Notes - How to Prove It - Chapter 2 : Quantificational Logic

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1 Introduction

Following our study of sentential logic, we now turn to a more advanced and powerful tool for mathematical reasoning: quantification. This chapter is dedicated to symbols that allow us to make precise statements about all elements within a set or the existence of at least one such element.

1.1 Quantifiers

The symbol $\forall$ is called the **universal quantifier**. We write $\forall x \in U\ P(x)$ to say that $P(x)$ is true for **every value** of x in the universe of discourse U.

The symbol $\exists$ is called the **existential quantifier**. We write $\exists x \in UP(x)$ to state that $P(x)$ is true for **at least one value** of x in the universe of discourse U

1.2 Equivalences involving quantifiers

Quantifier Negation laws

$\neg\exists\ x\ P(x)$ is equivalent to $\forall x\ \neg P(x)$

$\neg\forall\ x\ P(x)$ is equivalent to $\exists x\ \neg P(x)$

$\exists!\ x\ P(x)$ means that there is exactly one value of x such that $P(x)$ is true. We can think of $\exists!\ x\ P(x)$ as an abbreviation for the formula $\exists x\ (P(x) \wedge \neg\exists y\ (P(y) \wedge y \neq x))$

The quantifiers in these formulas are sometimes called **bounded quantifiers** because they place limits on the values of x to be considered. It is interesting to note that the quantifier negation laws work for bounded quantifiers as well. For example, $\neg\forall x \in A\ P(x)$ is equivalent to

$\neg\forall x(x \in A \longrightarrow P(x))$ (**Expanding abbreviation**)

which is equivalent to $\exists x\ \neg(x \in A \longrightarrow P(x))$ (**Quantifier Negation law**)

which is equivalent to $\exists x\ \neg(x \notin A \vee P(x))$ (**Conditional law**)

which is equivalent to $\exists x\ (x \in A \wedge \neg P(x))$ (**De Morgan's law**) which is equivalent to $\exists x \in A \ \neg P(x)$ (**Abbreviation**)

Using the quantifier negation laws, we can find other equivalences :

$\forall x \in A \ P(x)$

is equivalent to $\neg\neg\forall x \in A \ P(x)$ (**Double Negation law**)

which is equivalent to $\neg\exists x \in A \ \neg P(x)$ (**Quantifier Negation law**)

The **universal quantifier distributes over the conjunction**. For example,

$\forall x \ (E(x) \land P(x))$

is equivalent to $\forall x \ E(x) \land \forall x \ P(x)$

However, the **universal quantifier does not distribute over the disjunction**.
On the other hand, **the existential quantifier distributes over the disjunction**, for example,

$\exists x \ (E(x) \lor P(x))$

is equivalent to $\exists x \ E(x) \lor \exists x \ P(x)$

but does **not distribute over the conjunction**.

1.3 More operations on sets

Suppose that we want to define S as the set of all perfect squares. Perhaps the easiest way to describe this set is to say that all its elements are numbers of the form n^2, where n is a natural number. This is written as follows.

$S = \{n^2 \mid n \in \mathbb{N}\}$

But we can also define this set by writing

$S = \{x \mid \exists n \in \mathbb{N} \ (x = n^2)\}$

Similar notation is often used if the elements of the set have been numbered. For example, suppose that we wanted to form the set whose elements are the first hundred prime numbers. We might start by numbering the prime numbers, calling them $p_1, p_2, p_3, ...$,where $p_1 = 2, p_2 = 3, p_3 = 5$, etc. Then the set we are looking for would be the set $P = \{p_1, p_2, p_3, ..., p_{99}, p_{100}\}$. Another way of describing this set would be to say that it consists of all numbers p_i, for i an element of the set $I = \{1, 2, 3, ..., 99, 100\} = \{i \in \mathbb{N} \mid 1 \leq i \leq 100\}$

This could be written as

$P = \{p_i \mid i \in I\}$

Each element p_i in this set is identified by a number $i \in I$, called the **index of the element**. A set defined in this way is sometimes called **an indexed family**, and I is called the **index set**.

