Data Structures
Chapter 5 Tree

1. Introduction
2. Binary Tree
3. Binary Search Tree
4. **Balancing Tree**
   - AVL Tree
   - **Coding**

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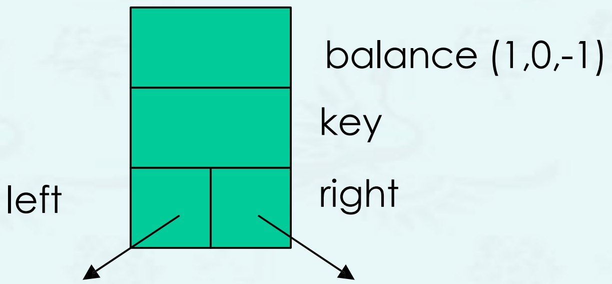
절관주

모든 성경은 하나님의 감동으로 된 것으로 교훈과 책망과 바르게 함과 의로 교육하기에 유익하니
이는 하나님의 사람으로 온전하게 하며 모든 선한 일을 행할 능력을 갖추게 하려 함이라 (딤후3:16-17)

우리는 그가 만드신 바라 그리스도 예수 안에서 선한 일을 위하여 지으심을 받은 자니 이 일은
하나님이 전에 예비하사 우리로 그 가운데서 행하게 하려 하심이니라 (엡2:10)

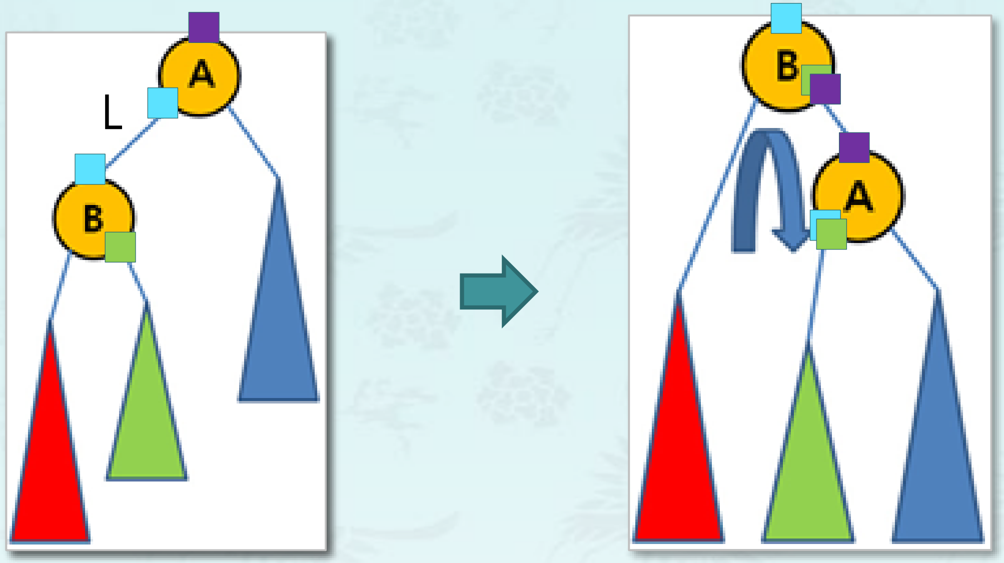
Coding

- You can either keep the height or just the difference in height, i.e. the balance factor; this has to be modified on the path of insertion even if you don't perform rotations.
  - Once you have performed a rotation (single or double) you won't need to go back up the tree for the computation.
- You may compute the balance factor **on the fly** after the insert is done during the recursion.



