Notes - How to Prove It - Chapter 3 : Proofs

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Contents

1 Introduction

Now that we have studied the language of sentential and quantificational logic, we are ready to tackle the core of mathematical reasoning: proofs.
This chapter is a practical guide to constructing rigorous arguments.

2 Proof strategies

2.1 To prove a goal of the form $P \longrightarrow Q$

First method : Assume that P is true, then prove Q.

Form of the final proof:

Suppose P.

[*Proof of Q goes here*]

Therefore $P \longrightarrow Q$. $\square$

Note: The small square symbol, either solid ($\blacksquare$) or empty ($\square$), means the proof is finished.

For example,

Suppose that a and b are real numbers. Prove that if $0 < a < b$ then $a^2 < b^2$

Scratch work
We are given as a hypothesis that a and b are real numbers. Our conclusion has the form $P \longrightarrow Q$, where P is the statement $0 < a < b$ and Q is the statement $a^2 < b^2$. Thus, we start with these statements as given and goal :

Given :

a and b are real numbers.

Goal :

$(0 < a < b) \longrightarrow (a^2 < b^2)$

According to our proof technique, we should assume that $0 < a < b$ and try to use this assumption to prove that $a^2 < b^2$. In other words, we transform the problem by adding $0 < a < b$ to the list of givens and making $a^2 < b^2$ our new goal.

Givens :

a and b are real numbers.

$0 < a < b$

Goal :

$a^2 < b^2$

Let us now use our givens to prove our goal. Our second given $(0 < a < b)$ will be pretty useful. As $a > 0$, we can multiply both sides of our second given by a, and we obtain $a^2 < ab$. As $b > a$ and $a > 0$, then $b > 0$, which means that we can also multiply both sides of our second given by b, and we obtain $ab < b^2$. As $a^2 < ab$ and $ab < b^2$, therefore $a^2 < ab < b^2$ which means that $a^2 < b^2$. Thus, our goal has been proven.

In mathematics, a statement is called a **theorem** as soon as it has been proven to be true.

Theorem 2.1.1. Suppose a and b are real numbers. If $0 < a < b$ then $a^2 < b^2$.
Proof. Suppose $0 < a < b$. Multiplying both sides of the inequality $a < b$ by the positive number a we can conclude that $a^2 < ab$, and similarly, multiplying by the positive number b, we get $ab < b^2$. Therefore, $a^2 < ab < b^2$, so $a^2 < b^2$, as required. Thus, if $0 < a < b$, then $a^2 < b^2$. □

2.2 To prove a goal of the form $P \longrightarrow Q$ (contrapositive law)

According to the contrapositive law, $P \longrightarrow Q$ is equivalent to $\neg Q \longrightarrow \neg P$. Therefore, we can prove $P \longrightarrow Q$ by proving $\neg Q \longrightarrow \neg P$. We will assume $\neg Q$, then prove $\neg P$.

Form of the final proof:

Suppose $\neg Q$.

[*Proof of $\neg P$ goes here*]

Therefore, $P \longrightarrow Q$.

For example,

Suppose a, b, c are real numbers and $a > b$. Prove that if $ac \leq bc$ then $c \leq 0$.

Scratch work

Givens :

a, b, c are real numbers.

$a > b$

Goal :

$(ac \leq bc) \longrightarrow (c \leq 0)$

Using our proof technique with the contrapositive law, we **rewrite** the goal as $\neg(c \leq 0) \longrightarrow \neg(ac \leq bc)$. We suppose $\neg(c \leq 0)$ and set $\neg(ac \leq bc)$ as our new goal.

Givens :

a, b, c are real numbers.

$a > b$

$\neg(c \leq 0)$

Goal :

$\neg(ac \leq bc)$

As $\neg(c \leq 0)$ is equivalent to $c > 0$ and $\neg(ac \leq bc)$ is equivalent to $ac > bc$, we can rewrite our givens and our goal :

Givens :

a, b, c are real numbers.

$a > b$

$c > 0$

Goal :

$ac > bc$

Let us now use our givens to prove the goal. As $c > 0$, we can multiply both sides of our inequality $a > b$ by c, it follows that $ac > bc$, the goal has been proven.

Theorem 2.2.1. Suppose a, b, and c are real numbers and $a > b$. If $ac \leq bc$ then $c \leq 0$.
Proof. We will prove the contrapositive. Suppose $c > 0$. Then we can multiply both sides of the given inequality $a > b$ by c and conclude $ac > bc$. Therefore, if $ac \leq bc$ then $c \leq 0$. □

2.3 To prove a goal of the form $\neg P$

Proving a positive statement is often simpler than proving a negative one. Therefore, whenever possible, converting a negative statement **into its positive equivalent can make a proof easier**.

For example,

Suppose $(A \cap C) \subseteq B$ and $a \in C$. Prove that $a \notin (A \setminus B)$.

Givens :

$(A \cap C) \subseteq B$

$a \in C$

Goal:

$a \notin (A \setminus B)$

Using our strategy, let us rewrite $a \notin (A \setminus B)$. The statement $a \notin (A \setminus B)$ is equivalent to $\neg(a \in (A \setminus B))$ which is equivalent to $\neg(a \in A \wedge a \notin B)$. Using De Morgan's law, this statement is also equivalent to $a \notin A \vee a \in B$. Using conditional law, the statement is also equivalent to $a \in A \longrightarrow a \in B$. Our goal can therefore be replaced by $a \in A \longrightarrow a \in B$ as they are equivalent.

Givens :

$(A \cap C) \subseteq B$

$a \in C$

Goal:

$a \in A \longrightarrow a \in B$

Our new goal has the form $P \longrightarrow Q$, we can therefore use the strategy seen previously. We suppose $a \in A$ and prove that $a \in B$.

Givens :

$(A \cap C) \subseteq B$

$a \in C$

$a \in A$

Goal:

$a \in B$

Looking at our givens, we see that $a \in A$ and $a \in C$ which means that $a \in A \cap C$. As $(A \cap C) \subseteq B$, we can conclude that $a \in B$. The goal has been proven.

Theorem 2.3.1 Suppose $(A \in C) \subseteq B$ and $a \in C$. Then $a \notin (A \setminus B)$
Proof. Suppose $a \in A$. Then since $a \in C$, $a \in A \cap C$. But then, since $(A \cap C) \subseteq B$, it follows that $a \in B$. Thus, it cannot be the case that a is an element of A but not an element of B, so $a \notin (A \setminus B)$. $\square$

2.4 To prove a goal of the form $\neg P$ (proof by contradiction)

It might happen that a goal of the form $\neg P$ cannot be rewritten as a positive statement. Therefore, a proof by contradiction might be a good approach. Start by assuming that P is true, and try to use this assumption to prove a statement that you know is false. Most of the time, this is done by proving a statement that **contradicts one of the givens**. Because you know that the statement you have proven is false, the assumption that P was true **must have been incorrect**. The only remaining possibility then is that P is false.

Form of the final proof :

Suppose P is true.

[*Proof of contradiction goes here*]

Thus, P is false.

For example,

Prove that if $x^2 + y = 13$ and $y \neq 4$ then $x \neq 3$

Scratch work

Our goal can be written as $((x^2 + y = 13) \land (y \neq 4)) \longrightarrow x \neq 3$. We therefore suppose $((x^2 + y = 13) \land (y \neq 4))$ and prove that $x \neq 3$.

Givens:

$x^2 + y = 13$

$y \neq 4$

Goal :

$x \neq 3$

Our goal is a negative statement that cannot be written as a positive statement. The strategy suggests us to suppose that $x = 3$ (although we know that the statement $x = 3$ is false) and try to reach a contradiction.

Givens:

$x^2 + y = 13$

$y \neq 4$

$x = 3$

Goal :

Contradiction.

Let us plug in $x = 3$ in our first given $x^2 + y = 13$. After plugging in, we get $3^2 + y = 13$ which is equivalent to $9 + y = 13$ which means that $y = 4$. But this contradicts the fact that $y \neq 4$ (second given). Therefore, $x \neq 3$.

Theorem 2.4.1. If $x^2 + y = 13$ and $y \neq 4$ then $x \neq 3$.
Proof. Suppose $x = 3$. Substituting this into the equation $x^2 + y = 13$, we get $9 + y = 13$, so $y = 4$. But this contradicts the fact that $y \neq 4$. Therefore, $x \neq 3$. Thus, if $x^2 + y = 13$ and $y \neq 4$ then $x \neq 3$. $\square$

2.5 To use a given of the form $\neg P$ (proof by contradiction)

If you are doing a proof by contradiction (**it only applies if so**), try making P your goal. If you can prove P, then the proof will be complete , because P contradicts the given $\neg P$.

For example,

> *Suppose $A, B,$ and C are sets, $(A \setminus B) \subseteq C$, and x is anything at all.*
>
> *Prove that if $x \in (A \setminus C)$, then $x \in B$.*

Scratch work

Our goal has the form $P \longrightarrow Q$ and can be written as $x \in (A \setminus C) \longrightarrow x \in B$. Therefore, we suppose $x \in (A \setminus C)$ and prove $x \in B$.

