Stochastic differential equations

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Ordinary Differential Equations (ODE's)

- An ODE consists in a functional relation between a function and its derivatives.

Example: A kth order ODE in implicit form

$$a\left(x(t), x'(t), x''(t), \dots, x^{(k)}(t)\right) = 0$$

- A first order ODE in explicit form is

$$\frac{dx(t)}{dt} = a\big(t, x(t)\big)$$

Formal solution of an ODE

- Consider the first order ODE in explicit form

$$\frac{dx(t)}{dt} = a\big(t, x(t)\big)$$

- The solution of this ODE for an initial condition $x(t_0) = x_0$ is

$$x(t) = x(t_0) + \int_{t_0}^{t} a\big(s, x(s)\big)\, ds$$

Closed-form solution of a simple ODE

- Consider the first order ODE

$$\frac{dx(t)}{dt} = r\,x(t)$$

- The solution of this ODE for an initial condition $x(t_0) = x_0$ is

$$x(t) = x_0\,exp\{r(t - t_0)\}$$

Numerical solution of an ODE

- Consider the alternative form of a first order ODE

$$dx(t) = a\big(t, x(t)\big)dt$$

- Using the definition of the differential:

$$dx(t) = x(t + dt) - x(t)$$

- It is possible to write

$$x(t + dt) = x(t) + a\big(t, x(t)\big)dt$$

Numerical solution of an ODE: Euler scheme

- Initial condition: $x(t_0) = x_0$
- Integration interval $[t_0\,, t_0 + T]$
- Grid of points for integration: $t_0 < t_1 < \cdots < t_N = t_0 + T$;
- Solution trajectory:

$$\boxed{x_{n+1} = x_n + a(t_n, x_n)\Delta t_n; \qquad n = 0,1, \dots, (N-1)}$$

$$x_n \stackrel{\text{def}}{=} x(t_n); \qquad \Delta t_n \stackrel{\text{def}}{=} t_{n+1} - t_n \longrightarrow 0^+$$

Euler scheme for a simple ODE

- Consider the first order ODE

$$\frac{dx(t)}{dt} = rx(t);$$

- The numerical solution of this ODE starting from the i. c. $x(t_0) = x_0$

$$x_{n+1} = x_n(1 + r\Delta t_n); \quad n = 0,1, \dots, (N-1)$$

- Solution at $t_N = t_0 + T$

$$x_N = x_0 \prod_{n=0}^{N-1}(1 + r\Delta t_n) \xrightarrow[\Delta t_n \to 0^+]{} x_0 exp\left(r \sum_{n=0}^{N-1} \Delta t_n\right) = x_0 exp(rT)$$

Euler scheme: Regular grid

- Initial condition: $x(t_0) = x_0$
- Integration interval $[t_0\,, t_0 + T]$
- Regular grid of integration points : $t_n = t_0 + n\Delta T; \quad n = 0, 1, \dots, N$
- Solution trajectory:

$$\boxed{x_{n+1} = x_n + a(t_n, x_n)\Delta T; \qquad n = 0,1, \dots, (N-1)}$$

$$x_n \stackrel{\text{def}}{=} x(t_n); \qquad \Delta T \stackrel{\text{def}}{=} \frac{T}{N} \longrightarrow 0^+$$

Stochastic Differential Equation (SDE)

- Consider the first order ODE in explicit form

$$dX(t) = a\big(t, X(t)\big)dt + b\big(t, X(t)\big)dW(t)$$

where $W(t)$ is the Wiener process (standard Brownian motion).

