MATHEMATICAL STATICS
WITH APPLICATIONS
7^{TH} EDITN

WACKERLY • MENDEHALL • SCHEAFFER

COMMON PROBABILITY DIST

APPENDIX

DISCRATE

- BINOMIAL: $p(y) = \binom{n}{y} p^y (1-p)^{n-y}$ $\quad \mu = np \quad \sigma^2 = np(1-p)$
- GEOMETRIC: $p(y) = p(1-p)^{y-1}$ $\quad \mu = \frac{1}{p} \quad \sigma^2 = \frac{1-p}{p^2}$
- HYPERGEO: $p(y) = \dfrac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{y}}$ $\quad \mu = \frac{nr}{N} \quad \sigma^2 = n\left(\frac{r}{N}\right)\left[\frac{N-r}{N}\right]\left[\frac{N-n}{n-1}\right]$
- POISSON: $p(y) = \dfrac{\lambda^y e^{-\lambda}}{y!}$ $\quad \mu = \lambda \quad \sigma^2 = \lambda$
- NEGATIVE BINOM: $p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r}$ $\quad \mu = \frac{r}{p} \quad \sigma^2 = \frac{r(1-p)}{p}$

CONTINUOUS:

- UNIFORM: $f(y) = \dfrac{1}{\theta_1 - \theta_1}$ $\quad \mu = \frac{\theta_1+\theta_2}{2} \quad \sigma^2 = \frac{(\theta_2-\theta_1)^2}{12}$
- NORMAL: $f(y) = \dfrac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(y-\mu)^2\right)$ $\quad \mu = \frac{1}{N}\sum_i y_i \quad \sigma^2 = \frac{1}{N}\sum_i (y_i-\mu)^2$
- EXPONENTIAL: $f(y) = \frac{1}{\beta} e^{-\frac{y}{\beta}}$ $\quad \mu = \beta \quad \sigma^2 = \beta^2$
- GAMMA: $f(y) = \dfrac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-\frac{y}{\beta}}$ $\quad \mu = \alpha\beta \quad \alpha\beta^2 = \sigma^2$
- CHI-SQUARE: $f(y) = \dfrac{y^{\frac{\nu}{2}-1} e^{-\frac{y}{2}}}{2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})}$ $\quad \mu = \nu \quad 2\nu = \sigma^2$
- BETA: $f(y) = \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}$ $\quad \dfrac{\alpha}{\alpha+\beta} \quad \dfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

SAMPLE SPACE & EVENTS 2.1

(1)

$S_1 = \{g, b\}$ → girl or boy

$S_2 = \{$All $7!$ permutations of $(1234567)\}$ → 7 Horse Race winner

$S_3 = \{(H,H), (H,T), (T,H), (T,T)\}$ → 2 Coins

$S_4^A = \{1\ 2\ 3\ 4\ 5\ 6\}$ → 1 Die

$S_4^B = (S_4^A)^2 = \{(i,j) \mid i,j \in 1, 2 \ldots 6\}$ → 2 Die.

(2)

$E = \{g\}$ = Event a Child is born as a girl

$F = \{b\}$ = Event of a child born as a boy.

$E = \{(H,H), (H,T)\}$ = Event of a Coin turns up Head 1^{st}.

$\boxed{E \cup F = \{g, b\}}$ → The whole sample space!

$\cup$: The Union of a sample space

$E_{H^1} = \{(H,H), (H,T)\}$

$E_{H^2} = \{(T,H), (H,H)\}$.

$\boxed{E_{H^1} \cup E_{H^2} = \{(H,H), (H,T), (T,H)\}}$ → $(H,T) \neq (T,H)$

Coin Lands on heads!

→ The event that at least one coin lands on heads!

(3)

$E \cap F = EF$

$E = \{(H,H), (H,T), (T,H)\}$ → 1 head occurs

$F = \{(H,T), (T,H), (T,T)\}$ → 1 tail occurs

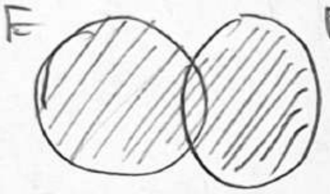
$\boxed{EF = E \cap F = \{(H,T), (T,H)\}}$ → Exactly 1 tails & 1 heads occur

$E \cup F = \{(H,H), (H,T), TT\}$.

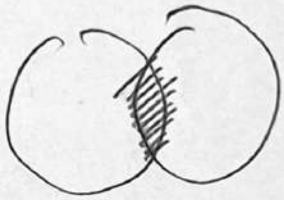
(4) ① $\bigcup_{n=1}^{\infty} E_n$: Event consists of $\boxed{\text{all}}$ outcomes.
that are in E_n for at least one value of $n \in \mathbb{Z}$.

② $\bigcap_{n=1}^{\infty} E_n$: Event consisting of $\boxed{\text{Those}}$ outcomes
that are in all event E_n

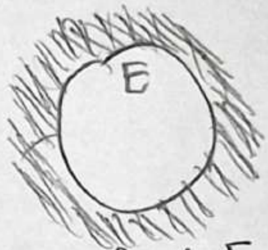
③ E^c : The complement of E.



$E \cup F$



$E \cap F = EF$



$E^c = 1 - E$.

* Examples of E^c;

① $E = \{(1,6), (2,5), (3,4)\ (4,3)\ (5,2)\ (6,1)\} \rightarrow$ Dices Equal to 7

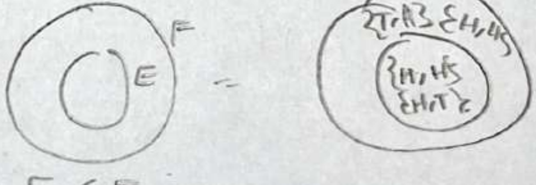
$E^c = 1 - \{(1,6) \ldots (6,1)\}$

= Does not equal to 7.

(5) $\boxed{E \subset F = F \supset E} \rightarrow$ E is contained in F

$E = \{(H,H), (H,T), \}$ $E \subset F =$

$F = \{(T,H), (H,H)\}$



$E \subset F$

SAMPLE POINT METHOD (28)

1. APPLICANT SELECTING

- 2 Applicants Selection / 5 Groups.
- Ranked by Competence: 1, 2, 3, 4, 5.
- 2 events:

A: Employer selects the best and one of the 2 poorest (1,4 or 1,5)
B: Employer Selects at least one of the 2 best

? Find the Probabilities of these events:

ANS,

i. Steps to Conduct

1) Define The Experiment & define how single event will play out (S)
2.) List the simple events associated w/ experiment
3.) Assign probabilities so that $P(E_i) \geq 0$, $\sum P(E_i) = 1$
4.) Define The event of Interest A as a spesific Collection of sample points
5.) Find $P(A)$ by Summing The probabilities of Sample Point A.

ii. Conducting the steps

1.) Experiment Involves randomly selecting 2 applicants out of 5.
Denote the selection of Applicants 3,5 by $\{3,5\}$:
Denote the selection of Applicants 3,5 by $\{3,5\}$.

2.) 10 Simple events $\{i,j\}$ w/ i,j = Applicants:

$E_1 = \{1,2\}$ $E_3 = \{1,4\}$ $E_5 = \{2,3\}$ $E_7 = \{2,5\}$ $E_9 = \{3,5\}$
$E_2 = \{1,3\}$ $E_4 = \{1,5\}$ $E_6 = \{2,4\}$ $E_8 = \{3,4\}$ $E_{10} = \{4,5\}$

3. A random selection gives out:

$$\boxed{P(E_i) = \frac{1}{10} = 0.1} \quad \forall i = 1 \ldots 10.$$

4. B-Occurs when $(E_1 \ldots E_7)$ occurs.

5. Finally:

- $P(B) = \sum_{i=1}^{7} P(E_i) = \sum_{i=1}^{7} 0.1 = 7 \cdot 0.1 = 0.7$
- $P(A) = 0.1 + 0.1 = 0.2$ w/ $\boxed{A \ni \{E_3 \cup E_4\}}$

∴ So:

$$\boxed{\begin{aligned} P(B) &= 0.7 \\ P(A) &= 0.2 \end{aligned}}$$

(2) 3 THROWS PROBLEM

Calculate the probability exactly 2 of 3 tosses Results In heads.

Ans: [i] Experiment consists of observing outcomes (H,T) ∀ tosses of a coin. Simple event of an experiment can be symbolized H,T

[ii] Sample Space:

$$\boxed{\begin{array}{llll} E_1 : (HHH) & E_3 : (HTH) & E_5 : (HTT) & E_7 : (TTH) \\ E_2 : (HHT) & E_4 : (THH) & E_6 : (THT) & E_8 : (TTT) \end{array}}$$

[iii] Because the coin is balanced, we expect them to be equally likely.

$$\boxed{P(E_i) = \frac{1}{8} \quad i = 1 \ldots 8}$$

[iv] Let A = Event that a head comes up EXACTLY

So: $\boxed{\{E_2, E_3, E_4\} = \{(HHT), (HTH), (THH)\}}$

(v) Finally:

$$\boxed{P(A) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}}$$

(3) TENNIS GAMES

(?) Odds two to 1 that (A, B) will play tennis, and A wins. Suppose A and B play 2 matches. Probability A wins at least one match.

Ans: [i] Experiment = winner (A or B) of the 2 matches
so we let = AB = A wins 1st match, B second.

[ii] The sample space is:

$E_1 : AA \quad E_2 : AB \quad E_3 : BA \quad E_4 : BB$

Refer section (2g)

[iii] The assigned probabilities: Let D_1, D_2 denote A wins ($1^{st}, 2^{nd}$) games respectively. So: $P(D_1 \cap D_2) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$. So:

$$\boxed{P(E_1) = \frac{4}{9},\ P(E_2) = \frac{2}{9},\ P(E_3) = \frac{2}{9},\ P(E_4) = \frac{1}{9}}$$

[iv] At least one match: (E_1 & E_2 & E_3)

$= \boxed{\frac{8}{9}}$

EXERCISE 2.5

∘ VOLKSWAGEN FALLACY , ∘ WINE CRITIQUE ∘ CRIMINAL TRIAL SS.

(25) VOLKSWAGEN FALLACY

A single car is randomly selected from all of those registered at a local tag agency. Evaluate the claim "All cars are either Volkswagen or not, so the probability it is $\frac{1}{2}$ for the Volkswagen selected."

Ans: Suppose we have 5 Volkswagens and 3 Mercedes w/ 1 BMW.

{V = Volkswagen, M = Mercedes, B = BMW}

Then: There are 9 cars meaning:

$$\boxed{P(V) = \frac{5}{9} ; P(M) = \frac{3}{9} ; P(B) = \frac{1}{9}}$$

It's valid since: $\sum_i P(i) = 1 \rightarrow \frac{5}{9} + \frac{3}{9} + \frac{1}{9} = \frac{8}{9} + \frac{1}{9} = \boxed{\frac{9}{9}}$ $\boxed{1 = 1}$

$\therefore$ $\boxed{\text{So NO! It's not}}$ → unless exactly $\frac{1}{2}$ of all the cars in the lot are Volkswagens

(26) WINE IMPORT CRITIQUE

3 Imported wines are to be ranked from lowest to highest by a wine expert. Ranks are {Best, 2nd Best, Worst}.

(b.) Describe sample s

(a.) Describe one sample point of this experiment.

Ans: Let $(W_1, W_2 W_3)$ be the wines that are imported. Then in accordance (Best, 2nd, worst)

3 Scenarios play out to Rank

$E_1 : (W_1 W_2 W_3)$ $\quad E_3 : (W_1 W_3 W_2)$ $\quad E_6 = (W_3 W_1 W_2)$

$E_2 : (W_2 W_1 W_3)$ $\quad E_4 = (W_2 W_3 W_1)$

$E_5 = (W_3 W_2 W_1)$

$3! = 6$

(b.) Suppose the "Expert" merely guesses the quality. One of wine is much better quality. Probability expert ranks the best wine as the no. worst than 2nd best?

Ans. Let W_1 = Best wine. Then we have the cases of $\{E_1\ E_2\ E_6\ E_3\}$ = {cases that either Rank 1 and Rank 2}

So: $P\{\text{case either Rank 1 or 2}\} = 4 \cdot \frac{1}{6} = \frac{4}{6} = \boxed{\frac{2}{3}}$

(2g.) CRIMINAL TRIAL & JUROR SAMPLE POINT

Two additional jurors are needed to complete a jury for a criminal trial. There's 6 probative jurors, 2 women 4 men. 2 jurors are selected from the 6 that's available:

(a) Define the experiment and describe one sample point. Assume you only need to describe the 2 jurors chosen and not the order they were selected.

ANS; Let $(M_1 M_2 M_3 M_4 W_1 W_2)$ be the set of men and women. Then there's $6! = 720$ ways. However since we only need 2 then:

$$\binom{6}{2} = \frac{6}{2!(6-2)!} = \frac{6!}{2!4!} = \boxed{15 \text{ ways}}$$

(b) List the sample space:

ANS. $E_1 = (M_1, M_2)$
$E_2 = (M_1, W_1)$ ...... E_{15}.

(c) Probability both jurors are women?

ANS $\boxed{P(W_1, W_2) = \frac{1}{15}}$

EXERCISES 2.5

0 MEDIAN FAMILY INCOME

(31.) BUREAU CENSUS & MEDIAN FAMILY INCOME.

- The Bureau of The Census reports that the Median Family Income for 2003 In the U.S is \$43.318.
- So half has Above this/equal or low/equal This Amount
- Suppose 4 families are Surveyed and that each one reveals whether it's Income exceeded \$43.318 in 2003

(a) List the Sample Space:

ANS.. Let A B C D be the 4 families and (L), M (s) to be the event that Their Incomes are Less and/or equal and More equal to \$43.318 Resp. Then:

$E_1 = \{A_L\ B_L\ C_L\ D_L\}$ ... $E(16): \{A_M B_M C_M D_M\}$

$\underbrace{\quad 2 \quad 2 \quad 2 \quad 2 \quad}$ $4 \times 4 = \boxed{16}$ events. or $\boxed{2^4 = 16}$

E	A	B	C	D
1	M	M	M	M
2	M	M	M	L
3	M	M	L	L
4	M	L	L	L
5	L	L	L	L
6	L	L	L	M
7	L	L	M	M
8	L	M	M	M
9	L	M	L	M
10	L	M	L	L
11	L	L	M	L
12	M	L	M	L
13	M	M	L	M
14	M	L	M	M
15	L	M	M	L
16	M	L	L	M

(b) Identify The simple events In each of the following events

A : At least 2 had Income exceeding 43.318
B : Exactly 2 had Incomes exceeding 43.318
C : Exactly 1 had Incomes Less/equal 43.318

ANS :

A: $\begin{pmatrix} E_1 & E_7 & E_{13} & E_{14} \\ E_2 & E_8 & E_{14} & E_{12} \\ E_3 & E_9 & E_{15} & \end{pmatrix}$ = 11 events

B: $(E_3\ E_7\ E_9\ E_{12}\ E_{15}\ E_{16})$ = 6 events

C: $(E_2, E_8, E_{13}, E_{14})$ = 4 events

(c) compute P(A), P(B), P(C). Sol.

ANS:

$P(A) = \frac{11}{16}$

$P(B) = \frac{6}{16} = \frac{3}{8}$

$P(C) = \frac{4}{16} = \frac{1}{4}$

so

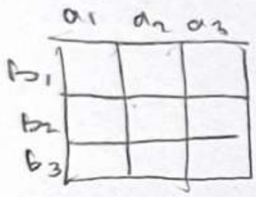
$P(A) = \frac{11}{16}$

$P(B) = \frac{3}{8}$

$P(C) = \frac{1}{4}$

TOOLS FOR SAMPLE POINTS (26)

(1.)



M×n Rule:

With m elements $a_1 \ldots a_m$ and n elements $b_1 \ldots b_n$, It's possible to form $\boxed{mn = m \times n}$ pairs containing one element from each group

[i] DICE EXAMPLE

Experiment of pair of dices thrown, w/ observation on the faces:

1st Die = $\{a_1 \ldots a_6\}$
2nd Die = $\{b_1 \ldots b_6\}$

So: $\boxed{m = n = 6 \Rightarrow 6 \times 6 = 36}$

[ii] COIN TOSSING EXPERIMENT

Suppose 3 coins and we want to find the probability exactly 2/3 tosses are heads. Then:

1st coin = $\{H, T\}$: $n = 2$
2nd coin = __,__ : $m = 2$
3rd coin = __,__ : $p = 2$

So: $\boxed{m \times n \times p = 2 \times 2 \times 2 = 8}$

[iii] Birthday problem

Suppose 20 randomly selected persons has distinct birthdays, w/ 365 possible ones. To find the number of sample points:

1.)

Days: $\{1 \ldots\ldots 365\}$

1.) $\boxed{E = \{\text{Person}(i)\}_{i=1}^{20}}$ wh/ 1st number = 1st person's bday, 2nd ... = 2nd person's bday

So: Person(1) : 365 possible birthdays
Person(2) : 365 possible birthdays

So by the m×n Rule: Person(1) × Person(2) ..... Person(n)

$$= \boxed{\prod_{i=1}^{n=20} \text{Person}(i) = \boxed{(365)^{20}}}$$

Look:
1st person = 365 (any days)
2nd person = 364 days left
3rd person = 363 days left
20th ... = 346 days
(19 days have been taken)!

2.) So by permutation, we can find the probability A

$$\boxed{P(A) = \frac{P^{365}_{20}}{(365)^{20}}} = \frac{365 \times \cdots 364}{365} = \boxed{0.5886}$$

(2.)

$$\boxed{P^n_r = \frac{n!}{(n-r)!}}$$

(PERMUTATION)

[i] PROOF:

- We want to fill r-positions w/ n-distinct objects
- Applying the extension of the $\boxed{m \times n \text{ Rule}}$, we see the 1st object can be chosen n ways
- The 2nd object is left $\boxed{(n-1)}$ ways --
- The 3rd object is left $\boxed{(n-2)}$ --
- Continuing, The r^{th} object is $\boxed{(n-r+1)}$ * Remember the 20 Patio Birthday Probby

So: $\boxed{P^n_r = n(n-1)(n-2)\cdots(n-r+1)}$

expressed in factorials: $P^n_r = n(n-1)\cdots(n-r+1)\frac{(n-r)!}{(n-r)!} = \boxed{\frac{n!}{(n-r)!}}$ QED

[ii] 3/30 EMPLOYEE PROBLEM

- 3 employees drawn from 30 employees.
- 1st name drawn wins \$100, 2nd \$50, 3rd \$25 ......
- ? How many sample points are Associated w/ experiment:

Ans: Using $P^n_r = P^{n=30}_{r=3}$ so: $\boxed{P^{30}_3 = \frac{30!}{27!} = \frac{30\times 29\times 28\times \cancel{27!}}{\cancel{27!}} = 24.360}$

(3.)

$$\boxed{N = \binom{n}{n_1 \cdots n_k} = \frac{n!}{n_1! \cdots n_k!}}$$ -> MULTINOMIAL COEFFICIENT.

N = number of distinct arrangement of n objects so Partitioning n distinct objects Into k groups

$$\boxed{\left(\sum_k y_k\right)^n = \sum_k N \prod_k y_k^{n_k}}$$

[i] LETTER Arrangement

$\boxed{abc|defg|hijk|}$

- We can see that $(n_1 = abc : 3)$, $(n_2 : defg : 4)$ and $(n_3 : hijk) : 4$.
- So The Permutation of these distinct groups whilst ignoring the order IS. P^n_n w/ orderings of $n_1 \cdots n_k$ ~~groups~~ elements ∀group
- by the m×n:

$$P^n_n = (N)\cdot(n_1! \cdots\cdots n_k!)$$

$$N = \frac{n!}{n_1!\cdots n_k!} \equiv \binom{n}{n_1 \cdots n_k}$$

[ii] LABOUR DISPUTE

- How do we assign 20 laborers to Jobs that group 1 until 4 needs (6, 4, 5, 5) Labors Respectively? well:

$$N = \binom{20}{6\,4\,5\,5} = \boxed{\frac{20!}{6!4!5!5!}}$$

- IF 4 laborers are Already put In the 1st Job: $\binom{20-4}{(6-4)!4!5!5!} = \boxed{\frac{16!}{2!4!5!5!}}$

EXERCISES 2.6

35 NY CA HWI FLIGHT problem

Airline w/ 6 flights from NY to CA, 7 flights from CA to HWI. If flight Is made on seperate days, (?) how many diffrent flight arrangement can be made from NY to Hawaii.

NY 6 CA 7 HWI

ANS

$$NY - CA : \{a_1 \dots a_6\} = 6 \text{ flight sequence}$$
$$CA - HWI = \{b_1 \dots b_7\} : 7 \text{ flight sequence}$$
$$NY - HWI = m \times n = 6 \times 7 = \boxed{42 \text{ Flights}}$$

36 Assembly OPERATIONS

(?) Assembly operation requires 3 steps that can be performed In any sequence. How may diffrent ways Can the assembly be performed?

ANS:

$$E_1 = S_1 S_2 S_3$$
$$E_2 = S_1 S_3 S_2$$
$$E_3 = S_2 S_1 S_3$$
$$E_4 = S_2 S_3 S_1$$
$$E_5 = S_3 S_2 S_1$$
$$E_6 = S_3 S_1 S_2$$

$\boxed{6 \text{ ways}}$ OR $\boxed{3! = 6 \text{ ways}}$

37 PHILADELPHIA BUSINESS WOMAN

Business woman In philadelphia Is preparing an Itenary for a visit to 6 major Cities. The distance travelled will depend on the trips that she plans:

(a) How many diffrent Itenaries (and trip costs) are possible?

ANS: $\boxed{6! = 720 \text{ ways}}$ 6 major cities

(b) Suppose 1 of 2 cities picked which is Denver and San Fransisco. What's the Probability she'll visit Dener befor SF?

w/ 2 = 2 cities Dener & San frasisco

ANS: $\boxed{P(A) = \frac{1}{2} = 0.5}$

w/ 360 Denver befor Sanfransiso $\rightarrow \frac{720}{2} = 360$

360 San fransisco before denver

38 APPETIZERS!

In a fixed entry menu w/ 4 appetizers, 3 salads, 4 entrees, 5 deserts. how may diffrent dinners IS available If The dinner Consists of 1 appetizer, 1 salad, 1 entree, 1 desert?

Ans:

Appetizers : $\{a_1, a_2, a_3, a_4\}$: $m = 4$

Salad : $\{s_1, s_2, s_3\}$: $n = 3$

Entree : $\{e_1, e_2, e_3, e_4\}$ = $p = 4$

Desserts : $\{d_1, d_2, d_3, d_4\}$ = $q = 5$

by the M×N Rule: $m \times n \times p \times q = 4 \times 3 \times 4 \times 5 = \boxed{240}$

(40.) CUSTOM CAR

Brand of Automobile comes w/ 5 styles, 4 engines, 2 transmissions, 8 colors

(a) How Many will the dealer have to stock If he Included one for each engine transmission Combination?

Ans: using M×N Rule only regarding the Engine & Transmis.

$\boxed{5 \times 4 \times 2 = 40 \text{ stocks}}$ → No color!

(b) Now How many would a distribution center have to carry if all colors of cars were stocked for each combination in part (a)?

Ans: $\boxed{5 \times 4 \times 2 \times 8 = 160 \text{ stocks}}$ → yes color!

(39) DIE SUM 7

In a tossing of a Pair of die, what's the probability that the sum appearing on the die is 7?

Ans: Die 1 = $\{a_1 \ldots a_6\}$: 6 = m

Die 2 = $\{b_1 \ldots b_6\}$: 6 = m

36 Sample Space possibilities

possible sums of 7: (6,1)(1,6)(5,2)(2,5)(4,3)(3,4)

so: $\boxed{P(i+j=7) = \frac{6}{36} = \frac{1}{6}}$

(43) TAXI FLEET W/AIRPORT

Fleet of taxis Is to be dispatched to 3 airports In such way 3 go to airport A, 5 go to B, and 1 goes to C. How Many distinct Ways can this be accomplished?

$\boxed{\binom{9}{3\ 5\ 1}}$

ANS: Total taxis = 9

Total Airports = 3

Grouping: $T_1 T_2 T_3 \mid T_4 T_5 T_6 T_7 T_8 \mid T_9 \mid$

$= \boxed{\frac{9!}{3!\,5!\,1!}} = 504$

EXERCISES 2.6

(49) FLORIDA UNIVERSITY APPLICANTS

- 130 Major Areas of study can be selected from study at the University at Florida
- Student's major is identified in the registrar's record w/ 2 or 3 letter code (ex: Stats Major = STA, Math Major = MS)
- Some students opted for a double major and complete the requirements for both majors before graduation
- Registrar was asked to consider assigning these double major a distinct 2 or 3 letter code, so that it can be identified.

(a) What's the maximum number of possible double majors available to university of Florida students?

ANS: • Students can either be double majors or single majors from the set of 130 studies. If we can simplify from 1 to 3 study by

1 study = 0 doubles ⟺ $\{x_1\}$
2 study = 1 doubles ⟺ $\{x_1, x_2\}$
3 study = 3 doubles ⟺ $\{x_1, x_2, x_3\}$ → $E_1 = x_1, x_2$; $E_2 = x_2, x_3$; $E_3 = x_1, x_3$

• Thus:

$$130 \text{ study} = \binom{130}{2} = \frac{130!}{128!\,2!} = \boxed{8.385} \text{ double majors}$$

to identify the majors or double majors.

(b) If any 2-or-3 letter code to identify the majors are available? How many major codes are available?

ANS: Suppose we picked from the alphabet $\{a \ldots z\}$. Then there's 26. So:

2 letter code $= 26^2 = 17.576$
3 letter code $= 26^3 = 676.$ +

∴ $\boxed{\text{Total for all} = 18.252}$

(c) How many major codes are required to identify students who have either a single major or a double major?

ANS: There's a maximum of 8.385 studies that can be double majors. This means that there's an additional 130 students who are single majors. Thus: $\boxed{(8.385 \text{ doubles} + 130 \text{ singles}) = 8515 \text{ required}}$

(d) Are there enough major codes to identify all the majors at the University of Florida?

ANS: $\boxed{\text{Yes!}}$. Why? Well there's 18.252 possible combinations and only 8.515 maximums of doubles / single majors. So yeah!

(80) RIGGING THE 1980 STATE LOTTERY IN PENNSYLVANIA

- Determining 3 digit numbers to win w/ numbers {1 ... 9}
  Placed on an one Ping-Pong ball
- 10 balls are blown into a Compartment, and the number selected for the digits is one of the balls that floats to the top of machine
- To alter the odds, The Conspirators injected a liquid into all balls used in the game except those numbers (4,6)
- By This, It's almost Certain that the lighter balls are selected and determine The digits In the winning numbers
- Then They bought lottery tickets bearing the potential winning numbers. How many potential winnings were there (winner is 666)

Ans:

- There were 10 balls w/ digits {0, ..., 9} and 3 balls were picked to satisfy a winning digit.
- So there were like $\binom{10}{3} = \frac{10!}{3!\,7!} = \boxed{120 \text{ ways}}$.
- Since (4,6) Is Lighter, Then the grouping of balls is:

  4 6 | 1 2 3 8 5 9 7 0

- Meaning that there's only $2 \times 2 \times 2 = 2^3 = \boxed{8 \text{ ways}}$ The ball appears, That is {444, 464, 646, 466, 664, 446, 666}
- hence, that's why the lottery winner is 666

CONDITIONAL PROBABILITY (2.9)

- Florida Sport fisher man Interested In Rain
- Probability of rain given two days of rain succession for tommorow ? i.e $P(\text{Rain tommorow} \mid \text{Rain yestarday})$?

and $P(\text{Rain}) = \dfrac{\text{days occured}}{\text{period of time}}$.

(1) $$\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$$

[I] simple example:

$A = \{1\ 2\ 3\}^n$: $n = 3$ so: $P(A) = \frac{1}{3}$

$B = \{\cancel{3}\ 2\ 6\}_m$: $m = 3$ so: $P(B) = \frac{1}{3}$

$A \cap B = \{3, 2\}_j$: $|j| = 2$ $P(A \cap B) = \frac{1}{2}$

so: $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{\frac{1}{2}}{\frac{1}{3}} = \dfrac{1}{2} \cdot \dfrac{3}{1} = \cancel{\boxed{\frac{3}{2}}}$? Wrong $p = 1.5 > 1 = P$

OR: $\dfrac{2}{\cancel{4}} \cdot \dfrac{\cancel{4}}{3} = \boxed{\dfrac{2}{3}}$

Instead: $P(A) = \dfrac{|A|}{|U|}$ $P(B) = \dfrac{|B|}{|U|}$ $P(C) = \dfrac{|C|}{U}$ w/ $C = A \cap B$.

so: $P(A \cap B) = \dfrac{2}{|U|}$ $P(B) = \dfrac{3}{|U|}$.

and: $$\boxed{P(A \mid B) = \frac{\frac{2}{|U|}}{\frac{3}{|U|}} = \frac{2}{\cancel{|U|}} \cdot \frac{\cancel{|U|}}{3} = \boxed{\frac{2}{3}}}$$

Makes more sense!, The probability since $|U| > \{A, B, C, \ldots \text{etc}\}$

[II] LARGE NUMBERS

Suppose an experiment is repeated w/ Large number N of times resulting in both A & B, $A \cap B$ n_{11} times, $\bar{A} \cap B$ n_{21}, $\bar{A} \cap B$, n_{12} times, $\bar{A} \cap \bar{B}$, n_{22} times. Then:

$P(A) = \dfrac{n_{11} + n_{21}}{N}$ $P(B) = \dfrac{n_{11} + n_{12}}{N}$ $P(A \mid B) : \dfrac{n_{11}}{N}$.

Table

	A	$\bar{A}$	
B	n_{11}	n_{12}	$n_{11}+n_{12}$
$\bar{B}$	n_{21}	n_{22}	$n_{21}+n_{22}$
	$n_{11}+n_{21}$	$n_{12}+n_{22}$	N

$P(B \mid A) = \dfrac{n_{11}}{n_{11} + n_{21}}$ $P(A \cap B) = \dfrac{n_{11}}{N}$

so: $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{n_{11}/N}{\frac{n_{11}+n_{21}}{N}} = \dfrac{n_{11}}{\cancel{N}} \cdot \dfrac{\cancel{N}}{n_{11}+n_{21}} = \boxed{n_{11} / n_{11}+n_{21}}$ ✓

[ii] Dice example

(3) Supposed a balance die is tossed once. Use Definition 2.4 to find the probability of a 1 given an odd number was obtained.

Ans:

$A = \{\text{Event of } 1\} = \{1 \ldots 6\} = 6$

$B = \{\text{ODD}\} = \{1, 3, 5\} = 3$

so. $P(A) = \frac{1}{6}$ and $A \cap B = \{1, 3, 5\} = \frac{3}{6} = \frac{1}{2}$

$P(B) = \frac{3}{6} = \frac{1}{2}$

so: $P(A \cap B) = P(B)$

so: $\boxed{\frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{6} \cdot \frac{2}{1} = \frac{2}{6} = \boxed{\frac{1}{3}}}$ ✓

(2.) Suppose the probability of A isn't affected by the event B. So, AND, those probabilities can be independent. $P(A)$ & $P(B)$ is independent.

Iff:

!! $\boxed{\begin{aligned} P(A \mid B) &= P(A) \\ P(B \mid A) &= P(B) \\ P(A \cap B) &= P(A)P(B) \end{aligned}}$

⚠ BIAS EXAMPLE : P(smoking) and P(Lung cancer) are actually independent but most people think it is!

