Hyperbolic Browsing

Scalable Hierarchy Browsing in Hyperbolic Space

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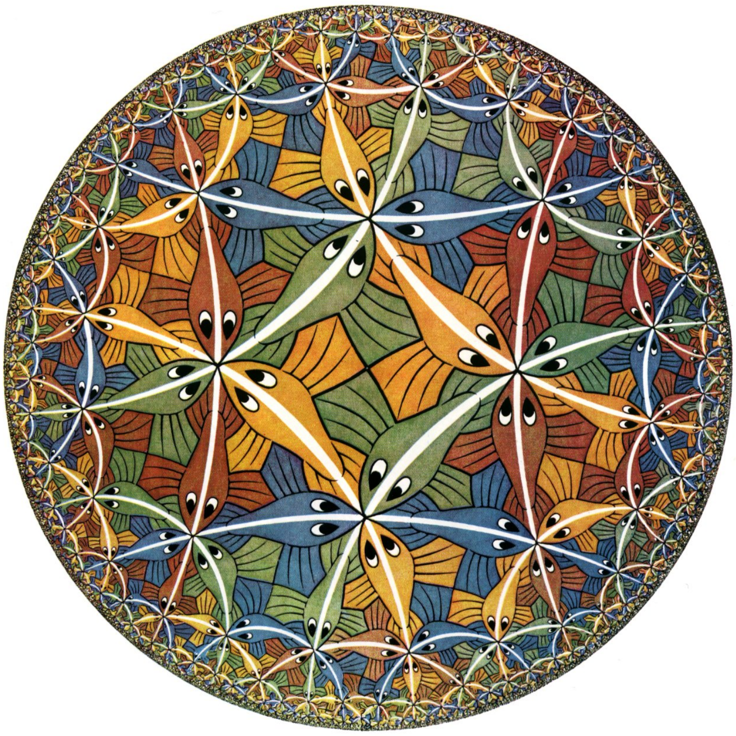

Abstract

Hyperbolic space can be used in conjunction with projections to euclidean space as a basis for visualising and exploring large hierarchies. This work will first present a brief history and overview of hyperbolic geometry, including projections and why they are useful for hierarchy visualisation. The work will then survey existing hyperbolic browsing UI components and applications. The work will conclude with a look at d3-hypertree, a new JavaScript component which implements hyperbolic tree browsing.

Oct 2018, updates on GitHub

Demo: hyperbolic-tree-of-life.github.io

Code: glouwa.github.io/d3-hypertree

Title image: Circle Limit III, M.C. Escher 1959.

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Chapter 1

Hyperbolic Geometry

1.1 Euclid

Ancient greek mathematicians produced books about geometry for hundreds of years, before Euclid summarized this knowledge in his book 'Elements'. This book enabled technology to flourish, was the second most replicated book before 15th century, and a main text book until the 19th century. Its applications ware numerous, but constrained to the five axioms stated by Euclid **[Euclid 300BC]**:

1. Any two points lie on a unique line.
2. Any straight line can be continued indefinitely in either direction.
3. You can draw a circle of any centre and any radius.
4. All right angles are equal.
5. If a straight line, crossing another two straight lines L, L',
   makes angles α, β with L, L' on one side,
   and if $\alpha + \beta < \pi$,
   then L, L' if extended sufficiently far meet on that same side.

Unfortunately the 5th axioms formulation is not best suited for our considerations, this is Playfair's version formulated and proven to be equivalent in 1795 **[Playfair 1795] [fig1.1]**:

5. In a plane, through a point not on a given straight line,
   at most one line can be drawn that never meets the given line.

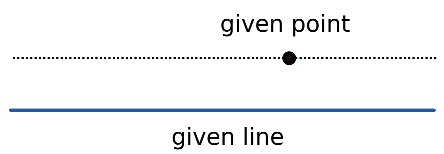


Figure 1.1: Playfair's 5th axiom shows clearly that there is only one line not crossing the given line in euclidean space. [This^{CC4}]

In short, there is just on parallel. But as we will see on the next pages, not all geometries are restricted to one parallel. For a better comparison we will use this formulation:

$$\text{Through } P \notin L \text{ there is } \textbf{one} \text{ line not meeting } L \tag{1.1}$$

1.2 Lobachevsky Bolyai

Although euclidean geometry was widely used and understood for more than 2000 years, its non euclidean derivations were not discovered before 1823. This is reasonable because the theory of complex numbers is necessary to formulate this kind of mathematics. Lobachevsky and Bolyai discovered independently that a slight modification of the 5th axiom has no influence of the correctness of all 28 theorems. The modifications of the 5th axiom are **[Lobachevsky 1823] [Bolyai 1823]**:

$$\begin{aligned}&\text{Through } P \notin L \in \mathbb{S} \text{ there is } \textbf{no} \text{ line not meeting } L\\&\text{Through } P \notin L \in \mathbb{H} \text{ there are } \textbf{infinitely many} \text{ lines not meeting } L\end{aligned} \tag{1.2}$$

This axioms can not be fulfilled in euclidean $\mathbb{R}^2$ or $\mathbb{R}^3$, but we can find a surface embedded in $\mathbb{R}^3$ fulfilling them . This embedding was found early on for spherical geometry, and 1955 for hyperbolic geometry. Table **[tab1.1]** shows possible explanations how to fulfill the modified axioms. But one of them is a embedding and one a projection. Embeddings are conform to a sheet of paper in reality, a $\mathbb{R}^2$ embedded in $\mathbb{R}^3$. This is the most intuitive way to approach such a geometry. Projections and models will be discussed later in section **[1.4]**, to get a basic understanding, i suggest to answer the question how a hyperbolic plane can be embedded in $\mathbb{R}^3$ first.

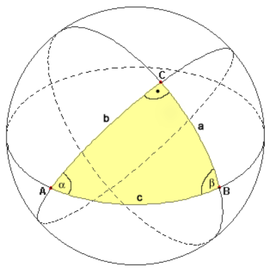
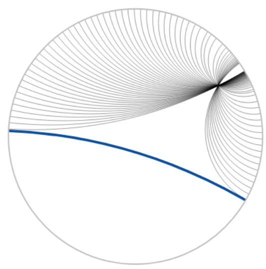
	Spherical geometry	Hyperbolic geometry
5th axiom	Through $P \notin L$ there is **no** line not meeting L	Through $P \notin L$ there are **infinitely many** lines not meeting L
Embeddings	The surface of a sphere is an isometric embedding of $\mathbb{S}^2$ in $\mathbb{R}^3$. Great circles always intersect.  [Peter Mercator$^{PublicDom.}$]	We do not know yet how to embed a $\mathbb{H}^2$ in $\mathbb{R}^3$.
Projections	$\mathbb{S}^2 \mapsto \mathbb{R}^2$ map projections are an appplication of spherical geometry.  [Michael Glatzhofer^{Obs}]	A projection from $\mathbb{H}^2 \mapsto \mathbb{R}^2$ shows that the 5th axiom is fulfilled. Hyperbolic projections will be discussed in section 1.4.  [Trevorgoodchild$^{PublicDom.}$]

Table 1.1: Early applications of Lobachevsky's axioms.

1.3 Embeddings

Embeddings are the idea of placing a space within a space, like a sheet of paper in reality. Each space has its own coordinates system, which can have a different number of dimensions, and each point of the embedding can be described by both coordinate systems. To map coordinates from one space to the other is called embedding. Note that a sheet of paper has one dimension less than its containing space. The paper is considered infinitely thin, mathematically this means a point on the paper is given by only two coordinates. We can also imagine such a embedded plane as surface of a sphere $\mathbb{S}^2$. The embedding of a $\mathbb{S}^2$ in $\mathbb{R}^3$ in fulfillment of the 5th axiom is easy to find, all great circles on the sphere cross each other at least on time. Using a sphere instead of a sheet of paper, and accepting great circles are lines, shows that this kind of geometry fulfills all axioms. This intuitive handling was maybe the reason why spherical geometry did get more attention first, and lead to improvements in mathematics of spherical projections like map projections. But wait, circles are considered to be a lines? (Flat earth alarm?)