Let us analyze the logical forms of a few statements by writing out definitions.

1. $y \in \{\sqrt[3]{x} \mid x \in \mathbb{Q}\}$

For y to belong to this set, y must be **equal to at least one of the elements of this set**. In other words, the fact that y belongs to this set means that $\exists x \in \mathbb{Q}\ (y = \sqrt[3]{x})$

2. $\{x_i \mid i \in I\} \subseteq A$

This statement means that the set $\{x_i \mid i \in I\}$ is a subset of the set A. Thus, for all possible x such that x is an element of $\{x_i \mid i \in I\}$ then x is also an element of A. In other words, $\forall x(x \in \{x_i \mid i \in I\} \longrightarrow x \in A)$. The fact that $x \in \{x_i \mid i \in I\}$ means that x is equal to at least one element of the set $\{x_i \mid i \in I\}$, which means that $\exists i \in I\ (x = x_i)$. Therefore, the statement $\{x_i \mid i \in I\} \subseteq A$ is equivalent to $\forall x\ (\exists i \in I\ (x = x_i) \longrightarrow x \in A)$.
Another equivalent statement for the statement $\{x_i \mid i \in I\} \subseteq A$ would be $\forall i \in I\ (x_i \in A)$.

3. $\{n^2 \mid n \in \mathbb{N}\}$ and $\{n^3 \mid n \in \mathbb{N}\}$ are not disjoint.

This means that those two sets have at least one element in common. In fact, this is true for example 0 and 1. In other words, $\exists x\ (x \in \{n^2 \mid n \in \mathbb{N}\} \wedge x \in \{n^3 \mid n \in \mathbb{N}\})$. One way of rewriting it would be to write out the definition of $x \in \{n^2 \mid n \in \mathbb{N}\}$ and the definition of $x \in \{n^3 \mid n \in \mathbb{N}\}$. After rewriting, we would get this equivalent statement $\exists x\ (\exists n_1 \in \mathbb{N}\ (x = n_1^2) \wedge \exists n_2 \in \mathbb{N}\ (x = n_2^3))$. This statement is correct but seems not as clear as we would like it to be. Note that n_1 and n_2 can be different natural numbers, for example $64 = 8^2$ and $64 = 4^3$. Another equivalent statement would be $\exists n_1 \in \mathbb{N}\ \exists n_2 \in \mathbb{N}\ (n_1^2 = n_2^3)$ which also means that there is at least one element that is common to both sets.

Anything at all can be an element of a set ; thus, a set can be an element of another set. Note that the empty set $\varnothing$ is a set included in all possible sets. For example, consider A, B and C are three sets. We can form another set $\mathcal{F}$ whose elements are A, B and C. Therefore, $\mathcal{F} = \{A, B, C\}$. Note that it would be incorrect to say that an element of A is also an element of $\mathcal{F}$. Only A, B and C should be considered as elements of $\mathcal{F}$. Sets such as $\mathcal{F}$, whose elements are sets, are sometimes called **families of sets**.

$\mathcal{P}(A)$ is the notation used to express which elements are all subsets of the set A. This set $\mathcal{P}(A)$ is called the **power set** of set A and is defined as follows.

$\mathcal{P}(A) = \{x \mid x \subseteq A\}$

For example, if $A = \{a, b\}$, then $\mathcal{P}(A) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$ which means that the set A has four subsets.

Let us analyze the logical forms of a few statements.

