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Part II

Convex Unconstrained and Constrained Optimization

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Problem 1. (1 point) We have worked out the elementary version of Lagrange multipliers assuming that from $g(x, y) = 0$ we can find a function $y = h(x)$ such that $g(x, h(x)) = 0$.
But sometimes what we get is that there is an h such that $g(h(y), y) = 0$. Rewrite the Lagrange multiplier analysis in the lecture slides under this assumption.

For $f, g : \mathbb{R}^2 \to \mathbb{R}$ consider the following minimization problem

$$\min f(x, y) \text{ s.t. } g(x, y) = 0.$$

Assuming the **Implicit Function Theorem** holds, we can find a function $x = h(y)$ s.t. $g(h(y), y) = 0$ and, thus, we can write

$$f(x, y) = f(h(y), y) = \Psi(y).$$

At a minimum y^* with $x^* = h(y^*)$ we thus have

$$0 = \Psi'(y^*) = \frac{\partial g}{\partial x}(x^*, y^*)h'(y^*) + \frac{\partial g}{\partial y}(x^*, y^*). \tag{1}$$

But since $g(h(y), y) = 0$, we also have

$$0 = \frac{\partial g}{\partial x}(x^*, y^*)h'(y^*) + \frac{\partial g}{\partial y}(x^*, y^*) \implies h'(y^*) = -\frac{\frac{\partial g}{\partial y}(x^*, y^*)}{\frac{\partial g}{\partial x}(x^*, y^*)}. \tag{2}$$

Putting together (1) and (2) we arrive at

$$0 = \frac{\partial f}{\partial y}(x^*, y^*)\frac{\partial g}{\partial x}(x^*, y^*) - \frac{\partial f}{\partial x}(x^*, y^*)\frac{\partial g}{\partial y}(x^*, y^*).$$

That is, at (x^*, y^*), $\nabla f \perp \left(-\frac{\partial g}{\partial y}, \frac{\partial g}{\partial x}\right)$ and, since $\left(-\frac{\partial g}{\partial y}, \frac{\partial g}{\partial x}\right) \perp \nabla g$, we have $\nabla f \| \nabla g$ i.e. $\nabla f(x^*, y^*) = -\lambda^* \nabla g(x^*, y^*)$ for some $\lambda^* \neq 0$.

Thus, for the **Lagrangian**

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda g(x, y),$$

we have a minimum (x^*, y^*) there is a $\lambda^* \neq 0$ s.t.

$$\nabla \mathcal{L}(x^*, y^*, \lambda^*) = \nabla f(x^*, y^*) + \lambda^* \nabla g(x^*, y^*) = 0.$$

Problem 2. (3 points) We want to solve the following constrained minimization problem:

$$\begin{aligned} \min \quad & f(x,y) = x^2 + 2xy + 2y^2 - 3x + y \\ \text{s.t.} \quad & x + y = 1, \\ & x \geq 0, y \geq 0. \end{aligned}$$

Argue first that f is convex and then:

- Write its Lagrangian with α, β the multipliers of the inequality constraints.
- Write the KKT conditions.
- Use them to solve the problem. For this consider separately the $(\alpha = \beta = 0), (\alpha > 0, \beta = 0), (\alpha = 0, \beta > 0), (\alpha > 0, \beta > 0)$ cases.

First of all, we verify that f is convex because its Hessian matrix is positive semidefinite, or equivalently its eigenvalues are non-negative. The Hessian matrix is

$$\mathbf{H}_f = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x \partial y} \\ \dfrac{\partial^2 f}{\partial y \partial x} & \dfrac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix},$$

whose eigenvalues are $\lambda_1 = 3 + \sqrt{5}$ and $\lambda_2 = 3 - \sqrt{5}$, both positive. As a result, we determine f is convex.

