

---

## Take home exam Part II

### *Convex Unconstrained and Constrained Optimization*

March 28, 2022

Gloria del Valle Cano  
gloria.valle@estudiante.uam.es

---

**Problem 1.** (1 point) We have worked out the elementary version of Lagrange multipliers assuming that from  $g(x, y) = 0$  we can find a function  $y = h(x)$  such that  $g(x, h(x)) = 0$ .

But sometimes what we get is that there is an  $h$  such that  $g(h(y), y) = 0$ . Rewrite the Lagrange multiplier analysis in the lecture slides under this assumption.

For  $f, g : \mathbb{R}^2 \rightarrow \mathbb{R}$  consider the following minimization problem

$$\min f(x, y) \text{ s.t. } g(x, y) = 0.$$

Assuming the **Implicit Function Theorem** holds, we can find a function  $x = h(y)$  s.t.  $g(h(y), y) = 0$  and, thus, we can write

$$f(x, y) = f(h(y), y) = \Psi(y).$$

At a minimum  $y^*$  with  $x^* = h(y^*)$  we thus have

$$0 = \Psi'(y^*) = \frac{\partial g}{\partial x}(x^*, y^*)h'(y^*) + \frac{\partial g}{\partial y}(x^*, y^*). \quad (1)$$

But since  $g(h(y), y) = 0$ , we also have

$$0 = \frac{\partial g}{\partial x}(x^*, y^*)h'(y^*) + \frac{\partial g}{\partial y}(x^*, y^*) \implies h'(y^*) = -\frac{\frac{\partial g}{\partial y}(x^*, y^*)}{\frac{\partial g}{\partial x}(x^*, y^*)}. \quad (2)$$

Putting together (1) and (2) we arrive at

$$0 = \frac{\partial f}{\partial y}(x^*, y^*)\frac{\partial g}{\partial x}(x^*, y^*) - \frac{\partial f}{\partial x}(x^*, y^*)\frac{\partial g}{\partial y}(x^*, y^*).$$

That is, at  $(x^*, y^*)$ ,  $\nabla f \perp \left(-\frac{\partial g}{\partial y}, \frac{\partial g}{\partial x}\right)$  and, since  $\left(-\frac{\partial g}{\partial y}, \frac{\partial g}{\partial x}\right) \perp \nabla g$ , we have  $\nabla f \parallel \nabla g$  i.e.  $\nabla f(x^*, y^*) = -\lambda^* \nabla g(x^*, y^*)$  for some  $\lambda^* \neq 0$ .

Thus, for the **Lagrangian**

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda g(x, y),$$

we have a minimum  $(x^*, y^*)$  there is a  $\lambda^* \neq 0$  s.t.

$$\nabla \mathcal{L}(x^*, y^*, \lambda^*) = \nabla f(x^*, y^*) + \lambda^* \nabla g(x^*, y^*) = 0.$$

**Problem 2.** (3 points) We want to solve the following constrained minimization problem:

$$\begin{aligned} \min \quad & f(x, y) = x^2 + 2xy + 2y^2 - 3x + y \\ \text{s.t.} \quad & x + y = 1, \\ & x \geq 0, y \geq 0. \end{aligned}$$

Argue first that  $f$  is convex and then:

- Write its Lagrangian with  $\alpha, \beta$  the multipliers of the inequality constraints.
- Write the KKT conditions.
- Use them to solve the problem. For this consider separately the  $(\alpha = \beta = 0)$ ,  $(\alpha > 0, \beta = 0)$ ,  $(\alpha = 0, \beta > 0)$ ,  $(\alpha > 0, \beta > 0)$  cases.

First of all, we verify that  $f$  is convex because its Hessian matrix is positive semidefinite, or equivalently its eigenvalues are non-negative. The Hessian matrix is

$$\mathbf{H}_f = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix},$$

whose eigenvalues are  $\lambda_1 = 3 + \sqrt{5}$  and  $\lambda_2 = 3 - \sqrt{5}$ , both positive. As a result, we determine  $f$  is convex.

