Handout 1
von Neumann Stability Analysis

Example 1:

Consider the following diffusion equation in 1D with periodic boundary conditions

$$\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2} \quad , \quad 0 < x < L \quad , \quad f(0) = f(L)$$

Using Forward Time Central Space (FTCS) discretization on a uniform mesh with node spacing Δx and time step Δt, we get the following discrete equation at node i

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} = \alpha \frac{f_{i+1}^n - 2f_i^n + f_{i-1}^n}{(\Delta x)^2} \tag{1}$$

which can be arranged into

$$f_i^{n+1} = \lambda f_{i+1}^n + (1 - 2\lambda) f_i^n + \lambda f_{i-1}^n \tag{2}$$

where $\lambda = \alpha \Delta t / (\Delta x)^2$.

Let f_e be the exact solution, f be the numerical solution with error ε be the error, satisfying the following relation

$$f_e = f + \varepsilon \quad \rightarrow \quad f = f_e - \varepsilon \tag{3}$$

Substituting this into Eq. (2) we get

$$\left(f_{e_i}^{n+1} - \varepsilon_i^{n+1}\right) = \lambda\left(f_{e_{i+1}}^{n} - \varepsilon_{i+1}^n\right) + (1 - 2\lambda)\left(f_{e_i}^{n} - \varepsilon_i^n\right) + \lambda\left(f_{e_{i-1}}^{n} - \varepsilon_{i-1}^n\right) \tag{4}$$

Since the exact solution f_e must satisfy the discrete equation, the same is true for the error ε, i.e.

$$\varepsilon_i^{n+1} = \varepsilon_i^n + \lambda(\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n) \tag{5}$$

The error at the new time level can be associated to the previous one as follows

$$\varepsilon_i^{n+1} = G \varepsilon_i^n \tag{6}$$

where G is the amplification factor, which is in general a complex constant. For stability, its magnitude should be less than or equal to one, so that the error does not grow unboundedly.

$$\text{For stability:} \quad |G| \le 1 \tag{7}$$

From Eq. (5) we see that ε_i^{n+1} not only depends on ε_i^n, but also on ε_{i-1}^n and ε_{i+1}^n. Therefore, ε_{i-1}^n and ε_{i+1}^n must be related to ε_i^n, so that Eq. (6) can be solved for G. This is done by expressing the error at node x_i at time level t^n, $\varepsilon(x_i, t^n)$, as a summation of sine and cosine functions or in general as the following complex Fourier series.

$$\varepsilon(x_i, t^n) = \sum_{k=-\infty}^{\infty} A_k^n e^{I\frac{\pi k}{L} x_i} \tag{8}$$

where A_k^n is the time dependent amplitude of component k, I is the complex number $\sqrt{-1}$ and $\frac{\pi k}{L}$ is the wave number giving various frequencies of individual error components. Note that

$$e^{Ia} = \cos(a) + I \sin(a) \tag{9}$$

Complex number notation is preferred because it yields simpler mathematics compared to using separate sine and cosine functions.

Due to the linearity of the PDE and the discretized Eq. (5), it is not necessary to consider all the components of Eq. (7). It is enough to work with a single node x_i and consider only the behavior of a single component of error given as

$$\varepsilon_i^n = A^n e^{I\frac{\pi k}{L}x_i} = A^n e^{I\theta x_i} \quad , \quad \text{where } \theta = \frac{\pi k}{L} \tag{10}$$

If all individual components at all grid nodes are stable, then the whole error will remain stable.

Substituting

$$\varepsilon_i^{n+1} = A^{n+1} e^{I\theta x_i}$$

$$\varepsilon_i^n = A^n e^{I\theta x_i}$$

$$\varepsilon_{i+1}^n = A^n e^{I\theta x_{i+1}} = A^n e^{I\theta x_i} e^{I\theta \Delta x}$$

$$\varepsilon_{i-1}^n = A^n e^{I\theta x_{i-1}} = A^n e^{I\theta x_i} e^{-I\theta \Delta x}$$

into Eqn. (5), we get

$$A^{n+1} e^{I\theta x_i} = \lambda A^n e^{I\theta x_i} e^{I\theta\Delta x} + (1-2\lambda) A^n e^{I\theta x_i} + \lambda A^n e^{I\theta x_i} e^{-I\theta\Delta x} \tag{11}$$

Cancel out the common $e^{I\theta x_i}$ terms and arrange into

$$A^{n+1} = A^n \left[\lambda e^{I\theta\Delta x} + (1-2\lambda) + \lambda e^{-I\theta\Delta x}\right] \tag{12}$$