Givens :

$(A \setminus B) \subseteq C$

$x \in (A \setminus C)$

Goal :

$x \in B$

The statement $x \in (A \setminus C)$ is equivalent to $x \in A \wedge x \notin C$. Therefore,

Givens :

$(A \setminus B) \subseteq C$

$x \in A$

$x \notin C$

Goal :

$x \in B$

The statement $x \notin C$ has the form $\neg P$. According to our strategy, we should prove by contradiction supposing that $x \notin B$ and proving that $x \in C$ which will contradict the fact that $x \notin C$ and therefore we will be able to conclude that x must belong to B. Thus, suppose $x \notin B$ (proof by contradiction).

Givens :

$(A \setminus B) \subseteq C$

$x \in A$

$x \notin C$

$x \notin B$

Goal :

$x \in C$ (*Contradiction.*)

Looking at our givens, we see that $x \in A$ and $x \notin B$ which means that $x \in (A \setminus B)$. Since $(A \setminus B) \subseteq C$, we can conclude that $x \in C$. But this contradicts the fact that $x \notin C$. Therefore, the assumption we made ($x \notin B$) must be incorrect, so $x \in B$.

Theorem 2.5.1. Suppose A, B, and C are sets, $(A \setminus B) \subseteq C$, and x is anything at all. If $x \in (A \setminus C)$, then $x \in B$.
Proof. Suppose $x \in (A \setminus C)$. This means that $x \in A$ and $x \notin C$. Then $x \in (A \setminus B)$, so since $(A \setminus B) \subseteq C$, $x \in C$. But this contradicts the fact that $x \notin C$. Therefore, $x \in B$. Thus, if $x \in (A \setminus C)$, then $x \in B$. $\square$

2.6 To use a given of the form $P \longrightarrow Q$ (modus ponens and modus tollens)

We are going to use two different strategies to use a given of the form $P \longrightarrow Q$. Many strategies for using givens suggest ways of drawing inferences from the given. Such strategies are called **rules of inference**. Both strategies that we are going to use are examples of those rules of inference. The first of these rules of inference says that if you know that both P and $P \longrightarrow Q$ are true, you can conclude that Q must also be true, this rule of inference is called **modus ponens**. The second rule of inference says that if you know that $\neg Q$ and $\neg Q \longrightarrow \neg P$ are true, then you can conclude that P is false (because $\neg P$ must be true), this rule of inference is called **modus tollens**.

Thus, to be able to use a given of the form $P \longrightarrow Q$, we will either use modus ponens or modus tollens.
Note that we **must** prove that P is true or to be given P to use modus ponens and the same applies to $\neg Q$, to use modus tollens.

For example,

Suppose $P \longrightarrow (Q \longrightarrow R)$. *Prove that* $\neg R \longrightarrow (P \longrightarrow \neg Q)$

Scratch work

We have the following situation :

Given :

$P \longrightarrow (Q \longrightarrow R)$

Goal :

$\neg R \longrightarrow (P \longrightarrow \neg Q)$

As our goal is a conditional statement, we start by assuming $\neg R$ and then prove $P \longrightarrow \neg Q$.

Givens :

$P \longrightarrow (Q \longrightarrow R)$

$\neg R$

Goal :

$P \longrightarrow \neg Q$

We once again have a conditional statement as our goal. Therefore, we assume P and prove $\neg Q$

Givens :

$P \longrightarrow (Q \longrightarrow R)$

$\neg R$

P

Goal :

$\neg Q$

Looking at our givens, we see that P is true and $P \longrightarrow (Q \longrightarrow P)$ is also true. Therefore, we can conclude that $Q \longrightarrow R$ must be true (modus ponens).

Givens :

$P \longrightarrow (Q \longrightarrow R)$

$\neg R$

P

$Q \longrightarrow R$

Goal :

$\neg Q$

Looking again at our givens, we see that R is false (as $\neg R$ is true) and $Q \longrightarrow R$ is also true. Therefore, using modus tollens, we can conclude that Q is false, so $\neg Q$ is true. The goal has been proven.

Theorem 2.6.1. Suppose $P \longrightarrow (Q \longrightarrow R)$. Then $\neg R \longrightarrow (P \longrightarrow \neg Q)$.
Proof. Suppose $\neg R$. Suppose P. Since P and $P \longrightarrow (Q \longrightarrow R)$, it follows that $Q \longrightarrow R$. But then, since $\neg R$, we can conclude $\neg Q$. Thus, $P \longrightarrow \neg Q$. Therefore, $\neg R \longrightarrow (P \longrightarrow \neg Q)$.$\square$

Another great example from the book :

Suppose that $A \subseteq B$, $a \in A$, and $a \notin (B \setminus C)$. Prove that $a \in C$.

Givens :

$A \subseteq B$

$a \in A$

$a \notin (B \setminus C)$

Goal :

$a \in C$

Let us write the definition of the negated statement $a \notin (B \setminus C)$ in order to prove our goal, $a \in C$. The statement $a \notin (B \setminus C)$ is equivalent to $\neg(a \in (B \setminus C))$ which is equivalent to $\neg(a \in B \wedge a \notin C)$. Using De Morgan's law, this statement also means $a \notin B \vee a \in C$. Using the conditional law, this statement is also equivalent to $a \in B \longrightarrow a \in C$. Therefore,

Givens :

$A \subseteq B$

$a \in A$

$a \in B \longrightarrow a \in C$

Goal :

$a \in C$

If we ever want to use the third given ($a \in B \longrightarrow a \in C$), we would need to be given or to prove that $a \in B$ or $a \notin C$. Looking at our other givens, we see that $a \in A$. As $A \subseteq B$, we can conclude that $a \in B$. And now we are finally able to use our last given, $a \in B \longrightarrow a \in C$. Using modus ponens, we can conclude that $a \in C$. The goal has been proven.

Theorem 2.6.2. Suppose that $A \subseteq B$, $a \in A$ and $a \notin (B \setminus C)$. Then $a \in C$.
Proof. Since $a \in A$ and $A \subseteq B$, we can conclude that $a \in B$. But $a \notin (B \setminus C)$, so it follows that $a \in C$. $\square$

2.7 To prove a goal of the form $\forall x P(x)$

Let x be an **arbitrary** object and prove $P(x)$.
Note that the letter x must be a **new** variable in the proof.

Form of the final proof:

Let x be arbitrary.

[*Proof of* $P(x)$ *goes here*]

Since x *was arbitrary, we can conclude that* $\forall x(Px)$.

For example,

Suppose $A, B,$ *and* C *are sets, and* $(A \setminus B) \subseteq C$. *Prove that* $(A \setminus C) \subseteq B$

Scratch work

Givens :

$(A \setminus B) \subseteq C$

Goal :

$(A \setminus C) \subseteq B$

Let us rewrite our goal. The statement $(A \setminus C) \subseteq B$, is equivalent to $\forall x\ (x \in (A \setminus C) \longrightarrow x \in B)$. According to our strategy, let x be arbitrary, and then suppose that $x \in (A \setminus C)$ and prove that $x \in B$.

Givens :

$(A \setminus B) \subseteq C$

$x \in (A \setminus C)$

Goal :

$x \in B$

The statement $x \in (A \setminus C)$ is equivalent to $x \in A$ and $x \notin C$. A proof of the goal $x \in B$ is possible by contradiction. Let us suppose that $x \notin B$.

Givens :

$(A \setminus B) \subseteq C$

$x \in A$

$x \notin C$

$x \notin B$

Goal :

Contradiction

Let us use our givens to find a contradiction. As $x \in A$ and $x \notin B$, we can conclude that

$x \in (A \setminus B)$. Since $(A \setminus B) \subseteq C$, it follows that $x \in C$, which contradicts the fact that $x \notin C$. Therefore, $x \in B$.

Theorem 2.7.1. Suppose A, B, and C are sets, and $(A \setminus B) \subseteq C$. Then $(A \setminus C) \subseteq B$.
Proof. Let x be arbitrary. Suppose $x \in (A \setminus C)$. This means that $x \in A$ and $x \notin C$. Suppose $x \notin B$. Then since $x \in (A \setminus B)$, $x \in C$. But this contradicts the fact that $x \notin C$. Therefore, $x \in B$. Thus, if $x \in (A \setminus C)$ then $x \in B$. Since x was arbitrary, we can conclude that $\forall x$ $(x \in (A \setminus C) \longrightarrow x \in B)$. So $(A \setminus C) \subseteq B$.

Another example,

> *Suppose A and B are sets. Prove that if $A \cap B = A$, then $A \subseteq B$*

Scratch work

Applying our strategy for a conditional statement, we should suppose $A \cap B = A$ and prove $A \subseteq B$.