Euclid's axioms use the term straight lines, but as visible in table **[tab1.1]** these lines are not straight. So what is a straight line? Mathematicians answered this question for spherical, and hyperbolic geometry. A line[Wiki] needs to be only locally straight, that means in the coordinate system of the embedded space, to be a line in terms of euclid's axioms. On earth, the shortest line between to points on the surface is always a part of a great circle. For spherical geometry that means great circles are straight lines, also called geodesics, even though the are curved if viewed from outside the sphere. A straight line in hyperbolic space must be locally straight and orthogonal to the edge of the plane. This was mathematically proven by lobachevsky. But this knowledge is still not enough to find an embedding of the hyperbolic plane in $\mathbb{R}^3$. **[Weeks 1985]**

Hyperbolic geometry was not widely recognized, since no known embedding limits its applications. A problem which remained until 1955. Hilbert showed 1899 that it is not possible to place a full hyperbolic plane in $\mathbb{R}^3$ **[Hilbert 1899]**, and finally 1955, Kuiper showed that there is a C^1 isometric embedding for $\mathbb{H}^2$ in $\mathbb{R}^3$ **[Kuiper 1955]**. This lead to a theory describing how to construct parts of a hyperbolic plane $\mathbb{H}^n$ embedded $\mathbb{R}^{n+1}$ **[fig1.2]**. One way to construct a $\mathbb{H}^2$ embedding in $\mathbb{R}^3$ is to put together 7 triangles at every vertex, this is called the polyhedral model, or schläfly $\{7, 3\}$ tiling **[Taimina 2009]** **[1.4.4]**. By adding additional space in comparison to a flat $\{3, 6\}$ tiling the plane gets 'curly', and all postulated metrics of hyperbolic geometry can be measured using the given line definition.

(a) Polyhedral model construction rules **(b)** Football model **(c)** Hyperbolic crochets models

Figure 1.2: Embedding construction rules for $\mathbb{H}^2$ in $\mathbb{R}^3$, and handcrafted models.
[Jeffrey R. Weeks[Book]] [Frank Sottile[CC3]] [Daina Taimina[web]]

With this knowledge it is possible for humans to analyse hyperbolic planes in a intuitive way, by using paper models. But such a hyperbolic paper plane is fragile in handling and not suited for measurements. Using knitted planes is an improvement by Taimina introduced in 2001, and recognized as a powerful visualization tool for mathematicians. It allows to identify straight lines by simply stretching the crochet, if it folds properly, the line between the handle points is straight. Also measurements can be taken of this line **[Henderson and Taimina 2001]**.

Now we have seen how a hyperbolic plane looks like and are able to identify objects in reality to which this kind of mathematic can be applied. Note, Hilbert showed it is only possible to embed parts of $\mathbb{H}^n$ in $\mathbb{R}^{n+1}$. Embeddings as visualizations are therefore possible for $\mathbb{H}^2$ and $\mathbb{H}^1$, for humans and other 3-dimensional beeings. $\mathbb{H}^2$ embeddings occur in nature, but are not relevant for our purposes.

(a) Lettuce [Frank Müller^{CC4}]

(b) Coral [Toby Hudson^{CC3}]

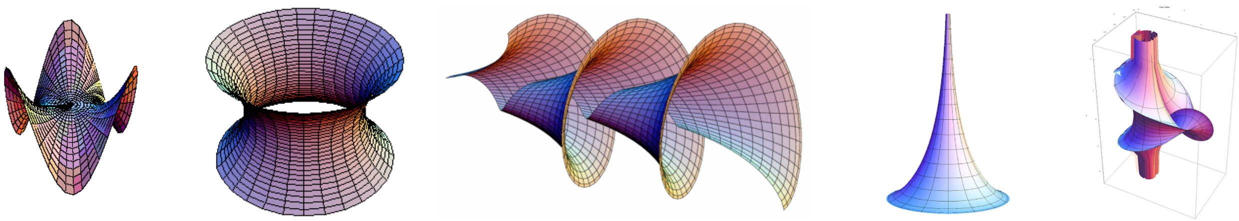
(c) Mushroom [This^{CC4}]

(d) Walnut [This^{CC4}]

Figure 1.3: Real objects approximating $\mathbb{H}^2$ embedded in $\mathbb{R}^3$.

Each of this objects in **[fig1.3]** is only a part a of a hyperbolic plane, according to Hilbert's theorem. If we would apply our construction rules further on, we would have to continue the construction at the border of the planes. 'Curls' would get smaller and smaller, until they need more space as available in $\mathbb{R}^3$ **[Hilbert 1899]**.

The walnut plane in **[fig1.3d]** has a different appearance. In terms of our construction rules this is because less iterations have been applied. This is related to curvature and curvature adopted layouts **[1.5]**. The objects are just approximations of the hyperbolic plane, this is also visible at the thickness the walnut, which is not considered at all mathematically.



(a) Monkey saddle **(b)** Catenoid **(c)** Dini's surface **(d)** Pseudosphere **(e)** Kuen surface

Figure 1.4: Surfaces with constant negative curvature.
[C.T.J. Dodson^{web}] [David Dumas^{web}] [Ermishin Fedor^{CC3}] [Krishnavedala^{CC3}]

The images in **[fig1.3]** do not show all surfaces to which hyperbolic geometry can be applied. In fact it can be applied to all surfaces with strictly negative curvature. **[fig2.8]** shows a selection of surfaces with this property. The monkey saddle **[fig1.3a]** is the mathematical equivalent of the images in **[fig1.3]**, note that also this surface is just a part of $\mathbb{H}^2$, and when extended it will get more 'curls' and result in a lettuce like surface. **[Cannon et al. 1997] [Series 2008]**

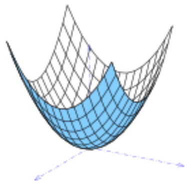
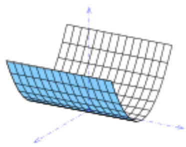
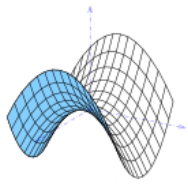
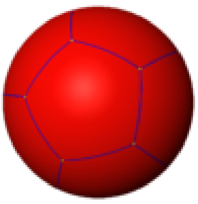
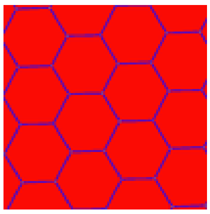
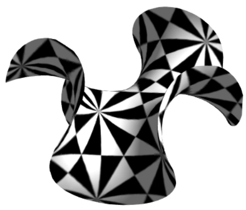
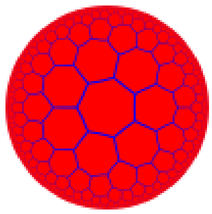
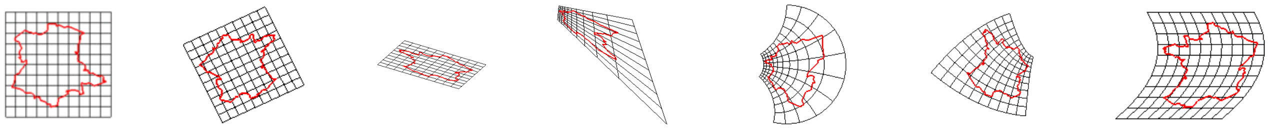
	Spherical	Euclidean	Hyperbolic
Parallels	through P $\notin$ L there is **no** line not meeting L	through P $\notin$ L there is **one** line not meeting L	through P $\notin$ L there are **infinitely many** lines not meeting L
Curvature	> 0	0	< 0
Model	$\mathbb{S}$	$\mathbb{E}$	$\mathbb{D}$ or $\mathbb{H}$
Lines	arcs of great circles	euclidean lines	arcs orthogonal to boundary
Applicable to	 also sphere, ellipsoid...Wiki [Ag2gaehCC4]	 also plane, cylinder...Wiki [Ag2gaehCC4]	 also hyperboloid...Wiki [Ag2gaehCC4]
Angle sum of △	$> \pi$	π	$< \pi$
Circumference of ○	$2\pi\ sin(r)$	$2\pi\ r$	$2\pi\ sinh(r)$
Area of ○	$4\pi\ sin^2(r/2)$	$\pi\ r^2$	$4\pi\ sinh^2(r/2)$
Embedding [1.3]	 Schläfli $\{5,3\}$ tiling [TomruenPublicDom.]	 Schläfli $\{6,3\}$ tiling [LasunnctyCC4]	 Schläfli $\{x,y\}$ tiling [Serfassshinyapps]
Projection [1.4.5]	 Spherical Lagrange projection. [Michael GlatzhoferObs]	transformation	 Schläfli $\{7,3\}$ tiling on poincaré disk [Parcly TaxelPublicDom.]

