再提思想来源:

$n=1$:

x	x_0	x_1
$f(x)$	$f(x_0)$	$f(x_1)$

x	x_0	x_1	$\cdots x_n$
$f(x)$	$f(x_0)$	$f(x_1)$	$\cdots f(x_n)$

2点式 $L_1(x)=\boxed{\dfrac{x-x_1}{x_0-x_1}f(x_0)+\dfrac{x-x_0}{x_1-x_0}f(x_1)}$ √ Lagrange 插值公式 $L_n(x)$

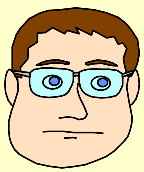
余项 $R_n(x)$

点斜式 $N_1(x)=\boxed{f(x_0)+\dfrac{f(x_1)-f(x_0)}{x_1-x_0}(x-x_0)}$? Newton 插值公式 $N_n(x)$

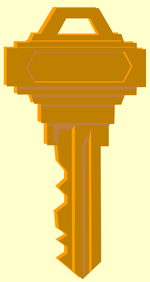
待定系数法 $y=P_1(x)=\dfrac{x_1f(x_0)-x_0f(x_1)}{x_1-x_0}+\dfrac{f(x_1)-f(x_0)}{x_1-x_0}x$ 唯一性 存在性

看来3个都有用。

§3 牛顿插值 /* Newton’s Interpolation */



Lagrange 插值虽然易算，但若要增加一个节点时，全部基函数 $l_i(x)$ 都需重新算过。



将 $L_n(x)$ 改写成 $\overset{?}{c_0}+\overset{?}{c_1}(x-x_0)+\overset{?}{c_2}(x-x_0)(x-x_1)+\cdots+\overset{?}{c_n}(x-x_0)\ldots(x-x_{n-1})$ 的形式，希望每加一个节点时，只附加一项上去即可。

➢ 牛顿插值 /* Newton’s Interpolation */

1. 公式构造: $N_n(x) = ?$

推导思想: 递增式构造, 故具有承袭性。

$n = 0$: $N_0(x_0) = f(x_0)$ $\Longrightarrow$ $N_0(x) = f(x_0)$ 0次多项式

$n = 1$:

x	x_0	x_1	x_2	x_3	$\cdots$	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$\cdots$	$f(x_n)$

$$\begin{cases} N_1(x_0) = f(x_0) \\ N_1(x_1) = f(x_1) \end{cases}$$

设 $N_1(x) = N_0(x) + A(x - x_0) = f(x_0) + A(x - x_0)$ $\xrightarrow{\text{则}}$ $\boxed{N_1(x_0) = f(x_0)}$

由 $N_1(x_1) = f(x_0) + A(x_1 - x_0)$

得 $A = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} \triangleq f[x_0, x_1]$

$$\boxed{f[a,b] \triangleq \frac{f(a) - f(b)}{a - b}}$$

有 $N_1(x) = f(x_0) + f[x_0, x_1](x - x_0)$

$n = 2$:

x	x_0	x_1	x_2	x_3	$\cdots$	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$\cdots$	$f(x_n)$

$$\begin{cases} N_2(x_0) = f(x_0) \\ N_2(x_1) = f(x_1) \\ N_2(x_2) = f(x_2) \end{cases}$$

设 $N_2(x) = N_1(x) + B(x - x_0)(x - x_1)$ $\xrightarrow{\text{则}}$ $\begin{cases} N_2(x_0) = f(x_0) \\ N_2(x_1) = f(x_1) \end{cases}$

由 $N_2(x_2) = N_1(x_2) + B(x_2 - x_0)(x_2 - x_1) = f(x_2)$

$n = 2$:

x	x_0	x_1	x_2	x_3	$\cdots$	x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$f(x_2)$	$f(x_3)$	$\cdots$	$f(x_n)$

$$\begin{cases} N_2(x_0) = f(x_0) \\ N_2(x_1) = f(x_1) \\ N_2(x_2) = f(x_2) \end{cases}$$

设 $N_2(x) = N_1(x) + B(x-x_0)(x-x_1)$ 则 $\begin{cases} N_2(x_0) = f(x_0) \\ N_2(x_1) = f(x_1) \end{cases}$

由 $N_2(x_2) = N_1(x_2) + B(x_2-x_0)(x_2-x_1) = f(x_2)$

得 $B = \dfrac{f(x_2) - N_1(x_2)}{(x_2-x_0)(x_2-x_1)} = \dfrac{f(x_2) - (f(x_0) + f[x_0,x_1](x_2-x_0))}{(x_2-x_0)(x_2-x_1)}$

$$= \frac{f(x_2) - f(x_0) - f[x_0,x_1](x_2-x_0)}{(x_2-x_0)(x_2-x_0)} = \frac{\dfrac{f(x_2)-f(x_0)}{x_2-x_0} - f[x_0,x_1]}{x_2-x_1}$$

$$= \frac{f[x_0,x_2] - f[x_0,x_1]}{x_2-x_1} \triangleq f[x_0,x_1,x_2]$$

$$\boxed{f[a,b,c] \triangleq \frac{f[a,c]-f[a,b]}{c-b}}$$

得 $N_2(x) = f(x_0) + f[x_0,x_1](x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$

一般地, 对于:

x	x_0	x_1	$\cdots$ x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$\cdots f(x_n)$

,

$$\boxed{N_n(x_i)=f(x_i), i=0,1,\cdots,n}$$

Newton插值是通过选取特殊的基 实现的，这时，

取

$\varphi_0(x)=1$

$$f(x_0)=N_n(x_0)=c_0$$

$$f(x_1)=N_n(x_1)=c_0+c_1(x_1-x_0)$$

$$=f(x_0)+c_1(x_1-x_0)$$

$$得c_1=\frac{f(x_1)-f(x_0)}{x_1-x_0}=f[x_0,x_1]$$

数

不超过 式 $N_n(x)$

$$N_n(x)=c_0+c_1(x-x_0)+c_2(x-x_0)(x-x_1)$$

$$+\cdots+c_n(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

其中 $c_0,c_1,\cdots,c_n$ 是待定系数，由插值条件 (2.1)决定。

➢ 差商(亦称均差) /* divided difference */

什么是差商???

先相减?

