Implementation of Diffusion-Reaction equation in FEniCS

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Introduction

The study investigates the convergence behavior of finite element solutions for the diffusion-reaction equation $-\kappa\Delta u + mu = f$, where $\kappa > 0$ and $m > 0$ implemented in FEniCS. The manufactured solution technique is employed, utilizing a cubic polynomial function, on a unit square mesh. We also aim to explore the impact of grid refinement and function approximation using piecewise linear and quadratic functions on the accuracy of the solutions and rate of convergence.

Numerical Method

FEniCS is utilized to discretize the diffusion-reaction equation on the unit square mesh. The finite element method divides the domain into finite elements and approximates the solution within each element using piecewise polynomial functions. Piecewise linear and quadratic functions are specifically chosen for this study.

Implementation

Firstly, the weak form of the equation is generate over the domain say, Ω and its given as:

$$\int_\Omega (\kappa\nabla u \cdot \nabla v + muv)\, dx = \int_\Omega fv\, dx$$

$\kappa = 1$ and $m = 1$, cubic polynomial $x^3 + y^3$ are used during implementation, which also represents our manufactured solution u satisfying the differential equation and boundary condition.
The source term for f is derived by solving the left hand side of the diffusion-reaction equation since we know u, κ and m. Hence;

$$f = -6(x + y) + (x^3 + y^3)$$

Dirichlet boundary condition is employed, so $u_D = u = x^3 + y^3$.

These values for f and u_D are passed into an Expression with degree 3 since we are using a cubic polynomial.

Results

The solutions for different grid sizes (8x8, 16x16, 32x32, 64x64) and function approximations (piecewise linear and quadratic) are computed. L^2 errors are calculated for each case to quantify the accuracy of the solutions. Visualization of the solutions and error analysis aid in understanding the convergence behavior. These results are crucial for evaluating the effectiveness of the numerical method. The L^2 *error* results are shown in the table below:

	Piecewise Linear Numerical Solution	**Piecewise Quadratic Numerical Solution**
8 x 8	8.6691×10^{-3}	9.5303×10^{-5}
16 x 16	2.1669×10^{-3}	1.1913×10^{-5}
32 x 32	5.417×10^{-4}	1.4891×10^{-6}
64 x 64	1.354×10^{-4}	1.8614×10^{-7}

Discussion and Conclusion

The convergence rates of piecewise linear and quadratic functions are analyzed based on the computed L^2 errors. Results indicate that finer grids lead to lower L^2 errors, demonstrating improved accuracy with mesh refinement. Furthermore, solutions obtained using Piece-wise quadratic functions exhibit faster convergence compared to piece-wise linear functions, particularly noticeable on coarser grids. Conclusively, this implementation provides valuable insights into the convergence behavior of finite element solutions for the diffusion-reaction equation implemented in FEniCS.

References

1. Class notes on Introduction to Computation of Fluid Dynamics.
2. GitHub link to FEniCS code: `www.github.com`