Single Rotation - LL case

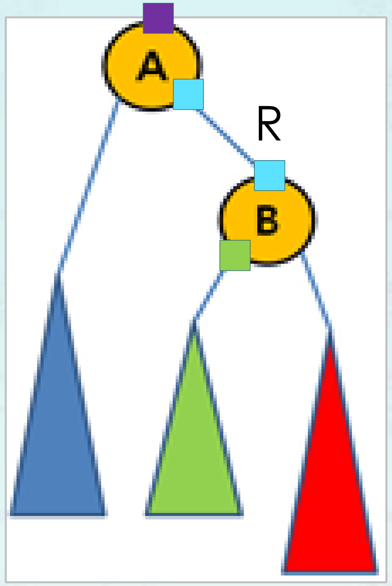
outside case

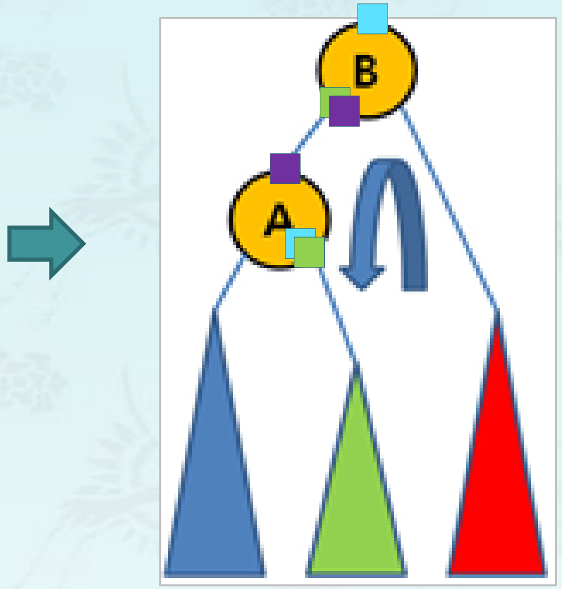



```
tree rotateLL(tree A)
{
  tree B    = A->left;
  A->left   = B->right;
  B->right = A;
  return B;
}
```

Single Rotation – RR case

outside case

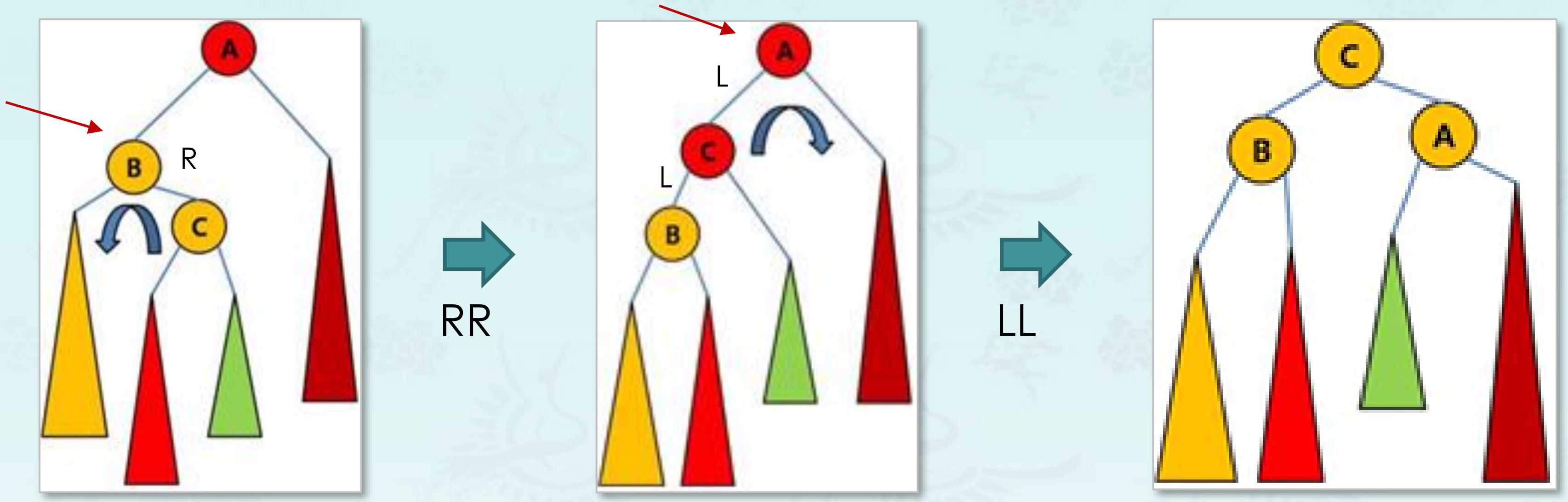





```
tree rotateRR(tree A)
{
  tree B    = A->right;
  A->right = B->left;
  B->left  = A;
  return B;
}
```

