STAT 1331 - Assignment 2

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Problem 1:

Let $\{x_t$ for t $= \ldots, $ -2, -1, 0, 1, 2, $\ldots \}$ be a time series process.

1a) What does it mean when we say x_t is weakly stationary?

Solution:

For x_t to be regarded as a weakly stationary process it must have constant mean and variance, and the covariance between x_t and x_{t_τ} depends only on the distance $|t - t_\tau|$, where $\tau = 1, 2, \ldots$

1b) Suppose $x_t = \sum_{j=0}^{4} \alpha_j \varepsilon_{t-j}$, where $\varepsilon_{t'} \sim$ **iid(0,σ_ε^2) for all t'. Show that x_t is weakly stationary.**

Solution:

For x_t to be regarded as a weakly stationary process it must have constant mean and variance, and the covariance between x_t and x_{t_τ} depends only on the distance $|t - t_\tau|$, where $\tau = 1, 2, \ldots, n$

Constant Mean:

First, we must prove that the time series process has constant mean, across time.

$$E(x_t) = \sum_{j=0}^{4} \alpha_j E(\varepsilon_{t-j})$$

Since $\varepsilon_{t'} \sim$ iid(0,σ_ε^2) for all t', we can say that $E(\varepsilon_{t-j}) = 0$. Thus we can conclude that:

$$E(x_t) = \sum_{j=0}^{4} \alpha_j E(\varepsilon_{t-j})$$

$$= \sum_{j=0}^{4} \alpha_j \times 0 = 0$$

Constant Variance:

Now, we must prove that the time series process has constant variance, across time.

$$var(X_t) = var(\sum_{j=0}^{4} \alpha_j E(\varepsilon_{t-j}))$$

Recall that $var(aX) = a^2 var(X)$. Thus, we have:

$$= \sum_{j=0}^{4} \alpha_j^2 var(\varepsilon_{t-j})$$

Since $\varepsilon_{t'} \sim iid(0, \sigma_\varepsilon^2)$ for all t', we can say that $var(\varepsilon_{t-j}) = \sigma_\varepsilon^2$. Thus we can conclude that:

$$\sum_{j=0}^{4} \alpha_j^2 var(\varepsilon_{t-j})$$

$$= \sum_{j=0}^{4} \alpha_j^2 \sigma_\varepsilon^2$$

Since σ_ε^2 has no dependency on the iteration variable, j, we can pull it outside of the summation. Thus, we have the following:

$$= \sigma_\varepsilon^2 \sum_{j=0}^{4} \alpha_j^2$$

Note that this final result has no variables attached, thus we can conclude that the variance is constant, and has no dependencies on time.

Covariance Depending Only on Distance Between Times:

Finally, we must show that the covariance between x_t and x_{t_τ} depends only on the distance $|t - t_\tau|$, where $\tau = 1, 2, ...$

Formally, we want to show that:

$$\gamma(t, t_\tau) = cov(x_t, x_{t-\tau})$$

$$= E(x_t, x_{t-\tau}) - E(x_t)E(x_{t-\tau})$$

$$= \gamma(\tau)$$

where $\tau = 1, 2, ...$

Starting out with the definition of covariance, we have:

$$cov(x_t, x_{t-\tau})$$

$$= E[(x_t)(x_{t-\tau})] - E(x_t)E(x_{t-\tau})$$

$$= E[(\sum_{j=0}^{4} \alpha_j \varepsilon_{t-j})(\sum_{k=0}^{4} \alpha_k \varepsilon_{t-\tau-k})] - E(\sum_{j=0}^{4} \alpha_j \varepsilon_{t-j})E(\sum_{k=0}^{4} \alpha_k \varepsilon_{t-\tau-k})$$

