Homework 7

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Problems

1. (a) $Var(X) = E(X^2) - E(X)^2$, or $9 = 13 - 2^2$, which is correct.

   (b) Let the distribution of X be discrete, with $P(X = 5) = \frac{1}{2}, P(X = -1) = \frac{1}{2}$. Then $Var(X) = 9$ and $E(X) = 2$; this satisfies the condition but is a counterexample.

   (c) Let $P(X = a) = p, P(X = b) = 1 - p$. Then using $p = 1/3$, and using the equations $pa + (1 - p)b = 2$ and $pa^2 + (1 - p)b^2 = 13$, we find that $a = 2 + 3\sqrt{2}, b = 2 - \frac{3}{\sqrt{2}}$ is a counter example.

   (d) True. Let A be the expected value using only the probabilities when $X \leq 1$. Then A is at most $p * 1$, where $p = P(X \leq 1)$. Let B be the expected value using the probabilities when $X > 1$. Then B at max is $10 * p$. We need $A + B = E = 2$, which means $p + 10(1 - p) \geq 2$. But this means that $p \leq \frac{8}{9}$.

   (e) By Cheybyshev, this is true when the difference is 4, since $P(X \geq 6) + P(X \leq -2) \leq 9/16$.

   (f) False, because since $9/32 = 0.28$, if we let $p = 0.29$ and plug in the equations $px + (1 - p)y = 2$ and $px^2 + (1 - p)y^2 = 13$, we get the solution $x = 6.69..., y = 0.08$, which is a counterexample.

2. (a) By linearity of expectation, it's $(3 + 4) * 5 = 35$.

   (b) By Markov, it's $p \leq \frac{35}{60} = \frac{7}{12}$.

   (c) Since they're independent, we can add up midterm and final variances. $Var(3X) + Var(4X) = 9 * Var(X) + 16 * Var(X) = 25 * Var(X)$.

   (d) $60 - 35$ is 25, so it's at most $Var/25^2 = Var(X)/25 = 1/25$.

3. (a) Yes, you have n identical variables with distribution P(X=1) = 1-p, P(X= 0) = p.

   (b) Yes, you have $n/2$ identical variables with distribution P(X = 2) = 1-p, P(X =0) = p.

(c) No, you only have two identical variables with distribution P(X = n/2) = 1-p, P(X = 0) = p.

(d) No, P(X = n) = 1-p, P(X=0) = p, sent only once.

4. (a) In Store 1, regardless of the first coupon she gets, she must wait until she gets the other one; the total would be $20/2 + 30/4 + ... = 30$ Expected dollars spent. In Store 2, we split it into cases; if she gets Coupon 1, she must keep buying until she gets Coupon 2, which is $(2/3)(20 * (1/3) + 30 * (1/3)^2 + ...) = 25/4$ Vice versa, it is $(1/3)(20 * (2/3) + 30 * (2/3)^2 + ... = 80/3$. $80/3 + 25/4 < 30$, so she should go to Store 2.

   (b) If she goes to Store 1 first: If she gets Coupon 1, then she should swap, and expected value is $(1/2)(2/3)(20 + 30 * (1/3) + 40 * (1/3)^2 + ...) = 37.5$ If she gets coupon 2, she should stay, which makes $EV = 30/2$ from before. Total for store 1 is 52.5 If she goes to Store 2 first: If she gets coupon 1, she should stay, $EV = 25/4$ from before. If she gets coupon 2, then she swaps, so $(2/3) * (1/2) * (20 + 30/2 + 40/4 + ...) = 20$. Total is 26.25. So she should go to Store 2 and use the following strategy.

5. (a) Begin at an arbitrary vertex; color it red. Now go to an arbitrary neighbor, and color it blue; then go to a neighboring vertex of the blue vertex and color it red... etc. Keep alternating. If at any point two points with an edge have the same color, then the graph is not bipartite, because applying this algorithm to a bipartite graph will make one independent set red, the other blue, contradiction. If we succeed, separate the blue and red points and make them the independent two sets.

   (b) The sum listed is the number of edges between L and R; one is calculated from the L side, another taken from the R side; thus they are equal.

6. (a) Consider the sum of the degrees over all vertices. This is already known to be twice the number of edges, and is even. If we have an odd number of odd degreed vertices, then the total degree over all odd vertices is odd (by multiplication of odds in mod 2). The total degree over all even vertices is even (again, by mod 2 multiplication). But adding these will give an odd result, contradiction.

   (b) We induct upwards on the number of edges E. In the base case, we will have a (not necessarily connected) set of disjoint trees with this condition. In this base case, for each tree, take a leaf vertex v, and arbitrarily move from v across the tree until it reaches another leaf, u. These are both of odd degree, and we may make the path we've taken as a single partition of the edges. Then repeat this algorithm downwards with $2c - 2$ odd degree vertices, until we are done.
   Now assume we have a graph G with the conditions. If G has no

cycles, then it is a tree because it is connected, which goes to the base case. If G has a cycle y, remove this cycle's edges from G, and append arbitrarily this cycle-pathing to any pathing in $G - y$. Clearly this preserves the number of odd vertices because the cycle has every vertex inside it with degree 2. Keep doing this until we have no more edges, and thus this will go to the base case, QED.