Hamilton - Logic for Mathematicians
Notes and Solutions

Contents

Chapter 1

Informal statement calculus

1.1 Statements and connectives

Solutions to exercises

1. (a) $(D \wedge P) \to T$

   (b) $J \to E$

   (c) $\sim J \to ((S \vee R) \wedge \sim E)$

   (d) $(X \wedge Y) \to \sim Z$

   (e) $M \vee H$

   (f) $(\sim M) \to H$

   (g) $S \leftrightarrow (E \vee O)$

   (h) $X \to (Y \to \sim Z)$

2. (a) The statements (a) and (d) have the same form.

   (b) The statements (d) and (h) have the same meaning ((a) also has the same form as (d), so it could also be interpreted as having the same meaning). The statements (e) and (f) have the same meaning.

1.2 Truth functions and truth tables

Solutions to exercises

3. Because truth tables are tedious to write, only (a) has been done. The rest are done similarly.

   (a)

$((\sim$	$p)$	$\wedge$	$(\sim$	$q))$
F	T	$\underline{\text{F}}$	F	T
F	T	$\underline{\text{F}}$	T	F
T	F	$\underline{\text{F}}$	F	T
T	F	$\underline{\text{T}}$	T	F

4. When p and q take on particular values, the values of $((\sim p) \vee q)$ and $(p \to q)$ are identical. This can be shown by constructing truth tables, but that process is

omitted here. Similarly, $((\sim p) \rightarrow (q \vee r))$ can be shown to give rise to the same truth function as $((\sim q) \rightarrow ((\sim r) \rightarrow p))$ by constructing truth tables.

5. The statement forms (a), (b), and (d) are tautologies.

6. Because truth tables are tedious to write, only (a) has been done. The rest are done similarly.

   (a) The truth table for $(p \rightarrow q)$:

$(p$	$\rightarrow$	$q)$
T	$\underline{\text{T}}$	T
T	$\underline{\text{F}}$	F
F	$\underline{\text{T}}$	T
F	$\underline{\text{T}}$	F

   The truth table for $((\sim q) \rightarrow (\sim p))$:

$((\sim$	$q)$	$\rightarrow$	$(\sim$	$p))$
F	T	$\underline{\text{T}}$	F	T
F	T	$\underline{\text{T}}$	T	F
T	F	$\underline{\text{F}}$	F	T
T	F	$\underline{\text{T}}$	T	F

   Notice that when p and q take on the same values in both truth tables, the underlined values, which indicate the values of the statement forms, are identical. Therefore, the two statement forms are logically equivalent.

7. When p and q are both true, the value $(((\sim p) \rightarrow q) \rightarrow (p \rightarrow (\sim q)))$ is false, so the statement form is not a tautology.

   Let $\mathscr{A} = \mathscr{B} = (p \rightarrow p)$. Notice that this is a tautology, and for that matter, any tautology can be substituted here. The value of $(((\sim \mathscr{A}) \rightarrow \mathscr{B}) \rightarrow (\mathscr{A} \rightarrow (\sim \mathscr{B})))$ can then be seen to be a contradiction by constructing the appropriate truth table or in the same way that the above statement form was seen to be false.

1.3 Rules for manipulation and substitution

Proposition 1.10. If $\mathscr{A}$ and $(\mathscr{A} \rightarrow \mathscr{B})$ are tautologies, then $\mathscr{B}$ is a tautology.
Proof. Suppose for a contradiction that $\mathscr{A}$ and $(\mathscr{A} \rightarrow \mathscr{B})$ are tautologies, while $\mathscr{B}$ is not a tautology. Then there is an assignment of truth values to the statement variables of $\mathscr{B}$ such that $\mathscr{B}$ is given the value F, while $\mathscr{A}$, being a tautology is necessarily given the value T. But then $(\mathscr{A} \rightarrow \mathscr{B})$ must have the value F, which contradicts it being a tautology.

Proposition 1.11. Let $\mathscr{A}$ be the statement form in which the statement variables $p_1, p_2, \ldots, p_n$ appear, and let $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n$ be any statement forms. If $\mathscr{A}$ is a tautology, then $\mathscr{B}$, the statement form obtained by replacing p_i in $\mathscr{A}$ with $\mathscr{A}_i$, is also a tautology.
Proof. By definition of $\mathscr{B}$, any assignment of truth values to $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n$ in $\mathscr{B}$, would result in the same truth value of $\mathscr{A}$ if the same truth values had been assigned to $p_1, p_2, \ldots, p_n$. The truth value of $\mathscr{A}$ is T, and so the truth value of $\mathscr{B}$ must also be T, making it a tautology as well.

Note. This proof is not entirely rigorous. See the note in Proposition 1.15.
Note. It is important to note that if $\mathscr{A}$ is not a tautology, then $\mathscr{B}$ may not be an equivalent truth function (see Exercise 7). However, if $\mathscr{A}$ is a contradiction, then $\mathscr{B}$ must be a contradiction, and the proof of this is nearly identical.

Proposition 1.12. For any statement forms $\mathscr{A}$ and $\mathscr{B}$, $(\sim(\mathscr{A} \wedge \mathscr{B}))$ is logically equivalent to $((\sim \mathscr{A}) \vee (\sim \mathscr{B}))$, and $(\sim(\mathscr{A} \vee \mathscr{B}))$ is logically equivalent to $((\sim \mathscr{A}) \wedge (\sim \mathscr{B}))$.
Proof. By Example 1.8 in the book or by creating tables, it can be easily seen that

$$(\sim (p \wedge q)) \leftrightarrow ((\sim p) \vee (\sim a)), \text{ and } (\sim (p \vee q)) \leftrightarrow ((\sim p) \wedge (\sim q))$$

is a tautology. By application of Proposition 1.11, we may conclude that if any statement form with p and q substituted for the statement forms $\mathscr{A}$ and $\mathscr{B}$ would be a tautology as well, i.e.,

$$(\sim (\mathscr{A} \wedge \mathscr{B})) \leftrightarrow ((\sim \mathscr{A}) \vee (\sim \mathscr{B})), \text{ and } (\sim (\mathscr{A} \vee \mathscr{B})) \leftrightarrow ((\sim \mathscr{A}) \wedge (\sim \mathscr{A}))$$

is a tautology, and therefore $(\sim(\mathscr{A} \wedge \mathscr{B}))$ is logically equivalent to $((\sim \mathscr{A}) \vee (\sim \mathscr{B}))$, and $(\sim(\mathscr{A} \vee \mathscr{B}))$ is logically equivalent to $((\sim \mathscr{A}) \wedge (\sim \mathscr{B}))$.

Proposition 1.14. Let $\mathscr{A}$ and $\mathscr{B}$ be logically equivalent statement forms. Let $\mathscr{A}'$ be a statement form in which $\mathscr{A}$ appears. Let $\mathscr{B}'$ be the statement form in which every instance of $\mathscr{A}$ is replaced by $\mathscr{A}'$. The statement forms $\mathscr{A}'$ and $\mathscr{B}'$ are logically equivalent.
Proof. We wish to show that $\mathscr{A}' \leftrightarrow \mathscr{B}'$ is a tautology, which is done by showing that the truth values of $\mathscr{A}'$ and $\mathscr{B}'$ always match under some arbitrary assignment of truth values to any statement variables. Consider the truth value of $\mathscr{A}'$ under some assignment of truth values. The truth value of $\mathscr{B}'$ must be the same, since it differs only in having $\mathscr{B}$ in place of $\mathscr{A}$, and $\mathscr{A}$ and $\mathscr{B}$ are logically equivalent. Therefore, $\mathscr{A}' \leftrightarrow \mathscr{B}$ must be a tautology, and so $\mathscr{A}'$ and $\mathscr{B}'$ must be logically equivalent.
Note. This proof is not entirely rigorous. See the note in Proposition 1.15.

For the remainder of Chapter 1, a statement form involving only the connectives $\sim$, $\wedge$ and $\vee$ will be called a *restricted statement form*.

Proposition 1.15. Let $\mathscr{A}$ be a restricted form and let $\mathscr{A}'$ be the statement form obtained from $\mathscr{A}$ by interchanging $\wedge$ and $\vee$ and replacing every statement variable p by $(\sim p)$. The statement forms $\mathscr{A}$ and $\mathscr{A}'$ are logically equivalent.
Proof. The proof is by strong induction on n, the number of connectives which appear in $\mathscr{A}$.

(base case) It may be the case that $n = 0$, i.e., $\mathscr{A}$ has no connectives, so $\mathscr{A}$ is p, where p is any statement variable. Then $\mathscr{A}'$ must be $(\sim p)$, which is logically equivalent to $(\sim \mathscr{A})$.

(inductive step) It may be the case that $n > 0$, i.e., $\mathscr{A}$ has one or more connectives. Suppose as an inductive hypothesis that any restricted statement form with fewer than n connectives is equivalent to the statement form obtained by interchanging $\wedge$ and $\vee$ and replacing each statement variable by its negation. Since $\mathscr{A}$ has one or more connectives, there are three cases to consider, based on the final connective in $\mathscr{A}$ to be evaluated.

1. It may be that $\mathscr{A}$ has the form $(\sim \mathscr{B})$, in which case $\mathscr{A}'$ must be $(\sim \mathscr{B}')$, since the only instances of $\wedge$, $\vee$, and any statement variables must necessarily appear in $\mathscr{B}$.

By the inductive hypothesis, since $\mathscr{B}$ has fewer than n connectives, $\mathscr{B}'$ is logically equivalent to $(\sim \mathscr{B})$. By Proposition 1.14, $(\sim \mathscr{B}')$ must be logically equivalent to $(\sim (\sim \mathscr{B}))$ which is $(\sim \mathscr{A})$. Since $(\sim \mathscr{B}')$ is $\mathscr{A}'$, we have proved that $\mathscr{A}'$ is logically equivalent to $(\sim \mathscr{A})$, as desired.

2. It may be that $\mathscr{A}$ has the form $(\mathscr{B} \vee \mathscr{C})$. Then $\mathscr{A}'$ must be $(\mathscr{B}' \wedge \mathscr{C}')$. By the induction hypothesis, $\mathscr{B}'$, which must have fewer than n connectives, is logically equivalent to $(\sim \mathscr{B})$. By proposition 1.14, $(\mathscr{B}' \wedge \mathscr{C}')$ must be logically equivalent to $((\sim \mathscr{B}) \wedge \mathscr{C}')$, which must again be equivalent to $((\sim \mathscr{B}) \wedge (\sim \mathscr{C}))$ by the same reasoning applied to $\mathscr{C}'$, which must finally be equivalent to $(\sim (\mathscr{B} \vee \mathscr{C}))$ by Proposition 1.11, and this final statement form is $(\sim \mathscr{A})$, as desired.

3. It may be that $\mathscr{A}$ has the form $(\mathscr{B} \wedge \mathscr{C})$. Then $\mathscr{A}'$ must be $(\mathscr{B}' \vee \mathscr{C}')$. By the induction hypothesis, $\mathscr{B}'$, which must have fewer than n connectives, is logically equivalent to $(\sim \mathscr{B})$. By proposition 1.14, $(\mathscr{B}' \vee \mathscr{C}')$ must be logically equivalent to $((\sim \mathscr{B}) \vee \mathscr{C}')$, which must again be equivalent to $((\sim \mathscr{B}) \vee (\sim \mathscr{C}))$ by the same reasoning applied to $\mathscr{C}'$, which must finally be equivalent to $(\sim (\mathscr{B} \wedge \mathscr{C}))$ by Proposition 1.11, and this final statement form is $(\sim \mathscr{A})$, as desired.

Verifying the desired property for all three cases finishes the inductive step, thereby concluding the induction and the proof.

Note. Whenever induction appears in the textbook, it is referring to strong induction. The general framework for induction seen in this proof will resemble all other inductive proofs seen throughout the manual. Strictly speaking, strong induction has no base case, but practically speaking, there are usually one or more special values of the number being inducted on which require special attention since they do not rely on the inductive hypothesis. These will be referred to as the base cases.

Note. The proofs of Proposition 1.14 and Proposition 1.10 could have been made rigorous by being done similarly, but an inductive proof would have been unnecessarily lengthy to prove the propositions, which were obvious.

Corollary 1.16. Let $p_1, p_2, \ldots, p_n$ be statement variables.

(i) The statement forms

$$(\sim (p_1 \wedge p_2 \wedge \cdots \wedge p_n)) \text{ and } ((\sim p_1) \vee (\sim p_2) \vee \cdots \vee (\sim p_n))$$

are logically equivalent.

(ii) The statement forms

$$(\sim (p_1 \vee p_2 \vee \cdots \vee p_n)) \text{ and } ((\sim p_1) \wedge (\sim p_2) \wedge \cdots \wedge (\sim p_n))$$

are logically equivalent.

Proof. For (i), let $\mathscr{A}$ be the statement form

$$(p_1 \wedge p_2 \wedge \cdots \wedge p_n).$$

Then interchanging $\wedge$ and $\vee$ and negating each statement variable results in

$$((\sim p_1) \vee (\sim p_2) \vee \cdots \vee (\sim p_n)).$$

By Proposition 1.15, this is equivalent to $(\sim \mathscr{A})$, which is

$$(\sim (p_1 \wedge p_2 \wedge \cdots \wedge p_n)),$$

as desired. Part (ii) is proved in the same way as (i).

Proposition 1.17 (De Morgan's Laws)**.** Let $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_n$ be statement forms.

(i) The statement forms

$$(\sim (\mathscr{A}_1 \wedge \mathscr{A}_2 \wedge \cdots \wedge \mathscr{A}_n)) \text{ and } ((\sim \mathscr{A}_1) \vee (\sim \mathscr{A}_2) \vee \cdots \vee (\sim \mathscr{A}_n))$$

are logically equivalent.

(ii) The statement forms

$$(\sim (\mathscr{A}_1 \vee \mathscr{A}_2 \vee \cdots \vee \mathscr{A}_n)) \text{ and } ((\sim \mathscr{A}_1) \wedge (\sim \mathscr{A}_2) \wedge \cdots \wedge (\sim \mathscr{A}_n))$$

are logically equivalent.

Proof. This is an application of Proposition 1.10 to Corollary 1.16.

Solutions to exercises

8. Since (a) - (d) are proved in the same way, only (a) will be done, and the rest will be omitted.

   (a) Let p, q, r be statement letters. The following truth table demonstrates that $((p \vee (q \vee r)) \leftrightarrow ((p \vee q) \vee r))$ is a tautology.

$((p$	$\vee$	$(q$	$\vee$	$r))$	$\underline{\leftrightarrow}$	$((p$	$\vee$	$q)$	$\vee$	$r))$
1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1	1	1	0
1	1	0	1	1	1	1	1	0	1	1
1	1	0	0	0	1	1	1	0	1	0
0	1	1	1	1	1	0	1	1	1	1
0	1	1	1	0	1	0	1	1	1	0
0	1	0	1	1	1	0	0	0	1	1
0	0	0	0	0	1	0	0	0	0	0

   By proposition 1.10, the statement form $((\mathscr{A} \vee (\mathscr{B} \vee \mathscr{C})) \leftrightarrow ((\mathscr{A} \vee \mathscr{B}) \vee \mathscr{C}))$ is a tautology, where $\mathscr{A}, \mathscr{B}, \mathscr{C}$ are any statement forms. Therefore, $((\mathscr{A} \vee (\mathscr{B} \vee \mathscr{C}))$ is logically equivalent to $((\mathscr{A} \vee \mathscr{B}) \vee \mathscr{C}))$.

9. (a) The statement form $((p \wedge q) \rightarrow p)$ can be seen to be a tautology by creating a truth table for it. By Proposition 1.10, $((\mathscr{A} \wedge \mathscr{B}) \rightarrow \mathscr{A})$ must be a tautology as well.

   (b) The statement form $((p \wedge q) \rightarrow q)$ can be seen to be a tautology by creating a truth table for it. By Proposition 1.10, $((\mathscr{A} \wedge \mathscr{B}) \rightarrow \mathscr{B})$ must be a tautology as well.

10. By part (a) of Example 1.4 in the book, it can be seen that $((\sim p) \vee q)$ is logically equivalent to $(p \to q)$, and so $(((\sim p) \vee q) \to (p \to q))$ is a tautology. By Proposition 1.10, for any statements forms $\mathscr{A}$ and $\mathscr{B}$, $(((\sim \mathscr{A}) \vee \mathscr{B}) \to (\mathscr{A} \to \mathscr{B}))$ must be a tautology as well, and so $(((\sim \mathscr{A}) \vee \mathscr{B})$ and $(\mathscr{A} \to \mathscr{B}))$ must be logically equivalent. This equivalence will be referred to as $(\star)$.