后相除?

$$f[x_i, x_j] = \frac{f(x_i) - f(x_j)}{x_i - x_j} \quad (i \neq j,\ x_i \neq x_j)$$

1阶差商

$$f[x_i, x_j, x_k] = \frac{f[x_i, x_j] - f[x_j, x_k]}{x_i - x_k} \quad (i \neq k)$$

2阶差商

$(k+1)$阶差商:

$$f[x_0, \ldots, x_{k+1}] = \frac{f[x_0, x_1, \ldots, x_k] - f[x_1, \ldots, x_k, x_{k+1}]}{x_0 - x_{k+1}}$$

$$= \frac{f[x_0, \ldots, x_{k-1}, x_k] - f[x_0, \ldots, x_{k-1}, x_{k+1}]}{x_k - x_{k+1}}$$

注: ☞ k阶差商必须由 $k+1$ 个节点构成, k个节点是构造不出 k阶差商的。

☞ 为统一起见,补充定义函数 $f(x_0)$ 为零阶差商。

用归纳法易证, 一般地, 对于:

x	x_0	x_1	$\cdots$ x_n
$f(x)$	$f(x_0)$	$f(x_1)$	$\cdots f(x_n)$

,

$$\boxed{N_n(x_i) = f(x_i), i = 0,1,\cdots,n}$$

通过插值条件运用数学归纳法（P29-30）可以求得

$$c_k = f[x_0, x_1, \cdots, x_k]$$

因此就得到下列的满足插值条件(2.1)的 n 次插值多项式

$$N_n(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots + f[x_0, x_1, \cdots, x_n](x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

$$N_n(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)(x-x_1) + \cdots + c_n(x-x_0)\cdots(x-x_{n-1})$$

$$\begin{cases} f(x) = f(x_0) + (x-x_0)\,f[x, x_0] & \cdots\cdots\cdots\cdots 1 \\ f[x, x_0] = f[x_0, x_1] + (x-x_1)\,f[x, x_0, x_1] & \cdots\cdots\cdots\cdots 2 \\ \quad \cdots\ \cdots\ \cdots\ \cdots & \\ f[x, x_0, \ldots, x_{n-1}] = f[x_0, \ldots, x_n] + (x-x_n)\,f[x, x_0, \ldots, x_n] & \cdots\cdots\ n-1 \end{cases}$$

$$1 + (x-x_0) \times 2 + \ldots\ \ldots + (x-x_0)\ldots(x-x_{n-1}) \times n-1$$

$$\Rightarrow f(x) = \boxed{f(x_0) + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1) + \ldots + f[x_0, \ldots, x_n](x-x_0)\ldots(x-x_{n-1})}$$
$$\boxed{+ f[x, x_0, \ldots, x_n](x-x_0)\ldots(x-x_{n-1})(x-x_n)}$$

$N_n(x)$

$$c_i = f[x_0, \ldots, x_i]$$

$R_n(x)$

差商的值与 x_i 的顺序无关!

差商性质:

1). k阶差商可表为函数值 $f(x_0), f(x_1), \cdots, f(x_k)$的线性组合，即

$$f[x_0,\cdots,x_k]=\sum_{j=0}^{k}\frac{f(x_j)}{(x_j-x_0)\cdots(x_j-x_{j-1})(x_j-x_{j+1})\cdots(x_j-x_k)}=\sum_{j=0}^{k}\frac{f(x_j)}{\omega'_{k+1}(x_j)}$$

2). 差商具有对称性，即

$$f[x_0,x_1,\cdots,x_k]=f[x_1,x_0,x_2\cdots,x_k]=\cdots=f[x_1,\cdots,x_k,x_0]$$

3). 若 $f(x)$是 n次多项式，则其k阶差商 $f[x_0,x_1\cdots,x_{k-1},x]$

当 $k\le n$时是一个 $n-k$次多项式，而当 $k>n$时恒为零.

4).若 $f(x)$在 $[a,b]$ 上有n阶导数,且节点 $x_i \in [a,b](i=0,1,\cdots n)$,
则 n 阶差商与n阶导数有如下关系式:

$$f[x_0,x_1\cdots,x_n]=\frac{f^{(n)}(\xi)}{n!} \qquad \xi \in [a,b]$$

注: ☞ 由唯一性可知 $N_n(x)\equiv L_n(x)$, 只是算法不同，故其余项也相同，即

$$f[x,x_0,\ldots,x_n]\omega_{n+1}(x)=\frac{f^{(n+1)}(\xi_x)}{(n+1)!}\omega_{n+1}(x)$$

$$\Rightarrow \quad f[x_0,\ldots,x_n]=\frac{f^{(n)}(\xi)}{n!}, \quad \xi\in(x_{\min},x_{\max})$$

差商计算可列差商表如下

x_i	$f(x_i)$	一阶差商		三阶差商
x_0				
x_1		$f[x_0,x_1]$		
x_2	$f(x_2)$	$f[x_1,x_2]$	$f[x_0,x_1,x_2]$	
x_3	$f(x_3)$	$f[x_2,x_3]$	$f[x_1,x_2,x_3]$	$f[x_0,x_1,x_2,x_3]$
$\cdots$	$\cdots$			$\cdots$

$\dfrac{f(x_1)-f(x_0)}{}$

$\dfrac{f(x_3)-f(x_2)}{x_3-x_2}$

$\dfrac{f(x_2)-f(x_1)}{x_2-x_1}$

$\dfrac{f[x_1,x_2]-f[x_0,x_1]}{}$

$\dfrac{f[x_2,x_3]-f[x_1,x_2]}{x_3-x_1}$

$$\frac{f(x_1,x_2,x_3)-f(x_0,x_1,x_2)}{x_3-x_0}$$

例 已知函数 $f(x)$ 在各节点处的函数值如下,用Newton插值法求 $f(0.596)$ 的值.

x_k	0.40	0.55	0.65	0.80
$f(x_k)$	0.41075	0.57815	0.69675	0.88811

解:

x_k	$f(x_k)$	一阶	二阶	三阶
0.40	0.41075			
0.55	0.57815	1.11600		
0.65	0.69675	1.18600	0.28000	
0.80	0.88811	1.27573	0.35893	0.19733

$$N_3(x) = 0.41075 + 1.11600(x-0.4) + 0.28000(x-0.4)(x-0.55) + 0.19733(x-0.4)(x-0.55)(x-0.65)$$

$$f(0.596) \approx 0.62836$$

§4 埃尔米特插值 /* Hermite Interpolation */

不仅要求函数值重合，而且要求若干阶导数也重合。
即：要求插值函数 $\varphi(x)$ 满足 $\varphi(x_i) = f(x_i)$, $\varphi'(x_i) = f'(x_i)$, …, $\varphi^{(m_i)}(x_i) = f^{(m_i)}(x_i)$.

注：☞ N 个条件可以确定 $N-1$ 阶多项式。

☞ 要求在1个节点 x_0 处直到 m_0 阶导数都重合的插值多项式即为Taylor多项式

$$\varphi(x) = f(x_0) + f'(x_0)(x-x_0) + \dots + \frac{f^{(m_0)}(x_0)}{m_0!}(x-x_0)^{m_0}$$

其余项为 $R(x) = f(x) - \varphi(x) = \dfrac{f^{(m_0+1)}(\xi)}{(m_0+1)!}(x-x_0)^{(m_0+1)}$

低次埃尔米特插值多项式

1.二点三次埃尔米特插值多项式

设给定区间 $[x_0, x_1]$两端点处的函数值与导数值如下：

x	x_0	x_1
$f(x)$	y_0	y_1
$f'(x)$	m_0	m_1

求插值多项式 $H_3(x)$,使满足

$$\begin{cases} H_3(x_0) = y_0, H_3(x_1) = y_1 \\ H_3'(x_0) = m_0, H_3'(x_1) = m_1 \end{cases}$$