- The formal solution of this equation, assuming that the process starts at the initial condition $X(t_0) = x_0$ with probability one, is

$$X(t) = x_0 + \int_{t_0}^{t} a\big(s, X(s)\big)\, ds + \int_{t_0}^{t} b\big(s, X(s)\big)\, dW(s)$$

Non-anticipating functions

Related to causality:
The present (t)
cannot be affected by
the future $(t+\tau)$

- The function $g(t,\omega)$ is non-anticipating if

$$\forall\, t,\tau > 0: \quad g(t,\omega) \perp \big(W(t+\tau,\omega) - W(t,\omega)\big)$$

- Examples of non-anticipating functions
  - $W(t,\omega)$, $\quad \int_{t_0}^{t} f(W(s,\omega))\, ds$, $\quad \int_{t_0}^{t} f(W(s,\omega))\, dW(s,\omega)$
  - Assuming that $g(t,\omega)$ is non-anticipating,

    the functions $\int_{t_0}^{t} g(s,\omega)\, ds$, $\int_{t_0}^{t} g(s,\omega)\, dW(s,\omega)$ are also non-anticipating.

Stochastic Integral (Itô)

- Consider the stochastic integral

$$I(\omega) = \int_{t_0}^{t} g(s,\omega)\, dW(s,\omega)$$

Non-anticipating function

$\omega \in \Omega$ indicates that it is a particular realization (trajectory) of the Wiener process. We will omit it when there is no ambiguity as to the source of randomness.

- Consider the Riemann sum in the grid $t_0 < t_1 < \cdots < t_N = t$

$$I_N(\omega) = \sum_{n=0}^{N-1} g(t_n,\omega)\left(W(t_{n+1},\omega) - W(t_n,\omega)\right)$$

The time at which g is evaluated matters!

- The stochastic integral is the limit $N \to \infty$ of this Riemann sum

$$I(\omega) = \lim_{N\to\infty} I_N\,(\omega)$$

Convergence in the mean square sense: $\lim_{N\to\infty} \mathbb{E}\left[(I_N - I)^2\right] = 0$

It is a random variable! Its value depends on the particular realization of the random process

A simple stochastic integral

- Consider the stochastic (Itô) integral

$$I = \int_{t_0}^{t} c \, dW(s)$$

- Applying the definition of the stochastic integral

$$I = \lim_{N \to \infty} \sum_{n=0}^{N-1} c \left(W(t_{n+1}) - W(t_n) \right) = c \lim_{N \to \infty} \sum_{n=0}^{N-1} \left(W(t_{n+1}) - W(t_n) \right)$$

$$= c(W(t) - W(t_0))$$

A more complex stochastic integral

- Consider the stochastic (Itô) integral

$$S = \int_{t_0}^{t} W(s)\, dW(s)$$

- Applying the definition of the stochastic integral

$$S = \lim_{N\to\infty} \sum_{n=0}^{N-1} W(t_n)\left(W(t_{n+1}) - W(t_n)\right)$$

$$= \frac{1}{2}\lim_{N\to\infty} \sum_{n=0}^{N-1} \left[\left(W(t_{n+1})^2 - W(t_n)^2\right) - \left(W(t_{n+1}) - W(t_n)\right)^2\right]$$

$$= \frac{1}{2}\left(W(t)^2 - W(t_0)^2\right) - \frac{1}{2}\lim_{N\to\infty} \sum_{n=0}^{N-1}\left[(t_{n+1} - t_n)Z_n^2\right]$$

$$= \frac{1}{2}\left(W(t)^2 - W(t_0)^2\right) - \frac{1}{2}(t - t_0)$$

$$\mathbb{E}\left[\lim_{N\to\infty} \sum_{n=0}^{N-1}\left[(t_{n+1} - t_n)Z_n^2\right]\right]$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N-1}\left[(t_{n+1} - t_n)\mathbb{E}\left[Z_n^2\right]\right]$$

$$= (t - t_0)$$

$$\mathrm{Var}\left[\lim_{N\to\infty} \sum_{n=0}^{N-1}\left[(t_{n+1} - t_n)Z_n^2\right]\right]$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N-1}\left[(t_{n+1} - t_n)^2\mathrm{Var}\left[Z_n^2\right]\right]$$