Table 1.2: Summary of curved space **[Series 2008]** **[Cannon et al. 1997]**. For details on tilings see **[1.4.4]**.

1.4 Transformations, Projections and Hyperbolic Models



(a) Isometry **(b)** Similarity **(c)** Affine **(d)** Projective **(e)** Inversion **(f)** Conformal **(g)** Equiareal

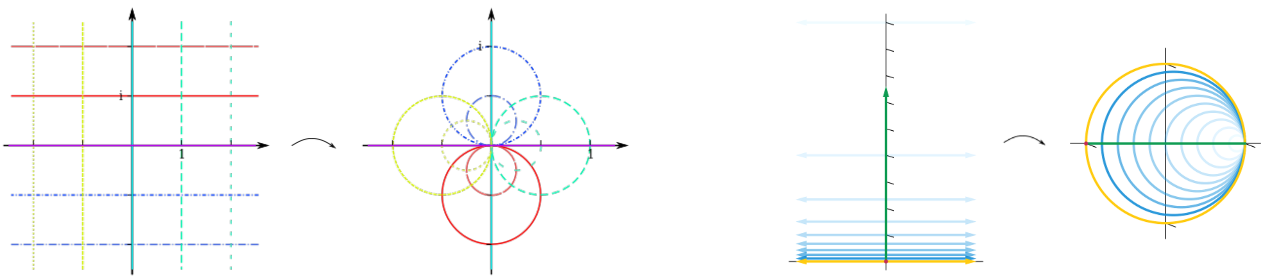
Figure 1.5: Transformation and projection classes. [Robert FERREOL[PublicDom.]]

Transformations[Wiki], coordinatesystems[Wiki], projections[Wiki] and tilings [1.4.4] are the base knowledge to compare hyperbolic models [1.4.5]. Projections map a point from one space to another, they were the first applications of hyperbolic geometry in physics in late 1800. Tilings are helpful tools to discuss layout properties.

$$\text{Transformation: } \mathbb{A}^n \mapsto \mathbb{A}^n \tag{1.3}$$

Transformations map a point to a point in the same space, but projections can do that too and are in this case comparable to transformations. Möbius transformations map from a riemann sphere to it self. They are inversion transformations and the mathematical foundation of the poincaré model. The following components of a möbius transformation are valid möbius transformations it self: inversion $f(z) = 1/z$, translation $f(z) = z + b$, rotation $f(z) = az$, $|a| = 1$. All möbius transformations have a defined inverse, and are in general cool. Caylay transformations map the upper half plane to the unit disk, this is useful to switch between poincare disk and poincaré half plane model **[Lamping et al. 2001]**[Pat].

$$\begin{aligned} \text{Caylay: } f(z) &= \frac{z-i}{z+i} \\ \text{Möbius: } f(z) &= \frac{az+b}{cz+d}, \quad a,b,c,d \in \mathbb{C} \cup \{\infty\}, \quad ad - bc \neq 0 \end{aligned} \tag{1.4}$$



(a) Möbius transformation inversion **(b)** Cayley transformation

Figure 1.6: Some new transformations and projections. [Chrislb[CC3]] [KSmrq[CC3]]

Next step will be to take a brief look on available coordinate systems and on the poincare model [1.4.3]. It serves us as example of a model to get a better understanding of what a model differs from a projection. With the help of tilings we will finally compare the main models of hyperbolic geometry from 1800 physics [1.4.5] [1.4.6], and show how they can be applied for hierarchy visualization [1.5].

1.4.1 Coordinate Systems

There are many hyperbolic coordinate systems[Wiki], polar-, axial- and cartesian-style coordinate systems, Lobachevsky, horocycle-based and quadrant coordinate systems, and model based coordinate systems. Interesting for hierarchy visualization is the horocycle[Wiki] based system, i suppose it is useful to implement schläfly tiling[Wiki] like layouts. Model based system like Beltrami, Weierstrass, Poincaré coordinate systems are actually not used in the according model, but representing it, like a reduced version of the model.

We will use polar coordinate systems for our discussion. A model is a theory of the relations of two spaces, therefore models define a coordinate system for both. Many models use complex numbers, the poincaré model allows only values on the complex unit disk, in both spaces **[1.4]**. The origin is located in center of embeddings, like in all azimuthal projections as the poincare disk model.

1.4.2 Projections

$$\text{Projection: } \mathbb{A}^n \mapsto \mathbb{B}^m \tag{1.5}$$

Projections map a point from a space to another, may have a defined inverse. We will only use azimuthal projections of 2-dimensional surfaces embedded in $\mathbb{R}^3$, except in **[fig1.7]**. In the image of azimuthal projections, geodesics always cross the center as straight line. The most common used projections in our context are:

- Gnomonic: from the center point of the sphere. Straight lines are preserved. Produces always infinite images.
- Stereographic: from a point on the sphere. Preserves angles and circles.
- Orthographic: uses parallel rays, or in other words: the light source is ∞ far away.

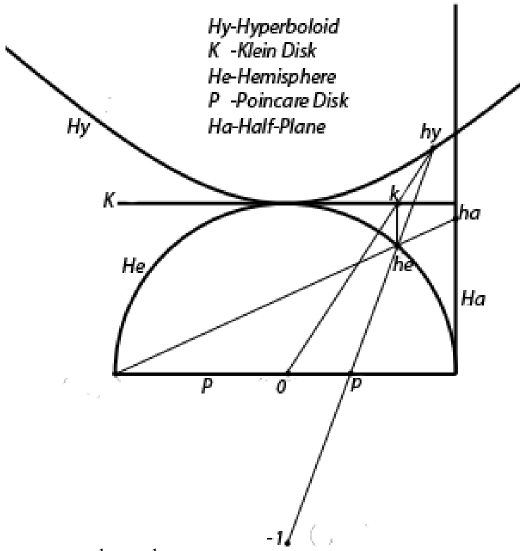


Figure 1.7: $\mathbb{D}^1$, $\mathbb{H}^1$ and other models embedded in $\mathbb{R}^2$. [Selfstudier[PublicDom.]]

Each hyperbolic model can be associated to a projection. Relations of hyperbolic models are shown in **[fig1.7]** as 1-dimensional embeddings in $\mathbb{R}^2$. Note that these projections may or may not fill the hole target plane. Each hyperbolic model has a spherical twin, obtained by replacing the hemisphere by a sphere **[1.4.5]**. d3-geo[Web] implements various projections, and is capable of handling projections of different topologies. This could be used as module in a hyperbolic browser **[fig2.2][2.4]**.

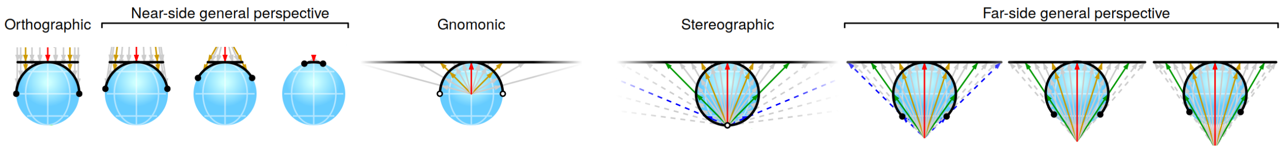


Figure 1.8: In azimuthal projections geodesics cross in origin. Stereographic projections are conformal and kreistreu. Gnomonic projections are isometric. [Cmglee[CC3+]]

1.4.3 Poincaré Model

The poincaré model might be considered famous, or at least its applications like Eschers art and Einsteins relativity. This and the fact that **[Lamping and Rao 1996]** used it for their work is the reason why it is the first model we discuss. Polar coordinates are used, projecting the hole hyperbolic plane to the euclidean unit disk, making it a finite model **[1.4.5]**.

$$\begin{aligned} \text{Hyperbolic Coordinate } z_h &\in \mathbb{C} \cup \mathbb{H} \quad |z_h| < 1 \\ \text{Euclidean coordinate } z_e &\in \mathbb{C} \cup \mathbb{E} \quad |z_e| < 1 \end{aligned} \tag{1.6}$$