由于给出了四个条件，故可唯一确定出一个三次多项式，不妨设

$$H_3(x) = \alpha_0(x)y_0 + \alpha_1(x)y_1 + \beta_0(x)m_0 + \beta_1(x)m_1$$

因此问题归结为构造

$$\alpha_0(x)与\beta_0(x),\alpha_1(x)与\beta_1(x)$$

首先, $\alpha_0(x)$与$\beta_0(x)$, $\alpha_1(x)$与$\beta_1(x)$应该为三次式。

再者，由插值条件易知 $\alpha_i(x_j),\beta_i(x_j)\ i,j=1,2$ 应该满足条件

$$\begin{cases} \alpha_i(x_j)=\delta_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases} & \alpha'_i(x_j)=0 \quad (i,j=0,1) \\ \beta_i(x_j)=0 & \beta'_i(x_j)=\delta_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases} \end{cases}$$

取

$$l^2_i(x)=\left(\frac{x-x_j}{x_i-x_j}\right)^2,(i,j=0,1;i\neq j)$$

令

$$\alpha_i(x) = (a_i x + b_i) l_i^2(x)$$

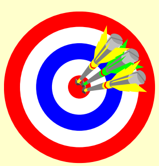
$$\beta_i(x) = c_i (x - x_i) l_i^2(x),\ i = 0,1;$$

其中a_i, b_i, c_i为待定系数，由$\alpha_0(x_0) = 1, \alpha'_0(x_0) = 0$知 a_0, b_0

满足

$$\begin{cases} a_0 x_0 + b_0 = 1 \\ a_0 + 2(a_0 x_0 + b_0) l_0'(x_0) = 0 \end{cases}$$

解之得



$$a_0 = -2 l_0'(x_0) = -2 \frac{1}{x_0 - x_1}$$

$$b_0 = 1 + 2 x_0 l_0'(x_0) = 1 + 2 x_0 \frac{1}{x_0 - x_1}$$

故

$$\alpha_0(x) = (1 - 2\frac{x - x_0}{x_0 - x_1})(\frac{x - x_1}{x_0 - x_1})^2$$

类似可以求得

$$\alpha_1(x) = (1 - 2\frac{x - x_1}{x_1 - x_0})(\frac{x - x_0}{x_1 - x_0})^2,$$

$$\beta_0(x) = (x - x_0)(\frac{x - x_1}{x_0 - x_1})^2,\ \beta_1(x) = (x - x_1)(\frac{x - x_0}{x_1 - x_0})^2$$

于是

$$H_3(x) = y_0(1 - 2\frac{x - x_0}{x_0 - x_1})\ l_0^2(x) + y_1(1 - 2\frac{x - x_1}{x_1 - x_0}) l_1^2(x)$$

$$+ m_0(x - x_0) l_0^2(x) + m_1(x - x_1) l_1^2(x)$$

2.三点三次带一个导数值的插值多项式

假设给定的函数表如下:

x	x_0	x_1	x_2
$f(x)$	y_0	y_1	y_2
$f'(x)$		m_1	

要求三次多项式$H_3(x)$,使

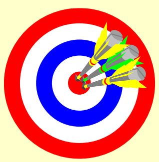
$$\begin{cases} H_3(x_i) = y_i \\ H_3'(x_1) = m_1 \end{cases} \qquad (i = 0,1,2)$$

利用满足三个条件的Newton插值多项式，我们设

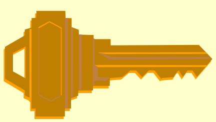
$$H_3(x) = y_0 + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + k(x - x_0)(x - x_1)(x - x_2)$$

其中 k 为待定系数，显然 $H_3(x)$ 满足前三个插值条件，利用第四个条件确定常数 k，于是

$$H_3'(x_1) = f[x_0, x_1] + f[x_0, x_1, x_2](x_1 - x_0)$$
$$+ k(x_1 - x_0)(x_2 - x_1) = m_1$$



$$k = \frac{m_1 - f[x_0, x_1] - f[x_0, x_1, x_2](x_1 - x_0)}{(x_0 - x_1)(x_1 - x_2)}$$



将其代入，即可得到 $H_3(x)$ 的表达式:

$$H_3(x) = y_0 + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$
$$+ \frac{m_1 - f[x_0, x_1] - f[x_0, x_1, x_2](x_1 - x_0)}{(x_0 - x_1)(x_1 - x_2)}(x - x_0)(x - x_1)(x - x_2)$$

例： 插值条件中有导数值，此时插值余项?

设有函数表

x	0	1	2
y	0	2	14
y'		5	

，求插值多项式，并写出余项表达式。

解:

设 $P_3(x)=a+bx+cx^2+dx^3$ 则 $P_3'(x)=b+2cx+3dx^2$

由插值条件得: $\begin{cases} P_3(0)=a=0 \\ P_3(1)=a+b+c+d \\ P_3'(1)=b+2c+3d=5 \\ P_3(2)=a+2b+4c+8d=14 \end{cases}$ 得 $\begin{cases} a=0 \\ b=1 \\ c=-1 \\ d=2 \end{cases}$

则 $P_3(x)=x-x^2+2x^3$

余项：类似Lagrange插值余项(只是把导数点也用上)

$$R_3(x)=f(x)-P_3(x)$$

则 $R_3(0)=R_3(1)=R_3(2)=0$，$R_3'(1)=f'(1)-P_3'(1)=0$

故可设 $R_3(x)=k(x)x(x-1)^2(x-2)$

设 $\varphi(t)=f(t)-P_3(t)-k(x)t(t-1)^2(t-2)=R_3(t)-k(x)t(t-1)^2(t-2)$

则当 $t=0,\ \ 1,\ \ 2,\ \ x$ 时 $\varphi(t)=0$

当 $t=\ \ \xi_0\ \ \xi_1\ \ \xi_2\ \ 1$ 时 $\varphi'(t)=0$

当 $t=\xi$ 时 $\varphi^{(4)}(t)=0,\ \xi\in[0,2]$

即 $\varphi^{(4)}(\xi)=0$

得 $f^{(4)}(\xi)-k(x)4!=0$

有 $k(x)=\dfrac{f^{(4)}(\xi)}{4!}$　　代入有: $R_3(x)=\dfrac{f^{(4)}(\xi)}{4!}x(x-1)^2(x-2)$

一般地，已知 $x_0, \dots, x_n$ 处有 $y_0, \dots, y_n$ 和 $y_0', \dots, y_n'$，求 $H_{2n+1}(x)$ 满足 $H_{2n+1}(x_i)=y_i$，$H'_{2n+1}(x_i)=y_i'$。

解：设 $H_{2n+1}(x)=\sum_{i=0}^{n} y_i\alpha_i(x)+\sum_{i=0}^{n} y_i'\beta_i(x)$

$l_i(x)=\prod_{j\neq i}\frac{(x-x_j)}{(x_i-x_j)}$

其中 $\alpha_i(x_j)=\delta_{ij}$, $\alpha_i'(x_j)=0$, $\beta_i(x_j)=0$, $\beta_i'(x_j)=\delta_{ij}$