$$\leq 2\max_n(t_{n+1} - t_n)\lim_{N\to\infty} \sum_{n=0}^{N}(t_{n+1} - t_n)$$

$$= 2\max_n(t_{n+1} - t_n)\,(t - t_0)$$

Properties of Itô integrals

- Consider the non-anticipating functions $g(s,\omega), g_1(s,\omega), g_2(s,\omega)$
  - $\mathbb{E}\left[\int_{t_0}^{t} g(s,\omega)\, dW(s,\omega)\right] = 0$
  - $\mathrm{Var}\left[\int_{t_0}^{t} g(s,\omega)\, dW(s,\omega)\right] = \int_{t_0}^{t} \mathbb{E}\left[g^2(s,\omega)\right] ds < \infty$
    (condition for the Itô stochastic integral to exist)
  - $\mathbb{E}\left[\int_{t_0}^{t} g_1(s,\omega)\, dW(s,\omega) \int_{t_0}^{t} g_2(s,\omega)\, dW(s,\omega)\right] = \int_{t_0}^{t} \mathbf{E}[g_1(s,\omega) g_2(s,\omega)]\, ds$
  - Linearity:

$$\int_{t_0}^{t} [a_1 g_1(s,\omega) + a_2 g_2(s,\omega)(s,\omega)]\, dW(s,\omega)$$
$$= a_1 \int_{t_0}^{t} g_1(s,\omega)\, dW(s,\omega) + a_2 \int_{t_0}^{t} g_2(s,\omega)\, dW(s,\omega)$$

$dW^2(t) = dt$

- Consider the non-anticipating, bounded function $g(t, \omega)$, then

$$\int_{t_0}^{t} g(s, \omega)\,[dW(s, \omega)]^2 = \int_{t_0}^{t} g(s, \omega)\,ds$$

$$dW^{2+n}(t) = 0; \qquad n > 0$$

- Consider the non-anticipating, bounded function $g(t, \omega)$, then

$$\int_{t_0}^{t} g(s, \omega)\,[dW(s, \omega)]^{2+n} = 0$$

General differentiation rules

- $[dW(t)]^2 \longrightarrow dt$
- $[dW(t)]^{2+n} \longrightarrow 0 \quad (n > 0)$
- $dt\, dW(t) \longrightarrow 0$
- $(dt)^{1+n} \longrightarrow 0 \quad (n > 0)$
- All higher powers vanish
- $\mathbb{E}\,[dW(t,\omega)\, dW(t',\omega)] \longrightarrow \delta(t - t')\, dt\, dt'$

$$d\varphi\big(t, W(t)\big) = \left(\varphi_t\big(t, W(t)\big) + \frac{1}{2}\,\varphi_{WW}\big(t, W(t)\big)\right) dt + \varphi_W\big(t, W(t)\big) dW(t)$$

Itô’s lemma

- Consider the SDE

$$dX(t) = a\big(t, X(t)\big)dt + b\big(t, X(t)\big)dW(t)$$

- The SDE for $Y(t) = \varphi\big(t, X(t)\big)$ is

$$dY(t) = \tilde{a}\big(t, X(t)\big)dt + \tilde{b}\big(t, X(t)\big)dW(t)$$

$$\tilde{a}\big(t, X(t)\big) = \varphi_t\big(t, X(t)\big) + \varphi_x\big(t, X(t)\big)a\big(t, X(t)\big) + \frac{1}{2}\varphi_{xx}\big(t, X(t)\big)\, b\big(t, X(t)\big)^2$$

$$\tilde{b}\big(t, X(t)\big) = \varphi_x\big(t, X(t)\big)b\big(t, X(t)\big)$$

General differentiation rules: Several variables

- $dW_i(t)$ behaves as a differential $(dt)^{1/2}$.
- $dW_i(t)dW_j(t) \longrightarrow \delta_{ij}dt; \quad i,j = 1,2,\ldots,D$
- $[dW_i(t)]^{2+n} \longrightarrow 0 \quad (n > 0)$
- $dt\, dW_i(t) \longrightarrow 0$
- $(dt)^{1+n} \longrightarrow 0 \quad (n > 0)$
- All higher powers vanish.