Givens :

$A \cap B = A$

Goal :

$A \subseteq B$

Our goal is equivalent to $\forall x$ $(x \in A \longrightarrow x \in B)$. Let x be an arbitrary element of A. Therefore, our new goal is $x \in B$.

Givens :

$A \cap B = A$

$x \in A$

Goal :

$x \in B$

Since $x \in A = A \cap B$, we can conclude that $x \in A \wedge x \in B$ and in particular $x \in B$. Our goal is proven.

Theorem 2.7.2. Suppose A and B are sets. If $A \cap B = A$, then $A \subseteq B$.
Proof. Suppose $A \cap B = A$ and suppose $x \in A$. Then, since $A \cap B$, $x \in B$. Since x was an arbitrary element of A, we can conclude that $A \subseteq B$. $\square$

2.8 To prove a goal of the form $\exists x P(x)$

Try to **find a value** of x for which you think $P(x)$ will be true. Then start your proof with "Let $x = ...$"

Note that x should be a new variable.

Form of the final proof:

Let $x =$ (the value you decided on)

[*Proof of $P(x)$ goes here*].

Thus, $\exists x\ P(x)$.

For example,

Prove that for every real number x, if $x > 0$, then there is a real number y

such that $y(y + 1) = x$.

Scratch work

Our goal is $\forall x \in \mathbb{R}\ ((x > 0) \longrightarrow \exists y \in \mathbb{R}\ (y(y + 1) = x))$. Therefore, we let x be an arbitrary real number, suppose $x > 0$ and prove that $\exists y \in \mathbb{R}\ (y(y + 1) = x))$.

Given :

$x > 0$

Goal :

$\exists y \in \mathbb{R}\ (y(y + 1) = x))$

In order to find the right y to prove our goal, let us focus on the equation $y(y + 1) = x$. Solving the equation for the variable y is the right approach to find the y we are looking for.

The answers to the equation are $y = \frac{-1\pm\sqrt{1+4x}}{2}$. Those two answers are two different possibilities for us to prove our goal. We only need one value of y to prove our goal, so let $y = \frac{-1+\sqrt{1+4x}}{2}$. After substituting our value of y we have found in our goal, we conclude that this value is a right answer. Our goal is proven.
Note that we do not have to worry that $1 + 4x$ cannot be negative for y to belong to $\mathbb{R}$ because our given says that $x > 0$.

Theorem 2.8.1. For every real number x, if $x > 0$, then there is a real number y such that $y(y + 1) = x$.
Proof. Let x be an arbitrary real number and suppose that $x > 0$. Let $y = \frac{-1+\sqrt{1+4x}}{2}$, which is defined as x>0. Then

$$y(y + 1) = \left(\frac{-1+\sqrt{1+4x}}{2}\right)\left(\frac{-1+\sqrt{1+4x}}{2} + 1\right) = \left(\frac{-1+\sqrt{1+4x}}{2}\right)\left(\frac{1+\sqrt{1+4x}}{2}\right) = \frac{1+4x-1}{4} = \frac{4x}{4} = x. \ \square$$

Sometimes, when not being able to find a value by simply looking at a statement $\exists x P(x)$, the givens might suggest a value. For example, a given of the form $\exists y P(y)$ suggests us to imagine a particular value for y, say x_0, so that $P(x_0)$ is true. We can therefore add $P(x_0)$ to our givens list. And this particular value x_0 (or something related to it) might be the value we where looking for to prove the statement $\exists x P(x)$.

2.9 To use a given of the form $\exists x P(x)$ (existential instantiation)

Introduce a new variable x_0 for which $P(x_0)$ is true. This rule is called **existential instantiation**.
Note that this rule is only about using the fact that **we know that some** x_0 **exists**, we do not give any value to x_0. If you have a given of this form, you should not hesitate to name it a particular variable.

2.10 To use a given of the form $\forall x P(x)$ (universal instantiation)

This given says that **no matter what value we give to** x**,** $P(x)$ **is must be true**. For example, if we plug in x_0, then we can conclude that $P(x_0)$ is true. This rule is called **universal instantiation**. Note that it will not be useful unless you have a particular value to plug in for x.

For example,

Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets and $\mathcal{F} \cap \mathcal{G} \neq \varnothing$. Prove that $\bigcap \mathcal{F} \subseteq \bigcup \mathcal{G}$.

Scratch work

Givens :

$\mathcal{F} \cap \mathcal{G} \neq \varnothing$

Goal :

$\bigcap \mathcal{F} \subseteq \bigcup \mathcal{G}$

Our goal means that $\forall x\ (x \in \bigcap \mathcal{F} \longrightarrow x \in \bigcup \mathcal{G})$. Therefore, we let x be arbitrary, suppose $x \in \bigcap \mathcal{F}$ and prove that $x \in \bigcup \mathcal{G}$.

Givens :

$\mathcal{F} \cap \mathcal{G} \neq \varnothing$

$x \in \bigcap \mathcal{F}$

Goal :

$x \in \bigcup \mathcal{G}$

According to the definition of $\bigcup \mathcal{G}$, our goal can be written as $\exists B \in \mathcal{G}(x \in B)$ and according to the definition of $\bigcap \mathcal{F}$, the given $x \in \bigcap \mathcal{F}$ can be written as $\forall A \in \mathcal{F}(x \in A)$.

Givens :

$\mathcal{F} \cap \mathcal{G} \neq \varnothing$

$\forall A \in \mathcal{F}(x \in A)$

Goal :

$\exists B \in \mathcal{G}(x \in B)$

Our goal has the form $\exists x\ P(x)$. Therefore, in order to prove our goal, we will need to find a value of $B \in \mathcal{G}$ such that $x \in B$. We have a given of the form $\forall x P(x)$. Thus, our strategy suggests us to use universal instantiation as soon as a relevant value of $A \in \mathcal{F}$ is suggested to us throughout the proof. Something pretty important has to be mentioned : the statement $\forall A \in \mathcal{F}(x \in A)$ is actually a conditional statement. In fact $\forall A \in \mathcal{F}(x \in A)$ can also be written as $\forall A(A \in \mathcal{F} \longrightarrow x \in A)$. Let us now, see what the statement $\mathcal{F} \cap \mathcal{G} \neq \varnothing$ hides. The statement $\mathcal{F} \cap \mathcal{G} \neq \varnothing$ means that there exists at least one $x \in \mathcal{F}$ such that $x \in \mathcal{G}$. In other words, $\exists x\ (x \in \mathcal{F} \wedge x \in \mathcal{G})$. Therefore,

Givens :

$\exists x\ (x \in \mathcal{F} \wedge x \in \mathcal{G})$

$\forall A(A \in \mathcal{F} \longrightarrow x \in A)$

Goal :

$\exists B \in \mathcal{G}(x \in B)$

The first given is probably the key to finish our proof. We immediately use existential instantiation and choose an x, say x_0. Therefore,we have two new givens.

Givens :

$x_0 \in \mathcal{F}$

$x_0 \in \mathcal{G}$

$\forall A(A \in \mathcal{F} \longrightarrow x \in A)$

Goal :

$\exists B \in \mathcal{G}(x \in B)$

This is the moment we were waiting for to use universal instantiation. It might not look like it but x_0 is a set. Using modus ponens (conditional statement as a given) and using universal instantiation, we can conclude that $x \in x_0$. The goal was to find a set such that this particular set belong to $\mathcal{G}$ and that the element x belongs to this particular set. We have just proven it finding the set x_0.
To avoid any confusion on the fact that x_0 is a set, let us call it A_0 for our proof.

Theorem 2.10.1. Suppose $\mathcal{F}$ and $\mathcal{G}$ are families of sets, and $\mathcal{F} \cap \mathcal{G} \neq \varnothing$. Then $\bigcap \mathcal{F} \subseteq \bigcup \mathcal{G}$.
Proof. Suppose $x \in \bigcap \mathcal{F}$. Since $\mathcal{F} \cap \mathcal{G} \neq \varnothing$, we can let A_0 be an element of $\mathcal{F} \cap \mathcal{G}$. Thus, $A_0 \in \mathcal{F}$ and $A_0 \in \mathcal{G}$. Since $x \in \bigcap \mathcal{F}$ and $A_0 \in \mathcal{F}$. It follows that $x \in A_0$. But we also know

that $A_0 \in \mathcal{G}$, so we conclude that $x \in \bigcup \mathcal{G}$. $\square$

One more example,

> *Suppose B is a set and $\mathcal{F}$ is a family of sets. Prove that if $\bigcup \mathcal{F} \subseteq B$, then $\mathcal{F} \subseteq \mathcal{P}(B)$*

Scratch work :

Our goal is a conditional statement. Therefore, we assume $\bigcup \mathcal{F} \subseteq B$ and we prove $\mathcal{F} \subseteq \mathcal{P}(B)$.

