A Curry-Howard Correspondence for $\lambda_\wedge^{BCD}$

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Introduction:

Intersection types are an exciting development in type theory and computer science. Originally formulated by Mario Cappo and Mariangiola Dezani in the 1970's, intersection types add a new type constructor "$\wedge$" to type-constructions on top of the existing constructor "$\rightarrow$" [2]. The constructor $\wedge$ can be thought of as analogous to intersections of sets. Cappo and Dezani also showed that a λ-term is strongly-normalizable if and only if it provably has an intersection type [5]. The type system created by Cappo and Dezani is called $\Lambda_\wedge^{BCD}$, and is the type-system we will be investigating in this paper.

Intersection-types allow for a richer type-system. For example, if a term Q has multiple types $B_1, ..., B_n$, then it suffices to say Q has the type $B_1 \wedge ... \wedge B_n$ [1]. Another example is that $\Lambda_\wedge^{BCD}$ has been used to capture the notion of a filter model with respect to type-theory [7]. This in effect has led to applications in domain theory and denotational semantics [4].

Naturally, it would be useful to find a formal-logic that corresponds to $\Lambda_\wedge^{BCD}$. Having the constructors $\rightarrow$ and $\wedge$ is initially comparable to propositional minimal logic with analogous logical connectives. It is natural to wonder if there is a Curry-Howard correspondence between the $\Lambda_\wedge^{BCD}$ and proposition minimal logic. However, the two do not share a Curry-Howard correspondence. We show can this in two different ways:

1. Counterexample: Let B be a basis, i.e. a set of variable declarations $\{x_0 : \tau_0, ..., x_n : \tau_n\}$ for some n (possibly 0). We define $|B|$ as the set of all types τ where $x : \tau \in B$ for some x. Consider the case where we have $|B| \vdash A \wedge B$. We want to show there exists a term X such that $B \vdash X : A \wedge B$. However, we can't be assured that any two types A and B are guaranteed the same term, so having $B \vdash X : A$ and $B \vdash X : B$ may be impossible. As a concrete example, consider a terms $\lambda x.x : A \rightarrow A$ and $\lambda xy.x : A \rightarrow (B \rightarrow A)$. There is no such term X where $\vdash X : (A \rightarrow A) \wedge (A \rightarrow (B \rightarrow A))$.

2. Determining if X has type A is undecidable, as shown in [6]. However, determining if a proposition A is provable in propositional minimal logic is decidable [8]. Thus there is a disconnect.

We therefore propose a fix by making alterations to our systems by introducing the concept of positive and negative sub-types and then showing a correspondence between a subset of terms in minimal logic and $\lambda_\wedge^{BCD}$. The objective of this paper is to find some connection between $\lambda_\wedge^{BCD}$ and propositional-minimal-logic.

Formalizing our type-system and logic:

The following thesis will involve a type system of lambda-calculus called $\lambda_{\wedge}^{BCD}$ (without top). $\lambda_{\wedge}^{BCD}$ is a type system with the defined type-constructor $\rightarrow$ and the added constructor $\wedge$. We let Π be our set of types defined as follows:

$$\Pi ::= U | \Pi \rightarrow \Pi | \Pi \wedge \Pi$$

where U is an infinite set of type-variables. The $\wedge$ is defined in terms of a semi-lattice $(\wedge, \sqsubseteq)$. The type system has the following rules of deduction:

$$\frac{}{\mathcal{B} \vdash (x : A)}\ (Ax.) \quad (if\ (x : A) \in \mathcal{B})$$

$$\frac{\mathcal{B} \vdash X : A \rightarrow B \quad \mathcal{B} \vdash Y : A}{\mathcal{B} \vdash (XY) : B}\ (\rightarrow E) \qquad \frac{\mathcal{B} \cup \{x : A\} \vdash X : B}{\mathcal{B} \vdash \lambda x.X : A \rightarrow B}\ (\rightarrow I)^{*}$$

$$\frac{\mathcal{B} \vdash X : A \quad \mathcal{B} \vdash X : B}{\mathcal{B} \vdash X : A \wedge B}\ (\wedge I) \qquad \frac{\mathcal{B} \vdash X : A \quad A \sqsubseteq B}{\mathcal{B} \vdash X : B}\ (\sqsubseteq)$$

* Where x is not free in the domain of $\mathcal{B}$[1].
Here α-conversion is permitted to prevent unnecessary bindings that may occur during substitution.

The axioms and rules of $\wedge$, $\rightarrow$ and $\sqsubseteq$ are the following:

Axioms:
A $\sqsubseteq$ A (Identity)
A $\wedge$ B $\sqsubseteq$ A
A $\wedge$ B $\sqsubseteq$ B
(A $\rightarrow$ B) $\wedge$ (A $\rightarrow$ C) $\sqsubseteq$ A $\rightarrow$ (B $\wedge$ C) (Distributivity)
Rules:
C $\sqsubseteq$ A and C $\sqsubseteq$ B implies C $\sqsubseteq$ A $\wedge$ B (Greatest Lower Bound)
C $\sqsubseteq$ B and B $\sqsubseteq$ A implies C $\sqsubseteq$ A (Transitivity)
A $\sqsubseteq$ C and D $\sqsubseteq$ B implies C $\rightarrow$ D $\sqsubseteq$ A $\rightarrow$ B (Covariance/Contravariance)

We will sometimes refer to these rules as the ”Traditional rules” later on.

[1] If $\mathcal{B}$ is a basis $\{X_1 : A_1, ..., X_n : A_n\}$, then the *domain* of $\mathcal{B}$ is $\{x_1, ..., x_n\}$

The set of propositions Ψ for propositional minimal logic is defined as

$$\Psi ::= PV | \Psi \to \Psi | \Psi \wedge \Psi$$

where PV is an infinite set of atomic-propositions[2]. The natural deduction rules of propositional minimal logic are defined as follows:

$$\frac{A \to B \quad A}{B}\ (\to E) \qquad \frac{\begin{array}{c}[A] \\ \vdots \\ B\end{array}}{A \to B}\ (\to I)$$

$$\frac{A \quad B}{A \wedge B}\ (\wedge I) \qquad \frac{A \wedge B}{A}\ (\wedge E_1) \qquad \frac{A \wedge B}{B}\ (\wedge E_2)$$

However for this paper, we will formalize propositional minimal logic in terms of an arbitrary context Γ that is a set of propositions. Specifically, for context Γ and proposition A, the relationship $\Gamma \vdash A$ is defined as follows:

$$\frac{}{\Gamma \vdash A}\ (Ax.) \quad (if\ A\ is\ in\ context\ \Gamma)$$

$$\frac{\Gamma \vdash A \to B \quad \Gamma \vdash A}{\Gamma \vdash B}\ (\to E) \qquad \frac{\Gamma \cup \{A\} \vdash B}{\Gamma \vdash A \to B}\ (\to I)$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ (\wedge I) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash A}\ (\wedge E_1) \qquad \frac{\Gamma \vdash A \wedge B}{\Gamma \vdash B}\ (\wedge E_2)$$

In order to allow for repeated propositions in a context Γ, we can assign to each arbitrary variable $A \in \Gamma$ a different variable x_A that can be identified at $(\to I)$

[2]We use $\wedge$ and $\to$ to refer both to the symbols formalized in $\lambda_{\wedge}^{BCD}$ and propositional minimal logic for the sake of convenience. Which system is being used from now on should be clear from context.

