ADVANCE MACROECONOMICS

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Solow Growth Model (1.2) (1.3) (1.4)

(2) Assumptions

[1] $\boxed{Y = F(K, AL)}$

Inputs & outputs + production

- AL = Effective labour : Augmented labour.
- AL = Labour Augmenting
- AK = Capital Augmenting
- $AF(K,L) = \boxed{\text{Hicks neutral}}$

Ass. 1) $Y = F(K, AL) \rightarrow \boxed{cY = F(cK, cAL) = cF(K, AL)}$ $\boxed{F = K^{\alpha}(AL)^{1-\alpha}}$

Cobb-Douglas Produce

Constant returns to scale

- The Economy is big enough that gains from specialization is exhausted
- Natural resources isn't really that important

Ass/ if $F(K, AL) = K^{\alpha}(AL)^{1-\alpha}$

then $F(cK, cAL) = (cK)^{\alpha}(cAL)^{1-\alpha} = c^{\alpha}K^{\alpha}c^{1-\alpha}AL^{1-\alpha} = c^{\alpha+1-\alpha}K^{\alpha}(AL)^{1-\alpha}$

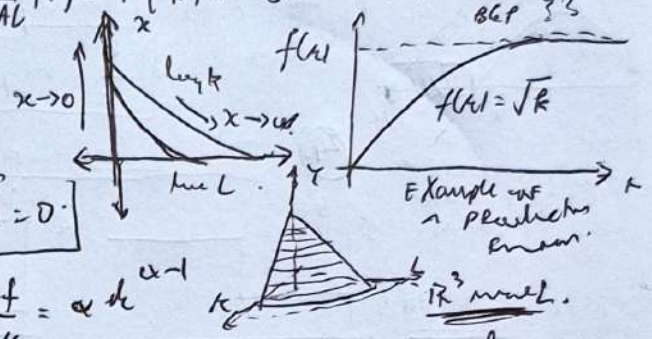
$= c^{1}K^{\alpha}(AL)^{1-\alpha} = cF(K, AL)$ here.

- The Solow model assumes if (K, L) doubles, then $F(K, L)$ also doubles.

2/) $F\left(\frac{K}{AL}, 1\right) = \frac{1}{AL}F(K, AL)$

Let $k = \frac{K}{AL}$, $y \equiv \frac{Y}{AL}$ $\rightarrow$ $F\left(\frac{K}{AL}, 1\right) = F(k, 1) = y(k) = f(k)$

here. $\boxed{y = f(k)}$



3.) The Inada conditions:

$\boxed{\lim_{k\to 0} f'(k) = \infty}$ $\boxed{\lim_{k\to\infty} f'(k) = 0}$

Proof: Let $f(k) = k^{\alpha}$ here. $f'(k) = \frac{df}{dk} = \alpha k^{\alpha-1}$

a) then. $\log f'(k) = \log\alpha + (\alpha-1)\log k = \log\alpha + \alpha\log k - \log k$.

but Now. $\lim_{k\to 0} f'(k) \equiv \lim_{k\to 0}[\log f'] = \alpha\lim_{k\to 0}(\log k - \log k) = \alpha(\log 0) - \log(\alpha)$

= Doesn't exist

Inputs

- Parameter summary
- Saving
- Depreciation
- Population
- Tech progress

Next: $\log Y = \beta_0 + \beta_1\log K + \beta_2\log L$ then $\frac{\partial\log Y}{\partial K} = \beta_1\frac{1}{K} \rightarrow \begin{cases}\lim_{K\to 0}\frac{\beta_1}{K} \equiv |\beta_1| = \boxed{\infty} \\ \lim_{K\to\infty}\frac{\beta_1}{K} = 0 \text{ since its inverse}\end{cases}$

and $\frac{\partial\log Y}{\partial L} = \frac{\beta_2}{L} \rightarrow \begin{cases}\lim_{L\to 0}\frac{\beta_2}{L} = \infty \text{ since it explodes.} \\ \lim_{L\to\infty}\frac{\beta_2}{L} = 0\end{cases}$

(ii) Evolution of Inputs

Exogenous

Since Savings (s, n, g, δ) is cons. In theory which also exogenous.

1) $\boxed{\frac{dL}{dt} = nL(t) \quad \frac{dA}{dt} = gA(t)}$ but $\boxed{\begin{matrix}L(t) = e^{nt}L_0 \\ A(t) = e^{gt}A_0\end{matrix}}$ and $\ln L = \ln L_0 + nt$ and $\ln A = \ln A_0 + gt$.

2) $\frac{dK}{dt} = (Y_t - C_t) - \delta K_t = sY_t - \delta K_t \quad \forall\, 0 \le s \le 1$

here: $\boxed{\frac{dK}{dt} = sY_t - \delta K_t}$ $\longrightarrow$ Since savings is Exogenous, Then If the goal is to analyze a certain policy that affects incentives but to Aggregate savings, Then its Too simple

(3.) Dynamics of Model

- Change in capital $= \dot{K}_t$
- Savings per output $= sY_t$
- Depreciation per input $= \delta K_t$

[i] Dynamics of K

if $\frac{dk}{dt} = \frac{d}{dt}\left[\frac{K}{AL}\right] = \frac{d\,K_t}{dt}u(L) + \frac{du(L_t)}{dt}K_t = \frac{dK_t}{dt}u(AL_t) + \frac{du(L_t)}{dt}u(A_t) + \frac{dU_2(A_t)}{dt}u(L_t)$.

$= \frac{dK_t}{dt}\frac{1}{A_tL_t} + K_t\left[-\frac{\dot{L}_tA_t}{(A_tL_t)^2} - \frac{\dot{A}_tL_t}{(A_tL_t)^2}\right] = \frac{\dot{K}_t}{A_tL_t} - \frac{K_t}{(A_tL_t)^2}\left[A_t\dot{L}_t + L_t\dot{A}_t\right]$.

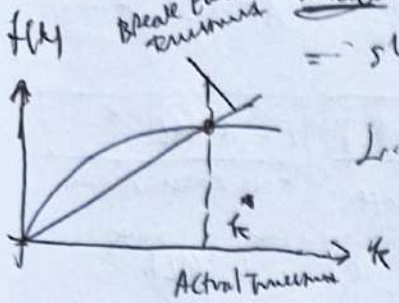
$= \frac{\dot{K}_t}{A_tL_t} - \frac{K_t}{A_tL_t}\frac{\dot{L}_t}{L_t} - \frac{K_t}{A_tL_t}\frac{\dot{A}_t}{A_t}$

2.) Recall that: $\boxed{\frac{\dot{A}_t}{A_t} = g \quad \frac{\dot{L}_t}{L_t} = n.}$

Lw: $\dot{k}(t) = \frac{\dot{K}_t}{A_t L_t} - \frac{K_t}{A_t L_t} n - \frac{K_t}{A_t L_t} g = \frac{\dot{K}_t}{A_t L_t} - \frac{K_t}{A_t L_t}(n-g).$

The essential graph of the Solow model

$= \frac{sY_t - \delta K_t}{A_t L_t} - k(t)(n-g) = \frac{sY_t}{A_t L_t} - \frac{\delta K_t}{A_t L_t} - k(t)(n-g)$

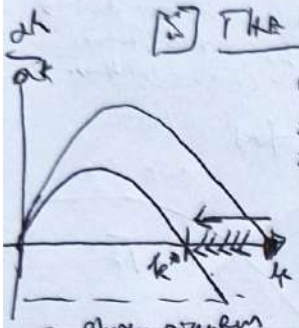


$= sy_t - \delta k(t) - k(t)(n-g) = sy_t - (n+g+\delta)k_t.$

Lw: $\boxed{\dot{k}(t) = sy_t - (n+g+\delta)k(t)}$ where $y_t = f(k_t)$

2.) If $f(0) = 0$, then: $\boxed{0 = 0 - 0}$ ✓

3 THE BALANCED GROWTH PATH



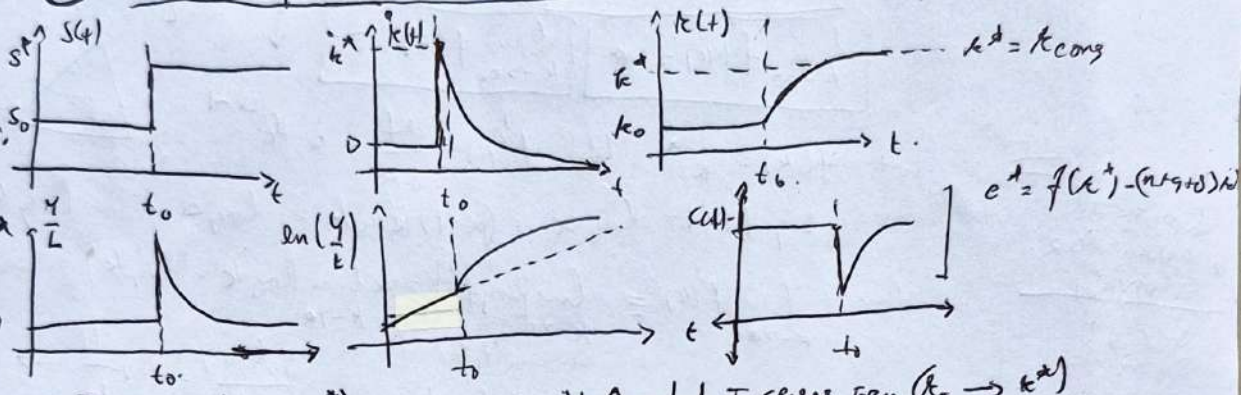
The Phase diagram of the Solow model

1) Since $(k \to k^*)$, then ? $\boxed{y_t = k^*}$?
2) The Labour & knowledge are growing at rates (n, g).
3) The capital stock $\boxed{K = AL \cdot k}$. Since $k = k^*$, then $(K \uparrow n+g)$.

Since: $\boxed{\frac{\dot{K}}{K} = n+g}$ →

4 THE IMPACT OF A CHANGE IN SAVINGS RATE

1) OUTPUT:



$c^* = f(k^*) - (n+g+\delta)k^*$

1.) Savings Increase $(s_0 \to s^*)$
2.) Δ(Capital) Increases Abruptly $(0 \to \dot{k}^*)$
Then Falls due to Decreased Investment
but $\boxed{(0 \to t_0 \to 0) \equiv (0 \to \dot{k}^* \to 0)}$

3) Capital Increases from $(k_0 \to k^*)$
Then stabilizes at $(k_{cons} = k^*)$
and $\left[\Delta \frac{Y}{L} \equiv \frac{dk}{dt}\right]$ s.t $\boxed{(t_0 \to g^* \to g)}$
s.t. Since Y/L Increases then $\ln(\frac{Y}{L})$ increases

- Permanent Increase in Savings = Temporary growth rate
- Temporary growth rate = Temporary output / increase.

5 Consumption:

1.) $\boxed{c^* = f(k^*) - (n+g+\delta)k^*}$ where $c = \frac{\text{Consumption}}{\text{Effective Labour}}$.

$(1-s)f(k) = c$

where $sy \to (1-s)y = c$
$y - sy = c \to -sy = c - y$
$sy = y - c$

then if $k^* = 0 \to sy - (n+g+\delta)k^*$
$0 = y - c - (n+g+\delta)k^*$

$\boxed{c^* = y^* - (n+g+\delta)k^*}$

2.) Let $k = k^*(n, g, s, \delta)$. Then

$\frac{\partial c^*}{\partial s} = \frac{\partial}{\partial s} f(k^*(n,g,s,\delta)) - (n+g+\delta)\frac{\partial k^*}{\partial s} = \frac{\partial f}{\partial k^*}\frac{\partial k^*}{\partial s} - (n+g+\delta)\frac{\partial k^*}{\partial s} = [f'(k^*) - (n+g+\delta)]\frac{\partial k^*}{\partial s}$

Lw: $\boxed{\frac{\partial c^*}{\partial s} = [f'(k^*) - (n+g+\delta)]\frac{\partial k^*}{\partial s}}$

[iii] SPEED OF CONVERGENCE

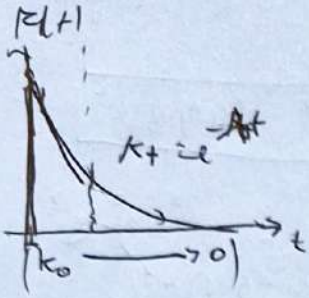
Convergence = How Rapidly Those Rates occur.

1) eh / $\dot{k} = s f(k) - (n+g+\delta)k$ or $\dot{k} = \dot{k}(k)$

and, $\dot{k}(k) \sim \left.\frac{\partial \dot{k}(k)}{\partial k}\right|_{k=k^*} (k-k^*) + \frac{1}{2!}\frac{\partial^2 \dot{k}(k)}{\partial k^2}(k-k^*)^2 + \dots$ → Taylor Series

$\equiv \left.\frac{\partial \dot{k}(k)}{\partial k}\right|_{k=k^*} (k-k^*)$ → 1st order.

$= \lambda(\dot{k}, k)(k-k^*)$.



Examples:

$\boxed{k_t \sim k^* + e^{-\lambda t}(k_0 - k^*)}$

$f' = \alpha k^{\alpha-1}$

$f = k^\alpha$

$\frac{f'}{f} = \frac{\alpha k^{\alpha-1}}{k^\alpha} = \alpha k^{\alpha-1-\alpha} = \alpha k^{\alpha-\alpha-1} = \alpha k^{-1}$

so $\boxed{\dot{k}(k) \sim \left.\frac{\partial \dot{k}}{\partial k}\right|_{k=k^*}(k-k^*) = \lambda(k-k^*)}$

2) $\lambda = -\left.\frac{\partial \dot{k}(k)}{\partial k}\right|_{k=k^*} = -[sf' - (n+g+\delta)] = (n+g+\delta) - sf'(k^*)$

$= (n+g+\delta) - \frac{(n+g+\delta)k^*}{f(k^*)} f'(k^*)$

$= (n+g+\delta)\left[1 - \frac{f'}{f}k^*\right] = n+g+\delta\left[1 - \alpha k^{-1}k^*\right]$

$= \left[1 - \alpha k(k^*)\right](n+g+\delta)$

so $\boxed{\lambda = (1-\alpha)[(n+g+\delta)]}$

2a Example:

	n_1	n_2
$n =$	1%	2%
$\delta =$	3%	4%
$g =$	1%	2%

say:

$n_1 + \delta_1 + g_1 = 1\% + 3\% + 1\% = 5\%$

$n_2 + \delta_2 + g_2 = 2\% + 4\% + 2\% = 8\%$

$\boxed{n_1 + \delta_1 + g = 0 = 4\%, \quad n_2 + \delta_2 + g = 0 = 6\%}$

a) $1 - \alpha\big|_{\alpha=\frac{1}{3}} = 1 - \frac{1}{3} = \frac{3-1}{3} = \frac{2}{3}$ → $\frac{2}{3} \cdot \frac{4}{100}, \frac{2}{3}\frac{6}{100}$ → ∴ 0.021, 0.04

so: $\boxed{\lambda\big|_{\alpha=\frac{1}{3}} = [0.02, 0.04]}$

which is

$T = \frac{70}{0.0\cancel{4}} = \frac{70 \cdot 0.04}{\cancel{4}\,x} = \frac{2.8}{x} = 17$

$x = \frac{2.8}{17} = 0.16$?

so $T = \frac{70\lambda}{0.16}$ or $\boxed{T = \frac{\ln 2}{\text{growth}} \sim 17 \text{ years}}$

HALF LIFE

TIME MATRIX

$\boxed{T = \frac{70}{\lambda}}$

In Theory

If here $T(\lambda) = \frac{70}{\lambda}$

$\boxed{T(\lambda) = \frac{\ln(0.5)}{\lambda}}$ and $e^{-\lambda T} = \frac{1}{2}$.

or $\boxed{y(t) - y^* \sim e^{-\lambda t}[y_0 - y(t)]}$

PROBLEMS 1

① Basic properties of Growth

show that : ⓐ If $Z_t = X_t Y_t$ Then $\frac{\dot{Z}_t}{Z_t} = \frac{\dot{X}_t}{X_t} + \frac{\dot{Y}_t}{Y_t}$

Ans: $Z_t = X_t Y_t \Rightarrow \log Z_t = \log X_t + \log Y_t$

$$\frac{\partial \log Z_t}{\partial t} = \frac{\partial \log X_t}{\partial t} + \frac{\partial \log Y_t}{\partial t}$$

$$\frac{\partial \log Z_t}{Z_t}\frac{\partial Z_t}{Z_t} = \frac{\partial \log X_t}{\partial X_t}\frac{\partial X_t}{\partial t} + \frac{\partial \log Y_t}{\partial Y_t}\frac{\partial Y_t}{\partial t}$$

$$\frac{1}{Z_t}\dot{Z}_t = \frac{1}{X_t}\dot{X}_t + \frac{1}{Y_t}\dot{Y}_t \Rightarrow \therefore \boxed{\frac{\dot{Z}_t}{Z_t} = \frac{\dot{X}_t}{X_t} + \frac{\dot{Y}_t}{Y_t}}$$