Itô's lemma: Geometric Brownian Motion

- Consider the SDE

$$dS(t) = \mu\, S(t)dt + \sigma\, S(t)\, dW(t)$$

- Change of variable: $Y(t) = \log S(t)$
  - $\varphi\big(t, S(t)\big) = \log S(t)$
  - $\varphi_t\big(t, S(t)\big) = 0; \quad \varphi_S\big(t, S(t)\big) = \frac{1}{S(t)}; \quad \varphi_{SS}\big(t, S(t)\big) = -\frac{1}{S(t)^2}$

$$dY(t) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW(t)$$

Geometric Brownian Motion SDE: Solution

- Consider the SDE: $dY(t) = \left(\mu - \frac{1}{2}\sigma^2\right)dt + \sigma dW(t)$

$$Y(t) = Y(t_0) + \left(\mu - \frac{1}{2}\sigma^2\right)(t - t_0) + \sigma(W(t) - W(t_0))$$

- Undoing the change of variable: $Y(t) = \log S(t)$

$$S(t) = S(t_0)\, exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)(t - t_0) + \sigma(W(t) - W(t_0))\right\}$$

Numerical solution of an SDE: Stochastic Euler scheme (order 1/2)

- Initial condition: $x(t_0) = x_0$
- Integration interval $[t_0\,, t_0 + T]$
- Grid of points for integration: $t_0 < t_1 < \cdots < t_N = t_0 + T$;
- Solution: $M \longrightarrow \infty$ trajectories

$$\boxed{\begin{aligned} x_0^{(m)} &= x_0; \quad m = 1, \dots, M. \\ x_{n+1}^{(m)} &= x_n^{(m)} + a\left(t_n, x_n^{(m)}\right)\Delta t_n + b\left(t_n, x_n^{(m)}\right)\sqrt{\Delta t_n}\, Z_n^{(m)} \\ & \qquad n = 0,1, \dots, (N-1); \quad m = 1, \dots, M. \end{aligned}}$$

$$x_n^{(m)} \overset{\text{def}}{=} x^{(m)}(t_n); \quad \Delta t_n \overset{\text{def}}{=} t_{n+1} - t_n \longrightarrow 0^+; \ \left\{Z_n^{(m)} \sim N(0,1)\right\} \ \text{iidrv's}$$

Value of the mth trajectory at instant t_n

Stochastic Euler scheme: Regular grid

- Initial condition: $x(t_0) = x_0$
- Integration interval $[t_0\ , t_0 + T]$
- Regular grid of integration points : $t_n = t_0 + n\Delta T; \quad n = 0, 1, \dots, N$
- Solution: $M \longrightarrow \infty$ trajectories

$$\boxed{\begin{aligned} x_0^{(m)} &= x_0; \quad m = 1, \dots, M. \\ x_{n+1}^{(m)} &= x_n^{(m)} + a\left(t_n, x_n^{(m)}\right)\Delta T + b\left(t_n, x_n^{(m)}\right)\sqrt{\Delta T} Z_n^{(m)} \\ &\qquad n = 0,1, \dots, (N-1); \qquad m = 1, \dots, M. \end{aligned}}$$

$$x_n^{(m)} \stackrel{\text{def}}{=} x^{(m)}(t_n); \qquad \Delta T \stackrel{\text{def}}{=} \frac{T}{N} \longrightarrow 0^+; \left\{Z_n^{(m)} \sim N(0,1)\right\} \text{ iidrv's}$$

Value of the mth trajectory at instant t_n

Numerical solution of an SDE: Milstein scheme (order 1)

