The Validity of Paradoxes of Implication Should be Preserved in Logic

Introduction

Paradoxes of implication refer to the valid sequents whose conclusions are irrelevant to premises [1]. Informally, these sequents are valid in classical logic but may not be valid intuitively. For example, consider the sequent $\bot \vdash p$, which is valid by the false constant ($\bot \vdash$) in sequent calculus. However, if the sequent becomes an English sentence like "If the sun is the moon, I am a good student.", it may seem strange. This is because whether "I" am a good student does not depend on the truth that the sun is the moon. If the sun indeed is the moon, it does not necessarily mean "I" am a good student. The rise of paradoxes of implication is because an implication is true so long as its antecedent is false or its consequent is true [2], and there is no requirement that the antecedent and the consequent must be related.

Response

Since the emergence of the implicit paradox, there have been two opposing responses. One tries to amend classical logic so that it can reject paradoxes of implication, while the other tends to accept and embrace them. In my opinion, the paradoxes of implication should remain valid in logic because they can reflect on some conversations in real life. People may sometimes want to say a sentence containing paradoxes of implication for some specific purposes. If all paradoxes of implication in logic are rejected, there will have no tools to classify and formalise these sentences since all paradoxes of implication are simply regarded as invalid.

For example, Mr Zhang was innocently charged with a crime. In court, he told the judge: "I swear I did not commit a crime. If I did, the sky would fall down." Let p be "I commit a crime" and q be "the sky falls down". The sequent of the sentence is $\neg p \vdash p \rightarrow q$, which is a paradox of implication [1]. In fact, Zhang can instantiate q with anything. However, the main point is that Zhang wants to use the paradox of implication to emphasise his words. Saying $\neg p \vdash p \rightarrow q$ seems more plausible than just saying $\neg p$, that is, "I did not commit a crime". In this case, paradoxes of implication may save Zhang. However, if we reject all paradoxes of implication and simply treat $\neg p \vdash p \rightarrow q$ as an invalid sentence, it does not make sense. This is because the court should not also reject Zhang's statement only because he says $\neg p \vdash p \rightarrow q$ instead of $\neg p$. Both $\neg p \vdash p \rightarrow q$ and $\neg p$ should be valid enough to convey the fact that he did not commit a crime.

There are also other cases like Mr Zhang in daily life where people add additional words with nonsense in logic to emphasise their main point in a sentence. As the rules of formal systems are all motivated by natural languages [3], I think the formal systems can be more adhesive to natural language. Therefore, these sentences with paradoxes from people's daily conversations should be adaptable by logic instead of rejected.

Evaluation

After accepting the validity of paradoxes of implication in logic, some thinkers may still find these paradoxes unacceptable [3]. Indeed, while some of the paradoxes of implication make

sentences better, others may be misleading or intuitively invalid. However, the phrase "intuitively invalid" here should be rephrased as "inappropriate" since it is still logically valid. As mentioned in the Logic Notes, while keeping these paradoxes valid, it is also necessary to distinguish between truth and appropriateness [3].

In my opinion, all sequent of paradoxes of implication may be classified into two main forms, those containing $\bot \vdash p$, such as $\neg p \vdash p \rightarrow q$ and those containing $p \vdash T$, such as $p \vdash q \rightarrow p$. The appropriateness depends on how we instantiate the variables such as p and q, not the form itself.

As for $p \vdash T$, consider the paradoxes sequent $p \vdash q \rightarrow p$. Let p be "I am a rich person" and let q be "the end of the world comes". The sequent becomes: "I am a rich person. Therefore, if the end of the world comes, I am a rich person.". The use of this paradox conveys that "I" am always a rich man, whether the end of the world comes or not. Although the words after "whether" are meaningless in propositional logic, it has the effect of emphasis in linguistics, which is a merit of using paradox sequent. However, other sentences with the same form can also be misleading. For example, if changing q into "I am poor", the sentence becomes "I am a rich person. Therefore, if I am poor, I am a rich person." In this case, some readers might be confused about whether the person is rich. This is because the additional $q \rightarrow p$ becomes distracting instead of helping readers understand. Therefore, using paradoxes with $p \rightarrow T$ may have both merit and demerit.