ⓑ $Z_t = \frac{X_t}{Y_t}$ Then $\frac{\dot{Z}_t}{Z_t} = \frac{\dot{X}_t}{X_t} - \frac{\dot{Y}_t}{Y_t}$

Ans: use ⓐ and $\log\left(\frac{X_t}{Y_t}\right) = \log X_t - \log Y_t \Rightarrow \boxed{\frac{\dot{X}_t}{X_t} - \frac{\dot{Y}_t}{Y_t}}$

ⓒ $Z_t = X(t)^{\alpha}$ Then $\dot{Z}_t / Z_t = \alpha \dot{X}_t / X(t)$

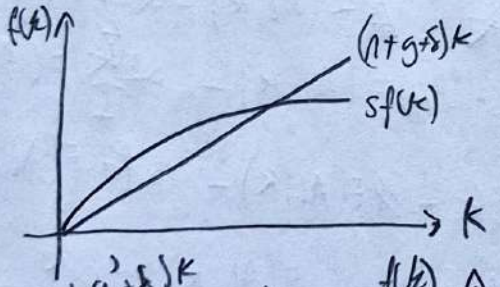
Ans: also same s.t : $\log X_t^{\alpha} = \alpha \log X_t \Rightarrow \frac{\partial \log X_t}{\partial X_t}\frac{\partial X_t}{\partial t}\alpha = \boxed{\alpha \frac{\dot{X}_t}{X_t}}$

③ Solow Growth Mechanics :

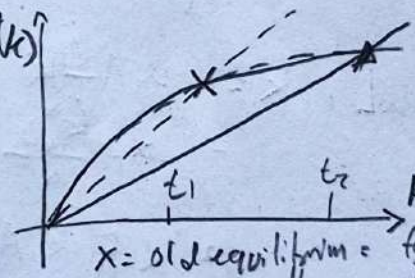
What happens if : ⓐ Depreciation falls? ⓑ Technological Rate Rises? ⓒ $f(k) = k^{\alpha}$ w/ k increase?, ⓓ Workers Exert more effort? (w/ k^{α})

Ans: The Solow model : $\dot{k}(t) = sf(k_t) - (n+g+\delta)k(t)$

our Graph:

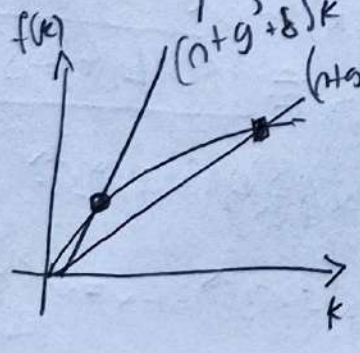


a.)

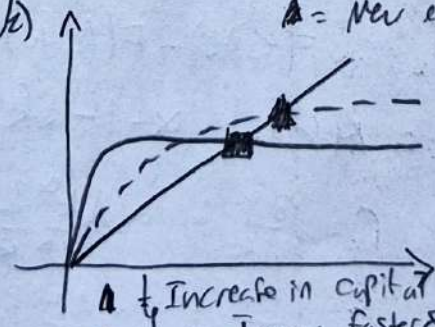


X = old equilibrium = faster BE
▲ = New equilibrium : slower BE

ⓑ



ⓒ



w/ $\frac{\partial f}{\partial k} = \alpha k^{\alpha-1} = \alpha \frac{k^{\alpha}}{k} = \alpha k^{\alpha}/k$

▲ ↑ Increase in capital = ~~Increase~~ faster δ.

(i) FACTOR PAYMENTS

Let $\boxed{W = \frac{\partial F(k,AL)}{\partial L}}$ and $\boxed{r = \frac{\partial F(k,AL)}{\partial k} - \delta}$.

(a) Show that The Marginal Product of Labour is $A[f(k) - kf'(k)]$:

ANS: $F(k,AL) = Ak^{\alpha}L^{1-\alpha} = Ak^{\alpha}\frac{L}{L^{\alpha}} = AL\frac{k^{\alpha}}{L^{\alpha}} = ALk^{\alpha} = ALf(k)$

So. $\frac{\partial F(k,AL)}{\partial L} = \frac{\partial}{\partial L}AL \cdot f(k) + \frac{\partial f(k)}{\partial L}AL = Af(k) + \frac{\partial}{\partial L}\frac{k^{\alpha}}{L^{\alpha}}AL =$

$= Af(k) - \alpha k^{\alpha}L^{-\alpha-2}AL = Af_k - \alpha k^{\alpha}\frac{L}{L^{\alpha}L}AL$

$= Af(k) - \frac{\alpha k^{\alpha}}{L^{\alpha}}A = Af(k) - Af'(k)$

$= A[f(k) - f'(k)k]$ so: $\boxed{W = A[f(k) - f'(k)k]}$

Algebra:

$\frac{\partial k^{\alpha}/L^{\alpha}}{\partial L}$

$= k^{\alpha}\frac{\partial L^{-\alpha}}{\partial L}$

$= -\alpha L^{-\alpha-2}k^{\alpha}$

$= -\alpha k^{\alpha}L^{-\alpha-2}AL$

$= -\alpha Ak^{\alpha}\frac{1}{L^{\alpha}L} = -\alpha\frac{Ak^{\alpha}}{L^{\alpha}}$

(b) show that under Cons. Returns: $wL + rk = F(k,AL) - \delta$

we know: $W = A(f(k) - f'(k)k)$. Now For Capital we take The derivative of F s.t:

ANS: $\frac{\partial F(k,AL)}{\partial k} = \frac{\partial}{\partial k}Ak^{\alpha}L^{1-\alpha} = \alpha Ak^{\alpha-1}L^{1-\alpha} = \alpha A\frac{k^{\alpha}}{k}\frac{L}{L^{\alpha}} = \alpha Af(k)\frac{L}{k}$

So: $r = \frac{\partial F(k,AL)}{\partial k} - \delta = \alpha Af(k)\frac{L}{k} - \delta$ and $rk = k\left(\alpha Af(k)\frac{L}{k} - \delta\right)$

$= \alpha Af(k)L - \delta k$

and $wL = (Af(k) - kf'(k))L = \left(A\frac{k^{\alpha}}{L^{\alpha}} - \alpha\frac{k}{L}\frac{k^{\alpha}}{L^{\alpha}}\right)L = A\left(k^{\alpha}L^{1-\alpha} - \alpha\frac{k}{L}k^{\alpha}L^{1-\alpha}\right)$

$= A\left(F(AL,k) - \alpha\frac{k}{L}F(AL,k)\right)$

So. $wL + rk = \alpha Af(k)\cdot L - \delta k - F(AL,k) - \alpha\frac{k}{L}F(AL,k)$

$= \alpha A\frac{k^{\alpha}}{L^{\alpha}}L - \delta k - F(AL,k) - \alpha\frac{k}{L}F(AL,k)$

$= \alpha F(k,AL) - \delta k - F(AL,k) - \alpha k\cdot F(AL,k)$

$= \alpha F(k,AL) - F(AL,k) - \alpha k F(AL,k) - \delta k$

$= F(k,AL)\{\alpha - 1 - \alpha k\} - \delta k \triangleq \boxed{F(AL,k) - \delta}$

so add δ and voila!

QUICK ANSWER:

Let $\cancel{A}\delta$. so: $r = \frac{\partial Y}{\partial k} = \alpha A\frac{k^{\alpha}}{L^{\alpha}}\frac{L}{k} \Rightarrow rk = \alpha A\frac{k^{\alpha}}{L^{\alpha}}L = \alpha Ak^{\alpha}L^{1-\alpha} = \alpha F$

$W = \frac{\partial Y}{\partial L} = 1-\alpha A\frac{k^{\alpha}}{L^{\alpha}} \Rightarrow wL = (1-\alpha)Ak^{\alpha}L^{1-\alpha} = 1-\alpha F$

$1 - \alpha F + F\alpha = F - \alpha F + \alpha F = \boxed{F}$

Quantitative Implications, Key Aspects & Empirical (1.5) (1.6)

(5.) Quantitative Implications

[1] Effect on Output in the long run:

1) Let $y^* = f(k^*)$ and $k^* = k^*(s, n, \delta, g)$. Then:

$$\frac{\partial y^*}{\partial s} = \frac{\partial f(k^*)}{\partial k^*}\frac{\partial k^*}{\partial s} = f'(k^*)\frac{\partial k^*(s,n,\delta,g)}{\partial s} \quad \text{so:} \quad \boxed{\partial_s y^* = f'(k^*)\,\partial_s k^*}$$

2) Law: $(1-s)y = c \Rightarrow y - sy = c \rightarrow -y + sy = -c \rightarrow sy = y - c$

In: at $(\dot{k}=0) \rightarrow 0 = sy^* - (n+g+\delta)k^* \quad -sy^* = (n+g+\delta)k^*$

$$\boxed{c^* = y^* - (n+g+\delta)k^*} \longleftrightarrow -y^* + c^* = (n+g+\delta)k^*$$

3) $sf^*(s,n,\delta,g) = sy^* = (n+g+\delta)k^*$ and if we take the derivative

$$\frac{\partial(sy^*)}{\partial s} = \frac{\partial k^*}{\partial s}(n+g+\delta) = \frac{\partial s}{\partial s}y^* + \frac{\partial y^*}{\partial s}s = y^* + f'(k)\frac{\partial k^*}{\partial s} = \frac{\partial k^*}{\partial s}(n+g+\delta)$$

In:
$$\boxed{y^* + f'(k^*)\frac{\partial k^*}{\partial s}s = \frac{\partial k^*}{\partial s}(n+g+\delta)}$$

4) Then:

$$y^* = \frac{\partial k^*}{\partial s}(n+g+\delta) - sf'(k^*)\frac{\partial k^*}{\partial s} = \frac{\partial k^*}{\partial s}\left[(n+g+\delta) - sf'(k)\right]$$

and finally:
$$\boxed{\frac{\partial k^*}{\partial s} = \frac{y^*}{\left[(n+g+\delta) - sf'(k)\right]}}$$

$$H = \begin{vmatrix} \frac{\partial^2 y^*}{\partial s^2} & \frac{\partial y}{\partial s} & \partial_{ns}y \\ \partial_{sg}y^* & \frac{\partial^2 y}{\partial g} & \frac{\partial^2 y}{\partial n^2} \\ \partial_{sn}y^* & \partial_{gsn}y^* & \frac{\partial^2 y}{\partial \delta^2} \\ \partial_{s\delta} & \partial_{g\delta n}y^* & \end{vmatrix}$$

5) Since $\partial_s y^* = f'(k^*)\,\partial_s k^*$. Then:

$$\frac{\partial y^*}{\partial s} = f'(k^*)\left[\frac{y^*}{(n+\delta+g) - sf'(k)}\right] = \frac{f'(k^*)\cdot f(k^*)}{(n+\delta+g) - sf'(k^*)}$$

Since Answer:

In:
$$\boxed{\partial_s y^* = y^* f'(k^*)\frac{1}{(n+\delta+g) - sf'(k)}}$$

6): $$\frac{s}{y^*}\frac{\partial y^*}{\partial s} = \frac{s}{y^*}y^* f'\frac{1}{(n+\delta+g) - sf'(k^*)} = \frac{sy^*}{y^*}f'\frac{1}{(n+\delta+g) - sf'(k)} = \frac{(n+g+\delta)k^* f'(k^*)}{f(k^*)\left[(n+g+\delta) - sf'(k^*)\right]}$$

a) The $$\frac{\cancel{sf(k^*)}\left[(n+g+\delta) - f'(k^*)\right]}{(n+g+\delta)k^* - f'(k^*)} = \frac{f(k^*)(n+\cancel{g}+\delta)}{\cancel{f(k)}(n+\cancel{g}+\delta)k^*} - \frac{f(k^*)f'(k^*)}{(n+g+\delta)k^*\cancel{f(k^*)}}$$

$$= \frac{f(k^*)}{f(k^*)}\frac{1}{k^*} - \frac{f(k^*)}{(n+\delta+g)}\frac{1}{k^*} \xrightarrow{\Delta} \frac{k^*(n+\delta+g) - f'(k^*)k^*}{f(k^*)}$$

b) Since: $sf(k^*) = (n+g+\delta)k^*$ and $\boxed{s = \frac{(n+g+\delta)k^*}{f(k^*)}}$

Then
$$\frac{s}{y^*}\frac{\partial y^*}{\partial s} = \frac{(n+\delta+g)k^* f'}{f\left[(n+g+\delta) - sf'\right]} = \frac{(n+\delta+g)k^* f'}{f\left[(n+g+\delta) - \frac{(n+g+\delta)k^* f'}{f}\right]}$$

$$= k^*\frac{f'(k^*)}{f(k^*)}\frac{1}{1 - k^*\left[\frac{f'(k^*)}{f(k^*)}\right]}$$

In:
$$\boxed{\frac{s}{y^*}\frac{\partial y^*}{\partial s} = k^*\frac{y'^*}{y^*}\frac{1}{1 - k^*\left[\frac{y'}{y^*}\right]}} \rightarrow 7)\ \boxed{\frac{s}{y^*}\frac{\partial y^*}{\partial s} = \frac{\alpha_k(k^*)}{1 - \alpha_k(k^*)}}$$

THE DIAMOND MODEL - BASICS

2.8 | 2.9

(8) Assumptions

- **UTILITY FORMULATION & CAPITAL**
  - Let (C_{1t}, C_{2t}) be the consumption of young and individuals at t. The utility of an individual born at t depends on C_{1t} and C_{2t+1} thus.

$$\boxed{U_t = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2t+1}^{1-\theta}}{1-\theta}} \quad \text{w/ } \theta > 0,\ \rho > -1.$$

  - Some properties:
    1.) If $(\rho > 0)$ individuals place greater weight on 1st period
    2.) If $(\rho < 0)$, they placed more on 2nd period.
  - The capital at $t+1$ will be its (Assets × wage − consumption) for every labor L_t thus.

$$\boxed{K_{t+1} = (w_t A_t - C_t) L_t}$$

- **Functional Assumptions**
  - The production function is: $Y_t = f(K_t, A_t, L_t)$.
  - Assets/knowledge grows at: $A_t = [1+g] A_{t-1}$.
  - The Cobb-Douglas Rent/wage: $r_t = f'(k_t)$ $\quad w_t = f(k_t) - k_t f(k_t)$
  - Initial capital is owned by the old: K_0 belongs to (t)

(9) Household Behavior

- **The Budget Formulation:**
  - The 2nd period consumption of an individual born at t is:

$$\boxed{C_{2t+1} = (1+r_{t+1})(w_t A_t - C_{1t})} \quad (2.44)$$

  - We solve for C_{1t} A_t

$$\text{w: } C_{2t+1} = (1+r_{t+1}) w_t A_t - (1+r_{t+1}) C_{1t} \Rightarrow \frac{C_{2t+1}}{1+r_{t-1}} - w_t A_t = -C_{1t}$$

$$\Rightarrow \boxed{w_t A_t = C_{1t} + \frac{1}{1+r_{t+1}} C_{2t+1}} \quad (2.46)$$

$\Rightarrow$ The present value of lifetime consumption equals wealth + present value labour income.

  - The maximization of this requires us to formulate:

i.) $\Delta C(1+r_{t+1})$ to route R_{2t+1} 2.) $(C_{1t}^{-\theta}) \left(\frac{1}{1+\rho} C_{2t+1}^{-\theta}\right)$ 3.) $\left\{ \begin{array}{l} C_{1t}^{-\theta} \Delta C \rightarrow \text{increment change in consumption} \\ \frac{C_{2t+1}^{-\theta} \Delta C}{1+\rho} \rightarrow \text{increment change in total cons} \end{array} \right.$

4.) The optimization is thus:

$$\boxed{C_{1t}^{-\theta} \Delta C = \frac{1}{1+\rho} C_{2t+1}^{-\theta} (1+r_{t+1}) \Delta C} \quad (2.47)$$

  - Thus, this culture: $C_{1t}^{-\theta} = (1+\rho)^{-1} C_{2t+1}^{-\theta} (1+r_{t+1}) \rightarrow \frac{C_{1t}^{-\theta}}{C_{2t+1}^{-\theta}} = \frac{1+r_{t+1}}{1+\rho} \rightarrow \frac{C_{2t+1}^{\theta}}{C_{1t}^{\theta}} = \frac{1+r_{t+1}}{1+\rho}$

$$\Rightarrow \boxed{\frac{C_{2t+1}}{C_{1t}} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}}} \quad (2.49)$$

- **The Lagrangian:**
  - We use the Lagrangian set:

$$\boxed{\mathcal{L} = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2t+1}^{1-\theta}}{1-\theta} + \lambda \left[A_t w_t - \left[C_{1t} + \frac{1}{1+r_{t+1}} C_{2t+1} \right] \right]} \quad 2 \leq 0$$

  - The FOC: $\frac{\partial \mathcal{L}}{\partial C_{1t}} = C_{1t}^{-\theta} - \lambda = 0 \quad \frac{\partial \mathcal{L}}{\partial C_{2t}} = \frac{1}{1+\rho} C_{2t+1}^{-\theta} + \lambda \frac{1}{1+r_{t+1}} C_{2t+1} = 0$

THE DIAMOND MODEL - BASICS

2.8 | 2.9

(8) Assumptions

- **UTILITY FORMULATION & CAPITAL**
  - Let (C_{1t}, C_{2t}) be the consumption of young and individuals at t. The utility of an individual born at t depends on C_{1t} and C_{2t+1} thus.