    Therefore, the following statement forms must be equivalent by the substitution on the right and Proposition 1.14. Note that $\mathscr{C} \equiv \mathscr{D}$ indicates that the statement forms $\mathscr{C}$ and $\mathscr{D}$ are logically equivalent.

$$\begin{array}{rr}
((p \to q) \to r) & \\
((\sim (p \to q)) \vee r) & ((p \to q) \to r) \equiv ((\sim (p \to q)) \vee r), \text{ by } (\star) \\
((\sim ((\sim p) \vee q)) \vee r) & (p \to q) \equiv ((\sim p) \vee q), \text{ by } (\star)
\end{array}$$

11. Let $\mathscr{A}, \mathscr{B}, \mathscr{C}$ be statement forms. The following equivalences will be used.

    (i) $(\mathscr{A} \vee \mathscr{B})$ is logically equivalent to $(\mathscr{B} \vee \mathscr{A})$.
    (ii) $(\mathscr{A} \to \mathscr{B})$ is logically equivalent to $((\sim \mathscr{A}) \vee \mathscr{B})$.
    (iii) $(\sim (\sim \mathscr{A}))$ is logically equivalent to $\mathscr{A}$.
    (iv) $(\mathscr{A} \to \mathscr{B})$ is logically equivalent to $((\sim \mathscr{B}) \to (\sim \mathscr{A}))$.
    (v) $(\mathscr{A} \vee \mathscr{A})$ is logically equivalent to $\mathscr{A}$.

    In the sub-exercises, successive statement forms are equivalent by substitution via Proposition 1.14 of an equivalent statement form indicated by the braces. The column on the right justifies the equivalence of substituted statement forms.

    (a) The statement form $((\sim (p \vee (\sim q))) \to (q \to r))$ is equivalent to:

$$\begin{array}{rr}
((\sim \underbrace{((\sim q) \vee p)}) \to (q \to r)) & \text{(i)} \\
((\sim \underbrace{(q \to p)}) \to (q \to r)) & \text{(ii)} \\
((\sim (q \to p)) \to \underbrace{((\sim q) \vee r)}) & \text{(ii)}
\end{array}$$

    (b) The statement form $((\sim (p \vee (\sim q))) \to (q \to r))$ is equivalent to:

$$\begin{array}{rr}
(\underbrace{((\sim p) \wedge (\sim (\sim q)))} \to (q \to r)) & \text{Proposition 1.17} \\
(((\sim p) \wedge \underbrace{\ q\ }) \to (q \to r)) & \text{(iii)} \\
(((\sim p) \wedge q) \to \underbrace{((\sim q) \vee r)}) & \text{(ii)} \\
(((\sim p) \wedge q) \to \underbrace{(\sim (\sim ((\sim q) \vee r)))}) & \text{(iii)} \\
(((\sim p) \wedge q) \to (\sim \underbrace{((\sim (\sim q)) \wedge (\sim r))})) & \text{Proposition 1.17} \\
(((\sim p) \wedge q) \to (\sim (\underbrace{\ q\ } \wedge (\sim r)))) & \text{(iii)}
\end{array}$$

(c) The statement form $((\sim (p \vee (\sim q))) \to (q \to r))$ is equivalent to:

$$\underbrace{((\sim (q \to r)) \to (\sim (\sim (p \vee (\sim q)))))} \qquad \text{(iv)}$$

$$((\sim (q \to r)) \to \underbrace{(p \vee (\sim q))}) \qquad \text{(iii)}$$

$$((\sim (q \to r)) \to \underbrace{((\sim q) \vee p)}) \qquad \text{(i)}$$

$$((\sim (q \to r)) \to \underbrace{(q \to p)}) \qquad \text{(ii)}$$

$$((\sim \underbrace{((\sim q) \vee r)}) \to (q \to p)) \qquad \text{(ii)}$$

(d) The statement form $((\sim (p \vee (\sim q))) \to (q \to r))$ is equivalent to:

$$\underbrace{((\sim ((\sim q) \vee r)) \to (q \to p))} \qquad \text{Exercise 11, part (c)}$$

$$(\underbrace{((\sim (\sim q)) \wedge (\sim r))} \to (q \to p)) \qquad \text{Proposition 1.17}$$

$$((\underbrace{q} \wedge (\sim r)) \to (q \to p)) \qquad \text{(iii)}$$

$$\underbrace{((\sim (q \wedge (\sim r))) \vee (q \to p))} \qquad \text{(ii)}$$

$$(\underbrace{((\sim q) \vee (\sim (\sim r)))} \vee (q \to p)) \qquad \text{Proposition 1.17}$$

$$(((\sim q) \vee \underbrace{r}) \vee (q \to p)) \qquad \text{(iii)}$$

$$(((\sim q) \vee r) \vee \underbrace{((\sim q) \vee p)}) \qquad \text{(ii)}$$

$$(\underbrace{(r \vee (\sim q))} \vee ((\sim q) \vee p)) \qquad \text{(i)}$$

$$\underbrace{(((\sim q) \vee (\sim q)) \vee (p \vee r)))} \qquad \text{associativity and commutativity of } \vee$$

$$(\underbrace{(\sim q)} \vee (p \vee r)) \qquad \text{(v)}$$

$$(q \to (p \vee r)) \qquad \text{(ii)}$$

Note. In (d), a few tedious steps were skipped when invoking the associativity and commutativity of $\vee$. An inductive proof could be used to prove that any parenthesization of $\mathscr{A}_1 \vee \mathscr{A}_2 \vee \ldots \mathscr{A}_n$ is equivalent to $\mathscr{A}_1 \vee \mathscr{A}_2 \vee \ldots \mathscr{A}_n$ to make the proof more rigorous, but it is more pedantic than necessary.

1.4 Normal forms

Solutions to exercises

1.5 Adequate sets of connectives

Solutions to exercises

1.6 Arguments and validity

Solutions to exercises

Chapter 2

Formal statement calculus

2.1 The formal system L

A *formal system* is a mathematical structure representing a deductive system. It consists of

1. A set of symbols called an *alphabet*.
2. A set of finite strings of these symbols representing the valid sentences in the system. Each string is called a *well-formed formula*, or *wf* for short.

   There is usually a formula, called a *grammar*, for determining which strings are wfs.
3. A subset of the set of wfs representing the axioms.
4. A finite set of *inference rules*, functions which take a set of *wf*s and return a *wf*. The returned *wf* is said to be deduced from the set of *wf*s.

This chapter is devoted to a particular formal system described below.

Definition 2.1. The *formal system L of statement calculus* is defined by the following:

1. The alphabet consists of the symbols $\sim, \rightarrow, ($, and $)$, along with the countably infinite set of symbols $p_1, p_2, p_3, \ldots$.
2. The set of *wf*s is defined recursively by the following rules:
   (a) For any i, p_i is a *wf*.
   (b) If $\mathscr{A}$ and $\mathscr{B}$ are *wf*s, then $(\sim \mathscr{A})$ and $(\mathscr{A} \rightarrow \mathscr{B})$ are *wf*s.
   This rule also defines the semantics of the parentheses, and, because one set of parentheses will always contain only one of $\sim$ or $\rightarrow$ not in parentheses, it also eliminates the need for an order of operations.
   (c) The set of all *wf*s is generated by the above rules.
3. Let $\mathscr{A}, \mathscr{B}, \mathscr{C}$ be *wf*s. All axioms take on one of the following forms:
   (a) Axiom scheme 1 (L1): $(\mathscr{A} \rightarrow (\mathscr{B} \rightarrow \mathscr{A}))$.
   (b) Axiom scheme 2 (L2): $((\mathscr{A} \rightarrow (\mathscr{B} \rightarrow \mathscr{C})) \rightarrow ((\mathscr{A} \rightarrow \mathscr{B}) \rightarrow (\mathscr{A} \rightarrow \mathscr{C})))$.

(c) Axiom scheme 3 (L3): $(((\sim \mathscr{A}) \to (\sim \mathscr{B})) \to (\mathscr{B} \to \mathscr{A}))$.

4. There is one rule of deduction known as *modus ponens* (MP): from $\mathscr{A}$ and $\mathscr{A} \to \mathscr{B}$, $\mathscr{B}$ is a direct consequence, where $\mathscr{A}, \mathscr{B}$ are any *wf*sof L.

Definition 2.2. A *proof of $\mathscr{A}_n$ in L*, or just a *proof in L*, is a sequence of *wf*s $\mathscr{A}_1, \ldots, \mathscr{A}_n$ such that for any i, $\mathscr{A}_i$ is an axiom of L or $\mathscr{A}_i$ follows from MP and two previous *wf*s in the sequence. The *wf* $\mathscr{A}_n$ is said to be a *theorem of L*.

Instead of using the word theorem to discuss a result about formal system, we will instead use the word *metatheorem* to prevent confusion with the word *theorem* in the sense of the previous definition.

Definition 2.5. Let Γ be a set of *wf*s of L. A proof in L with the members of Γ regarded as additional axioms is called a *deduction from* Γ. The last *wf* in the proof, call it $\mathscr{A}$, is said to be *deducible from* Γ or a *consequence of* Γ and is symbolized by $\Gamma \vdash_L \mathscr{A}$. If $\Gamma = \emptyset$ then we instead write $\vdash_L \mathscr{A}$, which is to say that $\mathscr{A}$ is a theorem of L.

Proposition 2.8 (The Deduction Theorem)**.** If $\Gamma \cup \{\mathscr{A}\} \vdash_L \mathscr{B}$, then $\Gamma \vdash_L (\mathscr{A} \to \mathscr{B})$, where $\mathscr{A}$ and $\mathscr{B}$ are *wf*s of L, and Γ is a set of *wf*s of L (possibly empty).

Proof. The proof is by strong induction on the number of *wf*s in the sequence forming the deduction of $\mathscr{B}$ from $\Gamma \cup \{\mathscr{A}\}$.

(base case) There is only one *wf* in the deduction of $\mathscr{B}$ from $\Gamma \cup \{\mathscr{A}\}$, which is to say that the proof consists of only $\mathscr{B}$, and there are two cases in which this can happen:

1. The *wf* $\mathscr{B}$ is an axiom of L or a member of Γ. In either case the deduction of $\mathscr{A} \to \mathscr{B}$ proceeds as follows:

1	$\mathscr{B}$	$\mathscr{B}$ is an axiom or a member of Γ
2	$(\mathscr{B} \to (\mathscr{A} \to \mathscr{B}))$	(L1)
3	$(\mathscr{A} \to \mathscr{B})$	1, 2, MP

The above is a deduction of $(\mathscr{A} \to \mathscr{B})$ from Γ.

2. The *wf* $\mathscr{B}$ is $\mathscr{A}$, and so $(\mathscr{A} \to \mathscr{B})$ is $(\mathscr{A} \to \mathscr{A})$.

1	$((\mathscr{A} \to ((\mathscr{A} \to \mathscr{A}) \to \mathscr{A})) \to ((\mathscr{A} \to (\mathscr{A} \to \mathscr{A})) \to (\mathscr{A} \to \mathscr{A})))$	(L2)
2	$(\mathscr{A} \to ((\mathscr{A} \to \mathscr{A}) \to \mathscr{A}))$	(L1)
3	$((\mathscr{A} \to (\mathscr{A} \to \mathscr{A})) \to (\mathscr{A} \to \mathscr{A}))$	1, 2, MP
4	$(\mathscr{A} \to (\mathscr{A} \to \mathscr{A}))$	(L1)
5	$(\mathscr{A} \to \mathscr{A})$	3, 4, MP

The above is a deduction of $(\mathscr{A} \to \mathscr{B})$ (which is $(\mathscr{A} \to \mathscr{A})$) from Γ. Note that it is also a general theorem of L.

(inductive step) Suppose that for any deduction of $\mathscr{C}$ from $\Gamma \cup \{\mathscr{A}\}$ with up to and including n members, it is possible to deduce $(\mathscr{A} \to \mathscr{B})$ from Γ alone. This is the hypothesis of strong induction.

Additionally suppose that there exists a deduction of $\mathscr{B}$ from $\Gamma \cup \{\mathscr{A}\}$ with $n + 1$ members. We now provide a proof that there exists a deduction of $(\mathscr{A} \to \mathscr{B})$ from Γ alone. There are three cases to consider:

1. The *wf* $\mathscr{B}$ is an axiom of L or a member of Γ. A deduction of $(\mathscr{A} \to \mathscr{B})$ from Γ is shown in case 1 in the base case.

2. The *wf* $\mathscr{B}$ is $\mathscr{A}$. A deduction of $(\mathscr{A} \to \mathscr{B})$ from Γ is shown in case 2 in the base case.

3. The *wf* $\mathscr{B}$ is obtained applying MP along with two *wf*s, which are necessarily of the form $\mathscr{C}$ and $(\mathscr{C} \to \mathscr{B})$, where $\mathscr{C}$ is any *wf*. The deduction of $\mathscr{B}$ from $\Gamma \cup \{\mathscr{A}\}$ must be the sequence

$$(\ldots, \mathscr{C}, \ldots, (\mathscr{C} \to \mathscr{B}), \ldots, \mathscr{B}) \text{ or } (\ldots, (\mathscr{C} \to \mathscr{B}), \ldots, \mathscr{C}, \ldots, \mathscr{B}),$$

from which it can be seen that the subsequences $(\ldots, \mathscr{C})$ and $(\ldots, (\mathscr{C} \to \mathscr{B}))$ are both deductions of $\mathscr{C}$ and $(\mathscr{C} \to \mathscr{B})$ from $\Gamma \cup \{\mathscr{A}\}$ with n members or less. Therefore, by the hypothesis, $\Gamma \vdash_L (\mathscr{A} \to \mathscr{C})$ and $\Gamma \vdash_L (\mathscr{A} \to (\mathscr{C} \to \mathscr{B}))$, which is to say that $(\mathscr{A} \to \mathscr{C})$ and $(\mathscr{A} \to (\mathscr{C} \to \mathscr{B}))$ are both deducible from Γ alone.

So appending the proof of $(\mathscr{A} \to (\mathscr{C} \to \mathscr{B}))$ to $(\mathscr{A} \to \mathscr{C})$, yields the sequence

$$(\ldots, (\mathscr{A} \to \mathscr{C}), \ldots, (\mathscr{A} \to (\mathscr{C} \to \mathscr{B}))).$$

which may have redundant *wf*s, but is nevertheless a valid deduction from Γ. The following proof builds on the sequence.

	$\vdots$	
k	$(\mathscr{A} \to \mathscr{C})$	deduced from Γ
	$\vdots$	
l	$(\mathscr{A} \to (\mathscr{C} \to \mathscr{B}))$	deduced from Γ
l + 1	$((\mathscr{A} \to (\mathscr{C} \to \mathscr{B})) \to ((\mathscr{A} \to \mathscr{C}) \to (\mathscr{A} \to \mathscr{B})))$	(L2)
l + 2	$((\mathscr{A} \to \mathscr{C}) \to (\mathscr{A} \to \mathscr{B}))$	l, l + 1
l + 3	$((\mathscr{A} \to (\mathscr{C} \to \mathscr{B})) \to ((\mathscr{A} \to \mathscr{C}) \to (\mathscr{A} \to \mathscr{B})))$	(L2)

And so in all three cases, $(\mathscr{A} \to \mathscr{B})$ can be deduced from Γ alone, which concludes the induction.

Proposition 2.9 (Converse of The Deduction Theorem)**.** If $\Gamma \vdash_L (\mathscr{A} \to \mathscr{B})$ then $\Gamma \cup \{\mathscr{A}\} \vdash_L \mathscr{B}$, where $\mathscr{A}$ and $\mathscr{B}$ are *wf*s of L and Γ is a (possibly empty) set of *wf*s of L.
Proof. The following is a deduction of $\mathscr{B}$ from $\Gamma \cup \{\mathscr{A}\}$, provided that $\Gamma \vdash_L (\mathscr{A} \to \mathscr{B})$.

