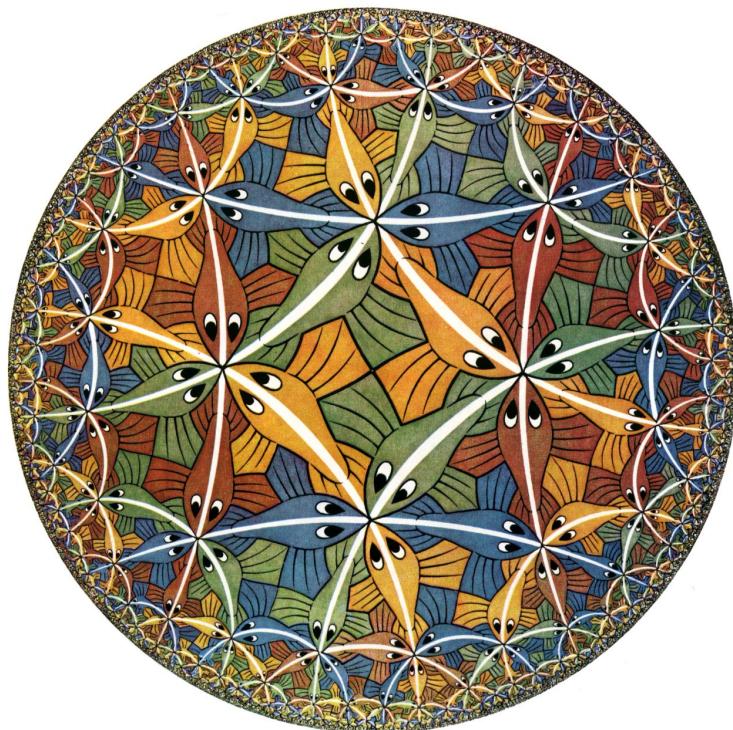


# **Hyperbolic Browsing**

Scalable Hierarchy Browsing in Hyperbolic Space  
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## **Abstract**

Hyperbolic space can be used in conjunction with projections to Euclidean space as a basis for visualising and exploring large hierarchies. This work will first present a brief history and overview of hyperbolic geometry, including projections and why they are useful for hierarchy visualisation. The work will then survey existing hyperbolic browsing UI components and applications. The work will conclude with a look at d3-hypertree, a new JavaScript component which implements hyperbolic tree browsing.

Oct 2018, updates on GitHub  
Demo: [hyperbolic-tree-of-life.github.io](https://hyperbolic-tree-of-life.github.io)  
Code: [glouwa.github.io/d3-hypertree](https://glouwa.github.io/d3-hypertree)  
Title image: Circle Limit III, M.C. Escher 1959.

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# **List of Listings**



# Chapter 1

## Hyperbolic Geometry

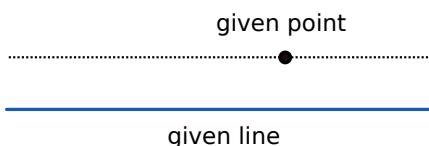
### 1.1 Euclid

Ancient greek produced mathematical books about geometry for hundreds of years, before Euclid summarized greek knowloedge in his book 'Elements'. This book enabled technology to flourish, was the second most replicated book before 15th century, and a main text book until the 19th century. Its applications ware numerous, but constrained to the five axioms stated by Euclid [Euclid 300BC]:

1. Any two points lie on a unique line.
2. Any straight line can be continued indefinitely in either direction.
3. You can draw a circle of any centre and any radius.
4. All right angles are equal.
5. If a straight line, crossing another two straight lines  $L, L'$ ,  
makes angles  $\alpha, \beta$  with  $L, L'$  on one side,  
and if  $\alpha + \beta < \pi$ ,  
then  $L, L'$  if extended sufficiently far meet on that same side.

Unfortunately the 5th axioms formulation is not best suited for our considerations, we will use Playfair's version formulated and proven to be equivalent in 1795 [Playfair 1795] [fig1.1]:

5. In a plane, through a point not on a given straight line,  
at most one line can be drawn that never meets the given line.



**Figure 1.1:** Playfair's 5th axiom shows clearly that there is only one line not crossing the given line in euclidean space. [This<sup>CC4</sup>]

Through  $P \notin L$  there is **one** line not meeting  $L$

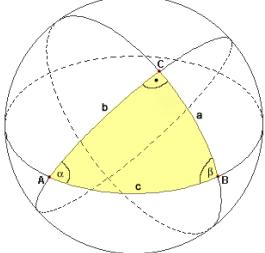
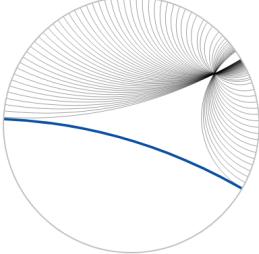
(1.1)

## 1.2 Lobachevsky Bolyai

Although euclidean geometry was widely used and understood for more than 2000 years, its non euclidean derivations were not discovered before 1823. This is reasonable because the theory of complex numbers is necessary to formulate this kind of mathematics. Lobachevsky and Bolyai discovered independently that a slight modification of the 5th axiom has no influence of the correctness of all 28 theorems. The modifications of the 5th axiom are [Lobachevsky 1823] [Bolyai 1823]:

- Through  $P \notin L \in \mathbb{E}$  there is **one** line not meeting  $L$
- Through  $P \notin L \in \mathbb{S}$  there is **no** line not meeting  $L$
- Through  $P \notin L \in \mathbb{H}$  there are **infinitely many** lines not meeting  $L$

This axioms can not be fulfilled in euclidean  $\mathbb{R}^2$  or  $\mathbb{R}^3$ , but we can find a surface embedded in  $\mathbb{R}^3$  fulfilling them. This embedding was found early on for spherical geometry, and 1955 for hyperbolic geometry. Table [tab1.1] shows possible explanations how to fulfill the modified axioms. But one of them is a embedding and one a projection. Embeddings are conform to a sheet of paper in reality, a  $\mathbb{R}^2$  embedded in  $\mathbb{R}^3$ . This is the most intuitive way to approach such a geometry. Projections and models will be discussed later in section [1.4], to get a basic understanding, i suggest to answer the question how a hyperbolic plane can be embedded in  $\mathbb{R}^3$  first.

	Spherical geometry	Hyperbolic geometry
5th axiom	Through $P \in L$ there is no line not meeting $L$	Through $P \in L$ there are infinitely many lines not meeting $L$
Embeddings	The surface of a sphere is an isometric embedding of $\mathbb{S}^2$ in $\mathbb{R}^3$ . Great circles always intersect.  [Peter Mercator <small>PublicDomain</small> ]	We do not know yet how to embed a $\mathbb{H}^2$ in $\mathbb{R}^3$ , and this would not visualize the 5th axiom.
Projections	$\mathbb{S}^2 \mapsto \mathbb{R}^2$ map projections are an application of spherical geometry.  [Michael Glatzhofer <small>Obs</small> ]	A projection from $\mathbb{H}^2 \mapsto \mathbb{R}^2$ shows that the 5th axiom is fulfilled. Hyperbolic projections will be discussed in section 1.4.  [Trevorgoodchild <small>PublicDomain</small> ]

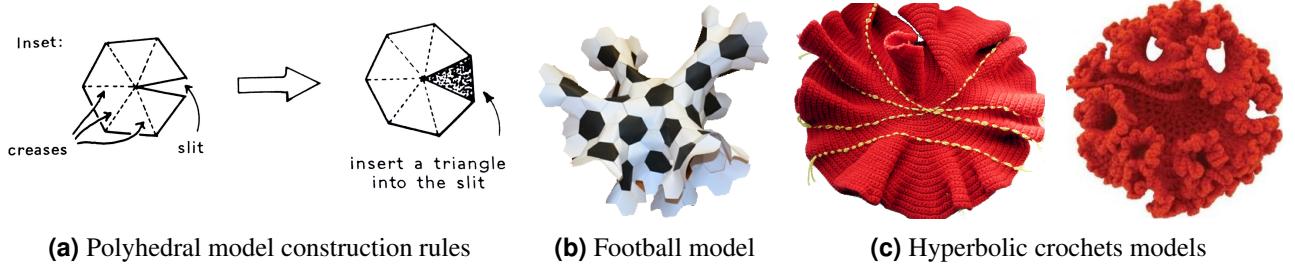
**Table 1.1:** Early applications of Lobachevsky's axioms.