Now we write its Lagrangian, with  $\alpha$  and  $\beta$  as multipliers of the inequality constraints and  $\lambda$  as equality constraint multiplier.

$$\mathcal{L}(x, y, \lambda, \alpha, \beta) = x^2 + 2xy + 2y^2 - 3x + y + \lambda(x + y - 1) - \alpha x - \beta y.$$

Assuming that the hypothesis of the KKT conditions theorem hold, the resulting KKT conditions on a local minimum  $(x^*, y^*)$  are the following:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial x}(x^*, y^*, \lambda, \alpha, \beta) \implies \lambda + 2x^* + 2y^* - 3 - \alpha = 0, \\ 0 &= \frac{\partial \mathcal{L}}{\partial y}(x^*, y^*, \lambda, \alpha, \beta) \implies \lambda + 2x^* + 4y^* + 1 - \beta = 0, \\ 0 &= \alpha x^*, \\ 0 &= \beta y^*. \end{aligned}$$

Then, we use them to solve the problem, considering the four possible cases below:

- Case  $\alpha = \beta = 0$ .

$$0 = \frac{\partial \mathcal{L}}{\partial x} = \lambda + 2x + 2y - 3$$

$$0 = \frac{\partial \mathcal{L}}{\partial y} = \lambda + 2x + 4y + 1$$

From both expressions we get  $2x + 2y - 3 = 2x + 4y + 1 \implies 2y = -4 \implies y = -2 \not\geq 0$ , so this point does not satisfy this constraint. Then,  $x = 1 - y = 3$  and  $\lambda = -1$ . So we have that  $(3, -2)'$  is not a feasible solution.

- Case  $\alpha > 0, \beta = 0$ . When  $\alpha > 0$ ,  $x = 0$ , so  $y = 1$  and  $\lambda = -5$ . Finally,  $\alpha = -6 \not\geq 0$ . Therefore we have that  $(0, 1)'$  is not a feasible point.
- Case  $\alpha = 0, \beta > 0$ . When  $\beta > 0$ ,  $y = 0$ , so  $x = 1$  and  $\lambda = 1$ . Finally,  $\beta = 4 > 0$ . Therefore we have a **feasible KKT point** on  $(1, 0)'$ .
- Case  $\alpha > 0, \beta > 0$ . This implies  $x = y = 0$ , so we have a contradiction because  $x + y \neq 1$ . This means that  $(0, 0)'$  is not a feasible point.

Given all points found, we can determine that our optimal solution is  $(1, 0)'$ , with an optimal value of  $\mathbf{f}(1, 0) = -2$ .

**Problem 3.** (1 point) Let  $f : S \subset \mathbb{R}^d \rightarrow \mathbb{R}$  be a convex function on the convex set  $S$  and we extend it to an  $\tilde{f} : \mathbb{R}^d \rightarrow \mathbb{R}$  as:

$$\begin{aligned}\tilde{f}(x) &= f(x) \text{ if } x \in S. \\ &= +\infty \text{ if } x \notin S.\end{aligned}$$

Show that  $\tilde{f}$  is a convex function on  $\mathbb{R}^d$ . Assume that  $a + \infty = \infty$  and that  $a \cdot \infty = \infty$  for  $a > 0$ .

We say that  $S$  is a **convex set** if for all  $x, x' \in S$  and  $\lambda \in [0, 1]$ ,

$$\lambda x + (1 - \lambda)x' \in S.$$

Let  $x, x' \in S$  and  $\lambda \in [0, 1]$ , so here we cover two cases:

- *First case.* If  $x, x' \in S$ :

$$\tilde{f}(\lambda x + (1 - \lambda)x') = f(\lambda x + (1 - \lambda)x') \leq \lambda f(x) + (1 - \lambda)f(x') = \lambda \tilde{f}(x) + (1 - \lambda)\tilde{f}(x').$$

Where the first equality holds given that  $S$  is convex. The inequality holds because  $f$  is convex. And, the last equality raises from the definition of  $\tilde{f}$ .

- *Second case.* If  $x \notin S$  or  $x' \notin S$ , we have that

$$\tilde{f}(\lambda x + (1 - \lambda)x') \leq \lambda \tilde{f}(x) + (1 - \lambda)\tilde{f}(x') = +\infty,$$

because  $\tilde{f}(y) \leq +\infty, \forall y \in \mathbb{R}^d$ .

Since both cases satisfy convexity definition holds, we conclude this function is convex.

