t_0}.$

Geometric Brownian motion (GBM)

- Geometric Brownian motion process is a stochastic process defined by the following Stochastic Differential Equation:

dt infinitesimal

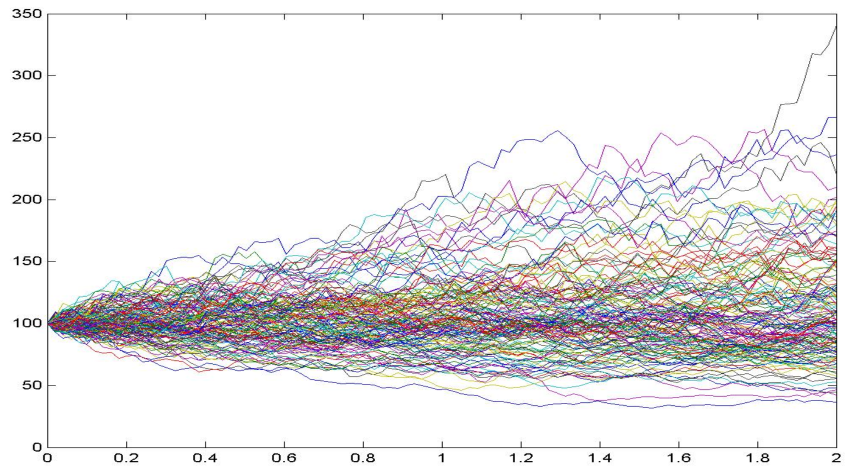
$$dW(t) = \sqrt{dt}\,Z \sim N\left(0, \sqrt{dt}\right)$$

$$Z \sim N(0,1); \quad Z \perp S(t)$$

$$S(t_0) = S_0$$

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t);$$

$$dS(t) = S(t + dt) - S(t)$$



GBM: simulation in a regular grid

$$S(t+\Delta t) = S(t) exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\, Z\right\}; \qquad Z \sim N(0,1);$$

Δt arbitrario

- Initial condition: $S(t_0) = S_0$
- Integration interval $[t_0, t_0 + T]$
- Regular grid of integration points : $t_n = t_0 + n\Delta T$; $n = 0, 1, \dots, N$
- Solution: $M \longrightarrow \infty$ trajectories

$$S_{n+1}^{(m)} = S_n^{(m)} exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\sqrt{\Delta t}\, Z_n^{(m)}\right\}; \qquad n = 0,1, \dots, N-1$$

$$m = 1, \dots, M$$

$$S_n^{(m)} \stackrel{\text{def}}{=} S^{(m)}(t_n); \qquad \Delta t \stackrel{\text{def}}{=} T/N; \qquad \left\{Z_n^{(m)} \sim N(0,1)\right\}$$ iidrv's

GBM: Simulation

- Simulation in the interval $[t_0, t_0 + T]$ in N steps of length $\Delta T = T/N$, M trajectories

$$t_n = t_0 + n\Delta T \qquad S_n^{(m)} \stackrel{\text{def}}{=} S^{(m)}(t_n) \qquad n = 0,1,2, \dots N \qquad m = 1,2, \dots, M$$

$$S_0^{(m)} = S_0$$

$$S_{n+1}^{(m)} = S_n^{(m)} e_{n+1}^{(m)} \qquad n = 0,1, \dots, (N-1) \qquad m = 1,2, \dots, M$$

$$e_{n+1}^{(m)} = exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)\Delta T + \sigma\sqrt{\Delta T} Z_{n+1}^{(m)}\right), \qquad Z_{n+1}^{(m)} \sim N(0,1)$$

	$S_0^{(1)} = S_0$	$S_1^{(1)} = S_0^{(1)}\, e_1^{(1)}$	$S_2^{(1)} = S_1^{(1)}\, e_2^{(1)}$	…	$S_N^{(1)} = S_{N-1}^{(1)}\, e_N^{(1)}$
Trajectory 2	$S_0^{(2)} = S_0$	$S_1^{(2)} = S_0^{(2)}\, e_1^{(2)}$	$S_2^{(2)} = S_1^{(2)}\, e_2^{(2)}$	…	$S_N^{(2)} = S_{N-1}^{(2)}\, e_N^{(2)}$
				…	
	$S_0^{(M)} = S_0$	$S_1^{(M)} = S_0^{(M)} e_1^{(M)}$	$S_2^{(M)} = S_1^{(M)}\, e_2^{(M)}$	…	$S_N^{(M)} = S_{N-1}^{(M)}\, e_N^{(M)}$
		Time t_1			