Definition: The distributive normal form (dnf) of a type is defined as follows:

$A^{dnf} = A$ if A is an atomic type.

$(A \wedge B)^{dnf} = A^{dnf} \wedge B^{dnf}$

$(A \rightarrow B)^{dnf} = (A^{dnf} \rightarrow B_1) \wedge ... \wedge (A^{dnf} \rightarrow B_k)$ where $B^{dnf} = B_1 \wedge ... \wedge B_k$.

where each B_i is either atomic or begins with $\rightarrow$.

Lemma 1: $B \sqsubseteq A$ implies $B^{dnf} \sqsubseteq A^{dnf}$.

Proof: We do the proof by induction on the derivation of $B \sqsubseteq A$.

Base Cases:

Identity: This case is immediate, since $A^{dnf} \sqsubseteq A^{dnf}$ is immediate.

A $\wedge$ B $\sqsubseteq$ A : Since $A^{dnf} \wedge B^{dnf} \sqsubseteq A^{dnf}$, and since $A^{dnf} \wedge B^{dnf} = (A \wedge B)^{dnf}$, then we know $(A \wedge B)^{dnf} \sqsubseteq A^{dnf}$.

A $\wedge$ B $\sqsubseteq$ B : This follows with the same reasoning as the previous case.

Distributivity: Assume $B \sqsubseteq A$ where $B = (C \rightarrow D) \wedge (C \rightarrow E)$ and $A = (C \rightarrow (D \wedge E))$. We want to show $((C \rightarrow D) \wedge (C \rightarrow E))^{dnf} \sqsubseteq (C \rightarrow (D \wedge E))^{dnf}$. Let

$D^{dnf} = D_1 \wedge .... \wedge D_j$

$E^{dnf} = E_1 \wedge .... \wedge E_k$.

Note also that $(D \wedge E)^{dnf} = D^{dnf} \wedge E^{dnf} = D_1 \wedge .... \wedge D_j \wedge E_1 \wedge .... \wedge E_k$

Then $((C \rightarrow D) \wedge (C \rightarrow E))^{dnf} = (C \rightarrow D)^{dnf} \wedge (C \rightarrow E)^{dnf}$.

$= (C^{dnf} \rightarrow D_1) \wedge ... \wedge (C^{dnf} \rightarrow D_j) \wedge (C^{dnf} \rightarrow E_1) \wedge ... \wedge (C^{dnf} \rightarrow E_k)$

$= (C^{dnf} \rightarrow D_1) \wedge ... \wedge (C^{dnf} \rightarrow D_j) \wedge (C^{dnf} \rightarrow E_1) \wedge ... \wedge (C^{dnf} \rightarrow E_k)$

$\sqsubseteq (C \rightarrow (D \wedge E))^{dnf}$ (by definition of dnf)

Induction Step:

Greatest Lower Bounds: $B \sqsubseteq A$ where $A = C \wedge D$ and derived from the fact that $B \sqsubseteq C$ and $B \sqsubseteq D$. We want to show $B^{dnf} \sqsubseteq (C \wedge D)^{dnf}$. We assume as induction hypothesis that $B^{dnf} \sqsubseteq C^{dnf}$ and $B^{dnf} \sqsubseteq D^{dnf}$. Then we know that $B^{dnf} \sqsubseteq C^{dnf} \wedge D^{dnf}$. Since $C^{dnf} \wedge D^{dnf} = (C \wedge D)^{dnf}$, then we know $B^{dnf} \sqsubseteq (C \wedge D)^{dnf}$

Transitivity: $B \sqsubseteq A$ derived from the facts that $B \sqsubseteq C$ and $C \sqsubseteq A$. Then we assume as induction hypothesis that $B^{dnf} \sqsubseteq C^{dnf}$ and $C^{dnf} \sqsubseteq A^{dnf}$. Then it trivially follows that $B^{dnf} \sqsubseteq A^{dnf}$.

Covariance/Contravariance: $B \sqsubseteq A$ where $B = C \rightarrow D$ and $A = E \rightarrow F$ such that $E \sqsubseteq C$ and $D \sqsubseteq F$. We assume as induction hypothesis that $E^{dnf} \sqsubseteq C^{dnf}$ and $D^{dnf} \sqsubseteq F^{dnf}$. Then we have the following:

$D^{dnf} = D_1 \wedge .... \wedge D_i$

$F^{dnf} = F_1 \wedge .... \wedge F_k$.

Then we have $(C \rightarrow D)^{dnf} = (C^{dnf} \rightarrow D_1) \wedge .... \wedge (C^{dnf} \rightarrow D_i)$

= $(C^{dnf} \rightarrow D_1 \wedge ... \wedge D_i)$ (by the distributive property)

= $(C^{dnf} \rightarrow D^{dnf})$

$\sqsubseteq (E^{dnf} \rightarrow F^{dnf})$

$= (E^{dnf} \rightarrow F_1 \wedge .... \wedge F_k)$

$= (E^{dnf} \rightarrow F_1) \wedge .... \wedge (E^{dnf} \rightarrow F_k)$

$$= (E \to F)^{dnf}$$

This covers all cases of the inference of $B \sqsubseteq A$. Thus $B \sqsubseteq A$ implies $B^{dnf} \sqsubseteq A^{dnf}$ ⊠

Lemma 2: Let $\mathcal{B} \vdash X : A$ in $\lambda_\wedge^{BCD}$. Then we can show $\mathcal{B}^{dnf} \vdash X : A^{dnf}$ where every type is in dnf. Here $\mathcal{B}^{dnf} = \{B^{dnf} | B \in \mathcal{B}\}$.[3]

Proof: By induction on the length of the proof.

Base Case (Ax.): Immediate.