Timothy Sauer. 2013. Computational solution of stochastic differential equations. WIREs Comput. Stat. 5, 5 (September 2013), 362–371. http://math.gmu.edu/~tsauer/pre/wires.pdf

- Solution: $M \longrightarrow \infty$ trajectories

$$x_0^{(m)} = x_0; \quad m = 1, \dots, M.$$

$$x_{n+1}^{(m)} = x_n^{(m)} + a\left(t_n, x_n^{(m)}\right)\Delta t_n + b\left(t_n, x_n^{(m)}\right)\sqrt{\Delta t_n} Z_n^{(m)}$$

$$+ \frac{1}{2} b\left(t_n, x_n^{(m)}\right) b_x\left(t_n, x_n^{(m)}\right)\left(\left(Z_n^{(m)}\right)^2 - 1\right)\Delta t_n$$

$$n = 0,1, \dots, (N-1); \quad m = 1, \dots, M.$$

$$x_n^{(m)} \stackrel{\text{def}}{=} x^{(m)}(t_n); \quad \Delta t_n \stackrel{\text{def}}{=} t_{n+1} - t_n \longrightarrow 0^+; \quad \left\{Z_n^{(m)} \sim N(0,1)\right\} \text{ iidrv's}$$

Value of the mth trajectory at instant t_n

Fokker-Planck equation (conditional probability density)

- Consider the SDE with the initial condition $X(t_0) = x_0$

$$dX(t) = A\big(t, X(t)\big)dt + \sqrt{B\big(t, X(t)\big)}dW(t)$$

- The corresponding Fokker-Planck equation is

$$\partial_t \, p(x, t \mid x_0, t_0) = -\partial_x[A(t, x)p(x, t \mid x_0, t_0)] + \frac{1}{2}\partial_x^2\big[B\big(t, X(t)\big)p(x, t \mid x_0, t_0)\big]$$

Proof: Apply Itô's lemma to compute $\frac{d}{dt}\mathbb{E}\big[f\big(X(t)\big)\big]$

$$\mathbb{E}\big[f\big(X(t)\big)\big] = \int f(x)p(x, t \mid x_0, t_0)\, dx$$

$$p(x, t_0 \mid x_0, t_0) = \delta(x - x_0)$$

Fokker-Planck equation (general probability density)

- Assuming that the distribution at t_0 is $p_0(x)$

$$p(x,t) = \int p(x,t;\ x_0,t_0)\, dx_0 = \int p(x,t \mid x_0,t_0) p_0(x_0) dx_0$$

- The corresponding Fokker-Planck equation is

$$\partial_t p(x,t) = -\partial_x[A(t,x)p(x,t)] + \frac{1}{2}\partial_x^2[B(t,x)p(x,t)]$$

$$p(x,t)|_{t=t_0} = p_0(x)$$

Ornstein-Uhlenbeck Process (Fokker-Planck)

k: rate,
inverse correlation time

D: diffusion constant

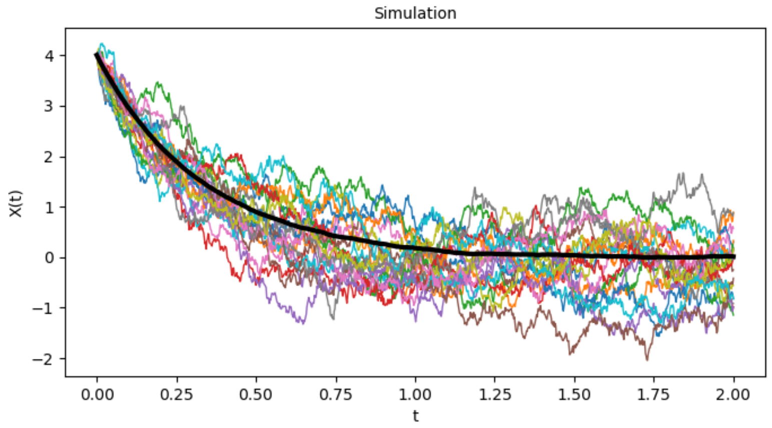
$$\partial_t p(x,t \mid x_0,0) = \partial_x[k\, x\, p(x,t \mid x_0,0)] + \frac{1}{2} D \partial_x^2 p(x,t \mid x_0,0)$$