[i] Dice example:

A = observing an odd number : $\{1, 3, 5\}$

B = observing an even number : $\{2, 4, 6\}$

C = observe a 1 or 2. : $\{1, 2\}$

$A \cap B = \emptyset$

$B \cap A = \emptyset$

so: $P(A) = 3/6$ $\quad P(C) = P(\{1, 2\}) = \frac{2}{6}$

$P(B) = 3/6$

and: $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{0}{3/6} = 0$ ✓

$P(B \mid A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)} = \frac{0}{3/6} = 0$ ✓.

and: $P(A \cap B) = P(A)P(B) = \frac{3}{6}\frac{3}{6} = \frac{9}{6} = \frac{3}{2} \neq 0$ ✗

by which, a contradiction! So B & A are

$\boxed{\text{DEPENDENT EVENTS}}$

EXERCISES 2.8

71. Simple Example

suppose $P(A) = 0.5$, $P(B) = 0.3$, and $P(A \cap B) = .1$

Find:

(a) $P(A|B)$ (b) $P(B|A)$ (c) $P(A|A\cup B)$ (d) $P(A|A\cap B)$ (e) $P\left(\frac{A\cap B}{A\cup B}\right)$

Ans: observe: $P(A) = \frac{50}{100} \rightarrow \{A_m\} =$ 5 members

$P(B) = \frac{30}{100} \rightarrow \{B_n\} =$ 3 members

so: $U =$ 10 members and $P(A\cap B) = \frac{1}{10} =$ 1 member

so: (a) $P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{1}{10}}{\frac{3}{10}} = \frac{1}{\cancel{10}} \cdot \frac{\cancel{10}}{3} = \boxed{\frac{1}{3}}$

(b) $P(B|A) = \frac{P(B\cap A)}{P(A)} = \frac{\frac{1}{10}}{\frac{5}{10}} = \frac{1}{\cancel{10}} \cdot \frac{\cancel{10}}{5} = \boxed{\frac{1}{5}}$

(c) $P(A|A\cup B) = \frac{P(A\cap(A\cup B))}{P(A\cup B)} = \frac{\frac{5}{10}}{\frac{7}{10}} = \frac{5}{10} \cdot \frac{10}{7} = \boxed{\frac{5}{7}}$

* Recall that: $A\cup B = \{A_m\} \cup \{B_n\} = 8$ members. $- A\cap B$

so: $A\cup B = A + B - (A\cap B) = \boxed{\frac{7}{10}}$ and $A\cap(A\cup B) = A\cap\{A_m, B_n\}$

$= A$. so $P(A) = \frac{5}{10}$

(d) $P(A|A\cap B) = \frac{P(A\cap(A\cap B))}{P(A\cap B)} = \frac{\frac{1}{10}}{\frac{1}{10}} = \frac{1}{10} \cdot \frac{10}{1} = \boxed{1}$

$A\cup B = A + B - A\cap B$.

(e) $P(A\cap B|A\cup B) = \frac{P((A\cap B)\cap(A\cup B))}{P(A\cup B)} = \frac{\frac{1}{10}}{\frac{8}{10}} = \frac{1}{10} \cdot \frac{10}{8} = \boxed{\frac{1}{7}}$

* Recall that: $A\cap B = \frac{1}{10} =$ 1 member
$A\cup B = \frac{8}{10} =$ 8 members

so: $(A\cap B)\cap(A\cup B)$
$= A\cap(B\cap A)\cup B$
$= (A\cap B)\cap A\cup B$
$= \frac{1}{10}$

(72) JUB Competency EXAM

%	Sex		total
Output	M	F	
A	24	36	60
$\bar{A}$	16	24	40
Total	40	60	100

w/ M = male, F = female
A = pass, $\bar{A}$ = fail

(a) Are A and M Independent?

Ans:

$P(A) = 60\%$

$P(M) = \frac{40}{100} = 40\%$

$P(A \cap M) = 24\%$

so: $P(A|M) = \frac{24\%}{40\%} = 60\% = P(A)$

$P(M|A) = \frac{24\%}{60\%} = 40\% = P(M)$

Thus both are Independent

so: $\boxed{P(A) \neq P(M)}$

(b) Are $\bar{A}$ and M Independent?

Ans:

$P(\bar{A}) = 40\%$

$P(F) = 60\%$

$P(\bar{A} \cap F) = 24\%$

so: $P(\bar{A}|F) = \frac{24}{60\%}\% = 40\% = P(\bar{A})$

$P(F|\bar{A}) = \frac{24}{40\%}\% = 60\% = P(F)$

so: $\boxed{P(\bar{A}) \neq P(F)}$

The Probability of failure is The same as the fraction of The female of The population

(c)+ How about A and F? and $\bar{A}$ & M?

(i)

$P(A) = 60\%$

$P(F) = 60\%$

$P(A \cap F) = 36\%$

so: $\frac{36\%}{60\%} = \boxed{0.6} = \boxed{60\%}$ = females

so: $P(A) = P(F)$

(ii)

$P(M) = 40\%$

$P(\bar{A}) = 40\%$

$P(\bar{A} \cap F) = 16\%$

so: $\frac{16\%}{40\%} = 0.4 = \boxed{40\%}$ = Males

BINOMIAL DISTRIBUTION (34)

(1)

$$\boxed{p(y) = \binom{n}{y} p^y q^{n-y}}$$

[i] Philosophy: $\underbrace{SSSSS \ldots SSS}_{y}\underbrace{FFF \ldots FF}_{n-y}$ w/ S = successes, F = failures

$$p(SSS \ldots S)p(F \ldots FF)$$
$$= pppp \ldots p(1-p)(1-p) \ldots (1-p) = \underbrace{pppp \ldots p}_{y \text{ times}}\underbrace{qqq \ldots q}_{(n-y) \text{ times}}$$
$$= p^y q^{n-y} \Rightarrow \binom{n}{y} p^y q^{n-y} \quad \text{w/ } \binom{n}{y} = \text{The number of distinct n-tuples}$$

[ii]
$$\boxed{\begin{aligned} \mu &= E(Y) = np \\ \sigma^2 &= V(Y) = np(1-p) \end{aligned}}$$

Proof
$$E(Y) = \sum_y y\,p(y) = \sum_{y=0}^{n} y\binom{n}{y} p^y q^{n-y} = \sum_{y=1}^{n} y \frac{n!}{n!(n-y)!} p^y q^{n-y}$$

1)
$$= np \sum_{i=1}^{n} \frac{(n-1)!}{(n-y)!(y-1)!} p^{y-1} q^{n-y} = np \sum_{i=1}^{n} \frac{(n-1)!}{(n-1-z)!\,z!} p^z q^{n-1-z}$$
$$= np \sum_{i=1}^{n} \binom{n-1}{z} p^z q^{n-1-z} = np(1) = \boxed{np}$$

2) using Moments:
$$M_Y(t) = E[e^{tY}] = \sum_{y=0}^{n} e^{tY} p(Y) \quad \text{w/ } p(Y) = p(Y=y) = p(y)$$

so:
$$M_Y(t) = \sum_{i=1}^{n} e^{tY} \sum_{i=1}^{n} \binom{n}{y} p^y q^{n-y} = \sum_{1}^{n} e^{ty}\binom{n}{y} p^y q^{n-y}$$
$$= (pe^{tY} + q)^n$$

w/ $\sum_y \binom{n}{y} p^y q^{n-y} = (p+q)^n$

&
$$\frac{dM_Y(t)}{dt} = n(pe^t + q)^{n-1} pe^t$$
$$\frac{d^2M_Y(t)}{dt^2} = n(n-1)(pe^t + q)^{n-1}(pe^t)^2 + n(pe^t + 1 - p)^{n-1} pe^t$$

Let
$$\begin{cases} \frac{dM}{dt} = M'_t \\ \frac{d^2M}{dt^2} = M''_t \end{cases} \Rightarrow \begin{cases} M'(0) \\ M''(0) \end{cases}$$

Then:
$$M'(0) = n(p+q)^{n-1} p = np$$
$$M''(0) = n(n-1)p^2 + np = n^2p^2 - p^2 + np$$
$$= (n^2-1)p^2 + np = p((n^2-1)p + n)$$
$$=$$

POISSON PROBABILITY 3-8

(1) THEORY: • Automobile accidents in particular day are recorded. $t \in [0,T]$

• $P(\text{No accidents}) = 1-p$

• $P(\text{Accidents}) = p$

DERIVATION:

Say $\boxed{\lambda = np}$, $\boxed{p = \frac{\lambda}{n}}$

$$\lim_{n\to\infty} \binom{n}{y} p^y (1-p)^{n-y} = \lim_{n\to\infty} \frac{n(n-1)\cdots(n-y+1)}{y!} \left(\frac{\lambda}{y}\right)^y \left(1-\frac{\lambda}{n}\right)^{n-y}$$

$$= \lim_{n\to\infty} \frac{\lambda^y}{y!}\left(1-\frac{\lambda}{n}\right)^n \frac{n(n-1)\cdots(n-y+1)}{} \left(1-\frac{\lambda}{n}\right)^{-y}$$

$$= \frac{\lambda^y}{y!} \lim_{n\to\infty} \left[1-\frac{\lambda}{n}\right]^n \left[1-\frac{\lambda}{n}\right]^{-y} \left\{\left[1-\frac{1}{n}\right]\left[1-\frac{2}{n}\right]\left[1-\frac{3}{n}\right] \times \cdots \left[1-\frac{y-1}{n}\right]\right\}$$

λ = Rate of frequency occurs, n = frequency

$$= \frac{\lambda^y}{y!} \lim_{n\to\infty} \left\{\left[1-\frac{\lambda}{n}\right]^n \left[1-\frac{\lambda}{n}\right]^{-y}\right\} \left[\prod_k \left(1-\frac{k}{n}\right)\right]_{n\to\infty}$$

$$= \frac{\lambda^y}{y!} e^{-\lambda} \cdot \left[\prod_k \left(1 - \lim_{n\to\infty}\frac{k}{n}\right)\right] = \frac{\lambda^y}{y!} e^{-\lambda} \left(\prod_k (1-0)\right)$$

$$= \frac{\lambda^y}{y!} e^{-\lambda}$$

we

$$\binom{n}{y} = \left(\frac{n}{n-y}\right)! = \prod_{n=0}^{y} \frac{n+1-y}{n} = \frac{\prod_y (n+1-y)}{\prod_y n} = \frac{n(n-1)\cdots(n+y-1)}{n^y}$$

def: $\boxed{\text{poiss}(y) = \lim_{n\to\infty} p^y (1-p)^{n-y} \binom{n}{y} = \frac{\lambda^y}{y!} e^{-\lambda}}$

* Poisson series $e = 2.718\ldots$

MOMENTS:

• $\boxed{\begin{aligned} \mu &= E(y) = \lambda \\ \sigma^2 &= V(y) = \lambda \\ \mu &= \sigma \end{aligned}}$

* Moments of Poisson

• $E(y) = \sum_y y \frac{\lambda^y e^{-y}}{y!}$

$$= y\sum_y \frac{y\lambda^y e^{-y}}{y(y-1)!} = \sum_y \frac{\lambda^y e^{-y}}{(y-1)!} = \lambda \sum_z \frac{\lambda^z e^{-\lambda}}{z!}$$

why $y-1 = z$, $y = z-1$

$$= \lambda e^{-\lambda} \sum_z \frac{\lambda^z}{z!} = \lambda e^{-\lambda} e^{\lambda} = \lambda e^{-\lambda+\lambda} = \lambda(1) = \boxed{\lambda}$$

• $$\sigma^2(y) = E(y^2) - E(y)^2 = E(y^2) - \lambda^2 = \sum_y y^2 \frac{e^{-\lambda}\lambda^y}{y!} = e^{-\lambda}\sum_y y^2 \frac{\lambda^y}{y!}$$

$$= e^{-\lambda} \sum \frac{(y(y-1)+y)\lambda^y}{y!} = e^{-\lambda}\left\{\sum_y \frac{y(y-1)\lambda^y}{y!} + \sum_y \frac{y\lambda^y}{y!}\right\}$$

Use the fact that: $y^2 + y = y(y-1) + y$

$$= e^{-\lambda}\left\{\sum_y \frac{\lambda^y}{(y-2)!} + \sum_y \frac{\lambda^y}{(y-1)!}\right\} = e^{-\lambda}\left\{\sum_{j_2} \frac{\lambda^{j_2+2}}{j_2!} + \sum_{j_1} \frac{\lambda^{j_1+2}}{j_1!}\right\}$$

Let $j_2 = y-2$, $j_1 = y-1$, Being $j_1 = j_2$?

$$= e^{-\lambda}\left\{\lambda^2 \sum_j \frac{\lambda^j}{j} + \lambda \sum_j \frac{\lambda^j}{j}\right\}$$

$$= e^{-\lambda}e^{\lambda}\left\{\lambda^2 + \lambda\right\} - \lambda^2 = \lambda^2 + \lambda - \lambda^2 = \boxed{\lambda}$$

(2) EXAMPLES: 3.122 INDUSTRIAL ACCIDENTS

Problem:

• Industrial accidents are Poisson process

• Last 2 months, 10 accidents occur

?/ Is highly improbable that mean number of accidents per month $\mu = 3$?

- $\lambda = np$ and: $\lambda^* = 2 \text{ month} \cdot 3 \frac{\text{Accidents}}{\text{month}} = 6$ Accidents
- $Poiss(y) = \sum_{n}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} = \sum_{n}^{\infty} \frac{6^y e^{-6}}{y!} = 0.06 \sum \frac{6^y}{y!}$

- $E(y) = \lambda = 6$ $\qquad = 0.06 \sum \frac{6^y}{y} = 0.06\, e^{\lambda} = 0.06 \cdot e^{6} = 0.06 \cdot 6$
- $\sigma(y) = \sqrt{6} = 2.45$ $\qquad = \boxed{0.97}$ but $\sum_{y=10} \frac{6^y e^{-6}}{y} \approx$
- $\mu \pm 2\sigma = \begin{cases} 6 + 2.45 \cdot 2 = 10.90 \\ 6 - 2.45 \cdot 2 = 1.1 \end{cases}$
- $\frac{E(\lambda)}{\sigma(y)} = \frac{6}{2.45} = 2.44$

• Does this indicate an Increase on Nursery Accidents? No since it's truly impossible, due it doesn't live more than 2σ.

3-20) TREE SEEDLINGS:

Problem:
- Seedlings are Randomly dispersed w/ mean density 5 per square yard.
- Suppose it counts a 10-square yard. ?) More regions will contain?

ANS: $\lambda = 5$

- $P(y=0) = p(0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-5} = 0.006 = 0.6\%$ } very small.

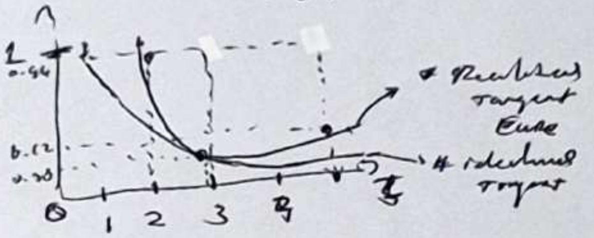
EXERCISES 3.8

(122) CUST. ARRIVAL I QUESTION:

- $\lambda = 7$ hour per Arrival. • Given Methods:

>> $P(Y \leq 3)$: at least / no more 3
>> $P(Y \geq 2)$: at least 2
>> $P(Y = 5)$: exactly 5

ANS: $\forall \lambda = 7 \begin{cases} P(Y \leq 3) = 0.0818 \\ P(Y \geq 2) = 0.9927 \\ P(Y = 5) = 0.1277 \end{cases}$



(123) CUST ARRIVAL II : QUESTION:

- Rest $\lambda \sim 10$ minutes. • (μ, σ^2) has total scheme for customers during 1-hour. $\lambda = \frac{10 \text{ minutes}}{60 \text{ minutes}} = \frac{1}{6}$ hours

?/ • Likely to exceed 25 ?

ANS: • $\boxed{\lambda = \mu_y = \sigma_y^2}$ so $P(Y \geq 2.5) = e^{\frac{1}{6}}$

• in minutes: $E_1(y) = 10 \quad \sigma_1 = 100$
$E_2(y) \sim 7 \quad \sigma_2 = 700$

(124) PROMOTION & SUPPLIES : QUESTIONS:

- Store owner overstocks and then ad decrease graph
- Item is marked \$100 price
- current Reduces price by factor \$ $\frac{1}{2}$
- 1st Customer price \$50, 2nd customer price \$25.
- Suppose number of cust has Poiss $\sim (\lambda = 2)$

?/ • Find Expected Cost of item on today
• AMT: the cost at end of day is: $100 \frac{1}{2^y}$

ANS: • Being $C(y) = 100 \frac{1}{2^y}$ is our cost func

• thus $P(y) = \frac{e^{-\lambda} \lambda^y}{y!}$

• Thus: $E(C(y)) = \sum_y C(y) P(y) = \sum_y \frac{100}{2^y} \frac{e^{-\lambda} \lambda^y}{y!} = \sum_y \frac{100}{2^y} \frac{e^{-2} 2^y}{y!}$

$= \sum_y \frac{100}{y!} e^{-2} 1^y \Rightarrow 100 e^{-1} \sum_y \frac{e^{-1} 1^y}{y!} = \boxed{100 e^{-1}}$

$\therefore \boxed{\text{Thus } E(C(y)) = 100 e^{-1} \sim \$36.79}$

w/ $\sigma = \sqrt{36.79} = \$6.0654$

$\mu/\sigma = 6.07$.

(127) COOKIES :

QUESTIONS:

- cookies yield \$200 per hour
- Extruder (machine) breaks down an average of 2 times every 60 minutes
- Let $\boxed{R = 1600 - 50y^2}$

?/ Expected Revenue:

$\lambda = 2$

ANS: $E(1600 - 5y^2) = E(1600) - 50 E(y^2)$ $\boxed{\$1300}$

$= 1600 - 50 \cdot (E(y)^2 + \sigma^2(y))$

$= 1600 - 50 \cdot (\lambda^2 + \lambda) = 1$

$1600 - 50(2^2 + 2) = 1600 - 50(6) = \boxed{1300}$

MOMENT GENERATING FUNCTIONS

(3.9)

(1)

$$\boxed{M(t) = E(e^{tY})}$$

PROBLEM
MGF can be the same but different distributions

SOLUTION
Moment generating functions

1.) The k^{th} moment Random Variable Y taken about origin is $\boxed{E(Y^k) \sim \mu'_k}$

2.) The k^{th} Moment Random Variable Y taken about it's mean is $\boxed{E[(Y-\mu)^k] := \mu_k}$

3.) Suppose (Y, Z) posses Random moments w/ convention μ'_k. Then Then it's finite moments are: $\mu'_{1Z}, \mu'_{2Z} = \mu_{2Z}, \ldots \mu_{jZ}$

(2)

$$\boxed{E(e^{tY}) = \sum_y e^{ty} p(y)}$$

s.t: $$E(e^{tY}) = \sum_y e^{ty} p(y) = \sum_y \left[1 + ty + \frac{(ty)^2}{2!} + \frac{(ty)^3}{3!} + \ldots\right] p(y)$$

$$= \sum_y \left\{ p(y) + ty\,p(y) + \frac{(ty)^3}{3!} p(y) + \frac{(ty)^4}{4!} p(y) + \ldots \right.$$

$$= \sum_y \frac{t^k}{k!} \mu'_k \quad \text{so} \quad \boxed{E(e^{tY}) = \sum_k \frac{t^k \mu'_k}{k!}}$$

(3) If $\exists M(t)$, Then

$$\boxed{\left.\frac{d^k M(t)}{dt^k}\right|_{t=0} = E(Y) = M^{(k)}(0)}$$

Proof:

1.) $$M(t) = E(e^{tY}) = \sum_{k=0} \frac{t^k \mu'_k}{k!}$$

$$M^{(1)}(t) = \frac{\partial E(e^{tY})}{\partial t} = \sum_{k=1} \frac{k t^{k-1}}{k!} \mu'_k$$

$$M^{(2)}(t) = \frac{\partial^2 E(e^{tY})}{\partial t^2} = \sum_{k=2} \frac{k(k-1) t^{k-2}}{k!} \mu'_1$$

2.) $$M^{(1)}(0) = \mu'_1 + \frac{2t}{2!} + \ldots \Big|_{t=0} = \mu_1$$

$$M^{(2)}(0) = \mu'_2 + \frac{2t}{2!} \mu'_3 + \frac{3t^2}{3!} \Big|_{t=0} = \mu_2$$

3.) So $\forall k$, $\boxed{m^{(k)}(0) = \mu_k \sim \mu}$

④ POISSON DISTRIBUTION:

1) $m(t) = E(e^{tY}) = \sum_y e^{ty} p(y) = \sum_y e^{ty} \left\{ \frac{\lambda^y}{y!} e^{-\lambda} \right\}$

MOMENT GEN-FN PROPERTIES:

$$\boxed{\begin{aligned} m^{(1)}(0) &= \mu \\ m^{(2)}(0) &= \sigma^2 \end{aligned}}$$

$$= \sum_y e^{ty-\lambda} \frac{\lambda^y}{y!} = \sum_y \frac{(\lambda e^t)^y e^{-\lambda}}{y!} = e^{-\lambda} \sum_{y=0}^{\infty} \frac{(\lambda e^t)^y}{y!}$$

So: $\boxed{\sum_{y=0}^{\infty} \frac{(\lambda e^t)^y}{y!} = e^{\lambda e^t}}$

2.) $\frac{\partial m(t)}{\partial t} = \frac{\partial}{\partial t} \exp(\lambda e^t) = e^{\lambda(e^t - 1)} \cdot \lambda e^t$

$\frac{\partial^2 m(t)}{\partial t} = \frac{\partial}{\partial t} \exp(\lambda e^t - 1) \lambda e^t = e^{\lambda(e^t-1)} (\lambda e^t)^2 + e^{\lambda(e^t-1)} \lambda e^t$

3.) $m^{(1)}(0) = e^{\lambda e^0 - \lambda} \lambda e^0 = e^{\lambda - \lambda} \lambda = \lambda$

$m^{(2)}(0) = e^{\lambda(e^0-1)} \cdot (\lambda e^0)^2 + e^{\lambda(e^0-1)} \lambda e^0$

$= e^0 \lambda^2 + e^0 \lambda = \lambda^2 + \lambda = \lambda(\lambda+1)$

So: $\boxed{\begin{aligned} E_{poisson}(y) &= \lambda \\ \sigma^2_{poisson}(y) &= \lambda(\lambda+1) \end{aligned}}$

EXERCISE 3.49

(148) BINOMIAL DIST:

Show that $M(t) = (pe^t + q)^n$

Ans: Recall: $Binom(p,q;x) = \binom{n}{x} p^x q^{n-x} \propto (x+y)^n$

1.) $M(t) = E(e^{tY}) = \sum_y^n e^{ty} p(y) = \sum_y e^{ty} \binom{n}{y} p^y q^{n-y}$

$$= \sum_y^n \binom{n}{y} p^y q^{n-y} e^{ty} = \sum_y \binom{n}{y} p^y e^{ty} q^{n-y}$$

2.) $p^y e^{ty} = (pe^t)^y$ so:

$$M(t) = \underbrace{\sum_y^n \binom{n}{y} (pe^t)^y q^{n-y}} = (pe^t + q)^n$$

$(x+y) = \binom{n}{k} x^k y^{n-k}$ Let $x = pe^t$, $y = q$, $k = y$

$= \binom{n}{y} (pe^t)^y q^{n-y}$

3.) Therefore: $\boxed{(pe^t + q)^n}$ ✓

(146) Find μ_y, σ_y^2 From (48):

Ans: 1) $\frac{\partial M(t)}{\partial t} = \frac{\partial}{\partial t}(pe^t + q)^n \Leftarrow$ Let $u = pe^t + q$

$$= \frac{\partial M}{\partial u}\frac{\partial u}{\partial t} = \frac{\partial}{\partial u} u^n \frac{\partial}{\partial t}(pe^t + q) = \frac{\partial}{\partial u} u^n \left(\frac{\partial}{\partial t} pe^t + \frac{\partial}{\partial t} q\right)$$

$$= n u^{n-1} pe^t = n(pe^t + q)^{n-1} pe^t$$

U^{n-1}, U^{n-2}
$= U^{n-1-1-2}$
$= U^{n-3}$
$= U^{n-n-1}$
$= U^{n(n-1)-1}$

$\frac{\partial^2 M}{\partial t^2} = \frac{\partial}{\partial t}\frac{\partial M}{\partial t}$
$= \frac{\partial}{\partial t} n U^{n-1}(t) pe^t$
$= \frac{\partial}{\partial t} n U^{n-1}(t) pe^t$
$+ n U^{n-1}(t) \frac{\partial}{\partial t} pe^t$
$= n(n-1)U^{n-2}(t) pe^t + n U^{n-1}(t) \frac{\partial}{\partial t} pe^t$
$n \frac{\partial}{\partial t}(pe^t + q)^{n-1}$
$\frac{\partial U}{\partial t}, \frac{\partial U'}{\partial t} =$

2) $\frac{\partial^2 M(t)}{\partial t^2} = \frac{\partial}{\partial t}\left(n U^{n-1}(t) pe^t\right) = \frac{\partial}{\partial t} n U^{n-1}(t) \cdot pe^t + \frac{\partial}{\partial t} pe^t \cdot n U^{n-1}(t)$

$= n U^{n-2} pe^t pe^t + pe^t n U^{n-1}$

$$= n \frac{\partial U^{n-1}}{\partial U}\frac{\partial U}{\partial t} pe^t + pe^t n U^{n-1}$$

$$= n U^{n-2}(pe^t)^2 - npe^t U^{n-1}$$

$$= n[pe^t + q]^{n-2}(pe^t)^2 + n[pe^t + q]^{n-1} pe^t$$

(3) $\left.\frac{\partial M}{\partial t}\right|_{t=0} = n[pe^0 + q]^{n-1} pe^0 = n[p+q]^{n-1} p$

$= \frac{n[(p+q)^n p]}{(p+q)} = \frac{n((p+(1-p))^n p}{(p+q)}$

$= n(n-1)p$

or:

$(p+q)^{n-1}$

$(p+(1-p))^{n-1}$

$n(1^{n-1})$

$= np$

so if: $n = 0 \Rightarrow 0(p+q)^{0-1} p = 0$

$n = 1 \Rightarrow 1(p+q)^{1-1} p = p$

$n = 2 \Rightarrow 2(p+q)p = \cancel{2p^2} + 2p - \cancel{2p^2}$

$= 2p$

$\equiv \boxed{np} \quad \forall n$

so: $\boxed{M^{(1)}(0) = np}$

4.) $\left.\frac{\partial^2 M}{\partial t}\right|_{t=0} = Var(X) = M''(0) - [M'(0)]^2$

$= n(n-1)p^2 + np - [np]^2$

$= n^2\cancel{p^2} - np^2 + np - \cancel{n^2p^2} = -np^2 + np$

$= np(1-p)$

so: $\therefore$ $\boxed{\begin{array}{l} E_Y\{Binom\} = np \\ \sigma_Y^2\{Binom\} = np(1-p) \end{array}}$ so: $\hat{Y} = \frac{\hat{X} - E(Y)}{\sqrt{\sigma_Y^2}} = \frac{X - np}{\sqrt{np(1-p)}}$

EXERCISE 3

(44) 3 PRISONERS PROBLEM

CASE:

- 3 prisoners are informed by their jailer will be chosen at random that 1 will be executed, 2 will
- A asks privately of which one of them will be executed since at least one of 2 will go free.
- Jailer says his probability of getting executed will rise from $\frac{1}{3}$ to $\frac{1}{2}$ since he would be 1 of the 2 prisoners.

(f) Question: What do you think of the jailer's reasoning?

Ans:

- probability of 1 of the 3 guys getting executed is $\frac{1}{3} = p(E)$.
- If A is to be certainly executed then before

$$P(\Omega) = P(A) + P(B) + P(C) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

$$\rightarrow P(\Omega') = 1 + P(B) + P(C) = 1$$

$$= \frac{3}{3} + (P(B) + P(C)) = 1$$

$$= \frac{3}{3} - 0 = 1$$

- If A more likely then (i.e. rather than certain)

$$P(\Omega'') = \frac{1}{3} + \left(\tfrac{1}{3}P^B + P^C_3 = \frac{2}{3}\right)$$

- The jailer says its $\frac{1}{2}$ this means that one prisoner isn't executed, one prisoner is half but ~~prisoner~~ A is $\frac{1}{2}$ which violates the split even between P^B & P^C which is $\boxed{\frac{2}{3}}$.

$\therefore$ So the probability is still $\frac{1}{3}$ and the jailer is wrong

(45) Suppose 10 coins such that i^{th} coin appear w/ probability of $\boxed{\frac{i}{10}}$ $\{i = 1 \ldots 10\}$.

When one of the coins is randomly selected and flipped it shows heads, whats the conditional probability of the 5th coin being flipped?

Ans:

- Suppose 2 coins. Then $\left(\frac{1}{10}, \frac{2}{10}\right) = (P_1, P_2)$

For 2 coins

$$P(C_i) = \frac{i}{10} \qquad P(H|C_i) = \frac{i}{10}$$

$$P(H) = \sum_i^{10} P(H|C_i) P(C_i)$$

$$= \sum_i^{10} \frac{i}{10} \times \frac{1}{10} = \frac{1}{100} \sum_i^{10} i$$

$$= \frac{1}{100} (1+2+3+4+5+ \ldots 10)$$

$$= \frac{1}{100} \left(\frac{10 \times 11}{2}\right) = \frac{1}{100} \frac{110}{2} = \frac{55}{100}$$

$$= \boxed{\frac{11}{20}}$$

So: $$P(C_5|H) = \frac{P(H|C_5) P(C_5)}{P(H)}$$

$$= \frac{\frac{5}{10} \times \frac{1}{10}}{0.55} = \frac{5}{100} \cdot \frac{100}{55} = \frac{5}{55} = \boxed{\frac{1}{11}}$$

So the probability of getting heads is $\frac{1}{11}$ or Approximately 0.09 or $\boxed{9.09\%}$

PROBABILITY & CUMULATIVE FUNCTIONS

$\boxed{4.2}$

(2) PROBABILITY of CONTINUOUS RANDOM VARIABLES:

- **Cumulative distr. funct.:**

  - Let Y denote any random variable. The distr function of Y is denoted by $F(y)$ and $\boxed{F(y) = P(Y \leq y)}$ for $-\infty \leq y \leq +\infty$ ... to $\boxed{F(y) = \text{DISB FUNC}}$

  - EXamples:
    - Binomial distribution. Suppose Y is Binomial $(n=2, p=\frac{1}{2})$. Then $F(y)$.

Ans: 1) The probability function is: $\boxed{p(y) = \binom{2}{y}\left(\frac{1}{2}\right)^y\left(\frac{1}{2}\right)^{2-y}}$ w/ $p(0) = \frac{1}{4}$, $p(2) = \frac{1}{4}$, $p(1) = \frac{1}{2}$

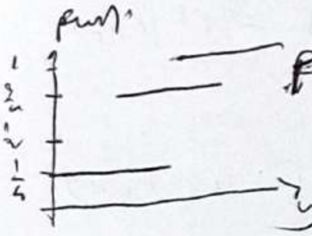
2) What is $F(-2) = P(Y \leq -2)$? Well because $Y := (0, 1, 2)$ and $(0,1,2) \nleq -2$, then $\boxed{F(-2) = 0}$. But for $F(1.5)$:

$P(1.5) = P(Y \leq 1.5) = P(Y=0) + P(Y=1) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$ w/ $Y \leq 1.5 = Y \leq 1 + 0 \Rightarrow (\frac{1}{4} + \frac{1}{2})$ then $= Y \in [0,1]$. $\boxed{1.5 \in [0,1]}$

3) The cumulative:

$F(y) = P(Y \leq y) = \{0, \frac{1}{4}, \frac{3}{4}, 1\}$ w/ $\frac{1}{4} \leftrightarrow 0 \leq y \leq 1$ and $\frac{3}{4} \leftrightarrow 1 \leq y < 2$, $1 \leftrightarrow y \geq 2$.

STEP FUNCTION



Discrete functions are usually step functions because they only increase by a finite substantial amount!