Given :

$\bigcup \mathcal{F} \subseteq B$

Goal :

$\mathcal{F} \subseteq \mathcal{P}(B)$

Our new goal, $\mathcal{F} \subseteq \mathcal{P}(B)$, can be written as $\forall x \ (x \in \mathcal{F} \longrightarrow x \in \mathcal{P}(B))$. Thus, we let x be arbitrary, suppose $x \in \mathcal{F}$ and prove that $x \in \mathcal{P}(B)$.

Givens :

$\bigcup \mathcal{F} \subseteq B$

$x \in \mathcal{F}$

Goal :

$x \in \mathcal{P}(B)$

The statement $x \in \mathcal{P}(B)$ can be written as $x \subseteq B$, which is equivalent to $\forall y \ (y \in x \longrightarrow y \in B)$. Thus, we let y be arbitrary, suppose $y \in x$ and prove $y \in B$.

Givens :

$\bigcup \mathcal{F} \subseteq B$

$x \in \mathcal{F}$

$y \in x$

Goal :

$y \in B$

$\mathcal{F}$ is a family of sets, and x is one of the sets that belong to $\mathcal{F}$. The union of a family of sets results in a set with all the elements of every set that belong to the family of sets. Therefore, as $x \in \mathcal{F}$, all its elements belong to $\bigcup \mathcal{F}$. In other words $y \in \bigcup \mathcal{F}$. Since $\bigcup \mathcal{F} \subseteq B$, it follows that $y \in B$. Our goal has been proven.

Theorem 2.10.2. Suppose B is a set and $\mathcal{F}$ is a family of sets. If $\bigcup \mathcal{F} \subseteq B$ then $\mathcal{F} \subseteq \mathcal{P}(B)$.
Proof. Suppose $\bigcup \mathcal{F} \subseteq B$. Let x be an arbitrary element of $\mathcal{F}$. Let y be an arbitrary element of x. Since $y \in x$ and $x \in \mathcal{F}$, by definition of $\bigcup \mathcal{F}$, $y \in \bigcup \mathcal{F}$. But then since $\bigcup \mathcal{F} \subseteq B$, $y \in B$. Since y was an arbitrary element of x, we can conclude that $x \subseteq B$, so $x \in \mathcal{P}(B)$ but x was an arbitrary element of $\mathcal{F}$ so this shows that $\mathcal{F} \subseteq \mathcal{P}(B)$, as required. □

2.11 To prove a goal of the form $P \wedge Q$

A goal of the form $P \wedge Q$ is treated as **two separate goals** : P, and Q.

2.12 To use a given of the form $P \wedge Q$

A given of the form $P \wedge Q$ is treated as **two separate givens** : P, and Q

For example,

Suppose $A \subseteq B$, and A and C are disjoint. Prove that $A \subseteq (B \setminus C)$.

Scratch work :

Our goal is the statement $A \subseteq (B \setminus C)$ which can be written as $\forall x\ (x \in A \longrightarrow x \in (B \setminus C))$. Therefore, we let x be arbitrary, suppose $x \in A$ and prove that $x \in (B \setminus C)$.

Givens :

$A \subseteq B$

$A \cap C = \varnothing$

$x \in A$

Goal :

$x \in (B \setminus C)$

Our new goal is equivalent to $x \in B \wedge x \notin C$. According to our strategy for a goal which is a conjunction, we will first prove that $x \in B$, then prove that $x \notin C$.

Givens :

$A \subseteq B$

$A \cap C = \varnothing$

$x \in A$

Goal :

$x \in B$

The goal follows from that fact that $x \in A$ and that $A \subseteq B$. Let us now prove that $x \notin C$.

Givens :

$A \subseteq B$

$A \cap C = \varnothing$

$x \in A$

Goal :

$x \notin C$

The statement $A \cap C = \varnothing$ is equivalent to $\forall y\ (y \in A \longrightarrow y \notin C)$. Using universal instantiation and modus ponens, we can conclude that $x \notin C$. Therefore, we have proven that $x \in B \wedge x \notin C$ which is equivalent to $x \in (B \setminus C)$

Theorem 2.12.1. Suppose $A \subseteq B$, and A and C are disjoint. Then $A \subseteq (B \setminus C)$.
Proof. Suppose $x \in A$. Since $A \subseteq B$, it follows that $x \in B$, and since A and C are disjoint, we must have $x \in C$. Thus, $x \in (B \setminus C)$. Since x was an arbitrary element of A, we can conclude that $A \subseteq (B \setminus C)$. $\square$

2.13 To prove a goal of the form $P \longleftrightarrow Q$

Prove $P \longrightarrow Q$ and $Q \longrightarrow P$ **separately**.

2.14 To use given of the form $P \longleftrightarrow Q$

Treat this as **two separate givens** : $P \longrightarrow Q$, and $Q \longrightarrow P$

For example, let us prove a statement using the following definition :

Definition. An integer x is even if $\exists k \in \mathbb{Z}(x = 2k)$, and x is odd if $\exists k \in \mathbb{Z}(x = 2k + 1)$.

Suppose x is an integer. Prove that x is even iff x^2 is even.

Scratch work :

Our goal is the statement : x is even $\longleftrightarrow$ x^2 is even. In other words, our goal is $\exists k \in \mathbb{Z}(x = 2k) \longleftrightarrow \exists l \in \mathbb{Z}(x^2 = 2l)$. Using our strategy, we will first prove the statement $\exists k \in \mathbb{Z}(x = 2k) \longrightarrow \exists l \in \mathbb{Z}(x^2 = 2l)$, then the statement $\exists l \in \mathbb{Z}(x^2 = 2l) \longrightarrow \exists k \in \mathbb{Z}(x = 2k)$. Therefore, we suppose $\exists k \in \mathbb{Z}(x = 2k)$ and prove that $\exists l \in \mathbb{Z}(x^2 = 2l)$.

Given :

$\exists k \in \mathbb{Z}(x = 2k)$

Goal :

$\exists l \in \mathbb{Z}(x^2 = 2l)$

In order to prove our goal, we will have to find a particular l such that $x^2 = 2l$. Using existential instantiation on our only given, we conclude that $k \in \mathbb{Z}$ and that $x = 2k$

Given :

$k \in \mathbb{Z}$

$x = 2k$

Goal :

$\exists l \in \mathbb{Z}(x^2 = 2l)$

Let us use our given $x = 2k$ to prove our goal. We can conclude from this given that $x^2 = (2k)^2 = 4k^2$. We want $x^2 = 2l$ for a particular l, and $4k^2 = 2(2k^2)$ so we can let $l = 2k^2$ (As $2k^2 \in \mathbb{Z}$). Our goal has been proven. Let us prove the other half of the proof, we could suppose $\exists l \in \mathbb{Z}(x^2 = 2l)$ and then prove that $\exists k \in \mathbb{Z}(x = 2k)$ but it requires to take the square root of $2l$. Let us process differently, the statement $\exists l \in \mathbb{Z}(x^2 = 2l) \longrightarrow \exists k \in \mathbb{Z}(x = 2k)$ is equivalent to $\neg\exists k \in \mathbb{Z}(x = 2k) \longrightarrow \neg\exists l \in \mathbb{Z}(x^2 = 2l)$ (using contrapositive laws). This last new statement means that if x is not even, then x^2 is not even. Being not even means being odd so our goal will be to prove that if x is odd, then x^2 is odd. In other words $\exists k \in \mathbb{Z}(x = 2k + 1) \longrightarrow \exists l \in \mathbb{Z}(x^2 = 2l + 1)$.

Givens :

$\exists k \in \mathbb{Z}(x = 2k + 1)$

Goal :

$\exists l \in \mathbb{Z}(x^2 = 2l + 1)$

Our goal is to find a particular $l \in \mathbb{Z}$ such that $x^2 = 2l + 1$. Our only given is an existential statement. Therefore, using existential instantiation, we have two more givens.

Givens :

$k \in \mathbb{Z}$

$x = 2k + 1$

Goal :

$\exists l \in \mathbb{Z}(x^2 = 2l + 1)$

As $x = 2k + 1$, we can conclude from that given that $x^2 = (2k + 1)^2 = 4k^2 + 4k + 1$. Let us get to the same form as the goal to find the particular l we need, $4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.

Therefore, we let $l = 2k^2 + 2k$ (As $l = 2k^2 + 2k \in \mathbb{Z}$). Our goal has been proven.
We can finally write the final form of the proof as both directions of the biconditional statement, x is even $\longleftrightarrow$ x^2 is even, have been proven.
Note: the two conditional statement we've proven can be though of representing two directions ($\longrightarrow$) and ($\longleftarrow$) of the biconditional symbol ($\longleftrightarrow$) in the original proof. These two parts of the proof are sometimes labeled with the symbol ($\longrightarrow$) and ($\longleftarrow$)

Theorem 2.14.1 Suppose x is an integer. Then x is even iff x^2 is even.
Proof.($\longrightarrow$) Suppose x is even. Then for some integer k, $x = 2k$. Therefore, $x^2 = 4k^2 = 2(2k^2)$, so since $2k^2$ an integer, x^2 is even. Thus, if x is even then x^2 is even.
($\longleftarrow$) Suppose x is odd. Then $x = 2k + 1$ for some integer k. Therefore, $x^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$, so since $2k^2 + 2k$ is an integer, x^2 is odd. Thus, if x^2 is even then x is even. $\square$

Let us do another example,

Prove that $\forall x \neg P(x) \longleftrightarrow \neg\exists x P(x)$.