GBM: $\frac{S(t)}{S_0} \sim LN\left(\left(\mu - \frac{1}{2}\sigma^2\right)(t - t_0), \sigma\sqrt{t - t_0}\right)$

- Assume $\frac{S(t)}{S_0} \sim LN\left(\left(\mu - \frac{1}{2}\sigma^2\right)(t - t_0), \sigma\sqrt{t - t_0}\right)$

$$S(t) = S_0 exp\left\{\left(\mu - \tfrac{1}{2}\sigma^2\right)(t - t_0) + \sigma\sqrt{t - t_0}\, Z\right\}; \quad Z \sim N(0,1)$$

$$S(t+\tau) = S(t) exp\left\{\left(\mu - \tfrac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}\, Z\right\}; \quad Z' \sim N(0,1); \quad Z' \perp Z$$

$$S(t+\tau) = S_0 exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)(t - t_0) + \sigma\sqrt{t - t_0}\, Z\right\} exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)\tau + \sigma\sqrt{\tau}\, Z\right\}$$

$$= S_0 exp\left\{\left(\mu - \tfrac{1}{2}\sigma^2\right)(t + \tau - t_0) + \sigma\left(\sqrt{t - t_0}\, Z + \sqrt{\tau}\, Z'\right)\right\}$$

$$= S_0 exp\left\{\left(\mu - \tfrac{1}{2}\sigma^2\right)(t + \tau - t_0) + \sigma(\sqrt{t + \tau - t_0}\, Z'')\right\}; \quad Z'' \sim N(0,1)$$

$\sqrt{t - t_0}\, Z + \sqrt{\tau}\, Z'$
$\sim N\left(0, \sqrt{t + \tau - t_0}\right)$

The process at $t + \tau$ has the assumed form

$$S(t+\tau)/S_0 \sim LN\left(\left(\mu - \frac{1}{2}\sigma^2\right)(t + \tau - t_0), \sigma\sqrt{t + \tau - t_0}\right)$$

GBM: Density

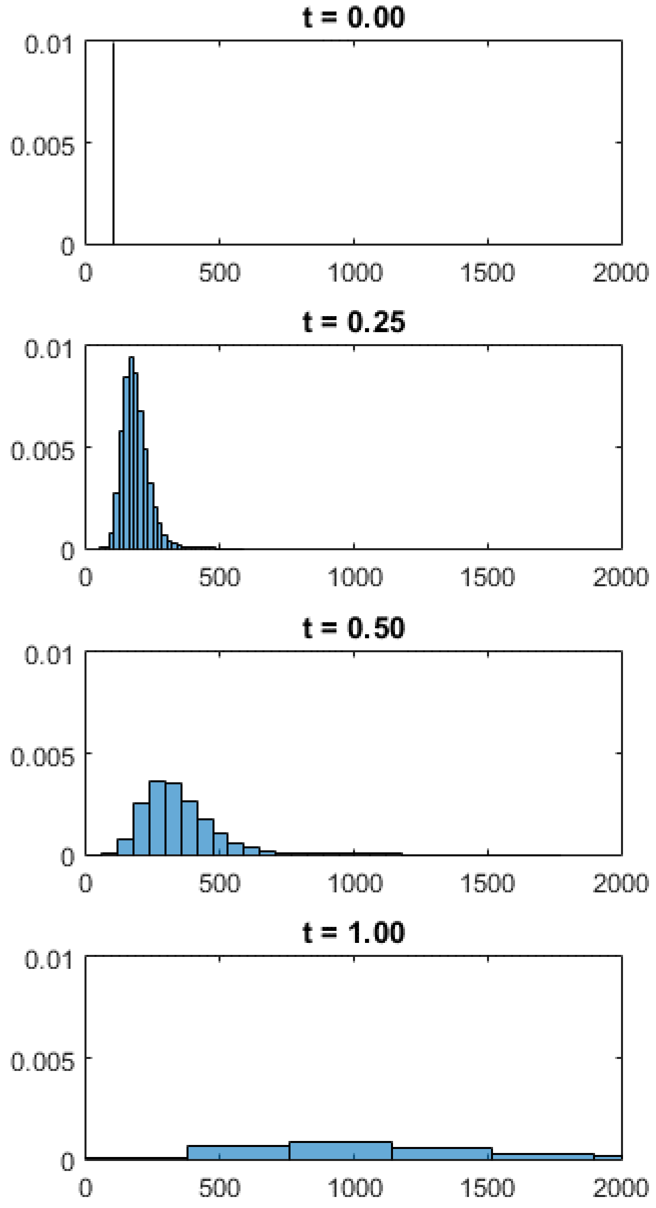
$$p(S(t) = x, t \,|S(t_0) = S_0, t_0)$$

$$= \frac{1}{x\sqrt{2\pi\sigma^2(t-t_0)}} exp\left\{-\frac{\left(\log\frac{x}{S_0} - \left(\mu - \frac{1}{2}\sigma^2\right)(t-t_0)\right)^2}{2\sigma^2(t-t_0)}\right\};$$