Double Rotation - LR case

inside case

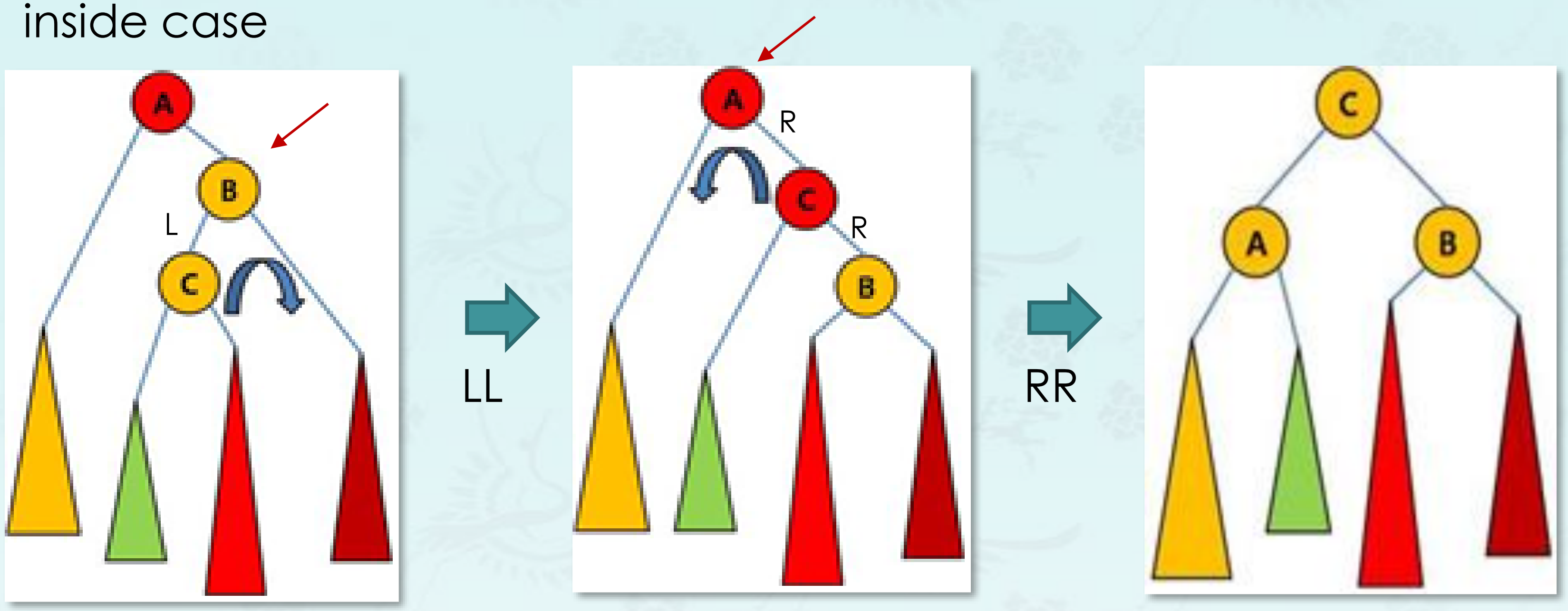



```
tree rotateLR(tree A) // RR and LL
{
   tree B  = A->left;
   A->left = rotateRR(B);
   return rotateLL(A);
}
```

What will return eventually?

Double Rotation - RL case

inside case

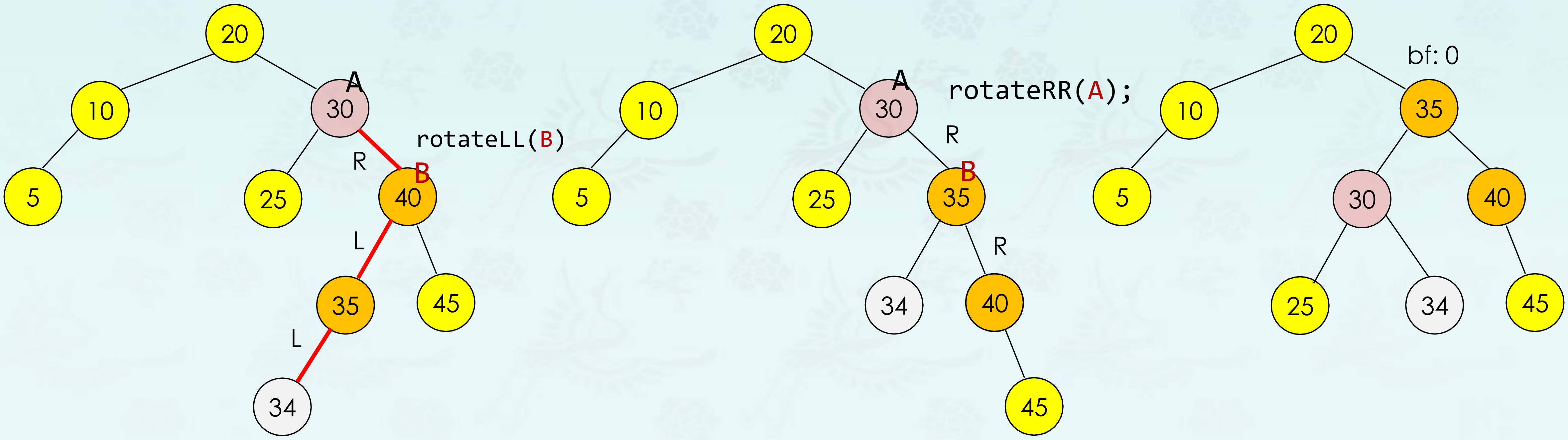



```
tree rotateRL(tree A) { // LL and RR
{
  tree B  = A->right;
  A->right = rotateLL(B);
  return rotateRR(A);
}
```

Double Rotation - RL case

- Insertion of 34
- Imbalance at 30
- Balance factor -2
- Rotation ___RL___ case

```
tree rotateRL(tree A) {
  tree B  = A->right;
  A->right = rotateLL(B);
  return rotateRR(A);
}
```