However, since $\varepsilon_{t'} \sim \text{iid}(0,\sigma_\varepsilon^2)$ for all t', we can say that $E(\varepsilon_{t-j}) = 0$. Thus we can conclude that:

$$= E[(\sum_{j=0}^{4} \alpha_j \varepsilon_{t-j})(\sum_{k=0}^{4} \alpha_k \varepsilon_{t-\tau-k})] - E(\sum_{j=0}^{4} \alpha_j \varepsilon_{t-j})E(\sum_{k=0}^{4} \alpha_k \varepsilon_{t-\tau-k})$$

$$= E[(\sum_{j=0}^{4} \alpha_j \varepsilon_{t-j})(\sum_{k=0}^{4} \alpha_k \varepsilon_{t-\tau-k})] - 0 \times E(\sum_{k=0}^{4} \alpha_k \varepsilon_{t-\tau-k})$$

$$= E[(\sum_{j=0}^{4} \alpha_j \varepsilon_{t-j})(\sum_{k=0}^{4} \alpha_k \varepsilon_{t-\tau-k})]$$

Further expanding on our Expectation, we get:

$$E[(\sum_{j=0}^{4} \alpha_j \varepsilon_{t-j})(\sum_{k=0}^{4} \alpha_k \varepsilon_{t-\tau-k})]$$

$$= \sum_{j=0}^{4}\sum_{k=0}^{4} \alpha_j \alpha_k E[(\varepsilon_{t-j}\varepsilon_{t-\tau-k})]$$

$$= \sum_{j=0}^{4} \alpha_j \alpha_{j-\tau} E[(\varepsilon_{t-j}^2)]$$

However, since $\varepsilon_{t'} \sim \text{iid}(0,\sigma_\varepsilon^2)$ for all t', we can say that $var(\varepsilon_{t-j}) = \sigma_\varepsilon^2$. Thus we can say that:

$$\sum_{j=0}^{4} \alpha_j \alpha_{j-\tau} E[(\varepsilon_{t-j}^2)]$$

$$= \sum_{j=0}^{4} \alpha_j \alpha_{j-\tau} (\sigma_\varepsilon^2)$$

$$= \sigma_\varepsilon^2 \sum_{j=0}^{4} \alpha_j \alpha_{j-\tau}$$

Note we have σ_ε^2, which is a constant, and α_j and $\alpha_{j-\tau}$, thus it is evident that the covariance solely depends on j and $j - \tau$. We must show that the covariance will be some number, and not yield ∞. In other words, we must show that $\sum_{j=0}^{4} \alpha_j \alpha_{j-\tau} < \infty$. Since our summation only extends to 4, we can guarantee that our summation will be less than ∞. We can say α_j and $\alpha_{j-\tau}$ will be $< \infty$. Thus, we can conclude that

$$\gamma(|j - (j - \tau)|) = \gamma(\tau)$$

Implying that the covariance does in fact only depend on the distances between two distances in time.

Thus, putting it all together, since the time series process, x_t has constant mean, constant variance, and covariance that solely depends on the distance between two times, we can say that x_t is a weakly stationary process.

1c) Suppose $x_t = \sum_{j=0}^{n} \alpha_j \varepsilon_{t-j}$, **where** $\varepsilon_{t'} \sim$ **iid(0,**σ_ε^2**) for all t' and n is a positive constant. Show that** x_t **is weakly stationary.**

Solution:

For x_t to be regarded as a weakly stationary process it must have constant mean and variance, and the covariance between x_t and x_{t_τ} depends only on the distance $|t - t_\tau|$, where $\tau = 1, 2, ..., n$

Constant Mean:

First, we must prove that the time series process has constant mean, across time.

$$E(x_t) = \sum_{j=0}^{n} \alpha_j E(\varepsilon_{t-j})$$

Since $\varepsilon_{t'} \sim$ iid(0,σ_ε^2) for all t', we can say that $E(\varepsilon_{t-j}) = 0$. Thus we can conclude that:

$$E(x_t) = \sum_{j=0}^{n} \alpha_j E(\varepsilon_{t-j})$$

$$= \sum_{j=0}^{n} \alpha_j \times 0 = 0$$

Constant Variance:

Now, we must prove that the time series process has constant variance, across time.

