
Take home exam Part II

Convex Unconstrained and Constrained Optimization

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Problem 1. (1 point) We have worked out the elementary version of Lagrange multipliers assuming that from $g(x, y) = 0$ we can find a function $y = h(x)$ such that $g(x, h(x)) = 0$. But sometimes what we get is that there is an h such that $g(h(y), y) = 0$. Rewrite the Lagrange multiplier analysis in the lecture slides under this assumption.

For $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$ consider the following minimization problem

$$\min f(x, y) \text{ s.t. } g(x, y) = 0.$$

Assuming the **Implicit Function Theorem** holds, we can find a function $x = h(y)$ s.t. $g(h(y), y) = 0$ and, thus, we can write

$$f(x, y) = f(h(y), y) = \Psi(y).$$

At a minimum y^* with $x^* = h(y^*)$ we thus have

$$0 = \Psi'(y^*) = \frac{\partial g}{\partial x}(x^*, y^*)h'(y^*) + \frac{\partial g}{\partial y}(x^*, y^*). \quad (1)$$

But since $g(h(y), y) = 0$, we also have

$$0 = \frac{\partial g}{\partial x}(x^*, y^*)h'(y^*) + \frac{\partial g}{\partial y}(x^*, y^*) \implies h'(y^*) = -\frac{\frac{\partial g}{\partial y}(x^*, y^*)}{\frac{\partial g}{\partial x}(x^*, y^*)}. \quad (2)$$

Putting together 1 and 2 we arrive at

$$0 = \frac{\partial f}{\partial y}(x^*, y^*)\frac{\partial g}{\partial x}(x^*, y^*) - \frac{\partial f}{\partial x}(x^*, y^*)\frac{\partial g}{\partial y}(x^*, y^*).$$

That is, at (x^*, y^*) , $\nabla f \perp \left(-\frac{\partial g}{\partial y}, \frac{\partial g}{\partial x}\right)$ and, since $\left(-\frac{\partial g}{\partial y}, \frac{\partial g}{\partial x}\right) \perp \nabla g$, we have $\nabla f \parallel \nabla g$ i.e. $\nabla f(x^*, y^*) = -\lambda^* \nabla g(x^*, y^*)$ for some $\lambda^* \neq 0$.

Thus, for the **Lagrangian**

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda g(x, y),$$

we have a minimum (x^*, y^*) there is a $\lambda^* \neq 0$ s.t.

$$\nabla \mathcal{L}(x^*, y^*, \lambda^*) = \nabla f(x^*, y^*) + \lambda^* \nabla g(x^*, y^*) = 0.$$

Problem 2. (3 points) We want to solve the following constrained minimization problem:

$$\begin{aligned} \min \quad & f(x, y) = x^2 + 2xy + 2y^2 - 3x + y \\ \text{s.t.} \quad & x + y = 1, \\ & x \geq 0, y \geq 0. \end{aligned}$$

Argue first that f is convex and then:

- Write its Lagrangian with α, β the multipliers of the inequality constraints.
- Write the KKT conditions.
- Use them to solve the problem. For this consider separately the $(\alpha = \beta = 0)$, $(\alpha > 0, \beta = 0)$, $(\alpha = 0, \beta > 0)$, $(\alpha > 0, \beta > 0)$ cases.

First of all, we verify that f is convex because its Hessian matrix is positive semidefinite, or equivalently its eigenvalues are non-negative. The Hessian matrix is

$$\mathbf{H}_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix},$$

whose eigenvalues are $\lambda_1 = 3 + \sqrt{5}$ and $\lambda_2 = 3 - \sqrt{5}$, both positive. As a result, we determine f is convex.

Now we write its Lagrangian, with α and β as multipliers of the inequality constraints and λ as equality constraint multiplier.

$$\mathcal{L}(x, y, \lambda, \alpha, \beta) = x^2 + 2xy + 2y^2 - 3x + y + \lambda(x + y - 1) - \alpha x - \beta y.$$

Assuming that the hypothesis of the KKT conditions theorem hold, the resulting KKT conditions on a local minimum (x^*, y^*) are the following:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial x}(x^*, y^*, \lambda, \alpha, \beta) \implies \lambda + 2x^* + 2y^* - 3 - \alpha = 0, \\ 0 &= \frac{\partial \mathcal{L}}{\partial y}(x^*, y^*, \lambda, \alpha, \beta) \implies \lambda + 2x^* + 4y^* + 1 - \beta = 0, \\ 0 &= \alpha x^*, \\ 0 &= \beta y^*. \end{aligned}$$

Then, we use them to solve the problem, considering the four possible cases below:

- Case $\alpha = \beta = 0$.

$$0 = \frac{\partial \mathcal{L}}{\partial x} = \lambda + 2x + 2y - 3$$

$$0 = \frac{\partial \mathcal{L}}{\partial y} = \lambda + 2x + 4y + 1$$

From both expressions we get $2x + 2y - 3 = 2x + 4y + 1 \implies 2y = -4 \implies y = -2$. Then, $x = 1 - y = 3$ and $\lambda = -1$. So we have $(\mathbf{3}, -\mathbf{2})'$ as **feasible KKT point**.