After insertion, AVL imbalanced tree

After insertion, AVL balanced tree

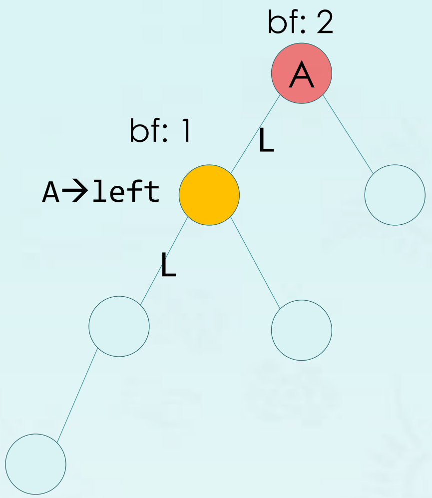
Balance Factor and Height

```
int height(tree node) {
  if (empty(node)) return -1;
  int left = height(node->left);
  int right = height(node->right);
  return max(left, right) + 1;
}
```

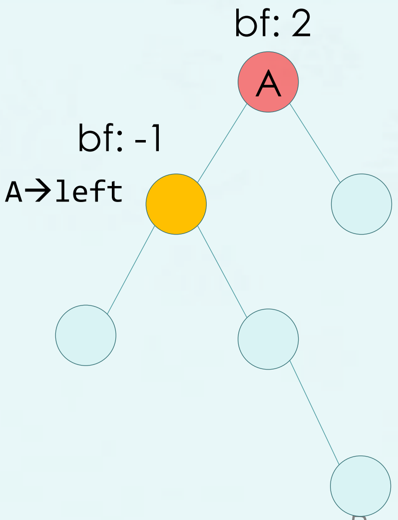
```
int balanceFactor(tree node) {
  if (node == NULL) return 0;
  int left  = height(node->left);
  int right = height(node->right);
  return left – right;
}
```

Rebalance

outside case



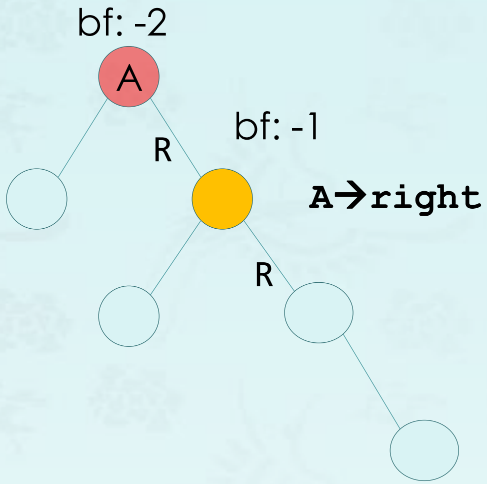
inside case



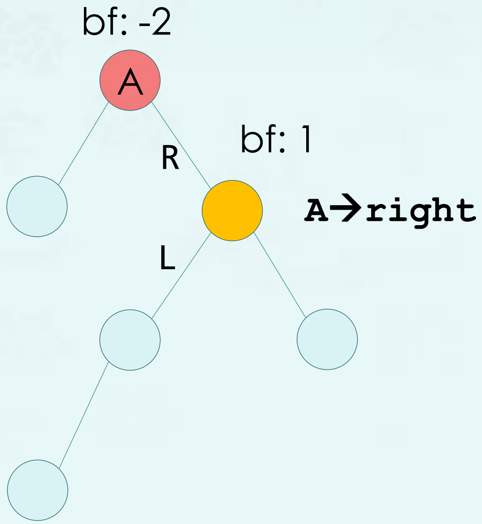

```
tree rebalance(tree A) {
  int bf = balanceFactor(A);
  if (bf == 2) {
    if (balanceFactor(A->left) == 1)



  }
  else if (bf == -2) {
    if (balanceFactor(A->right) == -1)



  }
  return A;  // no rebalanced needed
}
```

Observation: If A and its child have the same sign in bf's, a single rotation is needed, a double rotation otherwise.

outside case



inside case



growAVL() & trimAVL()

```
// inserts a key into the AVL tree and rebalance it.
tree growAVL(tree node, int key) {
  if (node == nullptr) return new TreeNode(key);

  // your code here

  return rebalance(node);      // O(log n)
}
```

almost same as grow()

AVL rotation if necessary

```
// deletes a key into the AVL tree and rebalance it.
tree trimAVL(tree node, int key) {
  if (node == nullptr) return new TreeNode(key);

  // your code here

  return rebalance(node);      // O(log n)
}
```

almost same as trim()

AVL rotation if necessary

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Height of an AVL Tree

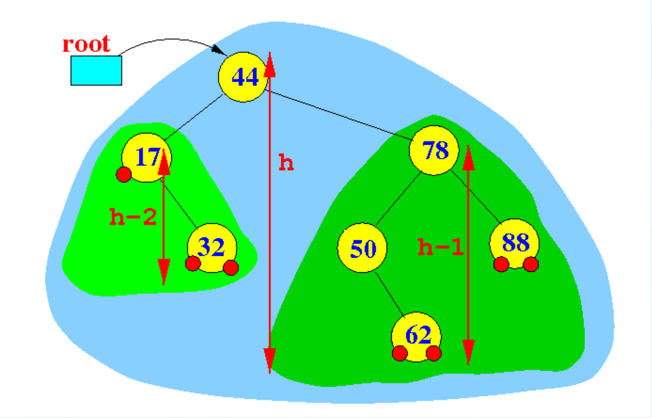
- What is the maximum height of an AVL tree having exactly n nodes?
  - To answer this question, we must ask this question first:
    What is the minimum number of nodes (sparsest possible AVL tree) an AVL tree of height h?

