1. Probability that I've survived for only k points:

$$P(k) = \binom{13}{k} \cdot k! \cdot \frac{4}{52} \cdot \frac{4}{51} \cdots \frac{4}{(53-k)} \cdot \frac{3 \cdot k}{(52-k)},$$ based on picking the k values in some permutation and finding the probability of k successes, one failure.

Then,

$$\text{Expectation} = 1 \cdot P(1) + 2 \cdot P(2) + \ldots + 13P(13)$$

Using python, this gives 4.6965, not a worthy bet for 5.

2.

(a) Counterexample: let $X = Z =$ first roll of fair coin being heads. $Y =$ second roll. Clearly (X, Y), (Y, Z) are independent, but $Z = X$, dependent.

(b) $X =$ first roll = heads, $Z =$ second roll being heads, $Y =$ rolled both heads; (X, Y) dependent, (Y, Z) dependent, but (X, Z) independent.

(c) $Var(X) = 0$, so $E((X-\mu)^2) = 0$, but since $(X-\mu)^2 \geq 0$, this means $(X-\mu)^2 = 0$ for all points from X, so $X = \mu$, a constant.

(d) $Var(X) \geq 0$, but $Var(X) = E(X^2) - E(X)^2 \geq 0$

Plug in $X = X^2$ instead, we have $E(X^4) - E(X^2)^2 \geq 0$,

or $E(X^4) \geq E(X^2)^2 \geq E(X)^4$

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3. (a) Let X = # floors not stopped at

Then $X = X_1 + X_2 + X_3 \dots + X_n$, the variables indicating floors not stopped at. Then

$$E(X) = E(X_1) + E(X_2) + \dots + E(X_n)$$
$$= n \cdot \left(1 - \frac{1}{n}\right)^n$$

$$E(X^2) = E(X_1^2) + \dots + E(X_n^2)$$
$$+ \left(2 E(X_1 X_2) + \dots\right)$$
$$= n \cdot \left(1 - \frac{1}{n}\right)^2 + 2 \cdot \binom{n}{2} \cdot \left(1 - \frac{2}{n}\right)^n$$

so $E(X^2) - E(X)^2 = n^2\left(1-\frac{1}{n}\right)^{2n} - n\left(1 - \frac{1}{n}\right)^2 + 2\binom{n}{2}\left(1-\frac{2}{n}\right)^n$

(b) Let $X = X_1 + X_2 + \dots + X_n$, $X_i = 1$ if i-th week all 3 friends are reading the same book, $X_i = 0$ otherwise.

Then

$$E(X) = E(X_1) + \dots + E(X_n) = n \cdot n \cdot \left(\frac{(n-1)!}{n!}\right)^3 = \frac{1}{n}$$

$$E(X^2) = \sum E(X_i^2) + \sum_{i \neq j} E(X_i X_j) = \frac{1}{n} + 2\binom{n}{2} \cdot \left[\binom{n}{2} \cdot \left(\frac{(n-2)!}{n!}\right)^3\right]$$

$$= \frac{1}{n} + 2\binom{n}{2}\left[\binom{n}{2} \cdot \left(\frac{1}{(n)(n-1)}\right)^3\right] = \frac{1}{n} + \frac{1}{2} \cdot \frac{1}{n(n-1)}$$

$$E(X^2) - E(X)^2 = \frac{1}{n} + \frac{1}{2} \cdot \frac{1}{n(n-1)} - \frac{1}{n^2}$$

4. (a) $P[A \cap B] = P[A]P[B]$ b/c independence.

$P[A] = P[B]$, so we need to prove $P[A] \geq (1 - n(1-\frac{1}{n})^k)$.

Note that the probability of a student not getting a cookie is $(1-\frac{1}{n})^k$. Then $P[X_1 \cup X_2 \dots \cup X_n] \leq P[X_1] + \dots + P[X_n]$ $= n \cdot (1-\frac{1}{n})^k$, where X_i = event student i gets no cookie.

so $P[A] = 1 - P[X_1 \cup X_2 \dots \cup X_n] \geq 1 - n(1-\frac{1}{n})^k$

(b) $(1-n(1-\frac{1}{n})^k)^2 \geq 0.64$, $(1-n(1-\frac{1}{n})^k) \geq 0.8$,

$\frac{0.2}{10} \geq (\frac{9}{10})^k$, so $k \geq 17.8$, $k = 18$.

