Take home exam
Part II

Convex Unconstrained and Constrained Optimization

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Problem 1. (1 point) We have worked out the elementary version of Lagrange multipliers assuming that from $g(x,y)=0$ we can find a function $y=h(x)$ such that $g(x,h(x))=0$.
But sometimes what we get is that there is an h such that $g(h(y),y)=0$. Rewrite the Lagrange multiplier analysis in the lecture slides under this assumption.

For $f,g:\mathbb{R}^2\to\mathbb{R}$ consider the following minimization problem

$$\min f(x,y) \text{ s.t. } g(x,y)=0$$

Assuming the **Implicit Function Theorem** holds, we can find a function $x=h(y)$ s.t. $g(h(y),y)=0$ and, thus, we can write

$$f(x,y)=f(h(y),y)=\Psi(y).$$

At a minimum y^* with $x^*=h(y^*)$ we thus have

$$0=\Psi'(y^*)=\frac{\partial g}{\partial x}(x^*,y^*)h'(y^*)+\frac{\partial g}{\partial y}(x^*,y^*) \tag{1}$$

But since $g(h(y),y)=0$, we also have

$$0=\frac{\partial g}{\partial x}(x^*,y^*)h'(y^*)+\frac{\partial g}{\partial y}(x^*,y^*) \implies h'(y^*)=-\frac{\frac{\partial g}{\partial y}(x^*,y^*)}{\frac{\partial g}{\partial x}(x^*,y^*)} \tag{2}$$

Putting together 1 and 2 we arrive at

$$0=\frac{\partial f}{\partial y}(x^*,y^*)\frac{\partial g}{\partial x}(x^*,y^*)-\frac{\partial f}{\partial x}(x^*,y^*)\frac{\partial g}{\partial y}(x^*,y^*)$$

That is, at (x^*,y^*), $\nabla f \perp \left(-\frac{\partial g}{\partial y},\frac{\partial g}{\partial x}\right)$ and, since $\left(-\frac{\partial g}{\partial y},\frac{\partial g}{\partial x}\right)\perp\nabla g$, we have $\nabla f\|\nabla g$ i.e. $\nabla f(x^*,y^*)=-\lambda^*\nabla g(x^*,y^*)$ for some $\lambda^*\neq 0$.
Thus, for the **Lagrangian**

$$\mathcal{L}(x,y,\lambda)=f(x,y)+\lambda g(x,y),$$

we have a minimum (x^*,y^*) there is a $\lambda^*\neq 0$ s.t.

$$\nabla\mathcal{L}(x^*,y^*,\lambda^*)=\nabla f(x^*,y^*)+\lambda^*\nabla g(x^*,y^*)=0.$$

Problem 2. (3 points) We want to solve the following constrained minimization problem:

$$\begin{aligned} \min \quad & f(x,y) = x^2 + 2xy + 2y^2 - 3x + y \\ \text{s.t.} \quad & x + y = 1, \\ & x \geq 0, y \geq 0. \end{aligned}$$

Argue first that f is convex and then:

- Write its Lagrangian with α, β the multipliers of the inequality constraints.
- Write the KKT conditions.
- Use them to solve the problem. For this consider separately the $(\alpha = \beta = 0), (\alpha > 0, \beta = 0), (\alpha = 0, \beta > 0), (\alpha > 0, \beta > 0)$ cases.

Problem 3. (1 point) Let $f : S \subset \mathbb{R}^d \to \mathbb{R}$ be a convex function on the convex set S and we extend it to an $\tilde{f} : \mathbb{R}^d \to \mathbb{R}$ as:

$$\begin{aligned} \tilde{f}(x) &= f(x) \text{ if } x \in S. \\ &= +\infty \text{ if } x \notin S. \end{aligned}$$

Show that $\tilde{f}$ is a convex function on $\mathbb{R}^d$. Assume that $a + \infty = \infty$ and that $a \cdot \infty = 1$ for $a > 0$.

We say that S is a **convex set** if for all $x, x' \in S$ and $\lambda \in [0,1]$,

$$\lambda x + (1-\lambda)x' \in S.$$

We know $S \subset \mathbb{R}^d$ is a non empty convex set. Let $x, x' \in S$ and $\lambda \in [0,1]$, so here we cover two cases:

- *First case.* If $x \in S$, we have by definition that

$$\tilde{f}(\lambda x + (1-\lambda)x') = f(\lambda x + (1-\lambda)x') \leq \lambda f(x) + (1-\lambda) = \lambda \tilde{f}(x) + (1-\lambda)$$

  is convex.

- *Second case.* If $x \notin S$, we have that

$$\lambda \tilde{f}(x) + (1-\lambda)\tilde{f}(x') = +\infty,$$

  which we can say it is directly convex, $\tilde{f}(y) \leq +\infty, \forall y$. Note that this also applies to x'.

Since both cases satisfy convexity, we proof that this function is convex.

Problem 4. (2 points) Prove **Jensen's inequality**: if f is convex on $\mathbb{R}^d$ and $\sum_1^k \lambda_i = 1$, with $0 \leq \lambda_i \leq 1$ we have for any $x_1, ..., x_k \in \mathbb{R}^d$

$$f\left(\sum_1^k \lambda_i x_i\right) \leq \sum_1^k \lambda_i f(x_i)$$

Hint: just write $\sum_1^k \lambda_i x_i = \lambda_1 x_1 + (1 - \lambda_1)$ *for an appropriate* v *and apply repeatedly the definition of a convex function. Start with* $k = 3$ *and carry on.*

Problem 5. (3 points) Prove that the following function is convex

$$\begin{aligned} f(x) &= x^2 - 1, & |x| > 1 \\ &= 0 & |x| \leq 1 \end{aligned}$$

and compute its proximal. Which are the fixed points of this proximal?