**Problem 4.** (2 points) Prove **Jensen's inequality**: if  $f$  is convex on  $\mathbb{R}^d$  and  $\sum_1^k \lambda_i = 1$ , with  $0 \leq \lambda_i \leq 1$  we have for any  $x_1, \dots, x_k \in \mathbb{R}^d$

$$f\left(\sum_1^k \lambda_i x_i\right) \leq \sum_1^k \lambda_i f(x_i)$$

*Hint: just write  $\sum_1^k \lambda_i x_i = \lambda_1 x_1 + (1 - \lambda_1) v$  for an appropriate  $v$  and apply repeatedly the definition of a convex function. Start with  $k = 3$  and carry on.*

We proceed using an inductive procedure:

- If  $k = 1$  then  $\lambda = 1$ , so we simply have  $f(x_1) \leq f(x_1)$ , which is true, and nothing to prove. If  $k = 2$  we have the definition of the convexity of  $f$ :

$$\lambda_1 + \lambda_2 = 1, \lambda_1, \lambda_2 \geq 0 \implies f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2).$$

- Considering  $k = 3$ , given  $\lambda_1 + \lambda_2 + \lambda_3 = 1$ , and  $\lambda_2, \lambda_3 > 0$ , we have the following inequality

$$\begin{aligned} f(\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3) &= f\left(\lambda_1 x_1 + (1 - \lambda_1) \left(\frac{\lambda_2}{\lambda_2 + \lambda_3} x_2 + \frac{\lambda_3}{\lambda_2 + \lambda_3} x_3\right)\right) \\ &\leq f(\lambda_1 x_1) + f\left((1 - \lambda_1) \left(\frac{\lambda_2}{\lambda_2 + \lambda_3} x_2 + \left(1 - \frac{\lambda_2}{\lambda_2 + \lambda_3}\right) x_3\right)\right) \\ &\leq \lambda_1 f(x_1) + (1 - \lambda_1) \left(\frac{\lambda_2}{\lambda_2 + \lambda_3} f(x_2) + \left(1 - \frac{\lambda_2}{\lambda_2 + \lambda_3}\right) f(x_3)\right) \\ &= \sum_{i=1}^3 \lambda_i f(x_i), \end{aligned}$$

where we applied the case  $k = 2$  twice.

- We assume the statement its true for  $k$  and consider  $k + 1$  points  $x_1, \dots, x_{k+1}$ , with coefficients  $\lambda_1, \dots, \lambda_{k+1} \geq 0$ ,  $\sum_{i=1}^{k+1} \lambda_i = 1$ . The evaluation of the linear combination can be decomposed as

$$f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) = f\left((1 - \lambda_1) \left(\sum_{i=2}^{k+1} \frac{\lambda_i}{1 - \lambda_1} x_i\right) + \lambda_1 x_1\right).$$

Using this, it is straightforward to use the Jensen's inequality on  $x_1$  and  $\sum_{i=2}^{k+1} \frac{\lambda_i}{1 - \lambda_1} x_i$  with coefficients  $\lambda_1$  and  $1 - \lambda_1$  respectively. That is,

$$f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) \leq (1 - \lambda_1) f\left(\sum_{i=2}^{k+1} \frac{\lambda_i}{1 - \lambda_1} x_i\right) + \lambda_1 f(x_1).$$

We may notice that  $\sum_{i=2}^{k+1} \frac{\lambda_i}{1-\lambda_1} x_i$  verifies the inductive hypothesis, thus,

$$f\left(\sum_{i=2}^{k+1} \frac{\lambda_i}{1-\lambda_1} x_i\right) \leq \sum_{i=2}^{k+1} \frac{\lambda_i}{1-\lambda_1} f(x_i).$$

Finally,

$$f\left(\sum_{i=1}^{k+1} \lambda_i x_i\right) \leq (1-\lambda_1) \sum_{i=2}^{k+1} \frac{\lambda_i}{1-\lambda_1} f(x_i) + \lambda_1 f(x_1) = \sum_{i=1}^{k+1} \lambda_i f(x_i).$$

**Problem 5.** (3 points) Prove that the following function is convex

$$\begin{aligned} f(x) &= x^2 - 1, & |x| > 1 \\ &= 0 & |x| \leq 1 \end{aligned}$$

and compute its proximal. Which are the fixed points of this proximal?

We note that  $f$  can be seen as the maximum of two functions  $f(x) = \max\{0, x^2 - 1\}$ . Both of these functions are convex. Then, we are going to show that the maximum of two convex functions is also convex.