$$var(X_t) = var(\sum_{j=0}^{n} \alpha_j E(\varepsilon_{t-j}))$$

Recall that $var(aX) = a^2 var(X)$. Thus, we have:

$$= \sum_{j=0}^{n} \alpha_j^2 var(\varepsilon_{t-j})$$

Since $\varepsilon_{t'} \sim$ iid(0,σ_ε^2) for all t', we can say that $var(\varepsilon_{t-j}) = \sigma_\varepsilon^2$. Thus we can conclude that:

$$\sum_{j=0}^{n} \alpha_j^2 var(\varepsilon_{t-j})$$

$$= \sum_{j=0}^{n} \alpha_j^2 \sigma_\varepsilon^2$$

Since σ_ε^2 has no dependency on the iteration variable, j, we can pull it outside of the summation. Thus, we have the following:

$$= \sigma_\varepsilon^2 \sum_{j=0}^{n} \alpha_j^2$$

Note that this final result has no variables attached, thus we can conclude that the variance is constant, and has no dependencies on time.

Covariance Depending Only on Distance Between Times:

Finally, we must show that the covariance between x_t and x_{t_τ} depends only on the distance $|t - t_\tau|$, where $\tau = 1, 2, ..., n$

Formally, we want to show that:
$$\gamma(t, t_\tau) = cov(x_t, x_{t-\tau})$$

$$= E(x_t, x_{t-\tau}) - E(x_t)E(x_{t-\tau})$$

$$= \gamma(\tau)$$

where $\tau = 1, 2, ..., n$

Starting out with the definition of covariance, we have:

$$cov(x_t, x_{t-\tau})$$

$$= E[(x_t)(x_{t-\tau})] - E(x_t)E(x_{t-\tau})$$

$$= E[(\sum_{j=0}^{n} \alpha_j \varepsilon_{t-j})(\sum_{k=0}^{n} \alpha_k \varepsilon_{t-\tau-k})] - E(\sum_{j=0}^{n} \alpha_j \varepsilon_{t-j})E(\sum_{k=0}^{n} \alpha_k \varepsilon_{t-\tau-k})$$

However, since $\varepsilon_{t'} \sim iid(0, \sigma_\varepsilon^2)$ for all t', we can say that $E(\varepsilon_{t-j}) = 0$. Thus we can conclude that:

$$= E[(\sum_{j=0}^{n} \alpha_j \varepsilon_{t-j})(\sum_{k=0}^{n} \alpha_k \varepsilon_{t-\tau-k})] - E(\sum_{j=0}^{n} \alpha_j \varepsilon_{t-j})E(\sum_{k=0}^{n} \alpha_k \varepsilon_{t-\tau-k})$$

$$= E[(\sum_{j=0}^{n} \alpha_j \varepsilon_{t-j})(\sum_{k=0}^{n} \alpha_k \varepsilon_{t-\tau-k})] - 0 \times E(\sum_{k=0}^{n} \alpha_k \varepsilon_{t-\tau-k})$$

$$= E[(\sum_{j=0}^{n} \alpha_j \varepsilon_{t-j})(\sum_{k=0}^{n} \alpha_k \varepsilon_{t-\tau-k})]$$

Further expanding on our Expectation, we get:

$$E[(\sum_{j=0}^{n} \alpha_j \varepsilon_{t-j})(\sum_{k=0}^{n} \alpha_k \varepsilon_{t-\tau-k})]$$

$$= \sum_{j=0}^{n} \sum_{k=0}^{n} \alpha_j \alpha_k E[(\varepsilon_{t-j} \varepsilon_{t-\tau-k})]$$

$$= \sum_{j=0}^{n} \alpha_j \alpha_{j-\tau} E[(\varepsilon_{t-j}^2)]$$

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However, since $\varepsilon_{t'} \sim \text{iid}(0, \sigma_\varepsilon^2)$ for all t', we can say that $var(\varepsilon_{t-j}) = \sigma_\varepsilon^2$. Thus we can say that:

