Notes on How to Prove It - Chapter 1 : Sentential Logic

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1 Introduction

To truly understand mathematics on a deeper level, one must first master its fundamental language. This is the very purpose of Daniel J. Velleman's How to Prove It. In this first post, I share my notes on Chapter 1, "Sentential Logic," which lays the groundwork for all mathematical reasoning.

1.1 Deductive reasoning and logical connectives

A statement is a declarative sentence that is true or false, but not both. For example, Clara's eyes are black" is a statement that is true if her eyes are black and false otherwise. "Can I borrow your book?" is not a statement because it has no truth value; it is a question.
There are different connective symbols that allow us to write correct mathematical proofs:

- $\wedge$: this symbol means ***and***
- $\vee$: this symbol means ***or***
- $\neg$: this symbol means ***not***

Thus, if P and Q stand for two statements, then we will write $P \vee Q$ to stand for the statement P ***or*** Q , $P \wedge Q$ stands for P ***and*** Q , and $\neg P$ stands for ***not P***
The statement $P \vee Q$ is sometimes called *the disjunction of P and Q* , $P \wedge Q$ is called *the conjunction of P and Q* and $\neg P$ is called *the negation of P*

1.2 Truth tables

Truth tables are a fundamental tool used to determine the truth value of a complex logical proposition by testing all possible combinations.

The truth table for the statement $P \wedge Q$ is as follows.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

For the statement (**conjunction** of P and Q) $P \wedge Q$ to be true, both premises P and Q **have to be true.**

For example, consider the statement: *"It is raining and the sun is shining."*

Let P be the statement: *"It is raining."*

Let Q be the statement: *"The sun is shining."*

For the entire statement $P \wedge Q$ to be true, it must be both raining and sunny **at the same time**. If it is only raining, or if it is only sunny, the statement is false.

The statement (or **disjunction** of P and Q) $P \vee Q$ has a different truth table as shown as follows.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

For the statement $P \vee Q$ to be true, at least one of the premises, P or Q, must be true.

For example, consider the statement: "*I will have coffee or I will have tea.*"

Let P be the statement: "*I will have coffee.*"

Let Q be the statement: "*I will have tea.*"

For the entire statement $P \vee Q$ to be true, you don't need to do both. It is enough for one of the conditions to be met:

You have coffee (and no tea).

You have tea (and no coffee).

You have both coffee and tea.

The only time the statement is false is if you have neither coffee nor tea.

Note: *In mathematics,* ***or*** *always means inclusive* ***or***

And for the statement (**or negation**) $\neg P$ the truth table is once again different from the others as follows.

P	$\neg P$
T	F
F	T

For the statement $\neg P$ to be true, the original premise P must be false. The negation of a statement simply reverses its truth value.

For example, consider the statement: "*It is raining.*"

Let P be the statement: "*It is raining.*"

Let $\neg P$ be the statement: "*It is **not** raining.*"

If the statement "It is raining" is true, then its negation, "It is not raining," must be false. If the statement "It is raining" is false, then its negation must be true.

A great example was presented in the book:
Consider this deductive argument.

It will either rain or snow tomorrow

It is too warm for snow

Therefore, it will rain

If we let P stand for the statement "*It will rain tomorrow*" an Q for the statement "*It will snow tomorrow*" then we can represent the argument symbolically as follows:

$$\begin{array}{l} P \vee Q \\ \neg Q \\ \hline \therefore P \end{array}$$

(The symbol $\therefore$ means therefore)

An argument is valid when its conclusion is guaranteed to be true whenever all its premises are true. To check an argument's validity, you can use a truth table that includes every premise and the conclusion. If you can find even one row where all premises are true but the conclusion is false, the argument is invalid.
Two statements are considered equivalent if and only if they have the same truth value in every possible case. This means that no matter what the individual truth values of the variables (like P and Q) are, the final result for both formulas will always match.
A key way to prove this is by building a truth table for each formula. If the final columns of the two truth tables are identical, the formulas are equivalent.

Here are some equivalent formulas :

De Morgan's laws:

$\neg(P \vee Q)$ is equivalent to $\neg P \wedge \neg Q$

$\neg(P \wedge Q)$ is equivalent to $\neg P \vee \neg Q$

Commutative laws:

$P \land Q$ is equivalent to $Q \land P$

$P \lor Q$ is equivalent to $Q \lor P$

Associative laws:

$P \land (Q \land R)$ is equivalent to $(P \land Q) \land R$

$P \lor (Q \lor R)$ is equivalent to $(P \lor Q) \lor R$

Idempotent laws :

$P \land P$ is equivalent to P

$P \lor P$ is equivalent to P

Distributive laws :

$P \land (Q \lor R)$ is equivalent to $(P \land Q) \lor (P \land R)$

$P \lor (Q \land R)$ is equivalent to $(P \lor Q) \land (P \lor R)$

Absorption laws :