- **PROPERTIES of CUMULATIVE DISTRIBUTION FUNCTION**

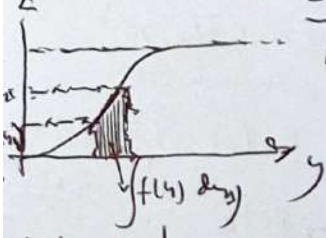
  - Some common properties:

    1) $\boxed{F(-\infty) \equiv \lim_{y \to -\infty} F(y) = 0}$ 2) $\boxed{F(\infty) \equiv \lim_{y \to \infty} F(y) = 1}$

    3) If (y_1, y_2) are any values s.t. $(y_1 < y_2)$, then $\boxed{F(y_1) \leq F(y_2) \leftrightarrow y_1 < y_2 \;\; \forall y_1, y_2}$

do: if $F(y) \in [0,1]$ for $\lim_{y \to \infty} F(y) = 1$, $\lim_{y \to -\infty} F(y) = 0$ satisfies Kolmogorov's 1st Axiom

THE CUMULATIVE vs. THE PROBABILITY DENSITY FUNCTION



- **CONTINUOUS SYMMETRY:**

  - Some random variable Y w/ $F(y) = \text{DISB}(Y)$ is continuous if $\boxed{\forall F(y),\; y \in [-\infty, \infty]}$. To differentiate difference see graph in the side

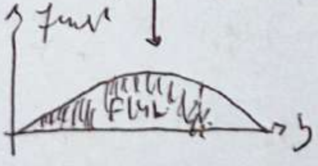
  - The fundamental theorem of calculus function:

$$\boxed{f(y) = \frac{dF(y)}{dy}} \text{ or } \boxed{F(y) = \int_{y_1}^{y_2} f(y) \cdot dy}$$

or: $\boxed{PDF(y) = \frac{\partial CDF(y)}{\partial y}}$

$f(y)$ = Probability density

$F(y)$ = Cumulative distribution



- **PROPERTIES:**

$\boxed{\begin{array}{l} 1)\; f(y) \geq 0 \;\; \forall y \in [-\infty, +\infty] \\ 2)\; \int_{-\infty}^{\infty} f(y)\, dy = 1 \end{array}}$

  - Some common applications:
    - Rainfall measurements: DISCRETE: $P(2 \text{ or } 3 \text{ inch})$; CONTINUOUS: $P(2.13 \text{ inch})$
    - Manufacturing of a Machine vs PDF.

Examples

EXAMPLE 4.2: Suppose that $F(y) = \begin{cases} 0 & y<0 \\ y & 0\le y<1 \\ 1 & y\ge 1 \end{cases}$. Find $f(y)$ of y and graph it.

Ans: • Basically we have 3 functions. So ∂f:

$$f(y) = \frac{\partial F(y)}{\partial y} = \begin{cases} \partial F_1/\partial y \leftrightarrow F_1 = 0 \\ \partial F_2/\partial y \leftrightarrow F_2 = y \\ \partial F_3/\partial y \leftrightarrow F_3 = 1. \end{cases}$$

• Then:

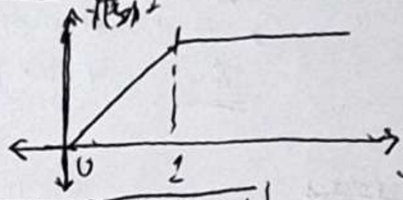
1) $\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}[0] = 0$ 2) $\frac{\partial F_2}{\partial y} = \frac{\partial}{\partial y}[y] = 1$ 3) $\frac{\partial F_3}{\partial y} = \frac{\partial}{\partial y}[1] = 0.$

2. $\boxed{\{F_1', F_2', F_3'\} = \{0, 1, 0\}.}$

• So:

$$f(y) = \begin{cases} 0 & y<0 \\ 1 & 0<y<1 \\ 0 & y>1 \end{cases}$$

and



EXAMPLE 4.3: Suppose that $\boxed{f(y) = \begin{cases} 3y^2 & 0\le y\le 1 \\ 0 & \text{elsewhere} \end{cases}}$ find $F(y)$ and graph (F, f).

Ans: • Basically $F(y)$ is:

$$\boxed{F(y) = \int_{-\infty}^{y} f(y)\,dt}$$

• So for $[0\le y\le 1]$:

$$F(y) = \int_{-\infty}^{y} f(y)\,dt = \int_{-\infty}^{0} f_1(t)\,dt + \int_{0}^{y} f_2(t)\,dt = \int_{-\infty}^{0} 0\,dt + \int_0^y 3y^2\,dy$$

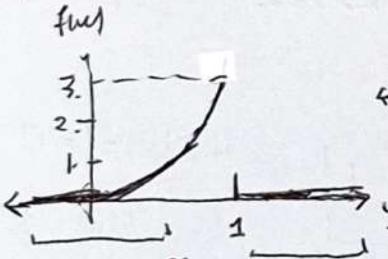
$$= \int_0^y 3y^2\,dy = 3\frac{1}{3}y^3\Big|_0^y = 3\tfrac{1}{3}y^3 = \boxed{y^3}$$

• and for $[1<y]$:

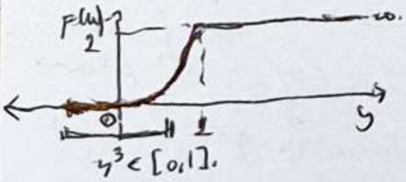
$$F(y) = \int_{-\infty}^{0} f_1(t)\,dt + \int_0^1 3t^2\,dt + \int_1^y f_3(t) = 0 + t^3\Big|_0^1 + 0 = 1^3 - 0^3 = \boxed{1}$$

∴ $$\boxed{F(y) = \begin{cases} 0 & y<0 \\ y^3 & 0\le y\le 1 \\ 1 & 1>y \end{cases}}$$

Graphing of F(y), f(y)

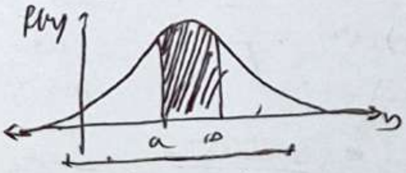


$(y, f(y)) = [3y^2]$ given $y^2 \in [0,1]$. $(y, f(y)) = [(1,y), 0] = [1, 0] \cup [\infty, 0]$



$y^3 \in [0,1]$.

Probability Distribution



$P(a\le y\le b)$.

The Probability Density Function

Definition 4.4:

• Let (Y) denote any random variable. 7 $(0<p<1)$, the p^{th} quantile of Y is denoted by ϕ_p, is the smallest value st $P(Y\le \phi_p) = F(\phi_p) \ge p$. If Y is continuous then $F(\phi_p) = P(Y\le \phi_p) = p$.

then $\boxed{F(\phi_p) = P(Y\le \phi_p) = p \\ \text{or } F(\phi_p) \ge p \Rightarrow P(Y \ge \phi_p)}$ → $F(\phi_p) = p^{th}$ quantile
$F(\phi_p) = 100^{th}$ pth percentile of Y.

Consequences

• Special case if $p = \frac{1}{2} \to \phi_{\frac{1}{2}} \Rightarrow F(\phi_{0.5}) = 0.5$
• It's also obvious that: $(\phi_{0.5})^3 = 0.5 \Rightarrow \therefore \phi_{0.5} = (0.5)^{\frac{1}{3}} = 0.7937$

The Probability Density: Basically it's just the PDF $\boxed{P(a\le Y\le b) = \int_a^b f(y)\,dy}$

EXAMPLE 4.4: $f(y) = cy^2 \to F(y) = \frac{1}{3}cy^3.$

Exercises 4.2

8. Suppose ... Suppose Y has PDF: $f(y) = \begin{cases} ky(1-y) & 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$

a) Find k that makes $f(y)$ a PDF!

Ans: $F(y) = \int_0^1 f(y)\,dy = \int_0^1 ky(1-y)\,dy = \int_0^1 ky - ky^2\,dy = k\left[\frac{1}{2}y^2 - \frac{1}{3}y^3\right]_0^1$

$= k\left(\frac{1}{2} - \frac{1}{3}\right)\Big|_0^1 = k\left(\frac{1}{2}\frac{3}{3} - \frac{1}{3}\frac{2}{2}\right) = k\left(\frac{3}{6} - \frac{2}{6}\right) = k\frac{1}{6}$

• Since $\int f(y)\,dy = 1$ then $k\frac{1}{6} = 1 \rightarrow \boxed{k = 6}$

$f(y) = ky(1-y) = 6y(1-y)$ so $\boxed{f(y) = 6y(1-y)}$

b) Find $P(0.4 \le Y \le 1)$..!

Ans: • Since: $P = \int f(y)\,dy \equiv k\left[\frac{1}{2}(b^2 - a^2) - \frac{1}{3}(b^3 - a^3)\right]$

• Then: $P(0.4 \le Y \le 1) = 6\left[\frac{1 - 0.4^2}{2} - \frac{1 - 0.4^3}{3}\right] = 6\left[0.42 - 0.31\right]$

$= 6\left[0.11\right] = \boxed{0.66}$

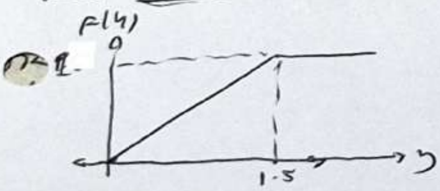
c) Find $P(Y \le 0.4 \mid Y \le 0.8)$

Ans: • $P(Y \le 0.4 \mid Y \le 0.8) = \dfrac{P(0.8 \ge Y \cap 0.4 \ge Y)}{P(Y \le 0.8)} =$

$= \dfrac{P(Y \in [-\infty, 0.4] \cap Y \in [\infty, 0.8])}{P(Y \in [0, 0.8])} = \dfrac{P(Y \le 0.4)}{P(Y \le 0.8)} = 0.3922457$

$0.8 \ge Y \cap 0.4 \ge Y$
$= Y \in [\infty, 0.4]$
$\cap\, Y \in [0.8, \infty]$

9. Kerosene supply: A supplier of kerosene has a 150-gallon tank at the beginning of each week. His demand shows a relatively frequency behaviour that increases steadily up to 100 gallons then levels off (100, 150) gallons. If (Y = weekly demand):



$$\boxed{f(y) = \begin{cases} y & 0 \le y \le 1 \\ 1 & 1 < y \le 1.5 \\ 0 & \text{else} \end{cases}}$$

∴ So after 100 gallons it stops it's filling up the gas tank.

A Find $F(y)$

Ans: $F(y) = \begin{cases} \int_0^y y\,dy = \frac{1}{2}y^2\Big|_0^y = y \quad \forall\, 0 < y < 1 \\ \int_0^1 y\,dy + \int_1^y 1\,dy = \frac{1}{2}y^2\Big|_0^1 + y\Big|_1^y = \frac{1}{2} + y - 1 \\ = \frac{1}{2} - 1 + y = \frac{1-2}{2} = y - \frac{1}{2} \quad \forall\, 1 < y \le 1.5 \end{cases}$

so $F(y) = \begin{cases} 0 & y < 0 \\ \frac{1}{2}y^2 & 0 \le y \le 1 \\ y - \frac{1}{2} & \forall\, 1 < y \le 1.5 \\ 1 & y > 1.5 \end{cases}$

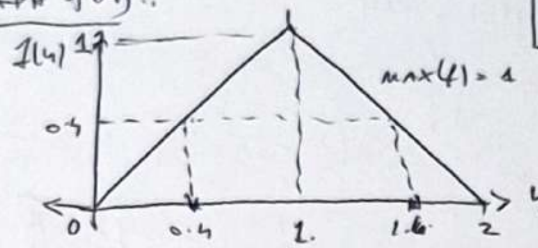
when $y = 1.5$ then $y - \frac{1}{2} = 1.5 - 0.5 = 1.00 = \boxed{1}$

so $y > 1.5 \Rightarrow F = 1$

1a GAS PUMP: Gas station operates w/ 2 pumps. Each can pump up to 10.000 gallons of gas in a month. The total amount of gas pumped at the station is a

$$f(y) = \begin{cases} y & 0 < y < 1 \\ 2-y & 1 \le y < 2 \\ 0 & \text{else} \end{cases}$$

[a] GRAPH $f(y)$..



$\max(y) = 1$

The maximum $f(y)$ is actually 1, meaning it's more likely to pump 10 gallons every week.

so $F(y) = \begin{cases} 0 & \text{else} \\ 2y - 1 - \frac{1}{2}y^2 & 1 \le y \le 2 \\ \frac{1}{2}y^2 & 0 \le y < 1 \end{cases}$

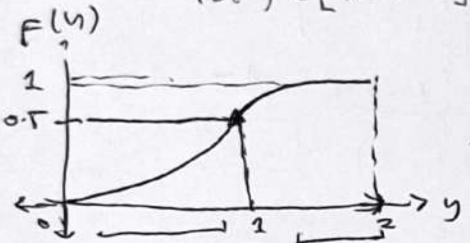
[b] Find $F(y)$ & Graph it:

Ans. $f_1(y) \in [0 < y < 1]$ so $F_1(y) = \int_0^1 y\,dy = \frac{1}{2}y^2\Big|_0^y = \frac{1}{2}\cdot y^2$

$f_2(y) \in [1 \le y < 2]$ so $F_2(y) = \int_0^1 y\,dy + \int_1^y 2 - y = \frac{y^2}{2} + 2y\Big|_1^y - \frac{y^2}{2}\Big|_1^y = \frac{4y - y^2}{2}\Big|_1^y = 4y - y$

$= \frac{1}{2} + 2(y-1) - \frac{1}{2}y^2 - \frac{1}{2}$

$= 2(y-1) - \frac{1}{2}y^2 = 2y - 1 - \frac{1}{2}y^2$



Gallons at 0 to 10.000 gallons will be pumped at at $\frac{1}{2} \times y^2$ in weekly

$\boxed{\max(y) = 10.000}$

If the gallons pumped more than required 10.000, then it will pump until it peaks at 20.000

$\boxed{\text{MAX}(y) = 20.000}$

So the gas will quickly fill up to 10.000 gallons, but after 10.000 gallons it starts to slow down until 2 gallons per week

[c] Find the probability it'll be pumped between (8000 & 12.000) gallons in a particular month

Ans > $P(8000 < y < 12000)$.

$= \int_{8.000}^{12.000} f(y)\,dy = F(1.2) - F(0.8)$

$= 0.68 - 0.32 = 0.36 = \boxed{36\%}$

EXERCISES 4.2

15 MICE INTELLIGENCE: Measuring mice intelligence is timed when you go through a maze to reach a reward of food. The time required in seconds = Y w/

$$\boxed{f(y) = \begin{cases} \frac{b}{y^2} & \forall y \geq b \\ 0 & \text{else} \end{cases}}$$

w/ b = minimum possible time needed to traverse maze

[a] Show that $f(y)$ has the properties of the density function that is required for a valid PDF.

Ans: • Observe that there are 2 conditions that

1) $\forall f(y),\ f(y) \geq 0$

2) $\forall f(y),\ \int_{-\infty}^{\infty} f(y)\,dy = 1$.

• The 1st condition is satisfied since

$f(y) = \frac{b}{y^2}$ and $y \geq b \Rightarrow y - b \geq 0$ be it by b then

$f(y) = \frac{b}{b^2} = b^{1-2} = \frac{1}{b}$ and $\boxed{b \neq 0}$ and $\boxed{b > 0}$ ✓

• The 2nd condition is then met.

$$\int_{-\infty}^{\infty} f(t)\,dt = \int_{b}^{\infty} \frac{b}{t^2}\,dt = b\int_{b}^{\infty} \frac{1}{t^2}\,dt = -b\,t^{-1}\Big]_b^{\infty} = -b(\infty)^{-1} - \left(-\frac{b}{b}\right)$$

since $y \geq b$

$$= -b \lim_{t\to\infty} \frac{1}{t} + \frac{b}{b} = -b(0) + 1 = \boxed{1 = 1}$$ ✓

[b] Find $F(y)$:

Ans: $F(y) = \int_b^y f(t)\,dt = -b\frac{1}{t}\Big|_b^y = -\frac{b}{y} + \frac{b}{b} = -\frac{b}{y} + 1 = 1 - \frac{b}{y}$

hence $\boxed{F(y) = 1 - \frac{b}{y}}$ and $\frac{dF}{dy} = -\left(-\frac{b}{y^2}\right) = \frac{b}{y^2} = f(y)$ ✓

[c] Find $P(Y > b+c),\ \forall c = \text{cons} \in \mathbb{R}^+$.

Ans: $P(Y > b+c) = 1 - F(b+c) = \boxed{\frac{b}{b+c}}$

[d] Find $P(Y > b+d \mid Y > b+c)$

Ans: $P(Y > b+d \mid Y > b+c) = \frac{P(Y > b+c)}{P(Y > b+d)} = \boxed{\frac{(b+c)}{(b+d)}}$

16 1-HOUR EXAM OF STUDENTS: The length of time required by students to complete a 1-hour exam is w/

$$\boxed{f(y) = \begin{cases} cy^2 + y, & 0 \leq y \leq 1 \\ 0 & \text{else} \end{cases}}$$

[a] Find c
[b] Find $F(y)$
[c] Graph $f(y)$, $F(y)$

[a] [b] [c]:

Ans: • we find c by integrating the density func and setting to 1.

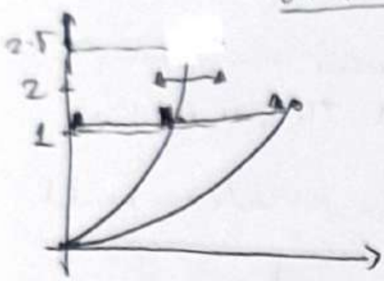
$$\int_0^1 cy^2 + y\,dy = c\frac{y^3}{3} + \frac{y^2}{2}\Big|_0^1 = c\left(\frac{1^3}{3}\right) + \left(\frac{1^2}{2}\right) = c\frac{1}{3}\frac{2}{2} + \frac{1}{2}\frac{3}{3} = c\frac{2}{6} + \frac{3}{6} = \frac{2c+3}{6} = 1$$

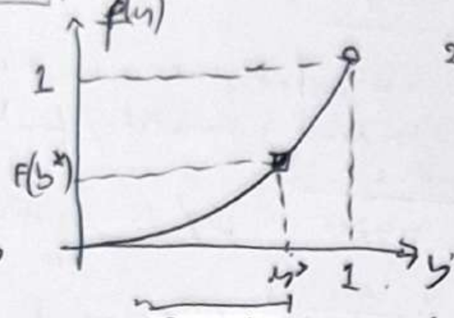
$2c + 3 = 6$, $2c = 3 \rightarrow \boxed{c = \frac{3}{2}}$ hence $\boxed{f(y) = \frac{3}{2}y^2 + y \quad \forall\, 0 \leq y \leq 1}$

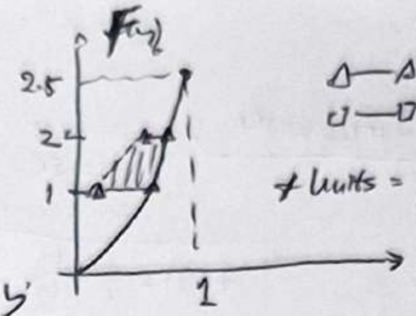
• The $F(y)$ is simply:

$$\int_0^y f(t)\,dt = \int_0^y ct^2 + t\,dt = \frac{3}{2}\left(\frac{t^3}{3}\right) + \left(\frac{t^2}{2}\right)\Big|_0^y = \frac{3}{2}\left(\frac{y^3}{3} + \frac{y^2}{2}\right) = \boxed{\frac{1}{2}y^3 + \frac{?}{2}y^2}$$

· Graphs:







Δ—Δ = Ceil(f(y))
□—□ = Ceil(F(y)).

units = $\frac{1}{N}$ Hours · & CDF → ∞ = $\frac{1}{1}$ Hour

The Density argumentum whats shown means it goes out as $f(y) \to \infty$, $f(y) = 1$, and it's much faster to complete the exam after $f(y^*)$

We see here that the cumulative function is much more faster near the activity: meaning that after 1 Hour, the student needs to be able to finish it w/max 2.5

[b] Use $F(y)$ to calculate $F(-1)$, $F(0)$, $F(1)$

Ans: $\boxed{F(y) = \frac{1}{2}y^3 + \frac{1}{2}y^2}$

b: $F(-1) = \frac{1}{2}(-1)^3 + \frac{1}{2}(-1)^2 = \frac{1}{2} - \frac{1}{2} = \boxed{0}$ ✓

$F(0) = \boxed{0}$

$F(1) = \frac{1}{2} + \frac{1}{2} = \boxed{1}$] The probability of finishing the exam in hour is 100%.

[c] Use $F(y)$ to find the probability that the student will finish in less than an hour.

Ans: $Pr(\text{Finish} < \frac{1}{2}) = P(y < \frac{1}{2}) = F(\frac{1}{2}) = \frac{1}{2}\frac{1}{2^3} + \frac{1}{2}\frac{1}{2^2} = \frac{1}{2}\frac{1}{8} + \frac{1}{2}\frac{1}{4} = \frac{1}{16} + \frac{1}{8}$

$= \frac{1}{16} + \frac{2}{16} = \frac{3}{16} = \boxed{18.75}\%$

[d] Given that a particular student needs at least 15 minutes for completion, find the probability that she will require 30 minutes to finish the exam

Ans: $Pr(30\text{ minutes} \mid 15\text{ minutes}) = Pr(Y \geq 30 \mid Y \geq 15) = \frac{P(Y \geq 30)}{P(Y \geq 15)} = \frac{F(0.50)}{F(0.25)}$

$= \frac{\frac{1}{2}0.5^3 + \frac{1}{2}0.5^2}{\frac{1}{2}0.25^3 + \frac{1}{2}0.25^2} = \frac{3/16}{341/1000} = \frac{104}{123} = \boxed{84.55\%}$

So the probability of the student finishing the exam under 1 hour given that it required min 15 minutes to finish is 84.55%. Meaning as the hour approaches 1, the students from 80%–100% of the students will go for that approach.

EXPECTED VALUE of CONTINUOUS RANDOM; THE UNIFORM DISTRIBUTION & THE NORMAL

4.3 | 4.4 | 4.5

(1) EXPECTED VALUE of Cont. R.V.

Formulation:

- Let Y be a random variable. Then the expected value of the function $f(y)$ given $y \in [-\infty, \infty]$, is:

$$\boxed{E(Y) = \int_{-\infty}^{\infty} y\, f(y)\, dy}$$

EXISTENCE: $E(Y)$ exists if $\int_{-\infty}^{\infty} |y| f(y)\, dy < \infty$

- where lim the Riemann sum theorem:

$$E_{CONT} = \lim_{\Delta y \to 0} E_{DISC}\, \Delta y = \lim_{\Delta y \to 0} \left[\sum_{i=1}^{N} y_i f(y_i) \right] \Delta y_i = \sum_{i=1}^{\infty} y_i f(y_i)\, \Delta y_i$$

$$= y_1 f(y_1) \Delta y_1 + y_2 f(y_2) \Delta y_2 + \dots \infty = \int_{-\infty}^{\infty} y f(y)\, dy \checkmark$$

- If $y \to g(Y)$ i.e. some function instead of some variable, then

$$\boxed{E[g(Y)] = \int_{-\infty}^{\infty} g(y) f(y)\, dy}$$

w/ $E[Y \to g(Y)] = \int_{-\infty}^{\infty} y \Big|_{y \to g(y)} f(y)\, dy = \int_{-\infty}^{\infty} g(y) f(y)\, dy \checkmark$

- Some common properties:

1) $\boxed{E(c) = c}$ — The Constant Rule-I

2) $\boxed{E[c\, g(Y)] = c\, E[g(Y)]}$ — The Constant Rule-II

3) $\boxed{E\left[\sum_i g_i(Y)\right] = \sum_i E[g_i(Y)]}$ — The Summation Rule

EXAMPLE 4.6

Suppose that $\boxed{f(y) = \frac{3}{8} y^2 \;\forall\, 0 \le y \le 2}$. If Y = Rand. Variable then find $\mu = E(Y)$, $\sigma^2(Y)$.

Ans: • The μ:

$$E(Y) = \int_{-\infty}^{\infty} y f(y)\, dy = \int_0^2 y \frac{3}{8} y^2 = \int_0^2 y^3 \frac{3}{8} = \frac{3}{8}\int_0^2 y^3\, dy = \frac{3}{8}\left[\frac{y^4}{4}\right]_0^2$$

$$= \frac{3}{8}\left[\frac{2^4}{4}\right] = \frac{3}{8}\frac{16}{4} = \frac{3}{8}\cdot 4 = \frac{12}{8} = \boxed{1.5}$$ So $\boxed{\mu = 1.5}$

• The variance follows:

$$V(Y) = E(Y^2) - E(Y)^2 = E(Y^2) - \mu^2 = E(Y^2) - 2.25 = \int_0^2 y^2 \frac{3}{8} y^2 - 2.25$$

$$= \frac{3}{8}\frac{y^5}{5}\Big|_0^2 - 2.25 = \frac{3}{8}\frac{2\cdot 16}{5} - 2.25 = \left(\frac{3}{8}\frac{32}{5} = \frac{32}{8}\frac{3}{5} = \frac{12}{5} = 2.4\right)$$

$$= 2.4 - 2.25 = \boxed{0.15}$$ So $\boxed{\sigma^2 = 0.15}$. So $\sigma = \sqrt{0.15} = 0.38$ and $\boxed{\sigma = 0.38}$

• The CV:

$$CV = \frac{\mu}{\sigma} = \frac{1.5}{0.38} = 3.94$$ times more mean than STD.

(2) UNIFORM PROBABILITY DISTRIBUTION

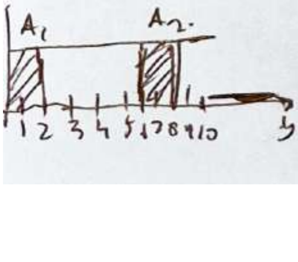
SCENARIO

- Suppose that a bus arrives at a particular stop between (8:00, 8:10) AM. Then:

$\boxed{P_r(\text{Arrive at subinterval of time}) \propto \text{Length of interval}}$

The bus will likely arrive between $\boxed{8:06 - 8:08}$ | $\boxed{8:00 - 0:2}$

The graph:



- If we measure in 8:00 am's minute interval like (8:00, 8:02) → (0, 2), then the relative frequency will be the same at

$\boxed{P_r(0 \le Y \le 2) = P(6 \le Y \le 8)}$

FORMULATION

Let $\theta_1 < \theta_2$ be a random variable that has a continuous uniform probability. Then it's continuous on (θ_1, θ_2) iff:

$$\boxed{f(y) = \begin{cases} \frac{1}{\theta_1 - \theta_2} & \forall \theta_1 \le y \le \theta_2 \\ 0 \end{cases}}$$

PARAMETERS: The cdf = parameters of cases depends further $f(y)$

Example 4.7:

ARRIVALS of customers at a CHECKOUT counter follow a POISSON DISTRIBUTION. It's assumed that between a 30 minute period, one customer arrived at the counter. Pr(Arrived last 5 minutes of 30 minute interval)

Ans: .. They follow a uniform distr:

$$P(30-5 \le y \le 30) = P(25 \le y \le 30) = \int_{25}^{30} \frac{1}{30}\,dy = \frac{30-25}{30} = \frac{5}{30} = \frac{1}{6}$$

so $\boxed{\Pr(\text{5 minutes in 30 minutes interval}) = \frac{1}{6} = 16.67\%}$

EXPECTATION & VARIANCE

- The expectation of a uniform distr is:

$$\mu = E(Y) = \int_{-\infty}^{\infty} y f(y)\,dy = \int_{\theta_1}^{\theta_2} y \frac{1}{\theta_2 - \theta_1}\,dy = \frac{1}{2} \frac{y^2}{\theta_2 - \theta_1}\Big|_{\theta_1}^{\theta_2}$$

$$= \frac{1}{2}\frac{(\theta_1 - \theta_2)^2}{\theta_1 - \theta_2} = \boxed{\frac{\theta_2 + \theta_1}{2}}$$

- The variance is:

$$V = \int_{\theta_1}^{\theta_2} y^2 \frac{1}{\theta_1 - \theta_2}\,dy - \frac{(\theta_1 + \theta_2)^2}{4} = \frac{1}{3}\frac{(\theta_2 - \theta_1)^3}{(\theta_2 - \theta_1)} - \frac{(\theta_1 + \theta_2)^2}{4} = \frac{1}{3}(\theta_2 - \theta_1)^2 - \frac{1}{4}(\theta_1 + \theta_2)^2$$

$$= (\theta_2 - \theta_1)^2\left(\frac{1}{3} - \frac{1}{4}\right) = (\theta_2 - \theta_1)^2 \left(\frac{4}{12} - \frac{3}{12}\right) = \boxed{\frac{(\theta_2 - \theta_1)^2}{12}}$$

so $\boxed{\mu = \frac{\theta_1 + \theta_2}{2}}$ $\boxed{\sigma^2 = \frac{(\theta_2 - \theta_1)^2}{12}}$

$(\theta_1 + \theta_2)^2 : \theta_1^2 + 2\theta_1\theta_2 + \theta_2^2 \to \frac{3(\cdot)}{12}$

$\frac{(\theta_2 - \theta_1)^2}{3} : \frac{\theta_1^2 - 2\theta_1\theta_2 + \theta_2^2}{3} \to \frac{4(\cdot)}{12}$

$3(\theta_1^2 + 2\theta_1\theta_2 + \theta_2^2) - (\theta_1^2 - 2\theta_1\theta_2 + \theta_2^2)4$

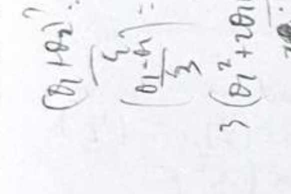
$3\theta_1^2 - 4\theta_1^2 + 6\theta_1\theta_2 + 12\theta_1\theta_2 \ldots$

3. THE NORMAL PROBABILITY DISTR.

FORMULATION:

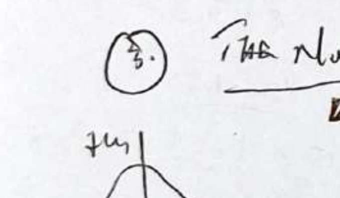
- Random variable Y has a normal distribution if:

$$\boxed{f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)} \quad \forall\ f(-\infty < y < \infty)$$



- w/ $E(\cdot)$ & $Var(\cdot)$ as:

$$\boxed{E(Y) = \mu} \quad \boxed{\sigma(Y) = \sigma^2}$$



NUMERICAL only:

- The integral doesn't have a closed form solution

$$\int_a^b \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{y^2}{2\sigma}}\,dy. \quad -y^2 = -(y-\mu)^2.$$

PROBLEMS 4.3

[29] THERMOSTAT AVERAGES: The Temperature Y at which a thermostatically controlled switch has a probability density f:

$$\boxed{f(y) = \begin{cases} \frac{1}{2} & 59 \le y \le 61 \\ 0 & \text{else} \end{cases}}$$

(?) Prove $E(Y) \le$ ...

Ans: $E(Y) = \int_{59}^{61} y f(y)\,dy = \int_{59}^{61} y \frac{1}{2}\,dy = \frac{1}{2}\frac{1}{2}(61^2 - 59^2) = 60°F$?

$V(Y) = \int_{59}^{61} y^2 \frac{1}{2}\,dy - 60^2 F = \frac{61^3 - 59^3}{3} - 60^2 = 3600.33 - 3600 = 0.3$

$= \boxed{\frac{1}{3}}$

[30] 40 HOUR/WEEK INDUSTRIAL ROBOT: The proportion of time Y that an industrial Robot is operating during a 40-hour week is a Random variable: $\boxed{f(y) = \begin{cases} 2y \\ 0 \end{cases}}$ $\begin{matrix} 0 \le y \le 1 \\ \text{else.} \end{matrix}$

[a] Find $E(Y), V(Y)$:

Ans. $E(Y) = \int_0^1 2y \cdot y\,dy = \int_0^1 2y^2\,dy = \left[2\frac{1}{3}y^3\right]_0^1 = \boxed{\frac{2}{3}}$

$V(Y) = \int_0^1 2y \cdot y^2\,dy - \left(\frac{2}{3}\right)^2 = \frac{2}{4}y^4\Big|_0^1 - \frac{4}{9} = \underbrace{\frac{2}{4}}_{E(Y^2)} - \underbrace{\frac{4}{9}}_{E(Y)^2} = \frac{1}{2} - \frac{4}{9} = \frac{9}{18} - \frac{8}{18} = \boxed{\frac{1}{18}}$

so $\boxed{\mu = \frac{2}{3}}$ $V(Y) = \boxed{\sigma^2 = \frac{1}{18}}$

[b] The Robot weekly profit is $X = 200Y - 60$ per week. Find $E(X)$ and $V(X)$.