- Case $\alpha > 0, \beta = 0$. When $\alpha > 0$, $x = 0$, so $y = 1$ and $\lambda = -5$. Therefore we have a **feasible KKT point** on $(\mathbf{0}, \mathbf{1})'$.
- Case $\alpha = 0, \beta > 0$. When $\beta > 0$, $y = 0$, so $x = 1$ and $\lambda = 1$. Therefore we have a **feasible KKT point** on $(\mathbf{1}, \mathbf{0})'$.
- Case $\alpha > 0, \beta > 0$. This implies $x = y = 0$, so we have a contradiction because $x + y \neq 1$. This means that $(\mathbf{0}, \mathbf{0})'$ is **not** a feasible KKT point.

Given all points found, we can determine that our optimal solution is $(\mathbf{1}, \mathbf{0})'$, with an optimal value $\lambda = \mathbf{1}$.

Problem 3. (1 point) Let $f : S \subset \mathbb{R}^d \rightarrow \mathbb{R}$ be a convex function on the convex set S and we extend it to an $\tilde{f} : \mathbb{R}^d \rightarrow \mathbb{R}$ as:

$$\begin{aligned}\tilde{f}(x) &= f(x) \text{ if } x \in S. \\ &= +\infty \text{ if } x \notin S.\end{aligned}$$

Show that \tilde{f} is a convex function on \mathbb{R}^d . Assume that $a + \infty = \infty$ and that $a \cdot \infty = \infty$ for $a > 0$.

We say that S is a **convex set** if for all $x, x' \in S$ and $\lambda \in [0, 1]$,

$$\lambda x + (1 - \lambda)x' \in S.$$

Let $x, x' \in S$ and $\lambda \in [0, 1]$, so here we cover two cases:

- *First case.* If $x, x' \in S$:

$$\tilde{f}(\lambda x + (1 - \lambda)x') = f(\lambda x + (1 - \lambda)x') \leq \lambda f(x) + (1 - \lambda)f(x') = \lambda \tilde{f}(x) + (1 - \lambda)\tilde{f}(x').$$

Where the first equality holds given that S is convex. The inequality holds because f is convex. And, the last equality raises from the definition of \tilde{f} .

- *Second case.* If $x \notin S$ or $x' \notin S$, we have that

$$\tilde{f}(\lambda x + (1 - \lambda)x') \leq \lambda \tilde{f}(x) + (1 - \lambda)\tilde{f}(x') = +\infty,$$

because $\tilde{f}(y) \leq +\infty, \forall y \in \mathbb{R}^d$.

Since both cases satisfy convexity definition holds, we conclude this function is convex.

Problem 4. (2 points) Prove **Jensen's inequality**: if f is convex on \mathbb{R}^d and $\sum_1^k \lambda_i = 1$, with $0 \leq \lambda_i \leq 1$ we have for any $x_1, \dots, x_k \in \mathbb{R}^d$

$$f\left(\sum_1^k \lambda_i x_i\right) \leq \sum_1^k \lambda_i f(x_i)$$

Hint: just write $\sum_1^k \lambda_i x_i = \lambda_1 x_1 + (1 - \lambda_1) v$ for an appropriate v and apply repeatedly the definition of a convex function. Start with $k = 3$ and carry on.

We proceed using an inductive procedure:

- If $k = 1$ then $\lambda = 1$, so we simply have $f(x_1) \leq f(x_1)$, which is true, and nothing to prove. If $k = 2$ we have the definition of the convexity of f :

$$\lambda_1 + \lambda_2 = 1, \lambda_1, \lambda_2 \geq 0 \implies f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2).$$

- Considering $k = 3$, given $\lambda_1 + \lambda_2 + \lambda_3 = 1$, and $\lambda_2, \lambda_3 > 0$, we have the following inequality

$$\begin{aligned} f(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3) &= f\left(\lambda_1 x_1 + (1 - \lambda_1) \left(\frac{\lambda_2}{\lambda_2 + \lambda_3} x_2 + \frac{\lambda_3}{\lambda_2 + \lambda_3} x_3\right)\right) \\ &\leq f(\lambda_1 x_1) + f\left((1 - \lambda_1) \left(\frac{\lambda_2}{\lambda_2 + \lambda_3} x_2 + \left(1 - \frac{\lambda_2}{\lambda_2 + \lambda_3}\right) x_3\right)\right) \\ &\leq \lambda_1 f(x_1) + (1 - \lambda_1) \left(\frac{\lambda_2}{\lambda_2 + \lambda_3} f(x_2) + \left(1 - \frac{\lambda_2}{\lambda_2 + \lambda_3}\right) f(x_3)\right) \\ &= \sum_{i=1}^3 \lambda_i f(x_i), \end{aligned}$$

where we applied the case $k = 2$ twice.