$P \lor (P \land Q)$ is equivalent to P

$P \land (P \lor Q)$ is equivalent to P

Double Negation law :

$\neg\neg P$ is equivalent to P

Formulas that are always true, such as $(P \lor \neg P)$, are called **tautologies**. On the other hand, formulas that are always false, such as $(P \land \neg P)$, are called **contradictions**

Tautology laws :

$P \land$ (a tautology) is equivalent to P

$P \lor$ (a tautology) is a tautology

$\neg$(a tautology) is a contradiction

Contradiction laws :

$P \wedge$ (a contradiction) is a contradiction

$P \vee$ (a contradiction) is equivalent to P

$\neg$(a contradiction) is a tautology

1.3 Variables and Sets

A set is a collection of objects, which are referred to as elements. The symbol $\in$ denotes the relationship "*is an element of*" or "*belongs to.*"
Two sets are considered equal if and only if they contain exactly the same elements. Sets can be defined using an elementhood test, which is a condition that determines whether an object belongs to the set.

For example :

$B = \{x \mid x \text{ is a prime number}\}$

In this notation, the set B consists of all elements x such that x is a prime number. The condition "x is a prime number", also written as a $P(x)$(a statement about x), serves as the elementhood test. We can use this test to determine membership: $2 \in B$ because 2 is a prime number, while $4 \notin B$ because 4 is not a prime number.
The simplest way to define a set is to list its elements explicitly within braces.
For example, the set B can also be specified as:

B $= \{2, 3, 5, 7, 11, ...\}$

In the statement $y \in \{x \mid x \text{ is a prime variable}\}$, y is a **free variable,** whereas x is a **bound variable** (or dummy variable). The free variable in a statement **stands for objects** (elements) that the statement says something about. Bound variables, on the other hand, are used as a convenience to help express an idea and should not be thought of as standing for any particular object.
A statement $P(x)$, containing a free variable x, may be true for some values of x and false for others. To separate the values of x that make $P(x)$ true from those that make $P(x)$ false, we could form the set of values of x that make the statement $P(x)$ true. This set is called "**the truth set of** $P(x)$ "
The set of all possible values for the free variable is called **the universe of discourse** for the statement. It is said that variables range over this universe.

Suppose A is the universe of discourse for the statement $P(x)$, then we can explicitly specify that for a set B,

B $= \{x \in A \mid P(x)\}$

This notation indicates that only elements of A are to be considered for elementhood in this set, and among elements of A, only those that pass the elementhood test will actually be in the set.

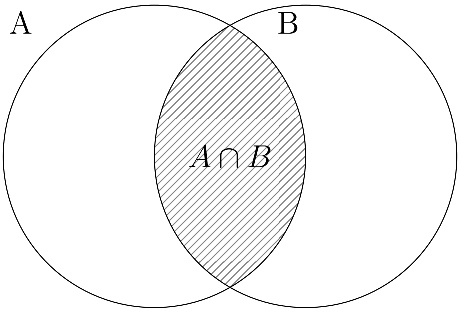
The statement $y \in \; \{x \in A \mid P(x)\}$ means the same thing as $y \in \text{A} \wedge P(y)$

There exists only one set that has no elements. This single set is called **the empty set** (or the null set) and is often denoted $\varnothing$

1.4 Operations on sets

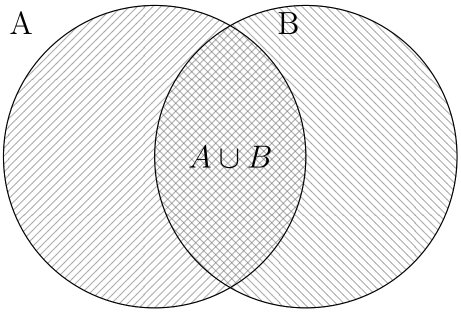
The **intersection** of two sets A and B is the set $A \cap B$ defined and represented using a Venn diagram as follows

$A \cap B = \{x \mid x \in A \text{ and } x \in B\} = \{x \mid x \in A \wedge x \in B\}$



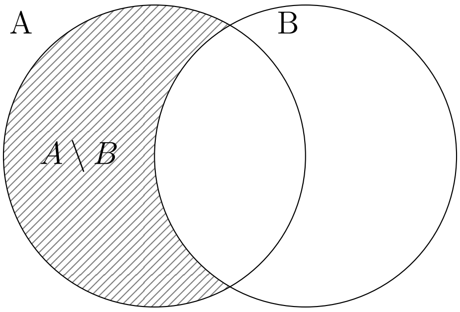
The **union** of A and B is the set $A \cup B$ defined and represented with Venn diagram as follows