Scratch work :

Our goal is a biconditional statement. Therefore, we will first prove $\forall x \neg P(x) \longrightarrow \neg\exists x P(x)$, then $\neg\exists x P(x) \longrightarrow \forall x \neg P(x)$. In order to prove the statement $\forall x \neg P(x) \longrightarrow \neg\exists x P(x)$, we suppose $\forall x \neg P(x)$ and prove $\neg\exists x P(x)$.

Givens :

$\forall x \neg P(x)$

Goal :

$\neg\exists x P(x)$

Our goal is a negated statement. We could obviously prove our goal using the quantifier negation law but let us suppose $\exists x P(x)$ and try to prove by contradiction.

Givens :

$\forall x \neg P(x)$

$\exists x P(x)$

Goal :

Contradiction.

Using existential instantiation (we choose $x = y$) on our second given, we can conclude that $P(y)$.

Givens :

$\forall x \neg P(x)$

$P(y)$

Goal :

Contradiction.

Now using universal instantiation on our first given, we conclude that $\neg P(y)$. As $\neg P(y)$ contradicts $P(y)$ our assumption that $\exists x P(x)$ must be false. Our goal has been proven. Let us now prove the other statement, $\neg\exists x P(x) \longrightarrow \forall x \neg P(x)$. We assume $\neg\exists x P(x)$ and prove $\forall x \neg P(x)$.

Givens :

$\neg\exists x P(x)$

Goal :

$\forall x \neg P(x)$

We let x be arbitrary and prove $\neg P(x)$.

Givens :

$\neg\exists x P(x)$

Goal :

$\neg P(x)$

Let us prove it by contradiction supposing $P(x)$.

Givens :

$\neg\exists x P(x)$

$P(x)$

Goal :

Contradiction.

Both of our givens contradict each other, as one says that there exists no x such that $P(x)$ and the other says $P(x)$. Therefore, our goal has been proven.

Theorem 2.14.2. $\forall x \neg P(x) \longleftrightarrow \neg\exists P(x)$.
Proof.($\longrightarrow$) Suppose $\forall x \neg P(x)$, and suppose $\exists x P(x)$. Then we can choose some y such that $P(y)$ is true. But since $\forall x \neg P(x)$, we can conclude that $\neg P(y)$, and this a contradiction. Therefore, $\forall x \neg P(x) \longrightarrow \neg\exists x P(x)$.
($\longleftarrow$) Suppose $\neg\exists P(x)$. Let x be arbitrary, and suppose $P(x)$. Since we have a specific x for

which $P(x)$ is true, it follows that $\exists x P(x)$, which is a contradiction. Therefore, $\neg P(x)$. Since x was arbitrary, we can conclude that $\forall x \neg P(x)$, so $\neg \exists x P(x) \longrightarrow \forall x \neg P(x)$. $\square$

2.15 To prove a goal of the form $P \longleftrightarrow Q$ (string of equivalences)

Sometimes, a proof of the form $P \longleftrightarrow Q$ can be simplified by writing it as a string of equivalences. We **suppose P and show that P is logically equivalent to some statement R, and then show that R is logically equivalent to Q**. The final form of the proof looks like $P \longleftrightarrow R \longleftrightarrow Q$.

For example,

Suppose $A, B,$ and C are sets. Prove that $A \cap (B \setminus C) = (A \cap B) \setminus C$

Scratch work :

Our goal can be written as $\forall x(x \in (A \cap (B \setminus C)) \longleftrightarrow x \in ((A \cap B) \setminus C))$. Therefore, we let x be arbitrary and prove $x \in (A \cap (B \setminus C)) \longleftrightarrow x \in ((A \cap B) \setminus C)$. Our strategy suggests that we find a statement R that logically connects the statements $x \in (A \cap (B \setminus C))$ and $x \in ((A \cap B) \setminus C)$. Let us then find that particular statement R developing both our statements $x \in (A \cap (B \setminus C))$ and $x \in ((A \cap B) \setminus C)$.

The statement $x \in (A \cap (B \setminus C))$ is equivalent to $x \in A \wedge x \in (B \setminus C)$, which is equivalent to $x \in A \wedge x \in B \wedge x \notin C$. The statement $x \in ((A \cap B) \setminus C)$ is equivalent to $x \in A \cap B \wedge x \notin C$, which is equivalent to $x \in A \wedge x \in B \wedge x \notin C$. Therefore, the particular statement R we were looking for is $x \in A \wedge x \in B \wedge x \notin C$.

Theorem 2.15.1. Suppose A, B, and C are sets. Then $A \cap (B \setminus C) = (A \cap B) \setminus C$.
Proof. Let x be arbitrary. Then $x \in A \cap (B \setminus C)$

iff $x \in A \wedge x \in (B \setminus C)$

iff $x \in A \wedge x \in B \wedge x \notin C$

iff $(x \in A \wedge x \in B) \wedge x \notin C$

iff $x \in A \cap B \wedge x \notin C$

iff $x \in (A \cap B) \setminus C$.

Thus, $\forall x(x \in (A \cap (B \setminus C)) \longleftrightarrow x \in ((A \cap B) \setminus C))$, so $A \cap (B \setminus C) = (A \cap B) \setminus C$. $\square$

Let us do another example,

Prove that for any real number a and b, $(a+b)^2 - 4(a-b)^2 = (3b-a)(3a-b)$.

Scratch work :

Our goal is the statement $\forall a \in \mathbb{R}\ \forall b \in \mathbb{R}\ ((a+b)^2 - 4(a-b)^2 = (3b-a)(3a-b))$. Therefore, we let a and b be arbitrary real numbers and prove that $(a+b)^2 - 4(a-b)^2 = (3b-a)(3a-b)$. Let us prove this by a string of equivalences.

$(a+b)^2 - 4(a-b)^2$

$= (a^2 + 2ab + b^2) - 4(a^2 - 2ab + b^2)$

$= a^2 + 2ab + b^2 - 4a^2 + 8ab - 4b^2$

$= -3a^2 + 10ab - 3b^2.$

$(3b-a)(3a-b)$

$= 9a^2 - 3b^2 - 3a^2 + ab$

$= -3a^2 + 10ab - 3b^2.$

Theorem 2.15.2. For any real numbers a and b, $(a+b)^2 - 4(a-b)^2 = (3b-a)(3a-b)$.
Proof. Let a and b be arbitrary real numbers. Then $(a+b)^2 - 4(a-b)^2$

$= (a^2 + 2ab + b^2) - 4(a^2 - 2ab + b^2)$

$= -3a^2 + 10ab - 3b^2$

$= 9a^2 - 3b^2 - 3a^2 + ab$

$= (3b-a)(3a-b)$. □

2.16 To use a given of the form $P \lor Q$

Break your proof into cases. For case 1, assume that P is true and use this assumption to prove the goal. For case 2, assume that Q is true and give another proof of the goal.

Form of the final proof:

Case 1. P is true.

[Proof of the goal goes here].

Case 2. Q is true.

[Proof of goal goes here].

Since we know $P \lor Q$, these cases cover all possibilities. Therefore, the goal must be true.

For example,

Suppose A, B and C are sets. Prove that if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

Scratch work :

Our goal can be written as $((A \subseteq C) \wedge (B \subseteq C)) \longrightarrow ((A \cup B) \subseteq C)$. Therefore, we assume $(A \subseteq C) \wedge (B \subseteq C)$ and prove $(A \cup B) \subseteq C$.

Givens :

$(A \subseteq C)$

$(B \subseteq C)$

Goal :

$(A \cup B) \subseteq C$

The statement $(A \cup B) \subseteq C$ can be written as $\forall x \ (x \in (A \cup B) \longrightarrow x \in C)$. Thus, we let x be arbitrary, suppose $x \in (A \cup B)$ and prove that $x \in C$.

Givens :

$(A \subseteq C)$

$(B \subseteq C)$

$x \in (A \cup B)$

Goal :

$x \in C$

The statement $x \in (A \cup B)$ is equivalent to $x \in A \vee x \in B$, which is a disjunction, so we will process by cases. Let us start by proving our goal with the case that $x \in A$.

Givens :

$(A \subseteq C)$

$(B \subseteq C)$

$x \in A$

Goal :

$x \in C$

Since $x \in A$ and $A \subseteq C$, it follows that $x \in C$. Now, let us prove our goal with the case $x \in B$.

Givens :

$(A \subseteq C)$

$(B \subseteq C)$

$x \in B$

Goal :

$x \in C$

Since $x \in B$ and $B \subseteq C$, it follows that $x \in C$. For all cases, we can conclude that $x \in C$. Therefore, our goal has been proven.

