Rings and Fields : A programmer's perspective

Exercise answers

1), The following is not necessarily true of a ring :

$$(a \square b) \star c = c \star (a \square b)$$

By a ring's definition, why is the above statement not always true? What assumption is it making about the ring?

This assumes the property of commutativity for multiplication which isn't required for a ring.

2), A ring with only $\{0\}$ under addition and multiplication is a trivial ring. Use the definition of a ring to show that the trivial ring is in fact a ring.

Firstly we can find the structure of the set under the operations.

Addition operation:

- The set is closed due to the operation always giving the 0 element i.e $0+0=0$
- It is associative

$$0+(0+0) = 0+(0+0)$$

- The identity is the single element 0

$$0+0=0$$

- Inverse - $0+0=0$
- In addition, the group is abelian due to the operation being commutative

Multiplication operation:

- The set is closed under multiplication $0 \times 0 = 0$
- It is associative
  $(0 \times 0) \times 0 = 0 \times (0 \times 0)$
- The identity is the same
  $0 \times 0 = 0$
- It dose not however have an inverse under multiplication due to there only being the single zero element so there is no distinct element to multiply by to get the identity

Therefore, we can consider the set under multiplication to be a monoid.

Lastly the properties of distribution

$$a \times (b + c) = (a \times b) + (a \times c)$$

$$(a + b) \times c = (a \times c) + (b \times c)$$

Substituting 0 shows that these are trivially held. Therefore it is a ring.

3), Square matrices of real numbers under addition and multiplication is a ring. Demonstrate this to be the case. Think carefully about what the identity elements are and whether an inverse always exists.

Matrices are closed under addition and multiplication. The sum of any $n \times n$ matrices $A + B$ gives another $n \times n$ matrix. The sum of the elements (i.e. real numbers) are themselves real numbers so is closed. The product $A \times B$ of $n \times n$ matrices is also an $n \times n$ matrix

Associativity is obeyed

Addition: $\forall A, B, C \quad (A+B)+C = A+(B+C)$

Multiplication: $\forall A, B, C \quad (A \times B) \times C = A \times (B \times C)$

Identity:

Addition: The identity matrix under addition is the zero matrix where every element is 0.

Multiplication: The identity matrix under multiplication has diagonal elements of 1 and the rest 0.

Inverse: There always exist a matrix B where $A + (-B) = I_0$ (I_0 being the zero matrix)

The multiplicative inverse is more complex as it will only exist sometimes. Specifically, there will only be an inverse iff the determinate is non zero.

Distribution:

$$\forall A, B, C \Rightarrow A \times (B + C) = (A \times B) + (A \times C)$$
$$\Rightarrow (A + B) \times C = (A \times C) + (B \times C)$$

4), There is no trivial field. It isn't possible to construct a trivial field using one element. Why is that the case?

A one element construct cannot have a distinct additive and multiplicative identity so therefore cannot be a field.