$E(X) = E(200Y) - E(60) = 200E(Y) - E(60) = 200\left(\frac{2}{3}\right) - 60 = \frac{220}{3}$

Ans: $E(X) = E(200Y) - E(60) = 200\,E(Y) - E(60) = 200\left(\frac{2}{3}\right) - 60 = \frac{220}{3}$

$V(X) = E(X^2) - \left(\frac{220}{3}\right)^2 = E\left((200Y-60)(200Y-60)\right) - \left(\frac{220}{3}\right)^2$

$= E\left(40000Y^2 - 12000Y - 12000Y + 3600\right) - \frac{48.400}{9}$

$= 40.000\,E(Y^2) - 24000\,E(Y) + 3600 - \frac{48.400}{9}$

$= 40.000\left(\frac{1}{2}\right) - 2400\left(\frac{2}{3}\right) + 3600 - \frac{48.400}{9}$

$= 20.000 - 1.600 + 3600 - 5.377 = 18.400 + 3600 - 5.777 = 21.994$

[32] LPV ACCOUNTING FIRM

METHOD OF DISTRIBUTIONS (6.3)

STEPS:

$$\boxed{\begin{array}{ll} 1.) & U = u \text{ in } y_1 \ldots y_n \text{ space} \\ 2.) & U \le u \\ 3.) & F_U(u) = P(U \le u) \Rightarrow \int_u^U f(y_1 \ldots y_n)\, dy_1 \ldots dy_n \\ 4.) & \frac{\partial F_U}{\partial u} \end{array}}$$

EXAMPLES:

1. SUGAR PRODUCTION

$$\boxed{f(y) = \begin{cases} 2y \\ 0 \end{cases} \quad w/\ U = 3Y - 1}$$

w/ $3Y$ = \$300 per ton production of sugar

Y = \$100 cost.

$$\begin{cases} 2y : 0 \le y \le 1 \\ 0 : \text{Elsewhere} \end{cases}$$

ANS:

1.) $F_U(u) = P(U \le u) = P(3Y - 1 \le u)$

$= P\left(Y \le \frac{u+1}{3}\right)$

$$\Rightarrow P\left(Y \le \frac{u+1}{3}\right) = \int_{-\infty}^{\frac{u+1}{3}} f(y)\, dy = \int_0^{\frac{u+1}{3}} 2y\, dy = \left(\frac{u+1}{3}\right)^2$$

So: $P\left(Y \le \frac{u+1}{3}\right) = \left(\frac{u+1}{3}\right)^2$

2.) $F_U(u) = \begin{cases} 0 \\ \left(\frac{u+1}{3}\right)^2 \\ 1 \end{cases} \Rightarrow \frac{\partial F}{\partial u} = \frac{\partial}{\partial u}\left(\frac{u+1}{3}\right)^2 = \frac{\partial}{\partial u}\left(\frac{u}{3} + \frac{1}{3}\right)\left(\frac{u}{3} + \frac{1}{3}\right)$

$$= \frac{\partial}{\partial u}\left(\frac{u^2}{9} + \frac{u}{9} + \frac{u}{9} + \frac{1}{9}\right) = \frac{\partial}{\partial u}\left(\frac{u^2 + 2u + 1}{9}\right)$$

$$= \frac{2u + 2}{9} = \frac{2(u+1)}{9}$$

$\therefore$ So $\boxed{\frac{\partial F_U}{\partial u} = \frac{2}{9}(u+1)}$ w/ $\boxed{P(U \le u) = P(3Y - 1 \le u) = \left(\frac{u+1}{3}\right)^2}$

2. PURE EXAMPLE

$$\boxed{U = h(Y) = Y^2}$$

1.) $F_U(u) = P(U \le u) = P(Y^2 \le u) = P(-\sqrt{u} \le Y \le \sqrt{u})$

$$\Rightarrow P(-\sqrt{u} \le Y \le \sqrt{u}) = \int_{-\sqrt{u}}^{\sqrt{u}} f(y)\, dy = F_Y(\sqrt{u}) - F_Y(-\sqrt{u})$$

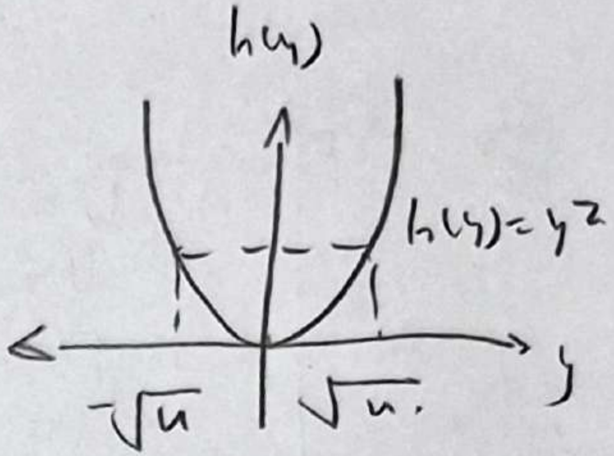
$$\Rightarrow F_U(u) = \begin{cases} F_Y(\sqrt{u}) - F_Y(-\sqrt{u}) & u > 0 \\ 0 & \text{otherwise.} \end{cases}$$

2.) $$\frac{\partial F}{\partial u} = \frac{\partial}{\partial u}\left[F_Y(\sqrt{u}) - F_Y(-\sqrt{u})\right]$$

$$= \frac{1}{2\sqrt{u}} f_Y(\sqrt{u}) - \frac{1}{2\sqrt{u}} f_Y(-\sqrt{u})$$

$$= \frac{1}{2\sqrt{u}}\left[f_Y(\sqrt{u}) - f_Y(-\sqrt{u})\right]$$

So: $$\boxed{\frac{\partial F_U}{\partial u} = \frac{1}{2\sqrt{u}}\left[f_Y(\sqrt{u}) - f_Y(-\sqrt{u})\right]}$$



Method of Transformations (6.4)

Theory

Let $U = h(Y)$ w/ $h(y)$ is either an increasing or decreasing function in y such that $f_Y(y) > 0$

$$\boxed{\begin{array}{ll} 1.) & y = h^{-1}(u) \\ 2.) & \dfrac{dh^{-1}}{du} = \dfrac{d[h^{-1}(u)]}{du} \\ 3.) & f_u = f_Y[h^{-1}(u)]\left|\dfrac{dh^{-1}}{du}\right|. \end{array}}$$

Examples

(1.) $f(y_1, y_2) = \begin{cases} 2(1-y_1) & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1. \\ 0 & 0 \end{cases}$

and $U = Y_1 Y_2$. (?) Find $E(U)$ & PDF(U)!

Ans. $U = h(Y_2) = y_1 y_2 \Rightarrow y_2 = \dfrac{u}{y_1} = h^{-1}(u).$

$$\Rightarrow \begin{cases} \dfrac{d[h^{-1}(u)]}{du} = \dfrac{d}{du}\dfrac{u}{y_1} = \dfrac{1}{y_1}. \\ f[y_1, h^{-1}(u)] = 2(1-y_1). \end{cases} \Rightarrow \boxed{g_u = 2(1-y_1)\left|\frac{1}{y_1}\right|.}$$

$$\Rightarrow f_U = \int_{-\infty}^{\infty} g_u\, du = \int_u^1 2(1-y_1)\left|\frac{1}{y_1}\right| dy_1 = \int_u^1 2\frac{1}{y_1} - 1\, dy_1 =$$

$$= \int_u^1 2\frac{1}{y_1} - \int_u^1 1\, dy_1 = 2\ln y_1\Big|_u^1 - y_1\Big|_u^1$$

$$= 2(\ln(1) - \ln(u)) - (\ln(1) - \ln u) = 2(u - \ln u - 1)$$

$$\therefore \text{So: } \boxed{f_U(u) = \begin{cases} 2(u - \ln u - 1) \\ 0 \end{cases}}$$

- For the Expected value:

$$E(u) = \int u f_u(u)\, du = \int_0^1 u\, 2(u - \ln u - 1)\, du = \int_0^1 2(u^2 - u\ln u - u)\, du.$$

$$= \left(\int_0^1 2u^2 - \int_0^1 2u\ln u - \int_0^1 2u\right) du$$

$$= \frac{2}{3}u^3\Big|_0^1 - 2\int_0^1 u\ln u\, du - u^2\Big|_0^1$$

$$= \frac{2}{3} - 2\left[\frac{u^2}{2}\ln u\Big|_0^1 - \int_0^1 \frac{u^2}{2}\left(\frac{1}{u}\right) du\right] - 1.$$

$$= \frac{2}{3} - 2\left[0 - \frac{u^2}{4}\Big|_0^1\right] - 1 = \frac{2}{3} - 2\frac{1}{4} - 1 = \frac{2}{3} - \frac{1}{2} - 1 = \boxed{\frac{1}{6}}$$

$$\therefore \text{So } \boxed{E(u) = \frac{1}{6}}$$

(2.) Sugar Production

$$\boxed{f_Y(y) = \begin{cases} 2y \\ 0 \end{cases} \quad \text{w/ Profit} : U = 3Y - 1}$$

(?) So find the density function w/ Transformation!

ANS: $U \Rightarrow h(u) = 3y - 1 \Rightarrow h^{-1}(u) = \frac{u+1}{3}$

$$\Rightarrow \frac{d[h^{-1}(u)]}{du} = \frac{d}{du}\frac{u+1}{3} = \frac{d}{du}\frac{u}{3} + \frac{1}{3} = \frac{1}{3}$$

$$\text{So: } g_u = f_Y(y)\left|\frac{d[h^{-1}(u)]}{du}\right| = 2\left(\frac{u+1}{3}\right)\left|\frac{1}{3}\right| = 2\left(\frac{u+1}{9}\right)$$

$$\text{So: } \boxed{f_U(u) = \begin{cases} 2\left(\frac{u+1}{9}\right) & 0 \le u \le 1 \\ 0 & \text{else} \end{cases}}$$

EXERCISES 6.3

(1.) $f(y) = \begin{cases} 2(1-y) & 0 \le y \le 1 \\ 0 & \end{cases}$

(a.) find Density $U_1 = 2Y - 1$
(b.) find Density $U_2 = 1 - 2Y$
(c.) Find Density $U_3 = Y^2$.
(d.) Find $E(U_1), E(U_2), E(U_3)$!

ANS:

(a.) $P(U_1 \le u) = P(2Y - 1 \le u) = P(2Y \le u-1) = P\left(Y \le \frac{u-1}{2}\right)$

$\Rightarrow P\left(Y \le \frac{u-1}{2}\right) = \int_y^{\frac{u-1}{2}} f(y)\,dy = \int_y^{\frac{u-1}{2}} 2(1-y)\,dy$

$= \int_y^{\frac{u-1}{2}} 2 - 2y\,dy = 2y - y^2 \Big|_0^{\frac{u-1}{2}}$

$= 2\left(\frac{u-1}{2}\right) - \left(\frac{u-1}{2}\right)^2 = u - 1 - \left(\frac{u-1}{2}\right)^2$

$\Rightarrow \frac{\partial F_u}{\partial u} = \frac{\partial}{\partial u}\left[u - 1 - \left(\frac{u-1}{2}\right)^2\right] = 1 - \frac{\partial}{\partial F}\frac{u^2 - 2u + 1}{4}$

$= 1 - \frac{2u-2}{4} = 1 - \frac{2(u-1)}{4}$

$= 1 - \frac{1}{2}u - 1 = \frac{2 - u - 1}{2} = \boxed{\frac{1-u}{2}}$

(b.) $P(U_2 \le u) = P(1 - 2Y \le u)$

$= P(-2Y \le u - 1)$

$= P\left(-Y \le \frac{u-1}{2}\right)$

$= P\left(Y \le -\left(\frac{1-u}{2}\right)\right) = \int_0^{\frac{1-u}{2}} f(y)\,dy = 2\left(\frac{1+u}{2}\right) - \left(\frac{1+u}{2}\right)^2$

$= 1 + u - \left(\frac{1+u}{2}\right)^2$

$\Rightarrow \boxed{\frac{\partial F_U}{\partial u} = \frac{u+1}{2}}$ Inverse of a)!

(c.) $P(U_3 \le u) = P(Y^2 \le u) = P(Y \le \sqrt{u}) = \int_0^{\sqrt{u}} f(y) = 2\sqrt{u} - u$

$\Rightarrow \frac{\partial F_U}{\partial u} = \frac{\partial}{\partial u}(2\sqrt{u}) - \frac{\partial}{\partial u}(u) = \cancel{2}\frac{1}{\cancel{2}}u^{\frac{1}{2}-\frac{2}{2}} - 1 = \frac{1}{u^{\frac{3}{2}}} - 1$

$= \frac{1}{\sqrt{u}} - 1$ So: $\boxed{\frac{\partial F_U}{\partial u} = \frac{1}{\sqrt{u}} - 1}$ wh $0 \le u < 1$.

(d.) $E(U_1) = E\left(U_1 \mid f_U = \frac{1-u}{2}\right) = \int u \cdot f_u\,du = \int_0^1 u\,\frac{1}{2}(u-1)\,du = \int_0^1 \frac{1}{2}u^2 - u\,du$. $\boxed{-\frac{1}{3}}$

$= \left(\int_0^1 \frac{1}{2}u^2 - \int_0^1 u\right)du = \frac{1}{2}\frac{u^3}{3}\Big|_0^1 - \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}\frac{1}{3} - \frac{1}{2}$

$= \frac{1}{6} - \frac{1}{2} = \frac{1}{6} - \frac{3}{6} = -\frac{2}{6}$

EXERCISES 6.4

(23) $f(y) = \begin{cases} 2(1-y) \; ; \; 0 \le y \le 1 \\ 0 \; ; \; \text{elsewhere} \end{cases}$

(a) $U = 2y - 1$?

ANS $h(u) = 2y-1 \Rightarrow U+1 = 2y \Rightarrow \boxed{y = \frac{U+1}{2}}$

$\therefore$ So $\boxed{h^{-1}(u) = \frac{u+1}{2}} = y$

$\cancel{u = 2-2y}$, $\cancel{\frac{-u+2}{2}} = y$

$\Rightarrow$ Then : $\frac{d\,h^{-1}(u)}{du} = \frac{d}{du}\left\{\frac{u}{2} + \frac{1}{2}\right\} = \frac{1}{2}$

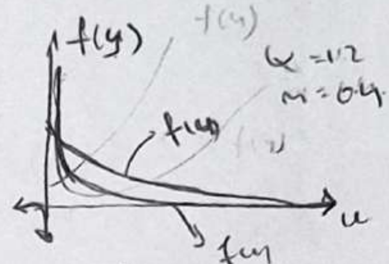
$\Rightarrow$ hence : $g_u = f(y)\,[h^{-1}(u)] \left|\frac{d\,h^{-1}(u)}{du}\right|$

$= 2(1-y)\left(\frac{1}{2}\right) = 2\left(1 - \frac{u+1}{2}\right)\frac{1}{2} = (2-u+1)\frac{1}{2}\boxed{\frac{u+1}{2}}$

$\Rightarrow$ So: $U = 2y-1 \Rightarrow \therefore \boxed{f_u = \frac{1-u}{2}}$

(26) WEIBULL DISTRIBUTION Form

$\boxed{f(y) = \frac{1}{\alpha} m y^{m-1} e^{-\frac{y^m}{\alpha}}}$, $y > 0$



? (a) $U = Y^m$ (b) $E(Y^k) \; \forall k \in \mathbb{Z}^+$?

ANS (a) $U = Y^m \Rightarrow \log U = m \log Y \Rightarrow \log Y = \log U^{\frac{1}{m}}$

$\Rightarrow e^{\log Y} = e^{\log U^{\frac{1}{m}}} \Rightarrow \boxed{Y = U^{m}}$ or $\boxed{Y = U^{\frac{1}{m}}}$.

$\Rightarrow$ So taking derivative: $\frac{dY}{dU} = \frac{1}{m} U^{\frac{1}{m} - \frac{m}{m}} = \frac{1}{m} U^{\frac{-m-1}{m}} =$

$\Rightarrow f_u = f_Y[h^{-1}(u)] \left|\frac{dy}{du}\right| = \frac{1}{\alpha} m y^{m-1} e^{-\frac{y^m}{\alpha}} \Big|_{Y = U^{\frac{1}{m}}} \left|\frac{1}{m} U^{-\frac{(m-1)}{m}}\right|$

$= \frac{1}{\alpha} m \left(u^{\frac{1}{m}}\right)^{m-1} e^{-\frac{\left(u^{\frac{1}{m}}\right)^m}{\alpha}} \left|\frac{1}{m} u^{\frac{m-1}{m}}\right| = \frac{1}{\alpha} \cancel{m}\, u^{\frac{m-1}{m}} e^{-\frac{u}{\alpha}} \left|\frac{1}{\cancel{m}} u^{-\frac{(m-1)}{m}}\right|$

$= \frac{1}{\alpha} \cancel{u^{\frac{m-1}{m}} u^{-\frac{m-1}{m}}} e^{-\frac{u}{\alpha}} = \frac{1}{\alpha} e^{-\frac{u}{\alpha}} \Rightarrow$ So: $\boxed{f_u = \frac{1}{\alpha} e^{-\frac{u}{\alpha}}}$

* See GRAPH above.

(b) $E(Y^k) = E\left(U^{\frac{k}{m}}\right) = \int_0^1 u^{\frac{k}{m}} f_Y \, du = \int_0^1 \frac{u^{\frac{k}{m}}}{\alpha} e^{\frac{u}{\alpha}} \, du$

$= \frac{1}{\alpha}\int_0^1 u^{\frac{k}{m}} e^{\frac{u}{\alpha}} \, du = \boxed{\Gamma\left(\frac{k}{m}+1\right) \alpha^{\frac{k}{m}}}$

(29) MOLECULE SPEED

$$\boxed{f(v) = av^2 e^{-bv^2}} \quad v > 0 \quad w/ \quad b = \frac{M}{2kT}, \; k = \text{Boltzmann's Constant}$$

(a) Derive Distribution of $W = \frac{1}{2}MV^2$, The kinetic energy of molecule

(b) Find $E(w)$

Ans (a) $W = \frac{1}{2}mv^2 \Rightarrow \sqrt{\frac{2W}{M}} = V$ so $V(W) = h^{-1}(W) = \sqrt{\frac{2W}{M}} \Rightarrow V'(w) =$

$$\Rightarrow f_W = f_V[h^{-1}(V)]\left|\frac{dV}{dW}\right|$$

$$= a\left(\sqrt{\frac{2W}{m}}\right)^2 e^{-b\left(\sqrt{\frac{2W}{m}}\right)^2} \cdot \frac{d}{dW}\left\{\sqrt{\frac{2}{M}}\sqrt{W}\right\} =$$

$$= a\frac{2W}{M} e^{-b\frac{2W}{m}} \frac{1}{\sqrt{2MW}} = a\frac{2W}{M\sqrt{2MW}} e^{-\frac{bW}{m}}$$

$$= a\frac{2W}{M^{\frac{3}{2}} 2W^{\frac{1}{2}}} e^{-\frac{bW}{M}} = a\frac{\sqrt{2W}}{M^{\frac{3}{2}}} e^{-\frac{bW}{m}}$$

$$\therefore \text{So: } \boxed{f_W = \frac{a\sqrt{2}}{M^{\frac{3}{2}}} W^{\frac{1}{2}} e^{-\frac{bW}{m}}}$$

(b) $$E(W) = \int W f_W(W)\,dW = \int_0^1 W \cdot \left\{\frac{a\sqrt{2}}{M^{\frac{3}{2}}} W^{\frac{1}{2}} e^{-\frac{bW}{m}}\right\} dW$$

$$= \int_0^1 \frac{a\sqrt{2}}{M^{\frac{3}{2}}} W^{\frac{3}{2}} e^{-\frac{bW}{M}} dW = \frac{1}{\Gamma\left(\frac{3}{2}\right)(kT)^{\frac{3}{2}}} W^{\frac{3}{2}} e^{\frac{bW}{M}}$$

$$= \frac{1}{(kT)^{\frac{3}{2}}} \frac{W^{\frac{3}{2}} e^{\frac{bW}{m}}}{\Gamma\left(\frac{3}{2}\right)} = \frac{1}{(kT)^{\frac{3}{2}}} \frac{W^{\frac{3}{2}} e^{\frac{M}{2kT}\frac{bW}{M}}}{\Gamma\left(\frac{3}{2}\right)}$$

$$= \frac{3}{2}kT \quad \text{So: } \boxed{E(W) = \frac{3}{2}kT}$$

→ Expected value is the Boltzmann Entropy.

MOMENT OF GENERATING FUNCTIONS

(6.5)

Probability PDF ?

$$\boxed{M_X(t) = M_Y(t)}$$

wh/ (X, Y) : Random Variables
t : Parameter

OR: $\boxed{M_U(t) = E(e^{tU})}$

AND in Respect to U :

$$\boxed{M_U(t) = \prod_i M_{Y_i}(t) \quad w/ \ U = \sum_i Y_i}$$

wh/ $M_U(t) = E\left(e^{t\sum_i Y_i}\right) = E\left(e^{tY_1}e^{tY_2}\cdots\right) = E(e^{tY_1})E(e^{tY_2})\cdots$

$= M_{Y_1} \times M_{Y_2} \times \cdots = \prod_i M_{Y_i}(t)$ QED

Expected Value and Variance

$$\boxed{\begin{aligned} U &= \sum_{i=1}^{n} a_i Y_i = a_1Y_1 + a_2Y_2 + \cdots + a_nY_n \\ E(U) &= \sum_{i=1}^{n} a_i \mu_i = a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n \\ \sigma^2(U) &= \sum_{i=1}^{n} a_i \sigma_i^2 = a_1\sigma_1^2 + a_2\sigma_2^2 + \cdots a_n\sigma_2^2 \end{aligned}}$$

$E(x)^2 \ (\sum x^2 \ldots)$ $x \sim N(\ldots)$

Proof: suppose: $U := M_U(t) = E(e^{tU})$

APPROX: $U \sim \sum_{i=1}^{n} \frac{t^n E(y_n)^n}{n!} = t^1E(y_n)^1 + \frac{1}{2}t^2E(y_n)^2 + \cdots$

$\sim tE(y_n) + \frac{1}{2}t^2E(y_n)^2 = t\mu_n + \frac{1}{2}t^2\sigma_n^2$

So: $M_U(t) = e^{Ut} = e^{\mu t + \frac{1}{2}\sigma^2 t^2} \Rightarrow \boxed{M_U(t) = \exp\left(\mu_i t + \frac{1}{2}\sigma_i^2 t\right)}$

$\Rightarrow M_{aY_i}(t) = E\left(e^{ta_iY_i}\right) = M_{Y_i}(a_i t)$

$= \exp\left(\mu_i a_i t + \frac{a_i^2\sigma_i^2 t^2}{2}\right) \Rightarrow \boxed{M_{a_iY_i}(t) = \exp\left(\mu_i a_i t + \frac{a_i^2\sigma_i^2 t^2}{2}\right)}$

QED.

$\Rightarrow \prod_i M_{Y_i}(t) = \prod_i \exp\left(\mu_i a_i t + \frac{1}{2}a_i^2\sigma_i^2 t^2\right).$

$= \exp\left(t\sum_{i=1}^{n} a_i\mu_i + \frac{t^2}{2}\sum_{i=1}^{n} a_i^2\sigma_i^2\right) \Leftrightarrow N\left(\sum_{i=1}^{n} a_i\mu_i, \sum_{i=1}^{n} a_i^2\sigma_i^2\right)$

So: $\boxed{\prod_i M_{Y_i}(t) = \exp\left(t\sum_{i=1}^{n} a_i\mu_i + \frac{t^2}{2}\sum_{i=1}^{n} a_i^2\sigma_i^2\right)}$

• Let $Y_1 \ldots Y_n$ be defined as above; form

$$\boxed{Z_i = \frac{Y_i - \mu_i}{\sigma_i}}$$

Then for: $\sum_{i=1}^{n} Z_i^2$ has a χ^2 distribution w/ n degrees of freedom.

EXAMPLES

① NORMAL DISTRIBUTION

• we wanna apply the Moment generating function for

$m_{z^2}(t)$

$$\boxed{f(z) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)}$$

• so:

$$M_{Z^2}(t) = E\left(e^{tZ^2}\right) = \int_{-\infty}^{\infty} e^{tz^2} f(z)\, dz = \int_{-\infty}^{\infty} e^{tz^2}\left[\frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-z^2}{2}\right)\right]$$

$$\text{So: } \boxed{M_{Z^2}(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(tz^2 - \frac{z^2}{2}\right)}$$

$$\Rightarrow M_{Z^2}(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(tz^2 - \frac{z^2}{2}\right) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(z^2\left(t - \frac{1}{2}\right)\right) dz$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2}\left(\cancel{+}2t+1\right)\right) dz = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{z^2}{2}(2t-1)\right) dz$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{-z^2/2}{(2t-1)^{-1}}\right) dz \quad \text{w/ } \begin{cases} \mu = 0 \\ \sigma^2 = (2t-1)^{-1} \end{cases} \rightarrow \sigma = (2t-1)^{-\frac{1}{2}}$$

$$= \frac{1}{(2t-1)^{-\frac{1}{2}}\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(\frac{z^2/2}{(2t-1)^{-1}}\right) dz$$

$$= \frac{1}{(2t-1)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \exp\left(\frac{z^2/2}{(2t-1)^{-1}}\right) \frac{1}{\sqrt{2\pi}(2t-1)^{\frac{1}{2}}}\, dz$$

$$= \frac{1}{(2t-1)^{\frac{1}{2}}} \{1\} = (1-2t)^{-\frac{1}{2}}$$

by Normalization: $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz = 1$

$$\Rightarrow f_U(u) = \begin{cases} \frac{u^{-\frac{1}{2}} e^{-\frac{u}{2}}}{\Gamma\left(\frac{1}{2}\right) 2^{\frac{1}{2}}} \end{cases}$$

CHAPTER 6.5 EXERCISES

BERNOULLI RANDOM VARIABLES

(37) Let $Y_1 \dots Y_n$ be Independent and Identically Distributed random Variables such That for $0<p<1$ $\begin{cases} P(Y_i=1) = p \\ P(Y_i=0) = q = 1-p \end{cases}$

(These are Bernolli Random variables)

(a) Moment Generating func ? (Y_1)

Ans $P(Y_i) = \begin{cases} p & \text{if } Y_i=1 \\ 1-q & \text{if } Y_i = 0 \end{cases} \Rightarrow P(Y_1) = \begin{cases} p & y_1 = 1 \\ 1-p & y_1 = 0 \end{cases}$

$\therefore$ so: $\boxed{P(Y_1) = \begin{cases} p \\ 1-p \end{cases}} \Rightarrow P(y_1) = p^y(1-p)^{1-y}$

$$m_Y(t) \Rightarrow E\left(e^{tY_1}\right) = \int^{\infty} e^{tY_1} P(Y_1)\,dY_1 = \int_0^{\infty} e^{tY} p^Y(1-p)^{1-y}\,dp$$

$$= \sum_{i=1}^{\infty} e^{ty} p^y (1-p)^{1-y} = 1-p+pe^t$$

so: $m_Y(t) = E\left(e^{tY_1}\right) = 1-p+pe^t$

(b) Find for Moment Generating func. $W = \sum_n Y_n$!

ANS $M_W(t) = E(e^{tW}) = E\left(e^{t\sum_i Y_i}\right) = \sum_{i=1}^{\infty} e^{t\sum Y_i} = \sum$

$$= \prod_{i=1}^{\infty} m_{Y_i}(t) = (1-p+pe^t)^n$$

$\therefore$ so: $\boxed{M_W(t) = \prod_{i=1}^{\infty} M_{Y_i}(t) = (1-p+pe^t)^n}$

(38) Prove that if $U = a_1Y_1 + a_2Y_2$ is a Moment Generating Function of $M_U(t) = mY_1(a_1t) \times mY_2(a_2t)$

Ans: $M_U(t) = E(e^{tU}) = E\left(e^{t(a_1Y_1 + a_2Y_2)}\right) = E\left(e^{ta_1Y_1} e^{ta_2Y_2}\right)$

$$= E\left(e^{t(a_1Y_1)}\right) E\left(e^{ta_2Y_2}\right) = m_{U_{Y_1}}(a_1t)\, m(Y_2(a_2t))$$ QED

so: $\boxed{M_U(t) = M_{U_{Y_1}}(a_1t) \times m_{Y_2}(a_2t) \Leftrightarrow U = a_1Y_1 + a_2Y_2}$

(40) Suppose Y_1 and Y_2 are Independent, standard normal Variables. Find the Density Function $U = Y_1^2 + Y_2^2$.

Ans:

$$M_u(t) = E(e^{ut}) = E(e^{Y_1^2+Y_2^2})$$
$$= M_{Y_1^2}(t) \cdot M_{Y_2^2}(t) \quad w/\ p(Y_n) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right)$$

$$\Rightarrow \begin{cases} M_{Y_1^2}(t) = \sum_{i=1}^{\infty} e^{Y_1^2 t} p(Y_1) = (1-2t)^{-1/2} \\ M_{Y_2^2}(t) = \sum_{i=1}^{\infty} e^{Y_2^2 t} p(Y_2) = (1-2t)^{-1/2} \end{cases} \text{by Example 1}$$

$$\Rightarrow M_{Y_1^2}(t) \times M_{Y_2^2}(t) = (1-2t)^{-\frac{1}{2}} \cdot (1-2t)^{-\frac{1}{2}} = (1-2t)^{-\frac{1}{2}+-\frac{1}{2}}$$
$$= (1-2t)^{-1} = \frac{1}{(1-2t)}$$

$$\therefore \text{So } \boxed{M_u(t) = \frac{1}{(1-2t)}}$$

for $\int_{-\infty}^{0} p(y) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{z^2}{2}} dz = \frac{1}{\sqrt{2\pi}}\sqrt{\pi} = \frac{\sqrt{\pi}}{\sqrt{2}\sqrt{\pi}} = \frac{1}{\sqrt{2}}$

(42) Elevator has a capacity Y_1 w/ mean 5000 pounds & SD 300 pounds. Elevator load has a capacity w/ mean 4000 pounds & SD 400 pounds. Assume it's Normally distributed, find the Probability that it will be overloaded!

Ans: Let: $U = Y_2 - Y_1$. Then by Theorem 3:

$$\begin{cases} E(U) = a_1\mu_1 \pm a_2\mu_2 = a_1 5000 \pm 4000 a_2 \\ \sigma^2(U) = a_1^2\sigma_1^2 \pm a_2\sigma_2^2 = a_1^2 300^2 \pm 400^2 a_2^2 \end{cases}$$

we: $\mu = 4000 - 5000 = -1000$
$\sigma^2 = 400^2 + 300^2 = 250.000$

$$\Rightarrow Z = \frac{Y - \mu}{\sigma} = \frac{Y - (-1000)}{\sqrt{250.000}} = \frac{Y + 1000}{\sqrt{250.000}}$$
$$= \frac{1000}{\sqrt{250.000}} \quad \text{So } \boxed{Z = \frac{1000}{\sqrt{250.000}}}$$

and: $P(Z: Y_1 - Y_2 > 0)$
$= P\left(Z > \frac{1000}{\sqrt{250.000}}\right) = P\left(Z > \frac{1000}{500}\right)$
$= P(Z > 2) = 0.0228$,

So the Probability that the Elevator will be overloaded is only 2.28%.

(?) why is it different?
$\{ \mu = \mu_2 - \mu_1 \Leftrightarrow Y_2 > Y_1$ = Weight of Content > Weight of Elevator

• How much to bid?

$$P\left(Z \ge \frac{c-\mu}{\sqrt{\sigma^2}}\right) = P\left(Z \ge \frac{c-182}{\sqrt{12.61}}\right) = P(Z \ge 2.33)$$

w/ $\boxed{c = 190.27}$ So the Construction worker needs to bid \$190.27 for minimal lost.