- Consider the equation for the characteristic function $\phi(s,t) = \int_{-\infty}^{\infty} e^{isx}\, p(x,t \mid x_0,0)\, dx$

$$\partial_t \phi(s,t) = -ks\, \phi(s,t) - \frac{1}{2} D s^2 \phi(s,t)$$

- The solution is

$$\phi(s,t) = exp\left[-\frac{Ds^2}{4k}\left(1 - e^{-2kt}\right) + isx_0 e^{-kt}\right]$$



Gardiner, C. W. (2004). Handbook of stochastic methods for physics, chemistry and the natural sciences (Vol. 13). Berlin: Springer-Verlag. ISBN: 3-540-20882-8

Stationary solution for the OU Process

- The stationary solution for the characteristic function of the OU process is

$$\phi_s(s) = \lim_{t\to\infty} \phi(s,t) = \lim_{t\to\infty} exp\left[-\frac{Ds^2}{4k}\left(1-e^{-2kt}\right) + isx_0e^{-kt}\right] = exp\left[-\frac{Ds^2}{4k}\right]$$

- Performing the inverse Fourier transform, we obtain the stationary solution of the OU process

$$p_s(x) = \left(\frac{k}{\pi D}\right)^{1/2} exp\left[-k\,\frac{x^2}{D}\right]$$

Ornstein-Uhlenbeck Process (SDE)

D: diffusion constant

$$dX(t) = -kX(t)dt + \sqrt{D}dW(t)$$

k: rate,
inverse correlation time

- Solution: $X(t) = x_0 e^{-kt} + \sqrt{D}\int_0^t e^{-k(t-s)} dW(s)$

$$\mathbb{E}[X(t)] = x_0 e^{-kt}$$

$$\mathrm{Var}[X(t)] = \frac{D}{2k}\left(1 - e^{-2kt}\right)$$

$$X(t) = x_0 e^{-kt} + \sqrt{\frac{D}{2k}\left(1 - e^{-2kt}\right)}\, Z; \quad Z \sim N(0,1)$$

$$\mathrm{Cov}[X(t), X(t')] = D \int_0^t \int_0^{t'} e^{-k(t+t'-s-s')} \mathbb{E}[dW(s)dW(s')] = D \int_0^{\min(t,t')} e^{-k(t+t'-2s)} ds$$

$$= -\frac{D}{2k} e^{-k(t+t')} + \frac{D}{2k} e^{-k|t-t'|}$$

Simulation scheme for Ornstein Uhlenbeck

- Initial condition: $x(t_0) = x_0$
- Integration interval $[t_0\ , t_0 + T]$
- Regular grid of integration points : $\quad t_n = t_0 + n\Delta T; \quad n = 0, 1, \dots, N$
- Solution: $M \longrightarrow \infty$ trajectories

$$\boxed{\begin{array}{l} x_0^{(m)} = x_0; \quad m = 1, \dots, M. \\ x_{n+1}^{(m)} = x_n^{(m)} e^{-k\Delta T} + \sqrt{\frac{D}{2k}\left(1 - e^{-2k\Delta T}\right)} Z_n^{(m)} \\ \qquad n = 0,1, \dots, (N-1); \qquad m = 1, \dots, M. \end{array}}$$

$x_n^{(m)} \stackrel{\text{def}}{=} x^{(m)}(t_n); \qquad \Delta T \stackrel{\text{def}}{=} \frac{T}{N}; \qquad \left\{Z_n^{(m)} \sim N(0,1)\right\}$ iidrv's