1. $x \in \mathcal{P}(A)$

This statement means that x belongs to the set of all subsets of A which means that x is itself a subset of A. In other words, $x \subseteq A$. As seen before, this means that if y is an element of x, then y is also an element of A. Therefore, this statement is equivalent to $\forall y$ $(y \in x \longrightarrow y \in A)$

2. $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

This statement is equivalent to $\forall x$ $(x \in \mathcal{P}(A) \longrightarrow x \in \mathcal{P}(B))$ (definition of subset) then according to 1. , this statement is equivalent to $\forall x$ $(x \subseteq A \longrightarrow x \subseteq B)$ which is also equivalent to $\forall x$ $[\forall y$ $(y \in x \longrightarrow y \in A) \longrightarrow \forall y$ $(y \in x \longrightarrow y \in B)]$

3. $B \in \{\mathcal{P}(A) \mid A \in \mathcal{F}\}$

This statement means that B is an element of a set whose elements have the form $\mathcal{P}(A)$. For B to belong to this set, B must be equal to at least one element of the form $\mathcal{P}(A)$. In other words, $\exists A \in \mathcal{F}$ $(B = \mathcal{P}(A))$. $B = \mathcal{P}(A)$ means that B and $\mathcal{P}(A)$ have the same elements. This can be equivalent to writing $B \subseteq \mathcal{P}(A) \wedge \mathcal{P}(A) \subseteq B$ because for two sets to be equal, they have to be a subset of each other. Writing out the definition of those two subsets would result in an equivalent statement, but let us think differently. $B = \mathcal{P}(A)$ also means that the elements of B are subsets of A. In other words $\forall x$ $(x \in B \longleftrightarrow x \subseteq A)$ and writing out the definition of subset, we get $\exists A \in \mathcal{F}$ $\forall x$ $(x \in B \longleftrightarrow \forall y$ $(y \in x \longrightarrow y \in A))$

4. $x \in \mathcal{P}(A \cap B)$

This statement means that x is a subset of $A \cap B$ is equivalent to $\forall y$ $(y \in x \longrightarrow y \in A \cap B)$ which is equivalent to $\forall y$ $(y \in x \longrightarrow y \in A \wedge y \in B)$

5. $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$

This statement is equivalent to $x \in \mathcal{P}(A) \wedge x \in \mathcal{P}(B)$ which is equivalent to $\forall y$ $(y \in x \longrightarrow y \in A) \wedge \forall y$ $(y \in x \longrightarrow y \in B)$. As the universal quantifier distributes over the conjunction, the statement is also equivalent to $\forall y$ $((y \in x \longrightarrow y \in A) \wedge (y \in x \longrightarrow y \in B))$. Let us go further and see if we can have only one $y \in x$ in our final answer. Using conditional law, the statement is also equivalent to $\forall y$ $((y \notin x \vee y \in A) \wedge (y \notin x \vee y \in B))$ which is also equivalent (using distributive law) to $\forall y$ $(y \notin x \vee (y \in A \wedge y \in B))$. And now, we get our final answer using the condition law, $\forall y$ $(y \in x \longrightarrow (y \in A \wedge y \in B))$

The set of all common elements of all sets in $\mathcal{F}$ is called the **intersection of the family** $\mathcal{F}$ and is denoted $\bigcap \mathcal{F}$
The set of all elements that are in at least one set in $\mathcal{F}$ is called the **union of the family** $\mathcal{F}$ and is denoted $\bigcup \mathcal{F}$

For example, let $\mathcal{F} = \{\{1, 2, 3\}, \{2, 3\}, \{1, 2, 3, 4\}\}$. Then

$\bigcap \mathcal{F} = \{1,2,3\} \cap \{2,3\} \cap \{1,2,3,4\} = \{2,3\}$

$\bigcup \mathcal{F} = \{1,2,3\} \cup \{2,3\} \cup \{1,2,3,4\} = \{1,2,3,4\}$

Therefore, for an element to belong to $\bigcap \mathcal{F}$, this element **must belong to all sets** $A \in \mathcal{F}$. In other words,

$x \in \bigcap \mathcal{F}$ is equivalent to $\forall A \in \mathcal{F}(x \in A)$ which is equivalent to $\forall A(A \in \mathcal{F} \longrightarrow x \in A)$

On the other hand, for an element to belong to $\bigcup \mathcal{F}$, this element just needs to **belong to at least one of the sets** $A \in \mathcal{F}$. In other words,

$x \in \bigcup \mathcal{F}$ is equivalent to $\exists A \in \mathcal{F}(x \in A)$ which is equivalent to $\exists A(A \in \mathcal{F} \land x \in A)$

The set $\bigcap \mathcal{F}$ is therefore the set of all x that belong to all the sets $A \in \mathcal{F}$ which are the elements of $\mathcal{F}$ and the set $\bigcup \mathcal{F}$ is the set that reunites all x's that are in at least one set A with $A \in \mathcal{F}$. Both sets are defined as follows.