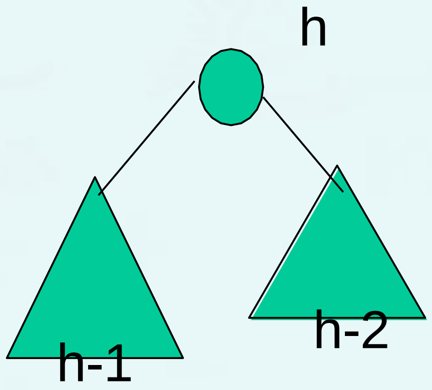
- Consider **the minimum number of nodes** in an AVL tree of height h:
- We can get the recurrence relationship:

$$n(0) = 1$$
$$n(1) = 2$$
$$n(2) = 4$$
$$n(h) = n(h-1) + n(h-2) + 1$$

where $h > 1$

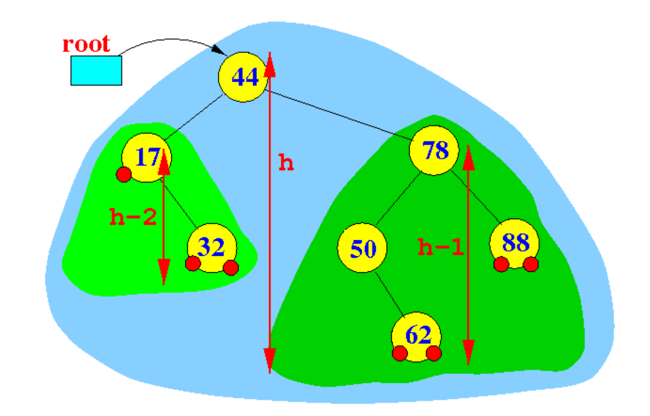
- This approximate solution of the recurrence is known as $\mathrm{n(h)} \cong 1.618^{h}$

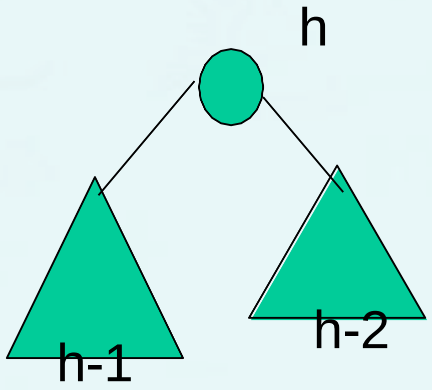




Height of an AVL Tree

- $n(h) = n(h-1) + n(h-2) + 1$
  where $h > 1$
- This approximate solution of the recurrence is known as $\mathrm{n(h)} \cong 1.618^{h}$
- Solve the equation above for h to get **the max height of an AVL tree** with n nodes?
  $\log_2 n \geq \mathrm{h} * \log_2 1.62$
  $\mathrm{h} \leq 1/\log_2 1.618 * \log_2 n$
  $\mathrm{h} \leq 1.44 * \log_2 n$





