Document Title

Author Name

January 15, 2026

Contents

1 Introduction

Fourier series provide a way to represent periodic functions as infinite sums of sines and cosines. This representation is fundamental in signal processing, differential equations, and mathematical physics.

2 Preliminaries

Definition 2.1 (Periodic Function)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is *periodic* with period $T > 0$ if $f(x + T) = f(x)$ for all $x \in \mathbb{R}$.

For simplicity, we consider functions with period 2π, so $f(x + 2\pi) = f(x)$.

Definition 2.2 (Inner Product on L^2)**.** For functions $f, g : [-\pi, \pi] \to \mathbb{R}$, we define the inner product:

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)g(x)\,\mathrm{d}x \tag{1}$$

3 Orthogonality of Trigonometric Functions

The key to deriving Fourier coefficients is the orthogonality of sines and cosines.

Proposition 3.1 (Orthogonality Relations)**.** *For integers $m, n \geq 0$:*

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,\mathrm{d}x = \begin{cases} 0 & m \neq n \\ \pi & m = n \neq 0 \\ 2\pi & m = n = 0 \end{cases} \tag{2}$$

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,\mathrm{d}x = \begin{cases} 0 & m \neq n \\ \pi & m = n \neq 0 \end{cases} \tag{3}$$

$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,\mathrm{d}x = 0 \quad \textit{for all } m, n \tag{4}$$

Proof. We prove eq. (2). Using the product-to-sum identity:

$$\cos(mx)\cos(nx) = \frac{1}{2}\left[\cos((m-n)x) + \cos((m+n)x)\right]$$

For $m \neq n$, both terms integrate to zero over $[-\pi, \pi]$. For $m = n \neq 0$:

$$\int_{-\pi}^{\pi} \cos^2(nx)\,\mathrm{d}x = \int_{-\pi}^{\pi} \frac{1 + \cos(2nx)}{2}\,\mathrm{d}x = \pi$$

The proofs for eqs. (3) and (4) are similar. □

4 Fourier Series Representation

Theorem 4.1 (Fourier Series)**.** *A periodic function f with period 2π can be represented as:*

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx)\right] \tag{5}$$

where the Fourier coefficients are given by:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x \quad (n \geq 0) \tag{6}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x \quad (n \geq 1) \tag{7}$$

Proof. Assume f has the form eq. (5). To find a_m, multiply both sides by $\cos(mx)$ and integrate:

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos(mx)\,\mathrm{d}x + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi}\cos(nx)\cos(mx)\,\mathrm{d}x + \cdots$$

By theorem 3.1, all terms vanish except when $n = m$, giving:

$$\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x = a_m \cdot \pi$$

Solving for a_m yields eq. (6). The derivation of b_n is analogous. □

5 Example: Square Wave

Example 5.1 (Square Wave)**.** Consider the square wave defined on $[-\pi, \pi]$:

$$f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$$

Since f is odd, all $a_n = 0$. For the sine coefficients:

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x = \frac{2}{\pi}\int_{0}^{\pi}\sin(nx)\,\mathrm{d}x \\ &= \frac{2}{\pi}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi} = \frac{2}{n\pi}\left(1 - \cos(n\pi)\right) \end{aligned}$$

Since $\cos(n\pi) = (-1)^n$, we have $b_n = 0$ for even n and $b_n = \frac{4}{n\pi}$ for odd n. Thus:

$$f(x) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\sin((2k+1)x)}{2k+1} = \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right) \tag{8}$$

6 Conclusion

The Fourier series decomposes periodic functions into a basis of orthogonal trigonometric functions. The orthogonality relations allow us to compute coefficients by projection, analogous to finding components in an orthonormal basis.