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  - Some properties:
    1.) If $(\rho > 0)$ individuals place greater weight on 1st period
    2.) If $(\rho < 0)$, they placed more on 2nd period.
  - The capital at $t+1$ will be its (Assets × wage − consumption) for every labor L_t thus.

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  - The production function is: $Y_t = f(K_t, A_t, L_t)$.
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$$\Rightarrow \boxed{w_t A_t = C_{1t} + \frac{1}{1+r_{t+1}} C_{2t+1}} \quad (2.46)$$

$\Rightarrow$ The present value of lifetime consumption equals wealth + present value labour income.

  - The maximization of this requires us to formulate:

i.) $\Delta C(1+r_{t+1})$ to route R_{2t+1} 2.) $(C_{1t}^{-\theta}) \left(\frac{1}{1+\rho} C_{2t+1}^{-\theta}\right)$ 3.) $\left\{ \begin{array}{l} C_{1t}^{-\theta} \Delta C \rightarrow \text{increment change in consumption} \\ \frac{C_{2t+1}^{-\theta} \Delta C}{1+\rho} \rightarrow \text{increment change in total cons} \end{array} \right.$

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  - Thus, this culture: $C_{1t}^{-\theta} = (1+\rho)^{-1} C_{2t+1}^{-\theta} (1+r_{t+1}) \rightarrow \frac{C_{1t}^{-\theta}}{C_{2t+1}^{-\theta}} = \frac{1+r_{t+1}}{1+\rho} \rightarrow \frac{C_{2t+1}^{\theta}}{C_{1t}^{\theta}} = \frac{1+r_{t+1}}{1+\rho}$

$$\Rightarrow \boxed{\frac{C_{2t+1}}{C_{1t}} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}}} \quad (2.49)$$

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  - We use the Lagrangian set:

$$\boxed{\mathcal{L} = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2t+1}^{1-\theta}}{1-\theta} + \lambda \left[A_t w_t - \left[C_{1t} + \frac{1}{1+r_{t+1}} C_{2t+1} \right] \right]} \quad 2 \leq 0$$

  - The FOC: $\frac{\partial \mathcal{L}}{\partial C_{1t}} = C_{1t}^{-\theta} - \lambda = 0 \quad \frac{\partial \mathcal{L}}{\partial C_{2t}} = \frac{1}{1+\rho} C_{2t+1}^{-\theta} + \lambda \frac{1}{1+r_{t+1}} C_{2t+1} = 0$

THE DIAMOND MODEL - BASICS

2.3 | 2.5

8. Assumptions

UTILITY FORMULATION & CAPITAL

- Let (C_{1t}, C_{2t}) be the consumption of young and individuals of t. Depends on C_{1t} or C_{2t+1} then.
  The utility of an individual born at t:

$$\boxed{U_t = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2t+1}^{1-\theta}}{1-\theta}} \quad \text{w/ } \theta > 0,\ \rho > -1.$$

- Some properties:
  1.) If $(\rho > 0)$, individuals place greater weight on 1st period
  2.) If $(\rho < 0)$, then placed more on 2nd period.
- The capital at $t+1$ will be its (assets × wage − consumption) for every labor L_t.
  Thus:

$$\boxed{K_{t+1} = (W_t A_t - C_t) L_t}$$

FUNDAMENTAL ASSUMPTIONS

- The production function is: $Y_t = F(K_t, A_t, L_t)$.
- Assets/knowledge grows at: $A_t = [1+g] A_{t-1}$.
- The Cobb-Douglas Rent/wage: $r_t = f'(k_t)$ $\quad w_t = f(k_t) - k_t f(k_t)$
- Initial capital is owned by the old: K_0 belongs to (R_t)

9. HOUSEHOLD BEHAVIOUR

THE BASIC FORMULATION:

- The 2nd period consumption of an individual born at t is:

$$\boxed{C_{2t+1} = (1+r_{t+1})(W_t A_t - C_{1t})} \quad (2.44)$$

- We solve for C_{1t} A_t

$$\text{w: } C_{2t+1} = (1+r_{t+1}) W_t A_t - (1+r_{t+1}) C_{1t} \Rightarrow \frac{C_{2t+1}}{1+r_{t-1}} - W_t A_t = -C_{1t}$$

$$\Rightarrow \boxed{W_t A_t = C_{1t} + \frac{1}{1+r_{t+1}} C_{2t+1}} \quad (2.46)$$

$\Rightarrow$ The present value of life time consumption equals wealth + present value labour income.

- The maximization of this requires us to formulate:
  1.) $\Delta C(1+r_{t+1})$ to raise C_{2t+1} 2.) $\left(C_{1t}^{-\theta}\right) A \left(\frac{1}{1+\rho} C_{2t+1}^{-\theta}\right)$ 3) $\begin{cases} C_{1t}^{-\theta}\Delta C \rightarrow \text{increment change in consumption} \\ \frac{C_{2t+1}\Delta C}{1+\rho} \rightarrow \text{increment change in mutual cons} \end{cases}$
  4.) The optimization is then:

$$\boxed{C_{1t}^{-\theta}\Delta C = \frac{1}{1+\rho} C_{2t+1}^{-\theta}(1+r_{t+1})\Delta C} \quad (2.47)$$

- Thus, this implies: $C_{1t}^{-\theta} = (1+\rho)^{-1} C_{2t+1}^{-\theta}(1+r_{t+1}) \Rightarrow \frac{C_{1t}^{-\theta}}{C_{2t+1}^{-\theta}} = \frac{1+r_{t+1}}{1+\rho} \Rightarrow \frac{C_{2t+1}^{\theta}}{C_{1t}^{\theta}} = \frac{1+r_{t+1}}{1+\rho}$

$$\Rightarrow \boxed{\frac{C_{2t+1}}{C_{1t}} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}}} \quad (2.49)$$

THE LAGRANGIAN:

- We use the Lagrangian set:

$$\boxed{\mathcal{L} = \frac{C_{1t}^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{2t+1}^{\theta}}{1-\theta} + \lambda\left[A_t W_t - \left[C_{1t} + \frac{1}{1+r_{t+1}} C_{2t+1}\right]\right]} \quad \lambda \leq 0$$

- The FOC: $\frac{\partial \mathcal{L}}{\partial C_{1t}} = C_{1t}^{+\theta} - \lambda = 0 \quad \frac{\partial \mathcal{L}}{\partial C_{2t}} = \frac{1}{1+\rho} C_{2t+1}^{\theta} + \lambda \frac{1}{1+r_{t+1}} C_{2t+1} = 0$

$\rightarrow C_{1t}^{-\theta} = \lambda$, $C_{2t+1}^{-\theta}\frac{1+r_{t+1}}{1+\rho} = \lambda$ (2.51)(2.52)

$\Rightarrow$ Thus substituting:

$$\boxed{C_{1t}^{-\theta} = C_{2t+1}^{-\theta}\frac{1+r_{t+1}}{1+\rho}} \quad (2.53) \quad \text{or} \quad \boxed{\frac{C_{2t+1}}{C_{1t}} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}}}$$

WEALTH & SAVINGS

- We can use the eqns of (2.46) via $C_{2t+1} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}} C_{1t}$ and Thus

$$C_{1t} + \frac{1}{1+r_{t+1}} C_{2t+1} = C_{1t} + \frac{1}{1+r_{t+1}}\left(\frac{1+r_t}{1+\rho}\right)^{\frac{1}{\theta}} C_{1t} = C_{1t} + \frac{(1+r_t)^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}} C_{1t} = A_t w_t$$

hence $\boxed{A_t w_t = C_{1t} + \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}} C_{1t}}$

NOTATION

$q_{r_j} = (1+r_{t+j})$

$q_{r_1} = (1+r_{t+1})$

$q_\rho = (1+\rho)$

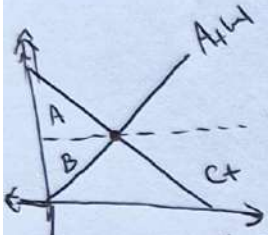
- To find the young's consumption we use a little more algebra:

$$C_{t+1}\left(1 + \frac{q_{r_1}^{\frac{1-\theta}{\theta}}}{q_\rho^{\frac{1}{\theta}}}\right) = A_t w_t \rightarrow C_{t+1} = \frac{A_t w_t}{\left(1 + \frac{q_{r_1}^{\frac{1-\theta}{\theta}}}{q_\rho^{\frac{1}{\theta}}}\right)} = A_t w_t\left(\frac{q_\rho^{\frac{1}{\theta}}}{q_\rho^{\frac{1}{\theta}} + q_{r_1}^{\frac{1-\theta}{\theta}}}\right)^{-1}$$

$$= A_t w_t\left(\frac{q_\rho^{\frac{1}{\theta}}}{q_\rho^{\frac{1}{\theta}} + q_{r_1}^{\frac{1-\theta}{\theta}}}\right) = A_t w_t\left(\frac{q_\rho^{\frac{1}{\theta}}}{q_\rho^{\frac{1}{\theta}} + q_{r_1}^{\frac{1-\theta}{\theta}}} + \frac{q_{r_1}^{\frac{1-\theta}{\theta}}}{q_\rho^{\frac{1}{\theta}} + q_{r_1}^{\frac{1-\theta}{\theta}}}\right) \Rightarrow A_t w_t\left(\frac{q_\rho^{\frac{1}{\theta}}}{q_\rho^{\frac{1}{\theta}} + q_{r_1}^{\frac{1-\theta}{\theta}}}\right) = C_{t+1}$$

so: $\boxed{C_{1t} = A_t w_t \frac{q_\rho^{\frac{1}{\theta}}}{q_\rho^{\frac{1}{\theta}} + q_{r_1}^{\frac{1-\theta}{\theta}}}}$ or $\boxed{C_{1t} = \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r)^{\frac{1-\theta}{\theta}}} A_t w_t}$

Supply & Demand



A: Consumption supplies of the Economy

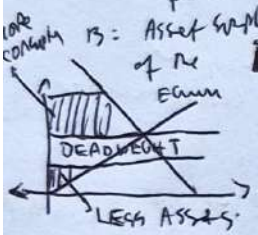
B: Asset supply of the Economy

- To define savings set SAVINGS = $\frac{\text{INCOME}}{\text{CONSUMPTION}}$, Then $\frac{A_t w_t}{C_t} = S_t$. Thus

$$S(t) = \frac{A_t w_t}{C_t} = \left(\frac{q_\rho^{\frac{1}{\theta}}}{q_\rho^{\frac{1}{\theta}} + q_r^{\frac{1-\theta}{\theta}}}\right)^{-1} = \frac{q_\rho^{\frac{1}{\theta}} + q_r^{\frac{1-\theta}{\theta}}}{q_\rho^{\frac{1}{\theta}}}$$

hence: $\boxed{S(r) = \frac{q_\rho^{\frac{1}{\theta}} + q_r^{\frac{1-\theta}{\theta}}}{q_\rho^{\frac{1}{\theta}}}}$ or it could also be: $\boxed{S(r) = \frac{q_r^{\frac{1-\theta}{\theta}}}{q_\rho^{\frac{1}{\theta}} + q_r^{\frac{1-\theta}{\theta}}}}$

That way. The function of



CONSEQUENCES

- The consumption is a strictly quasi:

$$\boxed{C_{1t} = [1 - S(r_{t+1})] A_t w_t}$$ and 2 comparatives, eg:

1) $q_{r,\theta} = (1+r)^{\frac{1-\theta}{\theta}}$ 2) $q_{\rho,\theta} = (1+\rho)^{\frac{1}{\theta}}$

- Taken the derivative in (r, ρ) resp:

1) $\frac{dq_{r,\theta}}{dr} = \frac{1-\theta}{\theta}(1+r)^{\frac{1-2\theta}{\theta}}$ and 2) $\frac{dq_{\rho,\theta}}{d\rho} = \frac{1}{\theta}(1+\rho)$

- We can draw out 2 consequences in the Regression via θ, s.t.

1) If $(\theta < 1)$, Then Savings Increase 2) If $(\theta > 1)$ Then savings Decrease.

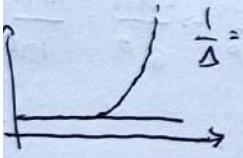
3) The rise in (r) has both Income & Substitution effect but:

a) 2nd period Favor = Substitution b) 2nd period. saving decrease = Income

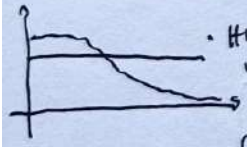
$\therefore$ So θ Low = Substitution = Rise in savings | θ High = Income = Rise in Income

$\theta = 1 \rightarrow$ INCOME = SUBSTITUTION

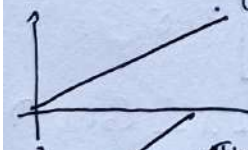
LIFE CYCLE AND MARGINAL RATES



$\frac{1}{\Delta}$ = Propensity to consume

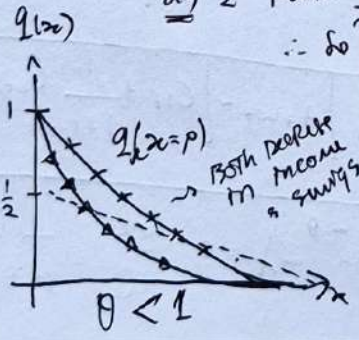


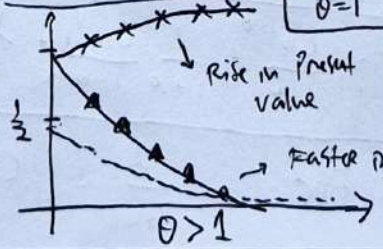
- Human wealth



- Consumption

- Financial assets





QUANTITATIVE IMPLICATIONS & DYNAMICAL ANALYSIS

(1) DYNAMICS

(i) EVOLUTION OR MOTION OF K

KAPITAL & SAVINGS.

- We can aggregate individual's behavior to characterize the dynamics
- Since the Euler curve fractions of $S(r_t)$, then:

$$\boxed{K_{t+1} = S(r_{t+1})\, A_t w_t \cdot L_t} \quad (2.58)$$

RENT - FORM

- Substituting for $r = f'(k)$ and $w = f - kf$ Then:

$$\boxed{k_{t+1} = \frac{1}{(1+n)(1+g)} S(f'(k)) \cdot [f(k) - k f(k)]} \quad (2.59)$$

CAPITAL PER UNIT OF EFFECTIVE LABOR

- We can divide each side for $L_{t+1} A_t$ get

$$\frac{K_{t+1}}{A_{t+1} L_{t+1}} = S(r_t) \frac{L_t}{L_{t+1}} = S(r_t) \frac{L_t}{L_{t+1}} \frac{A_t}{A_{t+1}} = S(r_t) \frac{w}{(1+n)(1+g)}$$

- So:

$$\boxed{k_{t+1} = \frac{1}{(1+n)(1+g)} S(r_{t+1}) w_t} \quad (2.60)$$

(ii) THE EVOLUTION OF k

FUNDAMENTAL EQUATION:

- Since: $k_{t+1} = \frac{1}{g_n g_g} S(r(k_t)) [w(k_t)] = w_t$ $\boxed{k_{t+1} = \frac{S(r(k_t)) \cdot w(k_t)}{g_n \cdot g_g}}$ $g_n \cdot g_g > 0$
- The value of $\boxed{k_{t+1} = k}$ is the balanced growth path, since once k reaches that value, it'll remain there

CONVERGENCE

(?) We wanna know whether there's a BGP of k and at least $k \to k_{t+1}$ or $k_t - k_{t+1}$?