	$\vdots$	
k	$(\mathscr{A} \to \mathscr{B})$	deduction from Γ
k + 1	$\mathscr{A}$	member of $\Gamma \cup \{\mathscr{A}\}$
k + 2	$\mathscr{B}$	k, k + 1, MP

Corollary 2.10 (The Hypothetical Syllogism (HS))**.** For any *wf*s $\mathscr{A}, \mathscr{B}, \mathscr{C}$ of L,

$$\{(\mathscr{A} \to \mathscr{B}), (\mathscr{B} \to \mathscr{C})\} \vdash_L (\mathscr{A} \to \mathscr{C}).$$

Proof. We will first prove $S \vdash_L \mathscr{C}$, where $S = \{(\mathscr{A} \to \mathscr{B}), (\mathscr{B} \to \mathscr{C})\} \cup \{\mathscr{A}\}$.

1	$\mathscr{A} \to \mathscr{B}$	member of S
2	$\mathscr{B} \to \mathscr{C}$	member of S
3	$\mathscr{A}$	member of S
4	$\mathscr{B}$	1, 3, MP
5	$\mathscr{C}$	2, 4, MP

Seeing as though we have proved

$$\{(\mathscr{A} \to \mathscr{B}), (\mathscr{B} \to \mathscr{C})\} \cup \{\mathscr{A}\} \vdash_L \mathscr{C}$$

we may use the deduction theorem to conclude

$$\{(\mathscr{A} \to \mathscr{B}), (\mathscr{B} \to \mathscr{C})\} \vdash_L (\mathscr{A} \to \mathscr{C}).$$

Proposition 2.11. For any *wf*s $\mathscr{A}$ and $\mathscr{B}$ of L, the following are theorems of L.

(a) $((\sim \mathscr{A}) \to (\mathscr{B} \to \mathscr{A}))$

(b) $(((\sim \mathscr{A}) \to \mathscr{A}) \to \mathscr{A})$

Proof.

(a)

1	$((\sim \mathscr{B}) \to ((\sim \mathscr{A}) \to (\sim \mathscr{B})))$	(L1)
2	$(((\sim \mathscr{A}) \to (\sim \mathscr{B})) \to (\mathscr{B} \to \mathscr{A}))$	(L3)

Application of HS and the above two lines yields $((\sim \mathscr{B}) \to (\mathscr{B} \to \mathscr{A}))$, as desired.

(b) We will first prove that $\{((\sim \mathscr{A}) \to \mathscr{A})\} \vdash_L \mathscr{A}$.

1	$((\sim \mathscr{A}) \to \mathscr{A})$	assumption
2	$((\sim \mathscr{A}) \to ((\sim (\sim ((\sim \mathscr{A}) \to \mathscr{A}))) \to (\sim \mathscr{A})))$	(L1)
3	$(((\sim (\sim ((\sim \mathscr{A}) \to \mathscr{A}))) \to (\sim \mathscr{A})) \to$ $(\mathscr{A} \to (\sim ((\sim \mathscr{A}) \to \mathscr{A}))))$	(L3)
4	$((\sim \mathscr{A}) \to (\mathscr{A} \to (\sim ((\sim \mathscr{A}) \to \mathscr{A}))))$	2, 3, HS
5	$(((\sim \mathscr{A}) \to (\mathscr{A} \to (\sim ((\sim \mathscr{A}) \to \mathscr{A})))) \to$ $(((\sim \mathscr{A}) \to \mathscr{A}) \to ((\sim \mathscr{A}) \to (\sim ((\sim \mathscr{A}) \to \mathscr{A})))))$	(L2)
6	$(((\sim \mathscr{A}) \to \mathscr{A}) \to ((\sim \mathscr{A}) \to (\sim ((\sim \mathscr{A}) \to \mathscr{A}))))$	4, 5, MP
7	$((\sim \mathscr{A}) \to (\sim ((\sim \mathscr{A}) \to \mathscr{A})))$	1, 6, MP
8	$(((\sim \mathscr{A}) \to (\sim ((\sim \mathscr{A}) \to \mathscr{A}))) \to$ $(((\sim \mathscr{A}) \to \mathscr{A}) \to \mathscr{A}))$	(L3)
9	$(((\sim \mathscr{A}) \to \mathscr{A}) \to \mathscr{A})$	7, 8, MP
10	$\mathscr{A}$	1, 9, MP

By the deduction theorem, we may conclude that $(((\sim \mathscr{A}) \to \mathscr{A}) \to \mathscr{A})$.

Note. Because of the use of HS in line 4, which is a metatheorem, the above proof is not technically a proof in L. An actual proof would require a few additional lines.

Solutions to exercises

Note. All formulas are fully parenthesized, so they may look different from how they appear in the book. Additionally, any metatheorems are referenced as if they were rules of deduction, just like in the book.

1. (a)

$$
\begin{array}{rrl}
1 & ((\sim p_1 \to \sim p_2) \to (p_2 \to p_1)) & \text{(L3)} \\
2 & (((\sim p_1 \to \sim p_2) \to (p_2 \to p_1)) \to & \\
 & ((p_1 \to p_2) \to ((\sim p_1 \to \sim p_2) \to (p_2 \to p_1)))) & \text{(L1)} \\
3 & ((p_1 \to p_2) \to ((\sim p_1 \to \sim p_2) \to (p_2 \to p_1))) & \text{1, 2, MP}
\end{array}
$$

(b)

$$
\begin{array}{rrl}
1 & ((\underbrace{(p_1 \to (p_2 \to p_3))}_{\mathscr{A}} \to (\underbrace{(p_1 \to p_2)}_{\mathscr{B}} \to \underbrace{(p_1 \to p_3)}_{\mathscr{C}})) \to & \\
 & (((p_1 \to (p_2 \to p_3)) \to (p_1 \to p_2)) \to ((p_1 \to (p_2 \to p_3)) \to (p_1 \to p_3)))) & \text{(L2)} \\
2 & ((p_1 \to (p_2 \to p_3)) \to ((p_1 \to p_2) \to (p_1 \to p_3))) & \text{(L2)} \\
3 & (((p_1 \to (p_2 \to p_3)) \to (p_1 \to p_2)) \to ((p_1 \to (p_2 \to p_3)) \to (p_1 \to p_3))) & \text{1, 2, MP}
\end{array}
$$

(c)

$$
\begin{array}{rrl}
1 & ((\underbrace{(p_1 \to (p_1 \to p_2))}_{\mathscr{A}} \to (\underbrace{(p_1 \to p_1)}_{\mathscr{B}} \to \underbrace{(p_1 \to p_2)}_{\mathscr{C}})) \to & \\
 & (((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_1)) \to ((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2)))) & \text{(L2)} \\
2 & ((p_1 \to (p_1 \to p_2)) \to ((p_1 \to p_1) \to (p_1 \to p_2))) & \text{(L2)} \\
3 & (((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_1)) \to ((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2))) & \text{1, 2, MP} \\
4 & (p_1 \to ((p_1 \to p_2) \to p_1)) & \text{(L1)} \\
5 & ((p_1 \to ((p_1 \to p_2) \to p_1)) \to ((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_1))) & \text{(L2)} \\
6 & ((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_1)) & \text{4, 5, MP} \\
7 & ((p_1 \to (p_1 \to p_2)) \to (p_1 \to p_2)) & \text{3, 6, MP}
\end{array}
$$

Lines 1-3 are identical to exercise (b) with p_1 substituted for p_2 and p_2 substituted for p_3.

Lines 4-6 are identical to example 2.4 in the book with $(p_1 \to p_2)$ substituted for p_2.

(d)

$$
\begin{array}{rrl}
1 & (p_2 \to (p_1 \to p_2)) & \text{(L1)} \\
2 & (\underbrace{(p_2 \to (p_1 \to p_2))}_{\mathscr{A}} \to (\underbrace{p_1}_{\mathscr{B}} \to \underbrace{(p_2 \to (p_1 \to p_2))}_{\mathscr{A}})) & \text{(L1)} \\
3 & (p_1 \to (p_2 \to (p_1 \to p_2)) & \text{1, 2, MP}
\end{array}
$$

2. (a)

$$
\begin{array}{rrl}
1 & (\sim \mathscr{A}) & \text{assumption} \\
2 & (((\sim \mathscr{B}) \to (\sim \mathscr{A})) \to (\mathscr{A} \to \mathscr{B})) & \text{(L3)} \\
3 & ((\sim \mathscr{A}) \to ((\sim \mathscr{B}) \to (\sim \mathscr{A}))) & \text{(L1)} \\
4 & ((\sim \mathscr{B}) \to (\sim \mathscr{A})) & \text{1, 3, MP} \\
5 & (\mathscr{A} \to \mathscr{B}) & \text{2, 4, MP}
\end{array}
$$

(b)

1	$(\sim (\sim \mathscr{A}))$	assumption
2	$(((\sim \mathscr{A}) \to (\sim (\sim (\sim \mathscr{A})))) \to ((\sim (\sim \mathscr{A})) \to \mathscr{A})$	(L3)
3	$(((\sim (\sim (\sim (\sim \mathscr{A})))) \to (\sim (\sim \mathscr{A}))) \to$ $((\sim \mathscr{A}) \to (\sim (\sim (\sim (\mathscr{A})))))$	(L3)
4	$((\sim (\sim \mathscr{A})) \to ((\sim (\sim (\sim (\sim \mathscr{A})))) \to (\sim (\sim \mathscr{A}))))$	(L1)
5	$((\sim (\sim (\sim (\sim \mathscr{A})))) \to (\sim (\sim \mathscr{A})))$	1, 4, MP
6	$((\sim \mathscr{A}) \to (\sim (\sim (\sim \mathscr{A}))))$	3, 5, MP
7	$((\sim (\sim \mathscr{A})) \to \mathscr{A})$	2, 6, MP
8	$\mathscr{A}$	1, 7, MP

(c)

1	$(\mathscr{A} \to \mathscr{B})$	assumption
2	$(\sim (\mathscr{B} \to \mathscr{C}) \to (\sim \mathscr{A}))$	assumption
3	$(((\sim (\mathscr{B} \to \mathscr{C})) \to (\sim \mathscr{A})) \to (\mathscr{A} \to (\mathscr{B} \to \mathscr{C})))$	(L3)
4	$(\mathscr{A} \to (\mathscr{B} \to \mathscr{C}))$	2, 3, MP
5	$((\mathscr{A} \to (\mathscr{B} \to \mathscr{C})) \to ((\mathscr{A} \to \mathscr{B}) \to (\mathscr{A} \to \mathscr{C})))$	(L2)
6	$((\mathscr{A} \to \mathscr{B}) \to (\mathscr{A} \to \mathscr{C}))$	4, 5, MP
7	$(\mathscr{A} \to \mathscr{C})$	1, 6, MP

(d)

1	$(\mathscr{A} \to (\mathscr{B} \to \mathscr{C}))$	assumption
2	$((\mathscr{A} \to (\mathscr{B} \to \mathscr{C})) \to ((\mathscr{A} \to \mathscr{B}) \to (\mathscr{A} \to \mathscr{C})))$	(L2)
3	$((\mathscr{A} \to \mathscr{B}) \to (\mathscr{A} \to \mathscr{C}))$	1, 2, MP
4	$(\mathscr{B} \to (\mathscr{A} \to \mathscr{B}))$	(L1)
5	$(\mathscr{B} \to (\mathscr{A} \to \mathscr{C}))$	3, 4, HS

3. (a) We apply the deduction theorem to exercise to the result of exercise 2(b) and get $\vdash_L (\sim (\sim \mathscr{A}) \to \mathscr{A})$, which will be referred to as $\star$.

1	$\mathscr{A}$	assumption
2	$(\sim (\sim (\sim \mathscr{A})) \to (\sim \mathscr{A}))$	$\star$
3	$((\sim (\sim (\sim \mathscr{A})) \to (\sim \mathscr{A})) \to (\mathscr{A} \to (\sim (\sim \mathscr{A}))))$	(L3)
4	$(\mathscr{A} \to (\sim (\sim \mathscr{A})))$	2, 3, MP
5	$(\sim (\sim \mathscr{A}))$	1, 4, MP

By the deduction theorem, $\vdash_L (\mathscr{A} \to (\sim (\sim \mathscr{A})))$.

(b) This solution relies on the solutions for 3(a) and 2(b).

1	$(\mathscr{B} \to \mathscr{A})$	assumption
2	$(\mathscr{A} \to (\sim (\sim \mathscr{A})))$	$\star$
3	$((\sim (\sim \mathscr{B})) \to \mathscr{B})$	exercise 3(a)
4	$(\mathscr{B} \to (\sim (\sim \mathscr{A})))$	1, 2, HP
5	$((\sim (\sim \mathscr{B})) \to (\sim (\sim \mathscr{A})))$	3, 4, HP
6	$(((\sim (\sim \mathscr{B})) \to (\sim (\sim \mathscr{A}))) \to ((\sim \mathscr{A}) \to (\sim \mathscr{B})))$	(L3)
7	$((\sim \mathscr{A}) \to (\sim \mathscr{B}))$	5, 6, MP

By the deduction theorem, $\vdash_L ((\mathscr{B} \to \mathscr{A}) \to ((\sim \mathscr{A}) \to (\sim \mathscr{B})))$.

(c)

1	$((\mathscr{A} \to \mathscr{B}) \to \mathscr{A})$	assumption
2	$((\sim \mathscr{A}) \to (\mathscr{A} \to \mathscr{B}))$	Proposition 2.11(a)
3	$(\sim \mathscr{A} \to \mathscr{A})$	1, 2, HS
4	$((\sim \mathscr{A} \to \mathscr{A}) \to \mathscr{A})$	Proposition 2.11(b)
5	$\mathscr{A}$	3, 4, MP

By the deduction theorem, $\vdash_L (((\mathscr{A} \to \mathscr{B}) \to \mathscr{A}) \to \mathscr{A})$.

(d)

1	$(\sim (\mathscr{A} \to \mathscr{B}))$	assumption
2	$((\mathscr{B} \to (\mathscr{A} \to \mathscr{B})) \to (\sim (\mathscr{A} \to \mathscr{B}) \to (\sim \mathscr{B})))$	exercise 3(b)
3	$(\mathscr{B} \to (\mathscr{A} \to \mathscr{B}))$	(L1)
4	$(\sim (\mathscr{A} \to \mathscr{B}) \to (\sim \mathscr{B}))$	2, 3, MP
5	$(\sim \mathscr{B})$	1, 4, MP
6	$(\sim \mathscr{B} \to (\mathscr{B} \to \mathscr{A}))$	Proposition 2.11(a)
7	$(\mathscr{B} \to \mathscr{A})$	5, 6, MP

By the deduction theorem, $\vdash_L ((\sim (\mathscr{A} \to \mathscr{B})) \to (\mathscr{B} \to \mathscr{A}))$.

4. (i)

1	$((\sim \mathscr{A}) \to (\sim \mathscr{B}))$	assumption
2	$((\sim \mathscr{A}) \to \mathscr{B})$	assumption
3	$(((\sim \mathscr{A}) \to (\sim \mathscr{B})) \to (\mathscr{B} \to \mathscr{A}))$	(L3)
4	$(\mathscr{B} \to \mathscr{A})$	1, 3, MP
5	$((\sim \mathscr{A}) \to \mathscr{A})$	2, 4, HS
6	$(((\sim \mathscr{A}) \to \mathscr{A}) \to \mathscr{A})$	2.11(b)
7	$\mathscr{A}$	5, 6, MP

By the deduction theorem, which is valid since its proof relies only on $(L'1)$ and $(L'2)$, $\vdash_{L'} (((\sim \mathscr{A}) \to (\sim \mathscr{B})) \to (((\sim \mathscr{A}) \to \mathscr{B}) \to \mathscr{A}))$.

(ii) We will prove $\{((\sim \mathscr{A}) \to (\sim \mathscr{B})), \mathscr{B}\} \vdash_{L'} \mathscr{A}$

1	$((\sim \mathscr{A}) \to (\sim \mathscr{B}))$	assumption
2	$\mathscr{B}$	assumption
3	$(((\sim \mathscr{A}) \to (\sim \mathscr{B})) \to (((\sim \mathscr{A}) \to \mathscr{B}) \to \mathscr{A}))$	(L'3)
4	$(((\sim \mathscr{A}) \to \mathscr{B}) \to \mathscr{A})$	1, 3, MP
5	$(\mathscr{B} \to ((\sim \mathscr{A}) \to \mathscr{B}))$	(L'2)
6	$((\sim \mathscr{A}) \to \mathscr{B})$	2, 5, MP
7	$\mathscr{A}$	4, 6, MP

By the deduction theorem twice, $\vdash_{L'} (((\sim \mathscr{A}) \to (\sim \mathscr{B})) \to (\mathscr{B} \to \mathscr{A}))$.