BINOMIAL

(49) Let Y_1 be a Binomial Random Variables w/ n-trials and Probability of success given by P. Let Y_2 be another Random Variable w/ n_2 trials and Prob. success by P.

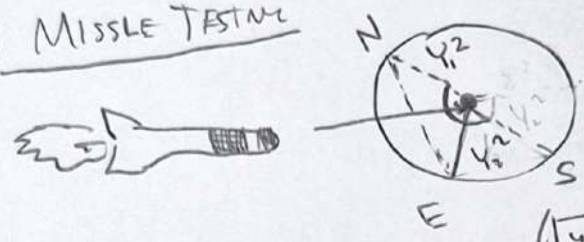
? Find Prob. fn of $Y_1 + Y_2$!

Ans: $U = P + (1-P) \Rightarrow m_U(t) = E(e^{Ut}) = E\left(e^{(P+(1-P))t}\right)$

$$= E\left(\exp([P+(1-P)]t)\right)$$

$$\Rightarrow \begin{cases} m_{Y_1}(t) = (e^tP + 1-P)^{n_1} \\ m_{Y_2}(t) = (e^tP + 1-P)^{n_2} \end{cases} \Rightarrow \boxed{Y_1 + Y_2 : \left[e^tP + (1-P)\right]^{n_1+n_2}}$$

(48) MISSLE TESTING



$U^2 = Y_1^2 + Y_2^2$

? Probability density Y_1, Y_2? Assume Y_1, Y_2 is Standard Normal!

Ans: $m_U(t) = E(e^{Ut}) = E\left(e^{\sqrt{Y_1^2+Y_2^2}\cdot t}\right) \equiv \chi^2$.

So: $\chi^2 = Y_1^2 + Y_2^2$

$$\Rightarrow f(V) = \frac{1}{2}e^{-V/2} \Leftrightarrow f(u) = ue^{-\frac{u^2}{2}}$$

w/ $F_V(u) = P(V \le u) = P(Y \le u^2) = F_V(u^2)$

EFFICIENCY

$$\boxed{\text{eff}(\hat\theta_1, \hat\theta_2) = \frac{V(\theta_2)}{V(\hat\theta_1)}}$$

- eff$\{V(\hat\theta_1), V(\hat\theta_2)\}$ w/ $\hat\theta_1$ = sample median, $\hat\theta_2$ = Sample mean
  (?) Estimate the mean of a Normal population

Ans: $\text{eff}(\hat\theta_1, \hat\theta_2) = \dfrac{V(\hat\theta_2)}{V(\hat\theta_1)} = \dfrac{\sigma^2/n}{1.2533\,\sigma^2/n} = \dfrac{1}{(1.2533)^2} = 0.6366 =$

So: 64% Represents the Variance of sample median sample mean.

- $\boxed{\hat\theta_1 = 2\bar{Y} \quad \hat\theta_2 = \left(\frac{n+1}{n}\right) Y_{(n)}}$ w/ $E(\hat{Y}_i) = \frac{\theta}{2}$, $\sigma^2(Y_i) = \frac{\theta^2}{12}$.

(1) $E(\hat\theta_1) = E(2\bar{Y}) = 2(\mu) = 2\frac{\theta}{2} = \theta$.

$$V(\theta_1) = V(2\bar{Y}) = 4V(\bar{Y}) = 4V\left[\frac{(Y_i)}{n}\right]$$
$$= \left(\frac{4}{n}\right)\left(\frac{\theta^2}{12}\right) = \frac{\theta^2}{3n}.$$

(2) $g_n(y) = n[F_Y(y)]^{n-1} f_Y(y) = \begin{cases} n\left(\frac{y}{\theta}\right)^{n-1}\left(\frac{1}{\theta}\right) \\ 0 \end{cases}$

$$E[Y_{(n)}] = \frac{n}{\theta^n}\int_0^\theta y^n\,dy = \left(\frac{n}{n+1}\right)\theta$$

$$V(Y_n) = E(Y_{(n)}^2) - [E(Y_n)]^2 = \left[\frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2\right]\theta^2.$$

$$V(\hat\theta_2) = V\left[\left(\frac{n+1}{n}\right)Y_{(n)}\right] = \left(\frac{n+1}{n}\right)^2 V(Y_{(n)}).$$
$$= \left[\frac{(n+1)^2}{n(n+2)} - 1\right]\theta^2 = \frac{\theta^2}{n(n+2)}.$$

(2.) $\text{eff}(\hat\theta_1, \hat\theta_2) = \dfrac{V(\hat\theta_2)}{V(\hat\theta_1)} = \dfrac{\theta^2/[n(n+2)]}{\theta^2/3n} = \boxed{\dfrac{3}{n+2}}$

EXERCISES 9.1

(1)

$$\boxed{f(y) = \begin{cases} 1/\theta \, e^{-y/\theta} \\ 0 \end{cases}}$$

$\hat{\theta}_1 = Y_1$

$\hat{\theta}_2 = \frac{(Y_1 + Y_2)}{2}$

$\hat{\theta}_3 = \frac{Y_1 + 2Y_2}{3}$

$\theta_5 = \bar{Y}$

ANS: $eff(\theta_1 \theta_5) = \frac{1}{3}$

$eff(\theta_2, \theta_5) = \frac{2}{3}$

$eff(\theta_3 \theta_5) = \frac{3}{5}$

(1) eff?

$$eff(\theta_1, \theta_5) = \frac{V(\hat{\theta}_1)}{V(\hat{\theta}_5)}$$

$$E(\hat{\theta}_1) = E[Y_1] = \int y_1 \frac{1}{\theta} e^{-\frac{y}{\theta}} dy$$

$$V(\hat{\theta}_1) = E(\hat{\theta}_1^2) - E(\hat{\theta}_1)^2$$

(2)

$$\boxed{\hat{\mu}_1 = \frac{1}{2}(Y_1 + Y_2) \quad \hat{\mu}_2 = \frac{1}{4} Y_1 + \frac{Y_2 + \cdots Y_{n-1}}{2(n-2)} + \frac{1}{4} Y_n \quad \mu_3 = \bar{Y}}$$

(a) (1) show that the estimator is unbiased. (ie) $(\forall = \mu)$!

ANS

$$E(\hat{\mu}_1) = \frac{1}{2}\left(E(Y_1) + E(Y_2)\right) = \frac{1}{2}(\mu + \mu) = \frac{1}{\cancel{2}} \cancel{2}\mu = \mu$$

$$E(\hat{\mu}_2) = \frac{1}{4} E(Y_1) + \frac{\sum_{i=2}^{n} E(Y_{n-1})}{2(n-2)} + \frac{1}{4} E(Y_n)$$

$$= \frac{1}{4}\mu + \frac{(n-\cancel{2})\mu}{2(\cancel{n-2})} + \frac{1}{4}\mu = \frac{1}{4}\mu + \frac{1}{4}\mu + \frac{1}{2}\mu$$

$$= \frac{2}{4}\mu + \frac{1}{2}\mu = \frac{2}{4}\mu + \frac{2}{4}\mu = \frac{4}{4}\mu = \mu$$

$$E(\hat{\mu}_3) = \mu$$

(b) Calculate eff $(\mu_3 ; \hat{\mu}_2 \, \hat{\mu}_1)$ respectively

$$V(\hat{\mu}_1) = \frac{1}{4}(\sigma^2 + \sigma^2) = \frac{1}{2}\sigma^2,$$

$V(\hat{\mu}_2) = \frac{?\sigma^2}{16} + \frac{(n-2)\sigma^2}{4(n-2)}$ $= \frac{\sigma^2}{8} + \sigma^2$

$$V(\hat{\mu}_1) = \frac{\sigma^2}{16} + \frac{\sigma^2}{16} + \frac{(n-2)\sigma^2}{4(n-2)^2}$$

$$= \frac{2\sigma^2}{16} + \frac{(n)\sigma^2}{4(n-2)^2}$$

$$= \frac{\sigma^2}{8} + \frac{\sigma^2}{4(n-2)^2}$$

$$V(\hat{\mu}_3) = \frac{\sigma^2}{n}.$$

$$\boxed{\text{eff}(\hat{\theta}_3 ; \hat{\theta}_1) = \frac{n^2}{8(n-2)}}$$

$$\boxed{\text{eff}(\hat{\mu}_3, \mu_1) = \frac{\frac{1}{n}\sigma^2}{\frac{1}{2}\sigma^2} = \frac{n}{2}}$$

(3) Let $Y_1 \dots Y_n$ denote a Random Sample from the Uniform Distribution on the Interval $(\theta, \theta+1)$. Let:

$$\boxed{\hat{\theta}_1 = \bar{Y} - \frac{1}{2} \qquad \hat{\theta}_2 = Y_{(n)} - \frac{n}{n+1}}$$

(a) Show that $(\hat{\theta}_1, \hat{\theta}_2)$ are Unbiased Estimators of θ.

(b) Find the efficiency of $(\hat{\theta}_1, \hat{\theta}_2)$.

Ans:

$$E(\hat{\theta}_1) = E(\bar{Y}) - \frac{1}{2} = \theta + \frac{1}{2} - \frac{1}{2} = \theta.$$

$$E(\hat{\theta}_2) = Y_{(n)} - \frac{n}{n+1} = n(y-\theta)^{n-1}$$

$$\text{w/ } g_n(y) = n(y-\theta)^{n-1} \quad \text{w/ } \hat{\theta}_2 = Y_{(n)}$$

EXERCISES 8.R

① $E[(\hat\theta - \theta)] = [\hat\theta - E(\hat\theta)] + [E(\hat\theta) - \theta]$
$= [\hat\theta - E(\hat\theta)] + B(\hat\theta)$

⑦ $MSE(\hat\theta) = E[(\hat\theta - \theta)^2] = V(\hat\theta) + (B(\hat\theta))^2$

ANS: $E[(\hat\theta - \theta)^2] = E[\hat\theta^2 - 2\hat\theta\theta - \theta^2]$

$\cancel{= E[\hat\theta^2] - 2E[\hat\theta\theta] - E[\theta^2]}$

$= ([\hat\theta - E(\hat\theta)] + B(\hat\theta))([\hat\theta - E(\hat\theta)] + B(\hat\theta))$

$= (\hat\theta - E[\hat\theta])^2 + 2B[\hat\theta - E(\hat\theta)] + B(\hat\theta)^2$

$= \hat\theta^2 - 2E(\hat\theta)\hat\theta - E(\hat\theta)^2 + 2\hat\theta - 2E\hat\theta + B(\hat\theta)^2$

$= V(\hat\theta) - B(\hat\theta)^2$. so $\boxed{MSE = V(\hat\theta) + B(\hat\theta)^2}$

⑥ $E(\hat\theta_1) = E(\hat\theta_2) = \theta$

$V(\hat\theta_1) = \sigma_1^2$

$V(\hat\theta_2) = \sigma_2^2$

$\theta_3 = a\hat\theta_1 + (1-a)\hat\theta_2$

(a) Show that $E(\theta_3)$ is unbiased!

$E(\theta_3) = aE(\hat\theta_1) + (1-a)E(\hat\theta)$
$= a\theta + (1-a)\theta$
$= a\theta + \theta - a\theta \longrightarrow \boxed{E(\theta_3) = \theta}$
$= \theta$

(b) $(\hat\theta_1, \hat\theta_2)$ indp, const a is to be minimize w/ $V(\hat\theta_3)$?

$V(\theta_3) = aV(\hat\theta_1 a) + (1-a)V(\hat\theta_2)$
$= a^2\sigma_1^2 + (1-a)\sigma_2^2 = a^2\sigma_1^2 + \sigma_2^2 - a\sigma_2^2$

$\frac{dV}{da} = 2a\sigma_1^2 + \sigma_2^2 - a\sigma_2^2 = 2a\sigma_1^2$

$\frac{d^2V}{da^2} \quad 2\sigma_1^2$

$$\frac{dV(\theta_1)}{da} = \frac{d}{da}\left\{a(\sigma_1^2 - \sigma_2^2) + \sigma_2^2\right\} = \sigma_1^2 - \sigma_2^2 + \sigma_1^2$$

$$= \frac{d}{da}\left(a^2\sigma_1^2 + \sigma_2^2 - a\sigma_1^2\right)$$

$$= 2a\sigma_1^2 - \sigma_1^2$$

CONSISTENCY

(9.3)

① Binomial Distribution of H w/ n tosses

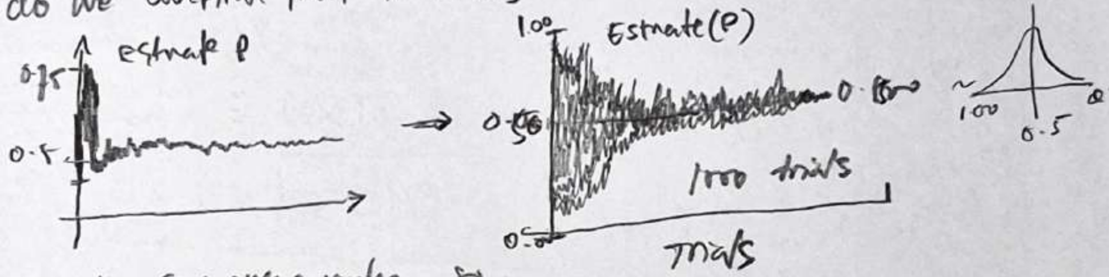
[i] ? what happens to the sample portion $\frac{Y}{n}$ w/ $\hat{p} = \frac{Y}{n}$ on 1000 Bernoulli trials

$p(Y_1) = p(Y_2) = \dots p(Y_n) = \frac{1}{2}$ so number of heads w/ Independent tosses grows exponentially if $p(Y) = \frac{1}{(2)^n}$

now $\hat{p} = \frac{1}{2}$, so: $\hat{p}: \frac{1}{2} = \frac{Y}{n}$ s.t $\boxed{2Y = n}$

[ii] ? How do we confirm that it actually converges 0.5?

for that:



Let ε be the convergence value. so:

$\hat{p} = \frac{Y}{n} \Rightarrow \hat{p} - \frac{Y}{n} = \frac{Y}{n} - p = 0$. so:

$$\boxed{p\left(\left|\frac{Y}{n} - p\right| \leq \varepsilon\right)}$$ If as $n \to \infty$ $p(\cdot) = 1$, Then $\frac{Y}{n}$ is a Consistent Estimator for e

[iii] Formal way:
$$\boxed{\begin{array}{l} \lim_{n \to \infty} p(|\hat{\theta}_n - \theta| \leq \varepsilon) = 1 \\ \lim_{n \to \infty} p(|\hat{\theta}_n - \theta| > \varepsilon) = 0 \end{array}}$$ Estimator $\hat{\theta}_n$ consistency

② UNBIASED ESTIMATOR

If $V(\hat{\theta}_n)$ is the variance of parameter $\hat{\theta}_n$, then

It's unbiased iff: $\boxed{\lim_{n \to \infty} V(\hat{\theta}_n) = 0}$

[i] PROOF: $P\left(\left|\frac{Y-\mu}{\sigma}\right| > k\right) = p(|Y - \mu| > k\sigma) = P(Y - \mu > k\sigma) \leq \frac{1}{k^2}$

↘ Tchebysheff's Theorem

2) Since $\hat{\theta}_n$ is an unbiased estimate for θ, it follows that $E(\hat{\theta}_n) = \theta$, $\sigma_{\hat{\theta}_n} = \sqrt{V(\hat{\theta}_n)}$ so:

$$P(|Y-\mu| > k\sigma) \rightarrow P\left(Y - E(\hat{\theta}_n) > k\sqrt{V(\hat{\theta}_n)}\right) = \boxed{P(|\hat{\theta}_n - \theta| > k\sigma_{\hat{\theta}_n}) \le \frac{1}{k^2}}$$

3) Let n be any fixed sample size. $\forall \varepsilon \in \mathbb{R}^+$

$$\boxed{k = \frac{\varepsilon}{\sigma_{\hat{\theta}_n}}}$$ w/ ε = Convergence, σ = volatility thus $k \propto$ efficiency

So: $P(|\hat{\theta}_n - \theta| > \varepsilon) = P\left(|\hat{\theta}_n - \theta| > \left[\frac{\varepsilon}{\sigma_{\hat{\theta}_n}}\right]\sigma_{\hat{\theta}_n}\right)$

$$= P(|\hat{\theta}_n - \theta| > \varepsilon) \le \frac{1}{(\varepsilon/\sigma_{\hat{\theta}_n})^2} = \frac{V(\hat{\theta}_n)}{\varepsilon^2}$$

Ergo for $\forall$ fixed (n): $0 \le P(|\hat{\theta}_n - \theta| > \varepsilon) \le \frac{V(\hat{\theta}_n)}{\varepsilon}$

4.) take the limit $n \to \infty$, w/ $\lim_{n\to\infty} V(\hat{\theta}_n) = 0$, so:

$$\lim_{n\to\infty} 0 \le \lim_{n\to\infty} P(|\hat{\theta}_n - \theta| > \varepsilon) \le \lim_{n\to\infty} \frac{V(\hat{\theta}_n)}{\varepsilon} = 0$$

Ergo: $\boxed{\lim_{n\to\infty} P(|\hat{\theta}_n - \theta| > \varepsilon) = 0}$

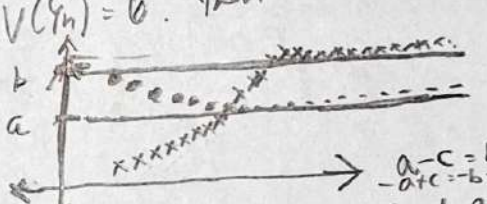
II Example - 9.2: $Y_1 \ldots Y_n$ is a sample distribution. Show that

(a) w/ $N(\mu, \sigma^2)$, $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^{n} Y_i$ is consistent w/ estimator μ.

Ans: $E(\bar{Y}_n) = E(Y_n) = \sum_{i=1}^{n} Y_i P(Y_i) = \sum_{i=1}^{n} Y_i \frac{1}{N} = \frac{1}{N}\sum_{i=1}^{n} Y_i = \boxed{\mu}$

$V(\bar{Y}_n) = \frac{\sigma^2}{n}$ and since $\lim_{n\to\infty} V(\bar{Y}_n) = 0$. Then

$$\lim_{n\to\infty} P(|\hat{\theta}_n - \mu| < \varepsilon) = 0$$



$a-c = b-c$
$-a+c = -b+c$
$a+b$

III Remark 9.2 / Th

1.) $\hat{\theta}_n + \hat{\theta}'_n \longrightarrow \theta + \theta'$
2.) $\hat{\theta}_n \times \hat{\theta}'_n \longrightarrow \theta \times \theta'$
3.) If $\theta' \neq 0$, then $\frac{\hat{\theta}_n}{\hat{\theta}'_n} \longrightarrow \frac{\theta}{\theta'}$

Resemblance of sequence in Real Numbers Converging to Real limits w/ s.t. $\begin{cases} a_n \to a \\ b_n \to b \end{cases}$ Then

$\boxed{a_n + b_n \longrightarrow a+b}$

IV Example - 9.3

Slutsky's Theorem

(9.3)

(1.) Book: Suppose U_n has a distribution func $f(u) \to N(\mu, \sigma)$ as $n \to \infty$. If $W_n \to 1$, then $\frac{U_N}{W_N} \to N(\mu, \sigma)$

Wiki: $\boxed{\text{Let } (X_n, Y_n) \in \Omega \text{ s.t } \Omega \text{ is a Random set. Then } X_n + Y_n \xrightarrow{d} X + C; \quad X_n Y_n \xrightarrow{d} XC; \quad X_n / Y_n \to C}$

[1] Proof: Let: $W_n \to \omega$. Since $U_n \to N(\mu, \sigma)$.

1.) This means that:

$$\lim_{N \to \infty} \frac{U_N}{W_N} = \frac{N(\mu, \sigma)}{\omega} = \boxed{N\left(\frac{\mu}{\omega}, \frac{\sigma^2}{\omega^2}\right)}^{*} \quad \text{w/ } E(W) = \omega, \ \sigma^2(W) = $$

*w/ By slutky's Th: $\frac{X_n}{Y_n} \to \frac{X}{C}$ s.t $X \mapsto N(\mu, \sigma)$, $C \mapsto \omega$

2.) But suppose W_n is a Rand. variable, then via the delta method, we setup:

$$\sqrt{n}\,(U_n - \mu) \xrightarrow{d} N(0, \sigma^2) \quad \text{w/ } (U_n, W_n) \text{ is jointly Asymptotically Normal.}$$

$$\sqrt{n}\,(W_n - \omega) \xrightarrow{d} N(0, \tau^2)$$

- The delta method states: $\sqrt{n}\,(h(\hat{\beta}) - h(\beta)) \xrightarrow{d} N(0, \nabla h(\beta)^T \Sigma \nabla h(\beta))$

If we define $g(\mu, \omega) = \frac{\mu}{\omega}$, then the variance is $\sigma^2(g) = \nabla g^T \Sigma \nabla g$.

Thus: $\boxed{\sqrt{n}\left(\frac{U_n}{W_n} - \frac{\mu}{\omega}\right) \xrightarrow{d} N(0, \nabla g^T \Sigma \nabla g)}$ → kinda like the portfolio theory in finance

w/ $\nabla g = \left(\frac{1}{\omega}, -\frac{\mu}{\omega^2}\right) = \left(\frac{\partial g}{\partial \mu}, \frac{\partial g}{\partial \omega}\right)$.

3.) This implies:

$$\boxed{\frac{U_n}{W_n} \approx N\left(\frac{\mu}{\omega}; \frac{\sigma^2}{\omega^2} + \frac{\mu^2 \tau^2}{\omega^4} - \frac{2\mu\rho\sigma\tau}{\omega^3}\right)}$$

$$\text{w/ } Var\left(\frac{U}{\omega}\right) = \nabla_\mu^2 g \cdot \sigma_U^2 + \nabla_\omega^2 g \cdot \sigma_W^2 + 2 \nabla_\mu^2 g \nabla_g^2 g \cdot \rho \sigma_U \cdot \sigma_W$$

$$= \frac{1}{w^2}\sigma_U^2 + \frac{\mu}{\omega^2}\frac{\tau^2}{\omega^4} - \frac{2}{\omega}\frac{\mu}{\omega^2}\rho\sigma_U\sigma_W$$

Then by slutsky's Th. $\boxed{G(U_n, W_n) \sim N(E(g), \sigma^2(g))}$

Example 9.4 - STD Deviation Sample

Let $Y_1, \ldots Y_n$. w/ $E(Y_i) = \mu$, $V(Y_i) = \sigma^2$. Define S_n^2 as:

$$\boxed{S_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y}_n)^2}$$

(Q) Show that the distb: $\sqrt{n}\left(\frac{Y_n - \mu}{S_n}\right)$ converges to a normal distribution

Ans: Suppose:

$$U_n := \sqrt{n}(U_n - \mu) \xrightarrow{d} N(0, \sigma^2)$$
$$W_n := \sqrt{n}(W_n - \omega) \xrightarrow{d} N(0, \sigma^2)$$

Then: $\frac{U_n}{W_n} \xrightarrow{d} \sqrt{n}\left(\frac{U_n}{W_n} - \frac{\mu}{\omega}\right) \xrightarrow{d} N\left(0, Var\left(\frac{U}{W}\right)\right)$

Now: $\sqrt{Var\left(\frac{U}{W}\right)} = \sqrt{\frac{S_n^2}{\sigma^2}} = \frac{S_n}{\sigma}$ and $Z = \frac{\bar{Y} - \mu}{\sigma}$

s.t if $U_n = \sqrt{n}\left(\frac{\bar{Y}_n - \mu}{\sigma}\right)$ Then:

$$\lim_{n \to \infty} \frac{U_n}{W_n} = \frac{\sqrt{n}\left(\frac{\bar{Y}_n - \mu}{\sigma}\right)}{\frac{S_n}{\sigma}} = \boxed{\sqrt{n}\left(\frac{\bar{Y}_n - \mu}{S_n}\right)}$$

wh $\lim_{n \to \infty} U_n = N(z)$ & $\sqrt{n} \cdot \left(\frac{\bar{Y}_n - \mu}{S_n}\right)$

EXERCISE 9.3

(21) Let $Y_1 \dots Y_n$ be a random sample size of n from a normal population mean μ and variance σ^2. Assuming $n = 2k$ for some Integer k, Consider the estimator:

$$\boxed{\hat{\sigma}^2 = \frac{1}{2k}\sum_{i=1}^{k}(Y_{2i} - Y_{2i-1})^2}$$

(a) Show that $\hat{\sigma}^2$ is an unbiased estimator for σ^2 (b.) Also show its consistent

Ans: (a) $E(\hat{\sigma}^2) = \frac{1}{2k}\sum_{i=1}^{k} E[(Y_{2i} - Y_{2i-1})^2] = \frac{1}{2k} k 2 Var(Y_{2i})$

$= \frac{1}{2k} k 2\sigma^2 = \frac{\cancel{2k}}{\cancel{2k}}\sigma^2 = \boxed{\sigma^2}$

(b) σ^2 is consistent since $\frac{1}{k}\sum_{i=1}^{k} X_i \xrightarrow{P} E[X_i] = \boxed{\sigma^2}$

or formally: $V(\hat{\sigma}^2) = E(\sigma^2) - E(\sigma^2)^2 = \frac{k 2\sigma^4}{k^2} = \frac{2\sigma^4}{k}$.

$= \frac{2\sigma^4}{\frac{n}{2}} = \frac{2}{n} 2\sigma^4 = \frac{4\sigma^4}{n}$.

Now as $n \to \infty$: $\frac{4\sigma^4}{\infty} = \boxed{0}$ so $\boxed{\lim_{n\to\infty} V(\hat{\sigma}^2) = 0}$ ✓

(22) Refer to (21), but now suppose it's a poisson distribution.

w/ $\boxed{\hat{\lambda} = \frac{1}{2k}\sum_{i=1}^{k}(Y_{2i} - Y_{2i-1})^2}$

(a) Show that $\hat{\lambda}$ is an unbiased estimator for λ (b.) Also show that it's Consistent

Ans: $f(y) = \frac{\lambda^y e^{-\lambda}}{y!}$ $E(Y) = \lambda$

$V(Y) = \lambda$

and $E(\hat{\lambda}) = \frac{2k}{2k} V(Y) = \frac{2k}{2k}\lambda = \boxed{\lambda}$

and: $\sigma_{\lambda} = \frac{2\lambda^4}{\frac{n}{2}} = \frac{2}{n} 2 \cdot \lambda^4 = \frac{4\lambda^4}{n}$ as $\lim_{n\to\infty} \frac{4\lambda^4}{n} = 0$

so $\boxed{V(\hat{\lambda}) = 0}$

SUFFICIENCY OF ESTIMATORS (9.4)

SUFFICIENCY ? = Summarize to (μ, σ^2).

(1) $\boxed{X_i = \begin{cases} 1 & \text{succ.} \\ 0 & \text{fail} \end{cases}}$

$X_i = \begin{cases} 1 & \text{w/ } p \\ 0 & \text{w/ } 1-p = q(p) \end{cases}$

$n \to \infty$, we obtain more information for y!

$$P(X_1 = x_1, \ldots, X_n = x_n \mid Y = y) = \frac{P(X_1 = x_1 \ldots X_n = x_n, Y = y)}{P(Y = y)}$$

$$p(X_1 = x_1 \ldots X_n = x_n \mid y = Y) = \begin{cases} \dfrac{\cancel{p^y (1-p)^{n-y}}}{\binom{n}{y} \cancel{p^y (1-p)^{n-y}}} = \dfrac{1}{\binom{n}{y}} & \text{if } \sum_{i=1}^{n} x_i = y \\ 0 & \end{cases}$$

conditional probability

So: $\boxed{P(x_1 \ldots x_n \mid y) = \dfrac{P(x_1 \ldots x_n ; y)}{P(y)}}$

(2) (1) Let $(Y_1 \ldots Y_n)$ Denote a Random sample from Probability Distribution θ. Then:

$\boxed{U = g(Y_1 \ldots Y_n) \text{ is Sufficient for } \theta}$

(2) Let $(y_1 \ldots y_n)$ be sample observations taken on corresponding Random Variables $(Y_1 \ldots Y_n)$ w/ θ Parameter. Then:

$\boxed{\begin{array}{l} L(Y_1 \ldots Y_n \mid \theta) \\ L(y_1 \ldots y_n \mid \theta) \end{array}}$ = Likelihood sample w/ $\{Y_n\}$ Discrete = Joint Density at $y_1 \ldots y_n$

Sufficiency (9.4)

1. Let $X_1 \ldots X_n$ Denote the trial of a Binomial experiment
w/ $X_i = \begin{cases} 1 & i^{th} \text{ trial success} \\ 0 & i^{th} \text{ trial fails} \end{cases}$

So: $\boxed{X_i = \begin{cases} p = 1 \\ 1-p = 0 \end{cases}}$

⚠ Sufficiency: means that the Parameters summarize all info

Suppose we're given $Y = \sum_{i=1}^{n} X_i$, The number of successes of n-trials.

? If we know Y, Can we gain any further Information about p by Looking conditional of $X_1 \ldots X_n$?

i. $P(\vec{X} | \vec{Y}) = \dfrac{P(\vec{X}, \vec{Y})}{P(\vec{Y})} = \dfrac{P(X_1 \ldots X_n = x_1 \ldots x_n ; Y = y)}{P(y)}$ w/ $y = Y$.

If $X_i \neq y$; $Y = y = \sum_{i=1}^{n} x_i = x_1 + \ldots + x_n$

So: $y = x_1 + \ldots + x_n \approx \dfrac{x(x+1)}{2}$ w/ $x_N \sim N$

$$\boxed{P(\vec{X} | \vec{Y}) = \frac{0}{P(y)} \text{ Iff } \sum_1 x_i \neq y}$$

Equ: $P(\vec{X} | \vec{Y} = y) = \dfrac{p^y (1-p)^{n-y}}{\binom{n}{y} p^y (1-p)^{n-y}} = \dfrac{1}{\binom{n}{y}}$ iff $\sum_{i=1}^{n} X_i = y$ otherwise 0.

This is Cool! , since $P(\vec{X} | \vec{Y})$, It doesn't depend on p anymore.

ii. THEOREM 9.3 :

1) Let $Y_1 \ldots Y_n$ denote a Sample w/ a Prob. dist. of θ.
Let $X_1 \ldots X_n$ will reveal Info on p s.t.
If Y is known, no further $X_1 \ldots X_n$ will

$\boxed{U = g(Y_1 \ldots Y_n) \text{ Is Sufficient for } \theta.}$

2) Now instead of we say that $p(y)$ = discrete or $f(y)$ for continuous
we know say: $\boxed{\begin{array}{l} p(y|\theta) = \text{value of } y \text{ given } \theta \text{ / discrete} \\ f(y|\theta) = \text{value of } y \text{ given } \theta \text{ / Continuous} \end{array}}$

iii. DEFINITION 9.4
Let $y_1 \ldots y_n$ be sample observations. Then

$\boxed{L(\vec{y}|\theta) = L(y_1 \ldots y_n | \theta) \text{ Is the Likelihood of sample.}}$

[1v] THEOREM 9.4

1) Let U be a statistic based on $Y_1 \dots Y_n$. Then U is a sufficient statistic for θ iff $L(\theta) = L(y_1 \dots y_n | \theta)$ can be factored into 2 func:

$$\boxed{L(y_1 \dots y_n | \theta) = g(u, \theta) \times h(y_1 \dots y_n)}$$

2) If $Y_1 \dots Y_n$ denotes a random sample s.t.

$$L(y_1 \dots y_n | \theta) = p(y_1 \dots y_n | \theta) = p(y_1 | \theta) \times p(y_2 | \theta) \times \dots \times p(y_n | \theta)$$

$$= \prod_{i=1}^{n} p(y_i | \theta) \quad \text{\# similar fashion for } f(y).$$

so
$$\boxed{L(\theta) = \begin{cases} \prod_{i=1}^{n} f(y_i | \theta) \\ \prod_{i=1}^{n} p(y_i | \theta) \end{cases}}$$

(2.) EXAMPLES:

[1] Let $Y_1 \dots Y_n$ be a rand. sample s.t. $f(y_i | \theta) = \frac{1}{\theta} e^{-\frac{y_i}{\theta}}$ if $0 < y_i < \infty$. w/ $\theta > 0$, $i = 1 \dots n$. Show that $\bar{Y}$ is a sufficient statistic.