As for $\bot \vdash p$, consider the English sentence "If the sun is the moon, I am a good student." as mentioned in the introduction. In classical logic, since the antecedent is always false, the consequent is simply ignored. Therefore, it is valid. However, it may not be appropriate to use the sentence. This is because what people see is not just a false statement implying anything, but actual words like "good student" and "the sun is the moon". Some people may capture the information "I am a good student" before identifying that "the sun is the moon" is false, or even ignore the $\bot$ part and wrongly treat $\bot \vdash p$ as p. This is why $\bot \vdash p$ can be misleading. Also, whenever seeing a conditional sentence, people may often tend to consider both sides, that is, think both the case where the sun is the moon and the sun is not the moon, to validate it. Even though the premise is indeed false, it does not stop some people from imagining. However, the paradox emerges when people start imagining a false antecedent to become true. Suppose people read the sentence in a classical logic way, which completely ignores the "I am a good student part" when they know the antecedent is false. In that case, the sentence may not be paradoxical anymore. However, the majority of people would probably not read like that. Therefore, some sentences containing $\bot \vdash p$ can be misleading and hence inappropriate.

In computer science, the code with $\bot \vdash p$ may be considered redundant and should be removed. Assume there is an "If" statement which checks the truth of the Boolean constant, "the sun is the moon", and prints out "I am a good student" whenever the constant is true (See Figure 1). Obviously, the "print" function will never run because "If" can never be satisfied. Although this part of code is valid in both logic and programming language (it does not crash the program), it occupies memory space.

```java
boolean theSunIsTheMoon = false;

if (theSunIsTheMoon){
    System.out.println("I am a good student.");
}
```

Figure 1: the code with the form $\perp \vdash p$ in Java.

However, consider $\neg p \vdash p \rightarrow q$ in which the right side also contains the form $\perp \rightarrow p$. Let p be "I save money", and q be "I am rich". The sentence becomes, "I didn't save money. If I'd saved money, I would have been rich". The only fact is that "I didn't save money", and the $p \rightarrow q$ part tells nothing in logic since it is just the person's imagination. q can, in fact, be anything and does not logically depend on p since p is false. Though it is a paradox containing $\perp \vdash p$, it conveys additional meaning, the person's regret, in everyday language compared to just $\neg p$. In this case, the use of paradoxes of implication, $\perp \vdash p$ has merit.

The observations above show that whether a paradox is appropriate depends on how we instantiate the variables such as p and q, not the form itself. The sequent of the form like $\perp \rightarrow p$ can be both inappropriate and appropriate. Therefore, it is necessary for people to use them carefully.

<u>Conclusion</u>

In short, the validity of paradoxes of implication should be preserved in logic. In every conversation, it is possible for people to make a valid sentence that contains paradoxes of implication, like in the Mr Zhang case. If paradoxes of implication are removed from logic, there would be no tools to adapt these valid sentences using formal language correctly. This is because the revised logic simply treats all these paradoxes as invalid. While accepting the validity of paradoxes of implication, the appropriateness of the paradoxes should still be carefully considered since the use of some paradoxes has merits, but others have demerits. The merits can be a sentence become more persuasive, while the demerits can be a sentence is misleading. What makes people think paradoxes of implication are paradoxical are probably those inappropriate and misleading paradoxes of implication, instead of all paradoxes of implication, since some appropriate paradoxes are beneficial.

Reference

[1] Maruyama, Y 2021, Introduction To Non-Classical Logic (Lecture Slides), viewed 29 May 2022, <https://wattlecourses.anu.edu.au/pluginfile.php/3017988/mod_resource/content/7/Logic_S1_Slides.zip>.

[2] En-Academic, 2020, Paradoxes of material implication, viewed 29 May 2022, <https://en-academic.com/dic.nsf/enwiki/7080848>

[3] Slaney, J 2021, Paradoxes of Implication (Logic Notes), viewed 29 May 2022, <http://users.cecs.anu.edu.au/~jks/LogicNotes/paradoxes.html>