(iii) LOGARITHMIC & COBB-DOUGLAS PRODUCTION FUNCTION

GOVERNING EQUATION

- Suppose that $\theta = 1$. Then savings via (2.56) is:

$$S(r)_{\theta=1} = \frac{(1+r)^{\frac{1-1}{1}}}{(1+\rho)^{\frac{1}{1}} + (1+r)^{\frac{1-1}{1}}} = \frac{(1+r)^0}{(1+\rho)^1 + (1+r)^0} = \left(\frac{(1+\rho)^0}{(1+r)^0} + \frac{(1+r)^0}{(1+r)^0}\right)^{-1} = ((1+\rho+1)^{-1} = (2+\rho)^{-1} = \frac{1}{(2-\rho)}$$

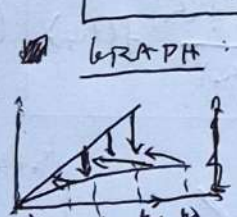
So: $\boxed{S_{\theta=1}(r) = \frac{1}{2-\rho}}$

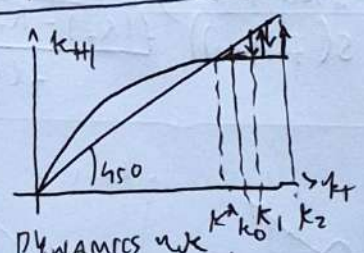
- Since by the Cobb Douglas that $f'(k) = \alpha k^{\alpha-1}$, $f(k) = k^{\alpha}$. Then

$$\boxed{k_{t+1} = \frac{1}{g(g)\, g(n)} \frac{1}{2+\rho} (1-\alpha) k_t^{\alpha}}$$

so if $\alpha = \frac{1}{3}$, $1-\alpha = \frac{2}{3}$: $\frac{1}{g_g g_n} \frac{1}{2+\rho} \frac{2}{3} k_t^{\frac{1}{3}}$

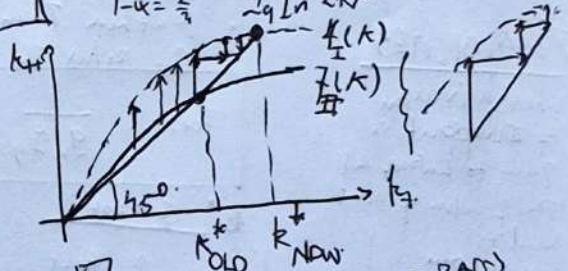
GRAPH:





DYNAMICS w/ k

- k^* is globally stable. *
- k_t will converge to k^*



EFFECTS IN DISCOUNT RATES

- We see that the stocks of capital of $f_{II}(k)$ is lower than f_I
- Since in an economic shocks, growth decreases and the young generations save capital (k^*_{New}) higher

(iv) SPEED OF CONVERGENCE

THE EQUILIBRIUM OUTPUT

- Recall that: $\boxed{k^* = \frac{1}{g_g g_n} \frac{1}{2+\rho} (1-\alpha) k^{*\alpha}}$

- Basically this means:

$$\frac{k^*}{k^{*\alpha}} = k^{*1-\alpha} = \frac{1}{g_n g_g}\frac{s(1-\alpha)}{2+\rho} \rightarrow k^* = \left[\frac{s(1+\alpha)}{g_n g_g(2+\rho)}\right]^{\frac{1}{1-\alpha}}$$

hence: $\boxed{k^* = \left[\frac{1-\alpha}{g_n g_g(2+\rho)}\right]^{\frac{1}{1-\alpha}}}$ or $h^* = \left[\frac{(1-\alpha)}{(1+n)(1+g)(2+\rho)}\right]^{\frac{\alpha}{1-\alpha}}$

TAYLOR APPROXIMATION & HAZARD RATE

- we take the taylor approximation around:

$$k_{t+1} \simeq k^* + \left[\frac{dk_{t+1}}{dk_t}\Big|_{k_t=k^*}\right]\left[k_t - k^*\right]$$

$$\simeq k^* + \lambda(t)[k_t - k^*] = k^* + \lambda^t(k_t - k^*)$$

$$\Rightarrow k_{t+1} - k^* = \lambda^t[k_t - k^*]$$

- we can assume that $k_{t+1} = \beta^{-1}k_t$ as it's a forward operator.

hence: $\beta^{-1}k_t - k^* = \lambda^t[\beta^{-1}k_t - k^*]\Big|_{j=0}$

SOLOW MODEL

DIFFERENCE

- Recall the Solow model:

$[1-\alpha_k](n+g+\delta) = \lambda$

- One model has only $\lambda = \alpha$

$$\Rightarrow \beta^{-1}\beta^{0}k_t - k^* = \lambda^t[\beta^{0-1}k_t - k^*]$$

$$\rightarrow k_t - k^* = \lambda^t[\beta^{-1}k_t - k^*] = \lambda^t(k_{t-1} - k^*) = \lambda^t(k_0 - k^*)$$

assume $t \geq 0$.

hence: $\boxed{k_t - k^* = \lambda^t[k_0 - k^*]}$

- To find the rate of change, we simply solve for $\dot{k}_{t+1}$ wrt.

$$\lambda = \frac{dk_{t+1}}{dk_t}\Big|_{k_t=k^*} = \frac{d}{dk_t}\left[\frac{k_t^{\alpha}}{g_n g_g}\frac{s}{2+\rho} - \frac{\alpha k_t^{\alpha}}{g_n g_g 2+\rho}\right]$$

$$= \frac{1}{g_n g_g}\frac{1}{2+\rho}\left(\frac{dk_t^{\alpha}}{dt} - \alpha\frac{dk_t^{\alpha}}{dt}\right) = \frac{1}{g_n g_g 2+\rho}\left((\alpha)k_t^{\alpha-1} - (\alpha^2)k_t^{\alpha-1}\right)$$

$$= \frac{1}{g_n g_g}\frac{1}{2+\rho}\alpha\left[k^{\alpha-1} - \alpha k^{\alpha-1}\right] = \frac{1}{g_n g_g}\frac{1}{2+\rho}\alpha\left[(1-\alpha)k^{\alpha-1}\right]$$

So: $\lambda = \frac{dk_{t+1}}{dk_t} = \alpha$

$$= \alpha\frac{1-\alpha}{g_n g_g(2+\rho)}\left(\frac{1-\alpha}{g_n g_g(2+\rho)}\right)^{\frac{\alpha-1}{1-\alpha}} = \boxed{\alpha}$$

or $\boxed{\lambda = \alpha}$

[V] THE GENERAL CASE:

RELAXING THE ASSUMPTIONS

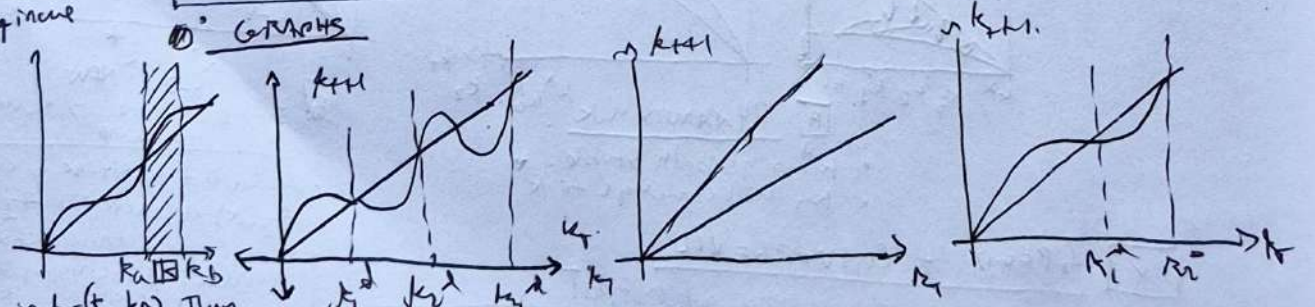
- Rather than a comprehensive Lagrangian utility analysis, we can relax the assumptions by [illegible]

$$\boxed{k_{t+1} = \frac{1}{g_n g_g}s(f'(k_{t+1}))\frac{f(k_t) - k_t f'(k_t)}{f(k_t)}f(k_t)} \quad (2.6.0)$$

POSSIBILITIES

- when a fraction of output is going to labour and the production income fund is constant, now

GRAPHS



if $k \in (k_a, k_b)$, then
1.) Saving dec. in (T) = high k_{t+1}
2.) If low k_{t+1} → lower
3.)

- stable = (k_1, k_3)

this is the case w/ $k_{t+1} \ll k_t$

PROBLEMS 2

① THE FIRM-WAGE PROBLEM

Consider N-firms w/ $Y_i = F(K, AL)$ or $Y = AL f(k)$. Assume $f' > 0, f'' < 0$. and assume that all firms can hire labor at wage wA and rent capital at r and all firms have technology A.

[A] Suppose a firm is trying to produce Y units of output at minimum cost. Show that the cost minimizing level of K is uniquely defined and is independent of Y.

Ans: Basically we formulate:

$$C = rK + wAL \quad \text{and} \quad Y = AL f\left(\frac{K}{AL}\right).$$

Let $x = \frac{K}{AL}$ so $f(x) = \frac{K}{AL}$

$$\frac{\partial f}{\partial K} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial K} = f'\frac{1}{AL}$$

- So we maximize:

$$\mathcal{L} = rK + wAL - \lambda\left[Y - AL f\left(\frac{K}{AL}\right)\right]$$

- FOC:

$$\frac{\partial \mathcal{L}}{\partial K} = r - \lambda \frac{\partial f}{\partial x}\frac{\partial x}{\partial K} = r - \lambda f'(x)\frac{\partial}{\partial K}\left(\frac{K}{AL}\right)AL = r - \lambda f'\left(\frac{K}{AL}\right)\frac{1}{AL}AL = 0$$

w/ $\frac{\partial \mathcal{L}}{\partial K}$

so: $r = \lambda f'\left(\frac{K}{AL}\right)\frac{1}{AL}AL \Rightarrow \boxed{r = \lambda f'\left(\frac{K}{AL}\right)}$

$$\frac{\partial \mathcal{L}}{\partial AL} = w - \lambda\left(\frac{\partial AL}{\partial AL}f(x) + \frac{\partial f}{\partial x}\frac{\partial x}{\partial AL}\right) = w - \lambda\left(AL f'(x)\left(-\frac{K}{AL^2}\right) + f\left(\frac{K}{AL}\right)\right)$$

so: $w = \left[-f'(k)k\frac{1}{AL} + f(k)\right] = \lambda\left[-k f'(k) + f(k)\right]$

so $\boxed{w = \lambda\left[f(k) - k f'(k)\right]}$

- Solve for λ:

$$\frac{w}{f(k) - k f'(k)} = \frac{r}{f'(k)} \rightarrow \boxed{\frac{f'(k)}{f(k) - k f'(k)} = \frac{r}{w}}$$

so The minimal cost is The one Bellman output if $\frac{r}{w} = f'(k)$

$$f'(k) = f(k) - k f'(k)$$
$$f'(k) + k f'(k) = f(k)$$
$$f'(k)(1+k) = f(k)$$
$$\boxed{\frac{df}{dk} = \frac{f(k)}{(1+k)}}$$

[B] Show that the total output of the N-cost minimizing firms equals the output that a single firm w/ the same production function has if it uses all the labor and capital used by N firms

Ans: $\sum_i Y_i = \sum_i AL_i f(k_i) = Af\sum_i L_i = \bar{A}\bar{L} f(k)$.

② The ELASTICITY of SUBSTITUTION w/ CRRA utility

Consider an individual who lives for 2 periods and whose utility is given by $\boxed{U_t = \frac{C_1^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_{t+1}^{1-\theta}}{1-\theta}}$ w/ $\theta > 0$, $\rho > -1$. Let (P_1, P_2) be the prices of goods. Let W be the life time income. Thus the budget constraint is $\boxed{P_1C_1 + P_2C_2 = W}$.

Consumption in the 2 periods are

[A] What are the individuals optimal choice of $(C_1, C_2 \mid P_2, P_1, W)$?

Ans: Basically we substitute/merge:

$$W - P_1C_1 = P_2C_2 \rightarrow \boxed{C_2 = \frac{W - P_1C_1}{P_2}} \quad \text{Thus} \quad \boxed{U_t = \frac{C_1^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{1}{1-\theta}\left(\frac{W - P_1C_1}{P_2}\right)^{1-\theta}}$$

- To Maximize, we take the derivative:

$$\frac{\partial U}{\partial C_1} = C_1^{-\theta} + \frac{1}{1-\theta}\frac{1}{(1+\rho)}\frac{\partial u(C_2)}{\partial C_2}\frac{\partial C_2}{\partial C_1} = C_1^{-\theta} + \frac{1-\theta}{1-\theta}C_2^{-\theta}\left(-\frac{P_1}{P_2}\right)\frac{1}{1+\rho} = C_1^{-\theta} - \frac{1}{1+\rho}\frac{P_1}{P_2}C_2^{-\theta} = 0$$

so $\boxed{C_1^{-\theta} = \frac{1}{1+\rho}\frac{P_1}{P_2}C_2^{-\theta}}$ or $\boxed{C_1 = (1+\rho)^{\frac{1}{\theta}}\left(\frac{P_2}{P_1}\right)^{\frac{1}{\theta}}C_2}$

• To find the wealth ~~basic~~ consumption we substitute the wealth, ~~Basic~~ set

$$\frac{W - P_1 C_1}{P_2} = \frac{1}{(1+\rho)^{\frac{1}{\theta}}}\left(\frac{P_1}{P_2}\right)^{\frac{1}{\theta}} C_1 \Rightarrow \frac{W_1 - P_1 C_1}{C_1} = \frac{1}{(1+\rho)^{\frac{1}{\theta}}} \frac{P_1^{\frac{1}{\theta}}}{P_2^{\frac{1}{\theta}}} P_2 \rightarrow \left\{ P_2 P_2^{-\frac{1}{\theta}},\ P_2^{1-\frac{1}{\theta}},\ P_2^{\frac{\theta-1}{\theta}} \right.$$

$$\frac{W}{C_1} - P_1 = \frac{1}{(1+\rho)^{\frac{1}{\theta}}} P_1^{\frac{1}{\theta}} P_2^{\frac{\theta-1}{\theta}} \Rightarrow \frac{W_1}{C_1} = \frac{1}{(1+\rho)^{\frac{1}{\theta}}} P_1^{\frac{1}{\theta}} P_2^{\frac{\theta-1}{\theta}} + P_1$$

$$\boxed{\frac{W_1}{C_1} = \frac{P_1^{\frac{1}{\theta}} P_2^{\frac{\theta-1}{\theta}} + P_1(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}}}$$

• Solve for C_1:

$$\frac{C_1}{W} = \frac{(1+\rho)^{\frac{1}{\theta}}}{P_1^{\frac{1}{\theta}} P_2^{\frac{\theta-1}{\theta}} + P_1(1+\rho)^{\frac{1}{\theta}}}$$

$$\boxed{C_1 = \frac{(1+\rho)^{\frac{1}{\theta}}}{P_1^{\frac{1}{\theta}} P_2^{\frac{\theta-1}{\theta}} + P_1(1+\rho)^{\frac{1}{\theta}}} W} \quad \text{or} \quad C_1 = \frac{W(1+\rho)^{\frac{1}{\theta}}}{P_1^{\frac{1}{\theta}} P_2^{\frac{\theta-1}{\theta}} + P_1(1+\rho)^{\frac{1}{\theta}}}$$

• Solving for C_2

$$C_2 = \frac{W}{P_2} - (1+\rho)^{\frac{1}{\theta}} \left[\frac{P_2}{P_1}\right]^{\frac{1}{\theta}} C_2 \left[\frac{P_1}{P_2}\right]$$

$$C_2 + C_2 \left[\frac{P_1}{P_2}\right]\left[\frac{P_2}{P_1}\right]^{\frac{1}{\theta}} (1+\rho) = C_2 \left[\left(\frac{P_1}{P_2}\right)^{\frac{1-\theta}{\theta}} (1+\rho)^{\frac{1}{\theta}} + 1\right] = \frac{W}{P_2}$$

$$\boxed{C_2 = \frac{W}{\left(\left(P_1/P_2\right)^{\frac{1-\theta}{\theta}} \left[(1+\rho)^{\frac{1}{\theta}} + 1\right]\right) P_2}}$$

Remark

If we took the Lagrangian: $\max_{C_1 C_2} \frac{C_1^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{C_2^{1-\theta}}{1-\theta}$ and $\partial_{C_1} \mathcal{L} = C_1^{-\theta} - \lambda P_1 = 0$, $\partial_{C_2} \mathcal{L} = \frac{C_2^{-\theta}}{1+\rho} - \lambda P_2 = 0$

Let $\frac{C_1^{-\theta}}{P_1} = \frac{C_2^{-\theta}}{(1+\rho) P_2}$ Set $W = P_1 C_1 + P_2 C_2$ from $\Rightarrow (C_1^{-\theta} P_1)^{-1} = (C_2^{\theta}(1+\rho) P_2)^{-1}$

$P_1 C_1^{\theta} = (1+\rho) P_2 C_2^{\theta}$ $\rightarrow$ $\boxed{\frac{P_2}{C_1} = \left(\frac{P_1}{P_2}\frac{1}{1+\rho}\right)^{\frac{1}{\theta}}}$ $\frac{C_1}{C_2} = \left((1+\rho)\frac{P_2}{P_1}\right)^{\frac{1}{\theta}}$ $C_1^{\theta} P_1 = C_2^{\theta}(1+\rho) P_2$

[B] Show that $\frac{1}{\theta}$ is the elasticity of consumption of 2 periods in terms of the.