ANS:

1) $$L(\theta) = \prod_{i=1}^{n} f(y_i | \theta) = \prod_{i=1}^{n} \frac{1}{\theta^n} e^{-\frac{y_i}{\theta}} = \frac{1}{\theta} \prod_{i=1}^{n} e^{-\frac{y_i}{\theta}} = \frac{e^{-\frac{y_1}{\theta}}}{\theta} \frac{e^{-\frac{y_2}{\theta}}}{\theta} \cdots \frac{e^{-\frac{y_n}{\theta}}}{\theta}$$

$$= \frac{1}{\theta^n} e^{-\frac{y_1 + y_2 + \dots y_n}{\theta}} = \frac{1}{\theta^n} e^{-\frac{\sum^n y_i}{\theta}} = \frac{1}{\theta^n} e^{-\frac{n\bar{y}}{\theta}}$$

2) see that: $g(u, \theta) = g(\bar{y}, \theta) = \frac{e^{-\frac{n\bar{y}}{\theta}}}{\theta^n}$ and $h(\vec{y}) = 1$.

so: $$\boxed{L(\theta) = g(\bar{y}, \theta) \times h(\vec{y})}$$

EXERCISE 9.3

(7) Suppose $X_1 \dots X_n$ & $Y_1 \dots Y_n$ are ind. rs w/ (μ_1, μ_2) + (σ_1^2, σ_2^2)

Show that $\bar{X} - \bar{Y}$ is a consistent estimator $\mu_1 - \mu_2$

Ans. $E[\bar{X}_1 - \bar{X}_2] = \mu_1 - \mu_2$

$V[\bar{X} - \bar{Y}] = V(\bar{X}) + V(\bar{Y}) = \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{n} = \frac{\sigma_1^2 + \sigma_2^2}{n}$

now $\lim_{n \to \infty} \frac{\sigma_1^2 + \sigma^2}{n} = 0$ so $\boxed{V(\bar{X} - \bar{Y}) \to 0 \text{ as } n \to \infty}$

(8) suppose populations are normally distributed $\sigma_1^2 = \sigma_2^2 = 0$.

show that: $\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2 + \sum_{i=1}^{n} (y_i - \bar{y})^2}{2n-2}$

is a consistent estimator of σ^2

Ans: $E\left[\frac{\sum^n (x_i - \bar{x})^2 + \sum^n (y_i - \bar{y}^2)}{2n-2}\right]$

$= \frac{1}{2n-2}\left(\sum^n E[(x_i - \bar{x})^2] + \sum E[(y_i - \bar{y})^2]\right)$

$= \frac{1}{2n-2}\left(\sum^n E[x_i^2 - 2x_i\bar{x} - \bar{x}^2] + \sum E(y_i^2 - 2y_i - \bar{y}^2)\right)$

$= \frac{1}{2n-2}\left(\sum^n [E(x_i^2) - 2E(x_i\bar{x}) - E(\bar{x}^2) + E(y_i^2) - 2E(y_i\bar{x}) + E(\bar{y}^2)\right)$

$\stackrel{\Delta}{=} \frac{1}{2n-2}\left(\sum^n Var(x_i) - Var(y_i)\right) = \frac{1}{2n-2}\left(\sigma^2 - \sigma^2\right) = \boxed{0}$

$\frac{1}{2n-2}\left(\sum^n Var(x_i) - Var(y_i)\right)$

and since $Var(x_i) = Var(y_i) = \sigma_1^2 = \sigma_1^2 = \sigma$

More formal: $E\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = \sigma^2$

w/ $S_X^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

$S_Y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$

so $\boxed{\hat{\sigma}^2 \xrightarrow{P} \sigma^2}$

and: $\hat{\sigma}^2 = \frac{(n-1)S_X^2 + n(-1)S_Y^2}{2n-2} \Rightarrow E(\hat{\sigma}^2) = \frac{2(n-1)\sigma^2}{2n-2} = \boxed{\sigma^2}$

(19) Let $Y_1 \dots Y_n$ denote a Random sample from the PDF of $\boxed{f(y) = \begin{cases} \theta y^{\theta-1} \\ 0 \end{cases}}$ w/ $\theta > 0$.

Show that $\bar{Y}$ is a consistent estimator $\frac{\theta}{\theta+1}$

Ans:

$$E(Y) = \theta \int_0^1 y \cdot y^{\theta-1} dy = \theta \int_0^1 y^{\theta-1+1} dy = \theta \int_0^1 y^{\theta} dy$$

$$= \theta \cdot \frac{y^{\theta+1}}{\theta+1} \Big|_0^1 = \boxed{\frac{\theta}{\theta+1}} \quad \forall \theta > 0$$

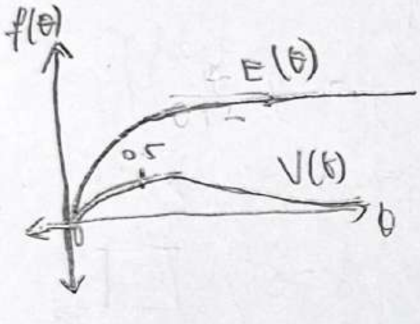
and: $V(Y) = E(Y^2) - E(Y)^2$

w/ $E(Y^2) = \theta \int_0^1 y^2 y^{\theta-1} dy = \theta \int_0^1 y^{2+\theta-1} = \theta \int_0^1 y^{\theta+1} = \theta \frac{y^{\theta+2}}{\theta+2} \Big|_0^1$

$$= \boxed{\frac{\theta}{\theta+2}}$$

so: $V(Y) = E(Y^2) - E(Y)^2 = \frac{\theta}{\theta+2} - \frac{\theta^2}{(\theta+1)^2}$

$$= \frac{\theta(\theta+1)^2 - \theta^2(\theta+2)}{(\theta+2)(\theta+1)^2}$$



$$= \frac{\theta(\theta^2+2\theta+1) - (\theta^3+2\theta^2)}{(\theta+2)(\theta+1)^2}$$

$$= (\theta^3 + 2\theta^2 + \theta - \theta^3 - 2\theta^2) / (\theta+2)(\theta+1)^2$$

$$= \frac{\theta}{(\theta+2)(\theta+1)^2}$$

and as $\theta \to \infty$: $\lim_{\theta \to \infty} \frac{\theta}{(\theta+2)(\theta+1)^2} = \boxed{0}$ ✓

(20.) If Y is a Binomial Distribution w/ n-trials and probability successes p, show that Y/n is a consistent estimator of p!

Ans

$E(Y) = \int y \cdot \binom{n}{y} p^{n-y} q^{y} \sim np$ so: $E\left(\frac{Y}{n}\right) = \frac{np}{n} = \boxed{p}$

$V(Y) = np(1-p) = npq$ $V\left(\frac{Y}{n}\right) = \frac{npq}{n^2} = \boxed{\frac{pq}{n}}$

and $\lim_{n \to \infty} V\left(\frac{Y}{n}\right) = \lim_{n \to \infty} \frac{pq}{n} = \boxed{0}$ ✓

s.t. the variance is an unbiased estimator

EXERCISE 9.4

(9.7). Let $(X_1 \ldots X_n)$ denote n - IID Bernoulli Random variable. s.t $P(X_i = 1) = p$, $P(X_i = 0) = 1-p$, $\forall i = 1 \ldots n$. Show That $\sum_{i=1}^{n} X_i$ is a sufficient estimator for p using Theorem 9.4

Ans:
$$\boxed{f(k; y) = y^k (1-y)^{1-k}}$$

[1] Now: $L(\theta) = \prod_{i=1}^{n} f(y|\theta) \triangleq \prod_{i=1}^{n} f(y_i|\theta)$

$$= \prod_{i=1}^{n} y_i^{\theta}(1-y_i)^{1-\theta} = y_1^{\theta}(1-y_1)^{1-\theta} \times y_1^{\theta}(1-y_2)^{1-\theta} \times \cdots$$

Let $(1-y_i) = x_i$ $= \prod_{i=1}^{n} y_i^{\theta} x_i^{1-\theta} \Rightarrow y_i^{\theta} \frac{x}{x^{\theta}}$ or $\prod_{i=1}^{n} y_i \frac{(1-y_i)}{(1-y_i)^{\theta}}$

$$\Rightarrow \left(\frac{y_i}{x_i}\right)^{\theta} x_i = \prod_{i=1}^{n} \left(\frac{y_i}{1-y_i}\right)^{\theta} 1-y_i = \left(\frac{y_1}{1-y_1}\right)^{\theta}(1-y_1) \times \left(\frac{y_2}{1-y_2}\right)^{\theta}(1-y_2) \times \cdots$$

$$= \left[\left(\frac{y_1}{1-y_1}\right)\left(\frac{y_2}{1-y_2}\right) \times \cdots \left(\frac{y_n}{1-y_n}\right)\right]^{\theta}\left[(1-y_1)(1-y_2) \cdots\right]$$

take logs? $\log L = \sum_{i=1}^{n} \theta \ell(y_i) + (1-\theta)\ell(1-y_i)$ w/ $\ell(y_i) = \log y_i$

$\log(1-y_i)$ $\log(-1) - \log y_i$

$$= \sum_{i=1}^{n} \theta \ell(y_i) + \ell(1-y_i) - \theta \ell(1-y_i).$$

$$= \sum_{i=1}^{n}\left[\theta \ell(y_i)\right] + \sum_{i=1}^{n} \ell(-y_i) + \sum \theta \ell(-y_i) = \sum_{i=1}^{n} \cancel{\theta \ell(y_i) - \theta \ell(y_i)} - \ell(y_i)$$

$$= + \sum_{i=1}^{n} \ell(y_i) \quad \text{So:} \quad \text{Log } L(\theta) = + \sum_{i=1}^{n} \ell(y_i) \Rightarrow \boxed{L(\theta) = \sum^{n} y_i}$$

[2] basically : $\ell(\theta) = \sum x_i \log p + (1-x_i)\log(1-p)$. $\rightarrow$ $\log(1-p) = \log(1) - \log(p) = -\log p$

$$= \sum x_i \ell(p) + \ell(1-p) - x_i \ell(1-p)$$

$$= \sum x_i \ell(p) - \ell(p) + x_i \ell(p)$$

$$= 2\sum x_i\left[\ell(p) - \ell(p)\right] = 2\left\{\sum_{i=1}^{n}[x_i] - 1\right\}\ell(p)$$

Mistake:

$L(p) = p^{\sum x_i}(1-p)^{n - \sum x_i}$

$g = p^{\sum x_i}(1-p)^{n - \sum x_i}$

w/ $h(y) = 1$

If $X_1 = 1 \Rightarrow \sum_{i=1}^{n}$

$X_0 = 0 \Rightarrow$

(39) Let $Y_1 \ldots Y_n$ be a random sample from a poisson distribution w/ parameter λ. Show that $\sum_i^n y_i$ is sufficient for λ.

Ans: [i] $f(y_i) = \frac{\lambda^k e^{-\frac{y_i}{\lambda}}}{k!} \sim \lambda e^{-\frac{y_i}{\lambda}}$ w/ $\frac{\lambda^k}{k!} e^{\lambda}$

$$f(y_i|\theta) = f(y_i|\lambda^k) = \prod_i \lambda^{u} e^{-\frac{y_i}{\lambda}} = \lambda e^{-\frac{y_1}{\lambda}} e^{-\frac{y_2}{\lambda}} \ldots \lambda e^{-\frac{1}{\lambda}\sum^n y_i \times n}$$

$$= \lambda e^{-\frac{n\bar{y}}{\lambda}}$$

w/ $y_1 \ldots y_n = \sum y_n$

an $\mu = \bar{y} = \frac{1}{n}\sum y_n$

so: $\bar{y} n = \sum y_n$

now $g(u,\lambda) = g(\bar{y},\lambda) = \lambda e^{-\frac{n\bar{y}}{\lambda}}$ w/ $h = 1$

[ii] or: $\prod \frac{\lambda^k}{k!} e^{-\frac{y_i}{\lambda}} \Rightarrow \prod \frac{\lambda^{y_i}}{y_i!} e^{-y_i} \Rightarrow \frac{p(Y)}{p(U)} = \frac{\frac{\lambda^{y_i} e^{-\lambda}}{\prod y_i!}}{\frac{ny^u e^{-n\lambda}}{u!}}$

if $\sum y_i = u$

$$\boxed{\frac{u!}{n^u \prod_i y_n}}$$

or: $\prod_i \frac{\lambda^k}{k!} e^{-\frac{y_i}{\lambda}} \Leftrightarrow \log\left(\frac{\lambda^{y_i}}{y_i!} e^{-\lambda}\right) = [y_i \log\lambda - \log(y_i!)] + [-\lambda n]$

so: $\sum_i [y_i \log\lambda - \log(y_i!) - \lambda n] = \ell(\theta)$

$$\cancel{\frac{\partial \ell(\theta)}{\partial y_i} = \sum_i \frac{1}{y_i} \log\lambda - \phi(y_i+1) - 1 - \lambda = 0}$$

$$\cancel{y_i = \frac{\log\lambda}{\phi(y_i+1)+1+\lambda}}$$

$$\cancel{\text{so: } \frac{1}{y_i}\log\lambda = \phi + 1 + \lambda}$$

$$\cancel{\Rightarrow y_i = \frac{\log\lambda}{1+\lambda}}$$

$$\frac{\partial \ell(\theta)}{\partial\lambda} = \sum \frac{\partial}{\partial\lambda} y_i \log\lambda - n\lambda = y_i \frac{1}{\lambda} - n = 0$$

so: $\frac{1}{\lambda} y_i = n$

$\Rightarrow \frac{1}{n} y_i = \lambda \rightarrow \lambda = \frac{1}{n}\sum_i y_i = \bar{y}$

so: $\boxed{\lambda = \bar{y} = \bar{x}}$

EXERCISE 4.4

(14) Let $Y_1, \ldots, Y_n$ be an IID Pareto Distribution s.t

$$\boxed{f(y|\alpha,\beta) = \alpha\beta^{\alpha} y^{-(\alpha+1)}}$$

If β is known, show that $\prod_{i=1}^{n} y_i$ is sufficient for y_i

Ans: $f(y|\alpha,\beta) \rightarrow f(y|\alpha) = \alpha\beta^{\alpha} y_i^{-(\alpha+1)}$

$$L(\theta) = \prod_{i=1}^{n} \alpha\beta^{\alpha} y^{-\alpha+1} = \alpha^n \beta^{\alpha n} \prod^{n} y^{-\alpha+1}$$

$$\alpha^n \beta^{\alpha n} y_1^{-\alpha+1} y_2^{-\alpha+1} \cdots = \alpha^n \beta^{\alpha n} \frac{1}{\left[\prod^{n} y_i\right]^{\alpha+1}}$$

Take logs: $\ell(\theta) = n \log \alpha + \alpha n \log \beta - (\alpha+1) \sum \ell(y_i)$

$$\frac{\partial \ell(\theta)}{\partial \alpha} = \frac{n}{\alpha} + n \log \beta - \sum \log(y_i) = 0$$

so: $\frac{n}{\hat{\alpha}} + n \log \beta = \sum^{n} \log(y_i)$

continue: $n\left(\frac{1}{\alpha} + \log \beta\right) = \sum^{n} \ell(y_i) \rightarrow \frac{1}{\alpha} + \log \beta = \frac{1}{n} \sum_{1} \ell(y_i)$

$$\rightarrow \frac{1}{\alpha} = \frac{1}{n} \sum \ell(y_i) - \log \beta \rightarrow \boxed{\hat{\alpha} = \left[\frac{1}{n} \sum \ell\left(\frac{y_i}{\beta}\right)\right]^{-1}}$$

Rao-Blackwell Theorem

9.5

(1) $\hat{\theta}^* = E(\hat{\theta} \mid U)$

$$\boxed{E(\hat{\theta}^*) = \theta \quad V(\hat{\theta}^*) \leq V(\hat{\theta})}$$ → Rao-Blackwell Theorem

$\hat{\theta}$ = unbiased Estimator

Proof:

$$E(\hat{\theta}^*) = E[E(\hat{\theta} \mid U)] = E(\hat{\theta}) = \theta$$

$$V(\hat{\theta}) = V[E(\hat{\theta} \mid U)] + E[V(\hat{\theta} \mid U)]$$

$$= V(\hat{\theta}^*) + E[V(\hat{\theta} \mid U)]$$

(2) (1) An unbiased estimator θ can be made to be a function of a sufficient statistic

(2). Applying statistic U, the $\hat{\theta}^* = E(\hat{\theta} \mid U)$ will be a function of statistic U, $\hat{\theta}^* = h(U)$

(3) $\delta_1(X)$ estimator of an unknown quantity θ is the conditional Expectation $E[\delta(X) \mid T(X)]$

Example 3 (W/K1)

$X_1 = 0$ → poisson process of Average rate λ, $X_1, \ldots X_N$

$$\delta_0 = \begin{cases} 1 \\ 0 \end{cases}$$

$$S_n = \sum_{i=1}^{n} X_i$$

$$\delta_1(S) = E[\delta_0 \mid S_n = s_n)$$

$$= E[\mathbb{1}_{\{X_1 = 0\}} \mid \sum s_n]$$

$$= P(X_1 = 0 \mid \sum X_i = s_n)$$

$$= P(X_1 = 0, \sum_{i=2}^{n} X_i = s_n) P(\sum_{i}^{n} X_i = s_n)^{-1}$$

$$= \frac{e^{-\lambda} [(n-1)\lambda]^{s_n} e^{-(n-1)\lambda}}{s_n!} \left(\frac{(n\lambda)^{s_n} e^{-n\lambda}}{s_n!} \right)^{-1}$$

$$= \frac{[(n-1)\lambda]^{s_n} e^{-n\lambda}}{s_n!} \cdot \frac{s_n!}{(n\lambda)^{s_n} e^{-n\lambda}}$$

$$= \left(1 - \frac{1}{n}\right)^{s_n} \approx \boxed{e^{}}$$

Expected Value of Random Variables

Theory

1) $$\begin{cases} E[g(Y_1 \dots Y_k)] = \sum_{\forall y_k} \dots \sum_{\forall y_1} g(y_1 \dots y_k)\, p(y_1 \dots y_k) \\ E[g(Y_1 \dots Y_k)]\Big|_{\infty} = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} g(y_1 \dots y_k) f(y_1 \dots y_k)\, dy_1 \dots dy_k \end{cases}$$

$\boxed{E(g(x)) = \int x\, y(x)\, dx}$

2) $$\boxed{E[g(Y_1)h(Y_2)] = E[g(Y_1)]\,E[h(Y_2)]}$$

Proof:
$$E[g(Y_1)h(Y_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y_1)h(y_2) f(y_1,y_2)\, dy_2\, dy_1$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(y_1)h(y_2) f(y_1) f(y_2)\, dy_2\, dy_1$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [g(y_1)f(y_1)][h(y_2)f(y_2)]\, dy_2\, dy_1$$
$$= E[g(y_1)]\,E[h(y_2)] \quad \text{QED}$$

Examples

① $$f(y_1,y_2) = \begin{cases} 2y_1 & 0 \le y_1 \le 1 \quad 0 \le y_2 \le 1 \\ 0 \end{cases}$$

Ans $$E(Y_1Y_2) = \int_0^1 (y_1\, 2y_1) \int_0^1 (y_2)\, dy_1\, dy_2 = \int_0^1\int_0^1 y_1 y_2 (2y_1)\, dy_1\, dy_2$$
$$= \int_0^1\int_0^1 2y_1^2 y_2\, dy_1\, dy_2 = \int_0^1 \frac{2y_1^3}{3}\Big|_0^1 dy_2 = \int_0^1 y_2 \frac{2}{3}\, dy_2$$
$$= \frac{1}{2} y_2^2 \frac{2}{3}\Big|_0^1 = \frac{1}{2}\frac{2}{3} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

② Chemical Industrial Products Yield Are modelled as

$$\boxed{f(y_1,y_2) = \begin{cases} 2(1-y_1) & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1 \\ 0 & \text{elsewhere} \end{cases}}$$

w/ Y_2 = Type I Impurities, Y_1 = Proportion of Impurities

(?) Expected value in proportion of Type 1 Impurities

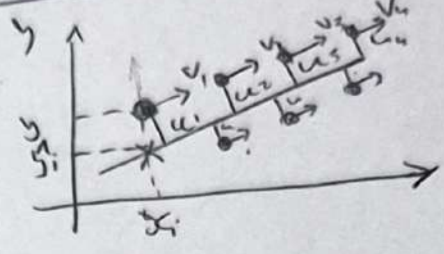
Ans : $$E(Y_1Y_2) = \int_0^1\int_0^1 2y_1y_2(1-y_1)\, dy_2\, dy_1 = 2\int_0^1 y_1(1-y_1)\frac{1}{2}\, dy_1$$
$$= \cancel{2}\cancel{\tfrac{1}{2}}\int_0^1 y_1 - y_1^2\, dy_1 = \frac{y_1^2}{2} - \frac{y_1^3}{3}\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{3-2}{6} = \boxed{\frac{1}{6}}$$

So the expected value is: $\boxed{\frac{1}{6}}$

METHOD OF LEAST SQUARES (11.3)

(1.) $$\boxed{SSE = \sum_{i=1}^{n}(y_i-\hat{y}_i)(y_i-\hat{y}_i)^* = \sum_{i=1}^{n}(y_i-\hat{y}_i)^2}$$

w/ $(y_i-\hat{y}_i)$ = "Main"
$(y_i-\hat{y}_i)^*$ = orthogonal part



(2.) $$\boxed{\min_{\beta_1,\beta_0}\{SSE\} : \begin{cases}\beta_1 = \dfrac{\sum_i^n x_i y_i - \frac{1}{n}\sum_i x_i \sum_i y_i}{\sum_i^n x_i^2 - \frac{1}{n}(\sum_i x_i)^2} \\ \beta_0 = \bar{y} - \hat{\beta}_1\bar{x}\end{cases}}$$

<u>ANS</u> : $$\min_{\beta_1\beta_0}\{SSE\} = \min_{\beta_1\,\beta_0}\sum_{i=1}^{n}\left[y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i)\right]^2$$

$$\Rightarrow \min_{\hat{\beta}_0}\{SSE\} : \frac{\partial SSE}{\partial \beta_0} = \frac{\partial}{\partial\hat{\beta}_0}\sum_{i=1}^{n}\left[y_i - (\hat{\beta}_0 + \hat{\beta}_1 + x_i)\right]^2$$
$$= 2\sum_{i=1}^{n}\left[y_i - \hat{\beta}_0 + \hat{\beta}_1 + x_i\right] = 0$$

$$\Rightarrow \min_{\hat{\beta}_1}\{SSE\} : \frac{\partial SSE}{\partial\beta_1} = \frac{\partial}{\partial\hat{\beta}_1}\sum_{i=1}^{n}\left[y_i - (\hat{\beta}_0 + \hat{\beta}_1 \cdot x_i)\right]^2$$
$$= 2\sum_{i=1}^{n}\left[y_i - (\hat{\beta}_0 + \hat{\beta}_1 + x_i)\right]x_i$$
$$= 2\sum_{i=1}^{n}\left[x_i y_i - x_i\hat{\beta}_0 + x_i^2\hat{\beta}_1\right] \overset{?}{=} 0$$

$$\Rightarrow \min_{\hat{\beta}_1}\{SSE\} = \min_{\hat{\beta}_0}\{SSE\}$$
$$\Rightarrow \cancel{2}\sum_{i=1}^{n}\left[y_i - \beta_0 + \hat{\beta}_1 \cdot x_i\right] = \cancel{2}\sum_{i=1}^{n}\left[x_i y_i - \hat{\beta}_0 x_i + \hat{\beta}_1 x_i^2\right]$$
$$\Rightarrow \frac{\sum_{i=1}^{n} y_i - \beta_0 + \hat{\beta}_1 + x_i}{\sum_{i=1}^{n} x_i y_i - \hat{\beta}_0 x_i + \beta_1 x_i^2} = 1.$$

or : $$\boxed{\min_{\beta_1\beta_1} SSE : \frac{\sum_{i=1}^{n} y_i - \sum_{i=1}^{n}\hat{\beta}_0 + \sum_{i=1}^{n}\beta_1 x_i}{\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n}\beta_0 x_i + \sum_{i=1}^{n}\hat{\beta}_1 x_i^2} = 1}$$

CHAPTER 1.4 EXERCISES

(15.) (a) Derive:

$$SSE = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \sum_{i=1}^{n}(\hat{y}_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2 = \sum_{i=1}^{n}(y_i - \bar{y})^2 - \beta_1 \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})$$

$$= S_{yy} - \hat{\beta} S_{xy} \Rightarrow \text{so: } \boxed{SSE = S_{yy} - \hat{\beta}_1 S_{xy}}$$

__Ans__ $SSE = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 = \sum_{i=1}^{n}\left[y_i - (\hat{\beta}_0 + \hat{\beta}_1 x_i)\right]^2$

$$= \sum_{i=1}^{n} y_i - \bar{y} - \beta_1 \sum_{i=1}^{n}(x_i + \bar{x})(y_i - \bar{y})$$

$$\Rightarrow \begin{cases} \dfrac{\partial SSE}{\partial \beta_1} = -\sum_{i=1}^{n}(x_i - \bar{x})(y - \bar{y}) = 0 \\ \dfrac{\partial SSE}{\partial \beta_0} = 0 \end{cases} \Rightarrow \frac{\partial SSE}{\partial \beta_1} = \frac{\partial SSE}{\partial \beta_0} = \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}). = S_{xy}.$$

$$\boxed{\min_{\beta_0 \beta_1}\{SSE\} : -\frac{\partial SSE}{\partial \beta_1} = \sum_{i=1}^{n}(x_i - \bar{5})(y_i - \bar{y}) = S_{xy}}$$

FOR $\{SSE = S_{xx} - \hat{\beta}_1 S_{xy}\}$

(b) PROOF THAT $\boxed{SSE \le S_{yy}}$ hint: $\left\{\hat{\beta}_1 = \dfrac{S_{xy}}{S_{xx}}\right\}$

__PROOF__: $SSE = S_{yy} - \hat{\beta}_1 S_{xy} = S_{yy} - \dfrac{S_{xy} S_{xy}}{S_{yy}} = S_{yy} - \dfrac{S_{xy}^2}{S_{yy}}$ 2.

$$= \frac{S_{yy}^2 - S_{xy}^2}{S_{yy}} \Rightarrow$$

OR: $SSE + \dfrac{S_{xy}^2}{S_{yy}} = S_{yy} \Rightarrow \begin{cases} S_{xy}^2 \ge 0 \\ S_{xx} > 0 \end{cases} \Rightarrow SSE + 0 = S_{yy} \Leftrightarrow \boxed{SSE \le S_{yy}}$

Q.E.D.

LEAST SQUARE ESTIMATOR PROPERTIES: (CA11.4)

(1.)

$$\beta_1 = \frac{S_{xy}}{S_{x}} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{n}(x_i-\bar{x})(x_i-\bar{x})} = \sum_{i=1}^{n}\frac{(x_i-\bar{x})(y_i-\bar{y})}{(x_i-\bar{x})^2}$$

$$\text{So: } \boxed{\beta_1 = \sum_{i=1}^{n}\frac{(x_i-\bar{x})(y_i-\bar{y})}{(x_i-\bar{x})^2}} \quad \text{w/ } Y_i = \beta_0+\beta_1 x_i+\varepsilon_i$$

(2.)

$$\boxed{E(\hat{\beta}_1) = \beta_1} \longrightarrow \text{Expected value of the Regression coef.}$$

Ans:

$$E(\hat{\beta}_1) = E\left(\sum_{i=1}^{n}\frac{(x_i-\bar{x})(y_i-\bar{y})}{S_{xx}}\right) = \frac{\sum_i(x_i-\bar{x})E(y)}{S_{xx}}$$

$$= \frac{\sum_i(x_i-\bar{x})(\beta_0+\beta_1 x_i)}{S_{xx}} = \beta_0\frac{\sum_i(x_i-\bar{x})}{S_{xx}} + \beta_1\frac{\sum_i(x_i-\bar{x})x_i}{S_{xx}}$$

$$= \frac{\beta_0\{0\}}{S_{xx}} + \beta_1\frac{\sum_i(x_i-\bar{x})^2}{S_{xx}} = 0 + \beta_1\frac{S_{xx}}{S_{xx}} = \boxed{\beta_1}$$

(3.)

$$\boxed{\sigma_x^2(\hat{\beta}_1) = \frac{\sigma_x^2}{S_{xx}}} \longrightarrow \text{Variance of the Regression curve}$$

ANS:

$$\sigma_x^2(\hat{\beta}_1) = \sigma_x\left[\frac{\sum_i(x_i-\bar{x})y_i}{S_{xx}}\right] = \left[\frac{1}{S_{xx}}\right]^2\sum_i\sigma_x\{(x_i-\bar{x})y_i\}$$

$$= \left[\frac{1}{S_{xx}}\right]^2\sum_i(x_i-\bar{x})^2\sigma_x^2 = \frac{1}{S_{xx}^2}S_{xx}\sigma_x^2 = \boxed{\frac{\sigma_x^2}{S_{xx}}}$$

$$\text{So: } \left\{\begin{array}{l}\beta_1 = \sum_{i=1}^{n}\frac{(x_i-\bar{x})(y_i-\bar{y})}{S_{xx}} \\ E(\hat{\beta}_1) = \beta_1 \\ \sigma_x^2(\hat{\beta}_1) = \frac{\sigma_x^2}{S_{xx}}\end{array}\right\} \Rightarrow \left(Y_i = \beta_0+\beta_1 x_i+\varepsilon_i\right)$$

(4.)

$$\boxed{\mathrm{Cov}(\bar{Y},\hat{\beta}_1) = 0} \longrightarrow \text{Covariance of the Cons. Regression}$$

Ans:

$$\mathrm{Cov}(\bar{Y},\hat{\beta}_1) = \mathrm{Cov}\left[\sum_i\left(\tfrac{1}{n}\right)y_i, \sum_i c_i y_i\right] \quad \text{w/ } c_i = \frac{x_i-\bar{x}}{S_{xx}}$$

$$= \sum_i\left(\frac{c_i}{n}\right)\sigma_x(Y_i) + \sum_{i\neq j}\sum\left(\frac{c_j}{n}\right)\mathrm{cov}(y_i,y_j)$$

$$= \sum_i\frac{c_i}{n}\sigma^2 + \sum_{i\neq j}\sum\frac{c_j}{n}\{0\} = \sum_i\frac{\sigma^2}{n}c_i = \frac{\sigma^2}{n}\sum_i c_i$$

$$= \frac{\sigma^2}{n} \sum_i^n \left(\frac{x_i - \bar{x}}{S_{xx}} \right) = \boxed{0}$$

(4.)

$$\boxed{\begin{cases} E(\hat{\beta}_0) = \beta_0 \\ \sigma^2(\hat{\beta}_0) = \dfrac{\sigma^2 \sum_i^n x_i^2}{n S_{xx}} \end{cases}}$$

Ans.

$$E(\hat{\beta}_0) = E(\bar{Y}) - E(\hat{\beta}_1)\bar{x} = \beta_0 + \beta_1\bar{x} - \beta_1\bar{x} = 0 + \beta_0 = \beta_0$$

$$\sigma^2(\hat{\beta}_0) = \sigma(\bar{Y}) + \bar{x}^2 \sigma(\hat{\beta}_1) - 2\bar{x}\, Cov(\bar{Y}, \hat{\beta}_1)$$

$$= \frac{\sigma^2}{n} + \bar{x}^2 \frac{\sigma^2}{S_{xx}} + 0 = \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right).$$

$$= \sigma^2\left(\frac{S_{xx} + \bar{x}^2 n}{n S_{xx}}\right) = \frac{\sigma^2 S_{xx} + \sigma^2 \bar{x}^2 n}{n S_{xx}}$$

$$= \sigma^2\left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right) = \boxed{\frac{\sigma^2 \sum_i^n x_i^2}{S_{xx}}}$$

CH 15.2, 15.3 : GENERAL 2 SAMPLE SHIFT MODEL & PAIRS TEST (MATCHED) 15.1

(1.) $\forall$ $X_1 \ldots\ldots X_n$, $Y_1 \ldots\ldots Y_n$ } Independent normal Random variables taken from Normal pop st $N(\mu, \sigma)$.