## 1.3 Embeddings

Embeddings are the ideas of placing a space within a space, like a sheet of paper in reality. Each space has its own coordinates system, which can have a different number of dimensions, and each point of the embedding can be described by both coordinate systems. To map coordinates from one space to the other is called embedding. Note that a sheet of paper has one dimension less than its containing space. The paper is considered infinitely thin, to adopt correctly to the fact that a point on the paper is given by only two coordinates. We can also imagine such a embedded plane as surface of a sphere  $\mathbb{S}^2$ . The embedding of a  $\mathbb{S}^2$  in  $\mathbb{R}^3$  in fulfillment of the 5th axiom is easy to find, all great circles on the sphere cross each other at least once. Using a sphere instead of a sheet of paper, and accepting great circles are lines, shows that this kind of geometry fulfills all axiom. This intuitive handling was maybe the reason why spherical geometry did get more attention first, and lead to improvements in mathematics of spherical projections like map projections. But wait, circles are considered to be a lines? (Flat earth alarm?)

Euclid's axioms use the term straight lines, but as visible in table [tab1.1] these lines are not straight. So what is a straight line? Mathematicians answered this question for spherical, and hyperbolic geometry. A line<sup>Wiki</sup> needs to be only locally straight, that means in the coordinate system of the embedded space, to be a line in terms of euclid's axioms. On earth, the shortest line between two points on the surface is always a part of a great circle. For spherical geometry that means a line which is locally straight like one of the great circles on a sphere is sufficient. A straight line in hyperbolic space must be locally straight and orthogonal to the edge of the plane. This was mathematically proven by lobachevsky. But this knowledge is still not enough to find an embedding of the hyperbolic plane in  $\mathbb{R}^3$ . [Weeks 1985]

Hyperbolic geometry was not widely recognised, since no known embedding limits its applications. A problem which remained until 1955. Hilbert showed 1899 that it is not possible to place a full hyperbolic plane in  $\mathbb{R}^3$  [Hilbert 1899], and finally 1955, Kuiper showed that there is a  $C^1$  isometric embedding for  $\mathbb{H}^2$  in  $\mathbb{R}^3$  [Kuiper 1955]. This lead to a theory describing how to construct a hyperbolic plane  $\mathbb{H}^n$  embedded  $\mathbb{R}^{n+1}$  [fig1.2]. One way to construct a  $\mathbb{H}^2$  embedding in  $\mathbb{R}^3$  is to put together 7 triangles at every vertex, this is called the polyhedral model, or schlafly  $\{7, 3\}$  tiling [Taimina 2009] [1.4.3]. By adding additional space in comparison to a flat  $\{3, 6\}$  tiling the plane gets 'curly', and all postulated metrics of hyperbolic geometry can be measured using the given line definition.



**Figure 1.2:** Embedding construction rules for  $\mathbb{H}^2$  in  $\mathbb{R}^3$ , and handcrafted models.

[Jeffrey R. Weeks<sup>Book</sup>] [Frank Sottile<sup>CC3</sup>] [Daina Taimina<sup>web</sup>]

With this knowledge it is possible for humans to analyse hyperbolic planes in an intuitive way, by using paper models. But such a hyperbolic paper plane is fragile in handling and not suited for measurements. Using knitted planes is an improvement by damia introduced in 2001, and regonicsed as an important tool for mathematicians. It allows to identify straight lines by simply stretching the crochet, if it folds properly, the line between the handle points is straight. Also measurements can be taken of this line [Henderson and Taimina 2001].

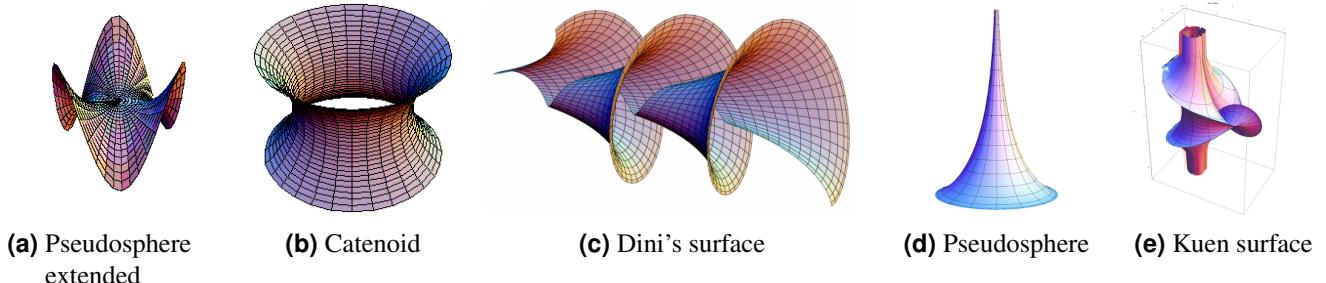
Now we have seen how a hyperbolic plane looks like and are able to identify objects in reality to which this kind of mathematic can be applied. Note, Kuiper showed it is only possible to embed parts of  $\mathbb{H}^n$  in  $\mathbb{R}^{n+1}$ . Embeddings as visualisations are therefore possible for  $\mathbb{H}^2$  and  $\mathbb{H}^1$ , for humans and other 3-dimensional beings.  $\mathbb{H}^2$  embeddings occur in nature, but are not relevant for our purposes.



**Figure 1.3:** Real objects approximating  $\mathbb{H}^2$  embedded in  $\mathbb{R}^3$ .

Each of these objects in [fig1.3] is only a part of a hyperbolic plane, according to Hilbert's theorem. If we would apply our construction rules further on, we would have to continue the construction at the border of the planes. 'Curls' would get smaller and smaller, until they need more space than available in  $\mathbb{R}^3$ . Does this hemisphere really converge to a sphere? Hilbert really? [Hilbert 1899]

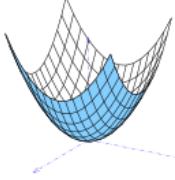
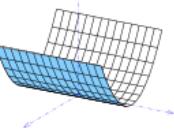
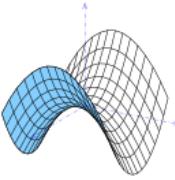
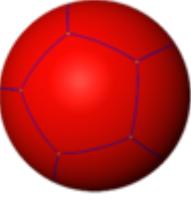
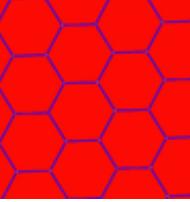
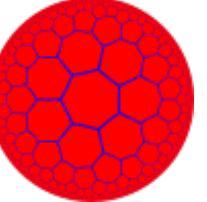
The walnut plane in [fig1.3d] stops extending after less iterations, thus has a different appearance. This is related to curvature and curvature adopted layouts. The objects are just approximations of the hyperbolic plane, this is also visible at the thickness of the walnut, which is not considered at all.