Ans: • Normally: $\ln\left(\frac{C_1}{C_2}\right) = \frac{1}{\theta}\ln\left(\frac{P_1}{P_2}\frac{1}{1+\rho}\right) = \frac{1}{\theta}\left[\ln\left(\frac{P_1}{P_2}\right) + \ln(1+\rho)\right]$.

• Notably if $\rho = 0$, $\ln\left(\frac{C_1}{C_2}\right) \ln\left(\frac{P_2}{P_1}\right) = \frac{1}{\theta}$

$$\ln\left(\frac{C_1}{C_2}\frac{P_2}{P_1}\right) = \ln\left(\frac{C_1}{C_2}\frac{P_2}{P_1}\right) \Rightarrow \ln\left(\frac{\Delta C_1}{\Delta C_2}\frac{1}{\frac{\Delta P_2}{P_1}}\right)$$

$$= \ln\left(\frac{\Delta C_1/\Delta C_2}{\Delta P_2/P_1}\right) = \boxed{\ln\left(\frac{\partial C_1/\partial C_2}{\partial P_2/\partial C_1}\right) = \frac{1}{\theta}}$$

• As $\theta \to 1$, then we will need to substitute. If $\theta \to -1$, then they aren't that useful

ROMER MODEL

(3)

(1) Assumptions: $\boxed{\text{Creation of knowledge}} \Leftarrow \boxed{\text{Returns on knowledge}}$

- The Market must Involve Departures from perfect competition
- If knowledge is sold at Marginal costs, Then Creators can negative Profits

[i] Proof $\dot{\Pi}(A) = \dfrac{\partial \Pi(A)}{\partial t} = -\dfrac{\partial C(A)}{\partial A}$ w/ $\dfrac{\partial \Pi}{\partial t} = \dfrac{\partial \Pi}{\partial C}\dfrac{\partial C}{\partial t} + \dfrac{\partial \Pi}{\partial R}\dfrac{\partial R}{\partial t}$

$= \Pi(C)\dot{C} + \Pi'(R)\dot{R}$

$= \Pi'[\dot{C} + \dot{R}]$

$= -\Pi'\dot{C} = \dfrac{\partial \Pi}{\partial t}$

$\dot{\Pi} = -\dot{C}$

$\dot{\Pi}/\Pi = -\dot{C}$

$\dot{\Pi}/\Pi = \dot{X}$

$\dot{X} = -\dot{C}(X)$

X = units produced

Since $\Pi(A) = R(A) - C(A)$ then:

$\dfrac{\partial \Pi}{\partial A} = \dfrac{\partial R}{\partial A} - \dfrac{\partial C}{\partial A}$ Since $R(A) = 0$, Then $\dfrac{\partial \Pi}{\partial A} = -\dfrac{\partial C}{\partial A}$

and $\int \partial \Pi = -\int \partial C \Rightarrow \boxed{\Pi(A) = -C(A)}$ ∎

[ii] model has no Transition dynamics since ignore monopoly.

(2) $\boxed{Y = \left[\int_{i=0}^{A} L(i)\, di\right]^{\frac{1}{\phi}}}$ $\Rightarrow$ Either Production Function

$L(i) = \dfrac{L_Y}{A}$

$Y = \left[A\left(\dfrac{L_Y}{A}\right)^{\phi}\right]^{\frac{1}{\phi}} = \left[A\dfrac{L_Y^{\phi}}{A^{\phi}}\right]^{\frac{1}{\phi}} = \dfrac{A^{\frac{1}{\phi}} L^{\phi\frac{1}{\phi}}}{A^{\phi}} = A^{\frac{1}{\phi}-\phi} L^{\phi}$

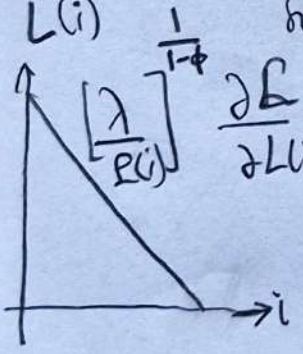
$= A^{\frac{1-\phi}{\phi}} L$ so: $\boxed{Y = A^{\frac{1-\phi}{\phi}} L}$ $\rightarrow$ (1) Cons. between L_Y (2) output Increasing in A

$\Pi(i) = \int_{i=0}^{A} P(i) L(i)\, di = \text{Profit}(i)$

$C(i) = 1 = \left[\int_{i=0}^{A} L(i)\, di\right]^{\frac{1}{\phi}} = \text{Cost}(i)$ s.t

$\mathcal{L} = \Pi(i) - \lambda C(i) = \int_{i=0}^{A} P_i L_i\, di - \lambda\left\{\left[\int_{i=0}^{A} [L(i)]^{\phi} di\right]^{\frac{1}{\phi}} - 1\right\}$

so: $\dfrac{\partial \mathcal{L}}{\partial L(i)} = \dfrac{\partial \Pi}{\partial L_i} - \lambda\dfrac{\partial C_i}{\partial L_i} = \int_{i=0}^{A} P_i\, di - \lambda\int_{0}^{A} L_i^{\phi-1} di = 0$



$\Rightarrow P_i = \lambda L(i)^{\phi-1} \Rightarrow L(i) = \left(\dfrac{P(i)}{\lambda}\right)^{\frac{1}{\phi-1}} = \left(\dfrac{\lambda}{P(i)}\right)^{\frac{1}{1-\phi}}$

so: $\boxed{L(i) = \left(\dfrac{\lambda}{P(i)}\right)^{\frac{1}{1-\phi}}}$

Take logs: $\ln L(i) = \dfrac{1}{1-\phi}\left[\ln\lambda - \ln P(i)\right]$

$(1-\phi)$ $\boxed{\phi(i) = \dfrac{\ln\lambda - \ln P_i - \ln L(i)}{\ln L(i)}}$ w/

CROSS COUNTRY INCOME DIFFERENCE – MARKET VALUE VAR [4.1]

$\dot{H}_t = n L_t \sigma(E)$

$L(t) = L_0 e^{nt}$

$H_t = L_0 e^{nt} \sigma(E)$

(i) SOLOW MODEL EXTENSION:

ASSUMPTIONS:

$$\boxed{Y(t) = K(t)^{\alpha}[A(t)H(t)]^{1-\alpha}}$$

$$\boxed{\dot{K}(t) = sY(t) - \delta K(t)}$$

- SOLOW'S MODEL on $H(t)$
- DEPRECIATION

$$\boxed{\dot{L}(t) = nL(t)}$$

- GROWTH

$$\boxed{H(t) = L(t)\sigma(E)}$$

GROWTH
- $\sigma(E)$ = Education
- $L(t)$ = Labor
- Education is the amount of Labor's multiplier.

$$\boxed{\sigma(E) = e^{\phi E}}$$

Impose: $\sigma'' < 0$, $\sigma' > 0$

EDUCATIONAL
- Education grows exponentially
- The ability to read / write, fashion matters

- $\sigma(E) = e^{\phi E}$
- $\sigma' = \phi\sigma(E) > 0$
- $\sigma'' = -\sigma(E) < 0$

MODEL:

$$\boxed{Y_t = K_t^{\alpha} A_t^{1-\alpha} H_t^{1-\alpha}}$$

- $\frac{Y_t}{(AH)_t} = \frac{K_t^{\alpha}}{(AH)_t^{\alpha}}$; $y_t = k_t^{\alpha}$; $f(k_t) = f\left(\frac{K_t}{A_t H_t}\right) = f\left(\frac{K_t}{L_t \sigma(E) A_t}\right)$

- Take Derivative:

$$\frac{\partial f(k_t)}{\partial t} = \frac{dy_t}{dt} = \frac{d}{dt}\frac{(K_t)^{\alpha}}{(A_t H_t)^{\alpha}} = \frac{df(k_t)}{dk_t}\frac{dk_t}{dt} = \frac{d}{dk_t}\left(k_t^{\alpha}\right)\frac{d}{dt}\left(\frac{K_t}{A_t H_t}\right) = f'(k_t)\left(\frac{\dot{K}_t}{A_t H_t} - \frac{\dot{A}_t K_t}{A_t H_t} - \frac{\dot{H}_t}{H_t}\frac{K_t}{A_t}\right)$$

$$= f'(k_t)\left[\frac{\dot{K}_t[A_t H_t] - \dot{A}_t H_t K_t - \dot{H}_t A_t K_t}{(A_t H_t)^2}\right]$$

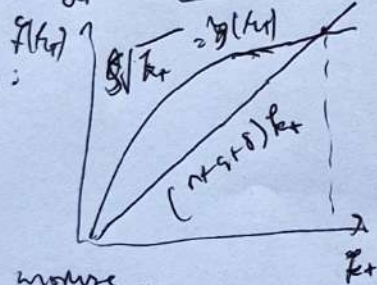
$$= f'(k_t)\left[\frac{\dot{K}_t}{A_t H_t} - g k_t - n k_t\right] = f'(k_t)\left[\frac{sY_t - \delta K_t}{A_t H_t} - g k_t - n k_t\right]$$

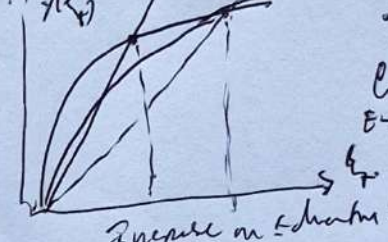
$$= f'(k_t)\left[s\frac{Y_t}{A_t H_t} - s\frac{K_t}{A_t H_t} - g k_t - n k_t\right] = f'(k_t)\left[s f(k_t) - (n+g+\delta)k_t\right]$$

$$\gg \frac{df}{dt}\frac{dk_t}{df} = \frac{dk_t}{dt} \Rightarrow \boxed{\dot{k}_t = s f(k_t) - (n+g+\delta)k_t}$$

- Same SOLOW MODEL GROWTH
- Same golden rule: $k^* = \left[\frac{s}{n+g+\delta}\right]^{\frac{1}{(1-\alpha)}}$

GRAPHICAL:

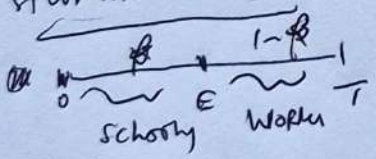




- If: $\lim_{E\to\infty} f(k_t) = \frac{K_t}{A_t L_t \sigma(E)}\Big|_{\infty} = 0$

Increase in education will adjust the cap

STUDENTS & WORKERS:



- $\mu(E) = \beta E + (1-\beta)(L-E) = \beta E + (1-\beta)L - (1-\beta)E$
  $= \beta E + (1-\beta)\cdot L - E + \beta E = 2\beta E - E + (1-\beta)L$
  $= (2\beta - 1)E + (1-\beta)L$
- $Var(E) = \beta(1-\beta)[2E - L]^2$

$\beta > \frac{1}{2}$: Risk outweighs return

$\beta < \frac{1}{2}$: Education is bad

- $\gg \frac{d\mu(E)}{dE} = 2\beta - 1$ $\gg \frac{dVar(E)}{dE} = 2\sqrt{\beta(1-\beta)}\,\text{sign}(2E - L)$ $\gg \boxed{\frac{d\mu/dE}{dVar/dE} = \frac{2\beta - 1}{2\sqrt{\beta(1-\beta)}}}$

GRAPHS,

• What if we take w β:

$$\Rightarrow \frac{\partial \mu(\beta)}{\partial \beta} = 2E - L$$

$$\Rightarrow \frac{\partial var(\beta)}{\partial \beta} = (2E-L)^2(1-2\beta)$$

$$\Rightarrow \frac{\partial \sigma(\beta)}{\partial \beta} = \frac{|2E-L||1-2\beta|}{2\sqrt{\beta(1-\beta)}}$$

$f(\beta)$

β

$$\Rightarrow S_\beta = \frac{2E-L}{\frac{(2E-L)(1-2\beta)}{2\sqrt{\beta(1-\beta)}}} = \frac{2\sqrt{\beta(1-\beta)}}{1-2\beta}$$

$$\therefore \boxed{S_E = \frac{2\beta-1}{2\sqrt{\beta(1-\beta)}}} \quad \boxed{S_\beta = \frac{2\sqrt{\beta(1-\beta)}}{1-2\beta}}$$

So compare: $S_E \gtreqless S_\beta = \frac{2\beta-1}{2\sqrt{\beta(1-\beta)}} \gtreqless \frac{2\sqrt{\beta(1-\beta)}}{1-2\beta}$

$$= (2\beta-1)(1-2\beta) \gtreqless 4(\beta(1-\beta))$$

$$= 2\beta - 4\beta^2 - 1 + 2\beta \gtreqless 4\beta(1-\beta)$$

$$= -4\beta^2 + 4\beta - 1 \gtreqless 4\beta - 4\beta^2$$

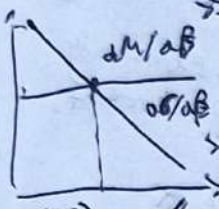
$$= 4\beta - 1 \gtreqless 4\beta - 4\beta^2 + 4\beta^2 = 4\beta$$

$$= 4\beta - 1 \gtreqless 4\beta$$

$$= -1 \gtreqless 0 \Rightarrow \boxed{S_E < S_\beta \quad (-1) < (0)}$$

• the implication of $S_E < S_\beta$:

$f(\beta)$



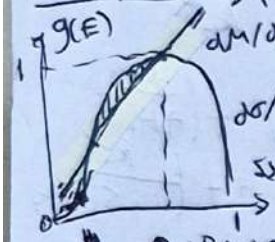
β

$\Rightarrow$ changing $E = \{\mu, \sigma\}$ moves together

$\Rightarrow$ changing $\beta = \begin{cases} [\Delta\mu > \sigma] : \text{Believers are near certainty} \\ [\Delta\sigma > \Delta\mu] : \text{Believers are frayne.} \end{cases}$

$\Rightarrow$ ASSET PRICE.

" If the market becomes more confident the schooling payoffs How much expected payoff typer i get per unit of Effort risk.



$\Rightarrow$ Risk Premium $\Longleftrightarrow$ $\Rightarrow$ State probabilities

$\Rightarrow$ Not how much firms must.

• RAND RENTS:

$$\cdot\ N(t) = \int_{t=0}^{T} \beta(t-\tau)\, d\tau = \int_{\tau=0}^{T} \beta(t)\, e^{-n\tau}\, d\tau = \frac{1-e^{-nT}}{n}\beta(t)$$

$$\cdot\ L(t) = \int_{t=E}^{T} \beta(t-\tau)\, d\tau = \int_{E}^{T} \beta(t)\, e^{-n\tau}\, d\tau = \frac{e^{-nE} - e^{-nT}}{n}\beta$$

$$\cdot\ \frac{L_t}{N_t} = \frac{e^{-nE} - e^{-nT}}{1-e^{-nT}} = \Delta \quad : \quad \boxed{\Delta = \frac{L_t}{N_t} = \frac{e^{-nE} - e^{-nT}}{1-e^{-nT}}}$$

$$y = k A L^{\frac{1}{2}} H^{\frac{1}{2}}$$

Human capital intensity

$$\cdot\ \boxed{\left(\frac{Y}{N}\right)^* = y^* A(t)\, \frac{e^{-nE} - e^{-nT}}{1-e^{-nT}}}$$

CROSS COUNTRY INCOME DIFFERENCES – EMPIRICAL

(2.) EA : ACCOUNTING CROSS DIFFERENCE .. MODEL :

- $Y_i = K_i^{\alpha}\left[A_i H_i\right]^{1-\alpha}$
- $\ln \frac{Y_i}{L_i} = \alpha \ln \frac{K_i}{L_i} + (1-\alpha)\ln \frac{H_i}{L_i} + (1-\alpha)\ln A_i$] Rise in the amount of education / measure of human capital productivity.