$\alpha_i(x)$ 有零点 $x_0, \dots, \hat{x}_i, \dots, x_n$ 且都是2重零点 $\Rightarrow \alpha_i(x)=(A_i x+B_i)l_i^2(x)$

由余下条件 $\alpha_i(x_i)=1$ 和 $\alpha_i'(x_i)=0$ 可解 A_i 和 B_i $\Rightarrow$

$$\alpha_i(x)=[1-2l_i'(x_i)(x-x_i)]l_i^2(x)$$

$\beta_i(x)$ 有零点 $x_0, \dots, x_n$ 且都是2重零点 $\Rightarrow \beta_i(x)=C_i(x-x_i)l_i^2(x)$

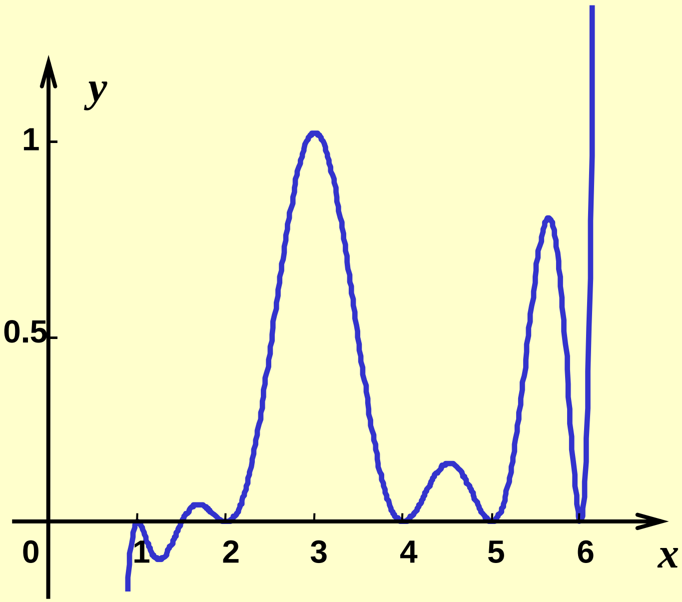
又由 $\beta_i'(x_i)=1 \Rightarrow C_i=1$

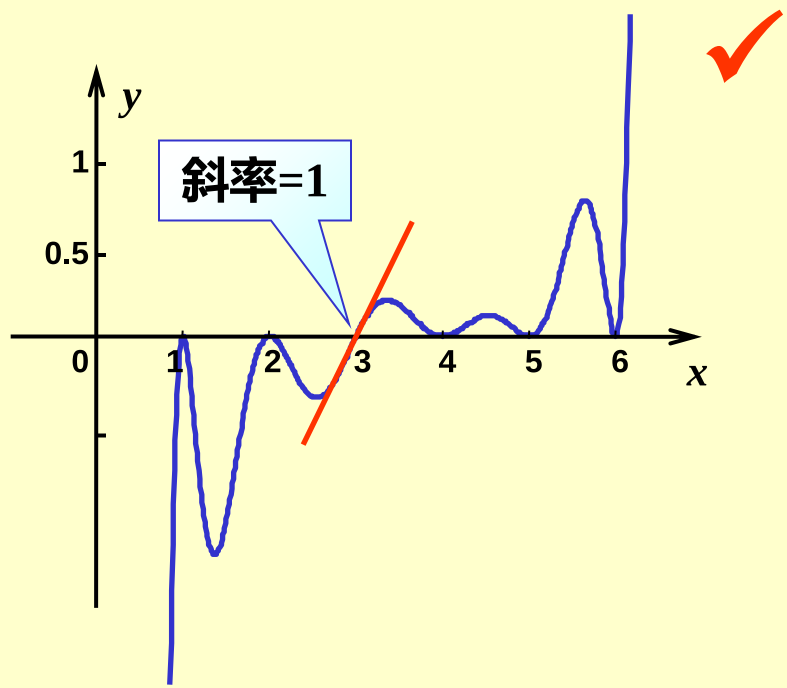
$$\beta_i(x)=(x-x_i)\,l_i^2(x)$$

这样的Hermite 插值唯一

设 $a=x_0<x_1<\dots<x_n=b$, $f\in C^{2n}[a,b]$ 则 $R_n(x)=\frac{f^{(2n+2)}(\xi_x)}{(2n+2)!}\left[\prod_{i=0}^{n}(x-x_i)\right]^2$

Quiz: 给定 $x_i = i + 1,\ i = 0, 1, 2, 3, 4, 5$. 下面哪个是 $\beta_2(x)$的图像?