Induction Step:

($\wedge$ I): We assume $\mathcal{B} \vdash X : A \wedge B$ and want to show $\mathcal{B}^{dnf} \vdash X : (A \wedge B)^{dnf}$. Assume $\mathcal{B} \vdash X : A$ and assume as induction hypothesis that we have the computation $\mathfrak{D}_0$ that gives us $\mathcal{B}_0^{dnf} \vdash X : A^{dnf}$ and the computation $\mathfrak{D}_1$ that give us $\mathcal{B}_1^{dnf} \vdash X : B^{dnf}$. Then we have

$$\dfrac{\overset{\mathfrak{D}_0}{\mathcal{B}^{dnf} \vdash X : A^{dnf}} \quad \overset{\mathfrak{D}_1}{\mathcal{B}^{dnf} \vdash X : B^{dnf}}}{\mathcal{B}^{dnf} \vdash X : A^{dnf} \wedge B^{dnf}} \wedge I$$

Since $A^{dnf} \wedge B^{dnf}$ = $(A \wedge B)^{dnf}$, then we have $\mathcal{B}^{dnf} \vdash X : (A \wedge B)^{dnf}$.

($\to$ I): We assume $\mathcal{B} \vdash \lambda x.X : A \to B$ and want to show $\mathcal{B}^{dnf} \vdash \lambda x.X : (A \to B)^{dnf}$. Assume $\mathcal{B} \cup \{A\} \vdash X : B$ and thus we have as the induction hypothesis the computation $\mathfrak{D}$ that gives us $\mathcal{B}^{dnf} \vdash X : B^{dnf}$. Then let $B^{dnf} = B_1 \wedge ... \wedge B_n$. Note also that $B_1 \wedge ... \wedge B_n \sqsubseteq B_i$ for $i \in \{1, ..., n\}$. Also let $\Delta^{dnf} = \mathcal{B}^{dnf} \cup \{x : A^{dnf}\}$, where x is free in $\mathcal{B}^{dnf}$. We can create the following proof:

$$\dfrac{\dfrac{\dfrac{\overset{\mathfrak{D}}{\Delta^{dnf} \vdash X : B^{dnf}}}{\Delta^{dnf} \vdash X : B_1} \sqsubseteq}{\mathcal{B}^{dnf} \vdash \lambda x.X : (A^{dnf} \to B_1)} (\to I) \quad \dfrac{\dfrac{\overset{\mathfrak{D}}{\Delta^{dnf} \vdash X : B^{dnf}}}{\Delta^{dnf} \vdash X : B_2} \sqsubseteq}{\mathcal{B}^{dnf} \vdash \lambda x.X : (A^{dnf} \to B_2)} (\to I)}{\mathcal{B}^{dnf} \vdash \lambda x.X : (A^{dnf} \to B_1) \wedge (A^{dnf} \to B_2)} (\wedge I) \quad \cdots \quad \dfrac{\dfrac{\overset{\mathfrak{D}}{\Delta^{dnf} \vdash X : B^{dnf}}}{\Delta^{dnf} \vdash X : B_n} \sqsubseteq}{\mathcal{B}^{dnf} \vdash \lambda x.X : (A^{dnf} \to B_n)} (\to I)$$

$$\vdots$$

$$\mathcal{B}^{dnf} \vdash \lambda x.X : (A^{dnf} \to B_1) \wedge ... \wedge (A^{dnf} \to B_n)$$

Since by the rules of dnf $(A^{dnf} \to B_1) \wedge ... \wedge (A^{dnf} \to B_n) = (A \to B)^{dnf}$, then we are done with this case.

($\to$ E): Assume $\mathcal{B} \vdash XY : B$ and show $\mathcal{B}^{dnf} \vdash XY : B^{dnf}$, where $\mathcal{B} \vdash XY : A$ is derived from $\mathcal{B} \vdash X : (A \to B)$ and $\mathcal{B} \vdash Y : A$. Assume as induction hypothesis that we have the computations $\mathfrak{D}_0$ and $\mathfrak{D}_1$ that give us $\mathcal{B}^{dnf} \vdash X : (A \to B)^{dnf}$ and $\mathcal{B}^{dnf} \vdash Y : A^{dnf}$ respectively. Since $(A \to B)^{dnf} = (A^{dnf} \to B_1) \wedge ... \wedge (A^{dnf} \to B_n)$ for $B^{dnf} = B_1 \wedge ... \wedge B_n$, then we have the following proof:

[3] Originally seen in [3]

$$
\begin{array}{c}
\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathfrak{D}_0\\ \mathcal{B}^{dnf} \vdash X : (A \to B)^{dnf}\end{array}}{\mathcal{B}^{dnf} \vdash X : A^{dnf} \to B_1}\sqsubseteq \qquad \begin{array}{c}\mathfrak{D}_1\\ \mathcal{B}^{dnf} \vdash Y : A^{dnf}\end{array}}{\mathcal{B}^{dnf} \vdash XY : B_1}(\to E) \qquad \begin{array}{c}\vdots\\ \mathcal{B}^{dnf} \vdash XY : B_2\end{array}}{\mathcal{B}^{dnf} \vdash XY : B_1 \wedge B_2}(\wedge I) \qquad \dots \qquad \begin{array}{c}\vdots\\ \mathcal{B}^{dnf} \vdash XY : B_n\\ \vdots\end{array} \\
\vdots \\
\mathcal{B}^{dnf} \vdash XY : B_1 \wedge \dots \wedge B_n
\end{array}
$$

Thus we have $\mathcal{B}^{dnf} \vdash XY : B_1 \wedge \dots \wedge B_n$ and since $B_1 \wedge \dots \wedge B_n = B^{dnf}$, we are done.

($\sqsubseteq$): We assume $\mathcal{B} \vdash X : B$ (derived from $\mathcal{B} \vdash X : A$ and $A \sqsubseteq B$) and want to show $\mathcal{B}^{dnf} \vdash X : B^{dnf}$. Since we assume as our induction hypothesis that there is a computation for $\mathcal{B}^{dnf} \vdash X : A^{dnf}$, and by **Lemma 1** we can say $A^{dnf} \sqsubseteq B^{dnf}$, then we can use ($\sqsubseteq$)-rule to get $\mathcal{B}^{dnf} \vdash X : B^{dnf}$.