$\Rightarrow (1-\alpha)\ln \frac{Y_i}{L_i} = (1-\alpha)\alpha \ln \frac{K_i}{L_i} + (1-\alpha)(1-\alpha)\ln \frac{H_i}{L_i} + (1-\alpha)(1-\alpha)\ln A_i$

$\Rightarrow (1-\alpha)\ln \frac{Y_i}{L_i} = \ln \frac{Y_i}{L_i} - \alpha \ln \frac{Y_i}{L_i} = \alpha \ln \frac{K_i}{L_i} + (1-\alpha)\ln \frac{H}{L} + (1-\alpha)\ln A$

$\alpha^2 \ln \frac{K_i}{L_i} + \alpha(1-\alpha)\ln \frac{H}{L} + \alpha(1-\alpha)\ln A$

$\ln \frac{K}{L} - \ln \frac{Y}{L}$
$= \ln K - \ln L - \ln Y + \ln L$
$= \ln K - \ln Y$

$\Rightarrow (1-\alpha)\ln \frac{Y}{L_i} = \left(\alpha \ln \frac{K}{L} - \alpha \ln \frac{Y}{L}\right) + (1-\alpha)\ln \frac{H}{L} + (1-\alpha)\ln A$

$= \alpha \ln \frac{K}{Y_i} + (1-\alpha)\ln \frac{H}{L} + (1-\alpha)\ln A$

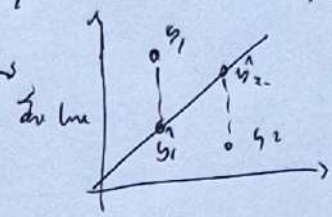
$\Rightarrow \boxed{\ln \frac{Y}{L_i} = \frac{\alpha}{1-\alpha}\ln \frac{K_i}{Y_i} + \ln \frac{H}{L_i} + \ln A_i}$

$\hat{\lambda}_0 \equiv \ln \frac{H}{L} = \text{Cons.}$

∴ It could be: $\ln y_i = \hat{\lambda}_1 \ln k_y + \hat{\lambda}_0 + \varepsilon$ + $\hat{\lambda}_1 = \ln A_i$ = Residual term

$\Rightarrow$ so: $\varepsilon = \hat{y} - y \propto \ln \frac{H}{L}$

$\Rightarrow$ and: $\frac{\alpha}{1-\alpha}\ln \frac{K}{Y}$ are η



} an increase in the rate of saving as the measure for accumulating of human capital!

RESULTS :

- SOURCE : • Penn World Tables • Hall & Jones • Klenow & Rodriguez.

- ANALYSIS : • years of schooling ⟷ stock of labour.
  - Human capital is: $e^{\phi(E_i)}L_i$ } Hall & Jones (1999) chose that the micro economic Evidence.
  - Linear function is $\phi(E) = \phi(E)$ ~ $\boxed{\Delta^{+}[\text{EARNINGS}] = \Delta^{-}[\text{SCHOOLING}]}$.

- STATS : • $\phi(E)$: $\hat{\beta}_1 = 0.09$ 1 4 years
  $\hat{\beta}_2 = 0.101$ 1 4 and 8 years
  $\hat{\beta}_3 = 0.063$ 1 above 8 years

- $\frac{\alpha}{1-\alpha}\ln\left(\frac{K}{Y}\right)$, $\ln \frac{H}{L}$, $\ln \frac{Y}{L}$:
  - $\ln\left(\frac{K}{Y}\right)$: 0.6 1 Accounts for 60% capital share of output between groups.
  - $\ln\left(\frac{K}{L}\right)$: 0.8 1 Human capital ratio to labour
  - Rich > Poor : 31.7 × 1 output per worker in group of rich is poor
  - $\rho\left(\ln \frac{H}{L}, \ln A\right) = 0.52$ ⇒ difference in capital intensity
  - $\rho\left(\ln \frac{K}{L}, \ln A\right) = 0.25$
  - $\rho\left(\ln \frac{K}{L}, \ln \frac{H}{L}\right) = 0.60$

} incorporating human capital adjust here $\boxed{\text{SIGNIFICANTS FOR ACCOUNTING}}$

NOMINAL RIGIDITY - FIX PRICES & BASIC MODELS

6.1 | 6.2

1 FIXED PRICES

- MOTIVATION:
  - » Y = output
  - » $\mathcal{U}$ = total utility
  - » $V(L_t)$ = value function of optimal consumption
  - » $\Gamma\left(\frac{M_t}{P_t}\right)$ = money holdings
  - » A_t = assets.

ASSUMPTIONS:

- Formulation of objective:

$$\mathcal{U} = \sum_t \beta^t \left[U(C_t) + \Gamma\left(\frac{M_t}{P_t}\right) - V(L_t) \right] \quad 0<\beta<1 \;\&\; \beta^t = e^{-\delta t}.$$

- FISHER RELATIONS:

$$1+\pi_t = \frac{P_{t+1}}{P_t}$$

$$1+r_t = \frac{(1+i_t)}{(1+\pi_t)}$$

$$i_t \approx r_t - \pi_t$$

Formulation of CRRA:

$$\text{ss} \quad \boxed{U(C_t) = \frac{C_t^{1-\theta}}{1-\theta}} \quad \theta \geq 0 \quad \text{ss} \quad \boxed{\Gamma\left(\frac{M_t}{P_t}\right) = \left(\frac{M_t}{P_t}\right)^{1-\chi} \frac{1}{1-\chi}}$$

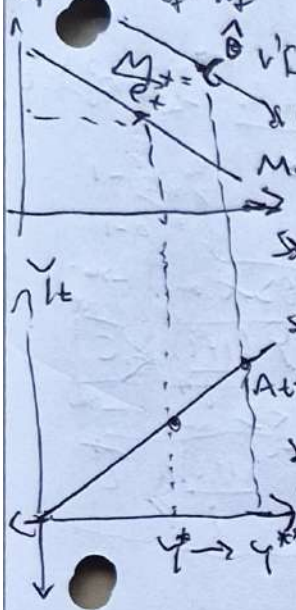
& CONSUMPTION & MONEY HOLDINGS.

Formulation of ASSETS:

$$\boxed{A_{t+1} = M_t + (A_t + W_t L_t - P_t C_t - M_t)(1+i_t)}$$

& ASSETS IN FUTURE

$$E_0 \sum_j \beta^j A_{t+j} = \frac{\text{Debt} + \text{Equity}}{\text{Price}}.$$



so we:

$$E_0 \sum_j \beta^j A_{t+j} = M_t + [A_0 + W_t L_t - P_t C_t - M_t] \Leftrightarrow$$

$$\Rightarrow \text{so that}: \quad V[P_t] = \sum_t \delta^t u(C_t) = u(C_t) + \sum_0^T V[M_t + (A_t + W_t L_t + \dots) e^{-\delta t}]$$

so we:

$$V[C_t] = \max_c \left(u(C_t) + \sum_0^T V[A_{t+1}] \right)$$

$$= u'(C_t) + \frac{\partial V}{\partial A_{t+1}} \frac{\partial A_{t+1}}{\partial P_t} = u'(C_t) - V' P_t C_t (1+r_t) = 0$$

so we:

$$u'(C_t) = \delta V'[A_{t+1}] P_t C_t (1+i_t) \Rightarrow C_t^{-\theta} = P_t C_t (1+i_t) [\delta V'[A_{t+1}]]$$

$$\Rightarrow \frac{C^{-\theta+1}}{(1+i)^t} = P_t [V'[A_{t+1}]] \Rightarrow P_t = e^{-\delta t} C_t^{-\theta-1} (V'[A_{t+1}])^{-1}$$

$$\Rightarrow \frac{1}{P_t} = e^{+\delta t} e^{+\theta(\theta+1)} V'[A_{t+1}]^{-1+1} \Rightarrow \frac{M_t}{P_t} = \frac{M_t C_t^{1-\theta}}{(1+r_t)^{-1}} V'[A_{t+1}].$$

$$e_x = \frac{\Delta M_t}{\Delta Y_t}$$

$$\text{Max}_t V = P_t Y_t$$

$$\frac{\Delta M_t}{P_t} = \Delta Y_t.$$

$$\boxed{F'(L) = \left(\frac{M_t}{P_t}\right)^* = (1+i_t) M_t C_t^{1+\theta} V'[A_{t+1}]}$$

& LABOUR AS MONEY & CONSUMPTION OUTPUT.

- Labor has the form of consumption and the money relationship as well the wealth of agents
- If $i \to \infty$, then cost of $M_t \to \infty$ and thus simplyfying consumption

$$\Rightarrow \text{Alternative way as}: \quad \boxed{P_t e^{it} C_t^{1+\theta} = V'[A_{t+1}]} \quad P_t (1+i_t) C_t^{1+\theta}$$

$$\boxed{C_t^{1-\theta} \frac{\Delta M_t}{\Delta Y_t} (1+i_t) = V'[A_{t+1}]}$$

- FUTURE assets now will have cost of inflation

& PRICE LEVEL TO MONEY & OUTPUT thus interest cost and objective future Assets.

$$i_t = r_t + \pi_t$$

$$\boxed{C_t^{1-\theta} \frac{\Delta M_t}{\Delta Y_t} [r_t + \pi_t]}$$

& RAISING INTEREST COST CAN LOWER INPUT if $\frac{\partial C_t}{\partial t} > \frac{\partial \pi_t}{\partial t}$

HOUSEHOLDS:

- Exploring Section 2.8:

$$\mathcal{L} = \frac{C_t^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{C_{t+1}^{1-\theta}}{1-\theta} + \lambda \left(A_t W_t - C_t + \frac{1}{1+r_{t+1}} C_{t+1} \right)$$

Solution: $\frac{\partial \mathcal{L}}{\partial C_t} = \boxed{\frac{C_{t+1}}{C_t} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}}}$

>> Thus $\left(\frac{C_{t+1}}{C_t}\right)^{\theta} = \frac{1+r}{\beta^{-1}}$

>> Take logs. $\ln\left(\frac{C_{t+1}}{C_t}\right) = \frac{1}{\theta}\ln\left(\frac{1+r}{\beta}\right) \Rightarrow \ln C_{t+1} - \ln C_t = \frac{1}{\theta}\left[\ln(1+r_t)\beta\right]$.

>> meaning: $\ln Y_t = a + \ln i_{t+1} - \frac{1}{\theta} r_t$. $a = -\frac{1}{\theta}\ln\beta$

$y_t = \phi_1 y_t^e - \phi_2 r_t$

$y_t = \sum \phi(r_{t-i}) - \sum \phi' r_{t-j}$

a REAL INTEREST to FUTURE OUTPUT

- Also consuming optimal holdings.
- observe that CURR OUTPUT = FULL OUTPUT − INTERESTS
- This says that ~~[illegible]~~ interest rates will presume the among y_t.

$$r'\left[\frac{M_t}{P_t}\right] = \frac{i_t}{1+i_t} U'(C_t)$$

$$\frac{M_t}{P_t} = Y^{\frac{\theta}{\chi}}\left[\frac{1+i_t}{i_t}\right]^{\frac{1}{\chi}}$$

$M_t = P_t Y_t$

- AND constitutionally imply:

>> meaning EQ: $\frac{M_t}{P_t} - \frac{M_t^*}{P_t^*} = 0$ >> $Y^{\frac{\theta}{\chi}}\left[\frac{1+i_t}{i_t}\right]^{\frac{1}{\chi}} - [1+i_t]P_t Y_t C_t^{1-\theta} V'[A_{t+1}] = 0$

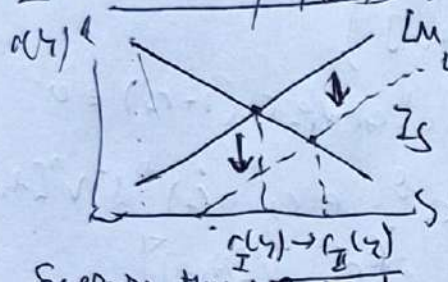
$\Rightarrow Y^{\frac{\theta}{\chi}}\left[\frac{1+i_t}{i_t}\right]^{\frac{1}{\chi}} - [1+i_t^*]P_t Y_t C_t^{1-\theta} V'[A_{t+1}]$ >> $Y^{\frac{\theta}{\chi}}\left[\frac{1+i_t}{i_t}\right]^{\frac{1}{\chi}} - P_t Y_t C_t^{1-\theta}[1+i_t] = 0$

$\Rightarrow Y^{\frac{\theta}{\chi}} \frac{1}{Y_t} \frac{(1+i_t)^{\frac{1}{\chi}}}{i_t^{\frac{1}{\chi}}} = P_t Y_t C_t^{\theta} \Rightarrow Y^{\frac{\theta-\chi}{\chi}} \frac{(1+i_t)^{\frac{1-\chi}{\chi}}}{i_t^{\frac{1}{\chi}}} = C^{1+\theta}$

$\lim_{\chi \to 0} \frac{1}{\chi} - 1 = 0 - 1$

- $\theta > \chi$ and $\chi \to 0$: $\theta = Y^{-1}(1+i) < C$
- $\theta < \chi$ and $\chi \to \infty$: $Y^{-1}\frac{(1+i)}{i} > C$

STOCKS w/ BONDS:



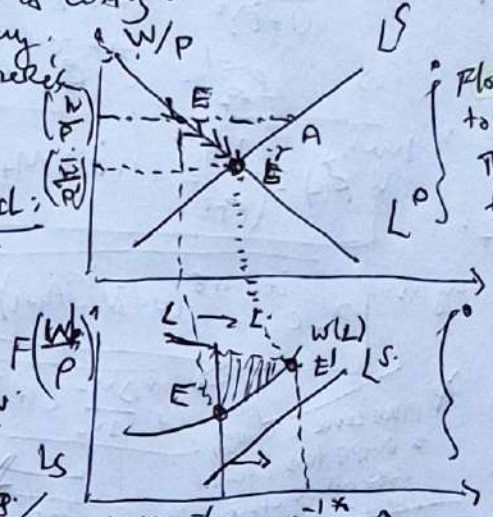
- At Equilibrium, if the endowment for money is smaller than consumption, consumption wins and interest rate is "less costly."
- At Equilibrium in the opposite, the endowment of money supply causes a smaller fraction of consumption to be "hold", thus borrowing is costly.

INCREASE AGGREGATE DEMAND = CROWD OUT PRIVATE SPENDING

Suppose that $P_t = \bar{P}$. Now Basically, we can see that the effects of an investment-shock will shift the interest to be more costly, since there's an Increase in Aggregate Demand.

(2.) PRICE RIGIDITY: Keynes's model:

- Aggregate nominal wage is: $W = \bar{W}$
  >> so nominal wage is unmovable
- Marginal Product of Labor via output will be: $F'(L) = \frac{W}{P}$



- Flow of wages / price to equilibrium will cause the supply of Labor to be equal to the demand of labor.
- When wages / price is at equilibrium, output is shifted to the right meaning that AGGREGATE DEMAND

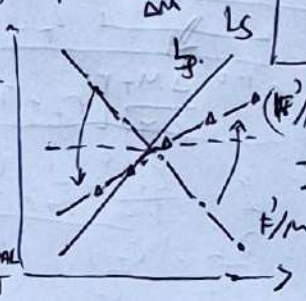
STICKY PRICES and STICKY WAGES ARE FLEXIBLE & MARKET IS IMPERFECT:

- If wages are sticky & prices are flexible, basically there's a markup price:

$$P = \mu(L)\frac{W}{F'(L)}$$

μ = MARKUP

$\frac{W}{P}$ = MARGINAL COST



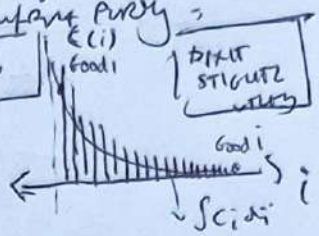
- Since this is going more labor, if firms have markup $\mu(L)$, then as output increases via labor too, the value of output will increase but it requires more labor to that due to [illegible]

MODEL of IMPERFECT COMPETITION

(5) MODEL W/ IMPERFECTS:

ASSUMPTIONS:

- Output: $\boxed{Y_i = L_i}$ >> $i \in [0,1]$ goods >> Each good produced by single firm i w/ monopoly pricing
- Consumption as C.E.S.: $\boxed{C = \left[\int_0^1 C_i^{\frac{\eta-1}{\eta}} di\right]^{\frac{\eta}{\eta-1}} \quad \forall \eta > 1}$ >> Indexed Household's consumption of Individual goods.
- Household's utility: $\boxed{U = C - \frac{1}{\gamma} L^{\gamma}}$ >> Consume = $\ddot{U}$ >> Laboring = $\ddot{U}$
- Measure of output purity: $\boxed{Y = C}$



- Money supply is inflexible: $\boxed{Y = \frac{M}{P}}$ >> cash in advance constraint >> Central bank conducts monetary policy

BEHAVIOR:

- Maximize Consumption s.t. Spending constraint: $\boxed{\mathcal{L} = \left[\int_{i=0}^1 C_i^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}} + \lambda\left[S - \int_{i=0}^1 P_i C_i \, di\right]}$ & $\mathcal{L} = f + \lambda g$