$A \cup B = \{x \mid x \in A \text{ or } x \in B\} = \{x \mid x \in A \vee x \in B\}$



The **difference** of A and B is the set $A \setminus B$ defined and represented with Venn diagrams as follows

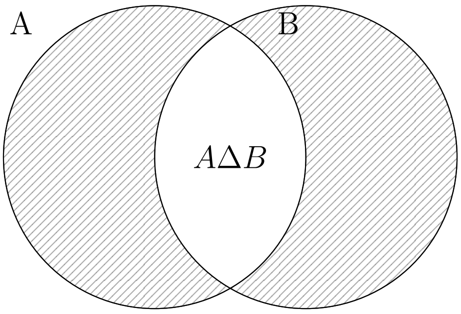
$A \setminus B = \{x \mid x \in A \text{ and } x \notin B\} = \{x \in A \wedge x \notin B\}$



The **symmetric difference** of A and B is defined and represented with Venn diagram as follows

$A \Delta B = \{x \mid (x \in A \text{ or } x \in B) \text{ and } x \notin A \cap B\}$

$A \Delta B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$

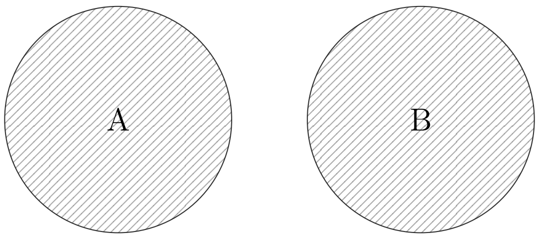


Suppose A and B are sets. We will say that A is a subset of B **if every element of A is also an element of B**. "A is a subset of B" is written as follows

$A \subseteq B$

Suppose A and B are two sets. A and B are said to be disjoint if they have **no elements in common**. In other words,

$A \cap B = \emptyset$. This can be represented with Venn diagram as follows



1.5 The Conditional and Biconditional connectives

Consider this example from the book:

If *today is Sunday,* ***then*** *I don't have to go to work today.*

Today is Sunday.

Therefore, I don't have to go to work today

It appears that there are some crucial words that occurs in the first premise

If....... , ***then*********

We therefore introduce a new logical connective "$\longrightarrow$", and write $P \longrightarrow Q$ to represent the statement "If P then Q ". This statement is sometimes called a conditional statement, with P as its antecedent and Q as its consequent.
If we let P stand for the statement "Today is Sunday " and Q for the statement "I don't have to go to work today ", then the logical form of the arguments would be

$$\begin{array}{l} P \longrightarrow Q \\ \underline{P \qquad\quad} \\ \therefore Q \end{array}$$

The statement "If P, then Q " is always true, with only one exception: It is false only when P is true and Q is false. For all other possibilities, the statement is considered true.
$P \longrightarrow Q$ is equivalent to $\neg P \vee Q$ (because they both have the same truth table)

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

$\neg P \vee Q$ is equivalent to $\neg P \vee \neg\neg Q$ (Double negation law)

which is equivalent to $\neg(P \wedge \neg Q)$ (De Morgan's law)

Condition laws

$P \longrightarrow Q$ is equivalent to $\neg P \vee Q$

$P \longrightarrow Q$ is equivalent to $\neg(P \wedge \neg Q)$

The formula $Q \longrightarrow P$ is called the converse of $P \longrightarrow Q$.
The formula $\neg Q \longrightarrow \neg P$ is called the contrapositive of $P \longrightarrow Q$ and it is equivalent to $P \longrightarrow Q$

Contrapositive law

$P \longrightarrow Q$ is equivalent to $\neg Q \longrightarrow \neg P$

The form "If P then Q " for a conditional statement can be expressed differently for the same idea in mathematics:

" P implies Q "

"Q, if P "

"P only if Q "

"P is a sufficient condition for Q "

"Q is a necessary condition for P "

The connective symbol " $\longleftrightarrow$ " is often used in mathematics to express that $P \longrightarrow Q$ and its converse $(Q \longrightarrow P)$ are both true. In other words
$P \longleftrightarrow Q$ is equivalent to $(P \longrightarrow Q) \wedge (Q \longrightarrow P)$ and can be thought as its abbreviation. A statement of the form $P \longleftrightarrow Q$ is called a biconditional statement, because it represents two conditional statements.
Here's its table truth

P	Q	$P \leftrightarrow Q$
V	V	V
V	F	F
F	V	F
F	F	V

By contrapositive law,

$P \longleftrightarrow Q$ is also equivalent to $(P \longrightarrow Q) \wedge (\neg P \longrightarrow \neg Q)$

$P \longleftrightarrow Q$ means "P if and only if Q " ,which can be written "P iff Q ", but $P \longleftrightarrow Q$ also means "P is a sufficient and necessary condition for Q ".