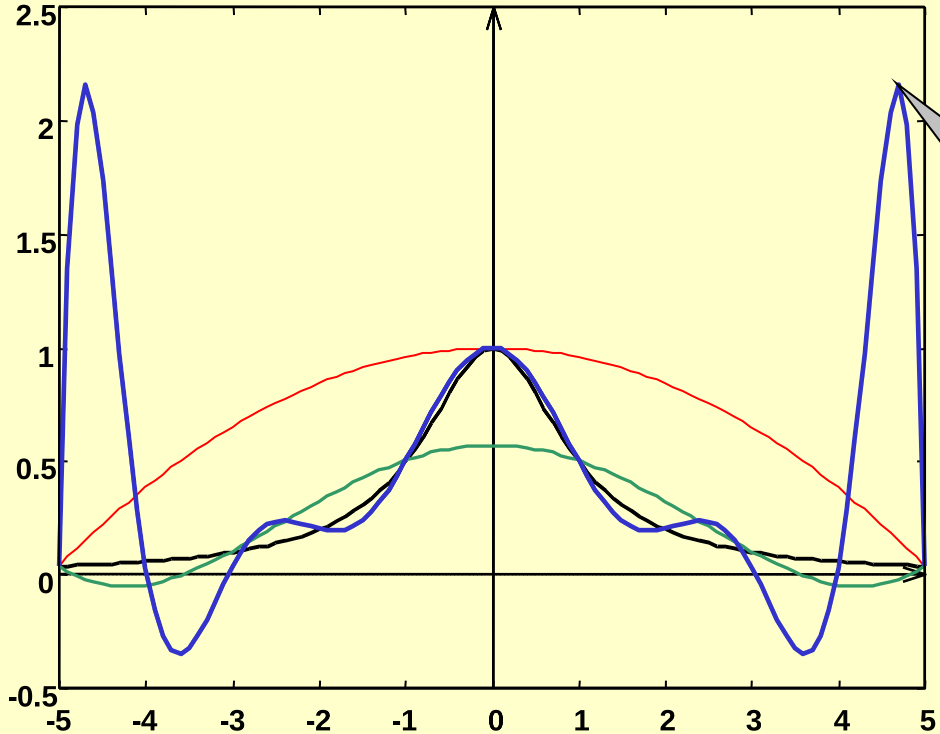




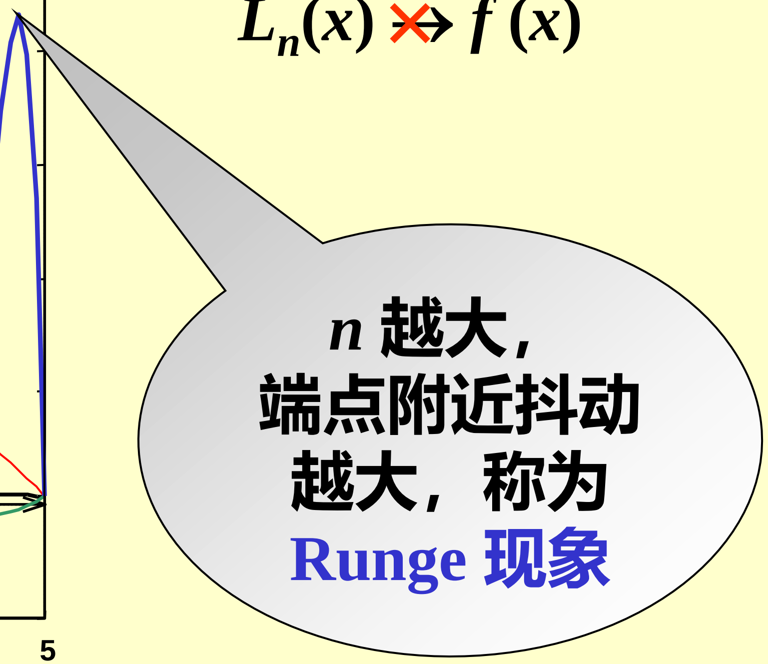
§5 分段低次插值 /* piecewise polynomial approximation */

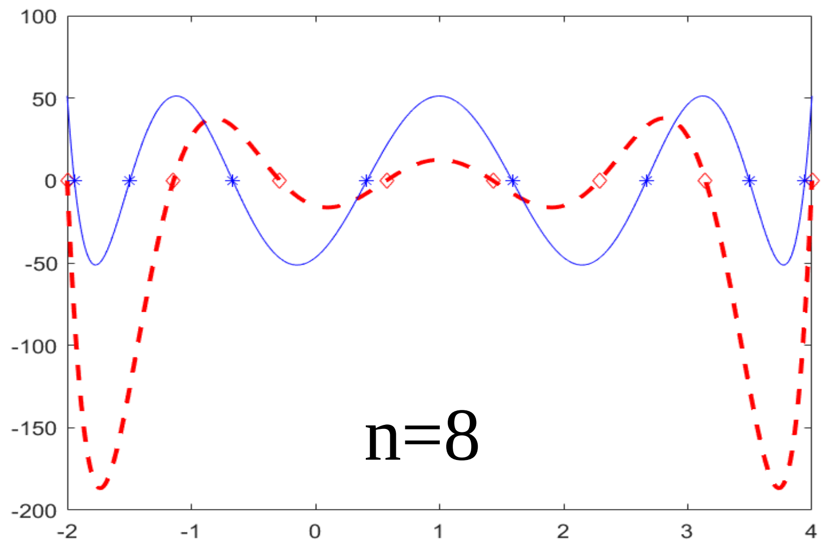
在区间 $[a,b]$ 上用插值多项式 $P_n(x)$ 近似函数 $f(x)$，是否 $P_n(x)$ 的次数越高，逼近效果越好呢，回答是否定的。由于次数越高计算工作量也越大，积累误差也越大；在整个区间上作高次多项式，当局部插值节点的值有微小误差时，就可能引起整个区间上函数值的较大变化，使计算不稳定。

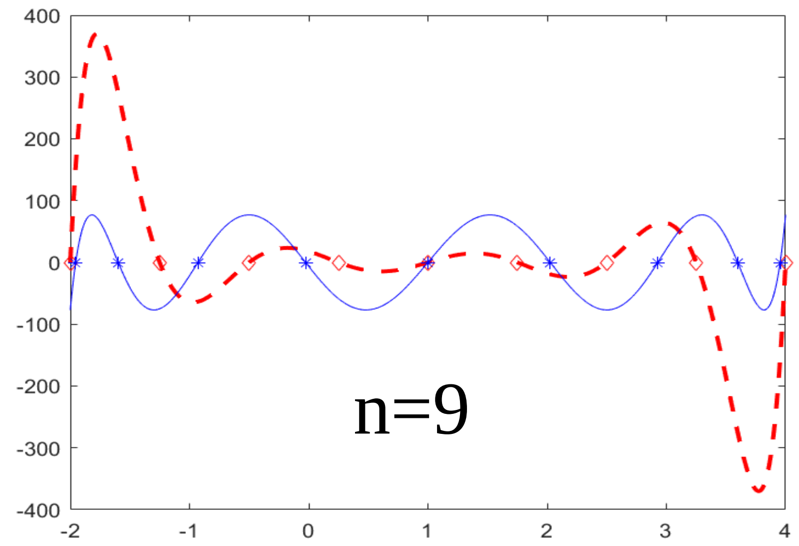
例： 在 $[-5, 5]$ 上考察 $f(x)=\dfrac{1}{1+x^2}$ 的 $L_n(x)$。取 $x_i=-5+\dfrac{10}{n}i \quad (i=0,\ldots,n)$