Value of the mth trajectory at instant t_n

Jump-diffusion SDE

$X(t^-) = \lim_{\substack{s\to t \\ s<t}} X(s)$

$J(t)$ is a jump process

$$dX(t) = a\big(t, X(t^-)\big)dt + b\big(t, X(t^-)\big)dW(t) + c\big(t, X(t^-)\big)dJ(t)$$

First jump at τ_1

Second jump at τ_2

- Jump process:
  - The jumps occur at random arrival times $t_0 < \tau_1 < \tau_2 < \cdots$
  - The jth jump occurs at time τ_j and has a magnitude $Y_j \sim f_Y(y)$.
  - Consider the counting process $N(t) = \sum_{j=1}^{\infty} \mathbb{I}\big[t > \tau_j\big]$

$$J(t) = \sum_{j=1}^{N(t)} Y_j$$

Jump-diffusion SDE: numerical integration

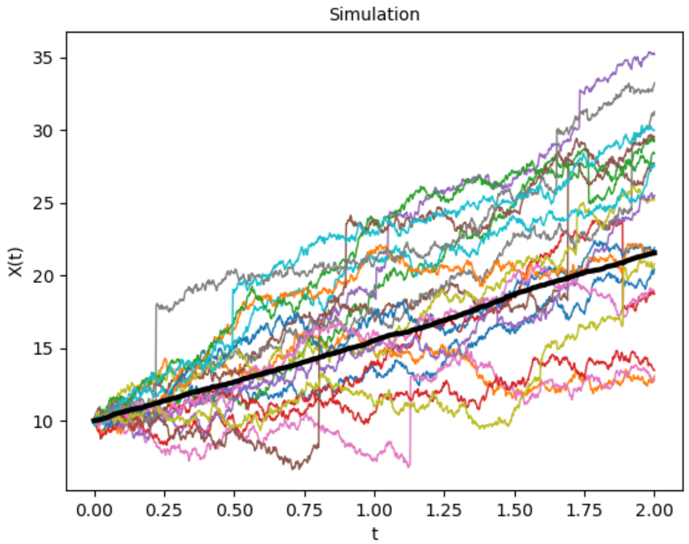
$$dX(t) = a\big(t, X(t^-)\big)dt + b\big(t, X(t^-)\big)dW(t) + c\big(t, X(t^-)\big)dJ(t)$$

- Integration in an interval without any jump $[t, t + \Delta T]$, $\quad \Delta T \to 0^+$

$$X(t + \Delta T) = a\big(t, X(t)\big)\Delta T + b\big(t, X(t)\big)\sqrt{\Delta T}\ Z; \qquad Z \sim N(0,1)$$

- Integration in interval with a jump $\left[\tau_j^-, \tau_j\right]$,

$$X(\tau_j) = X\left(\tau_j^-\right) + c\left(\tau_j, X\left(\tau_j^-\right)\right)Y_j; \qquad Y_j \sim f_Y(y)$$



Merton's jump-diffusion SDE

$$dS(t) = \mu\, S(t^-)dt + \sigma\, S(t^-)dW(t) + S(t)dJ(t)$$

iidrv's

$$S(t) = S(0)exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}\, Z\right\}\prod_{j=1}^{N(t)} Y_j \quad ; \qquad Z{\sim}N(0{,}1); Y_j{\sim}LN(\gamma, \delta)$$

- Integration in an interval without any jump $[t_n, t_{n+1}]$, $\quad \Delta T_n = t_{n+1} - t_n$

$$S(t_{n+1}) = S(t_n)exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)\Delta T_n + \sigma\sqrt{\Delta T_n}\, Z_n\right\}; Z_n{\sim}N(0{,}1)$$

- Integration in interval with a jump $\left[\tau_j^-, \tau_j\right]$,

$$S(\tau_j) = S\left(\tau_j^-\right)Y_j; \qquad Y_j{\sim}LN(\gamma, \delta)$$