Let  $m$  and  $n$  be two convex functions and  $h(x) = \max\{m(x), n(x)\}$ . Given  $\lambda \in [0, 1]$  and  $x, y \in \mathbb{R}$  we aim to show that

$$h(\lambda x + (1-\lambda)y) \leq \lambda h(x) + (1-\lambda)h(y).$$

On one hand, it is clear that

$$m(\lambda x + (1-\lambda)y) \leq \lambda m(x) + (1-\lambda)m(y) \leq \lambda h(x) + (1-\lambda)h(y),$$

where in the first inequality comes from the fact that  $m$  is convex and the second from the definition of  $h$ . The same inequalities holds for  $n$ . Given that both functions are upper bounded by the same value, the maximum is also upper bounded by this value, so  $h$  is convex. Using this auxiliary result,  $f$  is convex.

Now we compute the proximal operator of  $f$  as

$$\text{prox}_f(x) = \arg \min_z f(z) + \frac{1}{2}(z-x)^2 = \arg \min_z h(z)$$

with

$$h(z) = \begin{cases} z^2 - 1 + \frac{1}{2}(z-x)^2 & |z| > 1 \\ \frac{1}{2}(z-x)^2 & |z| \leq 1 \end{cases} \quad (3)$$

If the minimizer is attained at  $|z| \leq 1$ , then clearly  $z = x$ , meaning that  $\text{prox}_f(x) = x$  for  $|x| \leq 1$ . If it is attained at  $|z| > 1$ , we have

$$0 = h'(z) = 3z - x \implies z = \frac{1}{3}x,$$

which implies that  $\text{prox}_f(x) = \frac{1}{3}x$  for  $|x| > 3$ . The remaining values of the proximal must be studied separately: the only possible minimizers are the points of non differentiability of (3). That is  $-1$  and  $1$  with

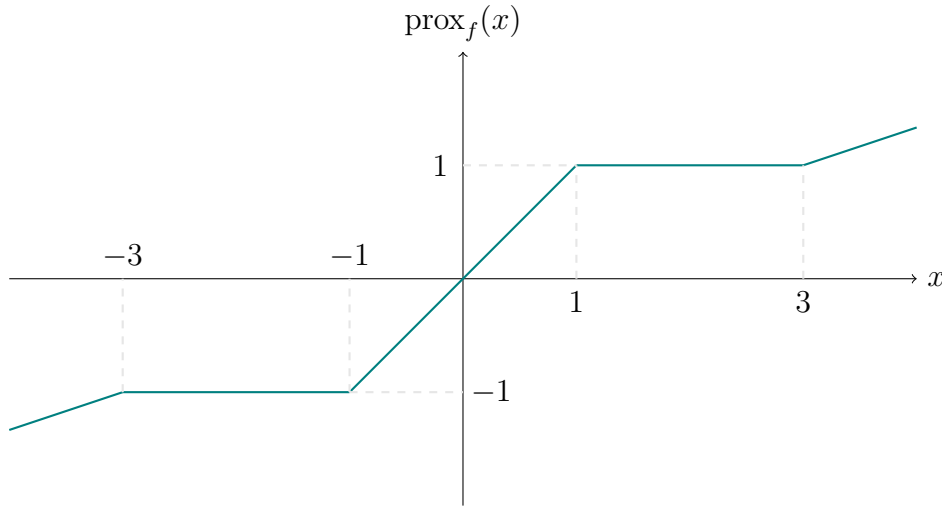
$$h(-1) = \frac{1}{2}(-1 - x)^2 \quad \text{and} \quad h(1) = \frac{1}{2}(1 - x)^2.$$

- When  $-3 \leq x < -1$ , the function is minimized at  $z = -1$ , that is, the proximal is  $\text{prox}_f(x) = -1$ .
- When  $1 < x \leq 3$ , the function is minimized at  $z = 1$ , that is, the proximal is  $\text{prox}_f(x) = 1$ .

As a result, the proximal is

$$\text{prox}_f(x) = \begin{cases} \frac{x}{3} & x \in (-\infty, -3), \\ -1 & x \in [-3, -1), \\ x & x \in [-1, 1], \\ 1 & x \in (1, 3], \\ \frac{x}{3} & x \in (3, \infty), \end{cases}$$

Furthermore, we illustrate  $\text{prox}_f(x)$  for a better comprehension.



Finally, we observe that our fixed points of this proximal are those that verify  $|x| \leq 1$ .