The poincaré model is a azimuthal and stereographic projection, and therefore conformal and great circles intersect in the origin. Geodesics[Wiki] are arcs in the image, the poincaré disk, and intersecting the unit circle perpendicular **[fig1.1]**. Circles stay circles, although the center of the circle in the euclidean image is not in the center of the euclidean circle. Parameter P defines the hyperbolic point mapped to the center of the image, we will call it 'focus point' in context of hierarchy visualization. θ is automatically set at some implementations for rotation compensation. The inverse is given as möbius inverse **[1.4]**.

$$\begin{aligned} \text{Rotation: } &\theta \in \mathbb{C} \quad |\theta| = 1 \\ \text{Translation: } &P \in \mathbb{C} \quad |P| < 1 \\ \text{Transformation: } &\left\langle P,\ \theta \right\rangle \\ z_e = &\ \frac{\theta z_h + P}{1 + \bar{P}\theta z_h} \\ \text{Inverse: } &\left\langle -\bar{\theta}P,\ \bar{\theta} \right\rangle \\ z_h = &\ \frac{\bar{\theta} z_e - \bar{\theta}P}{1 + \overline{-\bar{\theta}P}\bar{\theta} z_e} \end{aligned} \tag{1.7}$$

Vector based systems like SVG are using arcs or polylines to draw the potentially distorted image of a geodesic. Geo projection systems need to solve this problem too, and did, d3-geo[Web] uses adaptive sampling to interpolate a polyline. The poincaré model provides a analytical solution to calculate the circle center of the arc segment of a given line $\overline{AB}, \quad A, B \in \mathbb{C} \in \mathbb{E}$.

$$\begin{aligned} d &= \Re(A)\Im(B) - \Re(B)\Im(A) \\ \text{Arc center } c_e &= \frac{i}{2} \, \frac{A(1 + |B|^2) - B(1 + |A|^2)}{d} \\ \text{Orientaion} &= \begin{cases} \mathit{clockwise}, & \text{if } d > 0 \\ \mathit{counterclockwise}, & \text{otherwise} \end{cases} \end{aligned} \tag{1.8}$$

For mouse interactions like panning and pinching as well as animations the composition operator can be derived from the möbius transformation property. When implementing a hyperbolic browser, always keep in mind that a transformation $\left\langle -z_h, 1\right\rangle$ will place the point z_h in center of the projection image. θ can be used for möbius homothety, but they tend to cause motion sickness.

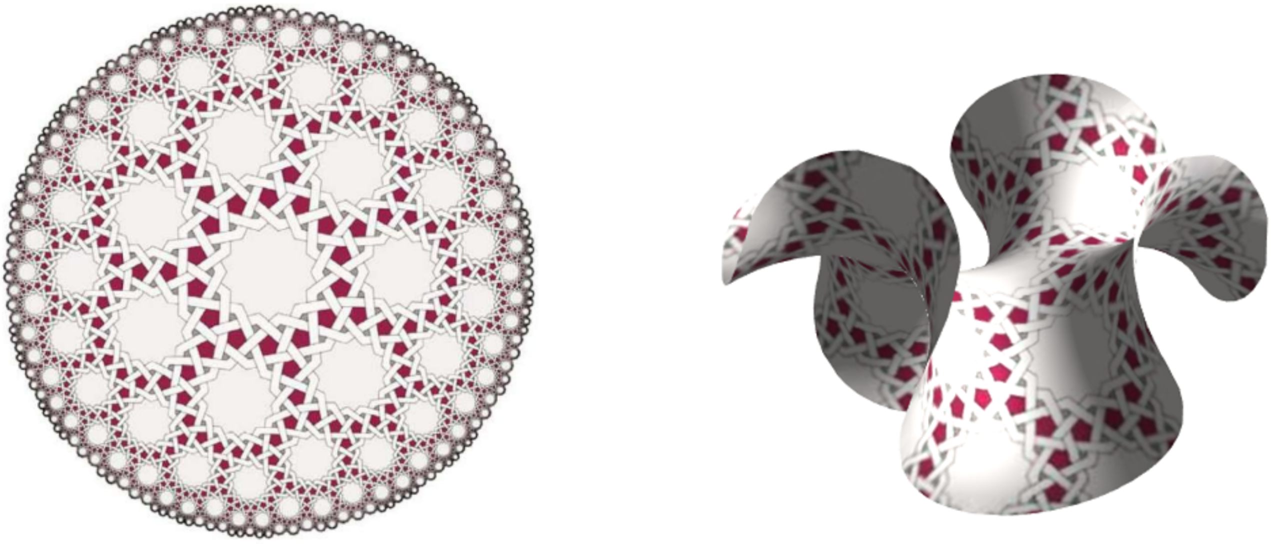
$$\begin{aligned} \text{Composition: } \left\langle P_1, \theta_1 \right\rangle \circ \left\langle P_2, \theta_2 \right\rangle &= \left\langle \frac{\theta_2 P_1 + P_2}{\theta_2 P_1 \overline{P_2} + 1}, \frac{\theta_1\theta_2 + \theta_1 \overline{P_1} P_2}{\theta_2 P_1 \overline{P_2} + 1} \right\rangle \\ \text{Identity transformation: } &\left\langle 0, 1 \right\rangle \\ \text{Focusing } z_h : &\left\langle -z_h, 1 \right\rangle \end{aligned} \tag{1.9}$$

1.4.4 Tilings

Tiling a.k.a. tessellations split a plane in one or more shapes without gaps or overlaps. They will help us to compare projections, and are closely related to most $\mathbb{H}^2$ layouts **[1.5]**. We will only use regular tilings, this means only one shape is used to fill the plane. This implies that tiles, the instances of this shapes have all the same area in hyperbolic space, but not necessarily in projection images. Isometric embeddings are an intuitive way to visualize hyperbolic planes, they preserve area, length, straightness, angles and circles, at least if viewed from the correct perspective.

We already know schläfli^{Wiki} tilings from the plane construction **[1.3]**. The most simple tilings are: $\{5,3\}$, $\{6,3\}$, $\{7,3\}$, respectively for spherical, euclidean and hyperbolic plane **[tab1.2]**. We will use the $\{7,3\}$ tiling if possible for comparison. Projecting a tiling shows in my opinion the properties of a projection even better than a Tissot's indicatrix^{Wiki}. The following pages will use this technique to compare finite and infinite models. Graticules may also be considered as source topology.

Comparing poincaré disk and embedding of a tiled plane, shows isometry and equiareality are not given in poincaré disk.**[fig1.9]**. The poincaré model is the most used for hyperbolic browsing, we see that the area of the hyperbolic plane grows exponentially with the distance to the center, since each tile has the same size in hyperbolic space. This is the key property to make it useful for hyperbolic browsing, because hierarchies worst case also grow exponentially.



(a) Poincaré projection **(b)** Embedding

Figure 1.9: Craig S. Kaplans^{Web} artistic schläfli $\{7,3\}$ tiling on poincaré disk (a), and its projection on an embedding by Loren Serfass^{shinyapps} (b). Tiles have the same size in hyperbolic space, get ∞ small at $r_e = 1,\ r_h = \infty$ on poincare disk, and appear to have the same size in embedding. For animation see Loren Serfass^{YouTube}

Loren Serfass^{Web} has created this beautiful visualization based on Craig S. Kaplan^{Web}s tiling. Remember, the embedding shows only a part of the hyperbolic plane, while the poincaré model shows the hole hyperbolic plane.