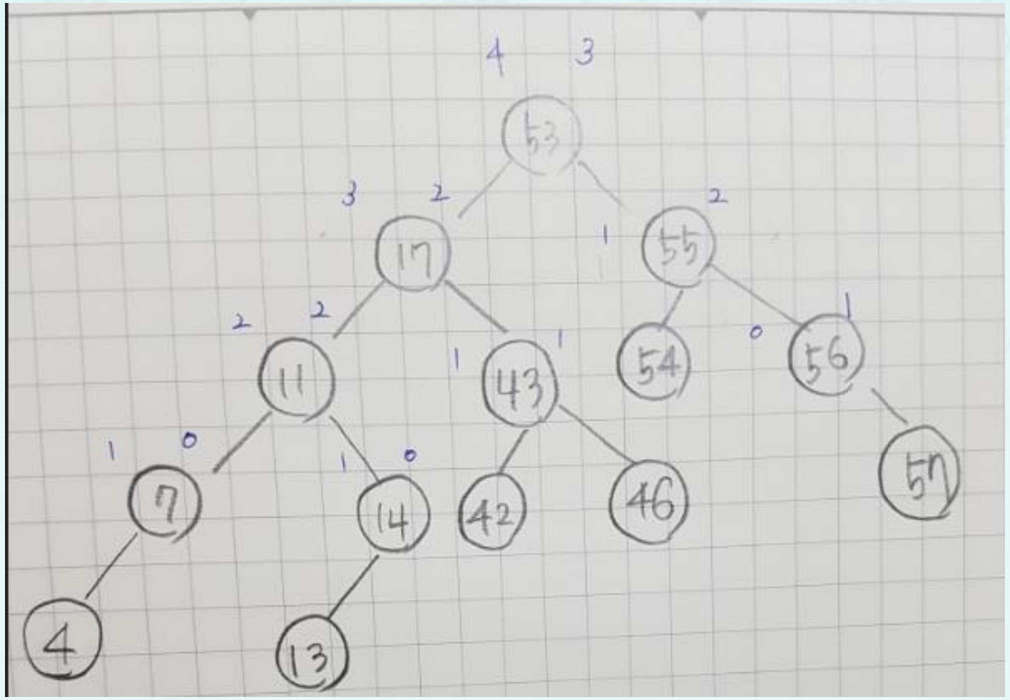
Height of an AVL Tree

- AVL trees are binary search trees that balances itself every time an element is **inserted or deleted**. **Each node** of an AVL tree has the property that the heights of the sub-tree rooted at its children differ **by at most one**.

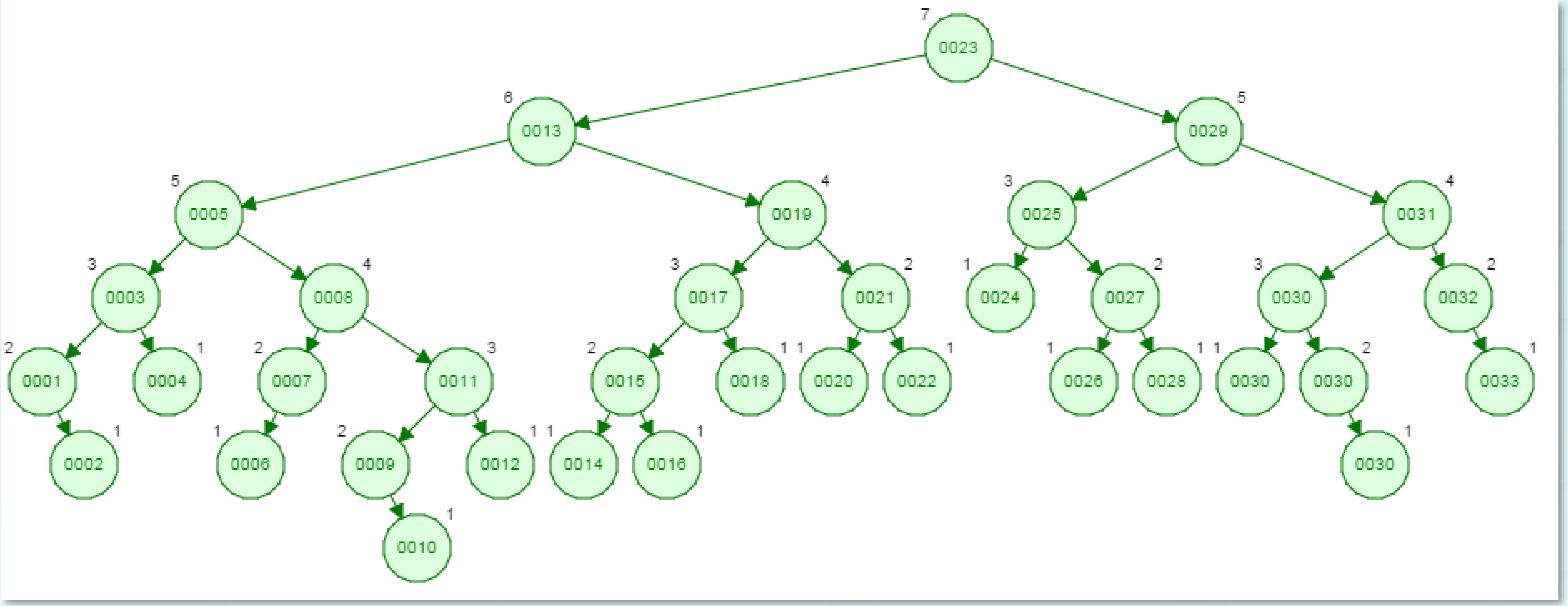
- If there are n nodes in AVL tree, minimum height of AVL tree is floor($\log_2 n$).
- If there are n nodes in AVL tree, maximum height can't exceed $1.44 * \log_2 n$.
- If height of AVL tree is h, maximum number of nodes can be $2^{h+1} - 1$.
- Minimum number of nodes in a tree with height h can be represented as:
  N(h) = N(h-1) + N(h-2) + 1, where N(0) = 1 and N(1) = 2.
- The complexity of searching, inserting and deletion in AVL tree is O($\log_2 n$).
- The cost of balancing AVL tree is O(1).
  What is the time complexity of adding N elements to an empty AVL tree?
  Time complexity: log(1) + log(2) + …. + log(n) <= log(n) + log(n) + … + log(n) = n log (n)

이것도 AVL tree가 될 수 있을까요?