- $\lim_{t\to 0^+} p(S(t) = x, t \,|S(t_0) = S_0, t_0) \; = \delta(x - S_0)$

Initially, all the probability is concentrated at $x = S_0$

- $\mathbb{E}[S(t)] = S_0 exp(\mu(t-t_0))$
- $\mathrm{Var}[S(t)] = S_0^2 \, exp(2\mu(t-t_0))(exp(\sigma^2(t-t_0)) - 1)$
- $\mathrm{Mode}[S(t)] = S_0 exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)(t-t_0)\right)$



Random vectors: multivariate normal

- Consider the random vector with a multivariate normal distribution

$$\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_D \end{pmatrix}$$

$$\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

$$\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1D} \\ \sigma_{21} & \sigma_2^2 & & \sigma_{2D} \\ \vdots & & \ddots & \vdots \\ \sigma_{D1} & \sigma_{D2} & \cdots & \sigma_D^2 \end{pmatrix}$$

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_D \end{pmatrix}$$

- Multivariate normal density

$$p(\mathbf{x}; \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{(2\pi)^{D/2}|\boldsymbol{\Sigma}|^{1/2}} exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\}$$

$|\boldsymbol{\Sigma}|$: Determinant of $\boldsymbol{\Sigma}$

Square of the Mahalanobis distance

Covariance matrix: properties

- Elements of the covariance matrices

$$(\mathbf{\Sigma})_{ii} = \sigma_i^2 = \mathrm{Var}[X_i]; \quad i = 1, \ldots, D$$

$$(\mathbf{\Sigma})_{ij} = \sigma_{ij} = \mathrm{Cov}\left[X_i, X_j\right]; \quad i, j = 1, \ldots, D; \; j \neq i.$$

Exception: If there are degenerate components or the components are not linearly independent, in which case the covariance matrix has eigenvalues equal to 0.

- The covariance matrix is symmetric and positive definite
  - $\sigma_{ij} = \sigma_{ji}$
  - $\forall\, \mathbf{x} \in \mathbb{R}^D, \;\; \mathbf{x} \neq \mathbf{0}: \;\; \mathbf{x}^T \mathbf{\Sigma}\, \mathbf{x} > 0$

All eigenvalues are positive:

$$\mathbf{\Sigma} \mathbf{v}_d = \lambda_d \mathbf{v}_d; \quad d = 1, \ldots, D$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_D > 0$$

- Correlations

$$(\boldsymbol{\rho})_{ii} = 1; \quad i = 1, \ldots, D$$

$$(\boldsymbol{\rho})_{ij} = \rho_{ij} = \mathrm{Corr}\left[X_i, X_j\right] = \frac{\sigma_{ij}}{\sigma_i \sigma_j}; \quad i, j = 1, \ldots, D; \; j \neq i.$$

$$-1 \leq \rho_{ij} \leq 1$$

Cholesky decomposition

- The Cholesky decomposition of a positive real-values matrix is

$$\boldsymbol{\Sigma} = \mathbf{L}\mathbf{L}^T$$

$$\mathbf{L} = \begin{pmatrix} L_{11} & 0 & \cdots & 0 \\ L_{21} & L_{22} & & 0 \\ \vdots & & \ddots & \vdots \\ L_{D1} & L_{D2} & \cdots & L_{DD} \end{pmatrix}$$

Lower triangular matrix

- Cholesky decomposition in 2 D

$$\boldsymbol{\rho} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} = \begin{pmatrix} L_{11} & 0 \\ L_{12} & L_{22} \end{pmatrix} \begin{pmatrix} L_{11} & L_{12} \\ 0 & L_{22} \end{pmatrix} = \begin{pmatrix} L_{11}^2 & L_{11}L_{12} \\ L_{11}L_{12} & L_{12}^2 + L_{22}^2 \end{pmatrix}$$

$$\begin{cases} L_{11}^2 = 1 & \Rightarrow L_{11} = 1 \\ L_{11}L_{12} = \rho & \Rightarrow L_{12} = \rho \\ L_{12}^2 + L_{22}^2 = 1 & \Rightarrow L_{22} = \sqrt{1-\rho^2} \end{cases} \qquad \mathbf{L} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}$$

Samples from a multivariate normal: Cholesky decomposition

For large dimensions the Cholesky decomposition becomes unstable. In that case, a decomposition based on singular value decomposition, which is computationally more costly, is preferred.