Suppose a *wf* is proved in L. By definition, it follows from (L1), (L2), and (L3) and MP. Since (L1), (L2) and MP are properties of L', and since we just proved that $\vdash_{L'}$ (L3), the *wf* can also be proved in L'. The other direction is argued for in the same way, so a *wf* is a theorem in L if and only if it is a theorem in L'.

5. The rule is valid, see example 2.6 (where $\mathscr{A}$ and $\mathscr{B}$ in the exercise appear switched in the example).

2.2 The Adequacy Theorem for L

Definition 2.12. A *valuation* of L is a function v whose domain is the set of *wf*s of L and whose range is the set $\{T, F\}$ such that, for any *wf*s $\mathscr{A}, \mathscr{B}$ of L,

(i) $v(\mathscr{A}) \neq v((\sim \mathscr{A}))$ and

(ii) $v((\mathscr{A} \to \mathscr{B})) = F$ if and only if $v(\mathscr{A}) = T$ and $v(\mathscr{B}) = F$.

This definition formalizes the previous idea of truth functions for statement forms. Note that (i) and (ii) simply define the behavior of the two logic operators in L such that they correspond to the truth tables introduced in the last chapter.

Definition 2.13. A *wf* $\mathscr{A}$ of L is a *tautology* if for every valuation v, $v(\mathscr{A}) = T$. If for every valuation v, $v(\mathscr{A}) = F$, it is a *contradiction*.
Note. The definition of a contradiction isn't in the book, but the term is used later on.

The goal of this chapter is to prove that every *wf* in L is a tautology if and only if it is a theorem in L. One direction can be done immediately.

Proposition 2.14 (The Soundness Theorem)**.** Every theorem of L is a tautology.
Proof. Let $\mathscr{A}$ be a *wf* in L. It is a theorem if and only if it is the last member of a proof in L. The proof is by strong induction on the number of *wf*s in the proof of $\mathscr{A}$.

Suppose that all theorems containing up to n *wf*s in their proofs are tautologies. Now suppose that $\mathscr{A}$ has n *wf*s in its proof. In the proof, any *wf* preceding $\mathscr{A}$ must necessarily be a tautology because it is a theorem with fewer than n *wf*s in its proof. So the only thing to prove is that $\mathscr{A}$ is a tautology, and $\mathscr{A}$ can either be an axiom, in which case it is a tautology (see exercise 6) or a product of MP and two previous *wf*s. These two *wf*s must necessarily be of the form $\mathscr{B}$ and $(\mathscr{B} \to \mathscr{A})$. By Proposition 1.9 (see the note below), $\mathscr{A}$ must be a tautology.
Note. In the above proof, since strong induction was used, no base case was necessary. An optional base case (when $n = 1$) would have verified that all axioms of L are tautologies, which would have been redundant to include.
Note. Proposition 1.9 is actually not strictly applicable here because it uses the previous informal notion of a tautology. But re-proving it using v would be nearly identical.
Note. A formal system is said to be sound if everything statement that is provable in it is true, and L has this property, as seen from this theorem, hence the name.

Definition 2.15. An *extension of* L is a formal system obtained by altering or enlarging the set of axioms so that all theorems of L remain theorems.
Note. By this definition, L is not an extension on L. Additionally, an extension of L may not actually extend the list of theorems of L.

Definition 2.16. An extension of L or L itself is *consistent* if for no *wf* $\mathscr{A}$ of L are both $\mathscr{A}$ and $(\sim \mathscr{A})$ theorems of the extension.
Note. This definition has been generalized slightly to be defined for L.

Proposition 2.17. L is consistent.
Proof. Suppose that L is not consistent. Then there exists a *wf* $\mathscr{A}$ such that $\mathscr{A}$ and $(\sim \mathscr{A})$ are both theorems of L. Since all theorems of L are tautologies (Proposition 2.14), $\mathscr{A}$ and $(\sim \mathscr{A})$ must be tautologies, meaning that for any valuation v, $v(\mathscr{A}) = v((\sim \mathscr{A})) = T$. But this contradicts v being a valuation (see Definition 2.12(i)).
Note. Therefore, consistency of L is a consequence of its soundness, Proposition 2.14.

Proposition 2.18. An extension L^* of L is consistent if and only if there is a *wf* which is not a theorem of L^*.
Proof. $\Rightarrow$ Let L^* be consistent. Then, for any *wf* $\mathscr{A}$, either $\mathscr{A}$ or $\sim \mathscr{A}$ is not a theorem.

$\Leftarrow$ For the other direction, we use contrapositive reasoning. Suppose that L^* is not consistent, which is to say that there exists a *wf* $\mathscr{B}$ such that $\mathscr{B}$ and $(\sim \mathscr{B})$ are both theorems of L^*. Now let $\mathscr{A}$ be any *wf* of L^*. Since, by Proposition 2.11(a), $((\sim \mathscr{B}) \to (\mathscr{B} \to \mathscr{A}))$ is a theorem of L, it is a theorem of L^*. So by MP and since $(\sim \mathscr{B})$ is a theorem, $(\mathscr{B} \to \mathscr{A})$ is a theorem. By MP and since $\mathscr{B}$ is a theorem, $\mathscr{A}$ is a theorem. Therefore, there any *wf* $\mathscr{A}$ is a theorem of L^*.
Note. The above proposition says that in a system with a contradiction in it, that contradiction can be used to vacuously prove anything, so a consistent system need only have one *wf* which is not a theorem.

Adding axioms to L or any of its extension can break its consistency. Consider a *wf* $\mathscr{A}$. Either it is true or false or maybe even both in L^* in the sense that $\vdash_{L^*} \mathscr{A}$ and/or $\vdash_{L^*} (\sim \mathscr{A})$, or it is undecidable in the sense that neither $\vdash_{L^*} \mathscr{A}$ or $\vdash_{L^*} (\sim \mathscr{A})$. In the latter case, would arbitrarily adding $\mathscr{A}$ or $(\sim \mathscr{A})$ break the consistency of L? The answer is no, as the following proposition shows.

Proposition 2.19. Let L^* be a consistent extension of L and let $\mathscr{A}$ be a *wf* of L which is not a theorem of L^*. Then L^{**} also consistent, where L^{**} is the extension of L obtained from L^* by including $(\sim \mathscr{A})$ as an additional axiom.
Proof. Suppose that L^{**} is not consistent. Then there exists some *wf* $\mathscr{B}$ such that both $\mathscr{B}$ and $(\sim \mathscr{B})$ are theorems of L^{**}. Now by Proposition 2.18 (see the note below), $\mathscr{A}$ must be a theorem of L^{**}. But since any theorem of L^{**} is a deduction from $(\sim \mathscr{A})$ in L^*, which is to say that $\{(\sim \mathscr{A})\} \vdash_{L^*} \mathscr{A}$ it follows from the deduction theorem that $\vdash_{L^*} ((\sim \mathscr{A}) \to \mathscr{A})$. From Proposition 2.11(b) and since all theorems of L are theorems of L^*, we have $\vdash_{L^*} ((\sim \mathscr{A}) \to \mathscr{A}) \to \mathscr{A})$, and so $\vdash_{L^*} \mathscr{A}$ by MP. But this contradicts $\mathscr{A}$ not being a theorem of L^*. Therefore, L^{**} must be consistent.
Note. Proposition 2.18 is not strictly applicable here, but its proof can be easily generalized to L^{**}.
Note. Clearly if $(\sim \mathscr{A})$ is a theorem of L^*, then adding it as an axiom in L^{**} will not break consistency. Only when $\mathscr{A}$ is "neither true nor false" is this theorem interesting.

Definition 2.20. An extension of L is *complete* if for each *wf* $\mathscr{A}$, either $\mathscr{A}$ or $(\sim \mathscr{A})$ is a theorem of the extension.
Note. Completeness is the converse of soundness.

Proposition. The set of *wf*s of L is countable.
Proof. Prove this.

Proposition 2.21. Let L^* be a consistent extension of L. Then there is a consistent complete extension of L^*.

Proof. Since the set of all *wf*s of L is countable, let $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3$ be an enumerations of the *wf*s. Define a sequence $J_0, J_1, J_2, \ldots$ by the following rules.

1. If $n = 0$, let J_0 be L^*.

2. If $n > 0$, let J_n be J_{n-1} if $\vdash_{J_{n-1}} \mathscr{A}_n$.

3. If $n > 0$, let J_n be J_{n-1} extended with $(\sim \mathscr{A}_n)$ as an additional axiom if $\mathscr{A}_n$ is not a theorem of J_{n-1}.

Notice that since $J_0 = L^*$ is consistent and every following member of the sequence is either the previous member or a consistent extension by Proposition 2.19, every member of the sequence is consistent.

Now define J to be an extension of L^* such that a *wf* is an axiom of J if and only if it is an axiom of J_n for any n. Notice that by construction of the sequence, for any k, either $\mathscr{A}_k$ or $(\sim \mathscr{A}_k)$ is a theorem of J_k. So J_k or $(\sim J_k)$ must be a theorem of J, which extends J_k. Therefore, J is complete.

Now suppose that J is not consistent. Then there is a *wf* $\mathscr{A}$ such that $\vdash_J \mathscr{A}$ and $\vdash_J (\sim \mathscr{A})$. Now in these proofs, there are a finite number of axioms used, and each axiom must of course appear in the list of all numbered *wf*s. Let $\mathscr{A}_k$ refer to the axiom with the highest index k. So both $\vdash_{J_k} \mathscr{A}_k$ and $\vdash_{J_k} (\sim \mathscr{A}_k)$, contradicting the consistency of J_k. Hence, J must be consistent.

Note. An extension of L^* is defined in the same way as an extension of L is.

Note. The set of *wf*s is purposely indexed starting from 1 instead of 0 like in the book so that $\mathscr{A}_n$ or $(\sim \mathscr{A}_n)$ is a theorem of $\mathscr{B}_n$.

Proposition 2.22. If L^* is a consistent extension of L then there is a valuation in which each theorem of L^* takes value T.

Proof. Let J be the consistent complete extension of L^* given in the proof of Proposition 2.21. Define v on *wf*s of L by $v(\mathscr{A}) = T$ if $\mathscr{A}$ is a theorem of J and $v(\mathscr{A}) = F$ otherwise.

Now it remains to be shown that v is a valuation consistent with Definition 2.12. Since J is complete, v is defined on all *wf*s. For any $\mathscr{A}$, $v(\mathscr{A}) \neq v((\sim \mathscr{A}))$, since J is consistent. It remains to show that $v(\mathscr{A} \to \mathscr{B}) = F$ if and only if $v(\mathscr{A}) = T$ and $v(\mathscr{B}) = F$.

$\Rightarrow$ Suppose that $v(\mathscr{A} \to \mathscr{B}) = F$ and that either $v(\mathscr{A}) = F$ or $v(\mathscr{B}) = T$. Since J is consistent, $\sim (\mathscr{A} \to \mathscr{B})$ must be a theorem of J and either $(\sim \mathscr{A})$ or $\mathscr{B}$ is also theorem of J. If $(\sim \mathscr{A})$, then

1	$(\sim \mathscr{A})$	assumption
2	$((\sim \mathscr{A}) \to ((\sim \mathscr{B}) \to (\sim \mathscr{A})))$	(L1)
3	$((\sim \mathscr{B}) \to (\sim \mathscr{A}))$	1, 2, MP
4	$(((\sim \mathscr{B}) \to (\sim \mathscr{A})) \to (\mathscr{A} \to \mathscr{B}))$	(L3)
5	$(\mathscr{A} \to \mathscr{B})$	3, 4, MP

or if $\mathscr{B}$, then

1	$\mathscr{B}$	assumption
2	$(\mathscr{B} \to (\mathscr{A} \to \mathscr{B}))$	(L2)
3	$(\mathscr{A} \to \mathscr{B})$	1, 2, MP

and so in either case, $(\mathscr{A} \to \mathscr{B})$ is a theorem of J along with $(\sim (\mathscr{A} \to \mathscr{B}))$, contradicting the consistency of J. Therefore, if $v(\mathscr{A} \to \mathscr{B})$, then $v(\mathscr{A}) = T$ and $v(\mathscr{B}) = F$.

$\Leftarrow$ Suppose that $v(\mathscr{A}) = T, v(\mathscr{B}) = F$ and that $v((\mathscr{A} \to \mathscr{B})) = T$. Then $\mathscr{A}$, $(\sim \mathscr{B})$, and $(\mathscr{A} \to \mathscr{B})$ are theorems of J. Then by MP, $\mathscr{A}$, and $(\mathscr{A} \to \mathscr{B})$, it follows that $\mathscr{B}$ is a theorem of J as well along with $(\sim \mathscr{B})$, contradicting the consistency of J. Therefore, if $v(\mathscr{A}) = T, v(\mathscr{B}) = F$ implies that $v((\mathscr{A} \to \mathscr{B})) = F$.

In conclusion, v is indeed a valuation and so if $\mathscr{A}$ is a theorem of L^*, then it must be a theorem of the extension J, in which case it takes the value T under the valuation v, making v a valuation in which each theorem of L^* takes value T.

Proposition 2.23 (The Adequacy Theorem for L)**.** If $\mathscr{A}$ is a *wf* of L and $\mathscr{A}$ is a tautology, then $\mathscr{A}$ is a theorem of L.
Proof. Let $\mathscr{A}$ be a tautology and suppose that it is not a theorem of L. Then $(\sim \mathscr{A})$ must be a theorem of the extension L^* by Proposition 2.21. Therefore, by Proposition 2.22, there exists a valuation v in which $v(\sim \mathscr{A}) = T$. But $v(\mathscr{A}) = T$, since $\mathscr{A}$ is a tautology. This contradiction demonstrates that $\mathscr{A}$ must be a theorem of L.

Proposition 2.24. L is *decidable*, i.e., there is an effective method for deciding, given any *wf* of L, whether it is a theorem of L.
Proof. The effective method of determining whether a *wf* is a tautology is to show that any valuation assigns the *wf* the value of T. If so, then it is a tautology, and by Proposition 2.23, it must be a theorem of L.
Note. Showing that any valuation assigns the *wf* the value of T can be done by creating truth tables like in the first chapter.

Solutions to exercises

6. The truth tables for each scheme of L are shown below, and since the values for any assignment of T or F to the *wf*s is T, the axioms must all be tautologies.

   For ($L1$),

$(\mathscr{A}$	$\rightarrow$	$(\mathscr{B}$	$\rightarrow$	$\mathscr{A}))$
T	T	T	T	T
T	T	F	T	T
F	T	T	F	F
F	T	F	T	F

   For ($L2$),

$((\mathscr{A}$	$\rightarrow$	$(\mathscr{B}$	$\rightarrow$	$\mathscr{C}))$	$\rightarrow$	$((\mathscr{A}$	$\rightarrow$	$\mathscr{B})$	$\rightarrow$	$(\mathscr{A}$	$\rightarrow$	$\mathscr{C})))$
T	T	T	T	T	T	T	T	T	T	T	T	T
T	F	T	F	F	T	T	T	T	F	T	F	F
T	T	F	T	T	T	T	F	F	T	T	T	T
T	T	F	T	F	T	T	F	F	T	T	F	F
F	T	T	T	T	T	F	T	T	T	F	T	T
F	T	T	F	F	T	F	T	T	T	T	F	F
F	T	F	T	T	T	F	T	F	T	F	T	T
F	T	F	T	F	T	F	T	F	T	F	T	F

   For ($L3$),

$(((\sim$	$\mathscr{A})$	$\rightarrow$	$(\sim$	$\mathscr{B}))$	$\rightarrow$	$(\mathscr{B}$	$\rightarrow$	$\mathscr{A}))$
F	T	T	F	T	T	T	T	T
F	T	T	T	F	T	F	T	T
T	F	F	F	T	T	T	F	F
T	F	T	T	F	T	F	T	F

7. Let $\mathscr{A}$ be a *wf* of L and let L^+ be the extension of L obtained by including $\mathscr{A}$ as a new axiom. It is to be proved that the set of theorems of L^+ is different from the set of theorems of L if and only if $\mathscr{A}$ is not a theorem of L.

   $\Rightarrow$ Proceeding by contrapositive, suppose that $\mathscr{A}$ is a theorem of L. Then let $\mathscr{B}$ any theorem of L^+. We will demonstrate that $\mathscr{B}$ is a theorem of L.