This concludes the proof. ⊠

Definition: : We define strictly positive, positive and negative occurrences as follows:

A is positive and strictly positive in A
If C is positive in B, then C is positive in A $\to$ B
If C is strictly positive in B, then C is strictly positive in A $\to$ B
If C is positive in B, then C is negative in B $\to$ A
If A is positive in B or C, then A is positive B $\wedge$ C.
If A is strictly positive in B or C, then A is strictly positive B $\wedge$ C.
If C is negative in B, then C is negative in A $\to$ B
If C is negative in B, then C is positive in B $\to$ A
If C is negative in B or A, then C is negative in A $\wedge$ B

A single occurrence of B as a subexpression of A will be indicated A(B). An expression can be thought of as a root-oriented binary tree with atoms at its leaves and either $\to$ or $\wedge$ at each internal vertex. For each subexpression B of A there is a unique path from the root of A to the root of B. If we remove this occurrence of B we have a context A(.) where we could just as easily have thought of this as the replacement of B by a new atom p. The rules of the Munich version are

Axioms:
1: A $\sqsubseteq$ B if A,B are congruent by the semilattice axioms for $\wedge$
2: A(D(B) $\wedge$ D(C)) $\sqsubseteq$ A(D(B $\wedge$ C)) if D(.) strictly positive
3: A(D(B $\wedge$ C)) $\sqsubseteq$ A(D(B) $\wedge$ D(C)) if D(.) strictly positive[4]
4: A(B $\wedge$ C) $\sqsubseteq$ A(B) if A(.) is positive
5: A(B) $\sqsubseteq$ A(B $\wedge$ C) if A(.) is negative
Rules:
Transitivity: A $\sqsubseteq$ B and B $\sqsubseteq$ C implies A $\sqsubseteq$ C

[4]This law is not in the list Munich rules previously provided in [10], but is derivable from the original Munich rules and is added as an axiom for convenience.

Lemma 3: (Munich)[5]. If by the Traditional rules A $\sqsubseteq$ B then by the Munich rules A $\sqsubseteq$ B.

Proof: We verify that the Munich rules are closed under the Traditional rules by simulating the Traditional proofs by Munich proofs. We case on the different axioms of the traditional rules that don't contain $\rightarrow$.

Identity: Show $A \sqsubseteq B$ for A and B congruent: This follows immediately from the Munich rules.

$A \wedge B \sqsubseteq B$: Show $A \wedge B \sqsubseteq B$: Since in munich rules, $(A \wedge B)$ is positive in itself, then we have $(A \wedge B)(A \wedge B) \sqsubseteq (A \wedge B)(B)$ which means $(A \wedge B) \sqsubseteq (B)$.

$A \wedge B \sqsubset B$: Show $A \wedge B \sqsubseteq A$: This holds from the same reasoning from the previous case and commutativity.

Distributivity: Show (A $\rightarrow$ B) $\wedge$ (A $\rightarrow$ C) $\sqsubseteq$ A $\rightarrow$ (B $\wedge$ C). Using our second Munich rule, since B is strictly positive in A $\rightarrow$ B and C is strictly positive in A $\rightarrow$ C, then we can conclude (A $\rightarrow$ B) $\wedge$ (A $\rightarrow$ C) $\sqsubseteq$ A $\rightarrow$ (B $\wedge$ C).

Greatest Lower Bound: We want to show if in Munich $C \sqsubseteq A$ and $C \sqsubseteq B$, then in Munich $C \sqsubseteq A \wedge B$. We can copy that proof into a proof that C$\wedge$C $\sqsubseteq$ A$\wedge$C by ignoring the second occurrence of C. We can copy the proof of $C \sqsubseteq B$ into a proof that A$\wedge$C $\sqsubseteq$ A$\wedge$B by ignoring the A. So by transitivity C$\wedge$C $\sqsubseteq$ A$\wedge$B and by idempotency C $\sqsubseteq$ A$\wedge$B.

Covariance/contravariance: Show A $\sqsubseteq$ C and D $\sqsubseteq$ B implies C $\rightarrow$ D $\sqsubseteq$ A $\rightarrow$ B.

Sublemma: If in the Munich rules $A \sqsubseteq B$ and $D(\cdot)$ is positive, then $D(A) \sqsubseteq D(B)$, and if $D(\cdot)$ is negative, then $D(B) \sqsubseteq D(A)$

Sublemma-proof: We assume $D(\cdot)$ is positive or negative, and proceed by induction on the formulation of $A \sqsubseteq B$:

We assume $D(\cdot)$ is positive and then case on the possible constructions of $A \sqsubseteq B$:

Axiom 1: If A and B are congruent by the semi-lattice axioms, then the result is immediate.

Axiom 2: If A is of the form $J(K(L) \wedge K(M))$ and B is of the form $J(K(L \wedge M))$, where $K(\cdot)$ is positive, then we still have $K(\cdot)$ positive in both $D(J(K(L) \wedge K(M)))$ and $D(J(K(L \wedge M)))$. Therefore $D(J(K(L) \wedge K(M))) \sqsubseteq D(J(K(L \wedge M)))$

Axiom 3: The same reasoning holds from the Axiom 2 case.

Axiom 4: Assume A is of the form $J(K \wedge L)$ and B is of the form $J(K)$ where $J(\cdot)$ is positive. Note that if some arbitrary M is positive in J and J is positive in D, then M is positive in D. Since we assumed $D(\cdot)$ is positive, then we know $D(J(K \wedge L)) \sqsubseteq D(J(K))$, and thus $D(A) \sqsubseteq D(B)$.

Axiom 5: Assume A is of the form $J(K)$ and B is of the form $J(K \wedge L)$ where $J(\cdot)$ is negative. Similar to the Axiom 4 case, since we know if an arbitrary M is negative in J and J is positive in D, then M is negative in D. Since we assume $D(\cdot)$ is positive, then we can conclude $D(J(K)) \sqsubseteq D(J(K \wedge L))$, and thus $D(A) \sqsubseteq D(B)$.

Rule 1 (Transitivity): We assume $A \sqsubseteq B$ where $A \sqsubseteq C$ and $C \sqsubseteq B$ for some C. We assume as induction hypothesis that $D(A) \sqsubseteq D(C)$ and $D(C) \sqsubseteq D(B)$. Then $D(A) \sqsubseteq D(B)$ result is immediate!

This covers all the cases for when $D(\cdot)$ is positive.

Now we assume $D(\cdot)$ is negative and do the same casing:

[5]This lemma and its proof were taken from [10].

Axiom 1: If A and B are congruent by the semi-lattice axioms, then the result is immediate.

Axiom 2: If A is of the form $J(K(L) \wedge K(M))$ and B is of the form $J(K(L \wedge M))$, where $K(\cdot)$ is positive, then we still have $K(\cdot)$ positive in both $D(J(K(L) \wedge K(M)))$ and $D(J(K(L \wedge M)))$. Therefore we can use axiom 3 to show $D(J(K(L \wedge M))) \sqsubseteq D(J(K(L) \wedge K(M)))$

Axiom 3: The same reasoning holds from Axiom 2 case, where we use Axiom 2 to flip A and B to show $D(B) \sqsubseteq D(A)$.