$$\sum_{j=0}^{n} \alpha_j \alpha_{j-\tau} E[(\varepsilon_{t-j}^2)]$$

$$= \sum_{j=0}^{n} \alpha_j \alpha_{j-\tau} (\sigma_\varepsilon^2)$$

$$= \sigma_\varepsilon^2 \sum_{j=0}^{n} \alpha_j \alpha_{j-\tau}$$

Note we have σ_ε^2, which is a constant, and α_j and $\alpha_{j-\tau}$, thus it is evident that the covariance solely depends on j and $j - \tau$. It is also important to note the differences between this result and the result from (b). In this case, we have our summation extending to n, which is a generalization of (b). We must show that $\sum_{j=0}^{n} \alpha_j \alpha_{j-\tau} < \infty$. Since our summation only extends to a positive constant, n, we can guarantee that our summation will be less than ∞. We can say α_j and $\alpha_{j-\tau}$ will be $< \infty$. Thus, we can conclude that

$$\gamma(|j - (j - \tau)|) = \gamma(\tau)$$

Implying that the covariance does in fact only depend on the distances between two distances in time.

Thus, putting it all together, since the time series process, x_t has constant mean, constant variance, and covariance that solely depends on the distance between two times, we can say that x_t is a weakly stationary process.

1d) Suppose $x_t = \sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j}$, where $\varepsilon_{t'} \sim \text{iid}(0, \sigma_\varepsilon^2)$ for all t'. Show that x_t is weakly stationary if $\sum_{j=0}^{\infty} \alpha_j^2 \leq c < \infty$, where c is a positive constant.

Solution:

For x_t to be regarded as a weakly stationary process it must have constant mean and variance, and the covariance between x_t and x_{t_τ} depends only on the distance $|t - t_\tau|$, where $\tau = 1, 2, ..., \infty$

Constant Mean:

First, we must prove that the time series process has constant mean, across time.

$$E(x_t) = \sum_{j=0}^{\infty} \alpha_j E(\varepsilon_{t-j})$$

Since $\varepsilon_{t'} \sim \text{iid}(0, \sigma_\varepsilon^2)$ for all t', we can say that $E(\varepsilon_{t-j}) = 0$. Thus we can conclude that:

$$E(x_t) = \sum_{j=0}^{\infty} \alpha_j E(\varepsilon_{t-j})$$

$$= \sum_{j=0}^{\infty} \alpha_j \times 0 = 0$$

Constant Variance:

Now, we must prove that the time series process has constant variance, across time.

$$var(X_t) = var(\sum_{j=0}^{\infty} \alpha_j E(\varepsilon_{t-j}))$$

Recall that $var(aX) = a^2 var(X)$. Thus, we have:

$$= \sum_{j=0}^{\infty} \alpha_j^2 var(\varepsilon_{t-j})$$

Since $\varepsilon_{t'} \sim \text{iid}(0, \sigma_{\varepsilon}^2)$ for all t', we can say that $var(\varepsilon_{t-j}) = \sigma_{\varepsilon}^2$. Thus we can conclude that:

$$\sum_{j=0}^{\infty} \alpha_j^2 var(\varepsilon_{t-j})$$

$$= \sum_{j=0}^{\infty} \alpha_j^2 \sigma_{\varepsilon}^2$$

Since σ_{ε}^2 has no dependency on the iteration variable, j, we can pull it outside of the summation. Thus, we have the following:

$$= \sigma_{\varepsilon}^2 \sum_{j=0}^{\infty} \alpha_j^2$$

Note that this final result has no variables attached, thus we can conclude that the variance is constant, and has no dependencies on time.

Covariance Depending Only on Distance Between Times:

Finally, we must show that the covariance between x_t and x_{t_τ} depends only on the distance $|t - t_\tau|$, where $\tau = 1, 2, ..., \infty$

Formally, we want to show that:
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$$= E(x_t, x_{t-\tau}) - E(x_t)E(x_{t-\tau})$$

$$= \gamma(\tau)$$

where $\tau = 1, 2, ..., \infty$

Starting out with the definition of covariance, we have:

$$cov(x_t, x_{t-\tau})$$

$$= E[(x_t)(x_{t-\tau})] - E(x_t)E(x_{t-\tau})$$

$$= E[(\sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j})(\sum_{k=0}^{\infty} \alpha_k \varepsilon_{t-\tau-k})] - E(\sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j})E(\sum_{k=0}^{\infty} \alpha_k \varepsilon_{t-\tau-k})$$