Now we write its Lagrangian, with α and β as multipliers of the inequality constraints and λ as equality constraint multiplier.

$$\mathcal{L}(x, y, \lambda, \alpha, \beta) = x^2 + 2xy + 2y^2 - 3x + y + \lambda(x + y - 1) - \alpha x - \beta y.$$

Assuming that the hypothesis of the KKT conditions theorem hold, the resulting KKT conditions on a local minimum (x^*, y^*) are the following:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial x}(x^*, y^*, \lambda, \alpha, \beta) \implies \lambda + 2x^* + 2y^* - 3 - \alpha = 0, \\ 0 &= \frac{\partial \mathcal{L}}{\partial y}(x^*, y^*, \lambda, \alpha, \beta) \implies \lambda + 2x^* + 4y^* + 1 - \beta = 0, \\ 0 &= \alpha x^*, \\ 0 &= \beta y^*. \end{aligned}$$

Then, we use them to solve the problem, considering the four possible cases below:

- Case $\alpha = \beta = 0$.

$$0 = \frac{\partial \mathcal{L}}{\partial x} = \lambda + 2x + 2y - 3$$
$$0 = \frac{\partial \mathcal{L}}{\partial y} = \lambda + 2x + 4y + 1$$

  From both expressions we get $2x+2y-3 = 2x+4y+1 \implies 2y = -4 \implies y = -2 \ngeq 0$, so this point does not satisfy this constraint. Then, $x = 1 - y = 3$ and $\lambda = -1$. So we have that $(\mathbf{3}, -\mathbf{2})'$ **is not a feasible solution**.

- Case $\alpha > 0, \beta = 0$. When $\alpha > 0$, $x = 0$, so $y = 1$ and $\lambda = -5$. Finally, $\alpha = 6 > 0$. Therefore we have a **feasible KKT point** on $(\mathbf{0}, \mathbf{1})'$.

- Case $\alpha = 0, \beta > 0$. When $\beta > 0$, $y = 0$, so $x = 1$ and $\lambda = 1$. Finally, $\beta = 4 > 0$. Therefore we have a **feasible KKT point** on $(\mathbf{1}, \mathbf{0})'$.

- Case $\alpha > 0, \beta > 0$. This implies $x = y = 0$, so we have a contradiction because $x+y \neq 1$. This means that $(0, 0)'$ **is not a feasible point**.

Given all points found, we can determine that our optimal solution is $(\mathbf{1}, \mathbf{0})'$, with an optimal value $\lambda = \mathbf{1}$.

Problem 3. (1 point) Let $f : S \subset \mathbb{R}^d \to \mathbb{R}$ be a convex function on the convex set S and we extend it to an $\tilde{f} : \mathbb{R}^d \to \mathbb{R}$ as:

$$\begin{aligned} \tilde{f}(x) &= f(x) \text{ if } x \in S. \\ &= +\infty \text{ if } x \notin S. \end{aligned}$$

Show that $\tilde{f}$ is a convex function on $\mathbb{R}^d$. Assume that $a + \infty = \infty$ and that $a \cdot \infty = \infty$ for $a > 0$.

We say that S is a **convex set** if for all $x, x' \in S$ and $\lambda \in [0, 1]$,

$$\lambda x + (1 - \lambda)x' \in S.$$

Let $x, x' \in S$ and $\lambda \in [0, 1]$, so here we cover two cases:

- *First case.* If $x, x' \in S$:

$$\tilde{f}(\lambda x + (1-\lambda)x') = f(\lambda x + (1-\lambda)x') \leq \lambda f(x) + (1-\lambda) f(x') = \lambda \tilde{f}(x) + (1-\lambda)\tilde{f}(x').$$

  Where the first equality holds given that S is convex. The inequality holds because f is convex. And, the last equality raises from the definition of $\tilde{f}$.

- *Second case.* If $x \notin S$ or $x' \notin S$, we have that

$$\tilde{f}(\lambda x + (1-\lambda)x') \leq \lambda \tilde{f}(x) + (1-\lambda)\tilde{f}(x') = +\infty,$$

  because $\tilde{f}(y) \leq +\infty$, $\forall y \in \mathbb{R}^d$.

Since both cases satisfy convexity definition holds, we conclude this function is convex.