$\bigcap \mathcal{F} = \{x \mid \forall A \in \mathcal{F}(x \in A)\}$

$\bigcup \mathcal{F} = \{x \mid \exists A \in \mathcal{F}(x \in A)\}$

Let us once again analyze a few statements.

1. $x \in \bigcap \mathcal{F}$

As mentioned above, this is equivalent to $\forall A \in \mathcal{F}(x \in A)$

2. $\bigcap \mathcal{F} \not\subseteq \bigcup \mathcal{G}$

This statement is the negation form of $\bigcap \mathcal{F} \subseteq \bigcup \mathcal{G}$ which means that it is equivalent to $\neg(\bigcap \mathcal{F} \subseteq \bigcup \mathcal{G})$. Using the definition of being a subset, this statement is also equivalent to $\neg(\forall x\ (x \in \bigcap \mathcal{F} \longrightarrow x \in \bigcup \mathcal{G}))$. This statement is therefore (using the definition of an element in $\bigcap \mathcal{F}$ and $\bigcup \mathcal{G}$) equivalent to $\neg(\forall x\ (\forall A \in \mathcal{F}(x \in A) \longrightarrow \exists B \in \mathcal{G}(x \in B))$. Now, using the quantifier negation law, the statement also means that $\exists x\ \neg(\forall A \in \mathcal{F}(x \in A) \longrightarrow \exists B \in \mathcal{G}(x \in B))$. According to conditional laws the statement is also equivalent to $\exists x\ \neg(\exists A \in \mathcal{F}(x \notin A) \lor \exists B \in \mathcal{G}(x \in B))$. Using once again the negation law, $\exists x\ (\forall A \in \mathcal{F}(x \in A) \land \forall B \in \mathcal{G}(x \notin B))$.

3. $x \in \mathcal{P}(\bigcup \mathcal{F})$

This statement means that x belongs to the set of all subsets of $\bigcup \mathcal{F}$ which means that $x \subseteq \bigcup \mathcal{F}$ which is equivalent to $\forall y\ (y \in x \longrightarrow y \in \bigcup \mathcal{F})$. Our final answer is $\forall y\ (y \in x \longrightarrow \exists A \in \mathcal{F}(y \in A))$

4. $x \in \bigcup\{\mathcal{P}(A) \mid A \in \mathcal{F}\}$

This means that x is an element of at least one of the sets $\mathcal{P}(A)$, with $A \in \mathcal{F}$ and then an equivalent statement would be $\exists A \in \mathcal{F}(x \in \mathcal{P}(A))$. The fact that $x \in \mathcal{P}(A)$ means that $\exists A \in \mathcal{F}(x \subseteq A)$ and, finally, from the subset relation $x \subseteq A$, we get $\exists A \in \mathcal{F}(\forall y$ $(y \in x \longrightarrow y \in A))$

An alternative notation is sometimes used for the union of the intersection of an indexed family of sets. Suppose $\mathcal{F} = \{A_i \mid i \in I\}$, where each A_i is a set. Then $\bigcap \mathcal{F}$ would be the set of all elements common to the A_i's for $i \in I$, and this is also written as $\bigcap_{i \in I} A_i$. In other words,

$$\bigcap \mathcal{F} = \bigcap_{i \in I} A_i = \{x \mid \forall i \in I(x \in A_i)\}$$

Therefore,

$$\bigcup \mathcal{F} = \bigcup_{i \in I} A_i = \{x \mid \exists i \in I(x \in A_i)\}$$