>> So:

$$\frac{\partial f}{\partial C_i} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial C_i} + \lambda\frac{\partial g}{\partial C_i} = \frac{\partial}{\partial x}\left[x^{\frac{\eta}{\eta-1}}\right]\frac{\partial}{\partial C_i}\left[C_i^{\frac{\eta-1}{\eta}}\right] - \lambda\frac{\partial}{\partial C_i}\left[\int_{i=0} P_i C_i \, di\right]$$

$$= \frac{\eta}{\eta-1} x^{\frac{\eta}{\eta-1}-1} \frac{\eta-1}{\eta} C_i^{\frac{\eta-1}{\eta}-1} - \lambda P_i = \frac{\eta}{\eta-1} x^{\frac{\eta-1-\eta-1}{\eta-1}} \frac{\eta-1}{\eta} C_i^{\frac{\eta-1-\eta}{\eta}} = \lambda\frac{\partial}{\partial C_i}$$

>> This F.O.C is: $\left[\int_{j=0}^1 C_j^{\frac{\eta-1}{\eta}} dj\right]^{\frac{1}{\eta-1}} C_i^{-\frac{1}{\eta}} = \lambda P_i$

>> j-index vanish since dependent on i: $C_i^{-\frac{1}{\eta}} = \lambda P_i \Rightarrow C_i = \lambda^{-\eta} P_i^{-\eta} = A P_i^{-\eta}$

>> So: $\boxed{C_i = A P_i^{-\eta}}$

- Plug in into Budget constraint:

$$S = \int_0^1 P_i C_i \, di = \int_0^1 P_i P_i^{-\eta} A \, di = A\int_0^1 P^{1-\eta} di \Rightarrow \boxed{A = \frac{S}{\int_{j=0}^1 P_j^{1-\eta} dj}}$$

- Plug this back to the objective:

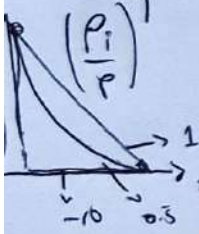
$$C = \left[\int_{i=0}^1 P_i C_i \, di\right]^{\frac{\eta-1}{\eta}} = \left[\int_{i=0}^1 \left(\frac{S}{\int_{j=0}^1 P_j^{1-\eta} dj} P_i^{-\eta}\right)^{\frac{\eta-1}{\eta}} di\right]^{\frac{\eta}{\eta-1}}$$

*ECH = $\frac{\text{SPEND}}{\text{TOTAL GOODS' PRICE}}$ ENDOWN

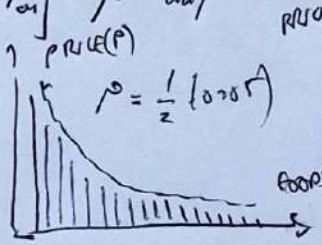
MICROECONOMIC PRICE ADJUSTMENT

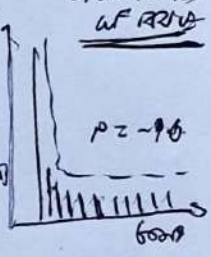
$$C_i(c) = \frac{S}{\int_{j=0}^1 P_j^{1-\eta} dj}\left[\int_{i=0}^1 P_i^{1-\eta} di\right] = \frac{S}{\left[\int_{i=0}^1 P_i^{1-\eta} di\right]^{\frac{1}{1-\eta}}} \quad \checkmark$$

$\frac{1}{1-\eta} = \frac{1}{\rho}$, so $\eta \rho$.



- So Basically: $\boxed{P = \left[\int_{i=0}^1 P_i^{1-\eta} di\right]^{\frac{1}{1-\eta}}}$ w/



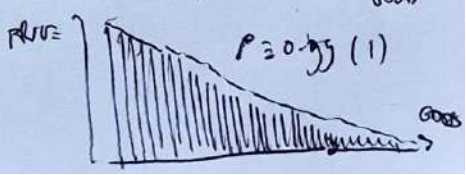


GRAPHICALS of PRICE

- Plug back in: $C_i = A P_i^{-\eta} = \left(\frac{P_i}{P}\right)^{-\eta}\frac{S}{P} = \left(\frac{P_i}{P}\right)^{-\eta} C$

If individual consumption advises, then its price of good i to total will "brought up" but if $P \to \infty$, then there'll be less consumption on them

hence $\boxed{C_i = \left[\frac{P_i}{P}\right]^{-\eta} C}$ and $\boxed{C = C_i\left[\frac{P_i}{P}\right]^{\eta}}$

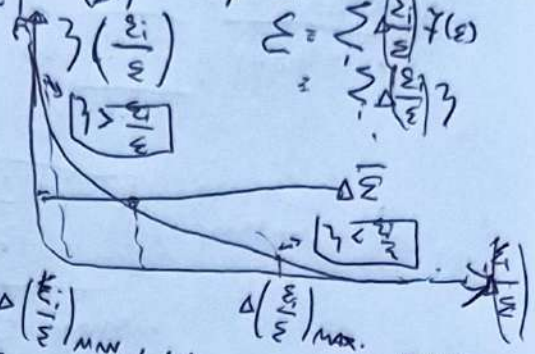


• Assume that:

$$\Rightarrow \ln C_i = -\eta \ln\left(\frac{P_i}{P}\right) + \ln C \Rightarrow \frac{\ln C_i}{\ln C} = -\eta \ln\left(\frac{P_i}{P}\right)$$

$$\Rightarrow \ln\left(\frac{C_i}{C}\right) = -\eta \ln\left(\frac{P_i}{P}\right) \Rightarrow -\ln\left(\frac{C_i}{C}\right) \ln\left(\frac{P}{P_i}\right) = -\eta$$

$$\Rightarrow \boxed{-\frac{\partial \ln C_i}{\partial \ln P_i} \frac{\partial \ln P}{\partial \ln C_i} = \eta}$$



$$\bar{\varepsilon} = \sum_i \Delta\left(\frac{\varepsilon_i}{\varepsilon}\right) \eta(\varepsilon) = \sum_i \Delta\left(\frac{\varepsilon_i}{\varepsilon}\right) \eta$$

• Spending is $WL + R$ by which W is wage, R is income, so that:

$$\boxed{V(L) = \max_L \left(\frac{WL + R}{P} - \frac{1}{\gamma} L^{\gamma} \right)}$$

↳ HOUSEHOLD OBJECTIVE

• The average elasticity is lower than the probability of all elasticities

• The average elasticity to it's individual aggregate is bigger due to its base individual utility is smaller than the aggregate ones

$\eta >$: Consumption and preferences are high in order to

$\eta <$: Consumption + prices come really slowly down due to using Thinnest Inelastic Demand.

NOMINAL RIGIDITY - USAGES

(4) USABLE MODELS:

64.

PERMANENT OUTPUT - INFLATION TRADE OFF ?

- Suppose that we model inflation as a Random Walk: $\boxed{W_t = A P_{t-1}}$
- Also for the labour to consumer we have:

$$\boxed{F'(L_t) = A \frac{P_{t-1}}{P_t} = \frac{A}{1+\pi_t}}$$

- In a Lagrangian, supposing 2 period consumption C_t, C_{t+1}; we have:

$$\max_{C_t, C_{t+1}} \left(\frac{1}{2}(C_{t+1}^2 + C_t^2) \right) \quad \text{s.t} \quad W = A P_{t-1}, \; F'(L_t) = \pi_t.$$

SS meaning that:

$$\mathcal{L}(c_t, \nu_t \mid \hat{\lambda}) = \hat{\lambda}\left(\frac{1}{2}C_t^2 + \frac{1}{2}\nu_t^2\right) + \hat{\lambda}\left[P_t F(L_t; \tfrac{1}{2}C_t^2) + A P_{t-1}\nu_t^2 - W\right].$$

REFORMULATED GRADIENT:

$$\mathcal{L}(x, y_x \mid \hat{\lambda}) = f(x) - \hat{\lambda} g(x)?$$

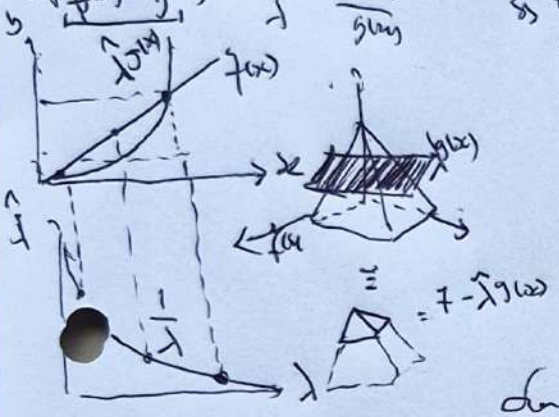
$$\begin{cases} \frac{\partial \mathcal{L}}{\partial C_t} : -\lambda C_t + \left[F'(L_t) C_t\right] = 0 \\ \frac{\partial \mathcal{L}}{\partial \nu_t} : -\lambda \nu_t + \left[A P_{t-1} \nu_t\right] = 0 \end{cases} \quad \begin{cases} C_t = \lambda P_t F'(L) \; \text{SA} \\ \nu_t = \lambda P_t A P_{t-1} \nu_t \end{cases}$$

$$\Rightarrow \frac{\nu_t}{\nu_t} = \frac{\lambda}{\lambda} \Rightarrow C_t F' = A P_{t-1}.$$

$$\boxed{F' = A \frac{P_{t-1}}{P_t}}$$

max $\mathcal{L} = 0$: $\hat{\lambda} = \frac{f(x)}{g(x)}$

$\lambda g(x) = f(x) \leftarrow \frac{1}{\lambda} = \frac{f(x)}{g(x)}$



- Concerts sat the Random walk.

SS Basically $F'(L_t) = A \frac{P_{t-1}}{P_t}$ means that wages are now:

$$F'(L_t) = \frac{W_t}{P_t} = A \frac{P_{t-1}}{P_t} \Leftrightarrow (1+\pi_t) = \frac{(1+i_t)}{(1+r_t)}$$

$$\frac{1}{1+\pi_t} = \frac{1+r_t}{1+i_t}$$

$$= A \frac{1+r_t}{1+\pi_t}$$

$$\sim A \frac{r_t}{r_t + \pi_t} = A \cdot P_r(r_t)$$

$P_r(i_t) \equiv P_r(r_t), P_r(\pi_t)$

$$\begin{cases} P_r(r_t) = \frac{r_t}{r_t + \pi_t} \\ P_r(\pi_t) = \frac{\pi_t}{r_t + \pi_t} \end{cases}$$

So. $\boxed{F'(L_t) = A \cdot \frac{r_t}{r_t + \pi_t} = A P_r(r_t)}$

SS If real interest rates are higher, the probability of the tech / innovation / endowment assets gets higher

SS meaning that

If $N(\mu, \sigma^2) \sim r_t$ then

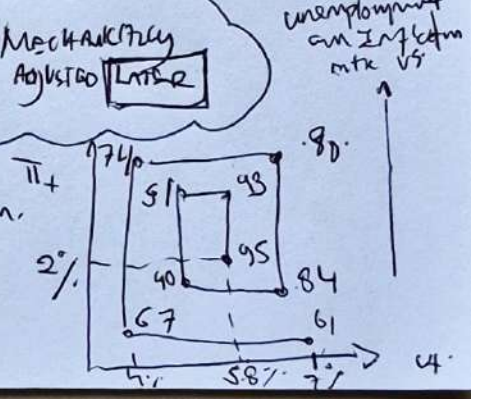
Intuitively: $\frac{N(r_t)}{N(r_t) + N(\pi_t)}$

$$\frac{1}{1+\pi} \sim \frac{P_{t-1}}{P_t} = \frac{B_{t-2}}{S^e_{t+1}} \frac{S^e_t}{B_{t-1}}$$

$$\Rightarrow \begin{cases} P_{t-1} = \frac{B_{t-2}}{S^e_{t+1}} \\ P_t = \frac{B_{t-1}}{S^e_t} \end{cases}$$

NATURAL RATE - MILTON FRIEDMAN

- Milton Friedman basically said: (MECHANICALLY ADJUSTED $\boxed{\text{NOW}}$ $\neq$ MECHANICALLY ADJUSTED $\boxed{\text{LATER}}$)
- The forces that's causing Employment are also present in the forces that's causing Inflation.
- Increase in oil prices for a given level of wages in the 70's are due to changing higher prices for a given level of $\boxed{\text{WAGES}}$



NOMINAL RIGIDITY - EMPIRICAL APPROACHES

[6.4]

3) CYCLICAL BEHAVIOR I REAL WAGE: ECONOMETRIC METHOD:

Δw = change in wage
Δu = change in unemployment
e_{it} = error term
X_{it} = control

- Regress: $\boxed{\Delta \ln w_{it} = a' X_{it} + b \Delta u_t + e_{it}}$

So in which $\Delta W = A \exp(aX_{it} + b\Delta u_t + e_{it})$

$rk \rightarrow \frac{\partial \Delta W}{\partial \Delta X_{it}} \Delta X_{it}$

$wL \rightarrow \frac{\partial \Delta W}{\partial \Delta u} \Delta u$

$\Rightarrow$ ans: $\boxed{\Delta W = A \Delta X_{it}^{a} \Delta u_t^{b} \varepsilon_{it}}$

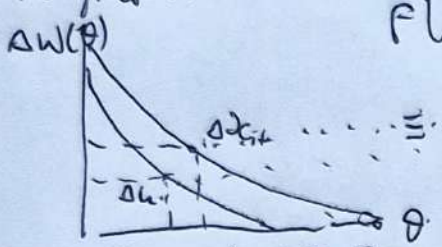
$\frac{\partial \Delta W}{\partial \Delta X_{it}} = \frac{a}{\Delta X_{it}} \Delta W$

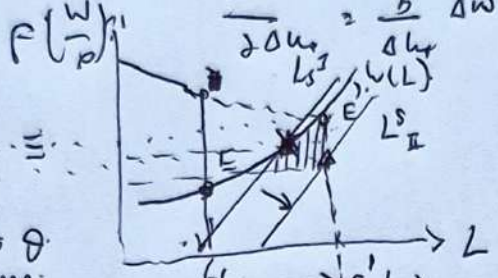
$\frac{\partial \Delta W}{\partial \Delta u_t} = \frac{b}{\Delta u_t} \Delta W$

$rk + wL = \underbrace{\frac{a}{\Delta X_{it}} \Delta W + \frac{b}{\Delta u_t} \Delta W}_{= \Delta W}$

$\boxed{\Delta W = \frac{a}{\Delta X_{it}} + \frac{b}{\Delta u_t}}$

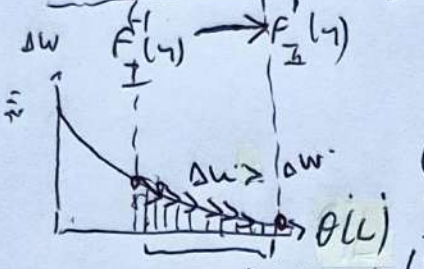
$\Rightarrow$ The graphs:





- observe that the micro-economic explains an aggregate behavioral mechanism.

for $\Delta(\Delta u, \Delta x) \rightarrow \infty$, then $\left(\lim_{\substack{\Delta u \to \infty \\ \Delta x \to \infty}} \Delta W = 0\right)$ since basically, the marginal value wage is ZERO.



$\Delta u \leftrightarrow \frac{\partial \Delta W}{\partial \Delta u} \sim F\left(\frac{W}{P}; L\right)$.

- observe that new output will raise unemployment as which more labour will be used on the production process.

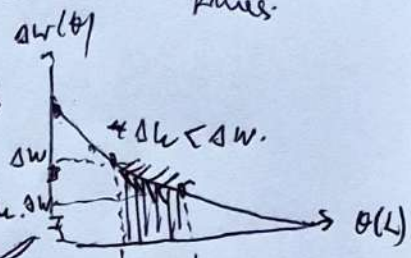
STAT RESULTS:

D-Variables	I-Variables	Sol?
ΔW	Δu	$\approx +1.2\%$
$\Delta \bar{W}$	Δu	$\approx +1.2\%$

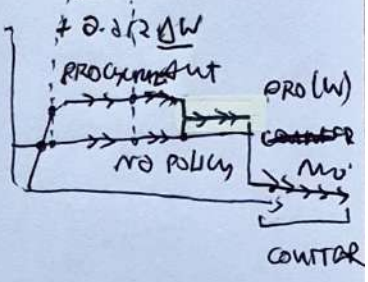
- Fall in the unemployment rate of 1% corresponds to rise of workers' wages of 0.012.