**Figure 1.4:** Surfaces with constant negative curvature.

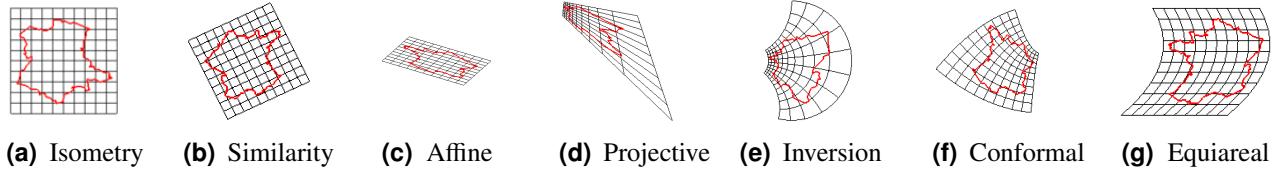
[C.T.J. Dodson<sup>web</sup>] [David Dumas<sup>web</sup>] [Wiki<sup>CC0</sup>] [Wiki<sup>CC4</sup>]

The images in [fig1.3] do not show all surfaces to which hyperbolic geometry can be applied. In fact, it can be applied to all surfaces with constant negative curvature. [fig2.8] shows a selection of surfaces with this property. [fig1.3a] is the mathematical equivalent of the images in [fig1.3], note that also this surface is just a part of  $\mathbb{H}^2$ , and when extended it will get more 'curls' and result in a lettuce-like surface. [Cannon et al. 1997] [Series 2008]

	Spherical	Euclidean	Hyperbolic
Parallels	through $P \in L$ there is no line not meeting $L$	through $P \in L$ there is a unique line not meeting $L$	through $P \in L$ there are infinitely many lines not meeting $L$
Curvature	$> 0$	$0$	$< 0$
Model	$\mathbb{S}$	$\mathbb{E}$	$\mathbb{D}$ or $\mathbb{H}$
Lines	arcs of great circles	euclidean lines	arcs orthogonal to boundary
Applicable to	 also sphere, ellipsoid... <a href="#">Wiki</a> <a href="#">[Ag2gaeh<sup>CC4</sup>]</a>	 also plane, cylinder... <a href="#">Wiki</a> <a href="#">[Ag2gaeh<sup>CC4</sup>]</a>	 also hyperboloid... <a href="#">Wiki</a> <a href="#">[Ag2gaeh<sup>CC4</sup>]</a>
Angle sum of $\Delta$	$> \pi$	$\pi$	$< \pi$
Circumference of $\circ$	$2\pi \sin(r)$	$2\pi r$	$2\pi \sinh(r)$
Area of $\circ$	$4\pi \sin^2(r/2)$	$\pi r^2$	$4\pi \sinh^2(r/2)$
Embedding [1.3]	 Schläfli $\{5, 3\}$ tiling [Tomruen <sup>PublicDomain</sup> ]	 Schläfli $\{6, 3\}$ tiling [Lasunncy <sup>CC4</sup> ]	 Schläfli $\{x, y\}$ tiling [Serfass <sup>shinyapps</sup> ]
Projection [1.4.4]	 Spherical Lagrange projection. [Michael Glatzhofer <sup>Obs</sup> ]	transformation	 Schläfli $\{7, 3\}$ tiling on poincaré disk [Parsly Taxel <sup>PublicDomain</sup> ]

**Table 1.2:** Summary of curved space [Series 2008] [Cannon et al. 1997]. For details on tilings see [1.4.3].

## 1.4 Transformations, Projections and Hyperbolic Models



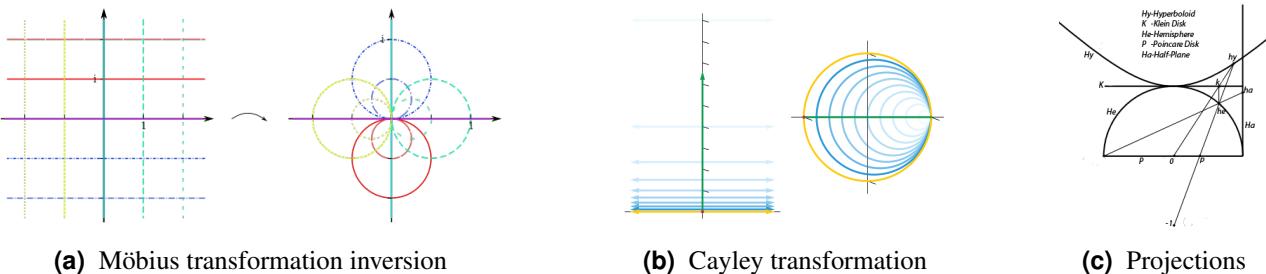
**Figure 1.5:** Transformation and projection classes. [Robert FERREOL *PublicDomain*]

Transformations<sup>Wiki</sup>, coordinatesystems<sup>Wiki</sup>, projections<sup>Wiki</sup> and tilings [1.4.3] are the base knowlege to compare hyperbolic models [1.4.4]. Projections map a Point from one space to another, they were the first applications of hyperbolic geometry in physics in late 1800. Tilings are helpfull tools to discuss layout properties.

$$\begin{aligned} \text{Transformation: } & \mathbb{A}^n \mapsto \mathbb{A}^n \\ \text{Projection: } & \mathbb{A}^n \mapsto \mathbb{B}^m \end{aligned} \tag{1.3}$$

Transformations map a point to a point in the same space, but projections can do that too and are in this case comparable to transformations. Möbis Transformations map from a riemann sphere to it self. They are inversion transformations and the mathematical foundation of the poincaré model. The following components of a möbius transformation are valid möbius tranformations it self: inversion  $f(z) = 1/z$ , translation  $f(z) = z+b$ , rotation  $f(z) = az$ ,  $|a| = 1$ . All möbius transformations have a defined inverse, and are in general cool. Caylay transformations map the upper half plane to the unit disk, this is usefull to switch between poincare disk and ponacare half plane model [J. O. Lamping et al. 2001]<sup>Pat</sup>.

$$\begin{aligned} \text{Caylay: } f(z) &= \frac{z-i}{z+i} \\ \text{Möbius: } f(z) &= \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C} \cup \{\infty\}, \quad ad - bc \neq 0 \end{aligned} \tag{1.4}$$



**Figure 1.6:** Some new transformations and projections. [Chrislb<sup>CC3</sup>] [KSmrq<sup>CC3</sup>] [Selfstudier *PublicDomain*]

Next step will be to take a brief look on available coordinate systems and on the poincare model [1.4.2]. It serves us as example of a model to get a better understanding of what a model differs from a projection. With the help of tilings we will finally compare the main models of hyperbolic geometry from 1800 physics [1.4.4] [1.4.5], and show how they can be applied for hierarchy visualisation [1.5].

### 1.4.1 Coordinate Systems

There are many hyperbolic coordinate systems<sup>Wiki</sup>, Polar, Axial and Cartesian-style coordinate system, Lobachevsky, Horocycle-based and Quadrant coordinate systems, and model based coordinate systems. Interesting for hierarchy visualisation is the horocycle<sup>Wiki</sup> based system, i suppose it is usefull to implement schläfli tiling<sup>Wiki</sup> like layouts. Model based system like Beltrami, Weierstrass, Poincaré coordinate systems are actually not used in the according model, but representing it, like a reduced version of the model.

We will use polar coordinate systems for our discussion. A model is a theory of the relations of two spaces, therefore models define a coordinate system for both. Many models use complex numbers, the poincaré model allows only values on the complex unit disk, in both spaces [1.4]. The origin is located in center of embeddings, like in all azimuthal projections as the poincare disk model.