- We assume the statement is true for k and consider $k + 1$ points x_1, \dots, x_{k+1} , with coefficients $\lambda_1, \dots, \lambda_{k+1} \geq 0$, $\sum_{i=1}^{k+1} \lambda_i = 1$. The evaluation of the linear combination can be decomposed as

$$f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) = f\left((1 - \lambda_1) \left(\sum_{i=2}^{k+1} \frac{\lambda_i}{1 - \lambda_1} x_i\right) + \lambda_1 x_1\right).$$

Using this, it is straightforward to use the Jensen's inequality on x_1 and $\sum_{i=2}^{k+1} \frac{\lambda_i}{1 - \lambda_1} x_i$ with coefficients λ_1 and $1 - \lambda_1$ respectively. That is,

$$f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) \leq (1 - \lambda_1) f\left(\sum_{i=2}^{k+1} \frac{\lambda_i}{1 - \lambda_1} x_i\right) + \lambda_1 f(x_1).$$

We may notice that $\sum_{i=2}^{k+1} \frac{\lambda_i}{1 - \lambda_1} x_i$ verifies the inductive hypothesis, thus,

$$f\left(\sum_{i=2}^{k+1} \frac{\lambda_i}{1 - \lambda_1} x_i\right) \leq \sum_{i=2}^{k+1} \frac{\lambda_i}{1 - \lambda_1} f(x_i).$$

Finally,

$$f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) \leq (1 - \lambda_1) \sum_{i=2}^{k+1} \frac{\lambda_i}{1 - \lambda_1} f(x_i) + \lambda_1 f(x_1) = \sum_{i=1}^{k+1} \lambda_i f(x_i).$$

Problem 5. (3 points) Prove that the following function is convex

$$\begin{aligned} f(x) &= x^2 - 1, & |x| > 1 \\ &= 0 & |x| \leq 1 \end{aligned}$$

and compute its proximal. Which are the fixed points of this proximal?

We note that f can be seen as the maximum of two functions $f(x) = \max\{0, x^2 - 1\}$. Both of these functions are convex. Then, we are going to show that the maximum of two convex functions is also convex.

Let m and n be two convex functions and $h(x) = \max\{m(x), n(x)\}$. Given $\lambda \in [0, 1]$ and $x, y \in \mathbb{R}$ we aim to show that

$$h(\lambda x + (1 - \lambda)y) \leq \lambda h(x) + (1 - \lambda)h(y).$$

On one hand, it is clear that

$$m(\lambda x + (1 - \lambda)y) \leq \lambda m(x) + (1 - \lambda)m(y) \leq \lambda h(x) + (1 - \lambda)h(y),$$

where in the first inequality comes from the fact that m is convex and the second from the definition of h . The same inequalities holds for n . Given that both functions are upper bounded by the same value, the maximum is also upper bounded by this value, so h is convex. Using this auxiliary result, f is convex.

Now we compute the proximal operator of f as

$$\text{prox}_f(x) = \arg \min_z f(z) + \frac{1}{2}(z - x)^2 = \arg \min_z h(z)$$

with

$$h(z) = \begin{cases} z^2 - 1 + \frac{1}{2}(z - x)^2 & |z| > 1 \\ \frac{1}{2}(z - x)^2 & |z| \leq 1 \end{cases} \quad (3)$$

If the minimizer is attained at $|z| \leq 1$, then clearly $z = x$, meaning that $\text{prox}_f(x) = x$ for $|x| \leq 1$. If it is attained at $|z| > 1$, we have

$$0 = h'(z) = 3z - x \implies z = \frac{1}{3}x,$$

which implies that $\text{prox}_f(x) = \frac{1}{3}x$ for $|x| > 3$. The remaining values of the proximal must be studied separately: the only possible minimizers are the points of non differentiability of (3).

That is -1 and 1 with

$$h(-1) = \frac{1}{2}(-1 - x)^2 \quad \text{and} \quad h(1) = \frac{1}{2}(1 - x)^2.$$

- When $-3 \leq x < -1$, the function is minimized at $z = -1$, that is, the proximal is $\text{prox}_f(x) = -1$.
- When $1 < x \leq 3$, the function is minimized at $z = 1$, that is, the proximal is $\text{prox}_f(x) = 1$.

As a result, the proximal is

$$\text{prox}_f(x) = \begin{cases} \frac{x}{3} & x \in (-\infty, -3), \\ -1 & x \in [-3, -1), \\ x & x \in [-1, 1], \\ 1 & x \in (1, 3], \\ \frac{x}{3} & x \in (3, \infty), \end{cases}$$

Finally, we illustrate $\text{prox}_f(x)$ for a better comprehension.

