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1 Introduction

Fourier series provide a way to represent periodic functions as infinite sums of sines and cosines. This representation is fundamental in signal processing, differential equations, and mathematical physics.

2 Preliminaries

Definition 2.1 (Periodic Function). A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is *periodic* with period $T > 0$ if $f(x + T) = f(x)$ for all $x \in \mathbb{R}$.

For simplicity, we consider functions with period 2π , so $f(x + 2\pi) = f(x)$.

Definition 2.2 (Inner Product on L^2). For functions $f, g : [-\pi, \pi] \rightarrow \mathbb{R}$, we define the inner product:

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x) \, dx \quad (1)$$

3 Orthogonality of Trigonometric Functions

The key to deriving Fourier coefficients is the orthogonality of sines and cosines.

Proposition 3.1 (Orthogonality Relations). For integers $m, n \geq 0$:

$$\int_{-\pi}^{\pi} \cos(mx) \cos(nx) \, dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \neq 0 \\ 2\pi & m = n = 0 \end{cases} \quad (2)$$

$$\int_{-\pi}^{\pi} \sin(mx) \sin(nx) \, dx = \begin{cases} 0 & m \neq n \\ \pi & m = n \neq 0 \end{cases} \quad (3)$$

$$\int_{-\pi}^{\pi} \sin(mx) \cos(nx) \, dx = 0 \quad \text{for all } m, n \quad (4)$$

Proof. We prove eq. (2). Using the product-to-sum identity:

$$\cos(mx) \cos(nx) = \frac{1}{2} [\cos((m - n)x) + \cos((m + n)x)]$$

For $m \neq n$, both terms integrate to zero over $[-\pi, \pi]$. For $m = n \neq 0$:

$$\int_{-\pi}^{\pi} \cos^2(nx) \, dx = \int_{-\pi}^{\pi} \frac{1 + \cos(2nx)}{2} \, dx = \pi$$

The proofs for eqs. (3) and (4) are similar. □

4 Fourier Series Representation

Theorem 4.1 (Fourier Series). A periodic function f with period 2π can be represented as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)] \quad (5)$$

where the Fourier coefficients are given by:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx \quad (n \geq 0) \quad (6)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx \quad (n \geq 1) \quad (7)$$

Proof. Assume f has the form eq. (5). To find a_m , multiply both sides by $\cos(mx)$ and integrate:

$$\int_{-\pi}^{\pi} f(x) \cos(mx) \, dx = \frac{a_0}{2} \int_{-\pi}^{\pi} \cos(mx) \, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos(nx) \cos(mx) \, dx + \dots$$

By theorem 3.1, all terms vanish except when $n = m$, giving:

$$\int_{-\pi}^{\pi} f(x) \cos(mx) \, dx = a_m \cdot \pi$$

Solving for a_m yields eq. (6). The derivation of b_n is analogous. □

5 Example: Square Wave

Example 5.1 (Square Wave). Consider the square wave defined on $[-\pi, \pi]$:

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$$

Since f is odd, all $a_n = 0$. For the sine coefficients:

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx = \frac{2}{\pi} \int_0^{\pi} \sin(nx) \, dx \\ &= \frac{2}{\pi} \left[-\frac{\cos(nx)}{n} \right]_0^{\pi} = \frac{2}{n\pi} (1 - \cos(n\pi)) \end{aligned}$$

Since $\cos(n\pi) = (-1)^n$, we have $b_n = 0$ for even n and $b_n = \frac{4}{n\pi}$ for odd n . Thus:

$$f(x) = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\sin((2k+1)x)}{2k+1} = \frac{4}{\pi} \left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots \right) \quad (8)$$

6 Conclusion

The Fourier series decomposes periodic functions into a basis of orthogonal trigonometric functions. The orthogonality relations allow us to compute coefficients by projection, analogous to finding components in an orthonormal basis.