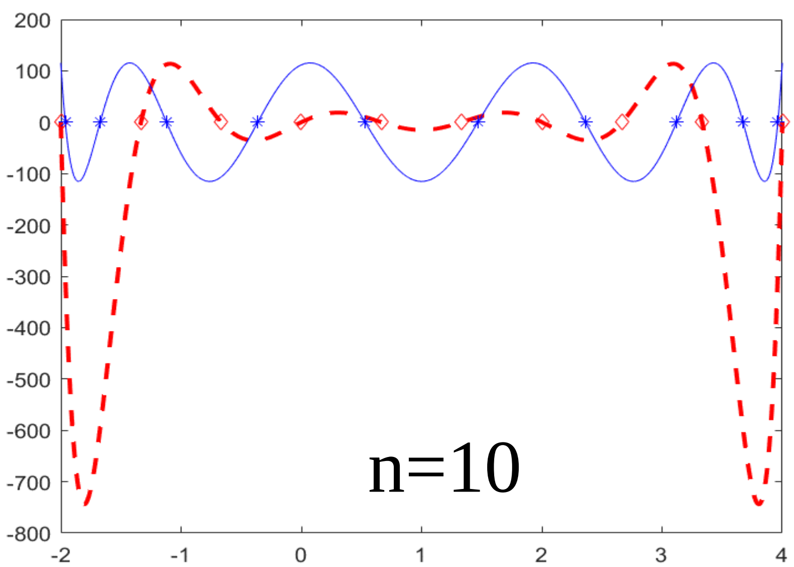


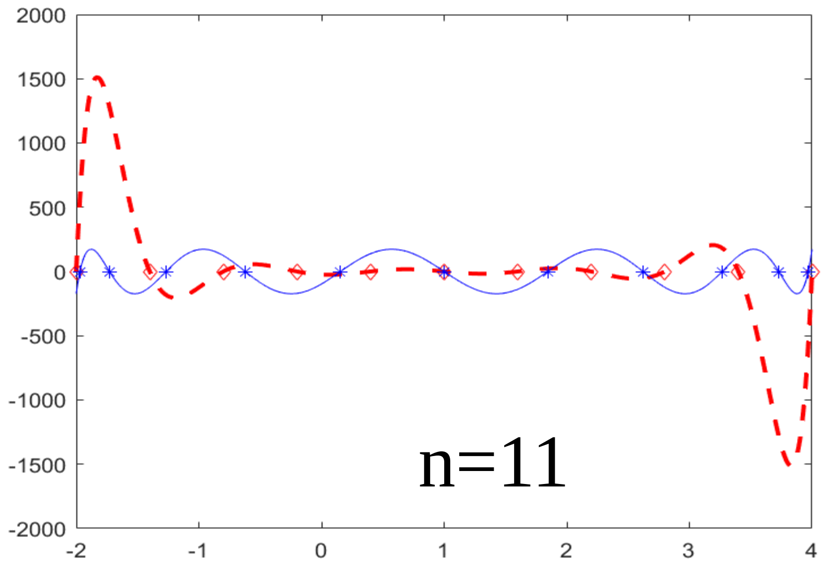
$L_n(x) \not\to f(x)$

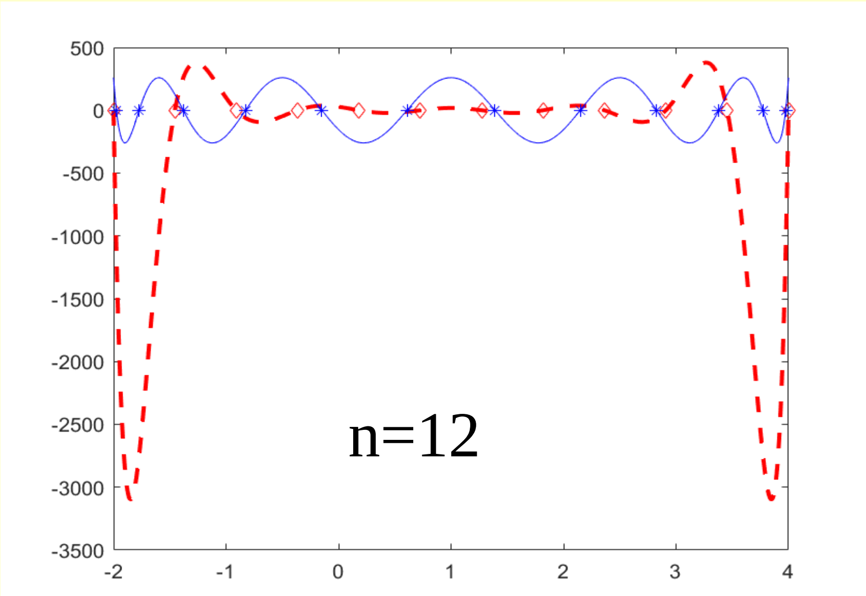


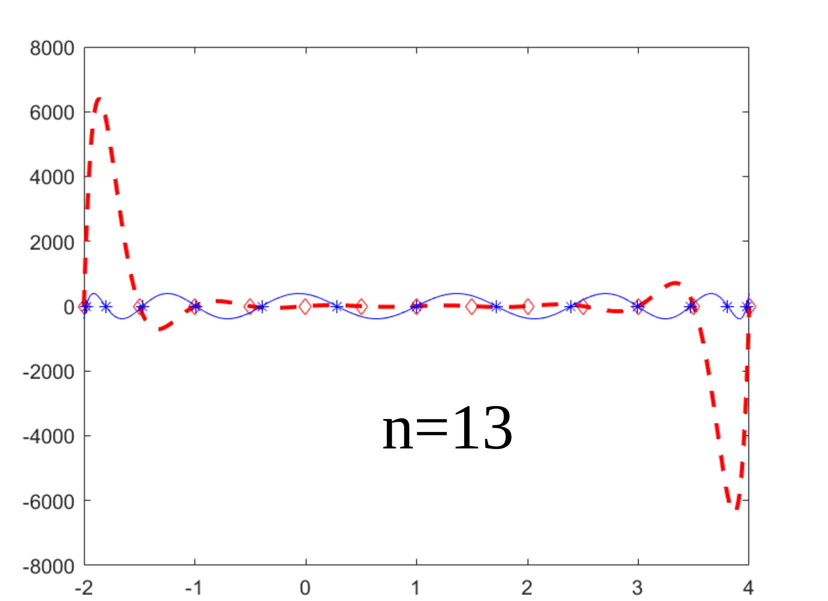

100
50
0
-50
-100
-150
-200
-2 -1 0 1 2 3 4
n=8

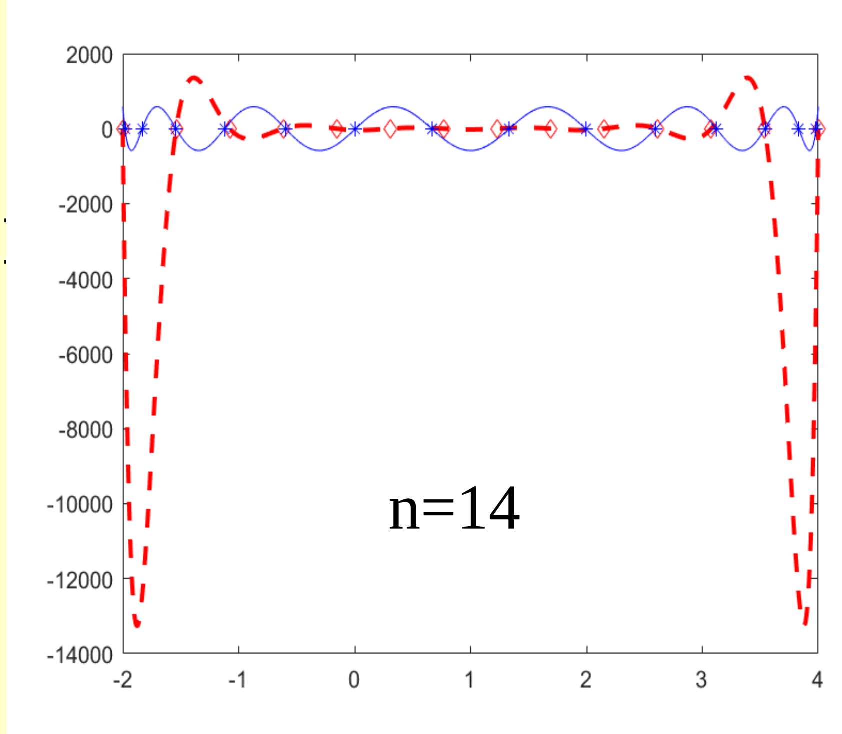

400
300
200
100
0
-100
-200
-300
-400
-2 -1 0 1 2 3 4
n=9

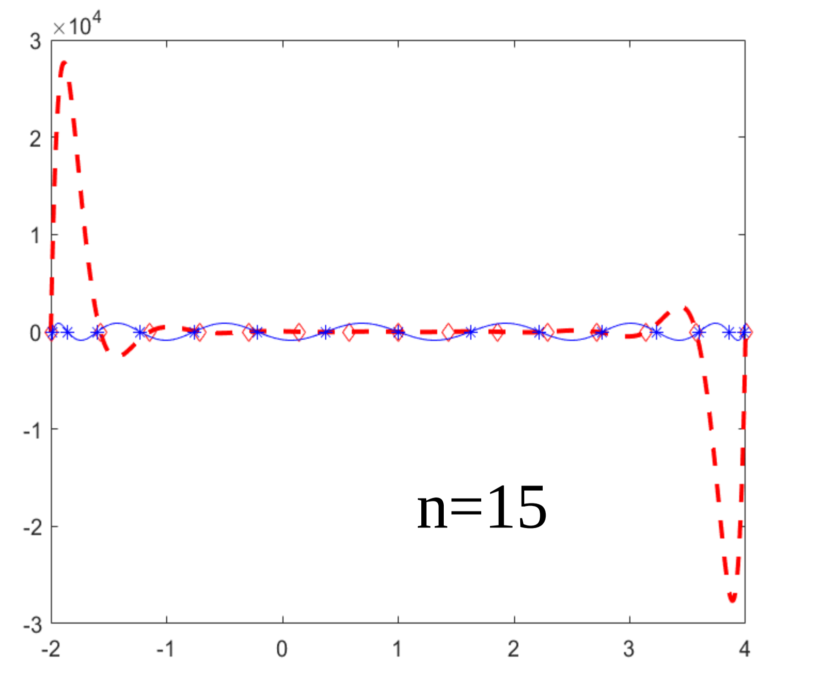

200
100
0
-100
-200
-300
-400
-500
-600
-700
-800
-2 -1 0 1 2 3 4
n=10


2000
1500
1000
500
0
-500
-1000
-1500
-2000
-2 -1 0 1 2 3 4
n=11


500
0
-500
-1000
-1500
-2000
-2500
-3000
-3500
-2 -1 0 1 2 3 4
n=12


8000
6000
4000
2000
0
-2000
-4000
-6000
-8000
-2 -1 0 1 2 3 4
n=13


2000
0
-2000
-4000
-6000
-8000
-10000
-12000
-14000
-2 -1 0 1 2 3 4
n=14


×10^4
3
2
1
0
-1
-2
-3
-2 -1 0 1 2 3 4
n=15

分段低次插值

➢ 分段线性插值 /* piecewise linear interpolation */