1.4.5 Finite Models

Finite model show the hole hyperbolic plane in a finite image, typically on the unit disk, or one or two half-planes. They were the first applications of hyperbolic geometry, the poincaré model is also on of them. It was used physics at the end of the 19th century. All models have a spherical twin, see table **[tab1.2]**. We will use hemisphere model as base since it is mathematically the most simple one.

$$\text{Hemisphere model } \mathbb{D}: \; x^2 + y^2 + z^2 = 1, \; z > 0 \tag{1.10}$$

The hemisphere is finite, therefore, for azimuthal projections, only the gnomonic fills a infinite plane. Half-plane and band model are not azimuthal. The half-plane model shows only one half of the hemisphere without extreme distortions. The Band model is similar but shows the other half on top to the first. The center horizontal line preserves length in the band model, similar to the mercator projection, its spherical twin.

$$\begin{aligned} \text{Beltrami-Klein m.} &= \text{Orthog. proj. of } \mathbb{D} \text{ from } \begin{pmatrix} 0 \\ 0 \\ \infty \end{pmatrix} \text{ onto plane } z = C \\ \text{Poincaré disk m.} &= \text{Stereog. proj. of } \mathbb{D} \text{ from } \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \text{ onto plane } z = 0 \\ \text{Poincaré half-plane m.} &= \text{Stereog. proj. of } \mathbb{D} \text{ from } \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \text{ onto plane } x = 1 \end{aligned} \tag{1.11}$$

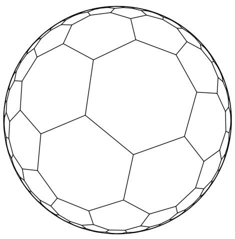
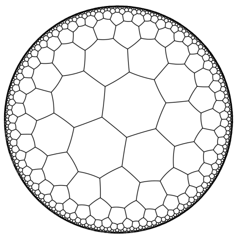
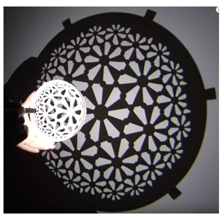
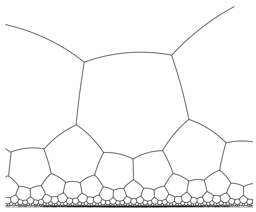
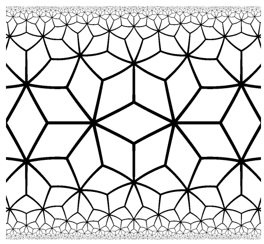
preserves 1.5	  1871 Beltrami–Klein	  1873 Poincaré disk	  1873 Poincaré half-plane	 Band model
Length				*
Area				
Straightness	✓			
Angle		✓	✓	✓
Circle		✓	✓	✓

Table 1.3: Selection of finite projections from $\mathbb{H}^2$ to $\mathbb{R}^2$. Empty cells indicate 'not preserved'. *The band model preserves length only along the 'equator'. [Theon^{CC3}] [Segerman Schleimer^{YouTube}]

1.4.6 Infinite Models

$$\text{Euclidean coordinate } z_e \in \mathbb{C} \in \mathbb{E} \tag{1.12}$$

Infinite models do not show the hole hyperbolic plane. They are used in physics, but may be a alternative for hierarchy browsing. Numerical problems should be less, because not all points are placed near the unit circle.

The gans model and the hyperboloid^{Wiki} model can can be obtained from the hemisphere model **[fig1.7]**. Both do not show the hole hyperbolic plane as visible in **[tab1.4]**. The hyperboloid in 4 dimensions part of he Minkowsky^{Wiki} space, also used in physics. **[Cannon et al. 1997]**.

$$\begin{aligned} \text{Gans m.} &= \text{Stereog. proj. of } \mathbb{D} \text{ from } \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \text{ onto plane } z = 1 (\text{gnomonic}) \\ \mathbb{H} : \text{Hyperboloid m.} &= \text{Stereog. proj. of } \mathbb{D} \text{ from } \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \text{ onto surface } x^2 + y^2 - z^2 = -1,\ z > 0 \end{aligned} \tag{1.13}$$

Gans model and hyperboloid form a geometric group, thus is is possible to obtain the gans models from the hyperboloid model. The gans models is used my mathematicians for its simplicity, but has no known real world applications.

$$\text{Gans m.} = \text{Stereog. proj. of } \mathbb{H} \text{ from } \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \text{ onto plane } z = 1? \tag{1.14}$$

David Madore^{YouTube} shows in his animations various classical map projections of the hyperboloid model. Applications for them are not mentioned.

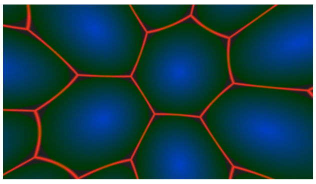
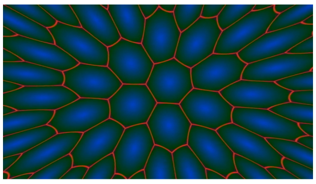
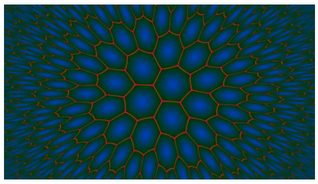
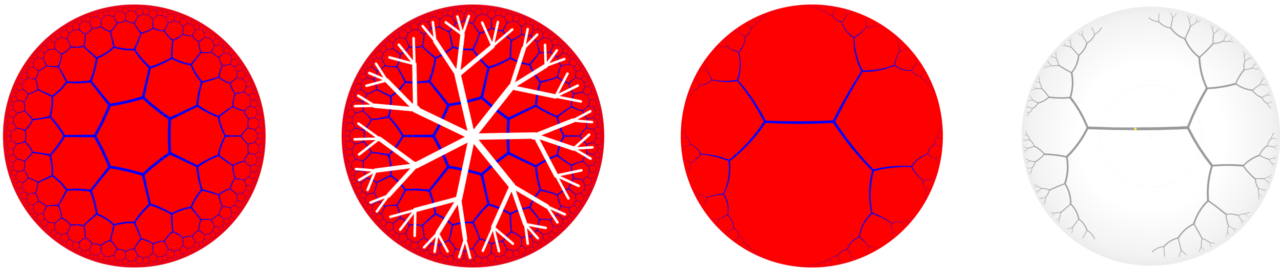
preserves 1.5	 1966 Gans model	 Lambert azimutal	 L. azimutal equidistant
Length			*
Area		✓	
Straightness			
Angle			
Circle			

Table 1.4: Selection of infinite projections from $\mathbb{H}^2$ to $\mathbb{R}^2$. Empty cells indicate that the 'row label' is not preserved. *The band model preserves length only along the 'equator'. [David Madore^{YouTube}]

1.5 Hyperbolic Space and Hierarchy Visualization



(a) Schläfli $\{3,7\}$ tiling on poincaré disk.

(b) Tree of all tiles has $2 \leq \lim_{n\to\infty} \overline{\delta} < 3$.

(c) Schläfli $\{\infty,3\}$ tiling on poincaré disk.

(d) Full binary tree in Bergé layout.

Figure 1.10: Poincaré projections of trees and tilings. [Parcly Taxel[PublicDom.]] [Anton Sherwood[PublicDom.]] [This[CC4]]

All images in **[fig1.10]** are poincaré projections for comparison. If the nodes of a tree are placed on equal area tiles, each tile a node, the degree of this tree shows the growth of area of the containing plain. Mathematicians showed that area of hyperbolic space grows exponentially **[Cannon et al. 1997]**, a property fitting to the growth of directed acyclic graphs[Wiki] aka trees or hierarchies **[fig1.10b]**.

Before we proceed some terms should be defined: Each tile in **[fig1.10c]** is a apeirogon[Wiki], a apeirogon has ∞ edges. The edges of a apeirogon lies on a circle, called horocycle[Wiki]. In a poincaré disk, all circles touching the unit circle are horocycles. We will need some other variables in the second part of this work, n will be used for the the number of nodes in a tree, and n_v will address the number of visible nodes on a poincaré disk. The poincaré disk shows the hole hyperbolic plane, 'visible' means in this case the edge occupies more than one pixel. The term degree and δ will be used for out degree, since the degree of a tree is almost constant with $\frac{n-1}{n}$ and not useful for our considerations.

$$\begin{aligned} \text{number of nodes of hole tree: } & n \\ \text{number 'visible' nodes: } & n_v \\ \text{mean out degree of hole tree: } & \overline{\delta} \\ \text{mean out degree of 'visible' tree: } & \overline{\delta}_v \\ \text{out degree of center node: } & \delta_c \\ \text{hyperbolic node distance: } & \lambda = \lambda_h \approx \lambda_{e,center} \end{aligned} \tag{1.15}$$