### 1.4.2 Poincaré Model

The poincaré model might be considered famouse, or at least its applications like Eschers art and Einsteins relativity. This and the fact that [J. Lamping and R. Rao 1996] used it for their work is the reason why it is the first model we discuss. Polar coordinates are used, projecting the hole hyperbolic to the euclidean unit disk, making it a finite model [1.4.4].

$$\begin{aligned} \text{Hyperbolic Coordinate } z_h &\in \mathbb{C} \cup \{\infty\} \cup \mathbb{H} \quad |z_h| < 1 \\ \text{Euclidean coordinate } z_e &\in \mathbb{C} \cup \{\infty\} \cup \mathbb{E} \quad |z_e| < 1 \end{aligned} \quad (1.5)$$

The poincaré model is a azimuthal and stereographic projection, and therefore conformal and great circles intersect in the origin. Geodesics<sup>Wiki</sup> are arcs in the image, the poincaré disk, and intersecting the unitcircle perpendicular [fig1.1]. Circles stay circles, althoug the center of the circle in the euclidean image is not in the center of the euclidean circle. Parameter  $P$  defines the hyperbolic point maped to the center of the image, we will call it 'focus point' in context of hierarchy visualisation.  $\theta$  is automatically set at some implementations for rotation compensation. The inverse is given as möbius inverse [1.4].

$$\begin{aligned} \text{Rotation: } \theta &\in \mathbb{C} \quad |\theta| = 1 \\ \text{Translation: } P &\in \mathbb{C} \quad |P| < 1 \\ \text{Transformation: } \langle P, \theta \rangle & \\ z_e &= \frac{\theta z_h + P}{1 + \bar{P}\theta z_h} \\ \text{Inverse: } \langle -\bar{\theta}P, \bar{\theta} \rangle & \\ z_h &= \frac{\bar{\theta}z_e - \bar{\theta}P}{1 + -\bar{\theta}P\bar{\theta}z_e} \end{aligned} \quad (1.6)$$

Vector based systems like SVG are using arcs or polylines to draw the potentially distorted image of a geodesic. Geo projection systems need to solve this problem too, and did, [d3-geo]<sup>Web</sup> uses adaptive sampling to interpolate a polyline. The poincaré model provides a analytical solution to calculate the circle center of the arc segment of a given line  $\overline{AB}$ ,  $A, B \in \mathbb{C} \in \mathbb{E}$ .

$$\begin{aligned} d &= \Re(A)\Im(B) - \Re(B)\Im(A) \\ \text{Arc center } c_e &= \frac{i}{2} \frac{A(1 + |B|^2) - B(1 + |A|^2)}{d} \\ \text{Orientation} &= \begin{cases} \text{clockwise,} & \text{if } d > 0 \\ \text{counterclockwise,} & \text{otherwise} \end{cases} \end{aligned} \quad (1.7)$$

For mouse interactions like panning and pinching as well as animations the composition operator can be derived from the möbius transformation property. When implementing a hyperbolic browser, always keep in mind that a transformation  $\langle -z_h, 1 \rangle$  will place the point  $z_h$  in center of the projection image.  $\theta$  can be used for möbius homothety, but they tend to cause motion sickness.

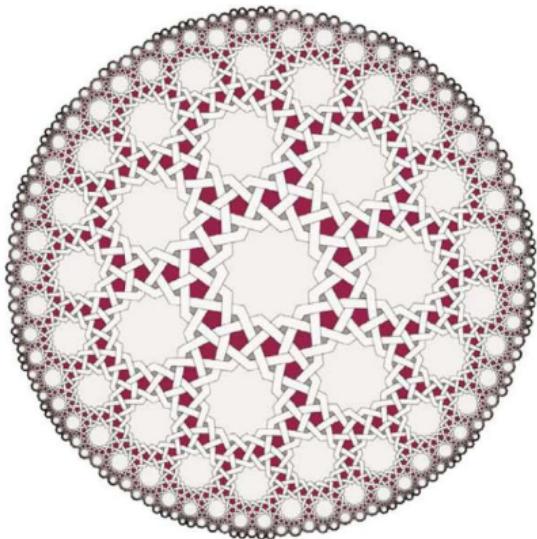
$$\begin{aligned} \text{Composition: } & \langle P_1, \theta_1 \rangle \circ \langle P_2, \theta_2 \rangle = \left\langle \frac{\theta_2 P_1 + P_2}{\theta_2 P_1 \overline{P}_2 + 1}, \frac{\theta_1 \theta_2 + \theta_1 \overline{P}_1 P_2}{\theta_2 P_1 \overline{P}_2 + 1} \right\rangle \\ \text{Identity transformation: } & \langle 0, 1 \rangle \\ \text{Focusing } z_h: & \langle -z_h, 1 \rangle \end{aligned} \quad (1.8)$$

### 1.4.3 Tilings

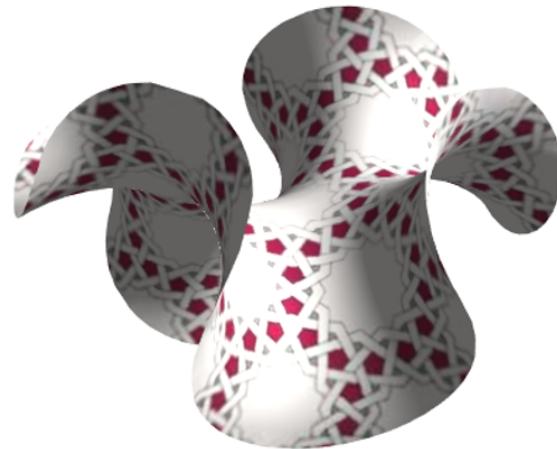
Tiling a.k.a. tessellations split a plane in one or more shapes without gaps or overlaps. They will help us to compare projections, and are closely related to most  $\mathbb{H}^2$  layouts [1.5]. We will only use regular tilings, this means only one shape is used to fill the plane. This implies that tiles, the instances of this shapes have all the same area in hyperbolic space, but not necessarily in projection images. Isometric embeddings are an intuitive way to visualize hyperbolic planes, they preserve area, length, straightness, angles and circles, at least if viewed from the correct perspective.

We already know schläfli<sup>Wiki</sup> tilings from the plane construction [1.3]. The most simple tilings are:  $\{5, 3\}$ ,  $\{6, 3\}$ ,  $\{7, 3\}$ , respectively for spherical, euclidean and hyperbolic plane [tab1.2]. We will use the  $\{7, 3\}$  tiling if possible for comparison. Projecting a tiling shows in my opinion the properties of a projection even better than a Tissot's indicatrix<sup>Wiki</sup>.

Comparing poincaré disk and embedding of a tiled plane, shows isometry and equiareality are not given in poincaré disk.[fig1.7]. Loren Serfass<sup>Web</sup> has created this beautiful visualisation based on Craig S. Kaplan<sup>Web</sup>'s tiling. Remember, the embedding shows only a part of the hyperbolic plane, while the poincaré model shows the whole hyperbolic plane.



(a) Poincaré projection



(b) Embedding

**Figure 1.7:** Craig S. Kaplan's artistic Schläfli  $\{7, 3\}$  tiling on poincaré disk (a), and its projection on an embedding by Loren Serfass<sup>shinyapps</sup> (b). Tiles have the same size in hyperbolic space, get  $\infty$  small at  $r_e = 1$ ,  $r_h = \infty$  on poincaré disk, and appear to have the same size in embedding. For animation see [Loren Serfass<sup>YouTube</sup>]

#### 1.4.4 Finite Models

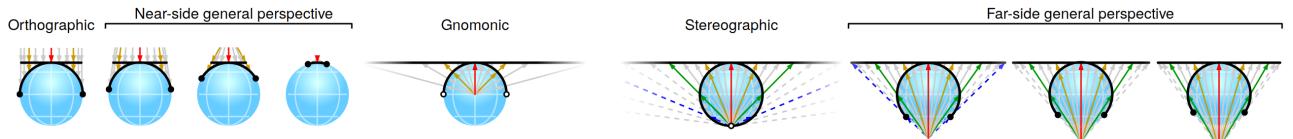
models have spherical twin, see table 1 From an distant azimuthal perspective the hemisphere is finite, therfore only the gnomonic projections fills a infinite plane.

fist finite,  $|C| < 1$  show the hole space, lets take a look on it with projected tilings: hemisphere model is the base for us. The hemisphere model is for its mathematical simplocity

$$\text{Hemisphere model } \mathbb{D} : x^2 + y^2 + z^2 = 1, z > 0 \quad (1.9)$$

next hyperboloid/Lorentz/Minkowski model not finite

$$\begin{aligned} \text{Beltrami-Klein m.} &= \text{Orthog. proj. of } \mathbb{D} \text{ from } \begin{pmatrix} 0 \\ 0 \\ \infty \end{pmatrix} \text{ onto plane } z = C \\ \text{Poincaré disk m.} &= \text{Stereog. proj. of } \mathbb{D} \text{ from } \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \text{ onto plane } z = 0 \\ \text{Poincaré half-plane m.} &= \text{Stereog. proj. of } \mathbb{D} \text{ from } \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \text{ onto plane } x = 1 \end{aligned} \quad (1.10)$$



**Figure 1.8:** In azimuthal projections geodesics cross in origin. Stereographic projections are conformal and kreisreu. Gnomonic projections are isometric. [Cmglee<sup>CC3+</sup>]

preserves 1.5				Band model
Length				*
Area				
Straightness	✓			
Angle		✓		✓
Circle		✓	✓	✓

**Table 1.3:** Selection of finite projections from  $\mathbb{H}^2$  to  $\mathbb{R}^2$ . Empty cells indicate that the 'row label' is not preserved. \*The band model preserves length only along the 'equator'.