Axiom 4: Assume A is of the form $J(K \wedge L)$ and B is of the form $J(K)$ where $J(\cdot)$ is positive. Note that if some arbitrary M is positive in J and J is negative in D, then M is negative in D. Since we assumed $D(\cdot)$ is negative, then we know $D(J(K)) \sqsubseteq D(J(K \wedge L))$ by axiom 5, and thus $D(B) \sqsubseteq D(A)$.

Axiom 5: Assume A is of the form $J(K)$ and B is of the form $J(K \wedge L)$ where $J(\cdot)$ is negative. Similar to the Axiom 4 case, since we know if an arbitrary M is negative in J and J is negative in D, then M is positive in D. Since we assume $D(\cdot)$ is negative, then we can conclude $D(J(K \wedge L)) \sqsubseteq D(J(K))$, and thus $D(B) \sqsubseteq D(A)$.

Rule 1 (Transitivity): Assume $A \sqsubseteq B$ where $A \sqsubseteq C$ and $C \sqsubseteq B$ for some C. We assume as induction hypothesis that $D(C) \sqsubseteq D(A)$ and $D(B) \sqsubseteq D(C)$. Then we have $D(B) \sqsubseteq D(A)$.

This completes the case where $D(\cdot)$ is negative, and thus we are done with the proof of the sublemma.

Using our sublemma, we can show that if $A \sqsubseteq C$, then for any X, $C \rightarrow X \sqsubseteq A \rightarrow X$ since C and A are negative in $C \rightarrow X$ and $A \rightarrow X$ respectively. We also know that if $D \sqsubseteq B$, then for any X, $X \rightarrow D \sqsubseteq X \rightarrow B$ since D and B are positive in $X \rightarrow D$ and $X \rightarrow B$ respectively. Putting these two facts together, if we assume $A \sqsubseteq C$ and $D \sqsubseteq B$, then we know $C \rightarrow D \sqsubseteq A \rightarrow B$.

End of proof ☒

Lemma 4: (Converse of Munich) If by the Munich rules A ⊑ B then by the Traditional rules A ⊑ B.

Proof idea: We follow the same proof idea as the first direction and simulate the Munich rules by Traditional proofs:

Case 1: Show A(B ∧ C) ⊑ A(B) where A(.) is positive using the traditional rules:
Basis Case: Let A(.) = .
Then B ∧ C ⊑ B is a BCD axiom, so this case is done.

Induction step:

Case i) Assume A(.) = D(.) ∧ E or A(.) = D ∧ E(.). In the former case, we have D(B ∧ C) ⊑ D(B) as induction hypothesis. Then we know that D(B ∧ C) ∧ E ⊑ D(B) ∧ E and thus A(B ∧ C) ⊑ A(B). The latter case is just the same since we have commutativity.

Case ii) Assume A(.) = D(.) → E or A(.) = D → E(.). In the former case, since A(.) is positive, then we have D(.) negative. So we have D(B ∧ C) ⊑ D(B) as induction hypothesis. Using this and the fact that E ⊑ E means we can use contravariance to gain D(B ∧ C)→ E ⊑ D(B) →E. Thus A(B ∧ C) ⊑ A(B).

In the latter case, we know that E(.) is positive. So we know that $E(B/\ C)[E(B)$ Then with D ⊑ D and using contravariance again, we have D → E(B ∧ C) ⊑ D →E(B). Thus A(B ∧ C) ⊑ A(B).

Case 2: Show A(B) ⊑ A(B ∧ C) where A(.) is negative using the traditional rules.

Baes case: Let A(.) = .
This is impossible, since then A(.) is positive. So this holds true vacuously.

Induction step: We break down the cases the same way as before:
Case i) A(.) = D(.) ∧ E or D ∧ E(.). In the former case, we assume as inductive hypothesis that D(B) ⊑ D(B ∧ C). Then we know that D(B) ∧ E ⊑ D(B ∧ C) ∧ E, and so this case is done. The latter case is the same.

Case ii) A(.) = D(.)→E or A(.) = D→E(.):
Subcase a; A(.) = D(.)→E. Since A(.) is negative, then D(.) is positive. So we have D(B ∧ C) ⊑ D(B) as induction hypothesis. Then using contravariance and the fact that E ⊑ E gives us D(B)→E ⊑ D(B ∧ C)→E. Thus A(B)⊑ A(B ∧ C).
Subcase b: A(.) = D→E(.). Here we know E(.) is positive. So by induction hypothesis, E(B ∧ C) ⊑ E(B). So again, we can apply covariance with D ⊑ to attain D → E(B) ⊑ D →E(B ∧ C). Thus A(B) ⊑ A(B ∧ C).
This covers all cases. ☒

Lemma 5 If A ⊑ B then in minimal logic we can derive a proof of $\vdash A \to B$.
Proof: By induction on a derivation of $A \sqsubseteq B$.
A ⊑ A: Show $\vdash A \to A$:

$$\dfrac{\dfrac{}{\{A\} \vdash A}\;(Ax.)}{\vdash A \to A}\;(\to I)$$

A ∧ **B** ⊑ **A**: Show ⊢ A ∧ B → A.

$$\dfrac{\dfrac{\dfrac{}{\{A \wedge B\} \vdash A \wedge B}\;(Ax.)}{\{A \wedge B\} \vdash A}\;(\wedge E_1)}{\vdash A \wedge B \to A}\;(\to I)$$

A ∧ **B** ⊑ **B**: Show ⊢ A ∧ B → B.

$$\dfrac{\dfrac{\dfrac{}{\{A \wedge B\} \vdash A \wedge B}\;(Ax.)}{\{A \wedge B\} \vdash B}\;(\wedge E_2)}{\vdash A \wedge B \to B}\;(\to I) \quad f$$

Distributive Law: We show $\vdash ((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\{(A \to B) \wedge (A \to C), A\} \vdash (A \to B) \wedge (A \to C)}\;(Ax.)}{\{(A \to B) \wedge (A \to C), A\} \vdash (A \to B)}\;(\wedge E_1) \qquad \dfrac{}{\{(A \to B) \wedge (A \to C), A\} \vdash A}\;(Ax.)}{\{(A \to B) \wedge (A \to C), A\} \vdash B}\;(\to E) \qquad \Psi}{\{(A \to B) \wedge (A \to C), A\} \vdash (B \wedge C)}\;(\wedge I)}{\{(A \to B) \wedge (A \to C)\} \vdash A \to (B \wedge C)}\;(\to I)}{\vdash ((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C))}\;(\to I)$$