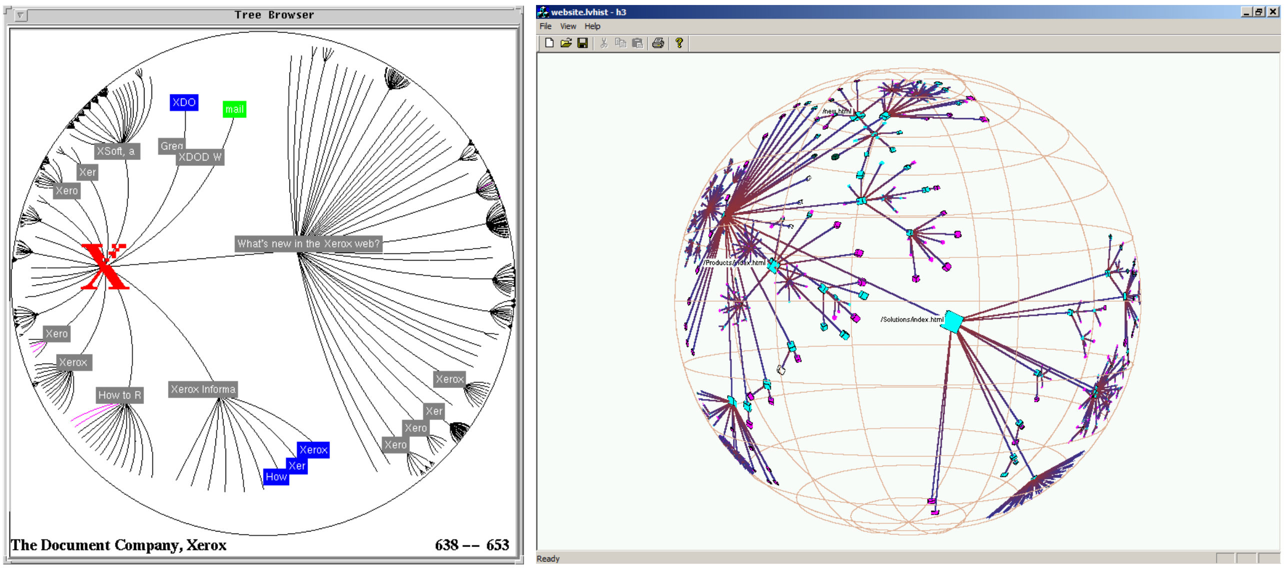
The schläfli $\{\infty,3\}$ tiling in **[fig1.10c]** looks almost like a full binary tree, in a hyperbolic browser **[fig1.10d]**. This particular layout is a Bergé layout, the only difference is the double length center edge **[2.4]**. I suggest to adopt the root edge length of Bergé layout to imitate schläfli $\{\infty,3\}$ tiling. Such a layout would have its nodes on horocycles, and all edges are part of a horogons. A horocycle or horocycle coordinate system based layout algorithm should have less numerical problems as Bergé or Lamping layout.

If a hierarchy layout in hyperbolic space is not adopted to curvature **[fig1.10b]**, the projection image may contains large empty areas. In other words: which tiling does fit to my tree degree, since tilings have a defined curvature. Apeirogonal tilings[Wiki] show that x defines the degree of the resulting tree in a $\{\infty, x\}$ tiling. A simple approximation could be $\{$number of edges on horocycle, degree$\}$ or $\{log_{\overline{\delta}}(n), \overline{\delta}\}$ Lamping and rao patented the other way, to adopt layout to a constant curvature **[Lamping and Rao 1998]**[Pat].

Chapter 2

Implementation Survey

2.1 Early Systems



(a) Hyperbolic Browser

(b) H3 Viewer

Figure 2.1: Hyperbolic Browser[YouTube] and H3 Viewer[YouTube], the first hyperbolic browsing systems. [Keith Andrews[??]]

The first hyperbolic browsing systems were implemented and studied in mid ’90s. The Hyperbolic Browser by **[Lamping et al. 1995]** lays out a tree in $\mathbb{H}^2$, whereas H3 Viewer by **[Munzner 1997]** uses $\mathbb{H}^3$ as layout space. The main difference of this projects is the dimension of layout space. They also use different projections, but that may be seen as a consequence of layout space dimensionality. Projectins are actually exchangealbe, although all following projects inherited the projection types according to this early systems. Both projects:

- push the boundary of scalability of hierrachy visualisation.
- use projection parameters to define a focus area in hyperbolic space, and modify this paramters by mouse interaction. A conzept still state of the are for spartial visuaisation pipelines.
- systems suffer from a lack of ability to show high degree nodes, espechially if mean degree is high. But this is a known limitation of Browsing solved by Searching.

pipeline: layout thn transform is same for both, variants.

achitecture, pipiele: layout, transform, transformt back, interaction, animation,

similarity to spartial systems. Hyperbolic browsing systems are even simples than spartial, no topo. [2.4] layout creates spartial data, from there on very simmilar to spartial visualisatoni system.

avoid relayout, but layout can be lazy,

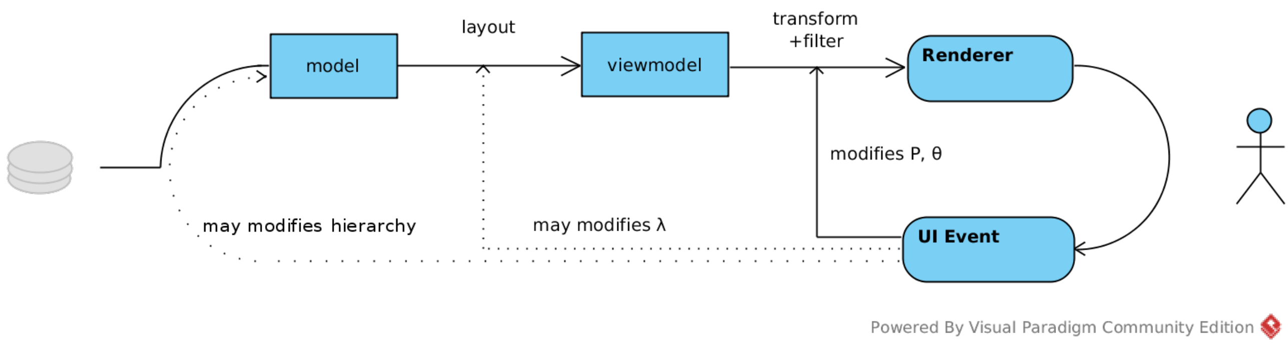


Figure 2.2: Common pipeline. [This^{CC4}]

model: formats, hierrachy, graph + spanning tree

2 impls are editable

layout:

H2 arranged on arc, H3 on a spherical cap (surface).

many implementations, h3, h2 bergé **[Bergé and Bouthier 2003]**, deviations

weights, lazy,

lambda, by place nodes as far outside as needed to ensure min node distance. possible to ensurses equal area for each node. simmilar to curvature adoption.

viewmodel: spartial data

P, θ

animation,

selection,

transformation: 2d, 3d, poincare, klein, technically psooisbel is hald so.

maybe klein is better suited to avoid hidden nodes in R3

rotation compensation,

multi focus

renderer: euclidean, even if arumented or VR.

input may be 2d or 3d

framerate statibilizede:

both are selecting nodes by radius. max informatin density

no sign of overwie by node reduction,

degree: n_v high degree problem. -» max information density. svg is slow, but enought on desktop.

2.2 H3 Implementations

most scaleable.

graph with spanning tree.

mathemeatics complex impl, less impls. see table. what a pitty.

layout: no line but plane S $\in$ cirle

additional dim gives best scalability, but also additional 3 rotailn of sphere nesessary. maybe eigenvetor. or virtualrealyty/aargumented reality. stand under tthe tree.

2.2.1 H3 Viewer[Code]

[Munzner 1997]

free prject used h3 and beltrami klein. maybe less overlaps because less ball more S.

c++ impl, with hge range of fetures inkl: selection, rotation, cyclic links.

spanning tree always necessary.

degrees up to 1k are beatuly fully visualized, but selection of single elements and labels gets hard. overview ok, selecting in overview not possible anyway.

2.2.2 Walrus[Code]

inckreases scalability, yeah mr. young!

implemented as non appletable java3d, runnning this tech nowadays is lottery.

framerate statbilized.

most are interactive, walruss shows that more nodea are percievable, at least in h3. node selection hard, overview/orientations good.

gifs available.