[Theon<sup>CC3</sup>] [Segerman Schleimer<sup>YouTube</sup>]

### 1.4.5 Infinite Models

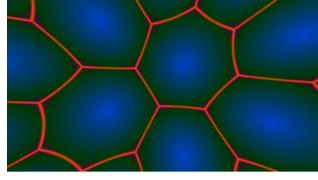
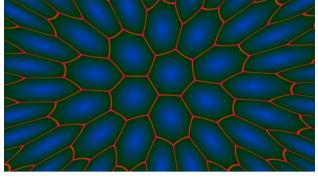
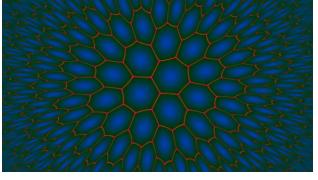
Il based on Hyperboloid model, lorenz, minkovsky, bla bla wer braucht des inifte models only physics, but should have less numerical problems. [wikiminkowski<sup>Wiki</sup>](#) [hyperboloid<sup>Wiki</sup>](#) hat vermutlich weniger numberische probleme [Cannon et al. 1997].

- all are azimuthal
  - lambert azimuthal areae proj
  - azimuthal distance (unflag)
- special thanks to madore

$$\begin{aligned} \text{Gans m.} &= \text{Stereog. proj. of } \mathbb{D} \text{ from } \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ onto plane } z = 1 \\ \mathbb{H} &= \text{Hyperboloid m.} = \text{Stereog. proj. of } \mathbb{D} \text{ from } \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \text{ onto surface } x^2 + y^2 - z^2 = -1, z > 0 \end{aligned} \quad (1.11)$$

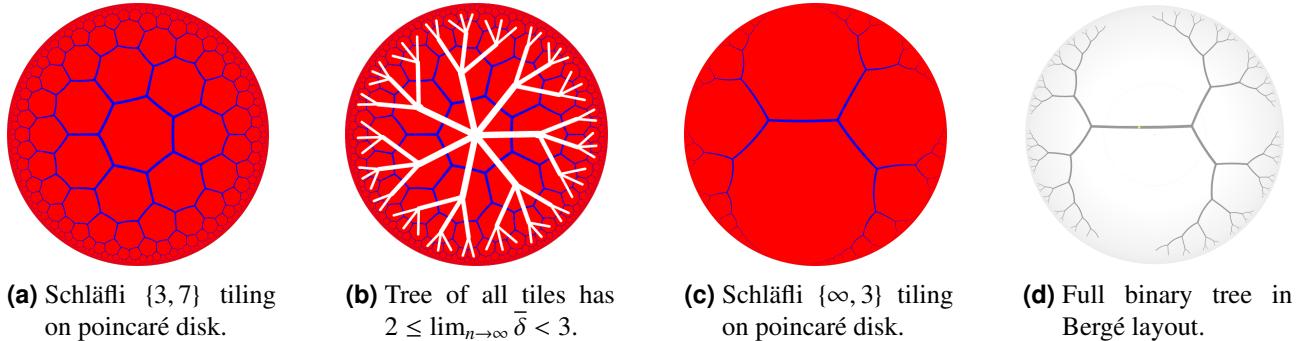
but it should be also possible to use gnomi hyperboloid to get to gans:

$$\begin{aligned} \text{Gans m.} &= \text{Stereog. proj. of } \mathbb{H} \text{ from } \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ onto plane } z = 1 \\ \text{Hyperboloid m.} &= \text{Stereog. proj. of } \mathbb{H} \text{ from } \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \text{ onto surface } x^2 + y^2 - z^2 = -1, z > 0 \end{aligned} \quad (1.12)$$

			
preserves 1.5	1966 Gans model	Lambert azimuthal	L. azimuthal equidistant
Length Area Straightness Angle Circle		✓	*

**Table 1.4:** Selection of infinite projections from  $\mathbb{H}^2$  to  $\mathbb{R}^2$ . Empty cells indicate that the 'row label' is not preserved. \*The band model preserves length only along the 'equator'. [David Madore [YouTube](#)]

## 1.5 Hyperbolic Space and Hierachy Visualisation



(a) Schläfli  $\{3,7\}$  tiling on poincaré disk.  
 (b) Tree of all tiles has  $2 \leq \lim_{n \rightarrow \infty} \bar{\delta} < 3$ .  
 (c) Schläfli  $\{\infty, 3\}$  tiling on poincaré disk.  
 (d) Full binary tree in Bergé layout.

**Figure 1.9:** Poincaré projections of trees and tilings.  
 [Parcly Taxel<sup>PublicDomain</sup>] [Anton Sherwood<sup>PublicDomain</sup>] [This<sup>CC4</sup>]

All images in [fig1.9] are poincaré projections for comparison. If a tree which nodes are placed on equal area tiles, is placed in a plane, the degree of this tree shows the growth of area of the containing plain. Mathematicians showed that area of hyperbolic space grows exponentially, a property fitting to the growth of directed acyclic graphs<sup>Wiki</sup> aka trees or hierarchies [fig1.9b].

Before we proceed some terms should be defined: Each tile in [fig1.9c] is a apeirogon<sup>Wiki</sup>, a apeirogon has  $\infty$  edges. The edges of a apeirogon lie on a circle, called horocycle<sup>Wiki</sup>. In a poincaré disk, all circles touching the unit circle are horocycles. We will need some other variables in the second part of this work,  $n$  will be used for the the number of nodes in a tree, and  $n_v$  will address the number of visible nodes on a poincaré disk. The poincaré disk shows the hole hyperbolic plane, 'visible' means in this case the edge occupies more than one pixel.  $\delta$  and the term degree will be used for out degree, since the degree of a tree is almost constant with  $\frac{n-1}{n}$  and not usefull for our considerations.

$$\begin{aligned}
 &\text{number of nodes of hole tree: } n \\
 &\text{number 'visible' nodes: } n_v \\
 &\text{mean out degree of hole tree: } \bar{\delta} \\
 &\text{mean out degree of 'visible' tree: } \bar{\delta}_v \\
 &\text{out degree of center node: } \delta_c
 \end{aligned} \tag{1.13}$$

[fig1.9c][fig1.9d] this looks like berge layout, but root edges half double length in berge.

circle around them called horocycle,like the corrdinate system.

sugest a horocycle/horogons based coordinatee system for layout algo. and correct line length anyway.

[https://en.wikipedia.org/wiki/Order-5\\_apeirogonal\\_tiling](https://en.wikipedia.org/wiki/Order-5_apeirogonal_tiling)

[fig1.9b] Wicht tree degree fits to the 73 tiling? which tiling do i need to embedd a d degree tree? wihch curvature, is this related? further analysis:  $\{\text{number of edges on horocyle, degree}\} \rightarrow \{\log_d(n), \text{degree}\}$  should fit to space..

A short answer is given by comparing the Bergeé layout with a schläfli  $\{\infty, 3\}$  tiling.