Cancel out $e^{I\theta x_i}$ terms and arrange to get

$$\frac{A^{n+1}}{A^n} = \lambda e^{I\theta\Delta x} + (1-2\lambda) + \lambda e^{-I\theta\Delta x} \tag{13}$$

Using the identity $e^{I\theta\Delta x} + e^{-I\theta\Delta x} = 2\cos(\theta\Delta x)$

$$\frac{A^{n+1}}{A^n} = 1 + 2\lambda[\cos(\theta\Delta x) - 1] \tag{14}$$

This ratio is nothing but the amplification factor G, because

$$G = \frac{\varepsilon_i^{n+1}}{\varepsilon_i^n} = \frac{A^{n+1} e^{I\theta x_i}}{A^n e^{I\theta x_i}} = \frac{A^{n+1}}{A^n} \tag{15}$$

Therefore, for stability we need to satisfy

$$|G| \le 1 \quad \rightarrow \quad |1 + 2\lambda[\cos(\theta\Delta x) - 1]| \le 1 \tag{16}$$

This should hold for any θ value, i.e. any k value. This gives two conditions

- $1 + 2\lambda[\cos(\theta\Delta x) - 1] \le 1 \quad \rightarrow \quad 2\lambda \underbrace{[\cos(\theta\Delta x) - 1]}_{\le 0} \le 0$ which is always true (Note that D is positive)

- $-1 \leq 1 + 2\lambda[\cos(\theta\Delta x) - 1] \quad \rightarrow \quad \lambda[1 - \cos(\theta\Delta x)] \leq 1 \quad \rightarrow \quad \lambda \leq \frac{1}{1-\cos(\theta\Delta x)}$. The limiting minimum value of λ is obtained when $1 - \cos(\theta\Delta x)$ is maximized, i.e. when it becomes 2. Therefore, $\lambda \leq 1/2$ is the condition to be satisfied.

$$\lambda \leq \frac{1}{2} \quad \rightarrow \quad \frac{\alpha\Delta t}{(\Delta x)^2} \leq \frac{1}{2} \quad \rightarrow \quad \boxed{\Delta t \leq \frac{(\Delta x)^2}{2\alpha}}$$

This is the limiting time step to get a stable solution of the unsteady diffusion equation using FTCS discretization.

It is important to note that von Neumann stability analysis is valid under the following conditions

- the PDE under analysis is linear with constant coefficients. The method can be applied to non-linear PDEs by first linearizing them.
- the boundary condition effects are neglected. Here we assumed periodic boundary conditions to simulate an infinitely long domain. The effect of other types BCs on stability cannot be analyzed with this method, but it is not that critical because BCs are known to change the stability condition only minimally.

Example 2:

Let's use von Neumann method to find the stability limit of Backward Time Central Space (BTCS) applied to the same diffusion equation. The discrete equation is

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} = \alpha \frac{f_{i+1}^{n+1} - 2f_i^{n+1} + f_{i-1}^{n+1}}{(\Delta x)^2} \tag{17}$$

where the differences between FTCS are shown in red color. Arrange this equation as follows

$$-\lambda f_{i-1}^{n+1} + (1 + 2\lambda) f_i^{n+1} - \lambda f_{i+1}^{n+1} = f_i^n \tag{18}$$

which is an equation also satisfied by the error ε as shown below

$$-\lambda \varepsilon_{i-1}^{n+1} + (1 + 2\lambda) \varepsilon_i^{n+1} - \lambda \varepsilon_{i+1}^{n+1} = \varepsilon_i^n \tag{19}$$

Substitute

$$\varepsilon_i^{n+1} = A^{n+1} e^{I\theta x_i}$$

$$\varepsilon_{i+1}^{n+1} = A^{n+1} e^{I\theta x_i} e^{I\theta\Delta x}$$

$$\varepsilon_{i-1}^{n+1} = A^{n+1} e^{I\theta x_i} e^{-I\theta\Delta x}$$

$$\varepsilon_i^n = A^n e^{I\theta x_i}$$

into Eq. (19) to get

$$A^{n+1} e^{I\theta x_i} \left[-\lambda e^{-I\theta\Delta x} + 1 + 2\lambda - \lambda e^{I\theta\Delta x}\right] = A^n e^{I\theta x_i} \tag{20}$$

Cancel out the common $e^{I\theta x_i}$ and get the following amplification factor

$$G = \frac{A^{n+1}}{A^n} = \frac{1}{-\lambda e^{-I\theta\Delta x} + 1 + 2\lambda - \lambda e^{I\theta\Delta x}} \tag{21}$$

Using the identity $e^{I\theta\Delta x} + e^{-I\theta\Delta x} = 2\cos(\theta\Delta x)$

$$G = \frac{1}{1 + 2\lambda[1 - \cos(\theta\Delta x)]} \tag{22}$$

$1 - \cos(\theta\Delta x)$ is greater or equal to zero and λ is also greater than zero. Therefore, the denominator of Eq. (22) is always ≥ 1. Thus, the stability condition $|G| \leq 1$ is always satisfied. There is no restriction on the time step for stability. BTCS scheme is unconditionally stable for the diffusion equation.