所谓的分段线性插值就是通过插值点用折线段连接起来逼近 $f(x)$ 设已知节点 $a < x_0 < x_1 < \cdots < x_n = b$ 上函数值 $f_0, f_1, \cdots, f_n$，记

$$h_k = x_{k+1} - x_k, h = \max_k h_k$$

求一折线函数 $I_h(x)$ 满足

$$(1) I_h(x) \in C[a,b],$$

$$(2) I_h(x_k) = f_k (k = 0,1,\cdots,n)$$

(3) $I_h(x)$ 在每个小区间 $[x_k, x_{k+1}]$ 上都是线性函数。

则称 $I_h(x)$ 为分段线性插值函数。

设

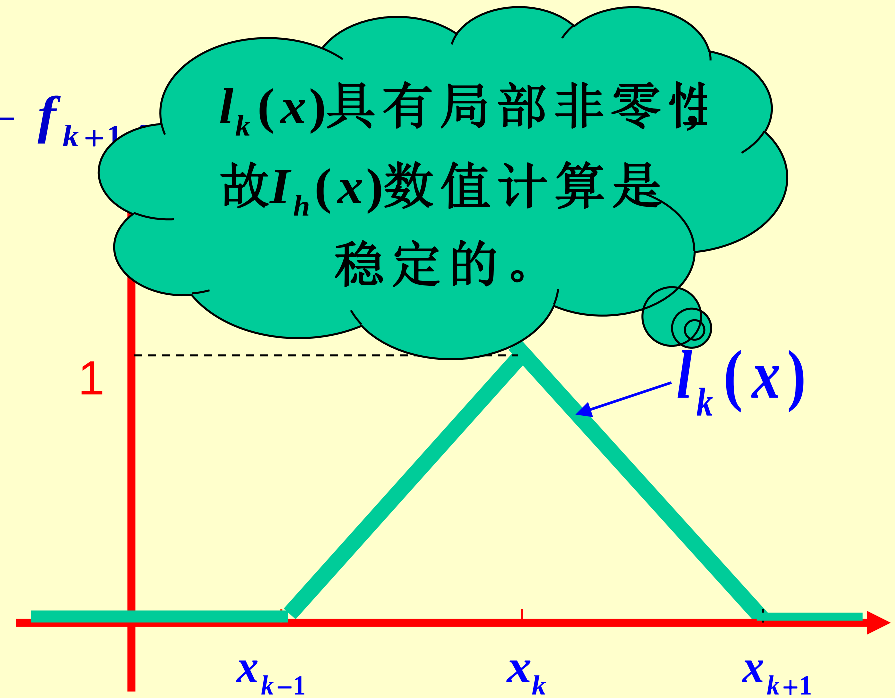
$$I_h(x) = \sum_{k=0}^{n-1} f_k l_k(x)$$

由定义可知$I_h(x)$在每个小区间上可表示为

$$I_h(x)=\frac{x-x_{k-1}}{x_k-x_{k-1}}f_k+\frac{x-x_k}{x_{k-1}-x_k}f_{k-1},\ x\in[x_{k-1},x_k]$$

$$I_h(x)=\frac{x-x_{k+1}}{x_k-x_{k+1}}f_k+\frac{x-x_k}{x_{k+1}-x_k}f_{k+1}$$

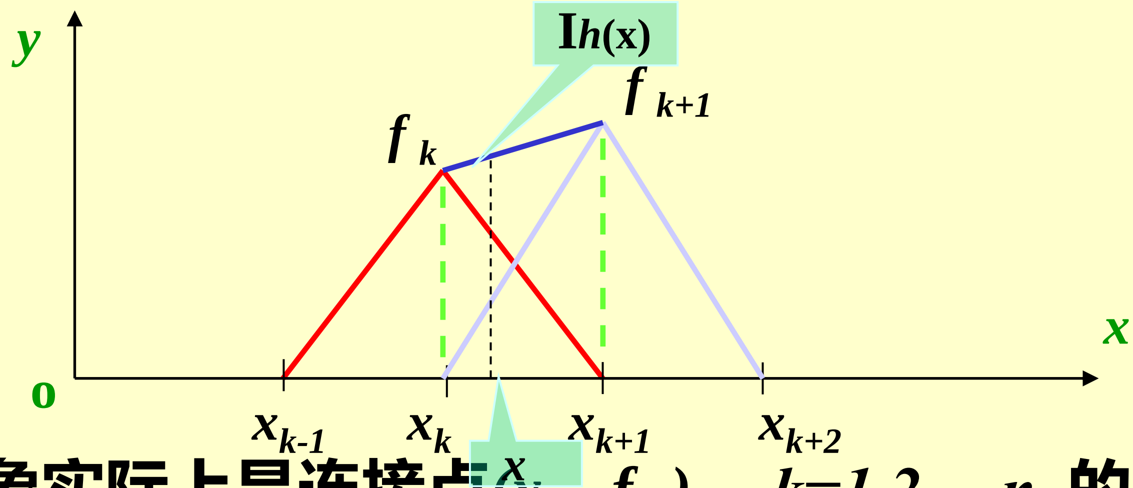
$$l_k(x)=\begin{cases}\dfrac{x-x_{k-1}}{x_k-x_{k-1}}, & x_{k\text{-}1}\le x\le x_k\,(k\ne 0)\\[2mm] \dfrac{x-x_{k+1}}{x_k-x_{k+1}}, & x_k\le x\le x_{k+1}\,(k\ne n)\\[2mm] 0, & x\in[a,b],\text{但}x\notin[x_{k-1},x_{k+1}].\end{cases}$$