Problem 4. (2 points) Prove **Jensen's inequality**: if f is convex on $\mathbb{R}^d$ and $\sum_1^k \lambda_i = 1$, with $0 \leq \lambda_i \leq 1$ we have for any $x_1, \ldots, x_k \in \mathbb{R}^d$

$$f\left(\sum_1^k \lambda_i x_i\right) \leq \sum_1^k \lambda_i f(x_i)$$

Hint: just write $\sum_1^k \lambda_i x_i = \lambda_1 x_1 + (1 - \lambda_1)$ for an appropriate v and apply repeatedly the definition of a convex function. Start with $k = 3$ and carry on.

We proceed using an inductive procedure:

- If $k = 1$ then $\lambda = 1$, so we simply have $f(x_1) \leq f(x_1)$, which is true, and nothing to prove. If $k = 2$ we have the definition of the convexity of f:

$$\lambda_1 + \lambda_2 = 1,\ \lambda_1, \lambda_2 \geq 0 \implies f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2).$$

- Considering $k = 3$, given $\lambda_1 + \lambda_2 + \lambda_3 = 1$, and $\lambda_2, \lambda_3 > 0$, we have the following inequality

$$\begin{aligned} f\left(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3\right) &= f\left(\lambda_1 x_1 + (1 - \lambda_1)\left(\frac{\lambda_2}{\lambda_2 + \lambda_3} x_2 + \frac{\lambda_3}{\lambda_2 + \lambda_3} x_3\right)\right) \\ &\leq f(\lambda_1 x_1) + f\left((1 - \lambda_1)\left(\frac{\lambda_2}{\lambda_2 + \lambda_3} x_2 + \left(1 - \frac{\lambda_2}{\lambda_2 + \lambda_3}\right) x_3\right)\right) \\ &\leq \lambda_1 f(x_1) + (1 - \lambda_1)\left(\frac{\lambda_2}{\lambda_2 + \lambda_3} f(x_2) + \left(1 - \frac{\lambda_2}{\lambda_2 + \lambda_3}\right) f(x_3)\right) \\ &= \sum_{i=1}^3 \lambda_i f(x_i), \end{aligned}$$

  where we applied the case $k = 2$ twice.

- We assume the statement its true for k and consider $k + 1$ points $x_1, \ldots, x_{k+1}$, with coefficients $\lambda_1, \ldots, \lambda_{k+1} \geq 0$, $\sum_{i=1}^{k+1} \lambda_i = 1$. The evaluation of the linear combination can be decomposed as

$$f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) = f\left((1 - \lambda_1)\left(\sum_{i=2}^{k+1} \frac{\lambda_i}{1 - \lambda_1} x_i\right) + \lambda_1 x_1\right).$$

  Using this, it is straightforward to use the Jensen's inequality on x_1 and $\sum_{i=2}^{k+1} \frac{\lambda_i}{1-\lambda_1} x_i$ with coefficients λ_1 and $1 - \lambda_1$ respectively. That is,

$$f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) \leq (1 - \lambda_1) f\left(\sum_{i=2}^{k+1} \frac{\lambda_i}{1 - \lambda_1} x_i\right) + \lambda_1 f(x_1).$$

We may notice that $\sum_{i=2}^{k+1} \frac{\lambda_i}{1-\lambda_1} x_i$ verifies the inductive hypothesis, thus,

$$f\left(\sum_{i=2}^{k+1} \frac{\lambda_i}{1-\lambda_1} x_i\right) \leq \sum_{i=2}^{k+1} \frac{\lambda_i}{1-\lambda_1} f(x_i).$$

Finally,

$$f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) \leq (1-\lambda_1) \sum_{i=2}^{k+1} \frac{\lambda_i}{1-\lambda_1} f(x_i) + \lambda_1 f(x_1) = \sum_{i=1}^{k+1} \lambda_i f(x_i).$$

Problem 5. (3 points) Prove that the following function is convex

$$\begin{aligned} f(x) &= x^2 - 1, & |x| > 1 \\ &= 0 & |x| \leq 1 \end{aligned}$$

and compute its proximal. Which are the fixed points of this proximal?

We note that f can be seen as the maximum of two functions $f(x) = \max\{0, x^2 - 1\}$. Both of these functions are convex. Then, we are going to show that the maximum of two convex functions is also convex.