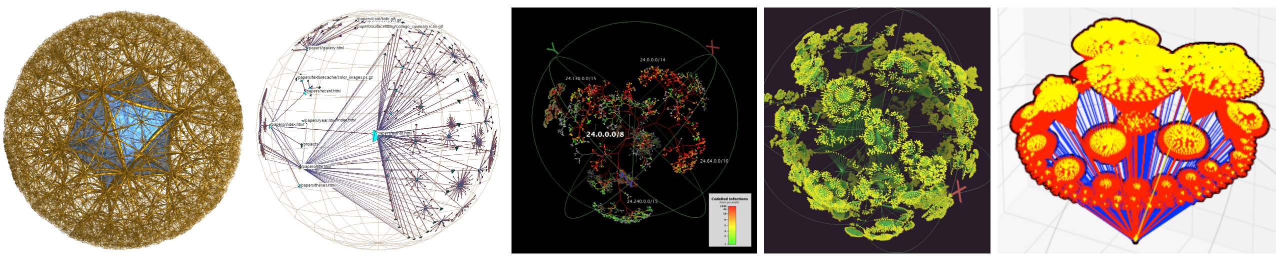
2.2.3 h3py[GitHub]

is a h3 layuot python plotting tool,

uses mathplotlib as backend.

easy to use with pip instal h3py.

link, supported formats:



(a) $\{3, 5, 3\}$ tiling **(b)** H3 Viewer **(c)** Walrus $\overline{\delta} = 2$ **(d)** Walrus $\overline{\delta} > 10$ **(e)** h3py demo

Figure 2.3: H3 implementations gallery, and icosahedral tiling in poincaré ball. [Claudio Rocchini[CC2.5]] [Keith Andrews[??]] [CAIDA / Young Hyun[??]] [Songxiao Zhang[GitHub]]

2.3 H2 Implementations

most used, implemented.

patent by Xerox

high degree problem worse than with h3.

h2 to r2 with poinvcare projection. no overlaps. only 1 pan necessary. uses pocare model line model to draw arcs. adaptive sampling like d3-geo would not need. perimeter culled node filter, crisp interaction with implementations in c?

rotation compensation for choosable node, mulitfocus, interpolation, animation, labels, 20k,

zoom=zieze of node area * diag(),

layout **[Lamping and Rao 1994]**[Pat]

[Lamping et al. 1995]

[Lamping and Rao 1996]

2.3.1 Inxight

1997 formed inxight and took patent.

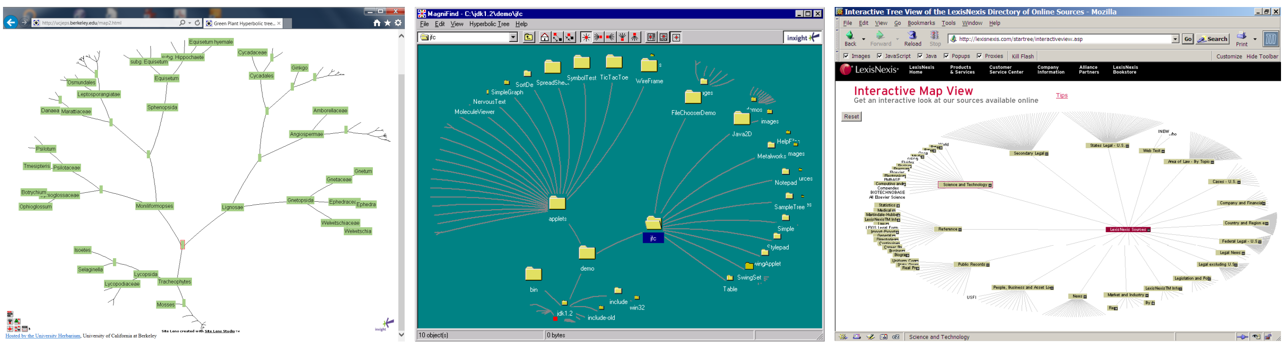
follow up in different languages java lisp

just some more patents

curvature, **[Lamping and Rao 1998]**[Pat]

caching, **[Lamping et al. 1998]**[Pat]

halfplane patent **[Lamping et al. 2001]**[Pat]



(a) Deep Green **(b)** MagniFind **(c)** LexisNexis

Figure 2.4: Inxight products gallery. [Keith Andrews[??]]

2.3.2 HyperTree[Web]

bioinforatics, taxonomy, protein kinase **[Bingham and Sudarsanam 2000]**

2.3.3 Hyptree[Web]

vermutlis aler Vladimir Bulatov link Yuezheng Zhou

2.3.4 HyperProf[Web]

bulgatov, art, is often cretdited.

http://bulatov.org/math/index.html

2.3.5 Roget2000

thesaurus vis credits bulgatov **[Baumgartner and Waugh 2002]**

2.3.6 HVS

master thesis. good rotation compensation? **[Nussbaumer 2005]**

2.3.7 Ontology4us[Web]

link, frauenhofer project ontologie browser.

2.3.8 JIT Hypertree[GitHub]

javascript, edit, simple. link

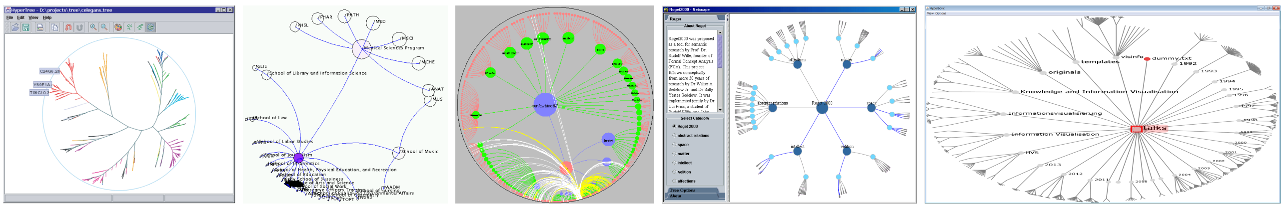
2.3.9 TreeBolic[Web]

comercial wordnet browser, free java compoent.

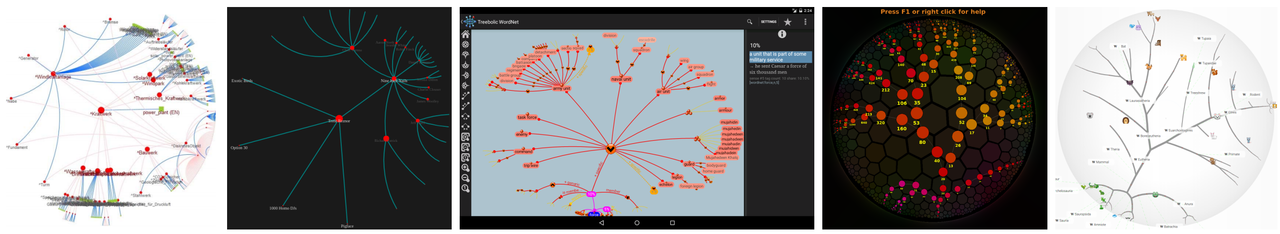
credits Vladimir Bulatov gpl meantion

2.3.10 RogueVis[Web]

sold as game, free for vis. links many papers.



(a) HyperTree **(b)** HypTree **(c)** HyperProf **(d)** Roget2000 **(e)** HVS Hyperbolic Tree



(f) Ontology4us **(g)** JIT **(h)** TreeBolic3 **(i)** RogueViz **(j)** d3-hypertree

Figure 2.5: Selected H2 implementations gallery. **[Bingham and Sudarsanam 2000]** [Yuezheng Zhou[Web]] [Vladimir Bulatov[Web]] **[Baumgartner and Waugh 2002]** [Keith Andrews[??]] [Ontology4us[Web]] [Nicolas Garcia Belmonte[Web]] [TreeBolic[Play]] [RougeVis[Web]]