   If the proof of $\mathscr{B}$ does not involve the axiom $\mathscr{A}$, then $\mathscr{B}$ is a theorem of L, since L differs from L^+ only in not having $\mathscr{A}$ as an axiom.

   Otherwise, the proof of $\mathscr{B}$ does involve the axiom $\mathscr{A}$, which is to say that $\{\mathscr{A}\} \vdash_L \mathscr{B}$, and by the deduction theorem $\vdash_L (\mathscr{A} \to \mathscr{B})$. By MP and $\mathscr{A}$ being a theorem in L, $\mathscr{B}$ is a theorem in L.

   Therefore, any theorem in L^+ is a theorem in L.

   $\Leftarrow$ Suppose that $\mathscr{A}$ is not a theorem of L. Then since $\mathscr{A}$ is a theorem of L^+ by virtue of all axioms being theorems, the set of theorems of L^+ must be different from the set of theorems of L^+.

8. Notice that $\mathscr{A}$ is neither a tautology nor a contradiction. Therefore, neither $\mathscr{A}$ nor $(\sim \mathscr{A})$ are theorems of L. Therefore, by the previous exercise, L^+, the extension of L with $\mathscr{A}$ as an axiom, has a larger set of theorems than L, in the sense that more *wf*s are theorems of L^+.

   Now suppose that L^+ is inconsistent. In an inconsistent system, every *wf* is a theorem, so $(\sim \mathscr{A})$ must be a theorem of L^+. But since L^+ is L with $\mathscr{A}$ as an additional axiom, it follows that $\{\mathscr{A}\} \vdash_L (\sim \mathscr{A})$, and by the deduction theorem, $(\mathscr{A} \to (\sim \mathscr{A}))$ is a theorem of L. But since $(\mathscr{A} \to (\sim \mathscr{A}))$ is not a tautology, it cannot be a theorem of L. With this contradiction, it is seen that L^+ must be consistent.

9. Suppose that $\mathscr{B}$ is a contradiction as well as a theorem in L^+. For a contradiction, suppose that L^+ is a consistent extension of L. Then by Proposition 2.22, there is a valuation v such that every theorem of L^* takes value T. So $v(\mathscr{B}) = T$, but this contradicts $\mathscr{B}$ being a contradiction.

   *Note.*See the note in Definition 2.13.

10. L^{++} must have the contradiction

$$(((\sim (p_1 \to p_1)) \to (p_1 \to p_1)) \to ((p_1 \to p_1) \to (\sim (p_1 \to p_1))))$$

   as a theorem, since it is an instance of the given axiom scheme. By the previous exercise, L^{++} cannot be consistent.

11. Let J be a consistent complete extension of L, and let $\mathscr{A}$ be a *wf* of L. Let J^+ be the extension of J obtained by including $\mathscr{A}$ as an additional axiom. It is to be proved that J^+ is consistent if and only if $\mathscr{A}$ is a theorem of J.

   $\Rightarrow$ Suppose that J^+ is consistent. For a contradiction, suppose that $\mathscr{A}$ is not a theorem of J. Then $(\sim \mathscr{A})$ must be a theorem of J, since J is consistent and complete. Since J^+ is an extension of J, $(\sim \mathscr{A})$ must also be a theorem of J^+, which contradicts the consistency of J^+, since $\mathscr{A}$ is an axiom and hence a theorem of J.

   $\Leftarrow$ Suppose that $\mathscr{A}$ is a theorem of J and let $\mathscr{B}$ be any theorem in J^+. We will demonstrate that $\mathscr{B}$ must be a theorem in J.

If the proof of $\mathscr{B}$ in J^+ does not rely on $\mathscr{A}$, then $\mathscr{B}$ is a theorem of J, since J^+ extends J with only $\mathscr{A}$ as an additional axiom.

The other possibility is that the proof of $\mathscr{B}$ does involve $\mathscr{A}$, which is to say that $\{\mathscr{A}\} \vdash_J \mathscr{B}$, and so by the deduction theorem $\vdash_J (\mathscr{A} \to \mathscr{B})$. By MP and $\mathscr{A}$ being a theorem in J, $\mathscr{B}$ is a theorem in J. Therefore, any theorem in J^+ is a theorem in J, and since J is consistent, J^+ must be so as well.

12. We will prove this using strong induction on the number of *wf*s in the proof of $\mathscr{A}$.

    Suppose as an induction hypothesis that for any theorem $\mathscr{A}$ in L in which statement letters appear and in which its proof involves less than n *wf*s, $\mathscr{B}$, a *wf* with any *wf*s substituted for the statement letters, is also a theorem of L.

    It may be the case that $\mathscr{A}$ is an instance of an axiom, in which case $\mathscr{B}$ is also an instance of axiom and hence a theorem of L.

    Otherwise, $\mathscr{A}$ proceeds from two prior *wf*s in the proof via MP. These two *wf*s have less than n *wf*s in their proof, and therefore by the induction hypothesis, there exist theorems in L with the same substitutions of *wf*s for the statement letters described above. By MP and these two statements, $\mathscr{B}$ is a theorem of L as well.

    Note. In other words, $\mathscr{B}$ can be proved in an identical manner as $\mathscr{A}$, except by substituting the proper *wf*s for each *wf* in the proof of $\mathscr{A}$.

Chapter 3

Informal predicate calculus

3.1 Predicates and sequences

A warning

The reader of this book, Hamilton's *Logic For Mathematicians*, of course already has knowledge of formal mathematics regarding quantified statements. There is common unspoken convention in math that statements are to quantified implicitly if quantifiers are absent. A statement like $x < y$ might be written to be short for $(\forall x)(\forall y)(x < y)$, where x and y are integers.

But this assumption cannot be held while reading Chapter 3 of the textbook or when studying first-order logic in general. Some strings of symbols, which will later be called *wf*s, are neither true nor false, and the reason for this is the absence of quantifiers. For example, the string $x < y$ is assumed to be indeterminate in the sense that it is neither true nor false, and likewise $(\forall x)x < y$ is also indeterminate. The truth of these formulas depend on how x and y are evaluated ($x = y = 3$ would yield a false evaluation, for example), which is an idea which will be formalized in the last chapter of this section.

We will see later that all *closed formulas*, which refer to formulas where all variables are quantified, are true or false, as expected. However, not all unquantified formulas are indeterminate. Consider $x = x$, which is true, despite the formula having no quantifiers.

Solutions to exercises

1. (a) $\sim (\forall x)(F(x) \to D(x))$
   (b) $(\exists x)(F(x) \land C(x) \land (\sim D(x)))$
   (c) $(\exists x)(T(x) \land L(x)) \to (\forall x)(T(x) \to L(x))$
   (d) $(\forall x)(E(x) \lor O(x))$
   (e) $\sim (\exists x)(E(x) \land O(x))$
   (f) $(\exists x)(P(x) \land (\forall y)(P(y) \to H(x, y)))$
   (g) $(\forall x)(E(x) \to (\forall y)(M(y) \to H(x, y)))$
2. (a) $\sim (\forall x)(C(x) \to T(x))$
   $(\exists x)(C(x) \land \sim T(x))$

(b) $\sim (\forall x)(P(x) \to (\sim L(x) \wedge \sim S(x)))$
$(\exists x)(P(x) \wedge (L(x) \vee S(x)))$

(c) $(\forall x)(\forall y)((M(x) \wedge E(y)) \to \sim H(x, y))$
$\sim (\exists x)(\exists y)(M(x) \wedge E(y) \wedge H(x, y))$

(d) $(\forall x)(N(x) \vee S(x))$
$\sim (\exists x)((\sim N(x)) \wedge (\sim S(x)))$

3.

3.2 First order languages

A *first order language* $\mathscr{L}$ will have as its alphabet of symbols:

- the countably infinite list of *variables* $x_1, x_2, \ldots$
- some or none of the *constants* $a_1, a_2, \ldots$
- some or none of the *predicate letters* $A_1^1, A_2^1, \ldots; A_1^2, A_2^2, \ldots; A_1^3, A_1^3, \ldots; \ldots$
- some or none of the *function letters* $f_1^1, f_2^1, \ldots; f_1^2, f_2^2, \ldots; f_1^3, f_1^3, \ldots; \ldots$
  - the predicate and function letters are countably infinite lists of countably infinite lists
  - the subscripts are used to distinguish different functions of the same arity, while the superscript indicates the arity
  - the function letters are not strictly necessary, as they can be expressed as relations as well, but the redundancy is kept for purposes of intuitive clarity
- the left and right parentheses (and) and the comma , as *punctuation symbols*
- the *connectives* $\sim$ and $\to$
- the quantifier $\forall$
  - the existential quantifier $\exists$ can be expressed in terms of the universal quantifier along with the $\sim$ connective, so it is not included

Definition 3.6. A *term* in a first order language $\mathscr{L}$ is defined as follows

(i) Variables and individual constants are terms.

(ii) If f_i^n is a function letter in $\mathscr{L}$, and $t_1, \ldots, t_2$ are terms in $\mathscr{L}$, then $f_i^n(t_1, \ldots, t_2)$ is a term in $\mathscr{L}$.

(iii) The set of all terms is generated as in (i) and (ii).

An *atomic formula* in $\mathscr{L}$ is defined by: if A_j^k is a predicate letter in $\mathscr{L}$ and $t_1, \ldots, t_n$ are terms in L, then $A_j^k(t_1, \ldots, t_n)$ is an atomic formula of $\mathscr{L}$.

A *well-formed formula* of $\mathscr{L}$ is defined by:

(i) Every atomic formula of $\mathscr{L}$ is a *wf* of $\mathscr{L}$.

(ii) If $\mathscr{A}$ and $\mathscr{B}$ are *wf*s, so are $(\sim \mathscr{A}), (\mathscr{A} \to \mathscr{B})$ and $(\forall x_i)\mathscr{A}$, where x_i is any variable.

(iii) The set of all *wf*s of $\mathscr{L}$ is generated as in (i) and (ii)

Terms are to be considered as objects in the language, a *wf* as a statement, and an atomic formula as the most simple kind of statement.

The symbols $\exists$, $\wedge$ and $\vee$ are treated as shorthand.

- $(\exists x_i)\mathscr{A}$ is an abbreviation for $(\sim ((\forall x_i)(\sim \mathscr{A}))$
- $(\mathscr{A} \wedge \mathscr{B})$ is an abbreviation for $(\sim (\mathscr{A} \to (\sim \mathscr{B})))$
- $(\mathscr{A} \vee \mathscr{B})$ is an abbreviation for $((\sim \mathscr{A}) \to \mathscr{B})$

Note. Unlike in chapter 2, this manual will follow the book's style for omitting parentheses. That is, a $\sim$ will be presumed to apply to the shortest possible subsequent *wf*. Also notice that this is how the $\forall$ quantifier is treated by definition with any omission of parentheses.

Definition 3.8. In the *wf* $(\forall x_i)\mathscr{A}$, we say that $\mathscr{A}$ is the *scope* of the quantifier. When $(\forall x_i)\mathscr{A}$ occurs as a subformula of a *wf* $\mathscr{B}$, the scope of the quantifier $(\forall x_i)$ is said to be $\mathscr{A}$ in $\mathscr{B}$.

A variable x_i in a *wf* is said to be *bound* if it occurs within the scope of a $(\forall x_i)$ in the *wf* or if it is the x_i in a $(\forall x_i)$. A variable which is not bound is said to be *free*.
Note. One point of confusion is that the meaning of a formula $(\forall x_i)\mathscr{A}$ is reliant on its *free* variables, not its bound ones, which might be contrary to one's intuitions about the words *free* and *bound*.

From here on out, if $\mathscr{A}(x_i)$ is a *wf* in which x_i occurs free, then $\mathscr{A}(t)$ will refer to $\mathscr{A}(x_i)$ with all *free* occurrences of x_i replaced with t. So if $\mathscr{A}(x_i)$ is $(\forall x_2)A_1^1(x_i) \to (\forall x_i)A_1^1(x_i))$, then $\mathscr{A}(t)$ is $(\forall x_2)A_1^1(t) \to (\forall x_i)A_1^1(x_i))$. We will want to only substitute t for x_i if it does not interact with quantifiers in $\mathscr{A}(x_i)$. In the previous example, x_2 would be a different substitution that any other variable. For this reason, we need the next definition, which is as important as its confusing. A few equivalent definitions will be provided.

Definition 3.11. Let $\mathscr{A}$ be any *wf* of $\mathscr{L}$. A term t is *free for x_i in $\mathscr{A}$* if x_i does not occur free in $\mathscr{A}$ within the scope of a $(\forall x_j)$, where x_j is any variable occurring in t.

Equivalently, a term t is *free for x_i in $\mathscr{A}$* if substituting t for any free instance of x_i in $\mathscr{A}$ would not introduce any new bound variables.

Equivalently, a term t is *free for x_i in $\mathscr{A}$* if any variable in t is free in $\mathscr{A}$ after substituting it for any free instance of x_i.

An algorithm for determining whether a term t is *free for x_i in $\mathscr{A}$* goes as follows:

1. Find all free instances of x_i in $\mathscr{A}$.
2. For each free instance x_j, repeat the following step:
   (a) For each variable x_k in t, repeat the following steps:
      i. Substitute x_k for x_j.

ii. If x_k is bound in $\mathscr{A}$, then t is not free for x_i, and terminate the algorithm.

3. Conclude that t is free for x_i.

Note. A term being free for a variable x_i does not necessarily indicate that it may be substituted for that variable, because the variable x_i may be bound. But if x_i only occurs free, then a term being free for it is equivalent to a term being substitutable for it. Therefore "t being free for x_i" can be thought of as "t being substitutable for free instances of x_i."
Note. It is easy to confirm that if x_i occurs only bound in $\mathscr{A}$, then any term is free for it. Also, for any *wf* and any variable x_i, x_i is free for itself in $\mathscr{A}$.

Solutions to exercises

4. The set of terms in a first order language with no function letters is just the set of the variables and the individual constants.

5. The set of terms is $f_1^1(x_1), f_1^1(x_2), f_1^1(x_3), \ldots$.

6. The formulas that are well-formed formulas are (a), (d), (e), (g), and (h).

   Note. The answer key in the book omits (d) as a well-formed formula, but this appears to be wrong.

7. (a) free

   (b) bound, bound

   (c) bound, bound, free

   (d) free, free, free, free

   Since all occurrences of x_2 in the *wf*s are bound, the term $f_1^2(x_1, x_3)$ (and any other term) is free for x_2 in the *wf*s.

8. Suppose that x_j is free for x_i in $\mathscr{A}(x_i)$. Then x_i does not occur free in the scope of a $(\forall x_j)$ in $\mathscr{A}(x_i)$. The goal is to show that x_j does not occur free in the scope of a $(\forall x_i)$ in $\mathscr{A}(x_j)$.

   Now consider an occurrence of an x_j in $\mathscr{A}(x_j)$, which is either (1) a substitution for an x_i in $\mathscr{A}(x_i)$ or (2) not a substitution for an x_i in $\mathscr{A}(x_i)$. If (1), then since x_1 was assumed to occur free in $\mathscr{A}(x_i)$, it must not be in the scope of a $(\forall x_i)$, and so the substituted x_j must also not be in the scope of a $(\forall x_i)$. If (2), then it must occur the same as it does in $\mathscr{A}(x_i)$, a *wf* in which it is assumed to occur bound, and therefore it does not occur free in the scope of a $(\forall x_i)$, or any quantifier for that matter.

   Since in both (1) and (2), x_j does not occur free in the scope of a $(\forall x_i)$ in $\mathscr{A}(x_j)$, it can be concluded that x_i is free for x_j in $\mathscr{A}(x_j)$.

   Note. The hypothesis that x_j is free for x_i is never used in the proof. It seems that there is either a mistake in the proof or in the exercise. The hint/proof in the back of the book seems to agree with the proof given here.

9. (a) Since $(\forall x_1)$ and $(\forall x_3)$ never occur, $\underline{t \text{ is free for } x_1}$.

(b) Since x_1 only occurs bound, $\underline{t \text{ is free for } x_1}$.

(c) Since x_1 occurs free in the scope of a $(\forall x_3)$, $\underline{t \text{ is not free for } x_1}$.

(d) Since x_1 occurs free in the scope of a $(\forall x_3)$, $\underline{t \text{ is not free for } x_1}$.