However, since $\varepsilon_{t'} \sim \text{iid}(0,\sigma_\varepsilon^2)$ for all t', we can say that $E(\varepsilon_{t-j}) = 0$. Thus we can conclude that:

$$= E[(\sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j})(\sum_{k=0}^{\infty} \alpha_k \varepsilon_{t-\tau-k})] - E(\sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j})E(\sum_{k=0}^{\infty} \alpha_k \varepsilon_{t-\tau-k})$$

$$= E[(\sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j})(\sum_{k=0}^{\infty} \alpha_k \varepsilon_{t-\tau-k})] - 0 \times E(\sum_{k=0}^{\infty} \alpha_k \varepsilon_{t-\tau-k})$$

$$= E[(\sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j})(\sum_{k=0}^{\infty} \alpha_k \varepsilon_{t-\tau-k})]$$

Further expanding on our Expectation, we get:

$$E[(\sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j})(\sum_{k=0}^{\infty} \alpha_k \varepsilon_{t-\tau-k})]$$

$$= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \alpha_j \alpha_k E[(\varepsilon_{t-j}\varepsilon_{t-\tau-k})]$$

$$= \sum_{j=0}^{\infty} \alpha_j \alpha_{j-\tau} E[(\varepsilon_{t-j}^2)]$$

However, since $\varepsilon_{t'} \sim \text{iid}(0,\sigma_\varepsilon^2)$ for all t', we can say that $var(\varepsilon_{t-j}) = \sigma_\varepsilon^2$. Thus we can say that:

$$\sum_{j=0}^{\infty} \alpha_j \alpha_{j-\tau} E[(\varepsilon_{t-j}^2)]$$

$$= \sum_{j=0}^{\infty} \alpha_j \alpha_{j-\tau} (\sigma_\varepsilon^2)$$

$$= \sigma_\varepsilon^2 \sum_{j=0}^{\infty} \alpha_j \alpha_{j-\tau}$$

Note we have σ_ε^2, which is a constant, and α_j and $\alpha_{j-\tau}$, thus it is evident that the covariance solely depends on j and $j-\tau$. In this case, we have our summation extending to ∞, we cannot guarantee that our summation is $< \infty$. So, we must show that $\sum_{j=0}^{\infty} \alpha_j \alpha_{j-\tau} < \infty$. By making use of the fact that $\sum_{j=0}^{\infty} \alpha_j^2 \leq c < \infty$, we can use the Cauchy-Schwartz Inequality to say:

$$\sum_{j=0}^{\infty} \alpha_j \alpha_{j-\tau} \leq (\sum_{j=0}^{\infty} \alpha_j^2)^{1/2}(\sum_{j=0}^{\infty} \alpha_{j-\tau}^2)^{1/2}$$

Making use of the observation that $\sum_{j=0}^{\infty} \alpha_j^2 \leq c < \infty$, we can say that the right-hand side, $(\sum_{j=0}^{\infty} \alpha_j^2)^{1/2}(\sum_{j=0}^{\infty} \alpha_{j-\tau}^2)^{1/2}$ will evaluate to a constant. This will inadvertently leave us with:

$$\sum_{j=0}^{\infty} \alpha_j \alpha_{j-\tau} \leq c < \infty$$

where c is a positive constant. Thus, this will allow to say that, by linearity, that $\sum_{j=0}^{\infty} \alpha_j \alpha_{j-\tau} < \infty$. So, we can say α_j and $\alpha_{j-\tau}$ will be $< \infty$. Thus, we can conclude that

$$\gamma(|j - (j - \tau)|) = \gamma(\tau)$$

Implying that the covariance does in fact only depend on the distances between two distances in time.

Thus, putting it all together, since the time series process, x_t has constant mean, constant variance, and covariance that solely depends on the distance between two times, we can say that x_t is a weakly stationary process.

Problem 2:

Consider the following AR(1) process,

$$y_t = 0.5 y_{t-1} + \varepsilon_t,$$

where $\varepsilon_t \sim \text{iid}(0,1)$.

