Take home exam
Part II

Convex Unconstrained and Constrained Optimization

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Problem 1. (1 point) We have worked out the elementary version of Lagrange multipliers assuming that from $g(x, y) = 0$ we can find a function $y = h(x)$ such that $g(x, h(x)) = 0$.
But sometimes what we get is that there is an h such that $g(h(y), y) = 0$. Rewrite the Lagrange multiplier analysis in the lecture slides under this assumption.

Problem 2. (3 points) We want to solve the following constrained minimization problem:

$$\begin{aligned} \min \quad & f(x, y) = x^2 + 2xy + 2y^2 - 3x + y \\ \text{s.t.} \quad & x + y = 1, \\ & x \geq 0, y \geq 0. \end{aligned}$$

Argue first that f is convex and then:

- Write its Lagrangian with α, β the multipliers of the inequality constraints.
- Write the KKT conditions.
- Use them to solve the problem. For this consider separately the $(\alpha = \beta = 0), (\alpha > 0, \beta = 0), (\alpha = 0, \beta > 0), (\alpha > 0, \beta > 0)$ cases.

Problem 3. (1 point) Let $f : S \subset \mathbb{R}^d \to \mathbb{R}$ be a convex function on the convex set S and we extend it to an $\tilde{f} : \mathbb{R}^d \to \mathbb{R}$ as:

$$\begin{aligned} \tilde{f}(x) &= f(x) \text{ if } x \in S. \\ &= +\infty \text{ if } x \notin S. \end{aligned}$$

Show that $\tilde{f}$ is a convex function on $\mathbb{R}^d$. Assume that $a + \infty = \infty$ and that $a \cdot \infty = 1$ for $a > 0$.

Problem 4. (2 points) Prove **Jensen's inequality**: if f is convex on $\mathbb{R}^d$ and $\sum_1^k \lambda_i = 1$, with $0 \le \lambda_i \le 1$ we have for any $x_1, ..., x_k \in \mathbb{R}^d$

$$f\left(\sum_1^k \lambda_i x_i\right) \le \sum_1^k \lambda_i f(x_i)$$

Hint: just write $\sum_1^k \lambda_i x_i = \lambda_1 x_1 + (1 - \lambda_1)$ *for an appropriate* v *and apply repeatedly the definition of a convex function. Start with* $k = 3$ *and carry on.*

Problem 5. (3 points) Prove that the following function is convex

$$\begin{aligned} f(x) &= x^2 - 1, \qquad && |x| > 1 \\ &= 0 && |x| \le 1 \end{aligned}$$

and compute its proximal. Which are the fixed points of this proximal?