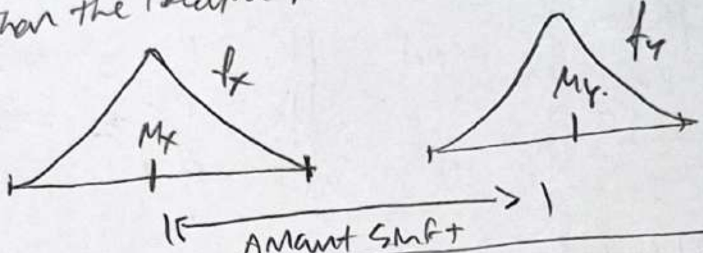
(i) Test: $\boxed{\begin{array}{l} H_0 : \mu_x - \mu_y = 0 \\ H_a : \mu_x - \mu_y < 0 \end{array}}$

If ($H_0 \triangleq$ True) => $N_x(\mu_x, \sigma_x)$ and $N_y(\mu_y, \sigma_y)$

If ($H_a \triangleq$ True) => $\mu_y > \mu_x$

So If The Alternative is true, the location parameter μ_y is bigger than the location parameter μ_x for X_1

Illustration:



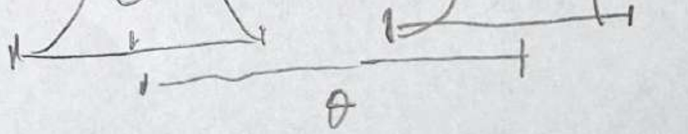
This = $\boxed{\text{TWO-SAMPLE PARAMETRIC SHIFT}}$!

(ii) Let $X_1 \ldots\ldots X_n$ be a Rand. Sample w/ $F(x)$ disb.
Let $Y_1 \ldots\ldots Y_n$ be a Rand sample w/ $G(y)$ dist.

Then: $\boxed{\begin{array}{l} H_0 : F(z) = G(z) \\ H_a : F(z) \neq G(z) \end{array}}$ w/ form unspecified

(?) How much Y_1 shifted via X_1 by Amount θ?

Illustration:



(?)

(5) [i] n-pairs of observation (X_i, Y_i)

(i) Test that $disb(X) = disb(Y)$

s.t. $H_0: D_i = X_i - Y_i = 0$, $D_i \in \mathbb{R}^{\pm}$

$H_a: D_i = X_i - Y_i \neq 0$

[ii] SIGN TEST FOR MATCHED PAIRS

$P = P(X, Y)$ and $(X_i, Y_i) \sim IID$

$H_0: p = \frac{1}{2}$

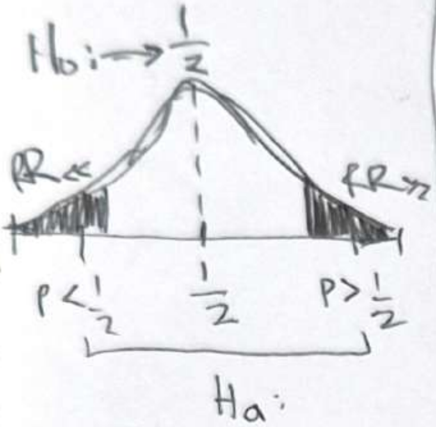
$H_a: p > \frac{1}{2}$; $p < \frac{1}{2}$ or $p \neq \frac{1}{2}$

Test : M = number of positive differences in $D_i = X_i - Y_i$

Reject : $H_a: p > \frac{1}{2} \Rightarrow$ Reject H_0 largest values M.

$H_a: p < \frac{1}{2} \Rightarrow$ Reject H_0 smallest values M

$H_a: p \neq \frac{1}{2} \Rightarrow$ Reject both.



[iv] ELECTRICAL fuse :

Day	1	2	3	4	5	6	7	8	9	10
A :	x_1	x_2	x_3	---		---		---		x_{10}
B :	y_1	y_2	y_4	---		---		---		y_{10}

Electrical fuse produced by each production lines - w/ Assumption the same daily output.

Ans: $H_0: p = \frac{1}{2} = 0.5$

So: $\alpha_1 = P(0) + P(10) = \binom{10}{0}(0.5)^{10} + \binom{10}{10}(0.5)^{10} = \boxed{0.002}$

$\alpha_2 = P(0) + P(1) + P(9) + P(10) = \boxed{0.22}$

$\alpha_3 = P(0) + P(1) + P(2) + P(8) + P(9) + P(10) = \boxed{0.11}$

$\alpha_4 = \text{---} / \text{---} = 0.344$

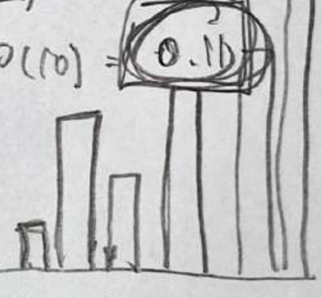
$\alpha_5 = 0.754$ $\quad \alpha_5 = 1$

$\alpha_1 = \{0, 10\}$

$\alpha_2 = \{0, \{1, 9\}, 10\}$

$\alpha_3 = \{0, 1, \{2, 8\}, 9, 10\}$

ooo



If RR = α_3, Then the probability Rejecting (H_0 = True) is 0.11 → Reject null hypothesis

CH. 13.3: TEST VALUES & $n > 25$

$\forall n \geq 25$, Z test is as follows:

(1) $$\boxed{Z = \frac{M - np}{\sqrt{npq}} = \frac{M - n/2}{\frac{1}{2}\sqrt{n}}}$$ w/ $p = \frac{1}{2}$.

Recall:

$E(\text{Binom}) = np$

$\sigma_\theta^2(\text{Binom}) = np(1-p) = np - np^2 = n(p - p^2)$

so: $\sigma_\theta = \sqrt{np - np^2}$ or $\sigma_\theta = \sqrt{np(1-p)} = \sqrt{npq}$

so: $Z_\theta = \frac{E(\cdot)}{\sigma_\theta(\cdot)} \stackrel{\wedge}{=} \frac{np}{\sqrt{npq}}$

now, since $H_a: p \lesseqgtr \frac{1}{2}$, Then we differ $M - NP$ on $E(\cdot)$.

so: $Z_\theta \longrightarrow \boxed{Z = \frac{M - np}{\sqrt{npq}}}$

why not σ_θ? well Random w/ test only by $M(\cdot)$! so σ_θ doesn't apply

(2) Sign test for $n > 25$

$H_0: p = 0.5$

$H_a: p \neq 0.5$

Test: $Z = \frac{M - np}{\sqrt{npq}}$

RR: $H_0: Z \geq Z_{\alpha/2}$

$H_0: Z \leq -Z_{\alpha/2}$

WILCOXON SIGNED RANKED

(15.4)

(1) $H_0: N_X = N_Y$

[i] H_a: ① N_X, N_Y differ in Location → Two Tail
② f_X / N_X shifted to the right or f_Y / N_Y

| Test: ① one-tail : Rank sum T^- for negative difference
| ② Two-tail : $T = \min\{T^+, T^-\}$

w/ $T^- = \sum_i^n \ominus_i$ and $\pm_i : D_i \in R^{\pm}$. $D \in R^-$ = if $D_i - B_i = -k$; $D \in R^+$ = if $D_i - B_i = +k$
$T^+ = \sum_i^n \oplus_i$

RR : TT: Reject H_0 if $T \le T_0$ w/ T_0 = crit. value Two tail
OT: Reject H_0 if $T^- \le T_0$ w/ T_0 = crit value one tail.

[ii] Basic Procedure:

*

1) Calculate differences $D_i \ \forall$ n-pairs
2) Rank absolute Values ($|D_i|$)
3) If 2 or more absolute values are tied for the same Rank

(2) CAKE EXPERIMENT :
Cakes prepared using Mix A & B and baked in 6 different ovens so There's 12 cakes w/ 6 in A, 6 in B.
(a) Test The Hypothesis that There's no difference in pop. distb. in Cake Densities

ANS : • our sample of $n = 12$ Cakes is Really Small. So:
$\alpha = 0.10$. and $T_0 = 2$.

• Thus: $\boxed{H_0 \text{ Rejected if } T \le 2}$

• There's 12 cakes so the Rank is the Arithmetic Sum of 6 s.t.:
$1 + \dots + 6 = \frac{6(6+1)}{2} = \frac{6 \cdot 7}{2} = \frac{42}{2} = 21$

So: $T^- = 21 - T^+ = 21 - 3 \text{ pairs} = \boxed{18}$

So we find the mean s.t.

$$\boxed{\min\{T^+, T^-\} = \min(3, 18) = 3}$$

(1) $3 = \frac{\cancel{6} \text{ cakes}}{2 \text{ pairs} \cdot \text{cakes}}$

and $3 \geq T = 2$ many exceeds crit value. So we can't Reject H_0. s.t. we conclude $\boxed{\cancel{\exists} \text{ Difference}}$

(3) Large samples?:

$$E(T^+) = \frac{n(n+1)}{4}$$

$$V(T^+) = \frac{n(n+1)(2n+1)}{24} \rightarrow \sigma_\theta = \sqrt{\frac{n(n+1)(2n+1)}{24}}$$

w/ $\delta f = T^+$

so.

$$\boxed{Z = \frac{E(T^+)}{V(T^+)} \triangleq \frac{T^+ - \frac{n(n+1)}{4}}{\sqrt{\frac{n(n+1)(2n+1)}{24}}}}$$

w/ RR = $\underbrace{\begin{matrix} Z \geq Z_{\frac{\alpha}{2}} \\ Z \leq -Z_{\frac{\alpha}{2}} \end{matrix}}_{\text{two tailed}}$ $\quad \underbrace{\begin{matrix} Z \geq Z_\alpha \\ Z < Z_\alpha \end{matrix}}_{\text{one tail}}$

Remarks:

$$T^+ + T^- = \sum_i R_i = \frac{n(n+1)}{2} \quad \text{since } P(T^+) = P(T^-) = \frac{1}{2}$$

so: $E\left(\sum_i R_i\right) = \sum_i P(R_i) \cdot \frac{n(n+1)}{2} = \frac{1}{2}\frac{n(n+1)}{2} = \boxed{\frac{n(n+1)}{4}}$

So: $\sigma_\theta^2 = E(R_i^2) - E(R_i)^2$

$= \frac{1}{2^2}\sum_i R_i^2 - \frac{n^2(n+1)^2}{16} + E(\cdot)^2 = \frac{1}{4}\sum R_i^2 - \frac{n^2(n+1)^2}{16} + E(\cdot)^2$

$= \frac{1}{4}\frac{n(n+1)(2n+1)}{6} - \frac{n^2(n+1)^2}{16} + \frac{n^2(n+1)^2}{16} = \boxed{\frac{n(n+1)(2n+1)}{24}}$

EXERCISE 15.3

(6) FLORIDA CITRUS

Protecting trees of Citrus from freezing conditions is expensive. Cold spots are from lower elevations (conjectured). Based on the table, is there sufficient evidence to support the conjecture?

Ans: Let T_H, T_L denote the temperatures on high and low elevations. If $N(T_H, T_L)$ is a function of night run

$$N_k = (T_{Hk}, T_{Lk})$$

Our hypothesis: $f(T_H) \neq f(T_L)$: H_a

$f(T_H) = f(T_L)$: H_0

S.t: $H_0 : p = \frac{1}{2}$

$H_a : p \neq \frac{1}{2} \Rightarrow p > \frac{1}{2}$

So: $n = 10$, $p = 0.05$, The table gives us

$\boxed{p\text{-val} = 2P(M \geq 9) = 0.11}$ wh/ $m = 9$ w/ df = 1.

(7) PSYCHOLOGICAL STIMULI

Comparison of response time between 2 different stimulis. find the p-val!

ANS: Let $S_i^1(x)$, $S_i^2(y)$ denote the effect of the stimuli. Then:

$H_0 : P\{S_i^1(x)\} = P\{S_i^2(x)\} = \frac{1}{2}$

$H_a : P\{S_i^1(x)\} \neq P\{S_i^2(x)\} \neq \frac{1}{2}$.

Excel result using (t): $P(T \leq t \mid \text{one-tail}) = 0.06$

$P(T \leq t \mid \text{2-tail}) = 0.138$

Thus: Failing Reject Null hypothesis

EXARCISE 15.3

Lead poisoning, devil's advocacy NP!

(1.) LEAD POISONING OF Indoor GUN RANGE

(7) claim of Indoor firing range Increases blood levels?

w/ Lead exposure experiments on 17 members of law enforcement
on 15/17 of the trainees post-firing elevated

Ans: small sample test since $n < 25 = 17$.
So: w/ $P = \frac{15}{17}$ $q = \frac{2}{17}$

$\alpha_1 = P(0) + P(17) = 0.119$
$\alpha_2 = P(0) + P(1) + P(2) + P(16) + P(17) = 0.389$
$\alpha_3 = \alpha_2 + P(3) + P(15) = 0.677$
$\therefore \alpha_6 = 0.99 = 1$ w/ $\alpha_4 = 0.95$, $\alpha_5 = 0.97$

But if $P(\cdot) = \frac{1}{2}$

(a) $\alpha_1 = P(M \geq 15) \cdot \binom{17}{15} 0.5^{17} + \binom{17}{16} 0.5^{17} + \binom{17}{16} 0.5^{17} = \boxed{0.0012}$

so: $P(M \geq 15) = 0.0012 \leq \frac{1}{2}$

(b) At $\alpha = 0.01$, Ho is Rejected an conclude that Indoor firing Increases blood lead!

(c) The normal Approximation: $P(M \geq 15) = P(M > 14.5) \sim P(Z > 2.91)$
$= 0.0018 \sim 0.002$ (Rounded) # very close to our previous Calcs

*Another method:

$\hat{p} = \frac{15}{17}$
$p_0 = 0.5$
$\hat{q} = \frac{2}{17}$

$Z = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0 q_0}{n}}} = 3.15$ so p-val $= 0.00081$

and: $\boxed{\begin{array}{l} np_0 \geq 5 = 17(0.5) = 8.5 \geq 5 \checkmark \\ n(1-p_0) \geq 5 = 17 \cdot (0.5) = 8.5 \geq 5 \checkmark \end{array}}$

$\Downarrow$ The normal Approximation is adequate.

(7) Drug Test :

wh/ $R = \left(\frac{ar}{at}, \frac{br}{bt}\right)$

a_r = Drug A recovery

b_r = Drug B recovery.

a_t = Drug A treatments

b_t = Drug B treatments

Drug A				Drug B			
Hospitals	Treat	rec	%	H	Treat	rec	%
1	a_{t_1}	a_{r_1}	75	1	b_{t_1}	b_{r_1}	85%
⋮	⋮	⋮	64.8	⋮	,	,	83%
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
10	$a_{t_{10}}$	$a_{r_{10}}$	77.4	10	$b_{t_{10}}$	$b_{r_{10}}$	80.4%

(a) P-Value For Rec. Diffrence

$(D_i \pm R^{+}_{-})$

$R_{a_i} = X$

$R_{b_i} = Y$.

$H_0 : p = \frac{1}{2}$

$H_a : p \lesseqgtr \frac{1}{2}$ or $p \neq \frac{1}{2}$.

Ans.

Hospitals	R_{a_i}	R_{b_i}	D_i	
1	R_{a_1}	R_{b_1}	$R_{a_1} - R_{b_1}$	75-85 = -
2	R_{a_2}	R_{b_2}	$R_{a_2} - R_{b_2}$	64-83 = -
⋮	⋮	⋮	⋮	⋮
10	$R_{a_{10}}$	$R_{b_{10}}$	$R_{a_{10}} - R_{b_{10}}$	77-80 = -

- Now suppose we test that for the difference's negativity. The table when calculated shows that hospital 3 and 6. shows a (+) sign
- Thus Hospital 1 > 3 has more success w/ Drug A.

the electric fuses so $\boxed{2P(M \leq 2) = 0.11}$

(b) We use the example on the electric fuses so we accept the null hypothesis

- Since $\alpha \geq 0.110$, α is fairly large. So we accept the null hypothesis such that $\boxed{\exists(\text{Different Recovery Rates for Drug A \& B})}$ s.t

* Another way :

$$Z = \frac{\hat{P}_A - \hat{P}_B}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n_a} + \frac{1}{n_B}\right)}}$$

wh/ $H_0 : P_A \neq P_B$

$H_a : P_A = P_B$.

or $H_0 : \hat{P}_a - \hat{P}_b = D_0$

$H_a : P_a - P_b \lesseqgtr D_0$

EXERCISE 18.4

(12) **SCORE DISTRIBUTION OF MATH & ART**

Student	Math	Art
1	22	53
2	37	68
3	36	42
4	38	49
5	42	51
6	58	65
7	58	57
8	60	71
9	62	55
10	65	74
11	66	68
12	56	64
13	66	67
14	67	73
15	62	65

(b) Wilcoxon's ranked test, to determine if the test scores have to differ. w/ $\alpha = 0.05$.

Ans: Let X_i = Math score student i

Y_i = Art score student i

So: $D_i = X_i - Y_i$ s.t $\rightarrow$ $|22 - 53| : \ominus$, $|37 - 68| : \ominus$, $|36 - 42| : \ominus$

...

The test is: $\frac{n(n+1)}{2} = \frac{15 \cdot 16}{2} = 120$ Pairs

(a) and there's 14 Pairs to test so:

$$\boxed{T^{+} + T^{-} = \frac{n(n+1)}{2} \Rightarrow T^{-} = 120 - 14 = 106}$$

and $\min(T^{-}, T^{+}) = (106, 14) = 14$

since $T = 14 < 16$, The p-val is < 0.01 so H_0 is Rejected.

w/ p-val $= 0.0006$ (1-tail)

p-val $= 0.012$ (2-tail)

(b.) H_0 : Math $\stackrel{\Delta}{=}$ Art

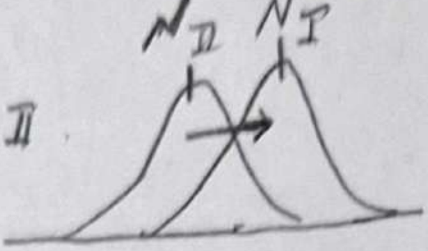
H_a : Math $\stackrel{\Delta}{\neq}$ Art.

So since H_0 rejected, Then Math and art test scores are different.

EXERCISES 15.6

(24) WORKABLE EX

p-values of $H_0: N_I = N_{II}$



(a) H_a: N_I shifted right of Pop II

$n_1 = 4,\ n_2 = 7,\ W_x = 34$

Ans: $n_1 n_2 = 4 \cdot 7 = 28$

$U_1 = 28 - \frac{4 \cdot 5}{2} - \cancel{34} = 4$

$U_2 = 28 - \frac{4 \cdot 8}{2} - \cancel{34} = 22$

$\min(U_1 U_2) = \min(4, 22) = \boxed{4} \to \boxed{P(U \leq 4) = 0.0364}$

w/ $U \geq n_1 n_2 - U_0$

$4 \leq 28 - U_0$

$U_0 \leq 28 - 4 = 24$

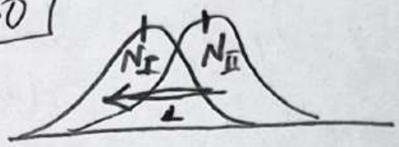
OR $U_0 \leq 28 - 22 = 6 +$ $\boxed{30}$

$(24, 6) \leq 30$

or

$(24, 6) \not\geq 30$

so Not different



(b) H_a: —/— left

w/ $n_1 = 5\ n_2 = 9,\ W = 38$

Ans: $U = 5 \cdot 9 + \frac{5 \cdot 6}{2} - 38 = 22$

p-val $= P(U \geq 22) =$

(c) $H_a: f(I) \neq f(II) \Rightarrow$ differ in loc $n_1 = 3,\ n_2 = 6,\ W = 23$

$U = \cancel{18} + \frac{3 \cdot 4}{2} - 23 = 18 + \frac{12}{2} - 23 = 18 + 6 - 23$

$= 24 - 23 = 1$

pv: $2P(U \leq 1) = 2 \cdot 0.0238 = \boxed{0.0476}$

$\boxed{\text{Reject } H_0}$

(22) ALZEMAR TEST FOR NEW MEMORY DRUG

test of Alzeihmer w/ CX 516 for early 20's and 65-70 men by Dr. Gary Lynch, univ of California

Age Group	Syllables Recalled
20's	11 7 6 8 6 9 2 10 3 6
65-70	1 9 6 8 7 8 5 7 10 3

w/ $N = 10$ men

H_0: CX-516 equal

H_a: CX-516 not equal

$$U_x = n_1 n_2 + \frac{n_1(n_1+1)}{2} - W$$

$$= 100 + 55 - W = 155 - W = 155 - 108 = \boxed{47}$$

*Sort for W = x : 2 3 6 6 6 7 8 9 10 11
y : 1 3 5 6 7 7 8 8 9 10

→ x : 20 11 7.5 14 9.5 16.5 2 18.5 3.5 7.5 W_A = 108
y : 1 16.5 7.5 14 11 5 11 18.5 3.5 W_B = 102

$$U_y = 100 + 55 - 102 = 155 - 102 = \boxed{53}$$

and $U_0 = |47 - 53| = \boxed{6}$

if : $U \geq n_1 n_2 - U_0 \rightarrow 102 \geq 100 - 6 \rightarrow \boxed{102 \geq 94}$

Failed Reject The null hypothesis

MANN WHITNEY U - TEST

① **Mann-Whitney U** : ordering all n_1+n_2 observations according to their magnitude and counting the number of observations in sample I that preceed each observation in sample II.

Ex/1a. :

$$\begin{array}{cccccccccc} 25 & 26 & 27 & 28 & 29 & 31 & 32 & 35 \\ \hline x_1 & x_2 & x_3 & y_1 & y_2 & x_4 & y_3 & y_4 \end{array}$$

$\underbrace{x_1 \; x_2 \; x_3}_{3\ x's}$ $\qquad$ $\underbrace{x_4}_{+1\ x's}$

- Smallest y_I : $y_1 = 28$
  x's preceding y_I : $\{x_1, x_2, x_3\} = 3 = u_1$
- Smallest y_{II} : $y_2 = 29$
  x's preceding y_I : $\{x_1\ x_2\ x_3\} = 3 = u_2$
- Smallest y_{III} : $y_3 = 32$
  x's preceding y_I : $\{x_1\ x_2\ x_3\ x_4\} = 4 = u_3$
- Smallest $y_{IV} = y_4$ w/ x's preceding $= 4 = u_4$

So: $U = \sum_i^7 u_i = u_1 + u_2 + u_3 + u_4 = 3+3+4+4 = \boxed{14}$

so $\boxed{U = 14 \text{ ; small/large? depends on the separation of } x \cdot y}$

② The Mann-Whitney U Test:

H_0 : $f_I = f_{II}$

H_a : $f_I \neq f_{II}$ (Left or right shift if one tail)

Assumption
- IID Selected
- Ranks are Average

Test : $\boxed{U = n_1 n_2 + \frac{n_1(n_1+1)}{2} + W}$

RR : $\forall \alpha$, Reject H_0 iff :

$\boxed{U \leq U_0 \text{ or } U \geq n_1 n_2 - U_0}$

w/ $P(U \leq U_0) = \frac{\alpha}{2}$

(3) EXAMPLES

1. KRAFT Paper Testing : Specified weight / chemical subs. w/ I = STD, II = TREAT. & if It's tested for Strength :

I	II
x_1	y_1
$\vdots$	$\vdots$
x_{10}	y_{10}

The Rank Sum : $W = 85.5$

$n = n_1 + n_2 = 10 + 10 = 20.$

we Reject H_0 if $U \leq U_0$ or $U \geq n_1 n_2 - U_0$

(?) What's U_0, ans: $n_2 = 10$ The $P(U_0)$ is 0.0526

so $U_0 = 28$. so:

$U \geq n_1 n_2 - U_0 = 10 \times 10 - 28 = 100 - 28 = \boxed{72}$

$U_x = n_x n_y + \frac{n_x(n_x+1)}{2} - R_x$

$U_y = n_x n_y + \frac{n_y(n_y+1)}{2} - R_y$

$100 + \frac{110}{2} - 85.5$

$- 14.49$

$U_x = 69.5$

$U_y = 140.51$

(4.)

$|U_y - U_x| = U_0$

$|140 - 69| = U_0$

$|71.01| = U_0$

$\sim 72 = U_0$

ergo:

$U = n_1 n_2 + \frac{n_1(n_1+1)}{2} - W$

$= 100 + \frac{10\cdot 11}{2} - 85.5 = 69.5$

and $\boxed{69.5 \not\geq 72} \rightarrow$ So we can't Reject H_0 null hypothesis

$\therefore$ not enough evidence.

$$E(U) = E\left(n_1 n_2 + \frac{n_1(n_1+1)}{2} - W\right) = \sum_k P_k(U_k) = \frac{1}{2} n_1 n_2$$

$$V(U) = E(U^2) - E(U)^2 = \frac{n_1 n_2 (n_1 + n_2 + 1)}{2}$$

so: $\boxed{Z = \frac{U - E(U)}{\sigma_U}}$

so: $n_1 > 10$, $n_2 > 10$:

$$\boxed{Z = \frac{U - \frac{n_1 n_2}{2}}{\sqrt{\frac{n_1 n_2 (n_1 + n_2 + 1)}{12}}}}$$

KRUSKAL-WALLIS TEST FOR ONE WAY LAYOUT

*SECTION 13.3 Assumptions:

IID Random Samples have Drawn from $N(\vec{\mu}, \vec{\Sigma})$ and w/ Dependent variables Y_{ij} for i^{th} sample & j^{th} experimental unit

$H_0: \mu_1 = \mu_k$ so: total $SS = SST + SSE = \sum_{i=1}^{k}\sum_{j=1}^{n_i}(Y_{ij} - \bar{Y})^2 = \sum_{i=1}^{k}\sum_{j=1}^{n_i} Y_{ij}^2 - CM$

$H_a: \mu_i \neq \mu_k$ $\leftarrow$ can $\left(S^2 = \frac{SSE}{n-k},\ MST = \frac{SST}{k-1}\right) \rightarrow \boxed{F = \frac{MST}{MSE} > F_\alpha}$

Q) What if the Distribution: IID $\cancel{\sim} N(\vec{\mu}, \vec{\Sigma})$

So that: $\boxed{\text{What if the Distribution isn't normal?}}$

A) To start, we use the same procedure, s.t Ranking $\sum_k^? n_k = n$ from smallest (1) to largest (n)

So: $\boxed{\bar{R}_i = \frac{R_i}{n_i}} \rightarrow$ Average of Ranks

$*\ \bar{R} = \frac{\frac{n(n+1)}{2}}{n} = \frac{n+1}{2}$

$$V = \sum_{i=1}^{k} n_i(\bar{R}_i - \bar{R})^2 = \sum_{i=1}^{k} n_i\left(\bar{R}_i - \frac{n+1}{2}\right)^2$$

$$= \sum_{i=1}^{k} n_i\left(\bar{R}_i^2 - 2\bar{R}_i\frac{n+1}{2} - \left(\frac{n+1}{2}\right)^2\right) = \sum_{i=1}^{k} n_i\left(\bar{R}_i^2 - 2\bar{R}_i\frac{n+1}{2} + \frac{n^2+2n+1}{4}\right)$$

$$= \sum_{i=1}^{k} n_i\left(\frac{R_i^2}{n_i^2} - 2\frac{R_i}{n_i}\frac{n+1}{2} + \frac{1}{4}(n+1)^2\right) = \sum_{i=1}^{k}\left(\frac{R_i^2}{n_i} - \cancel{2}R_i\frac{n+1}{\cancel{2}} + n_i\frac{(n+1)^2}{4}\right)$$

$$= \sum_{i=1}^{k}\left(\frac{R_i^2}{n_i} - R_i(n+1) + \frac{n_i(n+1)^2}{4}\right) = \sum_{i=1}^{k}\left(\frac{R_i^2}{n_i} - \frac{n(n+1)^2}{2} + \frac{n_i(n+1)^2}{4}\right)$$

$$= \sum_{n_i}^{k}\left(\frac{R_i^2}{n_i} + \frac{-2n(n+1)^2 - n_i(n+1)^2}{4}\right) = \sum_{i=1}^{k}\left(\frac{R_i^2}{n_i} + \frac{-(2n-n_i)(n+1)^2}{4}\right)$$

$$= \sum_{n_i}^{k}\left(\frac{R_i^2}{n_i} + \frac{-2n(n+1)^2 - \sum_i n_i(n+1)^2}{4}\right) = \sum_{i=1}^{k}\left(\frac{R_i^2}{n_i} - \frac{2n(n+1)^2 - n(n+1)^2}{4}\right)$$

$$= \sum_{i=1}^{?}\left(\frac{R_i^2}{n_i} - \frac{n(n+1)^2}{4}\right)$$

[ii] multiply by $\frac{12}{n(n+1)}$:

$$\frac{12}{n(n+1)}V = \frac{12}{n(n+1)}\sum_{i=1}^{k}\left(\frac{R_i^2}{n_i}\right) - \frac{\cancel{12}}{\cancel{n}(n+1)}\frac{\cancel{n}(n+1)^2}{\cancel{4}}$$

$$= \frac{12}{n(n+1)}\sum_{i=1}^{k}\left[\frac{R_i^2}{n_i}\right] - 3\cancel{n}(n+1)$$

Let $H = \frac{12}{n(n+1)}V$ so we can get:

$$\boxed{H = \frac{12}{n(n+1)}\sum_{i=1}^{k}\left[\frac{R_i^2}{n_i}\right] - 3(n+1)}$$ Kruskal-Wallis test

w/ H_0: k populations of distribution are identical

H_a: At least 2 differ in location

RR: Reject H_0 if $H > \chi^2_\alpha$ w/ $(k-1)$ df

(2) QUALITY CONTROL APPLICATION

Let L_1, L_2, L_3 be the 3 lines for the output of ten randomly selected hours of production was examined for defects w/ defects = D_i, Ranks = R_i $\forall i$ elements

(Table 15.6) page 767

so:

L_1: D_{i1} = 6 3 8 3 ---- $\rightarrow R_1 = 120$
R_{i1} = ---

L_2: D_{i2} = 34 28 42 --- $\rightarrow R_2 = 210.5$
R_{i2} =

L_3: D_{i3} = 13, 35 ---- $\rightarrow R_3 = 134.5$
R_{i3} = ---

[i] since $n_1 = 10 = n_2 = n_3$ and $n = n_1 + n_2 + n_3 = 3 \times 10 = 30$.

so: $H = \frac{12}{30 \cdot 31} + \left(\frac{120^2}{10} + \frac{210.5^2}{10} + \frac{(134.5)^2}{10}\right) - 3(31) = \boxed{6.097}$

w/ $\boxed{H > \chi^2_\alpha} = 6.097 > \chi^2_\alpha$ w/ $k-1 = 2$ df

since $k=3$

[ii] we see in Appendix 3 w/ $\alpha = 0.05$, $\chi^2_{0.05} = 5.99147$

so: $\boxed{6.097 > 5.99147}$

Concluding that we reject the null hypothesis to conclude at least 2/3 lines produce greater defects

BKNRUSPS 157

(29) MEAN LEAF Length :

4 sites to test 6 Randomly selected plants for the 4 swampy underdevelop sites

Ans : Using R

```
> len <- c(x1 ...... x24)
> site <- factor(c(Rep(1,6), Rep(2,6), Rep(3,6), Rep(4,6))
> kruskal.test(len ~ site)
```

output :

$\chi^2 = 16.974$

$df = 3$

$p\text{-val} = 0.0007155$ -> Reject Ho and conclude that $\exists$ difference in at least 2/4 sites

(30) Company Advert campaign

Product recognition ($\forall$ campaign) test, in which it was Tested on 15 market Areas, 5 Randomly assigned to $\forall$ campaign. At the end of Campaign, 400 adults Random Samples were Selected in each area

(a) Experimental Design?