2a) Find the coefficients α_j in the representation $y_t = \sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j}$. Use the result in part (d) of question 1 to conclude that y_t is weakly stationary.

Solution:

In order to see a pattern let's see what happens when we plug-in $0.5 y_{t-2} + \varepsilon_{t-1}$ for y_{t-1}:

$$y_t = 0.5 y_{t-1} + \varepsilon_t$$

$$y_t = 0.5(0.5 y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t$$

$$y_t = (0.5)^2 y_{t-2} + (0.5\varepsilon_{t-1} + \varepsilon_t)$$

If we further plug-in $0.5 y_{t-3} + \varepsilon_{t-2}$ for y_{t-2}, we have:

$$y_t = 0.5(0.5 y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t$$

$$y_t = 0.5(0.5[(0.5 y_{t-3} + \varepsilon_{t-2}) + \varepsilon_{t-1}]) + \varepsilon_t$$
$$y_t = (0.5)^3 y_{t-3} + (0.5^2 \varepsilon_{t-2} + 0.5\varepsilon_{t-1} + \varepsilon_t)$$

Thus, generally, if we interested in seeing the behavior of the process y_t, where n is the total amount of time elapsed so far, we have the following:

$$y_t = (0.5)^n y_{t-n} + \sum_{j=0}^{n-1} (0.5)^j \varepsilon_{t-j}$$

Thus, in the representation, $y_t = \sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j}$, we have:

$$y_t = \sum_{j=0}^{\infty} (0.5)^j \varepsilon_{t-j}$$

Thus, in this case, we have the coefficient α_j is $\sum_{j=0}^{\infty} (0.5)^j$.

2b) Find the mean of y_t.

Solution:

From part (a), we have:

$$y_t = \sum_{j=0}^{\infty} (0.5)^j \varepsilon_{t-j}$$

Since $\varepsilon_t \sim \text{iid}(0, 1)$, we can say that $\text{E}(\varepsilon_{t-j}) = 0$. Thus, we have:

$$E(y_t) = E(\sum_{j=0}^{\infty} (0.5)^j \varepsilon_{t-j})$$

$$= \sum_{j=0}^{\infty} (0.5)^j E(\varepsilon_{t-j})$$

$$= \sum_{j=0}^{\infty} (0.5)^j \times 0$$

$$= 0$$

Thus, we have that the mean of $y_t = 0$.

2c) Find the variance of y_t.

Solution:

From part (a), we have:

$$y_t = \sum_{j=0}^{\infty} (0.5)^j \varepsilon_{t-j}$$

Since $\varepsilon_t \sim \text{iid}(0, 1)$, we can say that $\text{var}(\varepsilon_{t-j}) = 1$. Thus, we have:

$$var(y_t) = var(\sum_{j=0}^{\infty} (0.5)^j \varepsilon_{t-j})$$

$$= var(\sum_{j=0}^{\infty} (0.5)^j \times 1)$$

$$= var(\sum_{j=0}^{\infty} (0.5)^j)$$

Recall from question 1, part (d), we derived that the variance for a time series process of the form $y_t = \sum_{j=0}^{\infty} \alpha_j \varepsilon_{t-j}$ has variance: $\sigma_\varepsilon^2 \sum_{j=0}^{\infty} \alpha_j^2$. In this case, we have $\alpha_j = \sum_{j=0}^{\infty} (0.5)^j$. Plugging in, we have the following:

$$var(\sum_{j=0}^{\infty}(0.5)^j)$$

$$= \sigma_{\varepsilon}^2 \sum_{j=0}^{\infty}(0.5)^j$$

Recall that $\varepsilon_t \sim$ iid$(0, 1)$, we can say that $\sigma_{\varepsilon}^2 = 1$. Thus, we have:

$$\sigma_{\varepsilon}^2 \sum_{j=0}^{\infty}(0.5)^j$$

$$= 1 \times \sum_{j=0}^{\infty}(0.5)^j$$

$$= \sum_{j=0}^{\infty}(0.5)^j$$

Note, that this is a sum of an infinite geometric series with a common ratio of $r = 0.5$. Thus, we can solve for the geometric series as:

$$\sum_{j=0}^{\infty}(0.5)^j$$

$$= \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

Thus, we can conclude that the variance of y_t is 1.