10. Here, the (a), (b), (c), and (d) refer to the *wf*s in the exercise 9 and not the terms in exercise 10.

Note. There is a mistake in the text, the exercise should read "Repeat Exercise 9..." instead of "Repeat Exercise 6...".

Let $t = x_2$.

(a) Since x_1 occurs free in the scope of a $(\forall x_2)$, $\underline{t \text{ is not free for } x_1}$.

(b) Since x_1 only occurs bound, $\underline{t \text{ is free for } x_1}$.

(c) Since x_1 does not occur in the scope of a $(\forall x_2)$, it does not occur free in it, so $\underline{t \text{ is free for } x_1}$.

(d) Since x_1 occurs free in the scope of a $(\forall x_2)$, $\underline{t \text{ is not free for } x_1}$.

Let $t = x_3$.

(a) Since a $(\forall x_3)$ never occurs, $\underline{t \text{ is free for } x_1}$.

(b) Since x_1 only occurs bound, $\underline{t \text{ is free for } x_1}$.

(c) Since x_1 occurs free in the scope of a $(\forall x_3)$, $\underline{t \text{ is not free for } x_1}$.

(d) Since x_1 occurs free in the scope of a $(\forall x_3)$, $\underline{t \text{ is not free for } x_1}$.

Let $t = f_1^2(a_1, x_1)$.

(a) Since a $(\forall x_1)$ never occurs, $\underline{t \text{ is free for } x_1}$.

(b) Since x_1 only occurs bound, $\underline{t \text{ is free for } x_1}$.

(c) Since a $(\forall x_1)$ never occurs, $\underline{t \text{ is free for } x_1}$.

(d) Since a $(\forall x_1)$ never occurs, $\underline{t \text{ is free for } x_1}$.

Let $f_1^3(x_1, x_2, x_3)$.

(a) Since x_1 occurs free in the scope of a $(\forall x_2)$, $\underline{t \text{ is not free for } x_1}$.

(b) Since x_1 only occurs bound, $\underline{t \text{ is free for } x_1}$.

(c) Since x_1 occurs free in the scope of a $(\forall x_3)$, $\underline{t \text{ is not free for } x_1}$.

(d) Since x_1 occurs free in the scope of a $(\forall x_2)$ and a $(\forall x_3)$, $\underline{t \text{ is not free for } x_1}$.

3.3 Interpretations

Definition 3.14. An *interpretation* I of $\mathscr{L}$is

- a non-empty set D_I called the *domain* of I together with a collection
- a collection of distinguished elements $(\overline{a}_1, \overline{a}_2, \dots)$ of D_I

- a collection of functions from D_I to D_I denoted by $(\overline{f}_1^n, i > 0, n > 0)$
- a collection of relations on D_I denoted by $(\overline{A}_1^n, i > 0, n > 0)$

An interpretation allows a *wf* of a first order language to be interpreted as a statement with a truth value, analogous to how a valuation of L allowed a *wf* in L to have a truth value. This concept will be formalized in the coming sections.

Solutions to exercises

11. The interpretation of $\mathscr{A}$ in I is the statement

$$(\forall x_1)(\forall x_2)(x_1 - x_2 < 0 \rightarrow x_1 < x_2)$$

which is a true statement in the integers. Consider the same interpretation I with $\overline{f}_1^2(x, y)$ as $x + y$. The interpretation of $\mathscr{A}$ in I is then

$$(\forall x_1)(\forall x_2)(x_1 + x_2 < 0 \rightarrow x_1 < x_2)$$

which is false.

12. Let I be the interpretation described above in the previous exercise with the addition of $\overline{f}_1^1, \overline{A}_1^1$ defined by $\overline{f}_1^1(x) = x - 1$ and $\overline{A}_1^1(x)$ if and only if $x > 0$. Then the statement corresponding to the *wf* under I is

$$(\forall x_i)(x_1 > 0 \rightarrow x_1 - 1 > 0)$$

which is false.

13. Let I again be the interpretation described in Exercise 11 with $\overline{A}_1^2(x, y)$ as $x < y$. Then the interpretation of the *wf* is

$$(\forall x_1)(x_1 < x_2 \rightarrow x_2 < x_1)$$

which is false, since it defies the law of trichotomy, a property of the integers.

3.4 Satisfaction, truth

In this chapter I will be an interpretation of the language $\mathscr{L}$ with notation consistent with Definition 3.14.

In the previous chapter, values of true or false were informally assigned to various *wf*s of $\mathscr{L}$ under some interpretation I. This chapter will formalize this process of evaluating the truth of a *wf*. The process will and must be similar to the informal process of determining the truth value of a *wf*. First, the terms must be assigned to values. A particular assignment is formally known as a *valuation*.

Definition 3.17. A *valuation* in I is a function v from the set of terms of $\mathscr{L}$ to the set domain of I, D_I, with the properties:

(i) $v(a_i) = \overline{a}_i$ for each constant a_i of $\mathscr{L}$.

(ii) $v(f_i^n(t_1, \ldots, t_n)) = f_i^n(v(t_1)), \ldots, v(t_n))$, where f_i^n is any function letter in $\mathscr{L}$, and $t_1, \ldots, t_n$ are any terms of $\mathscr{L}$.

Note. An interpretation will have as many different valuations as there are ways of assigning the variables in $\mathscr{L}$ to elements of D_I.
Note. A term in $\mathscr{L}$ may be a variable, a constant, or a function with terms as its arguments. The variables can be assigned to any elements in D_I and no property is needed to govern the valuation of a variable. By property (i), every valuation assigns constants of $\mathscr{L}$ to its corresponding constant in D_I. Property (ii) guarantees that the valuation of functions in $\mathscr{L}$ behave as expected.

Definition 3.19. Two valuations v and v' are *i-equivalent* if $v(x_j) = v'(x_j)$ for every $j \neq i$.
Note. The purpose of this definition will be better understood after reading further on.

Continuing from where we left off before Definition 3.17, after the terms are assigned values, the *wf*s can be evaluated as true or false, depending on the particular values of the terms. If a particular *wf* is to be interpreted as a true statement when its terms take on some particular values specified by a valuation, the valuation is said to satisfy the *wf*. Since a *wf* in $\mathscr{L}$ was defined recursively, and likewise the definition of satisfaction of a *wf* must also be defined recursively.

Definition 3.20. Let $\mathscr{A}$ be a *wf* of $\mathscr{L}$, and let I be an interpretation of $\mathscr{L}$. A valuation v in I is said to satisfy...

(i) the atomic formula $A_j^n(t_1, \ldots, t_n)$ if $\overline{A}_j^n(v(t_1), \ldots, v(t_n))$ is true in D_I,

(ii) the negation $(\sim \mathscr{B})$ if v does not satisfy $\mathscr{B}$,

(iii) the implication $(\mathscr{B} \rightarrow \mathscr{C})$ if either v satisfies $(\sim \mathscr{B})$ or v satisfies $\mathscr{C}$,

(iv) the quantified *wf* $(\forall x_i)\mathscr{B}$ if all valuations v' which are i-equivalent to v satisfy $\mathscr{B}$.

Note. The first three parts of the definition are straightforward. The last one should be explained further. It states that a valuation satisfies a quantified *wf* if any corresponding interpretation is true when the bound variable takes on any possible value.

Proposition 3.23. Let $\mathscr{A}(x_i)$ be a *wf* of $\mathscr{L}$ in which x_i appears free, and let t be a term free for x_i. Suppose that v is a valuation and v' is the valuation which is i-equivalent to v and has $v'(x_i) = v(t)$. Then v satisfies $\mathscr{A}(t)$ if and only if v' satisfies $\mathscr{A}(x_i)$.
Note. The condition that x_i appears free in $\mathscr{A}(x_i)$ can be relaxed if $\mathscr{A}(t)$ is defined as replacing all bound instances of x_i.
Proof. We will first prove a lemma.

Lemma. Let u be a term in which x_i occurs. Let u' be the term obtained by substituting t for x_i in u. Then $v(u') = v'(u)$.
Proof. The proof is by strong-induction on the number of sub-terms in u. Note that this number takes sub-terms of sub-terms into account, so $f_1^2(f_1^2(x_1, x_1), f_1^2(x_1, x_1))$ has four, not two, sub-terms.

Suppose as a hypothesis of strong induction that if a term has fewer than n sub-terms in it, then $v(u') = v'(u)$, where u and u' are defined as above.

(base case) It may be that $n = 1$, in which case $u = x_i$ and $u' = t$. Then $v'(u) = v'(x_i) = v(t) = v(u')$ by construction of v' in the premise.

(inductive step) Otherwise, $n > 1$, and so $u = f_i^k(u_1, \dots, u_k)$, where $u_1, \dots, u_k$ are sub-terms that necessarily have fewer than n sub-terms. In the same way that u' was defined for u, define $u'_1, \dots, u'_k$ as the terms obtained by substituting t for x_i in $u_1, \dots, u_k$. Finally, notice that $u' = f_i^k(u'_1, \dots, u'_k)$, so

$$\begin{aligned}
v(u') &= v(f_i^k(u'_1, \dots, u'_k)) && \text{definition of } u' \\
&= \overline{f}_i^k(v(u'_1), \dots, v(u'_k)) && \text{Definition 3.17} \\
&= \overline{f}_i^k(v'(u_1), \dots, v'(u_k)) && \text{Induction hypothesis} \\
&= v'(f_i^k(u_1, \dots, u_k)) && \text{Definition 3.17} \\
&= v'(u) && \text{Definition of } u
\end{aligned}$$

With this induction complete, we may conclude that $v'(u) = v(u')$ for any u.

Now we prove the proposition by another strong induction on the number of connectives and quantifiers of $\mathscr{A}(x_i)$.

(base case) It may be that $\mathscr{A}(x_i)$ has non quantifiers and connectives, and so it must be an atomic formula, say $A_i^n(u_1, \dots, u_n)$. Let $u'_1, \dots, u'_n$ be the terms $u_1, \dots, u_n$ with t substituted for x_i, so that $\mathscr{A}(t)$ must then be $\mathscr{A}(u'_1, \dots, u'_n)$. Then the following are all equivalent.

(a) v satisfies $\mathscr{A}(t)$, by assumption

(b) v satisfies $A_i^n(u'_1, \dots, u'_n)$, by definition of $\mathscr{A}(t)$

(c) $A_i^n(v(u'_1), \dots, v(u'_n))$ is true in I, by Definition 3.20

(d) $A_i^n(v'(u_1), \dots, v'(u_n))$ is true in I, by the lemma above

(e) v' satisfies $A_i^n(u_1, \dots, u_n)$, by Definition 3.20

(f) v' satisfies $A_i^n(x_i)$, by Definition 3.20

and the equivalence of (a) and (f) is what we desired to prove.

(inductive step) Otherwise, $\mathscr{A}(x_i)$ has k quantifiers and connectives. Suppose that $\mathscr{B}(x_i)$ has fewer than k quantifiers and connectives. Let w be a valuation and let w' be the valuation which is i-equivalent to w and has $w'(x_i) = w(t)$. Suppose, as an inductive hypothesis, that w satisfies $\mathscr{A}(t)$ if and only if w' satisfies $\mathscr{A}(x_i)$.

There are three cases to check.

1. The *wf* $\mathscr{A}(x_i)$ is $\sim \mathscr{B}(x_i)$, a *wf* with fewer than k quantifiers and connectives. Note that $\mathscr{A}(t)$ is $\sim \mathscr{B}(t)$. The following are equivalent.

   (a) v satisfies $\mathscr{A}(t)$, by assumption

   (b) v satisfies $\sim \mathscr{B}(t)$, by definition of $\mathscr{A}(t)$

   (c) v does not satisfy $\mathscr{B}(t)$, by Definition 3.20

   (d) v' does not satisfy $\mathscr{B}(t)$, by the induction hypothesis

   (e) v' satisfies $\sim \mathscr{B}(t)$, by definition of $\mathscr{A}(t)$

   (f) v' satisfies $\mathscr{A}(x_i)$, by Definition 3.20

The equivalence between (a) and (f) is what we desired to prove. Note that in (d), we used the equivalent negative form of the inductive hypothesis.

2. The *wf* $\mathscr{A}(x_i)$ is $\mathscr{B}(x_i) \to \mathscr{C}(x_i)$, where both $\mathscr{B}(x_i)$ and $\mathscr{C}(x_i)$ are *wf*s with fewer than k quantifiers and connectives. Note that $\mathscr{B}(t)$ is $\mathscr{B}(t) \to \mathscr{C}(t)$. The following are equivalent.

   (a) v satisfies $\mathscr{A}(t)$, by assumption
   (b) v satisfies $\mathscr{B}(t) \to \mathscr{C}(t)$, by definition of $\mathscr{A}(t)$
   (c) v satisfies $\sim \mathscr{B}(t)$ or v satisfies $\mathscr{C}(t)$, by Definition 3.20
   (d) v' satisfies $\sim \mathscr{B}(t)$ or v satisfies $\mathscr{C}(t)$, by the induction hypothesis
   (e) v' satisfies $\sim \mathscr{B}(t) \to \mathscr{C}(t)$, by Definition 3.20
   (f) v' satisfies $\mathscr{A}(x_i)$, by definition of $\mathscr{A}(x_i)$

3. The *wf* $\mathscr{A}(x_i)$ is $(\forall x_j)\mathscr{B}(x_i)$, where $i \neq j$ because x_i is assumed to occur free in $\mathscr{A}$. Note that $\mathscr{B}(x_i)$ has fewer than k quantifiers and that $\mathscr{A}(t)$ is $(\forall x_j)\mathscr{B}(t)$. Then the following are all equivalent.

   (a) v satisfies $\mathscr{A}(t)$, by assumption
   (b) v satisfies $(\forall x_j)\mathscr{B}(t)$, by definition of $\mathscr{A}(t)$
   (c) any j-equivalent valuation to v satisfies $\mathscr{B}(t)$, by Definition 3.20
   (d) any j-equivalent valuation to v' satisfies $\mathscr{B}(x_i)$, by the induction hypothesis, and the note below
   (e) v' satisfies $(\forall x_j)\mathscr{B}(x_i)$, by Definition 3.20
   (f) v' satisfies $\mathscr{A}(x_i)$, by definition of $\mathscr{A}(x_i)$

   Additional details needs to be given to show that (c) and (d) are equivalent.

   $\Rightarrow$ Suppose that any j-equivalent valuation to v satisfies $\mathscr{B}(t)$. Then let w' be a valuation j-equivalent to v'. Let w be a valuation j-equivalent to v with $w(x_j) = w'(x_j)$ that necessarily satisfies $\mathscr{B}(t)$. By construction, we have that w is i-equivalent to w'. Notice that $v'(x_i) = v'(t)$, so $w'(x_i) = w'(t)$, since w' is j-equivalent to v', and so we may apply the inductive hypothesis. Therefore, w' satisfies $\mathscr{B}(x_i)$.

   $\Leftarrow$ Suppose that any j-equivalent valuation to v' satisfies $\mathscr{B}(x_i)$. Then let w be a valuation j-equivalent to v. Let w' be a valuation j-equivalent to v' with $w'(x_j) = w(x_j)$ that necessarily satisfies $\mathscr{B}(x_i)$. By construction, we have that w' is i-equivalent to w. Notice that $v'(x_i) = v'(t)$, so $w'(x_i) = w'(t)$, since w' is j-equivalent to w, and so we may apply the inductive hypothesis. Therefore, w satisfies $\mathscr{B}(t)$.

Note. In the proof in the book, there is a mistake in Case 1, it should instead read: "$\mathscr{A}(x_i)$ is $\sim \mathscr{B}(x_i)$".

Definition 3.24. A *wf* $\mathscr{A}$ is true in an interpretation I if every valuation in I satisfies $\mathscr{A}$. It is *false* if there is no valuation in I which satisfies $\mathscr{A}$. If $\mathscr{A}$ is true in I, we write $I \models \mathscr{A}$.
Note. By part (ii) of Definition 3.20, if a given *wf* is satisfied by all valuations, then its negation is not satisfied by all valuations and vice versa. So no *wf* can be both true and false.

Note. Some *wf*s can be neither true nor false if there exists a valuation satisfying it and another one satisfying its negation.