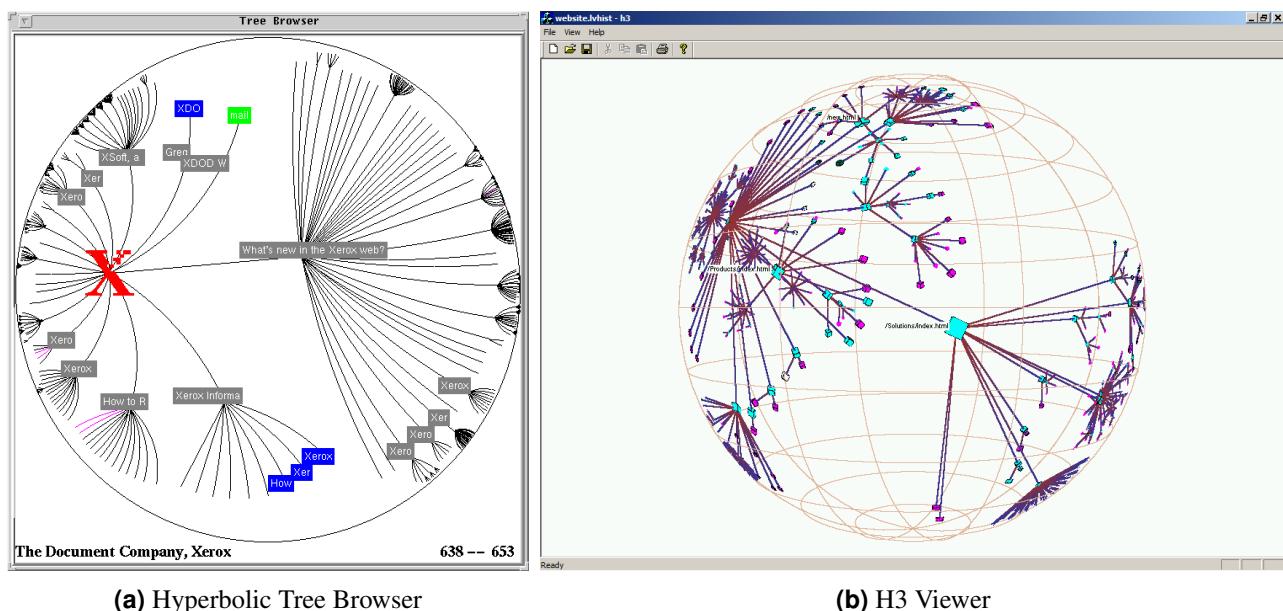
if you want to see it like this:  $d=2 \iff 73$  you get a tree like 19d, looks good.

interesting but patented.

# Chapter 2

# Implementation Survey

## 2.1 Early Systems

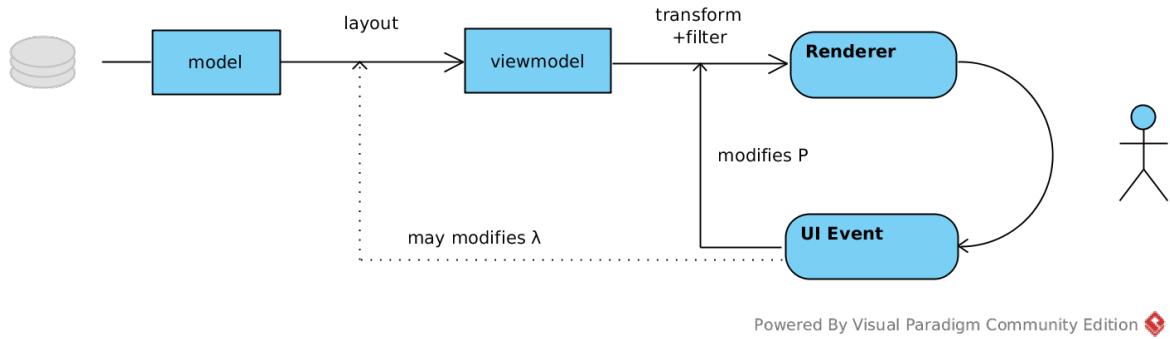


**Figure 2.1:** Hyperbolic Tree Browser and H3 Viewer, the first hyperbolic browsing systems.  
 [J. Lamping and R. Rao 1996] [Munzner 1997]

At dusk of silicon graphics and the dawn of the 9x flop, the first hyperbolic browsing systems were implemented and studied. both use projections parameter as focus. doable by graphics card  
both achieve good scalability  
layout thin transform is same for both, 2d, 3d,  
poincare, klein,

high degree problem.

most are interactive, walruss shows that more nodea are percieveable, at least in h3. node selection hard, overview good.



Powered By Visual Paradigm Community Edition

**Figure 2.2:** Common pipeline[This<sup>CC4</sup>]

achitecture, pipeline: layout, transform, transformt back, interaction, animation,  
 avoid relayout, but layout can be lazy,  
 layout creates spartial data, from there on very simmilar to spartial visualisatoni system.

no sign of overwie by node reduction,  
 no sign of editable  
 both are selecting nodes by radius.

## 2.2 H3 Implementations

most scaleable. graph with spanning tree.

mathemeatics complex impl, less impls. see table. what a pitty.

### 2.2.1 H3 Viewer

free project used h3 and beltrami klein.

additional dim gives best scalability, but also more user interactions nessesrry. c++ impl, with hge range of fetures inkl: selection, rotation, cyclic links.

spanning tree always necessary.

degrees up to 1k are beatuly fully visualized, but selection of single elements and labels gets hard. overview ok, selecting in overview not possible anyway.

3d rotation nescessary, would be better with virtualrealty/aargumented reality. stand under the tree.

[Munzner 1997]

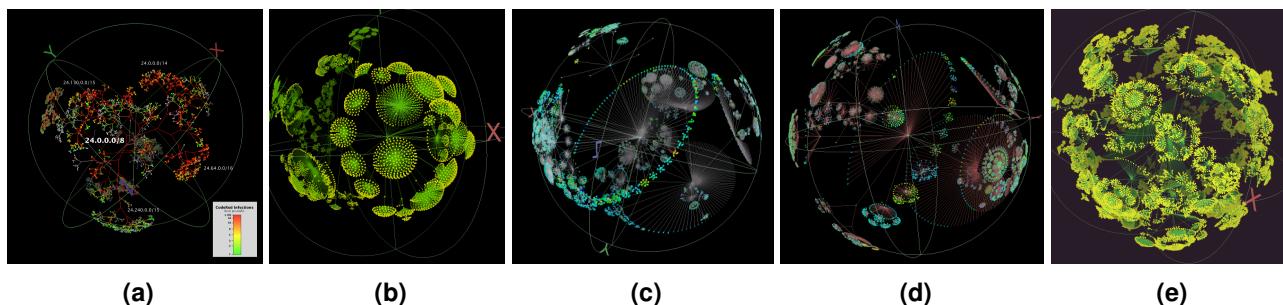
### 2.2.2 Walrus

increases scalability, yeah mr. young!

implemented as non appletable java3d, running this tech nowadays is lottery.

framerate statbilized.

gifs available.



**Figure 2.3:** Walrus H3 gallery

## 2.3 H2 Implementations

most used, implemented.

many patents.

high degree problem worse than with h3.

no overlaps, no additional rotation of sphere.

h2 to r2 with poinvcare projection. only pan necessary. require line as arc, or adaptive sampling like d3-geo, rao has analytical solution for pocare model. perimeter culled node filter, crisp interaction with implementations in c?