- 저는 AVL tree가 모든 노드의 왼쪽과 오른쪽의 height의 차이가 절대값 1을 넘어서지 않는 것이라고 알고있습니다. 근데 이 트리는 모든 노드에서 왼쪽과 오른쪽의 height의 차이가 절대값 1을 넘지는 않지만 55-54가 연결되어 있는 부분의 높이가 다른 쪽에 비해 2이상 차이나는 것을 보았습니다. 제가 아는 정의상으로는 AVL tree인거 같으면서도 저렇게 height가 2이상 차이가 나니... 결론을 내릴 수가 없어 질문 드립니다.
- 제가 AVL tree의 정의를 잘못 알고 있는건가요?



Example with leaf 24 on level 3 and leaf 10 on level 6:



- AVL maintain the maximum height difference of 1 between two children subtree, not any two leaves.
- The difference in levels of any two leaves can be any value!
  The definition of AVL describes height difference only on two sub-trees from one node.

https://stackoverflow.com/questions/28964971/height-difference-between-leaves-in-an-avl-tree

growAVL() & trimAVL()

```
// inserts a key into the AVL tree and rebalance it.
tree growAVL(tree node, int key) {
  if (node == nullptr) return new TreeNode(key);

  // your code here

  return rebalance(node);      // O(log n)
}
```

almost same as grow()

AVL rotation if necessary

```
// deletes a key into the AVL tree and rebalance it.
tree trimAVL(tree node, int key) {
  if (node == nullptr) return new TreeNode(key);

  // your code here

  return rebalance(node);      // O(log n)
}
```

almost same as trim()

AVL rotation if necessary

growN() & TrimN()

```
// removes randomly N numbers of nodes in the tree(AVL or BST).
// It gets N node keys from the tree, trim one by one randomly.
tree trimN(tree root, int N, bool AVLtree) {  // testing purpose
  vector<int> vec;

  // your code here

  delete[] arr;
  return root;
}
```

growN() & TrimN()

```
tree growN(tree root, int N, bool AVLtree) {     // coding a faster version
  int start = empty(root) ? 0 : value(maximum(root)) + 1;
  int* arr = new (nothrow) int[N];
  assert(arr != nullptr);
  randomN(arr, N, start);

#if 0  // use BST grow() first. then, if AVLtree, reconstruct it as AVL.
  for (int i = 0; i < N; i++) root = grow(root, arr[i]);
  if (AVLtree) root = reconstruct(root);
#else // use its own grow() function, respectively. it is too slow
  if (AVLtree)
    for (int i = 0; i < N; i++) root = growAVL(root, arr[i]);
  else
    for (int i = 0; i < N; i++) root = grow(root, arr[i]);
#endif

  delete[] arr;
  return root;
}
```

Reconstruct() – Building AVL tree from BST in O(n)

- **Goal:** Reconstruct a new AVL tree from BST in O(n).
- **Intuition:** Since we can get a sorted key values from the binary search tree, we take advantage of the sorted list to form a well balanced AVL tree faster.
- **Recreation Method**
  - Use an array of keys, using an existing inorder() function that returns a sorted keys in vector.
  - Clear the original tree since it is not used any more.
  - Since it goes through the tree twice only, the time complexity is O(n).
- **Recycling Method**
  - Use an array of nodes, simply reconstructs (or relink) the existing nodes.
  - Write a new inorder() that returns the sorted nodes of the tree.
  - Since it goes through the tree twice only, the time complexity is O(n).
- **For pedagogical purpose**, let us use the recycling method if the number of nodes are more than 10, otherwise use the recreation method.

Reconstruct() – Building AVL tree from BST in O(n)