Example 3:

Let's use von Neumann method to find the stability limit of FTCS applied to the following convection equation

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} = 0 \tag{23}$$

Discretizing with FTCS we get

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} + U\frac{f_{i+1}^n - f_{i-1}^n}{2\Delta x} = 0 \tag{24}$$

Defining $C = U\Delta t/\Delta x$, the solution and the error both satisfy the following equation

$$\varepsilon_i^{n+1} = \varepsilon_i^n - \frac{C}{2}(\varepsilon_{i+1}^n - \varepsilon_{i-1}^n) \tag{25}$$

Substituting

$$\varepsilon_i^{n+1} = A^{n+1}e^{I\theta x_i}$$

$$\varepsilon_{i+1}^n = A^n e^{I\theta x_i} e^{I\theta\Delta x}$$

$$\varepsilon_{i-1}^n = A^n e^{I\theta x_i} e^{-I\theta\Delta x}$$

into Eqn. (25), simplifying and arranging, we get the following amplification factor

$$G = \frac{A^{n+1}}{A^n} = 1 - \frac{C}{2}\left(e^{I\theta\Delta x} - e^{-I\theta\Delta x}\right) \tag{26}$$

using the relation $e^{I\theta\Delta x} - e^{-I\theta\Delta x} = 2I\sin(\theta\Delta x)$

$$G = 1 - C\,I\sin(\theta\Delta x) \tag{27}$$

This time the amplification factor is a complex number. The magnitude of G is

$$|G| = \sqrt{1 + C^2\sin^2(\theta\Delta x)}$$

which is always greater than 1. Therefore, the stability condition of $|G| \leq 1$ can never be satisfied. FTCS scheme is unconditionally unstable for the convection equation. FTCS scheme should not be used to solve pure convection problems.

Example 4:

Let's use von Neumann method to find the stability limit of forward time, first order upwind space scheme applied to the convection equation of the previous example. For $U > 0$, discretized equation is

$$\frac{f_i^{n+1} - f_i^n}{\Delta t} + U \frac{f_i^n - f_{i-1}^n}{\Delta x} = 0 \tag{28}$$

where the differences compared to the previous example are shown in red color.

Using $C = U\Delta t/\Delta x$, both the solution and the error satisfy the following equation

$$\varepsilon_i^{n+1} = \varepsilon_i^n - C(\varepsilon_i^n - \varepsilon_{i-1}^n) \tag{29}$$

Substituting

$$\varepsilon_i^{n+1} = A^{n+1} e^{I\theta x_i}$$

$$\varepsilon_i^n = A^n e^{I\theta x_i}$$

$$\varepsilon_{i-1}^n = A^n e^{I\theta x_i} e^{-I\theta\Delta x}$$

into Eqn. (29), simplifying and arranging, we get the following amplification factor

$$G = \frac{A^{n+1}}{A^n} = 1 - C\left(1 - e^{-I\theta\Delta x}\right) \tag{30}$$

using the relation $e^{-I\theta\Delta x} = \cos(-\theta\Delta x) + I\sin(-\theta\Delta x)$

$$= \cos(\theta\Delta x) - I\sin(\theta\Delta x)$$

$$G = 1 - C[1 - \cos(\theta\Delta x) + I\sin(\theta\Delta x)]$$

$$= \underbrace{1 - C + C\cos(\theta\Delta x)}_{\text{Real part}} + I \underbrace{C\sin(\theta\Delta x)}_{\text{Imaginary part}} \tag{31}$$