Theorem 2.16.1. Suppose that A, B and C are sets. If $A \subseteq C$ and $B \subseteq C$ then $(A \cup B) \subseteq C$.
Proof. Suppose $A \subseteq C$ and $B \subseteq C$, and let x be an arbitrary element of $A \cup B$. Then, either $x \in A$ or $x \in B$.
Case 1. $x \in A$. Then, since $A \subseteq C$, $x \in C$.
Case 2. $x \in B$. Then, since $B \subseteq C$, $x \in C$.
Since we know either $x \in A$ or $x \in B$, these cases cover all possibilities, so we can conclude that $x \in C$. Since x was an arbitrary element of $A \cup B$, this means that $(A \cup B) \subseteq C$. $\square$

2.17 To prove a goal of the form $P \vee Q$

Break your proof into cases. **In each case, prove P or prove Q.**

For example,

Suppose that A, B and C are sets. Prove that $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup C$.

Scratch work :

Our goal is the statement $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup C$, which is equivalent to $\forall x\, (x \in (A \setminus (B \setminus C)) \longrightarrow x \in (A \setminus B) \cup C))$. Therefore, we start by letting x be arbitrary, suppose $x \in (A \setminus (B \setminus C))$ and prove $x \in ((A \setminus B) \cup C)$.

Givens :

$x \in (A \setminus (B \setminus C))$

Goal :

$x \in ((A \setminus B) \cup C)$.

Our goal is equivalent to $x \in (A \setminus B) \vee x \in C$. Therefore, we will prove our goal using cases.

Givens :

$x \in (A \setminus (B \setminus C))$

Goal :

$x \in (A \setminus B) \vee x \in C.$

We have to find cases that are suggested by something in our proof. The natural place where such suggestions can be found is in our list of givens. Our only given is $x \in (A \setminus (B \setminus C))$, which is equivalent to $x \in A \wedge x \notin (B \setminus C)$, which means $x \in A \wedge \neg(x \in (B \setminus C))$. This last statement can be written as $x \in A \wedge \neg(x \in B \wedge x \notin C)$, and using negation laws, it is also equivalent to $x \in A \wedge (x \notin B \vee x \in C)$.

Givens :

$x \in A$

$x \notin B \vee x \in C$

Goal :

$x \in (A \setminus B) \vee x \in C.$

Let us take a look at our current situation, we have a given that is disjunction, and our goal is also a disjunction. In order to prove our goal, we will therefore need to prove either $x \in (A \setminus B)$ or $x \in C$ (a disjunction is true as soon as one of the premises is true). In order to use our given $x \notin B \vee x \in C$ we will have to break it into two cases : $x \notin B$ and $x \in C$, and show that no matter what the case is, the goal is proven. Let us start with the case $x \notin B$.

Givens :

$x \in A$

$x \notin B$

Goal :

$x \in (A \setminus B) \vee x \in C.$

Since $x \in A$ and $x \notin B$, it follows that $x \in (A \setminus B)$. Let us now go on to the case $x \in C$.

Givens :

$x \in A$

$x \in C$

Goal :

$x \in (A \setminus B) \vee x \in C.$

The proof of our goal follows directly from our given. The goal $x \in (A \setminus B) \vee x \in C$ has therefore been proven for all possible cases.

Theorem 2.17.1. Suppose that A, B and C are sets. Then $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup C$.
Proof. Suppose $x \in A \setminus (B \setminus C)$. Then $x \in A$ and $x \notin (B \setminus C)$. Since $x \notin (B \setminus C)$, it follows that $x \notin B$ or $x \in C$. We will consider these cases separately.
Case 1. $x \notin B$. Then since $x \in A$, $x \in (A \setminus B)$, so $x \in (A \setminus B) \cup C$.
Case 2. $x \in C$. Then clearly $x \in (A \setminus B) \cup C$.
Since x was an arbitrary element of $A \setminus (B \setminus C)$, we can conclude that $A \setminus (B \setminus C) \subseteq (A \setminus B) \cup C$.
□

2.18 To prove a goal of the form $P \vee Q$ (conditional statement)

Another way of proving a goal of the form $P \vee Q$ is to **rewrite it as a conditional statement**. Using conditional laws, a statement $P \vee Q$ is equivalent to $\neg P \longrightarrow Q$, but also $\neg Q \longrightarrow P$. Therefore, if we rewrite $P \vee Q$ as $\neg P \longrightarrow Q$, we suppose $\neg P$ and prove Q.

For example,

Prove that for every real number x, if $x^2 \geq x$ then $x \leq 0$ or $x \geq 1$.

Scratch work :

Our goal is the statement $\forall x \in \mathbb{R}\ (x^2 \geq x \longrightarrow (x \leq 0 \vee x \geq 1))$

Therefore, we start by letting x be an arbitrary real number, suppose $x^2 \geq x$ and prove $x \leq 0 \vee x \geq 1$

Given :

$x^2 \geq x$

Goal :

$x \leq 0 \vee x \geq 1$

Our goal is a disjunction. Let us write it as a conditional statement. Using conditional laws, the statement $x \leq 0 \vee x \geq 1$ is equivalent to $x > 0 \longrightarrow x \geq 1$ (also equivalent to $x < 1 \longrightarrow x \leq 0$). Therefore,

Given :

$x^2 \geq x$

Goal :

$x > 0 \longrightarrow x \geq 1$

As our new goal is a conditional statement, we suppose $x > 0$ and prove that $x \geq 1$

Givens :

$x^2 \geq x$

$x > 0$

Goal :

$x \geq 1$

The given $x > 0$ says that x is positive and different from 0, we can use this given on our other given, $x^2 \geq x$, and divide both sides by x. Therefore, it follows $x \geq 1$. Our goal has been proven.

Theorem 2.18.1. For every real number x, if $x^2 \geq x$, then $x \leq 0$ or $x \geq 1$.
Proof. Let x be an arbitrary real number. We want to prove that $x \leq 0 \lor x \geq 1$. Using conditional law, a disjunction $P \lor Q$ is equivalent to the implication $(\neg P) \longrightarrow Q$. Thus, our goal is equivalent to showing that $\neg(x \leq 0) \longrightarrow x \geq 1$, that is, $x > 0 \longrightarrow x \geq 1$. Suppose $x^2 \geq x$. Suppose further that $x > 0$. Because x is strictly positive, we may divide both sides of the inequality $x^2 \geq x$ by x without reversing the inequality. This gives $x \geq 1$. Hence the implication $x > 0 \longrightarrow x \geq 1$ holds. Therefore, in all cases, if $x^2 \geq x$, then $x \leq 0$ or $x \geq 1$. Since x was arbitrary, the theorem is proved. $\square$

3 Existence and Uniqueness proofs

In this section, we consider proofs with goals of the form $\exists!\ xP(x)$. This goal is said to be an abbreviation of $\exists x(P(x) \land \neg\exists y(P(y) \land y \neq x))$. Therefore, in order to prove that goal, we will have to find a particular value x such that $P(x) \land \neg\exists y(P(y) \land y \neq x)$ is true. The negated statement $\neg\exists y(P(y) \land y \neq x)$ is equivalent to $\forall y(\neg P(y) \lor y = x)$ which also means $\forall y(P(y) \longrightarrow y = x)$. Therefore, proving the statement $\exists!\ xP(x)$ is also proving the equivalent statement $\exists x(P(x) \land \forall y(P(y) \longrightarrow y = x))$. Other equivalent forms can be used in order to prove $\exists!\ xP(x)$. For example,

$\exists x \forall y(P(y) \longleftrightarrow y = x)$

$\exists x P(x) \land \forall y \forall z((P(y) \land P(z)) \longrightarrow y = z)$

Let us prove that all the following statements are equivalent.

1. $\exists x(P(x) \land \forall y(P(y) \longrightarrow y = x))$

2. $\exists x \forall y(P(y) \longleftrightarrow y = x)$

3. $\exists x P(x) \land \forall y \forall z((P(y) \land P(z)) \longrightarrow y = z)$

Scratch work.

Our goal has this form :(1. $\longleftrightarrow$ 2.) $\land$ (1. $\longleftrightarrow$ 3.) $\land$ (2 $\longleftrightarrow$ 3). Proving three biconditional statements would be pretty long. Let us use a shortcut that mathematicians usually use. We will first suppose 1. and prove 2., then suppose 2. and prove 3., then suppose 3. and prove 1. .

In other words $(1. \longrightarrow 2.) \wedge (2. \longrightarrow 3.) \wedge (3. \longrightarrow 1.)$. Therefore, we will prove only three conditional statements instead of six with the statement $(1. \longleftrightarrow 2.) \wedge (1. \longleftrightarrow 3.) \wedge (2 \longleftrightarrow 3)$. This strategy works because all statements 1., 2., and 3., will be shown to be equal to each others. For example, the equivalence of statement 1. and statement 2. is proven with the first direction $(1. \longleftrightarrow 2.)$ and then $(2. \longrightarrow 1.)$ is proven by the intermediary of 3. with $(2. \longrightarrow 3.) \wedge (3. \longrightarrow 1.)$. Same applies to all other equivalences. As you might have noticed, this is pretty similar to the proofs of statements in the form $P \longleftrightarrow Q$ with a string of equivalences as seen previously.