2d) Find the auto-covariance function of y_t.

Solution:

Recall in question 1, part (d), we derived that a time series process of the form $y_t = \sum_{j=0}^{\infty}\alpha_j\varepsilon_{t-j}$, has covariance: $\sigma_{\varepsilon}^2 \sum_{j=0}^{\infty}\alpha_j\alpha_{j-\tau}$. In this case, we have $\alpha_j = \sum_{j=0}^{\infty}(0.5)^j$. Plugging in, we have the following:

$$cov(y_t) = cov(\sum_{j=0}^{\infty}(0.5)^j\varepsilon_{t-j})$$

$$= \sigma_{\varepsilon}^2 \sum_{j=0}^{\infty}(0.5)^j(0.5)^{j-\tau}$$

Recall that $\varepsilon_t \sim$ iid$(0, 1)$, we can say that $\sigma_{\varepsilon}^2 = 1$. Thus, we have:

$$\sigma_{\varepsilon}^2 \sum_{j=0}^{\infty}(0.5)^j(0.5)^{j-\tau}$$

$$= 1 \times \sum_{j=0}^{\infty} (0.5)^j (0.5)^{j-\tau}$$

$$= \sum_{j=0}^{\infty} (0.5)^j (0.5)^{j-\tau}$$

Writing out the $(0.5)^{j-\tau}$ term, inside of the summation, we have:

$$\sum_{j=0}^{\infty} (0.5)^j (0.5)^{j-\tau}$$

$$= \sum_{j=0}^{\infty} (0.5)^j (0.5)^j (0.5)^{-\tau}$$

Since $(0.5)^{-\tau}$ does not depend on the iteration variable j, we can take it outside of the summation, and get:

$$\sum_{j=0}^{\infty} (0.5)^j (0.5)^j (0.5)^{-\tau}$$

$$= (0.5)^{-\tau} \sum_{j=0}^{\infty} (0.5)^j (0.5)^j$$

$$= (0.5)^{-\tau} \sum_{j=0}^{\infty} (0.5)^{2j}$$

$$= (0.5)^{-\tau} \sum_{j=0}^{\infty} (0.25)^j$$

Note, that this is a sum of an infinite geometric series with a common ratio of $r = 0.25$. Thus, we can solve for the geometric series as:

$$(0.5)^{-\tau} \sum_{j=0}^{\infty} (0.25)^j$$

$$= (0.5)^{-\tau} \left(\frac{\frac{1}{4}}{1 - \frac{1}{4}} \right)$$

$$= (0.5)^{-\tau} \left(\frac{\frac{1}{4}}{\frac{3}{4}} \right) = (0.5)^{-\tau} \left(\frac{1}{3} \right)$$

Thus, we can say that the auto-covariance function of y_t is: $(0.5)^{-\tau} (\frac{1}{3})$, where $\tau = 1, 2, ..., \infty$.

Problem 3:

Consider the following autoregressive processes,

a) $y_t = 3 - 0.3y_{t-1} + 0.04y_{t-2} + \varepsilon_t$

b) $y_t = 5 - 3.1y_{t-1} + 0.03y_{t-2} + \varepsilon_t$

c) $y_t = 2.5y_{t-1} - 2y_{t-2} + 0.5y_{t-3} + \varepsilon_t$

where $\varepsilon_t \sim \text{iid}(0, \sigma^2)$.

Which ones are weakly stationary?

Solution:

I believe that (a) is a weakly stationary autoregressive process, while (b) and (c) are not weakly stationary processes. Upon inspection, as we begin to plug in terms recursively for each process, we see that (a) will begin to get smaller and smaller, and converge to a certain value. However, for processes (b) and (c) it appears indeterminate as to the whereabout, as it seems to grow larger and larger, implying that the series will diverge, and inadvertently fail the property of a weakly stationary process determined in question 1, part (d)