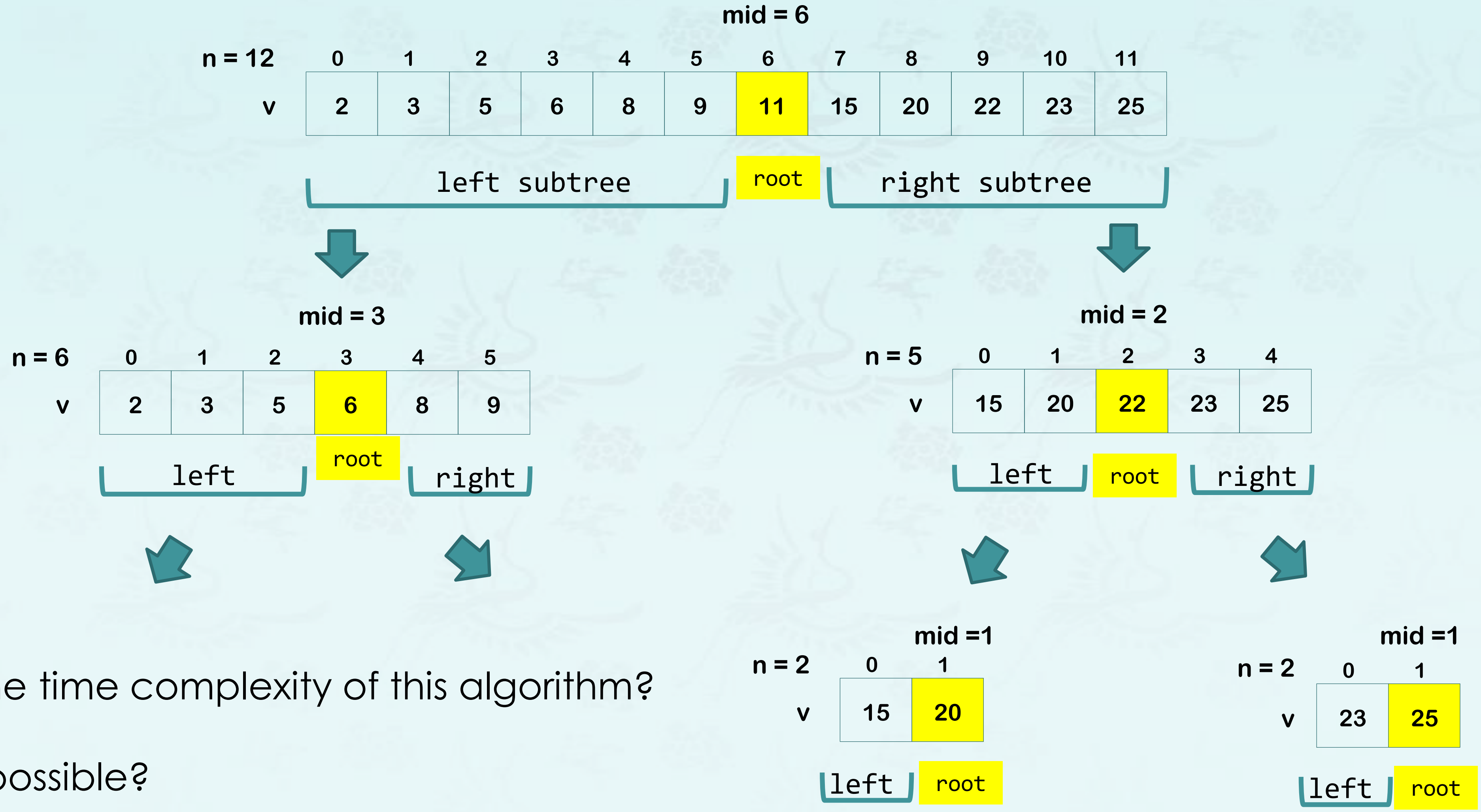
```
// reconstructs a new AVL tree from BST in O(n).
tree reconstruct(tree root) {
  if (root == nullptr) return nullptr;

  if (size(root) > 10) {           // recycling method

        cout << your code here

  }
  else {                           // recreation method

        cout << your code here

  }
  return root;
}
```

mid = 6

n = 12	0	1	2	3	4	5	6	7	8	9	10	11
v	2	3	5	6	8	9	11	15	20	22	23	25

left subtree (indices 0–5) | root (index 6) | right subtree (indices 7–11)

Reconstruct() – Building AVL tree from BST in O(n)



What is the time complexity of this algorithm?
O(n)
How is it possible?

Building AVL tree from BST in O(n) – recreation method

```
// rebuilds an AVL tree with a list of keys sorted.
// v - an array of keys sorted, n - the array size
tree buildAVL(int* v, int n) {
  if (n <= 0) return nullptr;
  int mid = n / 2;

  // create a root node

  // recursive buildAVL() calls for left & right, return it to root->left & root->right



  return root;
}
```

mid = 6

n = 12	0	1	2	3	4	5	6	7	8	9	10	11
v	2	3	5	6	8	9	11	15	20	22	23	25

left subtree (indices 0–5) | root (index 6) | right subtree (indices 7–11)

Building AVL tree from BST in O(n) – recycling method

```
// rebuilds an AVL tree using a list of nodes sorted, no memory allocations
// v - an array of nodes sorted, n - the array size
tree buildAVL(tree* v, int n) {
  if (n <= 0) return nullptr;
  int mid = n / 2;

  // v[mid] becomes the root; don't call new TreeNode.

  // set leaf nodes to null for recycling.

  // recursive buildAVL() calls for left & right, return it to root->left & root->right

  return root;
}
```

mid = 6

n = 12	0	1	2	3	4	5	6	7	8	9	10	11
v	2	3	5	6	8	9	11	15	20	22	23	25

$\underbrace{\texttt{2 3 5 6 8 9}}_{\text{left subtree}}$ root $\underbrace{\texttt{15 20 22 23 25}}_{\text{right subtree}}$

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