Therefore, let us start proving $(1. \longrightarrow 2.)$. In other words, $[\exists x(P(x) \wedge \forall y(P(y) \longrightarrow y = x)] \longrightarrow [\exists x \forall y(P(y) \longleftrightarrow y = x)]$

We suppose $\exists x(P(x) \wedge \forall y(P(y) \longrightarrow y = x)$ and prove $\exists x \forall y(P(y) \longleftrightarrow y = x)$

Givens :

$\exists x(P(x) \wedge \forall y(P(y) \longrightarrow y = x)$

Goal :

$\exists x \forall y(P(y) \longleftrightarrow y = x)$

Our current goal requires a specific x that makes the statement $\forall y(P(y) \longleftrightarrow y = x)$ come out true. Let us look at our list of given to find such x. Let us use existential instantiation on our given, we choose $x = x_0$.

Givens :

$P(x_0)$

$\forall y(P(y) \longrightarrow y = x_0)$

Goal :

$\exists x \forall y(P(y) \longleftrightarrow y = x)$

Therefore, our given suggested that x_0 was the specific x we were looking for to prove our goal. Let us plug in $x = x_0$ in our current goal.

Givens :

$P(x_0)$

$\forall y(P(y) \longrightarrow y = x_0)$

Goal :

$\forall y(P(y) \longleftrightarrow y = x_0)$

Let y be arbitrary. Our new goal is $P(y) \longleftrightarrow y = x_0$, which is equivalent to $(P(y) \longrightarrow y = x_0) \wedge (y = x_0 \longrightarrow P(y))$. Let us first prove $(P(y) \longrightarrow y = x_0)$

$(\longrightarrow)$

Givens :

$P(x_0)$

$\forall y(P(y) \longrightarrow y = x_0)$

Goal :

$P(y) \longrightarrow y = x_0$

Suppose $P(y)$. Therefore, let us prove $y = x_0$

Givens :

$P(x_0)$

$\forall y(P(y) \longrightarrow y = x_0)$

$P(y)$

Goal :

$y = x_0$

Our goal follows from using modus ponens as we are given $\forall y(P(y) \longrightarrow y = x_0)$ and $P(y)$ (there was no actual need to suppose $P(y)$ as $\forall y(P(y) \longrightarrow y = x_0)$, our goal was already proven). Let us now prove the other direction of the biconditional statement : $y = x_0 \longrightarrow P(y)$.

$(\longleftarrow)$

Givens :

$P(x_0)$

$\forall y(P(y) \longrightarrow y = x_0)$

Goal :

$y = x_0 \longrightarrow P(y)$

Suppose $y = x_0$. Let us prove $P(y)$.

Givens :

$P(x_0)$

$\forall y(P(y) \longrightarrow y = x_0)$

$y = x_0$

Goal :

$P(y)$

Our first given says $P(x_0)$ and the third says $x_0 = y$. Therefore, $P(y)$.

Let us now prove (2. $\longrightarrow$ 3.). In other words, $[\exists x \forall y (P(y) \longleftrightarrow y = x)] \longrightarrow [\exists x P(x) \wedge \forall y \forall z ((P(y) \wedge P(z)) \longrightarrow y = z)]$. We suppose $\exists x \forall y (P(y) \longleftrightarrow y = x)$ and prove that $\exists x P(x) \wedge \forall y \forall z ((P(y) \wedge P(z)) \longrightarrow y = z)$.

Given :

$\exists x \forall y (P(y) \longleftrightarrow y = x)$

Goals :

$\exists x P(x)$

$\forall y \forall z ((P(y) \wedge P(z)) \longrightarrow y = z)$

We have two goals, let us first prove $\exists x P(x)$. It requires to find a specific x. Let us take a look at our list of givens and see what it suggests. We use existential instantiation on our given and choose $x = x_0$.

Given :

$\forall y (P(y) \longleftrightarrow y = x_0)$

Goals :

$\exists x P(x)$

$\forall y \forall z ((P(y) \wedge P(z)) \longrightarrow y = z)$

We also plug in $x = x_0$ in the statement $\exists x P(x)$. Therefore, we have to prove $P(x_0)$. Plugging in x_0 in our given would directly prove $P(x_0)$. But let us seek for some more givens proving $\forall y \forall z ((P(y) \wedge P(z)) \longrightarrow y = z)$. We let y and z be arbitrary, suppose $P(y) \wedge P(z)$ and prove $y = z$.

Givens :

$\forall y (P(y) \longleftrightarrow y = x_0)$

$P(y)$

$P(z)$

Goals :

$P(x_0)$

$y = z$

The statement $\forall y(P(y) \longleftrightarrow y = x_0)$ is equivalent to $\forall y((P(y) \longrightarrow y = x_0) \wedge (y = x_0 \longrightarrow P(y))$. As the universal quantifier can be distributed over conjunction, the statement $\forall y(P(y) \longleftrightarrow y = x_0)$ is equivalent to $\forall y(P(y) \longrightarrow y = x_0) \wedge \forall y(y = x_0 \longrightarrow P(y))$.

Givens :

$\forall y(P(y) \longrightarrow y = x_0)$

$\forall y(y = x_0 \longrightarrow P(y))$

$P(y)$

$P(z)$

Goals :

$P(x_0)$

$y = z$

Let us plus in z for y in our first given $\forall y(P(y) \longrightarrow y = x_0)$. Then plugging in the same first statement y for y we get :

Givens :

$P(y) \longrightarrow y = x_0$

$P(z) \longrightarrow z = x_0$

$\forall y(y = x_0 \longrightarrow P(y))$

$P(y)$

$P(z)$

Goals :

$P(x_0)$

$y = z$

As we have $P(y)$ and $P(y) \longrightarrow y = x_0$, we use modus ponens to conclude $y = x_0$. As we have $P(z)$ and $P(z) \longrightarrow z = x_0$, we use modus ponens to conclude $z = x_0$. Since $y = x_0 = z$, $y = z$. Anything that we have proved can now be used to prove something else. Therefore, if needed, $y = z$ can be added to our list of given.

Givens :

$y = x_0$

$z = x_0$

$\forall y(y = x_0 \longrightarrow P(y))$

$P(y)$

$P(z)$

Goals :

$P(x_0)$

Let us plug in z for y in the statement $\forall y(y = x_0 \longrightarrow P(y))$. Therefore, $z = x_0 \longrightarrow P(z)$ and since we have $z = x_0$, we can conclude $P(z)$. Our goal follows from the fact that $z = x_0$, thus $P(z) = P(x_0)$.

The last conditional statement to prove is (3. $\longrightarrow$ 1.). In other words, $[\exists x P(x) \wedge \forall y \forall z((P(y) \wedge P(z)) \longrightarrow y = z)] \longrightarrow [\exists x(P(x) \wedge \forall y(P(y) \longrightarrow y = x))]$. We suppose $\exists x P(x) \wedge \forall y \forall z((P(y) \wedge P(z)) \longrightarrow y = z)$ and prove $\exists x(P(x) \wedge \forall y(P(y) \longrightarrow y = x))$.

Givens :

$\exists x P(x)$

$\forall y \forall z((P(y) \wedge P(z)) \longrightarrow y = z)$

Goal :

$\exists x(P(x) \wedge \forall y(P(y) \longrightarrow y = x))$

Once again, our goal requires to find a specific x such that $P(x) \wedge \forall y(P(y) \longrightarrow y = x)$ is true. Let us see what our givens suggest for a specific x. Using existential instantiation on our first given, we choose $x = x_0$ and also plug x_0 for x in our goal.

Givens :

$P(x_0)$

$\forall y \forall z((P(y) \wedge P(z)) \longrightarrow y = z)$

Goal :

$P(x_0)$

$\forall y(P(y) \longrightarrow y = x_0)$

The goal $P(x_0)$ follows directly from our list of given. Let us prove $\forall y \forall z((P(y) \wedge P(z)) \longrightarrow y = z)$. We let y be arbitrary, suppose $P(y)$, and prove $y = x_0$.

Givens :

$P(x_0)$

$\forall y \forall z((P(y) \wedge P(z)) \longrightarrow y = z)$

$P(y)$

Goal :

$y = x_0$

Using universal instantiation on $\forall y \forall z((P(y) \wedge P(z)) \longrightarrow y = z)$, we plug in y for y and x_0 for z.

Givens :

$P(x_0)$

$(P(y) \wedge P(x_0)) \longrightarrow y = x_0$

$P(y)$

Goal :

$y = x_0$

Our goal follows from using all our givens and modus ponens.