1. Generate $\mathbf{Z} \sim N(\mathbf{0}, \mathbf{I})$

   $\mathbb{E}[\mathbf{Z}] = 0$
   $\mathbb{E}[\mathbf{Z}\,\mathbf{Z}^T] = \mathbf{I}$

2. Cholesky decomposition of the covariance matrix $\boldsymbol{\Sigma} = \mathbf{L}\mathbf{L}^T$
3. Generate $\mathbf{X} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

$$\mathbf{X} = \boldsymbol{\mu} + \mathbf{L}\,\mathbf{Z}$$

- $\mathbb{E}[\mathbf{X}] \;= \mathbb{E}[\boldsymbol{\mu} + \mathbf{L}\,\mathbf{Z}] = \boldsymbol{\mu} + \mathbf{L}\,\mathbb{E}[\mathbf{Z}] = \boldsymbol{\mu}$
- $\mathrm{Cov}[\mathbf{X}] \;= \mathbb{E}\left[(\mathbf{X} - \mathbb{E}[\mathbf{X}])(\mathbf{X} - \mathbb{E}[\mathbf{X}])^T\right] = \mathbb{E}\left[(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^T\right]$

$$= \mathbb{E}\left[(\mathbf{L}\,\mathbf{Z})(\mathbf{L}\,\mathbf{Z})^T\right] = \mathbf{L}\,\mathbb{E}\left[\mathbf{Z}\,\mathbf{Z}^T\right]\mathbf{L}^T = \mathbf{L}\,\mathbf{I}\,\mathbf{L}^T = \mathbf{L}\,\mathbf{L}^T = \boldsymbol{\Sigma}$$

Samples from a 2D normal distribution

1. Generate $Z_1 \sim N(0,1); \quad Z_2 \sim N(0,1); \quad Z_1 \perp Z_2$
2. Cholesky decomposition of the correlation matrix $\boldsymbol{\rho} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$

$$\mathbf{L} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}$$

3. Generate $\mathbf{X} \sim N(\mathbf{0}, \boldsymbol{\rho})$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} Z_1 \\ \rho Z_1 + \sqrt{1-\rho^2} Z_2 \end{pmatrix} \Rightarrow \begin{cases} X_1 = Z_1 \\ X_2 = \rho Z_1 + \sqrt{1-\rho^2} Z_2 \end{cases}$$

$$X_1 \sim N(0,1); \quad X_2 \sim N(0,1); \qquad \mathrm{Corr}[X_1, X_2] = \rho$$

Correlated BM in D dimensions: simulation

$$\mathbf{B}(t+\Delta t) = \mathbf{B}(t) + \boldsymbol{\mu}\Delta t + \sqrt{\Delta t}\,\mathbf{L}\,\mathbf{Z}; \qquad \mathbf{Z} \sim N(\mathbf{0}, \mathbf{I});$$

$\mathbf{B}(t) \in \mathbb{R}^D$

$\boldsymbol{\Sigma} = \mathbf{L}\mathbf{L}^T$

- Initial condition: $\mathbf{B}(t_0) = \mathbf{B}_0$
- Integration interval $[t_0, t_0 + T]$
- Regular grid of integration points : $t_n = t_0 + n\Delta T; \quad n = 0, 1, \dots, N$
- Solution: $M \longrightarrow \infty$ trajectories

$$\mathbf{B}_{n+1}^{(m)} = \mathbf{B}_n^{(m)} + \boldsymbol{\mu}\,\Delta t + \sqrt{\Delta t}\,\mathbf{L}\,Z_n^{(m)}; \qquad n = 0,1, \dots, N-1$$

$$m = 1, \dots, M$$

$$\mathbf{B}_n^{(m)} \stackrel{\text{def}}{=} \mathbf{B}^{(m)}(t_n); \qquad \Delta t \stackrel{\text{def}}{=} T/N; \qquad \left\{\mathbf{Z}_n^{(m)} \sim N(\mathbf{0}, \mathbf{I})\right\} \text{ iidrv's}$$