Proposition 3.26. If, in an interpretation I, the *wf* $\mathscr{A}$ and $(\mathscr{A} \to \mathscr{B})$ are true, then $\mathscr{B}$ is also true.
Proof. Let v be a valuation in I. The *wf*s $\mathscr{A}$ and $(\mathscr{A} \to \mathscr{B})$ are true in I, which is to say that they are true for any valuation, and thus true for v. Since v satisfies $(\mathscr{A} \to \mathscr{B})$, it either satisfies $\mathscr{B}$ or $(\sim \mathscr{A})$. But it cannot satisfy $(\sim \mathscr{A})$, or else it would not satisfy $\mathscr{A}$. Therefore, it satisfies $\mathscr{B}$. Since v was chosen as an arbitrary valuation, every valuation satisfies $\mathscr{B}$, and so it is true in I.

Proposition 3.27. Let $\mathscr{A}$ be a *wf* of $\mathscr{L}$, and let I be an interpretation of $\mathscr{L}$. Then $I \models \mathscr{A}$ if and only if $I \models (\forall x_i)\mathscr{A}$, where x_i is any variable.
Proof. $\Rightarrow$ Suppose that $I \models \mathscr{A}$. Let v be any valuation in I and let v' be any i-equivalent valuation to v. Since all valuations satisfy $\mathscr{A}$, v' satisfies $\mathscr{A}$. Therefore, v, which was chosen to be an arbitrary valuation, satisfies $(\forall x_i)\mathscr{A}$, and so all valuations in I satisfy $(\forall x_i)\mathscr{A}$.

$\Leftarrow$ Suppose that $I \models (\forall x_i)\mathscr{A}$. Let v be any valuation in I. Since v is i-equivalent to v, it must satisfy $\mathscr{A}$. Since v was chosen as an arbitrary valuation, all valuations in I satisfy $\mathscr{A}$.

Corollary 3.28. Let $y_1, \ldots, y_n$ be variables in $\mathscr{L}$, let $\mathscr{A}$ be a *wf* of $\mathscr{L}$, and let I be an interpretation. Then $I \models \mathscr{A}$ if and only if $I \models (\forall y_1) \ldots (\forall y_n)\mathscr{A}$.
Proof. By repeated application of Proposition 3.27.

The above corollary is significant because it states that implicit quantification of variables is legitimate when a statement of an interpretation is known to be true. For example, $x = x$ as a statement about the integers does not need be quantified because it is known that the statement alone is true. Similarly, if a true statement already has all of its variables quantified, then the quantifiers can be omitted with no loss of meaning. It also implies that adding quantifiers to a false or indeterminate *wf* cannot "upgrade" its truth value so that the new quantified *wf* is true. However, adding quantifiers can turn an indeterminate *wf* into one which is false (there are many examples of this).

Proposition 3.29. In an interpretation I, a valuation v satisfies the formula $(\exists x_i)\mathscr{A}$ if and only if there is at least one valuation v' which is i-equivalent to v and which satisfies $\mathscr{A}$.
Proof. This proof is done by mechanically applying the definitions. Let v be a valuation in an interpretation I.

$\Rightarrow$ Suppose v satisfies the formula $(\exists x_i)\mathscr{A}$, which is to say that v satisfies $\sim (\forall x_i)(\sim \mathscr{A})$, and therefore v does not satisfy $(\forall x_i)(\sim \mathscr{A})$. Therefore there must exist some v' which i-equivalent to v which does not satisfy $\sim \mathscr{A}$, and so this v' must satisfy $\mathscr{A}$.

$\Leftarrow$ Suppose that v' is an i-equivalent valuation to v that satisfies $\mathscr{A}$. Then this v' does not satisfy $(\sim \mathscr{A})$, and so v does not satisfy $(\forall x_i)(\sim \mathscr{A})$, and so v must satisfy $\sim (\forall x_i)(\sim \mathscr{A})$, i.e., v satisfies $(\exists x_i)\mathscr{A}$.

A *wf* of L and a *wf* of $\mathscr{L}$ are both formed by possibly using the connectives $\sim$ and $\to$. If we take a *wf* $\mathscr{A}$ in L and replace all of its statement letters by the same *wf* in $\mathscr{L}$, the new formula is now a *wf* in $\mathscr{L}$, and we call it a *substitution instance* of $\mathscr{A}$ in $\mathscr{L}$.

Note that a *wf* in $\mathscr{L}$ can be a substitution instance of more than one *wf* in L, depending on how its sub-formulas are replaced. For instance,

$$(\underbrace{(\forall x_i)A_1^1(x_1)}_{p_1} \to (\underbrace{(\forall x_i)A_1^2(x_1)}_{p_2} \to \underbrace{(\forall x_i)A_1^2(x_1)}_{p_3}))$$

may be considered as a substitution instance of $(p_1 \to (p_2 \to p_3))$. Also,

$$(\underbrace{(\forall x_i)A_1^1(x_1)}_{p_1} \to \underbrace{((\forall x_i)A_1^2(x_1) \to (\forall x_i)A_1^2(x_1))}_{p_2})$$

may be considered as a substitution instance of $(p_1 \to p_2)$.

The use of the term tautology may be expanded to $\mathscr{L}$, and it has the expected property of being true regardless of its valuation in any interpretation.

Definition 3.30. A *wf* $\mathscr{A}$ of $\mathscr{L}$ is a *tautology* if it is a substitution instance in $\mathscr{L}$ of a tautology in L.

Proposition 3.31. A *wf* of $\mathscr{L}$ which is a tautology is true in any interpretation of $\mathscr{L}$.

Proof. Let $\mathscr{A}_{\mathscr{L}}$ be a tautology in $\mathscr{L}$ and let $\mathscr{A}_L$ be its corresponding tautology in L. Then $\mathscr{A}_L$ consists of the statement letters $p_1, \ldots, p_n$ whose replacements in L are the *wf*s that we shall label $\mathscr{A}_1, \ldots, \mathscr{A}_n$.

Now let $v_{\mathscr{L}}$ be a valuation in any interpretation I. The goal is to prove that $v_{\mathscr{L}}$ satisfies $\mathscr{A}_{\mathscr{L}}$.

First notice that $\mathscr{A}_L$ can only be evaluated as true or false if its statement letters have values, and to this end. So let v_L be the valuation in L defined on the statement letters $p_1, \ldots, p_n$ in an expected way:

$$v_L(p_i) = \begin{cases} T & \text{if } v_{\mathscr{L}} \text{ satisfies } \mathscr{A}_i \\ F & \text{if } v_{\mathscr{L}} \text{ does not satisfy } \mathscr{A}_i \end{cases}$$

The values of v_L for all other statement letters not appearing in $\mathscr{A}_L$ are arbitrarily set to T so that v_L is indeed a valuation.

Now we will prove that $v_{\mathscr{L}}$ satisfies $\mathscr{A}_{\mathscr{L}}$ if and only if $v_L(\mathscr{A}_L) = T$. From this result, the proof of the proposition will immediately follow. We proceed by strong induction on the number of connectives $\sim$ and $\to$ in $\mathscr{A}_L$:

Suppose as a hypothesis of string induction that if a *wf* in L has fewer than k connectives, then $v_{\mathscr{L}}$ satisfies the *wf* if and only if the value of v_L when applied to the substitution instance in $\mathscr{L}$ is T.

Now let the number of connectives of $\mathscr{A}_L$ be j. If $j = 0$, then $\mathscr{A}$ consists of a statement letter only, say p. By the definition of v_L, $v_L(p) = T$ if and only if $v_{\mathscr{L}}$ satisfies $\mathscr{A}_{\mathscr{L}}$, as desired. If $j > 0$, then there are two cases two consider:

1. The *wf* $\mathscr{A}_L$ is of the form $\sim \mathscr{B}_L$. Then $\mathscr{A}_{\mathscr{L}}$ is of the form $\sim \mathscr{B}_{\mathscr{L}}$, where $\mathscr{B}_{\mathscr{L}}$ is the substitution instance of $\mathscr{B}_L$. Since $\mathscr{B}_L$ has fewer than j connectives, by the induction hypothesis, v satisfies $\mathscr{B}$ if and only if $v'(\mathscr{B}_{\mathscr{L}}) = T$, which is equivalent to saying that v does not satisfy $\mathscr{B}$ if and only if $v'(\mathscr{B}_{\mathscr{L}}) = F$, which is once again equivalent, by Definition 3.20 (ii) and 2.12 (i) to saying that v satisfies $\mathscr{A}_{\mathscr{L}}$ if and only if $v(\mathscr{A}_L) = T$.

2. The *wf* $\mathscr{A}_L$ of the form $\mathscr{B}_L \to \mathscr{C}_L$, and so $\mathscr{A}$ is of the form $\mathscr{B}_{\mathscr{L}} \to \mathscr{C}_{\mathscr{L}}$, where $\mathscr{B}_{\mathscr{L}}$ and $\mathscr{C}_{\mathscr{L}}$ are the substitution instances of $\mathscr{B}_L$ and $\mathscr{C}_L$ respectively. Note that $\mathscr{B}_L$ and $\mathscr{C}_L$ both have fewer than j connectives. The following assertions are all equivalent:

   (a) $v_{\mathscr{L}}$ satisfies $\mathscr{A}_{\mathscr{L}}$

   (b) $v_{\mathscr{L}}$ satisfies $\mathscr{B}_{\mathscr{L}} \to \mathscr{C}_{\mathscr{L}}$

   (c) either $v_{\mathscr{L}}$ satisfies $\sim \mathscr{B}_{\mathscr{L}}$ or $\mathscr{C}_{\mathscr{L}}$ (by Definition 3.20 (iii))

   (d) either v does not satisfy $\mathscr{B}_{\mathscr{L}}$ or satisfies $\mathscr{C}_{\mathscr{L}}$ (by Definition 3.20 (ii))

   (e) either $v_L(\mathscr{B}_L) = F$ or $v_L(\mathscr{C}_L) = T$ (by the strong induction hypothesis)

   (f) $v_L(\mathscr{B}_L \to \mathscr{C}_L) = T$ (by Definition 2.12)

   (g) $v_L(\mathscr{A}_L) = T$

   and the equivalence of (a) and (g) is what we desired to prove for this case.

Now with the induction complete, we can prove the original proposition. Recall that $\mathscr{A}_{\mathscr{L}}$ is a tautology in $\mathscr{L}$, $\mathscr{A}_L$ is a tautology in L, and $v_{\mathscr{L}}$ is an arbitrary valuation in an arbitrary interpretation I. From the above, we know that $v_{\mathscr{L}}$ satisfies $\mathscr{A}_{\mathscr{L}}$ if and only if $v_L(\mathscr{A}_L) = T$. But $\mathscr{A}_L$ is a tautology in L, so indeed $v_L(\mathscr{A}_L) = T$, and so $v_{\mathscr{L}}$ satisfies $\mathscr{A}_{\mathscr{L}}$. Thus $\mathscr{A}_{\mathscr{L}}$ is true in any interpretation I.
Note. This proof is long, but mostly just straightforward applications of definition. Its length comes from having to define v_L
Note. The need for strong induction comes from case 2. Normal induction would not be sufficient because both of the *wf*s in L might have fewer than $j-1$ connectives.

As stated in the warning at the beginning of this chapter (in the manual, not the textbook), if a *wf* has all of its variables quantified, then it must be either true or false. We will prove this shortly, but first we introduce a short definition and a proposition.

Definition 3.32. A *wf* $\mathscr{A}$ of $\mathscr{L}$ is said to be *closed* if all variables in $\mathscr{A}$ occurs bound.

Proposition 3.33. Let I be an interpretation of $\mathscr{L}$ and let $\mathscr{A}$ be a *wf* of $\mathscr{L}$. If v and w are valuations such that $v(x_i) = w(x_i)$ for every free variable x_i of $\mathscr{A}$, then v satisfies $\mathscr{A}$ if and only if w satisfies $\mathscr{A}$.
Note. This is stating the obvious fact that if two valuations "plug in" the same values for variables, then the resulting truth values will be the same.
Proof. The proof follows from strong induction on the numbers of connectives and quantifiers in $\mathscr{A}$.

As a hypothesis of strong induction, suppose that for any *wf* $\mathscr{A}$ of $\mathscr{L}$ with fewer than n connectives, v satisfies $\mathscr{A}$ if and only if w satisfies $\mathscr{A}$, where v and w are valuations such that $v(x_i) = w(x_i)$ for any x_i of $\mathscr{A}$.

It may be the case that $n = 0$, in which case the *wf* is an atomic formula with j terms of the general form $A_i^j(t_1, \ldots, t_j)$. A term t can either be a constant, in which case $v(t) = w(t)$, since all are defined to have the same values for constants, or the term can be a variable or a function which takes terms. Since v and w agree for variables, since any variable in the atomic formula occurs free, they must also agree for functions (this can be formalized via another induction, but that is tedious). Therefore, for any atomic formula $\mathscr{A}$, v satisfies $\mathscr{A}$ if and only if w satisfies $\mathscr{A}$.

It may be the case that $n > 0$, in which case the inductive hypothesis must be employed to prove the three distinct cases which may occur.

1. The *wf* $\mathscr{A}$ is of the form $\sim \mathscr{B}$. Notice that $\mathscr{B}$ has fewer than n connectives, so v satisfies $\mathscr{B}$ if and only if w satisfies $\mathscr{B}$, which is to say that v does not satisfy $\mathscr{B}$ if and only if w does not satisfy $\mathscr{B}$, which is once again equivalent to stating that v satisfies $\sim \mathscr{B}$ if and only if w satisfies $\sim \mathscr{B}$, by Definition 3.20 (ii). And since $\sim \mathscr{B}$ is $\mathscr{A}$, we have proved the desired property for this case.

2. The *wf* $\mathscr{A}$ is of the form $\mathscr{B} \rightarrow \mathscr{C}$. The following are all equivalent:

   (a) v satisfies $\mathscr{A}$

   (b) v satisfies $\mathscr{B} \rightarrow \mathscr{C}$

   (c) v satisfies $\sim \mathscr{B}$ or v satisfies $\mathscr{C}$, by Definition 3.20 (iii)

   (d) w satisfies $\sim \mathscr{B}$ or w satisfies $\mathscr{C}$, by the induction hypothesis, and the fact that that both $\sim \mathscr{B}$ and $\mathscr{C}$ have fewer than n connectives

   (e) w satisfies $\mathscr{B} \rightarrow \mathscr{C}$, again by definition 3.20 (ii)

   (f) w satisfies $\mathscr{A}$

   and the equivalence between (a) and (f) is what we desired to prove for this case.

3. The *wf* $\mathscr{A}$ is of the form $(\forall x_i)\mathscr{B}$. We are to prove that v satisfies $\mathscr{A}$ if and only if w satisfies $\mathscr{A}$.

   $\Rightarrow$ Suppose that v satisfies $\mathscr{A}$. Then for any i-equivalent valuation to v, v' satisfies $\mathscr{B}$. To show that w satisfies $\mathscr{A}$, which is $(\forall x_i)\mathscr{B}$, we must show that any i-equivalent valuation to w satisfies $\mathscr{B}$. So let w' be i-equivalent to w, and let v' be the particular valuation i-equivalent to v which satisfies $v'(x_i) = w'(x_i)$. Now let y be a free variable of $\mathscr{B}$. There are two cases to consider.

   (a) If $y = x_i$, then $v'(x_i) = w'(x_i)$, since v' was chosen in this way.

   (b) If $y \neq x_i$, then it is a free variable of $\mathscr{A}$, since $\mathscr{B}$ differs from $\mathscr{A}$ in that only x_i may potentially be free in $\mathscr{B}$, and so

   $$\begin{array}{rl} v'(y) = v(y) & v' \text{ and } v \text{ are } i\text{-equivalent} \\ v(y) = w(y) & v(x) = w(x) \text{ for any free variable } x \text{ in } \mathscr{A} \\ w(y) = w'(y) & w' \text{ and } w \text{ are } i\text{-equivalent} \end{array}$$

   with the conclusion in this case being that $v'(y) = w'(y)$.

   Therefore, whenever y is a free variable of $\mathscr{B}$, a *wf* with fewer than n connectives, $v'(y) = w'(y)$, and so by the induction hypothesis and since v' satisfies $\mathscr{B}$, w' satisfies $\mathscr{B}$. Since w' was chosen to be an arbitrary i-equivalent valuation to w, it follows that w satisfies $(\forall x_i)\mathscr{B}$, i.e., w satisfies $\mathscr{A}$.

   $\Leftarrow$ This direction is proved in precisely the same way as the above direction except with the occurrences of v and w switched.

With the induction complete, we have proved the proposition.