### 2.3.1 Xerox/Inxight

follow up in different languages java lisp

multifocus, zoom, 10k,

layout curvature, caching, halfplane patent [**J. O. Lamping and R. B. Rao 1994**] <sup>Pat</sup>

[J. O. Lamping and R. B. Rao 1998]<sup>Pat</sup>

[J. O. Lamping et al. 1998]<sup>Pat</sup>

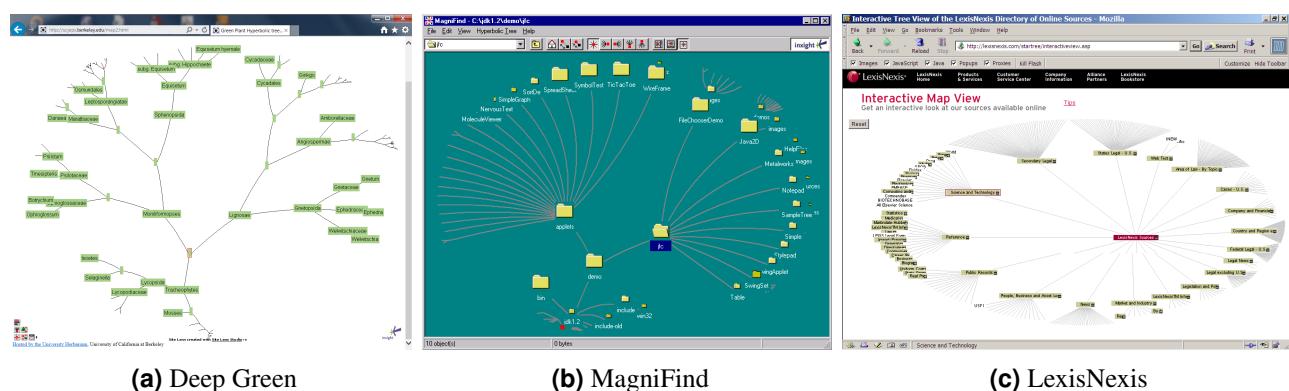
[J. O. Lamping et al. 2001]<sup>Pat</sup>

interpolation, animation,

labels,

[J. Lamping et al. 1995]

[J. Lamping and R. Rao 1996]



**Figure 2.4:** Inxight products gallery.

### 2.3.2 HyperTree

[Bingham and Sudarsanam 2000]

### 2.3.3 Hyptree

[Baumgartner and Waugh 2002]

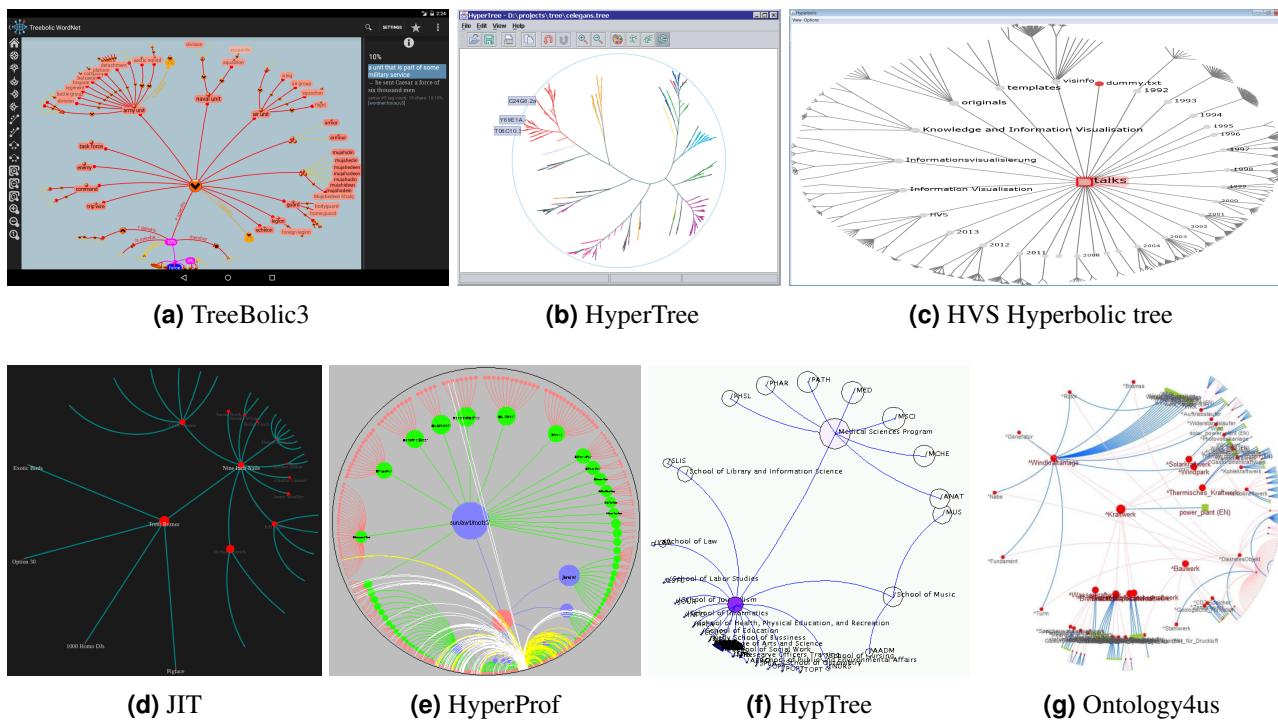
### 2.3.4 HVS

### 2.3.5 Ontology4us

### 2.3.6 JIT Hypertree

### 2.3.7 HyperProf

### 2.3.8 TreeBolic



**Figure 2.5:** Selected H2 implementations gallery.

### 2.3.9 d3-hypertree

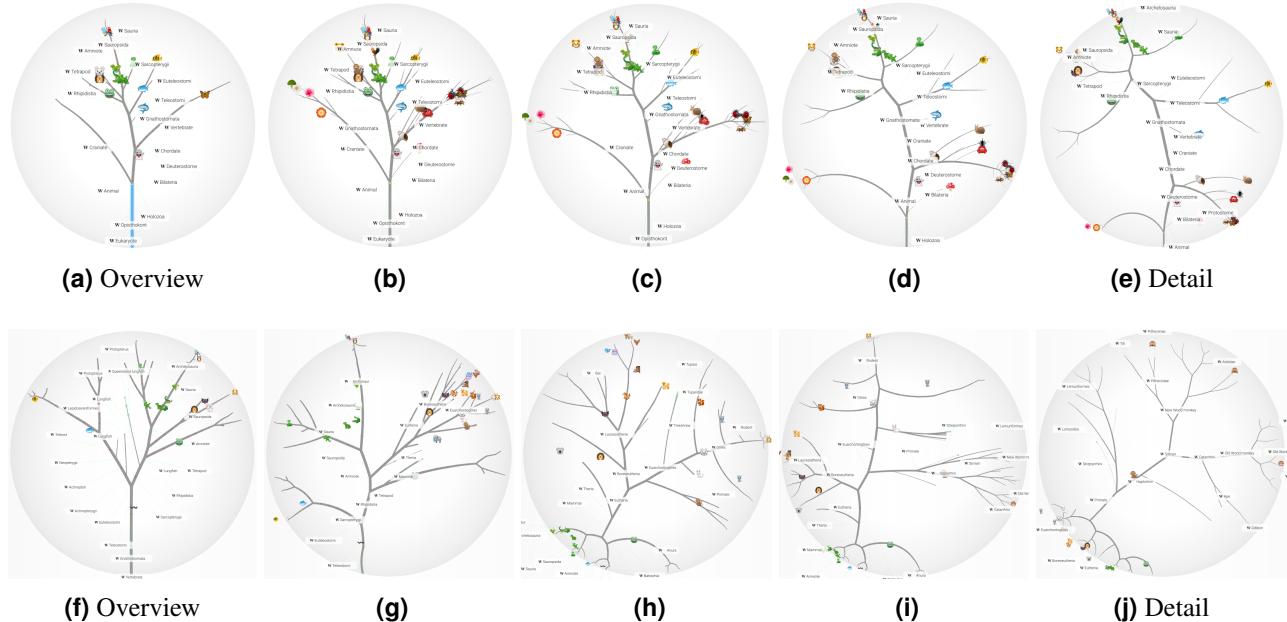
assuming html5 is no lterry for a long time. lettuce of life. d3-geo-hyperbolic.

see it like a map. overview must be available. ok with some constraints.

d3-architecture and performatnce . api, conteext div. fileformats.

config options: landmarks, labels, touch, imgs, colors, with infovis props.