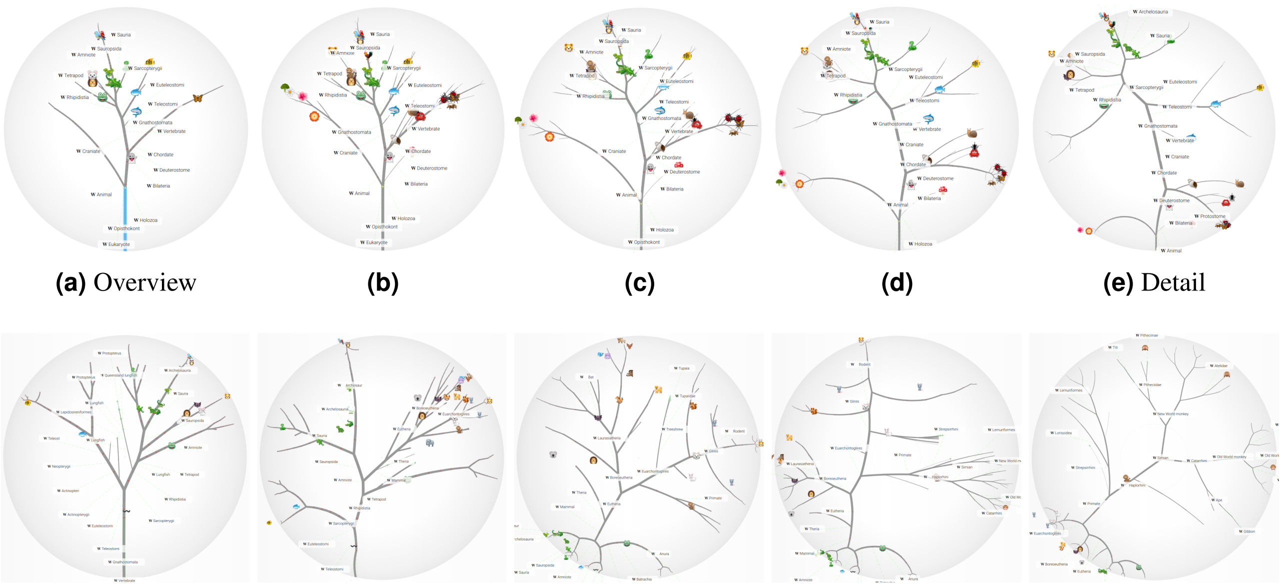
2.3.11 d3-hypertree

MIT, assuming html5 is no lttery for a long time. lettuce of life. d3-geo-hyperbolic.

see it like a map. overview must be available. ok with some constraints. fileformats. d3-hierarchy. d3-geo.

adaptive drawing [Hop97]

config options: layers, layouts, landmarks, labels, touch, imags, colors, with infovis props.



(a) Overview **(b)** **(c)** **(d)** **(e)** Detail

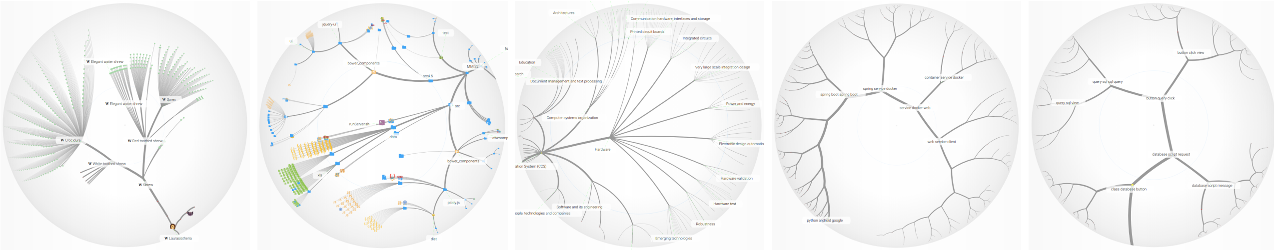
(f) Overview **(g)** **(h)** **(i)** **(j)** Detail

Figure 2.6: Comparison of a small hierarchy $n = 200$ (a-e), and a large hierarchy with $n = 70k$ (f-j). Low λ values on the left show overview/context, high λ values on the right show details/focus. If $\lambda < .3$ weight culling is necessary. Works for weighted datasets with average degree < 5.

detail view wrks as for other hyperbolic browsers.

examples in fig **[fig??]** show best case examples for $\lambda < .1$.

the following exampels show resons for a fail overview layout/filter.

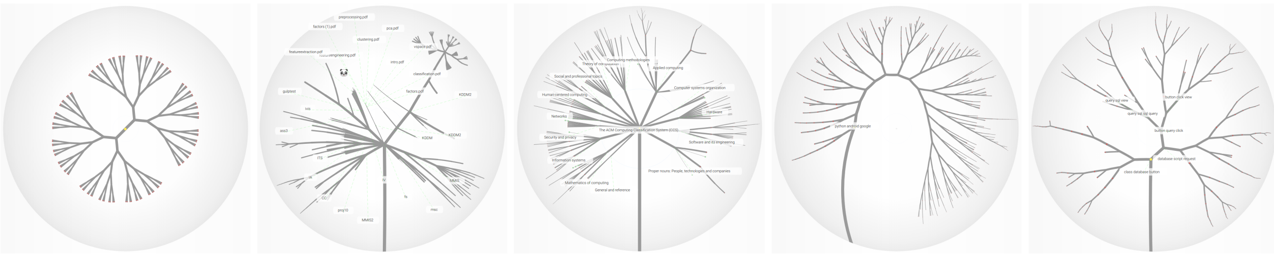


(a) $\delta > 500$ **(b)** FS **(c)** $\overline{\delta} > 50$ **(d)** $\overline{\delta} > 5$ **(e)** Path?

Figure 2.7: $\lambda > .4 \rightarrow$ Detailview. Failes if: $\delta > 100$. [This^{CC4}]

degree problem persists also in overview.

detail view examples.



(a) uniform **(b)** FS **(c)** $\overline{\delta} > 10$ **(d)** sorted **(e)** ok

Figure 2.8: $\lambda < .2 \rightarrow$ Overview. Failes if: weight uniform $\vee$ $(\overline{\delta} > 10 \wedge var(\delta) < e)$. [This^{CC4}]

overview fails to show nodes out to the perimeter if weights uniform distributed, but also in other cases, ie if one mega node consumes all available n_v. diffeent culling weights like entropy, angle, or by data given weights may overcome this problem. quadratic, logarithmic weight seen in images above. also laoutout algo must be considered when seraching for a culling weight.

2.4 Implementation Comparison

Layout space	$\mathbb{H}^3$			$\mathbb{H}^2$									
Max δ_v	$\sim 10^3$			$\sim 10^2$									
Projection	Beltrami−Klein			Poincaré disk									
	H3 Viewer	Walruss	pyh3	Inxight	HyperTree	Hyptree	HVS	Ontology4us	JIT	HyperProf	TreeBolic	RougeVis	d3-hypertree
Pan navigation	✓			✓	✓~		✓	×	×				✓
Click navigation	✓			✓			✓		✓				✓
Pseudo-Zoom		!		✓	✓~	!	✓	×	×	!	!		✓
Pinch Pseudo-Zoom	×						×	×	×				✓
Rotation compensation	✓			✓									✓
Layout	m	m	m	l			b						b
Cyclic links	✓							✓	✓	✓			×
Multi focus				✓									×
Selection	1^s	1^s			N^k		1^s	1^s	1^s		1^s		N^{st}
Editable					✓				✓				×
Max n	100k	500k		20k									70k
Max n_v													70k
Scalable if $\lambda < .1$													✓^{w}
Webpage	→	→	→		→	→		→	→	→	→	→	→
PDF	9p	-		8p	2p	7p	-		-				
Application code	C++ $\not\$$	Java GPL2		$	Java $\not\$$		Java			Java	Java $		
Libary code			Phyton MIT						JS $\not\$$		Java GPL3	C++ GPL2	JS MIT
First appeared	’97	’01		’95	’99	’02		’09		’96	’08		’17
Last update	’03	’05	’15	’09	’12		’05	’14	’13	’13	’17	’18	’18

Table 2.1: Empty cells indicate unknown value $ commercial $\not\$$ free for noncommercial use
$^!$ group not evaluated $^\sim$ strange behaviour in demo ^{b} Bergé layout. Similar to $\{\infty, \delta\}$
^{m} Munzner $\mathbb{H}^3$ layout ^{l} Lamping and Rao layout ^{k} multiselection with keyboard
^{s} single selection only ^{t} toggle multi selection ^{w} with given weights, see 2.3.11

Future work

- improofe bergé layout to imitate tiling.
- measure values in prototype.
- find better (generic) culling algo.
- solve numerical problems by layout origin shift.
- iplement horocycle layout.
- implement topojson in/as viewmodel, and use d3-geo to project.
- more projectins, with d3-geo easy.
- more layers for high degree.
- ui event transformation maybe also by d3 solved?
- cleanup infovis props, css, add more renderers.
- python, jupyter, mathplotlib, R, js cmoponent api.

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