Note. In case 3, x_i need not be free in $\mathscr{B}$, in which the quantifier $(\forall x_i)$ appears in $\mathscr{B}$, but in that case x_i would not be considered as a possible free variable y in $\mathscr{B}$, so it would be disregarded.

Note. In case 1, the free variables of $\mathscr{A}$ were the same as the free variables of $\mathscr{B}$. Similarly, in case 2, the free variables of $\mathscr{B}$ and $\mathscr{C}$ were the same as $\mathscr{A}$. Therefore, in both cases, v and w could be applied to the inductive hypothesis regarding $\mathscr{B}$ in case 2 or $\mathscr{B}$ and $\mathscr{C}$ in case 3. In case 3, on the other hand, the free variables of $\mathscr{A}$ and $\mathscr{B}$ differed in that x_i need not have been free in $\mathscr{B}$. Instead, i-equivalent valuations of v and w were shown to agree for any free variable in $\mathscr{B}$ so that the inductive hypothesis could be applied.

Corollary 3.34. If $\mathscr{A}$ is a closed *wf* of $\mathscr{L}$ and I is an interpretation of $\mathscr{L}$, then either $I \models \mathscr{A}$ or $I \models (\sim \mathscr{A})$.
Proof. Let v and w be any valuations. Since $\mathscr{A}$ has no free variables, $v(y) = w(y)$ for any free variable y, vacuously. So v satisfies $\mathscr{A}$ if and only if w satisfies $\mathscr{A}$, by Proposition 3.33. So, either every valuation satisfies $\mathscr{A}$ or every valuation does not satisfy $\mathscr{A}$, which is to say that $\mathscr{A}$ is either true or false in I. So either $I \models \mathscr{A}$ or $I \models (\sim \mathscr{A})$.

Definition 3.35. A *wf* $\mathscr{A}$ of $\mathscr{L}$ is *logically valid* if $\mathscr{A}$ is true in every interpretation of $\mathscr{L}$ and is *contradictory* if $\mathscr{A}$ is false in every interpretation of $\mathscr{L}$.

These terms are the analogues of tautology and contradiction in L. However, there are more logically valid *wf*s in $\mathscr{L}$ than there are tautologies in $\mathscr{L}$ in the sense that all tautologies in $\mathscr{L}$ are logically valid (Proposition 3.31), but there are some logically valid *wf*s that are not tautologies, i.e., their logical validity comes not from their form involving but $\sim$ and $\to$ but rather from the relationship between quantifiers and terms. The goal of the next chapter is to find all of these logically valid *wf*s.

Solutions to exercises

14. (a) The corresponding statement in N is $x_1 + x_1 = x_2 \times x_3$. Any valuation v with $v(x_1) = v(x_2) = v(x_3) = 0$ will satisfy the *wf*, and any valuation v with $v(x_1) = v(x_2) = v(x_3) = 1$ will not satisfy the *wf*.
    (b) The corresponding statement in N is $x_1 + 0 = x_2 \to x_1 + x_2 = x_3$. Any valuation v with $v(x_1) = v(x_2) = v(x_3) = 0$ will satisfy the *wf*, and any valuation v with $v(x_1) = v(x_2) = v(x_3) = 1$ will not satisfy the *wf*.
    (c) The corresponding statement in N is $\sim (x_1x_2 = x_2x_3)$. Any valuation v with $v(x_1) = 0, v(x_2) = v(x_3) = 1$ will satisfy the *wf*, and any valuation v with $v(x_1) = v(x_2) = v(x_3) = 1$ will not satisfy the *wf*.
    (d) The corresponding statement in N is $(\forall x_1)x_1x_2 = x_3$. Any valuation v with $v(x_2) = v(x_3) = 0$ will satisfy the *wf*, and any valuation v with $v(x_2) = v(x_3) = 1$ will not satisfy the *wf*.
    (e) The corresponding statement in N is $((\forall x_1)x_1 \times 0 = x_1) \to x_1 = x_2$. Since $(\forall x_1)x_1 \times 0 = x_1$ is false in N, any valuation will vacuously satisfy the *wf*, and so no valuation will not satisfy the *wf*.

15. (a) The corresponding statement is $x_1 < 0$. Any valuation v with $v(x_1) = -1$ will satisfy the *wf*, and any valuation v with $v(x_1) = 1$ will not satisfy the *wf*.
    (b) The corresponding statement is $x_1 - x_2 < x_1 \to 0 < x_1 - x_2$. Any valuation v with $v(x_1) = v(x_2) = 0$ will satisfy the *wf*, and any valuation v with $v(x_1) = v(x_2) = 1$ will not satisfy the *wf*.

(c) The corresponding statement is $\sim (x_1 < x_1 - (x_1 - x_2))$. Any valuation v with $v(x_1) = v(x_2) = 0$ will satisfy the *wf*, and any valuation v with $v(x_1) = 0, v(x_2) = 1$ will not satisfy the *wf*.

(d) The corresponding statement is $(\forall x_1)x_1 - x_2 < x_3$, which is false, so no valuation will satisfy the *wf*, and any valuation will not satisfy the *wf*.

(e) The corresponding statement is $((\forall x_1)x_1 - 0 < x_1) \to x_1 < x_2$. Since $((\forall x_1)x_1 - 0 < x_1)$ is false, any valuation will vacuously satisfy the *wf*, and no valuation will not satisfy the *wf*.

16. Only the *wf*s (b), (c), and (d) are true in the interpretation.

17. Only the *wf*s (c) and (d) are true in the interpretation.

18. We are to prove that in an interpretation I, a *wf* $(\mathscr{A} \to \mathscr{B})$ is false if and only if $\mathscr{A}$ is true and $\mathscr{B}$ is false. Let v be a valuation in I. The following are all equivalent statements.

(a) $(\mathscr{A} \to \mathscr{B})$ is false

(b) v does not satisfy $(\mathscr{A} \to \mathscr{B})$, by (a)

(c) v does not satisfy $(\sim \mathscr{A})$ and v does not satisfy $\mathscr{B}$, by Definition 3.20 (iii)

(d) v satisfies $\mathscr{A}$ and v does not satisfy $\mathscr{B}$, by Definition 3.20 (ii)

(e) $\mathscr{A}$ is true and $\mathscr{B}$ is false, by Definition 3.24

Note that (e) is true since v is an arbitrary valuation. The equivalence between (a) and (e) is what we set out to prove.

19. In the following lemma and sub-exercises, let I be any interpretation and let v be any valuation in I.

Lemma. Let $\mathscr{A}$ and $\mathscr{B}$ be *wf*s of $\mathscr{L}$. Let $\star$ be the implication "if v satisfies $\mathscr{A}$, then v satisfies $\mathscr{B}$". If $\star$ is true, then $\mathscr{A} \to \mathscr{B}$ is logically valid.

Proof. Suppose that the implication $\star$ is true. Then there are two cases to consider.

(i) The valuation v satisfies $\mathscr{A}$. By $\star$, v satisfies $\mathscr{B}$, and therefore by Definition 3.20 (iii), v satisfies $\mathscr{A} \to \mathscr{B}$.

(ii) The valuation v does not satisfy $\mathscr{A}$. By Definition 3.20 (ii), v satisfies $\sim \mathscr{A}$. By Definition 3.20 (iii), v satisfies $\mathscr{A} \to \mathscr{B}$.

We have proved that v, an arbitrary valuation in an arbitrary interpretation, always satisfies $\mathscr{A} \to \mathscr{B}$. Therefore, $\mathscr{A} \to \mathscr{B}$ is logically valid.

Note. This lemma can be more succinctly stated as "if $\mathscr{A}$ being true in any I implies that $\mathscr{B}$ is true in any I, then $\mathscr{A} \to \mathscr{B}$ is logically valid." Also notice that the relationship is an "if and only if", and a little more work could be done to prove the other direction.

This lemma confirms that the logical validity of a *wf* of the form $\mathscr{A} \to \mathscr{B}$ can be proved in the expected way. Using this lemma, we now prove that the *wf*s in (a), (b), (c) and (d) are logically valid.

(a) Suppose that v satisfies $(\exists x_1)(\forall x_2)A_1^2(x_1, x_2)$. By Proposition 3.29, there is a valuation v' which is 1-equivalent to v which satisfies $(\forall x_2)A_1^2(x_1, x_2)$. By Definition 3.20 (iv), any 2-equivalent valuation to v' satisfies $A_1^2(x_1, x_2)$ (*).

Now, let w be 2-equivalent to v. The goal is to show the existence of a valuation which is 1-equivalent to w and satisfies $A_1^2(x_1, x_2)$. So let w' be the valuation which is 2-equivalent to v' with $w'(x_2) = w(x_2)$. By (*), w' satisfies $A_1^2(x_1, x_2)$. Now let x be any element in the domain of w' which is not x_1. There are two cases to consider.

i. It may be that $x = x_2$, in which case, by construction of w', $w'(x) = w(x)$.
ii. Otherwise, $x \neq x_2$. Since w' is 2-equivalent to v', $w'(x) = v'(x)$. Since v' is 1-equivalent to v, and since x is assumed not to be x_1, we have $v'(x) = v(x)$. Since w is 2-equivalent to v, we have $v(x) = w(x)$. Finally, by the chaining the equalities, $w'(x) = v'(x) = v(x) = w(x)$.

Therefore, w' is 1-equivalent to w and satisfies $A_1^2(x_1, x_2)$, and by Proposition 3.29, $(\forall x_2)(\exists x_1)A_1^2(x_1, x_2)$. By the lemma above, we may conclude that $((\exists x_1)(\forall x_2)A_1^2(x_1, x_2) \to (\forall x_2)(\exists x_1)A_1^2(x_1, x_2))$.

Note. In the following sub-exercises for the sake of brevity, the lemma, Definition 3.20, and Proposition 3.29 will not be explicitly referenced when they are used.

(b) We will first demonstrate that if $(\forall x_1)A_1^1(x_1)$ is true in I, then $(\forall x_2)A_1^1(x_2)$ is true in I.

Suppose that v satisfies $(\forall x_1)A_1^1(x_1)$. Then any 1-equivalent valuation to v satisfies $A_1^1(x_1)$. Let v_2 be a 2-equivalent valuation to v. Let v_1 be the 1-equivalent valuation with $v_1(x_1) = v_2(x_2)$, which must necessarily satisfy $A_1^1(x_1)$, which is a *wf* in which x_2 is free for x_1. By Proposition 3.23, v_2 satisfies $A_1^1(x_2)$ if and only if v_1 satisfies $A_1^1(x_1)$. Since v_1 does indeed satisfy $A_1^1(x_1)$, v_2 must satisfy $A_1^1(x_2)$. Therefore, v satisfies $(\forall x_2)A_1^1(x_2)$, as desired.

Now, we will prove that the original *wf* is logically valid. Suppose that v satisfies $(\forall x_1)A_1^1(x_1)$. By the above, we know that v satisfies $(\forall x_2)A_1^1(x_2)$, and therefore it satisfies $((\forall x_1)A_2^1(x_1) \to (\forall x_2)A_1^1(x_2))$, and so we may conclude that $(\forall x_1)A_1^1(x_1) \to ((\forall x_1)A_2^1(x_1) \to (\forall x_2)A_1^1(x_2))$ is logically valid.

(c) Suppose that v satisfies $(\forall x_1)(\mathscr{A} \to \mathscr{B})$. There are two cases to consider.

i. It may be that v satisfies $\sim (\forall x_i)\mathscr{A}$. Then v satisfies $(\forall x_i)\mathscr{A} \to (\forall x_i)\mathscr{B}$.
ii. Otherwise, v satisfies $(\forall x_i)\mathscr{A}$. Now let v' be 1-equivalent to v. It must satisfy $\mathscr{A}$, and therefore it cannot satisfy $\sim \mathscr{A}$. But since v also satisfies $(\forall x_1)(\mathscr{A} \to \mathscr{B})$, v' must satisfy $\mathscr{B}$ if it does not satisfy $\sim \mathscr{A}$. Therefore, v must satisfy $(\forall x_i)\mathscr{B}$, and so v must satisfy $(\forall x_i)\mathscr{A} \to (\forall x_i)\mathscr{B}$.

In both cases, $(\forall x_i)\mathscr{A} \to (\forall x_i)\mathscr{B}$, and so we may conclude that

$$(\forall x_1)(\mathscr{A} \to \mathscr{B}) \to ((\forall x_1)\mathscr{A} \to (\forall x_1)\mathscr{B}).$$

Note. This proof relies on applying Definition 3.20 (iii) repeatedly.

(d) Suppose that v satisfies $(\forall x_1)(\forall x_2)\mathscr{A}$. Then

(i) any valuation v' which is 1-equivalent to v satisfies $(\forall x_2)\mathscr{A}$ and...

(ii) any valuation which is 2-equivalent to v' satisfies $\mathscr{A}$.

Now let w be a valuation which is 2-equivalent to v and let w' be a valuation that is 1-equivalent to w. The goal is to show that w' satisfies $\mathscr{A}$, from which we may deduce that w satisfies $(\forall x_1)\mathscr{A}$ and hence v satisfies $(\forall x_2)(\forall x_1)\mathscr{A}$.

Let v' be a valuation which is 1-equivalent to v with $v'(x_1) = w'(x_1)$. Then by (i), v' satisfies $(\forall x_2)\mathscr{A}$. Now let x be any element in the domain of w' that is not x_2. There are two cases to consider.

(1) It may be that $x = x_1$, in which case, by construction of v', $v'(x) = w'(x)$.

(2) Otherwise, $x \neq x_1$, so

$$\begin{aligned} v'(x) &= v(x), && \text{since } v' \text{ is 1-equivalent to } v \\ v(x) &= w(x), && \text{since } w \text{ is 2-equivalent to } w \\ w(x) &= w'(x), && \text{since } w' \text{ is 1-equivalent to } w \end{aligned}$$

and thus, $v'(x) = w'(x)$ in this case as well.

In both cases, we can see that $v'(x) = w'(x)$, and therefore w' is 2-equivalent to v'. By (ii), we may conclude that w' satisfies $\mathscr{A}$, as desired.

20. One example is $A_1^1(x_1) \to A_1^1(x_1)$. It is not closed, but it is a tautology since it is a substitution instance of $p_1 \to p_1$. Therefore, it is logically valid.

21. Suppose that v is a valuation in an interpretation I that satisfies $\mathscr{A}(t)$. Let v' be i-equivalent to v with $v'(x_i) = v(t)$. By Proposition 3.23, v' must satisfy $\mathscr{A}(x_i)$. By Proposition 3.29, v must satisfy $(\exists x_i)\mathscr{A}(x_i)$. By the lemma in Exercise 19, $\mathscr{A}(t) \to (\exists x_i)\mathscr{A}(x_i)$ is logically valid.

22. Let I be the interpretation with the integers as the domain, $\overline{a}_0 = 0$, the relation $\leq$ as A_1^2, and the relation $= 0$ as A_1^1. Note that A_1^1 does not involve $\overline{a}_0 = 0$, since the relation is just the set $\{(0,0)\}$. Then the given *wf*s (a) - (d) correspond to the following *wf*s.

(a) $((\forall x_1)(\exists x_2)\ x_1 \leq x_2) \to ((\exists x_2)(\forall x_1)\ x_1 \leq x_2)$

(b) $(\forall x_1)(\forall x_2)(x_1 \leq x_2 \to x_2 \leq x_1)$

(c) $(\forall x_1)(\sim (x_1 = 0)) \to (\sim (x_1 = 0))$

(d) $(\forall x_1)(x_1 \leq x_1) \to ((\exists x_2)(\forall x_1)\ x_1 \leq x_2)$

These statements in I are all easily seen to be false, and therefore, none of the *wf*s are logically valid.

23. This follows immediately from Proposition 3.23, since if v satisfies $v(x_i) = v(t)$, then it itself is an i-equivalent valuation v' to v with $v'(x_i) = v(t)$.

Note. In the textbook, Proposition 3.23 is not proved fully, and the remainder is left as an exercise, but in this manual it is proved fully, so there is no need to elaborate more on it here.

Chapter 4

Formal predicate calculus

Chapter 5

Mathematical systems

Chapter 6

The Gödel Incompleteness Theorem

Chapter 7

Computability, unsolvability, undecidability

Appendix A

Additional propositions

This chapter contains proofs of exercises left to the reader in the textbook or otherwise useful propositions.

Proposition A.1. The set of *wf*s of L is countable.

Appendix B

Additional exercises