Ans : Random design

CAMPAIGN		
1	2	3
0.33	0.28	0.21
0.29	0.41	0.30
0.21	0.34	0.20
0.32	0.39	0.23
0.25	0.27	0.21

(b) Sufficiency of evidence that a difference in locations of Distributions of Recogs?

Ans R Code :

```
> prop <- c[0.33 ..... 0.21]
> Region <- factor(c(Rep(1,5), Rep(2,5), Rep(3,5)))
> kruskal.test(prop ~ Region)
```

output:

$\chi^2 = 2.5491$

$df = 2$

$p\text{val} = 0.2796$

$\boxed{0.2796 > 0.05}$

Fail to Reject the null hypothesis

$\therefore$ NO Sufficient evidence.

(c) Campaign 2 & 3 are expensive. evidence very well catch that 2 >> 3 ?

Ans • R-lang:

```
> Wilcoxon.test( Prop[6:10], Prop[11:16], alt = "greater")
```

• output: $W = 19$

p-val = 0.1111 → 0.1111 > 0.05

still fail to reject s.t. not enough evidence that campaign 2 >> 3 !

w/ prop[6:10] =

2	
0.28	6
⋮	↓
0.27	10

in column

FRIEDMAN TEST (15.8)

(1) • Random Block Design = Equal variances, Normally distributions

• Friedman test = If 2 or more observations in the same block are tied for the same rank, then the average of the ranks that would've been assigned to this observations is assigned to each member of the tied group

[i] Let: R_i = Sum of Ranks corresp. treat. i

$\bar{R}_i = \frac{R_i}{b}$ w/ b = observations and bk = total observations.

Then: $1 + \dots + k = \frac{k(k+1)}{2}$ = sum of Ranks assigned

Since bk = total, Then: $\frac{k(k+1)}{2} \to b\frac{k(k+1)}{2}$

Thus:

$$\boxed{W = b\sum_{i=1}^{k}(\bar{R}_i - R)^2}$$

= Is the SST of a Randomized block design.

[ii] To make this visually easy, It's basically the same w/ the Kruskall-Wallis test, only w/ a b in it s.t.:

$$\boxed{\begin{aligned} &\text{Friedman: } F_r = \frac{12}{bk(k+1)}\sum_{i=1}^{k} R_i^2 - 3b(k+1) \\ &\text{Kruskall-wallis: } H = \frac{12}{k(k+1)}\sum_{i=1}^{k} \frac{R_i^2}{k} - 3(k+1) \end{aligned}}$$

w/ RR: $F_r > \chi_\alpha^2$ w/ $(k-1)$ df.

(2.) TASK-TEST

Let $R_1 = 7.5$, $R_2 = 16.5$, $R_3 = 12$ be the sum of Ranks for the completion time of task A, B, and Task C. We want to test that probability distributions differ in loc w/ $n = 6$ samples. So:

ans. $b = 6$ blocks
$k = 3$ treatments
$k-1 = 2$ df

$$\Rightarrow F_r = \frac{12}{6(3)(4)}\left[7.5^2 + 16.5^2 + 12^2\right] - 3(6)(4)$$

$$= \boxed{6.75}$$

Meanwhile the χ^2 is:

$$\chi^2_{0.05} \Big| 2df = 5.99147$$

So: $F_r > \chi^2_{0.05} = \boxed{6.75 > 5.99147}$ ∴

∴ So we Reject the null hypothesis and Conclude that Atleast 2/3 tastes differ in location

EXERCISE 15.4

(43) The 2^{2}

Consider F_r w/ $k=2$, $b=n$. Then $\boxed{F_r = \frac{2}{n}(R_1^2+R_2^2) - 9n}$

Let M be the number of blocks (pairs) in which treatment one

has rank 1. If there's no ~~~~ ties, then treatment 1 has rank

2 in the remaining $n-M$ pairs. Thus:

$$\boxed{\begin{aligned} R_1 &= M + 2(n-M) = 2n - M \\ R_2 &= n + M \end{aligned}}$$

$\boxed{1}\boxed{2} = M_1$

$\boxed{3}\boxed{4} = M_2$

$\cdots \sum_k M_k = M$

(?) Sub. this value into the previous value of F_r to show that

$$\boxed{F_r = \frac{4(M - 0.5n)^2}{n}}$$

Compare this result w/ Z stat of

$$\boxed{Z = \frac{T^{+} - \left[\frac{n(n+1)}{4}\right]}{\sqrt{\frac{n(n+1)(2n+1)}{24}}}}$$

Ans:

$$\boxed{F_r = \frac{2}{n}(R_1^2 + R_2^2) - 9n}$$

$\boxed{i}$ $R_1^2 = (2n-M)^2 = (2n-M)(2n-M) = 4n^2 - 2nM - 2nM - M^2$

$= 4n^2 - 4nM - M^2$

$R_2^2 = (n+M)^2 = (n+M)(n+M) = n^2 + 2nM + M^2$

so: $R_1^2 + R_2^2 = (4n^2 - 4nM + M^2) + (n^2 + 2nM + M^2)$

$= (4n^2 + n^2) + 2nM - 4nM + 2M^2$

$= 5n^2 - 2nM + 2M^2$

so: $F_r = \frac{2}{n}(5n^2 - 2nM + 2M^2) - 9n$

$= \frac{10n^2 - 4nM + 4M^2}{n} - 9n = \frac{10n^2 - 9n^2 - 4nM + 4M^2}{n}$

$= \frac{n^2 - 4nM + 4M^2}{n} = n - 4M + \frac{4M^2}{n} = n - 4\left(M + \frac{M^2}{n}\right)$

$= \frac{4}{n}\left(M^2 - nM + \frac{n^2}{4}\right) - \frac{n^2}{n} + n = \boxed{\frac{4}{n}\left(M - \frac{n}{2}\right)^2}$

$\left(M + \frac{M^2}{n}\right)4$

$= \left(M^2 - nM + \frac{n^2}{4}\right)\frac{4}{n}$

$= \frac{4M^2}{n} - 4M + n$

$= \boxed{\frac{4}{n}\left(M^2 - nM + \frac{n^2}{4}\right)}$

$\boxed{x}$ The Wilcoxon is:

$$\cancel{W} = \frac{M - \frac{1}{2}n}{\sqrt{\frac{n}{4}}} = \frac{2(M - 0.5n)}{\sqrt{n}}$$

which is $\boxed{Z^2 = \frac{4(M - 0.5n)^2}{n}}$

So: $\boxed{F_r = Z^2}$

(46) FRIEDMAN TEST DERIVATION

Derive F_r w/ $\bar{R}_i = \frac{R_i}{b}$, $\bar{R} = \frac{(k+1)}{2}$, $\sum_{i=1}^{k} R_i = b\frac{k(k+1)}{2}$.

Ans:

$$\sum_{i=1}^{k}(\bar{R}_i - \bar{R})^2 = \sum_{i=1}^{k}\left(\frac{R_i}{b} - \frac{(k+1)}{2}\right)^2$$

$$= \sum_{i=1}^{k}\left(\frac{R_i}{b}\right)^2 - \frac{k+1}{2}\frac{R_i}{b} - \frac{k+1}{2}\frac{R_i}{b} + \frac{(k+1)^2}{4} = \sum_{i=1}^{k}\left(\frac{R_i}{b}\right)^2 - \cancel{2}\frac{k+1}{\cancel{2}}\frac{R_i}{b} + \frac{(k+1)^2}{4}$$

$$= \sum_{i=1}^{k}\frac{R_i^2}{b^2} - \frac{(k+1)R_i}{b} + \frac{(k+1)^2}{4} = \sum_{i=1}^{k}\frac{bR_i^2 - (k+1)R_i + 4b(k+1)^2}{b}$$

$$= \sum_{i=1}^{k}\left(\frac{R_i^2}{b^2} - \frac{(k+1)R_i}{b} + \frac{(k+1)(k+1)}{4}\right)\frac{12}{bk(k+1)} = \frac{12b}{k(k+1)}\left(\sum\frac{R_i}{b^2}\right) - \left(\frac{12}{k}\frac{bk(k+1)}{2} + \frac{12b}{4k}\right)$$

$$\cancel{= \sum_{i=1}^{k}\frac{12R_i^2}{b^3k(k+1)} - \frac{12R_i(k+1)}{b(bk(k+1))} + \frac{(k+1)(k+1)12}{4bk(k+1)}} = \sum_{i=1}^{k} 12\left(\frac{R_i^2}{b^3k(k+1)} - \frac{R_i}{b^2k(k+1)} - \ldots\right)$$

$$= \frac{12}{bk(k+1)}\left(\sum_{i=1}^{k}\frac{R_i^2}{b^2} - \sum_{i=1}^{k}\frac{(k+1)R_i}{b} + \sum_{i=1}^{k}\frac{(k+1)(k+1)}{4}\right)$$

$$= \frac{12}{bk(k+1)}\left(\frac{b^2k(k+1)^2}{4b^2} - \sum\frac{\cancel{b}k(k+1)(k+1)}{\cancel{b}} + \frac{(k+1)(k+1)}{4}\right)$$

$$= \frac{12}{bk(k+1)}\left(\frac{k^2(k+1)^2}{4} - \frac{4k(k+1)(k+1)}{4} + \frac{k(k+1)(k+1)}{4}\right)$$

$$= \frac{12}{bk(k+1)}\left(\sum_i R_i^2 - \frac{(4k-k)(k+1)^2}{4}\right) = \frac{12}{bk(k+1)}\left(\sum R_i^2 - \frac{3k(k+1)^2}{4}\right)$$

$$= \boxed{\frac{12}{bk(k+1)}\sum_i R_i^2 - 3b(k+1)}$$

INTRODUCTION to BAYESIAN PRIORS - [1]

[16.1] [16.2]

① INTRODUCTION

PRIOR PROBABILITY

- Suppose we have a Binomial Distribution of:

$$\boxed{p(y|p) = \binom{n}{y} p^y q^{n-y}} \quad w/ \quad p(y|p) = \frac{p(p \cap y)}{p(y)}$$

- To make this concrete we suppose that p = single person responds to refer them.
  1.) n = sample size 2.) Y = respondents & p = parameter.

  So: $\boxed{p(y|p) = \text{PRIOR DISTRIBUTION}}$

- We know that $(0 < p < 1)$ applies to the Beta distribution of $\boxed{BETA(\alpha, 1)}$
  The question is ⓘ $\boxed{\text{WHICH BETA SHOULD WE USE?}}$. We can assume and elaborate:
  1.) $\mu = \frac{\alpha}{\alpha+\beta}$ 2.) $p \in [0.25] \rightarrow \mu = 0.25$ 3.) $\begin{cases}\alpha = 1 \\ \beta = 3\end{cases}$ so $\frac{1}{1+3} = \boxed{\frac{1}{4}} = \boxed{.25}$

- Thus the assigned density for p is:

$$\boxed{g(p) = \frac{1}{3}(1-p)^2} \quad w/ \quad g(p) = \frac{\alpha}{\beta}(1-p)^k = \frac{1}{3}(1-p)^2.$$

POSTERIOR

- Since we've conditioned $p|y$ w/ p-dist, we've also separated (y, p) so: $\boxed{Y|p \rightarrow (y, p)}$ and thus being able to determine the conditional distribution of $p|Y$. Such
  1.) Conditionals = $p|Y$ 2.) $Y = y$ 3.) $g^*(p|y)$.

- We will show that:

$$g^*(p|y) = \frac{\Gamma(n+4)}{\Gamma(y+1)\Gamma(n-y+3)} \, p^y (1-p)^{n-y+2} \quad w/$$

$p|y$ is beta-Bently of $\boxed{\begin{matrix}\alpha = y+1 \\ \beta = n-y+3\end{matrix}}$

② BAYESIAN PRIORS, POSTERIORS, ESTIMATORS

[1] THE LIKELIHOOD.

FORMULATIONS:

- Suppose $(Y_1 \text{---} Y_n)$ denote random variables w/ sample size n. Then θ is a parameter. Then:

$$\boxed{L(y_1 \ldots y_n | \theta)} \text{ and } \boxed{p(Y_1 = y_1, Y_2 = y_2, \ldots\ldots Y_n = y_n)}$$

The likelihood — The DISCRETE case

- Given the likelihood ~~data~~ and the prior of θ, it follows $(Y_1 \text{--} Y_n, \theta)$'s joint likelihood is:

$$\boxed{f(y_1, \text{---} y_n, \theta) = L(y_1 \text{---} y_n | \theta) \times g(\theta)}$$

- The marginal density / ~~mass function~~ of $Y_1 \ldots Y_n$ is:

$$\boxed{m(y_1 \text{---} y_n) = \int_{-\infty}^{\infty} L(y_1 \ldots y_n | \theta) \cdot g(\theta) \, d\theta}$$

INTRODUCTION TO
BAYESIAN PRIORS [II] [16.7]

(2) BAYESIAN PRIORS, POSTERIORS & ESTIMATORS.

[i] LIKELIHOOD

■ EXAMPLE - VIRULENT DISEASE (EX 16.2)

[Q] Consider the virulent disease and (16.1) Results. Compare the values of the whole Relation {w | p = Responders to New Therapy}

[a] $\alpha = 1,\ \beta = 3,\ n = 5$, $\sum_i^n y_i = 2$ [b] $\alpha = 1\ \beta = 3\ n = 25$, $\sum_i^n y_i = 10$ [c] $\alpha = 10,\ \beta = 30$, $n = 5,\ \sum_i^n y_i = 2$ [d] $\alpha = 10\ \beta = 30$, $n = 25$, $\sum y_i = 20$

Ans: 1) Notice that:

$$\mu = \frac{\alpha}{\alpha+\beta} = \frac{1}{1+3} = \frac{1}{4} = 0.25.$$

2) The maximum likelihood (MLE):

$$\hat{p} = \frac{1}{n}\sum_i^n y_i = \frac{2}{5} = \frac{10}{25} = \text{value}$$

3) and: a) $P(Y_i = 1) = p$, $P(Y_i = 0) = 1 - p$ b) $\theta_\alpha = \sum_i^n y_i + \alpha = \alpha^*$, $\theta_\beta = n - \sum_i^n y_i + \beta = \beta^*$

[a]: $\theta_\alpha = 2 + 1 = 3$, $\theta_\beta = 25 - 10 - 10 = 6$ [b] $\theta_\alpha = 11$, $\theta_\beta = 18$ [c] $\theta_\alpha = 12$, $\theta_\beta = 53$ $\theta_\alpha = 12$, $\theta_m = 33$. [d] $\theta_\alpha = 30$, θ_β ...

so:

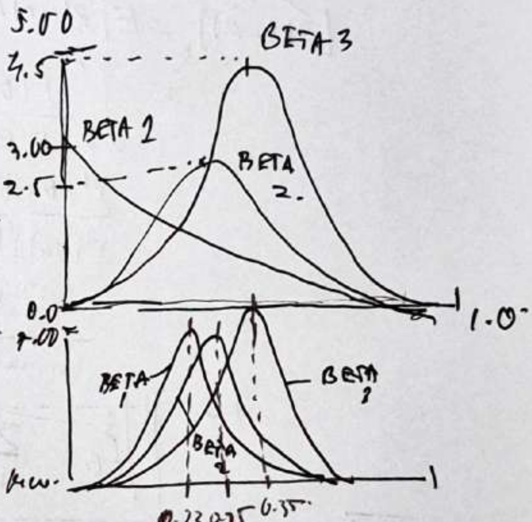
	a	b	c		
θ_α	3	11	12	12	20
θ_β	6	18	13	38	45

4) Since we know that:

$$\boxed{\mu = \frac{\alpha}{\alpha+\beta}} \quad \text{and} \quad \boxed{\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}}$$

5) We can construct a table:

DIST	n	$\sum_i^n y_i$	PARAMETER	μ	σ^2
• PRIOR	—	—	$\alpha = 1\ \beta = 3$	0.25	0.0375
• POSTER	5	2	$\alpha^* = 3\ \beta^* = 6$	0.33	0.0222
• POSTER	25	10	$\alpha^* = 11\ \beta^* = 18$	0.40	0.0078
			$\alpha = 10\ \beta = 30$	0.25	0.046
• PRIOR	—	2	$\alpha^* = 12\ \beta^* = 33$	0.266	0.043
• POSTER	5	10	$\alpha^* = 20\ \beta^* = 45$	0.307	0.032
• POSTER	25				



• DIFFERENCES a) $|\mu_{11} - \mu_{12}| = |0.33 - 0.26| = 0.05$

$|\sigma_{11} - \sigma_{12}| = |0.022 - 0.043| = 0.0210.$

b) $|\mu_{21} - \mu_{22}| = |0.4 - 0.37| = 0.03.$

$|\sigma_{21} - \sigma_{22}| = |0.078 - 0.032| = 0.04.$

[ii] The THEORIES of CONJUGATE PRIORS & POSTERIORS

■ CONJUGATE PRIORS

• Prior distributions that result in posterior distributions that are the same functional form as the prior but w/ altered parameter values are conjugate priors.

Thus:

If $\phi(\text{PRIOR}(x) \to \text{POSTERIOR}(x)) = \phi(\text{PRIOR}(x)) \to \phi(\text{POSTERIOR}\,x)$

Then $\boxed{\forall \phi(\text{PRI}(x), \text{POST}(x)),\ \phi \text{ is a Conjugate [illegible]}}$

Consequences:

- $\forall\, \text{Beta}(\alpha, \beta)$: $\phi\left(\text{Bernoulli}(\hat\theta) \xrightarrow{L(y|\theta)} \text{BETA}(\hat\theta_\alpha, \hat\theta_\beta)\right)$
  $= \phi\left(\text{Bernoulli}(\hat\theta)\right) \xrightarrow{L(y|\theta)} \phi\left(\text{Beta}(\hat\theta_\alpha, \hat\theta_\beta)\right)$

  i.e. $\forall\, \text{Beta}(\alpha, \beta)$ is a Conjugate prior for the Bernoulli.
- This is convenient since we can reform the exact formula for the posterior in terms of the new parameter based properties of a familiar distribution.

[iii] ESTIMATION

Formulation of $\widehat{t(\theta)}_B$

Let $(Y_1 \ldots Y_n)$ be a Random sample $L(y_1 \ldots y_n|\theta)$ and θ w/ $g(\theta)$ dist.

Then the posterior is:

$$\boxed{\widehat{t(\theta)}_B = E[t(\theta) \mid Y_1 \ldots Y_n]}$$

Examples:

[1] THE ESTIMATE of $\phi(\text{Bernoulli}(\theta), \text{BETA}(\hat\theta_\alpha, \hat\theta_\beta))$

- we say that:

$$g^* = \frac{\Gamma(\alpha^*+\beta^*)}{\Gamma(\alpha^*)\Gamma(\beta^*)} p^{\alpha^*-1}(1-p)^{\beta^*-1} \quad \forall\, 0 < p < 1$$

- and also

$$\hat t(\theta) \equiv \hat p_B = E(p|y_1 \ldots y_n) = \frac{\alpha^*}{\alpha^*+\beta^*} = \frac{\sum_1^n y_i + \alpha}{\sum_1^n y_i + \alpha + n - \sum_1^n y_i + \beta} = \frac{\sum_1^n y_i + \alpha}{n + \alpha + \beta}$$

- So:

$$\widehat{[p(1-p)]}_B = E[p(1-p)|y] = \int_0^1 p(1-p) \frac{\Gamma(\alpha^*+\beta^*)}{\Gamma(\alpha^*)\Gamma(\beta^*)} p^{\alpha^*-1}(1-p)^{\beta^*-1}\, dp$$

$$= \int_0^1 \frac{\Gamma(\theta_\alpha+\theta_\beta)}{\Gamma(\theta_\alpha)\Gamma(\theta_\beta)} p^{\theta_\alpha-1}(1-p)^{\theta_\beta-1}\, dp = \frac{\Gamma(\theta_\alpha+\theta_\beta)}{\Gamma(\theta_\alpha)\Gamma(\theta_\beta)} \frac{\Gamma(\theta_\alpha+1)\Gamma(\theta_\beta)}{\Gamma(\theta_\alpha+\theta_\beta+2)}$$

$$= \frac{\Gamma(\theta_\alpha+\theta_\beta)}{\Gamma(\theta_\alpha)\Gamma(\theta_\beta)} \cdot \frac{\theta_\alpha \Gamma(\theta_\alpha)\Gamma(\theta_\beta)\theta_\beta}{(\theta_\alpha+\theta_\beta+\cancel{1})(\theta_\alpha+\theta_\beta)\Gamma(\theta_\alpha+\theta_\beta)} = \frac{\theta_\alpha\theta_\beta}{(\theta_\alpha+\theta_\beta+1)(\theta_\alpha+\theta_\beta)}$$

$$= \frac{(\sum_1^n y_i + \alpha)(n - \sum_1^n y_i + \beta)}{(n+\alpha+\beta+1)(n+\alpha+\beta)} \checkmark$$

Thus: $\boxed{\hat p_B = \frac{\sum_1^n y_i + \alpha}{n+\alpha+\beta}}$ $\boxed{\widehat{[p(1-p)]}_B = \frac{(\sum_1^n y_i + \alpha)(n - \sum_1^n y_i + \beta)}{(n+\alpha+\beta+1)(n+\alpha+\beta)}}$

[iv] REPARAMETRIZATION by THE LIKELIHOOD

Let $L(y|\theta) = k(u, \theta)h(y)$ where u is suff-stat for $Y_1 \ldots Y_n$ and $\theta \notin h$ as $(\theta, u) \in k$.

Then:

$$g^*(\theta|y) = L(y|\theta)g(\theta)\left[\int_{-\infty}^{\infty} L(y|\theta)g(\theta)\,d\theta\right] = k(u,\theta)h(y)g(\theta)\left[\int_{-\infty}^{\infty} k(u|\theta)h(y)g(\theta)\right]$$

$$= \frac{k(u|\theta)g(\theta)}{\int_{-\infty}^{\infty} k(u|\theta)g(\theta)\,d\theta}$$

i.e. $\boxed{g^*(\theta, y) = k(u,\theta)g(\theta)\left[I_\theta(k(u|\theta)g(\theta))\right]^{-1}}$

Bayesian [illegible] Notes

[illegible] EXPONENTIAL DIST w/ GAMMA CONJUGATE

Suppose that $y_1, \ldots y_n$ is a random sample w/ $\boxed{f(y\mid\theta) = \theta e^{-\theta y}}$ and that $\mu = \frac{1}{\theta}$ w/ PRIOR := GAMMA(α, β)

[a] Show that the joint density is $\boxed{f(y\mid\theta) = \frac{\theta^{n+\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} \exp\left[-\frac{\theta}{\beta/\beta\sum_1^n y_i + 1}\right]}$

ANS. • Observe that: $L(y\mid\theta) \equiv f(y\mid\theta) = f(y_1, \ldots y_n\mid\theta) = \theta e^{-\theta y_1}\,\theta e^{-\theta y_2} \ldots\ldots \theta e^{-\theta y_n}$

$$= \theta^n \exp\left(-\theta \sum_1^n y_i\right).$$

• [illegible] Gamma$(\alpha, \beta \mid \theta) = g(\alpha, \beta\mid\theta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} \exp\left(-\frac{y}{\beta}\right)\theta^{\alpha-1}.$

• So:

$$f(y, \theta) = f(y\mid\theta)\,g(\alpha,\beta\mid\theta) = \theta^n e^{-\theta\sum_1^n y_i} \times \frac{1}{\Gamma(\alpha)\beta^{\alpha}} e^{-\frac{\theta}{\beta}}\theta^{\alpha-1} = \frac{\theta^{n+\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} e^{-\left(\theta\sum_1^n y_i + \frac{\theta}{\beta}\right)}.$$

• and the exponent's inside

$$-\left(\theta\sum_1^n y_i + \frac{\theta}{\beta}\right) = -\theta\left(\frac{\beta\sum_1^n y_i}{\beta} + \frac{1}{\beta}\right) = -\frac{\theta\left(\beta\sum_1^n y_i + 1\right)}{\beta} = -\frac{\theta\left(\sum_1^n y_i\,\beta + 1\right)}{\beta}.$$

$$\equiv -\theta\,\frac{\beta}{\beta\sum_1^n y_i + 1}$$

∴ Thus:

$$\boxed{f(y\mid\theta) = \frac{\theta^{n+\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} \exp\left[-\frac{\theta}{\beta/\beta\sum_1^n y_i + 1}\right]}$$

[b] Show that the Marginal Density is $\boxed{m(y_1 \ldots y_n) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\beta^{\alpha}}\left[\frac{\beta}{\beta\sum_1^n y_i + 1}\right]^{\alpha+n}}$

ANS. • Observe:

$$\Gamma(x) = \int_{-\infty}^{\infty} e^{-t} t^{x-1}\,dt \longrightarrow \Gamma(n+\alpha) = \int_{-\infty}^{\infty} e^{-t} t^{n+\alpha-1}\,dt$$

$$\text{let } (t \to \theta) \longrightarrow \Gamma(n+\alpha) = \int_{-\infty}^{\infty} e^{-\theta}\,\theta^{n+\alpha-1}\,d\theta$$

• And also: let $Z^{-1} = (\beta/\beta\sum_1^n y_i + 1)^{-1}$. Then:

$$\frac{d\Gamma}{d\theta}\frac{1}{\theta^{n+\alpha}}$$

$$e^{-\frac{\theta}{Z}} = \left(e^{-\theta}\right)^{\frac{1}{Z}} = \left(\frac{d\Gamma}{d\theta}\frac{1}{\theta^{n+\alpha-1}}\right)^{\frac{1}{Z}} = \frac{d\Gamma^{\frac{1}{Z}}}{d\theta}\frac{1}{(\theta^{n+\alpha-1})^{\frac{1}{Z}}}$$

$$\ldots\ m(y_1 \ldots y_n) = \int_{-\infty}^{\infty} \frac{\theta^{n+\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} \exp\left(-\frac{\theta}{Z}\right) d\theta = \int_{-\infty}^{\infty} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\beta^{\alpha}}\left(\frac{d\Gamma}{d\theta}\,\theta^{-(n+\alpha-1)}\right)^{\frac{1}{Z}} d\theta \ldots$$

$$= \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\beta^{\alpha}}\,Z^{n+\alpha} = \boxed{\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)\beta^{\alpha}}\left(\frac{\beta}{\beta\sum_1^n y_i + 1}\right)^{n+\alpha}}$$

[c] Show that $\mu = \frac{1}{\theta}$ is $\boxed{\hat{\mu}_B = \frac{\sum_1^n y_i}{n+\alpha-1} + \frac{1}{\beta(n+\alpha-1)}}$

ANS. Since: $\theta_{\alpha} = n+\alpha$, $\theta_{\beta} = \frac{\beta}{\beta\sum_1^n y_i + 1} = Z$. Then: $E(\mu\mid Y) = E\left(\frac{1}{\theta}\middle| Y\right) = \frac{1}{\theta} = \frac{1}{\theta_{\beta}(\theta_{\alpha}-1)}$

$$= \left[\theta_{\beta}(\theta_{\alpha}-1)\right]^{-1} = \left[\frac{\beta}{\beta\sum_1^n y_i + 1}(n+\alpha-1)\right]^{-1} = \frac{1}{(n+\alpha-1)\beta}\left(\beta\sum_1^n y_i + 1\right) = \boxed{\frac{\sum_1^n y_i}{n+\alpha-1} + \frac{1}{(n+\alpha-1)\beta}}$$

PROBLEMS 1b.d

(W) Poisson distribution w/ gamma

Let $Y_1, \ldots, Y_n$ denote a sample from the Poisson distributed population w/ mean λ. In this case: $U = \sum_i^n Y_i$ is a sufficient statistic for λ and U has a Poisson distribution w/ mean $n\lambda$. Take the conjugate Gamma (α, β) prior for λ then

[A] Show that the joint likelihood L of is

$$\boxed{L(u, \lambda) = \frac{n^u}{u!\,\beta^\alpha \Gamma(\alpha)} \lambda^{u+\alpha-1} \exp\left[-\lambda \Big/ \frac{\beta}{n\beta+1}\right]}$$

Ans: • Observe that:

$$L(y|\theta) \Rightarrow L(u|\lambda) = \frac{(n\lambda)^u e^{-n\lambda}}{u!}$$

and $G(\alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \lambda^{\alpha-1} \exp\left(-\frac{\lambda}{\beta}\right)$.

• Now $L(u, \theta) = L(y|\theta)\, G(\alpha\beta) = \left(\frac{(n\lambda)^u e^{-n\lambda}}{u!}\right) \cdot \left(\frac{1}{\beta^\alpha \Gamma(\alpha)} \lambda^{\alpha-1} e^{-\frac{\lambda}{\beta}}\right)$

$$= n^u \lambda^u \lambda^{\alpha-1} \cdot \frac{1}{u!\,\beta^\alpha \Gamma(\alpha)} \cdot e^{-\lambda/\beta} e^{-n\lambda} = \lambda^{n+\alpha-1} \frac{n^u}{u!\,\beta^\alpha \Gamma(\alpha)} e^{-\left(n\lambda + \frac{\lambda}{\beta}\right)}$$

• Include the exponents:

$$-\left(\frac{\lambda}{\beta} + \lambda n\right) = -\lambda\left(\frac{1+n\beta}{\beta}\right) = -\lambda\left(\frac{1+n\beta}{\beta}\right) = \frac{\lambda}{\frac{\beta}{1+n\beta}} = \lambda \Big/ \frac{\beta}{1+n\beta}$$

$\therefore$ have: $L(u, \theta) = \frac{n^u}{u!\,\beta^\alpha \Gamma(\alpha)} \lambda^{u+\alpha-1} \exp\left(\lambda \Big/ \frac{\beta}{1+n\beta}\right)$.

[B] Show that the marginal mass function of U is:

$$\boxed{m(u) = \frac{n^u \Gamma(u+\alpha)}{u!\,\beta^\alpha \Gamma(\alpha)} \left(\frac{\beta}{n\beta+1}\right)^{u+\alpha}}$$

$$e^{-\lambda / \frac{\beta}{n\beta+1}} = e^{-\lambda\left[\frac{\beta}{n\beta+1}\right]^{-1}} = \left(e^{-\lambda}\right)^{\frac{1}{2}}$$

Ans: • If:

$$m(u) = \int_0^\infty \frac{n^u}{u!\,\beta^\alpha \Gamma(\alpha)} \lambda^{u+\alpha-1} \exp\left(\frac{\beta}{n\beta+1}\right) d\lambda$$

• Then if: $\Gamma(u+\alpha) = \int e^{-\theta} \theta^{u+\alpha} d\theta = \int e^{-\lambda} \lambda^{u+\alpha-1}$ where $\frac{\partial \Gamma}{\partial \lambda} = e^{-\lambda} \lambda^{u+\alpha-1}$

$$\frac{1}{(u+\alpha)} \frac{1}{2}\left(\frac{\partial \Gamma}{\partial \lambda}\right)^{\frac{1}{2}} = e^{-\lambda\frac{1}{2}} \cancel{\lambda^{u+\alpha-1}\frac{1}{2}}$$

• Then $u+\alpha$ → shape parameter

$$m(u) = \frac{n^u \Gamma(u+\alpha)}{u!\,\beta^\alpha \Gamma(\alpha)} \underbrace{\left(\frac{\beta}{n\beta+1}\right)}_{\text{scale}}$$

[C] Show that $\boxed{\begin{aligned}\alpha^* &= u+\alpha \\ \beta^* &= \beta / n\beta+1\end{aligned}}$

[D] Show that:

$$\boxed{\hat{\lambda}_B = \frac{\left(\sum_i^n Y_i + \alpha\right)\beta}{n\beta+1}}$$

Ans: $\lambda_B = E(\lambda | U) = \alpha^* \beta^* = \frac{(u+\alpha)\beta}{n\beta+1} = \frac{\left(\sum Y_i + \alpha\right)\beta}{n\beta+1}$

Ans: $\alpha^* = u+\alpha \in \Gamma(u+\alpha)$, $\beta^* = \frac{\beta}{n\beta+1}$; parameter