Theorem 3.1. The followings are equivalent :
1. $\exists x(P(x) \wedge \forall y(P(y) \longrightarrow y = x))$
2. $\exists x \forall y(P(y) \longleftrightarrow y = x)$
3. $\exists x P(x) \wedge \forall y \forall z((P(y) \wedge P(z)) \longrightarrow y = z)$

Proof. (1. $\longrightarrow$ 2.). By statement 1., we can let x_0 be some object such that $P(x_0)$ and $\forall y(P(y) \longrightarrow y = x_0)$. To prove statement 2., we will show that $\forall y(P(y) \longleftrightarrow y = x_0)$. Let y be arbitrary. We already know the ($\longrightarrow$) direction of the biconditional. For the ($\longleftarrow$) direction, suppose $y = x_0$. Then, since we know $P(x_0)$, we can conclude $P(y)$
(1. $\longrightarrow$ 2.). By statement 2, choose x_0 such that $\forall y(P(y) \longleftrightarrow y = x_0)$. Then, in particular $P(x_0) \longleftrightarrow x_0 = x_0$, and since clearly $x_0 = x_0$, it follows that $P(x_0)$ is true. Thus, $\exists x P(x)$. To prove the second half of statement 3, let y and z be arbitrary and suppose $P(y)$ and $P(z)$. Then by our choice of x_0, it follows that $y = x_0$ and $z = x_0$, so $y = z$.
(3. $\longrightarrow$ 1.). By the first half of statement 3, let x_0 be some object such that $P(x_0)$. Statement 1 will follow only if we can show $\forall y(P(y) \longrightarrow y = x_0)$. So suppose $P(y)$. Since we now have both $P(x_0)$ and $P(y)$, by the second half of statement 3 we can conclude that $y = x_0$, as required. □

Now that we have proven that all these statements are equivalent, we can use any of them to prove a goal or use any of them as given of the form $\exists! x P(x)$. The choice of the best statement to prove a goal (or to use as given) of the form $\exists! x P(x)$ is crucial. Although there are all equivalent, one might be better for a specific proof than the others.

3.1 To prove a goal of the form $\exists! x P(x)$ (Existence and Uniqueness)

The statement 3. $\exists x P(x) \wedge \forall y \forall z((P(y) \wedge P(z)) \longrightarrow y = z)$ might be one of the best options. **It is a conjunction**, which means that we will have to prove $\exists x P(x)$ and $\forall y \forall z((P(y) \wedge P(z)) \longrightarrow y = z)$ **separately**. Proving $\exists x P(x)$ shows that there exists at least one x such that $P(x)$ is true. Proving it shows the **existence**. Proving $\forall y \forall z((P(y) \wedge P(z)) \longrightarrow y = z)$ shows that for all specific y and z plugged in for x in $P(x)$, if they both make $P(x)$ come out true, then they are equal. In other words, there is only one specific value that can make $P(x)$ come out true. Proving it shows the **uniqueness**.

Form of the final proof :

Existence : [Proof of $\exists x P(x)$ goes here].

Uniqueness : [Proof of $\forall y \forall z((P(y) \wedge P(z)) \longrightarrow y = z)$ goes here]

For example,

Prove that there is a unique set A such that for every set B, $A \cup B = B$

Scratch work :

Our goal is $\exists! A\ \forall B(A \cup B = B)$, which is a statement of the form $\exists! x P(x)$. Let us rewrite it as a statement of the form $\exists x P(x) \wedge \forall y \forall z((P(y) \wedge P(z)) \longrightarrow y = z)$. The corresponding part of $P(x)$ from $\exists! x P(x)$ in $\exists! A\ \forall B(A \cup B = B)$ is $\forall B(A \cup B = B)$. Therefore, our goal can be rewritten as $\exists A[\forall B(A \cup B = B)] \wedge \forall C \forall D[(\forall B(C \cup B = B \wedge D \cup B = B)) \longrightarrow C = D]$. Let us start by proving the existence part, proving $\exists A[\forall B(A \cup B = B)]$. This goal requires to find a specific A and we do not have any givens that can make suggestions. Therefore, let us have a closer look at the property that the specific A that we are looking for has. For any B, $A \cup B = B$. The empty set fits the description. Letting $A = \varnothing$ proves the existence.
Note that B would normally also fit the description without our current context but the statement requires finding a single, fixed set A that works universally. If you claim $A = B$, then A is not a single, unique set; its value changes depending on which B you are currently considering.

Let us now prove the uniqueness part, proving $\forall C \forall D[(\forall B(C \cup B = B \wedge D \cup B = B)) \longrightarrow C = D]$. We let C and D be arbitrary, suppose $\forall B(C \cup B = B \wedge D \cup B = B)$ and prove $C = D$.

Givens :

$\forall B(C \cup B = B \wedge D \cup B = B)$

Goal :

$C = D$

The statement $\forall B(C \cup B = B \land D \cup B = B)$ is equivalent to $\forall B(C \cup B = B) \land \forall B(D \cup B = B)$ (universal quantifier distribution over conjunction).

Givens :

$\forall B(C \cup B = B)$

$\forall B(D \cup B = B)$

Goal :

$C = D$

Using universal instantiation, we plug in D in the first given and C in the second given.

Givens :

$C \cup D = D$

$D \cup C = C$

Goal :

$C = D$

As $C \cup D$ and $D \cup C$ are equivalent, it follows that $C = D$. Our goal has been proven.

Theorem 3.1.1. There is a unique set A such that for every set B, $A \cup B = B$
Proof. Existence : $\forall B(\varnothing \cup B = B)$, so $\varnothing$ has the required property.
Uniqueness : Suppose $\forall B(C \cup B = B)$ and $\forall B(D \cup B = B)$. Applying the first of these assumptions to D, we see that $C \cup D = D$ we see that $C \cup D = D$, and applying the second to C we get $D \cup C = C$. As $C \cup D = D \cup C$, it follows that $C = D$. $\square$
Note that statement 2. and statement 3. **would also lead to the same conclusion.**

3.2 To use a given of the form $\exists! x P(x)$

Conjunctions are quite comfortable to use. Once again, statement 3. might suit what we are looking for. In other words, let us use $\exists x P(x) \land \forall y \forall z((P(y) \land P(z)) \longrightarrow y = z)$ as a given. For example,

Suppose A, B and C are sets, A and B are not disjoint, A and C are not disjoint, and A has exactly one element. Prove that B and C are not disjoint.

Scratch work :

Givens :

$A \cap B \neq \varnothing$

$A \cap C \neq \varnothing$

$\exists! x(x \in A)$

Goal :

$B \cap C \neq \varnothing$

Our goal means that there exists at least one element that belongs to B and C. In other words, $\exists y(y \in B \wedge y \in C)$. Thus, this goal requires a particular y such that $y \in B \wedge y \in C$. Let us take a look at our list of givens.

Givens :

$A \cap B \neq \varnothing$

$A \cap C \neq \varnothing$

$\exists! x(x \in A)$

Goal :

$\exists y(y \in B \wedge y \in C)$

All our givens can be rewritten. $A \cap B \neq \varnothing$ is equivalent to $\exists x(x \in A \wedge x \in B)$. The second given, $A \cap C \neq \varnothing$ can similarly be rewritten as $\exists x(x \in A \wedge x \in C)$. Our third given $\exists! x(x \in A)$ is, as seen previously, equivalent to $\exists x(x \in A) \wedge \forall y \forall z((y \in A \wedge z \in A) \longrightarrow y = z)$.

Givens :

$\exists x(x \in A \wedge x \in B)$

$\exists x(x \in A \wedge x \in C)$

$\exists x(x \in A)$

$\forall y \forall z((y \in A \wedge z \in A) \longrightarrow y = z)$

Goal :

$\exists y(y \in B \wedge y \in C)$

Let us use the existential for the first three givens of our list introducing new variables.

Givens :

$x_0 \in A \wedge x_0 \in B$

$x_1 \in A \wedge x_1 \in C$

$x_2 \in A$

$\forall y \forall z((y \in A \wedge z \in A) \longrightarrow y = z)$

Goal :

$\exists y(y \in B \wedge y \in C)$

Using universal instantiation for our last given, we can conclude that $x_0 = x_1 = x_2$. Therefore, let us introduce x such that $x = x_0 = x_1 = x_2$. Thus, $x \in B \wedge x \in C$, our goal follows.
Note that statement 2. and statement 3. **would also lead to the same conclusion.**

Theorem 3.2.1. Suppose A, B, and C are sets, A and B are not disjoint, A and C are not disjoint, and A has exactly one element. Then B and C are not disjoint.
Proof. Since A and B are not disjoint, we can let x_0 be some object such that $x_0 \in A$ and $x_0 \in B$. Similarly, since A and C are not disjoint, we can let x_1 be some object such that $x_1 \in A$ and $x_1 \in C$. As A has a unique element, say x_2, we can conclude that $x_0 = x_1 = x_2$. Thus, $x_2 \in B \cap C$ and therefore B and C are not disjoint. $\square$