Let m and n be two convex functions and $h(x) = \max\{m(x), n(x)\}$. Given $\lambda \in [0,1]$ and $x, y \in \mathbb{R}$ we aim to show that

$$h(\lambda x + (1-\lambda)y) \leq \lambda h(x) + (1-\lambda)h(y).$$

On one hand, it is clear that

$$m(\lambda x + (1-\lambda)y) \leq \lambda m(x) + (1-\lambda)m(y) \leq \lambda h(x) + (1-\lambda)h(y),$$

where in the first inequality comes from the fact that m is convex and the second from the definition of h. The same inequalities holds for n. Given that both functions are upper bounded by the same value, the maximum is also upper bounded by this value, so h is convex. Using this auxiliary result, f is convex.

Now we compute the proximal operator of f as

$$\operatorname{prox}_f(x) = \arg\min_z f(z) + \frac{1}{2}(z-x)^2 = \arg\min_z h(z)$$

with

$$h(z) = \begin{cases} z^2 - 1 + \frac{1}{2}(z-x)^2 & |z| > 1 \\ \frac{1}{2}(z-x)^2 & |z| \leq 1 \end{cases} \tag{3}$$

If the minimizer is attained at $|z| \leq 1$, then clearly $z = x$, meaning that $\operatorname{prox}_f(x) = x$ for $|x| \leq 1$. If it is attained at $|z| > 1$, we have

$$0 = h'(z) = 3z - x \implies z = \frac{1}{3}x,$$

which implies that $\operatorname{prox}_f(x) = \frac{1}{3}x$ for $|x| > 3$. The remaining values of the proximal must be studied separately: the only possible minimizers are the points of non differentiability of (3). That is -1 and 1 with

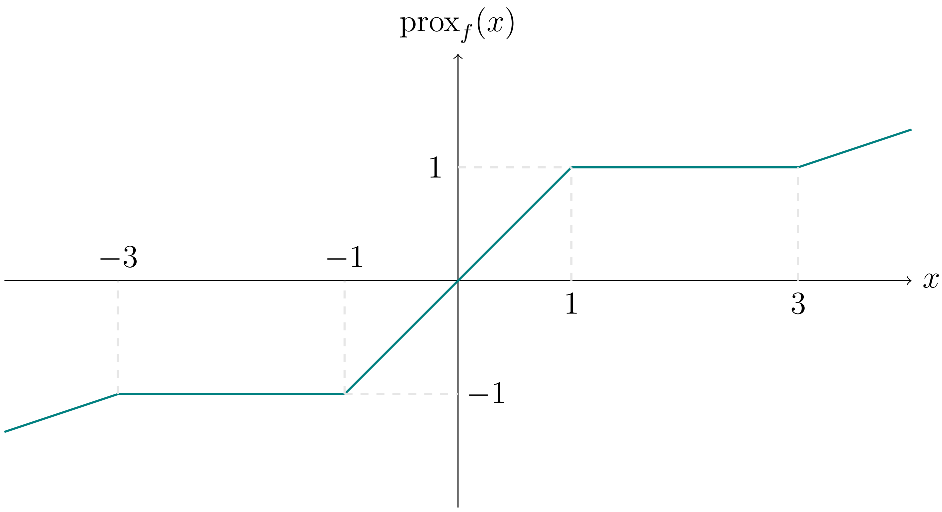
$$h(-1) = \frac{1}{2}(-1-x)^2 \quad \text{and} \quad h(1) = \frac{1}{2}(1-x)^2.$$

- When $-3 \leq x < -1$, the function is minimized at $z = -1$, that is, the proximal is $\operatorname{prox}_f(x) = -1$.
- When $1 < x \leq 3$, the function is minimized at $z = 1$, that is, the proximal is $\operatorname{prox}_f(x) = 1$.

As a result, the proximal is

$$\operatorname{prox}_f(x) = \begin{cases} \frac{x}{3} & x \in (-\infty, -3), \\ -1 & x \in [-3, -1), \\ x & x \in [-1, 1], \\ 1 & x \in (1, 3], \\ \frac{x}{3} & x \in (3, \infty), \end{cases}$$

Furthermore, we illustrate $\operatorname{prox}_f(x)$ for a better comprehension.



Finally, we observe that our fixed points of this proximal are those that verify $|x| \leq 1$.