Where Ψ is

$$\dfrac{\dfrac{\dfrac{}{\{(A \to B) \wedge (A \to C)), A\} \vdash (A \to B) \wedge (A \to C)}\,(Ax.)}{\{(A \to B) \wedge (A \to C)), A\} \vdash (A \to C)}\,(\wedge E_2) \quad \dfrac{}{\{(A \to B) \wedge (A \to C), A\} \vdash A}\,(Ax.)}{\{(A \to B) \wedge (A \to C), A\} \vdash C}\,(\to E)$$

Greatest Lower Bound: Assume $A \sqsubseteq B$ and $A \sqsubseteq C$ (and thus we have as induction hypothesis the proof $\mathfrak{D}_0$ that give us $\vdash (A \to B)$ and the proof $\mathfrak{D}_1$ that gives us $\vdash (A \to C)$). We have $A \sqsubseteq (B \wedge C)$ and show $\vdash (A \to (B \wedge C))$:

$$\dfrac{\dfrac{\dfrac{\overset{\mathfrak{D}_0}{\{A\} \vdash (A \to B)} \quad \dfrac{}{\{A\} \vdash A}\,(Ax.)}{\{A\} \vdash B}\,(\to E) \quad \dfrac{\overset{\mathfrak{D}_1}{\{A\} \vdash A \to C} \quad \dfrac{}{\{A\} \vdash A}\,(Ax.)}{\{A\} \vdash C}\,(\to E)}{\{A\} \vdash (B \wedge C)}\,(\wedge I)}{\vdash (A \to (B \wedge C))}\,(\to I)$$

Transitivity: Assume $A \sqsubseteq B$ and $B \sqsubseteq C$. We formulate our induction hypothesis to be the proofs $\mathfrak{D}_0$ and $\mathfrak{D}_1$ that give us $\vdash (A \to B)$ and $\vdash (B \to C)$ respectively. Then we show $A \sqsubseteq C$ implies $\vdash (A \to C)$:

$$\dfrac{\dfrac{\overset{\mathfrak{D}_1}{\{A\} \vdash (B \to C)} \quad \dfrac{\overset{\mathfrak{D}_0}{\{A\} \vdash (A \to B)} \quad \dfrac{}{\{A\} \vdash A}\,(Ax.)}{\{A\} \vdash B}\,(\to E)}{\{A\} \vdash C}\,(\to E)}{\vdash (A \to C)}\,(\to I)$$

Covariance/Contravariance: We assume $A \sqsubseteq C$ and $D \sqsubseteq B$, and thus as induction hypothesis the proofs $\mathfrak{D}_0$ and $\mathfrak{D}_1$ that give us $\vdash A \to C$ and $\vdash D \to B$ respectively. We know $(C \to D) \sqsubseteq (A \to B)$ and show $\vdash (C \to D) \to (A \to B)$:

$$\dfrac{\dfrac{\dfrac{\overset{\mathfrak{D}_1}{\{C \to D, A\} \vdash (D \to B)} \quad \dfrac{\dfrac{}{\{C \to D, A\} \vdash C \to D}\,(Ax.) \quad \dfrac{\overset{\mathfrak{D}_0}{\{C \to D, A\} \vdash A \to C} \quad \dfrac{}{\{C \to D, A\} \vdash A}\,(Ax.)}{\{C \to D, A\} \vdash C}\,(\to E)}{\{C \to D, A\} \vdash D}\,(\to E)}{\{C \to D, A\} \vdash B}\,(\to I)}{\{C \to D\} \vdash (A \to B)}\,(\to I)}{\vdash (C \to D) \to (A \to B)}\,(\to I)$$

This completes the proof. ⊠.

The sixth lemma we include is motivated by the following idea: We have a proof with every type in dnf, and we want to convert the proof into an equivalent proof where all applications of $(\sqsubseteq)$ are at the top level of the proof-tree, and not in the middle of the tree.

Lemma 6: If we have a proof where every type is in dnf, then this proof can be converted into a similar one where every $\sqsubseteq$ is applied to an axiom.

Proof:

Throughout this proof, we use the following sub-lemma:

Sublemma:

Let $A = A_1 \wedge ... \wedge A_n$ where $A_i = A_{(i,1)} \to (...(A_{(i,m(i))} \to a_i)...)$, where for $i = 1, ..., n$ a_i is an atom (where m(i) may be 0), and each $A_{(i,j)}$ is in distributive normal form, and let $B = B_1 \wedge ... \wedge B_k$ where for $i = 1, ..., k$, $B_i = B_{(i,1)} \to (...(B_{(i,l(i))} \to b_i)...)$, b_i is an atom (where $l(i)$ may be 0), each $B_{(i,j)}$ is in distributive normal form. Then $A \sqsubseteq$ B iff for each $i = 1, ..., k$ there exists $j = 1, ..., n$ such that $b_i = a_j, l(i) = m(j)$ and for $r = 1, ..., l(i)$ we have $B_{(i,r)} \sqsubseteq A_{(j,r)}$.

The proof for this sub-lemma is provided in [9].

We proceed by induction on the proof that $\mathcal{B} \vdash X : A$ and $A \sqsubseteq$ B then there is a proof of $\mathcal{B} \vdash X : B$ with the desired properties:

Base Case(Axiom): Then there are no applications of $(\sqsubseteq)$ to consider, and so this case holds true.

Case $(\to I)$: Given as induction hypothosis

$$\dfrac{\overset{\mathfrak{D}}{\mathcal{B} \cup \{x : A\} \vdash X : B}}{\mathcal{B} \vdash \lambda x.X : A \to B}\ (\to I)$$

Now let us also define type C = $C_1 \wedge ... \wedge C_n$ (since C is in dnf) and we assume $A \to B \sqsubseteq C$. Each C_i has the form $C_{(i,1)} \to (...(C_{(i,f(i))} \to c_i)...)$ (here f is simply a function that maps the i to the number of arrows in C_i).

We let $(D_i \to E_i) = C_i$ such that $D_i = C_{(i,1)}$ and $E_i = C_{(i,2)} \to (...(C_{(i,f(i))} \to c_i)...)$ for i=1,...n.

By our first sublemma, $D_i \sqsubseteq A$, and since B and E_i are the the tails of $A \to B$ and $D_i \to E_i$. Then we also know $B \sqsubseteq E_i$.

We want to show that C can be proven in a way where all applications of $\sqsubseteq$ are applied to axioms.