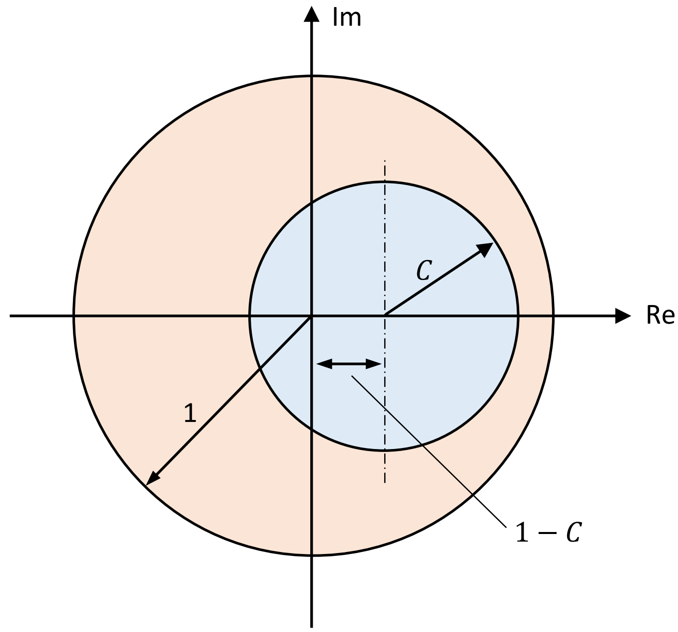
Again, the amplification factor is a complex number. It is easier to find the condition for $|G| \le 1$ graphically. Defining $\beta = \theta\Delta x$, the amplification factor becomes

$$G = 1 - C + C[\cos(\beta) + I\ \sin(\beta)] \tag{32}$$

For different values of β this equation gives a circle in the complex plane, with its center is at $(1 - C, 0)$ and radius C. It is plotted below in pink color. For stability this circle should be inside the larger blue circle, which shows the stable region $|G| \le 1$. This happens if $C \le 1$. Therefore, the stability condition is

$$C \le 1 \quad \rightarrow \quad \frac{U\Delta t}{\Delta x} \le 1 \quad \rightarrow \quad \boxed{\Delta t \le \frac{\Delta x}{U}}$$

The number C is known as the Courant number and the stability conditions in the form $C \le C_{max}$ is known as the Courant-Freidrichs-Lewy (CFL) condition. In this example the limiting C_{max} value is 1.



Example 5:

Let's determine the stability limit of FTCS applied to the following convection-diffusion equation

$$\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x} = \alpha\frac{\partial^2 f}{\partial x^2} \tag{33}$$

Discretizing with FTCS it is possible to show that the error should satisfy the following equation

Defining, the solution and the error both satisfy the following equation

$$\varepsilon_i^{n+1} = \varepsilon_i^n - \frac{C}{2}(\varepsilon_{i+1}^n - \varepsilon_{i-1}^n) + \lambda(\varepsilon_{i+1}^n - 2\varepsilon_i^n + \varepsilon_{i-1}^n) \tag{34}$$

where $\lambda = \alpha\Delta t/(\Delta x)^2$ and $C = U\Delta t/\Delta x$

Substituting

$$\varepsilon_i^{n+1} = A^{n+1}e^{I\theta x_i}$$

$$\varepsilon_i^n = A^n e^{I\theta x_i}$$

$$\varepsilon_{i+1}^n = A^n e^{I\theta x_i}e^{I\theta\Delta x}$$

$$\varepsilon_{i-1}^n = A^n e^{I\theta x_i}e^{-I\theta\Delta x}$$

into Eq. (34), simplifying and arranging, we get the following amplification factor

$$G = \frac{A^{n+1}}{A^n} = \left[1 - \frac{C}{2}\left(e^{I\theta\Delta x} - e^{-I\theta\Delta x}\right) + \lambda e^{I\theta\Delta x} - 2\lambda + \lambda e^{-I\theta\Delta x}\right] \tag{35}$$

Using the relations $e^{I\theta\Delta x} - e^{-I\theta\Delta x} = 2I\sin(\theta\Delta x)$ and $e^{I\theta\Delta x} + e^{-I\theta\Delta x} = 2\cos(\theta\Delta x)$

$$G = 1 - 2\lambda + 2\lambda\cos(\theta\Delta x) - IC\sin(\theta\Delta x) \tag{36}$$

The requirement of $|G| \leq 1$ gives the following two conditions

$$C^2 \leq 2D \leq 1 \qquad \rightarrow \qquad \left(\frac{U\Delta t}{\Delta x}\right)^2 \leq \frac{2\alpha\Delta t}{(\Delta x)^2} \leq 1 \tag{37}$$

These conditions can be arranged as

$$\boxed{\Delta t \leq \frac{2\alpha}{U^2} \quad , \qquad \Delta t \leq \frac{(\Delta x)^2}{2\alpha}} \tag{38}$$

The first condition is the limiting one for a convection dominated problem, i.e. high U and low α. These are the actual challenging problems. The second condition is the limiting one for a diffusion dominated problem, i.e. high α and low U.

Question: In the limit of pure convection and pure diffusion cases, are these results consistent with those obtained in Examples 1 and 3?