(c) by independence it would be $(1-n(1-\frac{1}{n})^k)^3$.

(d) $E(X_1) = E(Y_1) + \dots E(Y_k)$, where Y_i is giving i-th cookie to student 1. Then $E(X_1) = \frac{k}{n}$.

(e) $E(\text{first student}) = 0 \cdot (\frac{1}{2})^1 + 1 \cdot (\frac{1}{2})^2 + 2 \cdot (\frac{1}{2})^3 + \dots + (k-1)(\frac{1}{2})^k + k \cdot (\frac{1}{2})^k$

based on finding T, HT, HHT, ... HH...T, HH...H.

The other expected values are not the same, and are difficult to write out.

5. (a) # cases where all 3 are hashed to the same entry: $3 \cdot 3! = 18$

cases where 2 are hashed to same place only:

$3 \cdot \binom{3}{2} \cdot 2! \cdot 2 = 36$

cases where no collisions occur: $3! = 6$

Total $= 18 + 36 + 6 = 60$

(b) $E[X] = E[X_1] + E[X_2] + E[X_3]$ where X_i is 1 if i-th element hashes to entry 1.

Then $E[X] = 3 \cdot \frac{1}{3} = 1$

Also, $P(X=0) = \left(\frac{2}{3}\right)^3$, $P(X=1) = \binom{3}{1}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^2$

$P(X=2) = \binom{3}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)$, $P(X=3) = \binom{3}{3}\left(\frac{1}{3}\right)^3$

(c) $P(Y=1) = \frac{6}{3^3}$, no collisions,

$P(Y=2) = \frac{36}{3^3}$, $P(Y=3) = \frac{18}{3^3}$, using answers from (a)

Then $E[Y] = \frac{6}{3^3} \cdot 1 + \frac{36}{3^3} \cdot 2 + \frac{18}{3^3} \cdot 3 = \frac{44}{3^2}$

(d) $E[Y] > E[X]$ since $\frac{44}{9} > 1$.

The only time that $X \geq Y$ is when entry 1 contains the longest list, but that means $X = Y$ at that case. All other cases are $X < Y$, so taking the sum of the cases gives $E[X] < E(Y)$.

(e) $E[X] = \frac{m}{n}$, and $P(X=0) = \binom{m}{0}\left(\frac{1}{n}\right)^0\left(\frac{n-1}{n}\right)^m$

$P(X=1) = \binom{m}{1}\left(\frac{1}{n}\right)^1\left(\frac{n-1}{n}\right)^{m-1}, \ldots$ $P(X=m) = \binom{m}{m}\left(\frac{1}{n}\right)^m\left(\frac{n-1}{n}\right)^0$,

essentially a binomial distribution.

6. (a) That's equal to probability that all top students are a permutation of the n students; for a given permutation, this prob is $\left(\frac{(n-1)!}{n!}\right)^n = \left(\frac{1}{n}\right)^n$. Then total $= n! \cdot \left(\frac{1}{n}\right)^n$.

(b) $P(\text{Alice not in } K_1 \cap \text{Alice not in } K_2 \cap \ldots)$

$= \prod_{i=1}^{n} P(\text{Alice Not in } K_i) = \prod_{i=1}^{n} \frac{(n-K_i)\cdot(n-1)!}{n!}$

$= \prod_{i=1}^{n} \left(1 - \frac{K_i}{n}\right)$.

Note that $\left(1-\frac{K_i}{n}\right) < e^{-\frac{K_i}{n}}$, so $P < \prod_{i=1}^{n} e^{-\frac{K_i}{n}} = e^{-2\ln(n)} = \frac{1}{n^2}$.

(c) $P[\text{Student 1 not admitted} \cup \text{student 2 not admitted} \ldots]$

$\leq P[\text{student 1 not admitted}] + P[\text{student 2 not}] \ldots$

$= n \cdot P(\text{student 1 not admitted}) < n \cdot \frac{1}{n^2} = \frac{1}{n}$.