注意 表达式 $I_h(x)=\sum_{k=0}^{n-1}f_k l_k(x)$在区间$[x_k,x_{k+1}]$上，只有$l_k(x),l_{k+1}(x)$是非零的，其它基函数均为零。即

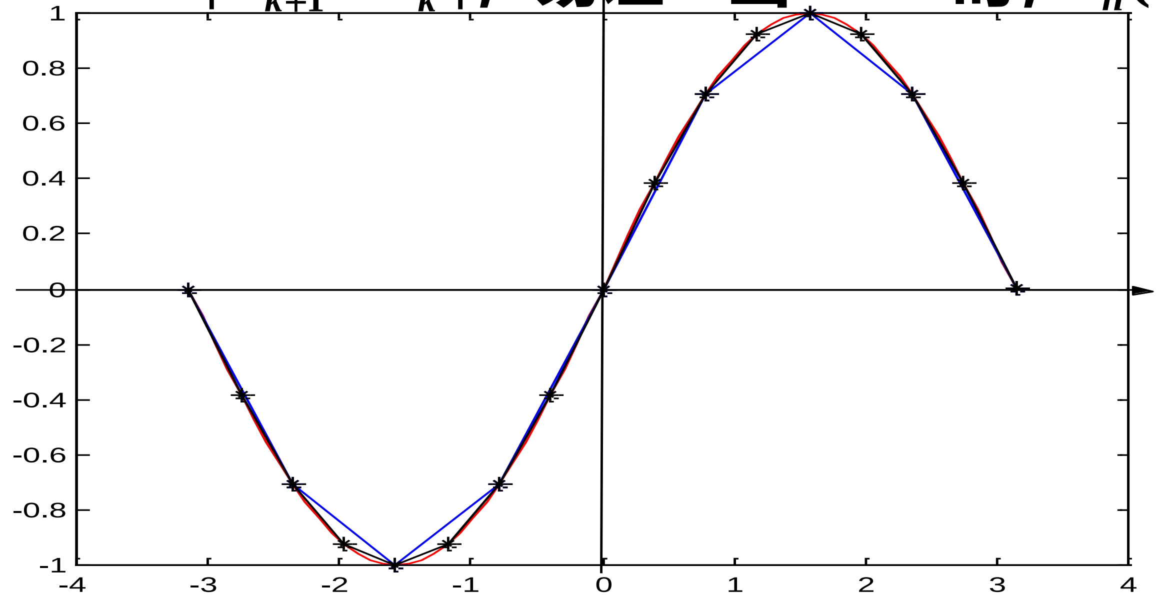
$$I_h(x)=f_k l_k(x)+f_{k+1}l_{k+1}(x)\qquad x\in[x_k,x_{k+1}]$$

如图：



$y=I_h(x)$的图象实际上是连接点(x_k, f_k)，$k=1,2,\ldots n$ 的一条折线，也称**折线插值**，如下图。如果增加节点数量，会改善插值效果，即：

记$h=\max|x_{k+1}-x_k|$，易证：当 $h\to 0$ 时，$I_h(x)\xrightarrow{\text{一致}} f(x)$



在节点处有尖点，失去了原函数的光滑性。

分段线性插值的误差估计

根据拉格朗日1次插值函数的余项,可以得到分段线性插值函数的插值误差估计:

对$x\in[a,b]$,当$x\in[x_k, x_{k+1}]$时,

$$R(x)=f(x)-I_h(x)=\frac{f''(x)}{2}(x-x_k)(x-x_{k+1})$$

则 $$|R(x)|\le\frac{h^2}{8}M$$ 其中 $$h=\max_{0\le i\le n-1}|x_{i+1}-x_i|$$ $$M=\max_{x\in(a,b)}|f''(x)|$$

可以加密插值结点，缩小插值区间，使h减小，从而减小误差

思考： 要构造对数表$\log_{10}x$，$10\le x<100$,怎样选择步长h，才能使得分段线性插值具有六位有效数字?

➢ 分段三次Hermite插值 /* Hermite piecewise polynomials */

分段线性插值函数$I_h(x)$的导数是间断的，若在节点$x_k(k=0,1,\cdots,n)$上除已知函数值f_k外还给出导数值$f'_k=m_k$，这样就可以构造一个导数连续的分段插值多项式函数$I_h(x)$，它满足

$$(1)\ I_h\in C^1[a,b],$$

$$(2)\ I_h(x_k)=f_k,\ I'_h(x_k)=f'_k\ (k=0,1,\cdots,n)$$

(3) $I_h(x)$在每个小区间$[x_k,x_{k+1}]$是三次多项式

设

$$I_h(x)=\sum_{k=0}^{n}[f(x_k)\alpha_k(x)+f'(x_k)\beta_k(x)]$$

根据两点三次插值多项式。可知，在区间$[x_k, x_{k+1}]$上 $I_h(x)$的表达式为

$$
\begin{aligned}
I_h(x) = {} & \left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2\left(1+2\frac{x-x_k}{x_{k+1}-x_k}\right)f_k \\
& + \left(\frac{x-x_k}{x_{k+1}-x_k}\right)^2\left(1+2\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)f_{k+1} \\
& + \left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2(x-x_k)f'_k \\
& + \left(\frac{x-x_k}{x_{k+1}-x_k}\right)^2(x-x_{k+1})f'_{k+1}
\end{aligned}
$$

于是:

$$\alpha_k(x)=\begin{cases}\left(\dfrac{x-x_{k-1}}{x_k-x_{k-1}}\right)^2\left(1+2\dfrac{x-x_k}{x_{k-1}-x_k}\right) & x\in[x_{k-1},x_k]\\[2ex] \left(\dfrac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2\left(1+2\dfrac{x-x_k}{x_{k+1}-x_k}\right) & x\in[x_k,x_{k+1}]\\[2ex] 0 & else\end{cases}$$

$$\beta_k(x)=\begin{cases}\left(\dfrac{x-x_{k-1}}{x_k-x_{k-1}}\right)^2(x-x_k) & x \\[2ex] \left(\dfrac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2(x-x_k) & \\[2ex] 0 & else\end{cases}$$

How can we make a smooth interpolation without asking too much from *f* ? Headache …

导数一般不易得到。