For each i, we have

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\mathcal{B} \cup \{x : D_i\} \vdash x : D_i}\ (Ax.) \quad D_i \sqsubseteq A}{\mathcal{B} \cup \{x : D_i\} \vdash x : A}\ (\sqsubseteq) \quad \dfrac{\overset{\mathfrak{D}}{\mathcal{B} \cup \{x : D_i, x : A\} \vdash X : B}}{\mathcal{B} \cup \{x : D_i\} \vdash \lambda x.X : A \to B}\ (\to I)}{\mathcal{B} \cup \{x : D_i\} \vdash X : B}\ (\to E) \quad B \sqsubseteq E_i}{\dfrac{\mathcal{B} \cup \{x : D_i\} \vdash X : E_i}{\mathcal{B} \cup \{x : D_i\} \vdash \lambda x.X : D_i \to E_i}\ (\to I)}\ (\sqsubseteq)$$

Since C is in dnf, the above proof only has $(\sqsubseteq)$ applied to axioms. The induction hypothesis gives

$$\overset{\mathfrak{D}_i}{\mathcal{B} \vdash \lambda x.X : D_i \to E_i}$$

with the desired property. Now we can combine each $\overset{\mathfrak{D}_i}{\mathcal{B} \vdash \lambda x.X : D_i \to E_i}$ together to obtain C with enough applications of $(\to I)$ and $(\wedge I)$:

$$\dfrac{\overset{\mathfrak{D}_1}{\mathcal{B} \vdash \lambda x.X : D_1 \to E_1} \quad \overset{\mathfrak{D}_2}{\mathcal{B} \vdash \lambda x.X : D_2 \to E_2}}{\lambda x.X : (D_1 \to E_1) \wedge (D_2 \to E_2)}\ (\wedge I) \quad ... \quad \overset{\mathfrak{D}_n}{\mathcal{B} \vdash \lambda x.X : D_n \to E_n}$$

$$\vdots$$

$$\lambda x.X : (D_1 \to E_1) \wedge ... \wedge (D_n \to E_n)$$

This gives us our desired C.

Case $(\wedge I)$:
Given as induction hypothesis:

$$\dfrac{\overset{\mathfrak{D}_0}{\mathcal{B} \vdash X : A} \quad \overset{\mathfrak{D}_1}{\mathcal{B} \vdash X : B}}{\mathcal{B} \vdash X : A \wedge B} \ (\wedge I)$$

Let C = $C_1 \wedge ... \wedge C_l$
A = $A_1 \wedge ... \wedge A_m$
B = $B_1 \wedge ... \wedge B_n$.

We assume $A \wedge B \sqsubseteq C$.Let $D_1 \wedge ... \wedge D_{m+n}$ represent $A_1 \wedge ... \wedge B_k$. Using our sublemma, we can say for each C_i for i$i = 1, ...l$, there exists a D_j such that for each $d_j = c_i$ and each $C_{(i,k)} \sqsubseteq D_{(j,k)}$ for $k = 1, ..., m(i)$, and $c_i = d_j$. However this immediately means that $D_j \sqsubseteq C_i$.

Using this, we can partition the intersection-fragments of C into $E_1, ..., E_r$ and $F_1, ..., F_s$ in a manner where $A_j \sqsubseteq E_i$ for some A_j and $B_j \sqsubseteq F_i$ for some B_j. Since $A \sqsubseteq A_j$, we can use transitivity to say $A \sqsubseteq E_i$. The same holds respectively that $B \sqsubseteq F_i$.

We then have

$$\dfrac{\overset{\mathfrak{D}_0}{\mathcal{B} \vdash X : A} \quad A \sqsubseteq E_i}{\mathcal{B} \vdash X : E_i} \qquad \dfrac{\overset{\mathfrak{D}_1}{\mathcal{B} \vdash X : B} \quad B \sqsubseteq F_i}{\mathcal{B} \vdash X : F_i}$$

We can then combine our $X : E_1, ..., X : E_m$ and $X : F_1, ..., X : F_n$ in the proper order with enough applications of $(\wedge I)$ to obtain C.

Case $(\to E)$
Given the following deduction,

$$\dfrac{\overset{\mathfrak{D}_0}{\mathcal{B} \vdash X : A \to B} \quad \overset{\mathfrak{D}_1}{\mathcal{B}' \vdash Y : A}}{\mathcal{B} \vdash (XY) : B}$$

we use as induction hypothesis that the proofs $\overset{\mathfrak{D}_0}{\mathcal{B} \vdash X : A \to B}$ and $\overset{\mathfrak{D}_1}{\mathcal{B} \vdash Y : A}$ match our criteria.
Let $C = C_1 \wedge ... \wedge C_n$. We assume $B \sqsubseteq C$.
Since $A \to B$ is in dnf, then we know that B has the form $B_1 \to B_2 \to (B_m \to b)...)$.
Using the sublemma again, for each $C_i = C_{(i,1)} \to (... \to c_i)...)$, we know $C_{(i,j)} \sqsubseteq B_j$ and $c_i = b$.
We then have the proof:

$$\dfrac{\dfrac{\overset{\mathfrak{D}_0}{\mathcal{B} \vdash X : A \to B} \quad A \to B \sqsubseteq A \to C_i}{\mathcal{B} \vdash X : A \to C_i} \sqsubseteq \quad \overset{\mathfrak{D}_1}{\mathcal{B} \vdash Y : A}}{\mathcal{B} \vdash XY : C_i} \to E$$

We can repeat this with each piece of C and then use $(\wedge I)$ to obtain $\mathcal{B} \vdash (XY) : C$.

Case $(\sqsubseteq)$:
This is immediate.

This covers all cases of the proof. ⊠

Proposition 0: If a term has a type, then that term has at least one type with no positive occurrence of ∧.[6].

Proof: Consider a term X : A, where A is a type. If (B ∧ C) is positive in A, then since A(B ∧ C) ⊑ A(B) by the third Munich rule, then we can use (⊑) on the proof of X and for some basis $\mathcal{B}$

$$\dfrac{\begin{array}{c}\vdots\\ \mathcal{B} \vdash X : A(B \wedge C)\end{array} \quad A(B \wedge C) \sqsubseteq A(B)}{\mathcal{B} \vdash X : A(B)} \sqsubseteq$$

since **lemma 3** and **lemma 4** allow us to simulate the Munich rules in the Traditional rules. We can apply this process to X : A until all positive occurrences of ∧ in A are gone. Thus we know that there exists a type for X with no positive occurrence of ∧. ☒

Proposition 1: Let $\mathcal{B}$ be a basis such that in BCD $\mathcal{B} \vdash X : A$. Then in minimal logic $|\mathcal{B}| \vdash A$.

Proof: First by subject reduction we can assume that X is normal [3]. We suppose that we have a proof of $B \vdash X : A$ where every ⊑ inference has an axiom for a axiom, which we can create using **lemma 2** and **lemma 6**. The proof is by induction on the length of such a proof.