MIT



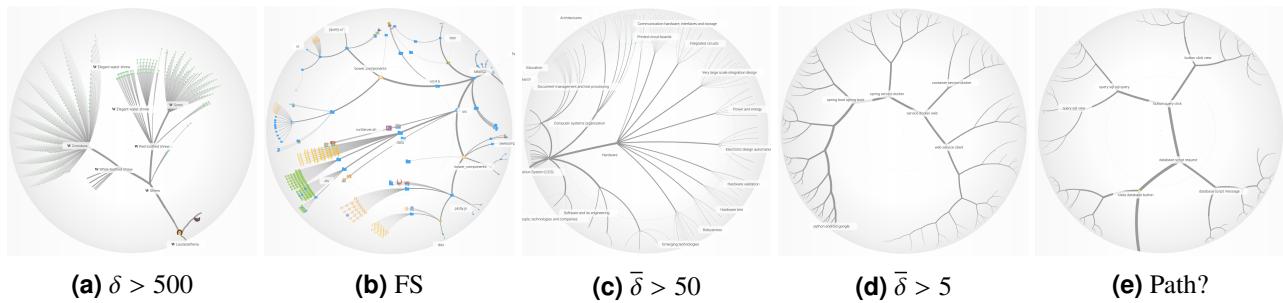
**Figure 2.6:** Comparison of a small hierarchy  $n = 200$  (a-e), and a large hierarchy with  $n = 70k$  (f-j). Low  $\lambda$  values on the left show overview/context, high  $\lambda$  values on the right show details/focus. If  $\lambda < .3$  weight culling is necessary. Works for weighted datasets with average degree  $< 5$ .

[This<sup>CC4</sup>]

Lorem Ipsum is simply dummy text of the printing and typesetting industry. Lorem Ipsum has been the industry's standard dummy text ever since the 1500s, when an unknown printer took a galley of type and scrambled it to make a type specimen book. It has survived not only five centuries, but also the leap into electronic typesetting, remaining essentially unchanged. It was popularised in the 1960s with the release of Letraset sheets containing Lorem Ipsum passages, and more recently with desktop publishing software like Aldus PageMaker including versions of Lorem Ipsum.

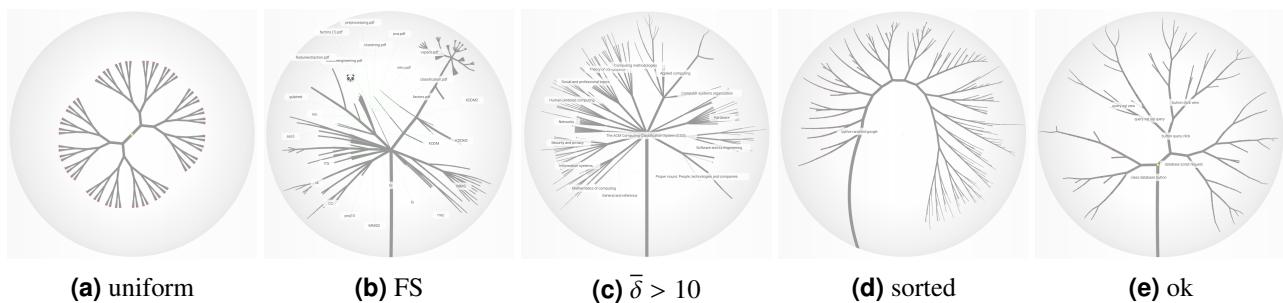
Lorem Ipsum is simply dummy text of the printing and typesetting industry. Lorem Ipsum has been the industry's standard dummy text ever since the 1500s, when an unknown printer took a galley of type and scrambled it to make a type specimen book. It has survived not only five centuries, but also the leap into electronic typesetting, remaining essentially unchanged. It was popularised in the 1960s with the release of Letraset sheets containing Lorem Ipsum passages, and more recently with desktop publishing software like Aldus PageMaker including versions of Lorem Ipsum.

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**Figure 2.7:**  $\lambda > .4 \rightarrow$  Detailview. Failes if:  $\delta > 100$ .

[This<sup>CC4</sup>]



**Figure 2.8:**  $\lambda < .2 \rightarrow$  Overview. Failes if: weight uniform  $\vee (\bar{\delta} > 10 \wedge var(\delta) < e)$ .

[This<sup>CC4</sup>]

Letraset sheets containing Lorem Ipsum passages, and more recently with desktop publishing software like Aldus PageMaker including versions of Lorem Ipsum.

## 2.4 Implementation Comparison

Layout space	$\mathbb{H}^3$		$\mathbb{H}^2$								
Projection	Beltrami-Klein		Poincaré disk								
	H3 Viewer	Walruss	Inxight	HyperTree	HypTree	HVS	Ontology4us	JIT	HyperProf	TreeBolic	d3-hypertree
Pan navigation	✓		✓	✓~		✓	✗	✗			✓
Click navigation	✓	!	✓	✓~	!	✓	✓	✓	!	!	✓
Zoom			✓	✓~		✓	✗	✗	!	!	✓
Pinch Zoom	✗					✗	✗	✗			✓
Rotation compensation	✓		✓								✓
Layout	<i>m</i>	<i>m</i>	<i>l</i>			<i>b</i>					<i>b</i>
Cyclic links	✓						✓	✓	✓		✗
Multi focus				✓							✗
Selection					<i>N<sup>k</sup></i>		<i>1<sup>s</sup></i>	<i>1<sup>s</sup></i>		<i>1<sup>s</sup></i>	<i>N<sup>t</sup></i>
Editable					✓			✓			✗
Scales up to ~	100k	500k	20k								70k
Scalable if $\lambda < .1$											✓ <sup>w</sup>
Webpage	→	→		→	→		→	→	→	→	→
PDF	9p	-	8p	2p	7p	-		-			
Application code	C++ \$	Java GPL2	\$	Java \$		Java			Java	Java \$	
Library code									JS \$	Java GPL3	JS MIT
First appeared	'97	'01	'95	'99	'02		'09			'08	'17
Last update	'03	'05	'09	'12		'05	'14	'13	'13	'17	'18

**Table 2.1:** Empty cells indicate unknown value      \$ commercial

<sup>!</sup> group not evaluated

<sup>m</sup> Munzner  $\mathbb{H}^3$  layout

<sup>s</sup> single selection only

~ strange behaviour in demo

<sup>l</sup> Lamping and Rao layout

<sup>t</sup> toggle multi selection

\$ free for noncommercial use

<sup>b</sup> Bergé layout. Similar to  $\{\infty, \delta\}$

<sup>k</sup> multiselection with keyboard

<sup>w</sup> with given weights, see 2.3.9

Hyperbolic coordinate	$z_h \in \mathbb{C} \quad  z_h  < 1$
Euclidean coordinate	$z_e \in \mathbb{C} \quad  z_e  < 1$
Translation	$P \in \mathbb{C} \quad  P  < 1$
Rotation	$\theta \in \mathbb{C} \quad  \theta  = 1$
Transformation	$\langle P, \theta \rangle$ $z_h \mapsto z_e$ $z_e = \frac{\theta z_h + P}{1 + \bar{P}\theta z_h}$
Transformation $^{-1}$	$\langle -\bar{\theta}P, \bar{\theta} \rangle$ $z_e \mapsto z_h$ $z_h = \frac{\bar{\theta}z_e - \bar{\theta}P}{1 + -\bar{\theta}P\bar{\theta}z_e}$
Composition	$\langle P_1, \theta_1 \rangle \circ \langle P_2, \theta_2 \rangle = \left\langle \frac{\theta_2 P_1 + P_2}{\theta_2 P_1 \bar{P}_2 + 1}, \frac{\theta_1 \theta_2 + \theta_1 \bar{P}_1 P_2}{\theta_2 P_1 \bar{P}_2 + 1} \right\rangle$
Arc center	$d$
Identity transformation	$\langle 0, 1 \rangle$
Focusing $\alpha_h$	$\langle -\alpha_h, 1 \rangle$

**Table 2.2:** Selected formulas of Poincaré model.

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