Base case: Assume $\mathcal{B} \vdash X : A$. Then there are no derivation further terms to derive $\mathcal{B} \vdash X : A$ and so we should have the derivation:

$$\frac{}{\mathcal{B} \vdash X : A}\ (Ax.)$$

where $(X : A) \in \mathcal{B}$. Then we know that $A \in |\mathcal{B}|$, and so we know $|\mathcal{B} \vdash A$.

Induction step: Assume $\mathcal{B} \vdash X : A$.

Case (∧ I): Assume $A \equiv (B \wedge C)$ such that $\mathcal{B} \vdash X : (B \wedge C)$ was derived from $\mathcal{B} \vdash X : B$ and $\mathcal{B} \vdash X : C$ using $(\wedge I)$. Our induction hypothesis tells us that $|\mathcal{B}| \vdash B$ and $|\mathcal{B}| \vdash C$. Then we know that $|\mathcal{B}| \vdash B \wedge C$ with the rule $(\wedge I)$.

Case (→ I): Assume $A \equiv (B \to C)$, such that $\mathcal{B} \vdash X : (B \to C)$ was derived from $\mathcal{B} \cup \{x : A\} \vdash P : B$. Note also that $|\mathcal{B} \cup \{x : A\}| = |\mathcal{B}| \cup A$. By our induction hypothosis, we know that $|\mathcal{B}| \cup \{A\} \vdash B$. Then we use the $(\to I)$-rule to derive $|\mathcal{B}| \vdash (A \to B)$.

Case (→ E): Assume $\mathcal{B} \vdash XY : A$ as computed from $\mathcal{B} \vdash (X : C \to A)$ and $\mathcal{B} \vdash (Y : C)$. By the induction hypothesis, we know that $|\mathcal{B}| \vdash (C \to A)$ and $|\mathcal{B}| \vdash C$. So then we use $(\to E)$-rule to derive $|\mathcal{B}| \vdash A$.

Case (⊑): Assume we derive $\mathcal{B} \vdash X : A$ from $\mathcal{B} \vdash X : B$ and $B \sqsubseteq A$ using the (⊑)-rule.

$$\frac{\mathcal{B} \vdash X : B \quad B \sqsubseteq A}{\mathcal{B} \vdash X : A}\ (\sqsubseteq)$$

and we want to show $|\mathcal{B}| \vdash A$. Then we know by the induction hypothesis that $|\mathcal{B}| \vdash B$. We are also given by induction hypothesis that $B \sqsubseteq A$. Our strategy is the following: By **lemma 5** we have $\mathcal{B} \vdash B \to A$, and thus we have $\mathcal{B} \vdash A$.

[6] A positive occurrence of ∧ of some arbitrary type D is a subtype of the form $A \wedge B$ such that $A \wedge B$ is positive in D

This completes the proof! ⊠

Proposition 2: Suppose $\mathcal{B}$ be a basis such that $|\mathcal{B}|$ has the property that no member has a negative occurrence of $\wedge$, and A has no positive occurrence of $\wedge$. Then if $|\mathcal{B}| \vdash A$ in minimal logic then there exists X such that $\mathcal{B} \vdash X : A$ in BCD.

Proof:

First, for each minimal logic assumption $A \in \mathcal{B}$, we assign a different variable x_A that can be identified at $(\rightarrow I)$.

By induction on the derivation of $|\mathcal{B}| \vdash A$.

Base Case: We have the derivation

$$|\mathcal{B}| \vdash A$$

where $A \in |\mathcal{B}|$

Then we can formulate $\mathcal{B} \vdash x_A : A$ since there must be an $x_A : A \in \mathcal{B}$.

Then we're done with this case.

Case ($\wedge$ I):

This case we can skip, since if we have $|\mathcal{B}| \vdash A \wedge B$ since $A \wedge B$ is positive in $A \wedge B$.

Case ($\rightarrow$ I):

Assume we have a proof of $|\mathcal{B}| \vdash A$ where $A \equiv B \rightarrow C$, and we derived this from $|\mathcal{B}| \cup \{B\} \vdash C$.

We case on the following:

(a) $B \in |\mathcal{B}|$. Then we can create $\mathcal{B} \cup \{y : B\} \vdash X : C$ where x_B does not occur in the domain of $\mathcal{B}$, and then using ($\rightarrow$ I) to get $\mathcal{B} \vdash \lambda y.X : (B \rightarrow C)$.

(b) $B \notin |\mathcal{B}|$. Then we straightforwardly create $\mathcal{B} \cup \{x_B : B\} \vdash X : C$ as induction hypothesis and then use ($\rightarrow$ I) to obtain $\mathcal{B} \vdash \lambda x_B.X : (B \rightarrow C)$.

Note that in either case, we don't have to worry about a positive occurance of $\wedge$ in $B \rightarrow C$ since there is necessarily no negative occurrence of $\wedge$ in B.

Case ($\rightarrow$ E):

Assume we have a proof of $|\mathcal{B}| \vdash A$ so that we derived it from $|\mathcal{B}| \vdash B \rightarrow A$ and $|\mathcal{B}| \vdash B$.

Then we have that $\mathcal{B} \vdash X : B \rightarrow A$ and $\mathcal{B} \vdash Y : B$ by the induction hypothesis. Then we can use ($\rightarrow$ E) to get $\mathcal{B} \vdash (XY : A)$

Thus we have $\mathcal{B} \vdash X : A$

Case ($\wedge E_1$ and $\wedge E_2$):

We show the proof with ($\wedge E_1$): Assume we have a proof of $|\mathcal{B}| \vdash B$ so that we derived it from $|\mathcal{B}| \vdash B \wedge A$.

Then we have that $\mathcal{B} \vdash X : B \wedge A$ by the induction hypothesis. Note that $(A \wedge B) \sqsubseteq B$. Then we use ($\sqsubseteq$) to derive $\mathcal{B} \vdash X : B$.

The same reasoning will then allow us to ($\wedge E_2$): Assume we have a proof of $|\mathcal{B}| \vdash A$ so that we derived it from $|\mathcal{B}| \vdash B \wedge A$. Then we have that $\mathcal{B} \vdash X : B \wedge A$ by the induction hypothesis. Since $(A \wedge B) \sqsubseteq B$, we use ($\sqsubseteq$) to derive $\mathcal{B} \vdash X : A$.

This completes the proof. ⊠.

Finally **Proposition 0** allows us to not need to worry about the terms with types that have a positive occurrence, allowing our restrictions in **proposition 2** to be unimportant. This completes the correlation between $\lambda_{\wedge}^{BCD}$ and propositional minimal logic.

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