$\Rightarrow$ Instead: $\frac{\partial \Delta W}{\partial \Delta u_t} = \frac{b}{\Delta u_t} \Delta W = +0.012 \frac{\Delta W}{\Delta u_t}$

for: as $\Delta W > \Delta u$, then $+0.012$



- The problem: $\boxed{\text{COMPOSITION BIAS}}$



CHAPTER 7 - theory

DSGE models of fluctuation

BUILDING BLOCKS.

- HOUSEHOLDS.

1) $\sum_{t=0}^{\infty} \beta^t [U(C_t) - V(L_t)]$ # OBJECTIVE

2) $U(C_t) = \frac{C_t^{1-\theta}}{1-\theta}$ # CRRA

3) $V(L_t) = \frac{1}{\gamma} B L_t^{\gamma}$ # LABOUR DEMAND

4) $\frac{W_t}{P_t} = \frac{V'(L_t)}{C'(C_t)} \equiv \frac{V'(Y_t)}{C'(Y_t)}$

5.) $\frac{W_t}{P_t} = \frac{dV/dL_t}{dU/dC_t} = B L_t^{\gamma-1} C_t^{-\theta} = B Y_t^{\theta+\gamma-1}$

- FIRMS

1) $Y_{it} = L_{it} = Y_t \left(\frac{P_{it}}{P_t}\right)^{-\eta}$: so $\boxed{Y_t = Y_{it}\left(\frac{P_{it}}{P_t}\right)^{-\eta}}$ ①

2) $R_t = \left(\frac{P_{it}}{P_t}\right) Y_{it} - \left(\frac{W_t}{P_t}\right) Y_{it} = \boxed{Y_{it}\left[\left(\frac{P_{it}}{P_t}\right)^{1-\eta} - \left(\frac{W_t}{P_t}\right)\left(\frac{P_{it}}{P_t}\right)^{-\eta}\right]}$ ②

3) $A = E\left[\sum_{t=0}^{\infty} q_t \lambda_t R_t\right] = E\left[\sum_{t=0}^{\infty} q_t \lambda_t \left\{\left[\left(\frac{P_{it}}{P_t}\right)^{1-\eta} - \frac{W_t}{P_t}\left(\frac{P_{it}}{P_t}\right)^{-\eta}\right] Y_{it}\right\}\right]$

$= E\left[\sum_{t=0}^{\infty} q_t \lambda_t Y_t P_t^{\eta-1} \cdot F(P_i, P_t^*)\right]$ ∴ so $\boxed{A = E\left[\sum_{t=0}^{\infty} q_t \lambda_t Y_t P^{\eta-1} \cdot F(P_i, P_t^*)\right]}$

4.) Assumptions : a.) LOW INFLATION; b.) ECONOMY CLOSE to its PRICE FLEXIBLE EQUILIBRIUM ③; c.) 2nd order Approximation

$\Rightarrow P_i^* = P_t \Rightarrow \frac{\partial F(P_i, P^*)}{\partial P_i} = 0, \quad \frac{\partial^2 F(P_i, P^*)}{\partial P_i^2} \leq 0$

$\Rightarrow F(P_i P_t) \leq F(P_t^*, P_t^*) - K(P_i - P_t^*)^2 \; w/ \; K > 0$

5.) $\boxed{\min \sum_{t=0}^{\infty} q_t E[(P_i - P_t^*)^2]} \Rightarrow \boxed{\sum_{t=0}^{\infty} q_t \{(P_i - E[P_t^*])^2 + Var(P_t^*)\}}$ ⑤

⑥ $\Rightarrow P_i = \sum_{t=0}^{\infty} \omega_t E[P_t^*]$ # ω_t = probability of price set at 0.

$\Rightarrow \min \sum_{t=0}^{\infty} q_t \beta^t \{(P_i - E[P_t^*])^2 + Var(P_t^*)\}$

⑥ $\boxed{\omega_t = \frac{q_t}{\sum_{\tau}^{\infty} q_\tau}}$

$\Rightarrow \boxed{P_i = \sum_{t=0}^{\infty} \tilde{\omega}_t E[P_t^*]}$ ⑦, $\tilde{\omega}_t = \frac{\beta^t q_t}{\sum_{\tau}^{\infty} \beta^\tau q_\tau}$

⑥. $\boxed{E[(P-P)^2] = P - E[P] - E^2[P]}$

statistics.

6.) $P^* = P + \ln[\eta/(\eta-1)] + \ln B + (\theta + \gamma - 1) y$

$\Leftrightarrow P_t^* = \phi M_t + (1-\phi) P_t$

$\Rightarrow \boxed{P_i = \sum_{t=0}^{\infty} \tilde{\omega}_t E[\phi M_t + 1 - \phi P_t]}$ ⑦

CENTRAL BANK.

1.) Central bank follows some rule to set real interest rates in correspondence with Macroeconomic Conditions

2.) Central banks target Nominal GDP by setting Interest rate paths

FISCHER MODEL

1.) $P_t = \frac{1}{2}(P_t^1 + P_t^2)$ => Average PRICE At period 1 & 2

2) $$\begin{cases} P_t^1 = E_{t-1}[\phi M_t + (1-\phi)P_t] = \phi E_{t-1} M_t + (1-\phi)\frac{1}{2}(P_t^1 + P_t^2) \\ P_t^2 = E_{t-2}[\phi M_t + (1-\phi)P_t] = \phi E_{t-2} M_t + (1-\phi)\frac{1}{2}(E_{t-2}P_t^1 + P_t^2) \end{cases}$$

$$\therefore \text{so} \quad P_t^1 = \phi E_{t-1} M_t + (1-\phi)\frac{1}{2}(P_t^1 + P_t^2)$$
$$P_t^2 = \phi E_{t-2} M_t + (1-\phi)\frac{1}{2}(E_{t-2}P_t^1 + P_t^2).$$

3) $$\boxed{P_t^1 = \phi E_{t-1} M_t + (1-\phi)\frac{1}{2}(P_t^1 + P_t^2)}$$

$$\Rightarrow P_t^1 - (1-\phi)\frac{1}{2}P_t^1 = (1-\phi)\frac{1}{2}P_t^2 + \phi E_{t-1} M_t$$

$$\Rightarrow P_t^1\left(1 - (1-\phi)\frac{1}{2}\right) = (1-\phi)\frac{1}{2}P_t^2 + \phi E_{t-1} M_t$$

$$\Rightarrow \boxed{P_t^1 = \frac{1}{\left(1-(1-\phi)\frac{1}{2}\right)}(1-\phi)\frac{1}{2}P_t^2 + \phi E_{t-1} M_t}$$

$$\rightarrow P_t^1 = \frac{1}{2-(1-\phi)}(1-\phi)\frac{1}{2}P_t^2 + \phi E_{t-1} M_t$$

$$= \frac{1}{2-(1-\phi)}(1-\phi)\frac{1}{2}P_t^2 + \phi E_{t-1} M_t \frac{2}{2-(1-\phi)}$$

$$= \frac{1-\phi}{1+\phi}P_t^2 + E_{t-1}\frac{2\phi}{(1+\phi)} \quad \therefore \text{so} \quad \boxed{P_t^1 = \frac{1-\phi}{1+\phi}P_t^2 + E_{t-1}\frac{2\phi}{(1+\phi)}} \quad (1)$$

4) $$\boxed{P_t^2 = \phi E_{t-2} M_t + (1-\phi)\frac{1}{2}(E_{t-2}P_t^1 + P_t^2)}$$

$$\Rightarrow P_t^2 = \phi E_{t-2} M_t + (1-\phi)\frac{1}{2}\left(\frac{2\phi}{1+\phi}E_{t-2}M_t + \frac{1-\phi}{1+\phi}P_t^2 + P_t^2\right)$$

$$\Rightarrow \therefore P_t^2 = \phi E_{t-2} M_t + (1-\phi)\frac{1}{2}\left(\frac{2\phi}{1+\phi}E_{t-2}M_t + \frac{1-\phi}{1+\phi}P_t^2 + P_t^2\right)$$

$$P_t^2 = \phi E_{t-2} M_t + \frac{(1-\phi)}{(1+\phi)}E_{t-2}M_t + \frac{(1-\phi)^2}{2(1+\phi)}P_t^2 + \frac{1}{2}(1-\phi)P_t^2$$

$$P_t^2 - \frac{(1-\phi)^2}{2(1+\phi)}P_t^2 - \frac{1}{2}(1-\phi)P_t^2 = \phi E_{t-2} M_t (1-\phi) + \frac{(1-\phi)}{(1+\phi)}E_{t-2}M_t$$

$$P_t^2\left[\frac{(1-\phi)^2}{2(1+\phi)} - \frac{1}{2}(1-\phi) + 1\right] = E_{t-2}M_t\left[\phi(1-\phi)M_t + \frac{(1-\phi)}{(1+\phi)}E_{t-2}M_t\right]$$

$$\Rightarrow \therefore \boxed{P_t^2 = E_{t-2} M_t} \quad (2)$$

5) $$\boxed{\begin{aligned} P_t^1 &= \frac{1-\phi}{1+\phi}P_t^2 + E_t\frac{2\phi}{1+\phi} \\ P_t^2 &= E_{t-2} M_t \end{aligned}}$$ => At EQUILIBRIUM:

$$\boxed{P_t^1 = E_{t-2} M_t + \frac{2\phi}{1+\phi}(E_{t-1}M_t - E_{t-2}M_t)}$$

$$\boxed{P_t = E_{t-2}M_t + \frac{\phi}{1+\phi}E_{t-1}M_t - E_{t-2}M_t} \Rightarrow \boxed{y_t = \frac{1}{1+\phi}E_{t-1}M_t - E_{t-2}M_t + M_t - E_{t-1}M_t}$$

- IMPLICATIONS

$$y_t = \frac{1}{1+\phi}\left(\underbrace{E_{t-1}m_t - E_{t-2}m_t}_{\text{UNANTICIPATED OUTPUT}}\right) + \left(\underbrace{m_t - E_{t-1}m_t}_{\text{UNANTICIPATED DEMAND SHIFTS}}\right)$$

1.) UNANTICIPATED REAL EFFECTS : unanticipated shifts have real
=> effects by which price setters are assumed to NOT know m_t
=> shocks are passed one - one for output

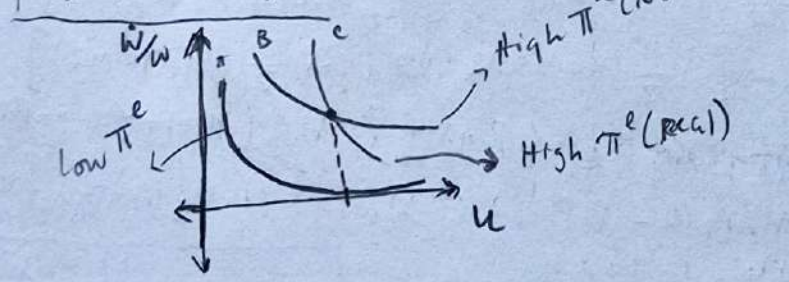
2.) UNANTICIPATIONS AFTER FIRST PRICE EFFECTS OUTPUT
=> Survey Results and Indicators in correspondence to Economic output
=> Surveys Shift Monetary Policy => (PRICE & MONEY)
=> Portions of $\frac{1}{1+\phi}$ ARRIVES at $(t-2, t-1)$ AND INTO OUTPUT
AND THE REMAINDER INTO PRICES. I.e : $\boxed{w_p P_t + w_y y_t = P}$

3.) PRICE SETTERS CAN EITHER INCREASE / DECREASE EFFECTS of MICROECONOMIC PRICE STICKINESS
=> REMEMBER! Prices are half to adjust ($w_p P_t$) and others are to be set ($w_y y_t$), half of that Information about m_t comes between $(t-2)(t-2)$
=> So : $w_p\left\{E_{t-2}m_t + \frac{\phi}{1+\phi}(E_{t-1}m_t - E_{t-2}m_t)\right\} + w_y\left\{\frac{1}{1+\phi}(E_{t-1}m_t - E_{t-2}m_t) + (m_t - E_{t-1}m_t)\right\} = E(w_p, w_y)$

4) RIGIDITY (ϕ) : $\uparrow\phi$ = VARIATIONS IN PRICES , $\downarrow\phi$ = REAL RIGIDITY
=> RECALL that real Rigidity is where Real Prices and wages failed to adjust to the level indicated by Equilibrium or natural.
=> NOMINAL = Not changing quickly in Response to changes
(COFFEE shop owners not adjusting their coffee to harvest falls)
=> REAL = How is it resistant to changes? (Resistance of wage cuts as Price falls, results in real high wages $\boxed{\downarrow W_{Nom} \gg \downarrow P}$

5) MODEL DOSEN'T DEPEND ON $E_{t-2}m_t$
ANY INFORMATION about Aggregate Demand that all price setters have had a chance to Respond has no effect on output

6) PHILIPS CURVE



TAYLOR MODEL

- MODEL / ASSUMPTIONS:

1) $m_t = m_{t-1} + u_t$

2) $x_t = \frac{1}{2}(p_t^* + E_t p_{t-1}^*) - \frac{1}{2}\{[\phi m_t + (1-\phi)p_t] + [\phi E_t m_{t-1} + (1-\phi) E_t p_{t+1}]$

$$\therefore \text{SO } x_t = \frac{1}{2}\Big\{\underbrace{[\phi m_t + (1-\phi)p_t]}_{\text{CURRENT ADJUSTMENTS}} + \underbrace{[\phi E_t m_{t-1} + (1-\phi) E_t p_{t+1}]}_{\text{EXPECTED ADJUSTMENTS}}\Big\}$$

3) $x_t = \phi m_t + \frac{1}{4}(1-\phi)(x_{t-1} + 2x_t + E_t x_{t+1})$

4) $\boxed{x_t = A(x_{t-1} + E x_{t+1}) + (1-2A) m_t}$ ①, $A \equiv \frac{1}{2}\frac{1-\phi}{1+\phi}$

- UNDETERMINED COEFFICIENTS

1.) $\boxed{x_t = \mu + \lambda x_{t-1} + v m_t}$ w/ $\left\{\begin{matrix}\mu \\ \lambda \\ v\end{matrix}\right\}$ SOLUTIONS TO the MODEL

2) $\boxed{m_t = \mu + \lambda m_t + V m_t.}$ ② w/ PRICE SETTERS' PRICE IS ADJUSTED w/ EQUILIBRIUM FLEXIBLE PRICES to y.

3.) $x_t = \lambda x_{t-1} + (1-\lambda) m_t$

4) $E_t x_{t+1} = \lambda[\lambda x_{t-1} + (1-\lambda) m_t] + (1-\lambda) m_t$

$= \lambda^2 x_{t-1} + (1-\lambda^2) m_t$ $\therefore$ SO $\boxed{E_t x_{t+1} = \lambda^2 x_{t-1} + (1-\lambda^2) m_t}$

5.) ③ $\boxed{x_t = A[x_{t-1} + E x_{t+1}] + (1-2A) m_t}$ ← sub w/ 4

$\Rightarrow x_t = A\left[x_{t-1} + \{\lambda^2 x_{t-1} + (1-\lambda^2) m_t\}\right] + (1-2A) m_t)$

$= A x_{t-1} + A\lambda^2 x_{t-1} + A(1-\lambda^2) m_t + (1-2A) m_t$

$= (A + A\lambda^2) x_{t-1} + [A(1-\lambda^2) + (1-2A) m_t]$

$\therefore$ SO: $\boxed{x_t = A + A\lambda^2 x_{t-1} + [A(1-\lambda^2) + (1-2A) m_t]}$ ④

6.) $\boxed{A + A\lambda^2 = \lambda}$; $A + A\lambda^2 = \lambda \Rightarrow A(1-\lambda^2) + (1-2A) = 1-\lambda$

$\Rightarrow A - A\lambda^2 + 1 - 2A \Rightarrow -A\lambda^2 - A + 1 - 1 - \lambda = \boxed{-A\lambda^2 - A - \lambda}$

$\Rightarrow \boxed{\lambda = \frac{1 \pm \sqrt{1-4A^2}}{2A}}$ ⑤ $\Rightarrow \boxed{\left\{\begin{matrix}\lambda_1 = \frac{1-\sqrt{\phi}}{1+\sqrt{\phi}} \\ \lambda_2 = \frac{1+\sqrt{\phi}}{1-\sqrt{\phi}}\end{matrix}\right.}$

7) $y_t = m_t - \frac{1}{2}\{[\lambda x_{t-2} + (1-\lambda) m_{t-1}] + [\lambda x_{t+1} + (1-\lambda) m_t].\}$

a) $m_t = m_{t-1} + u_t$

b) $\frac{x_{t-1} + x_{t-2}}{2} = p_t$

$\Rightarrow y_t = m_t - \frac{1}{2}\{[\lambda x_{t-2} + (1-\lambda) m_t] + [\lambda x_{t-1} + (1-\lambda) m_t]\}$

$= m_t - [\lambda \frac{1}{2}(x_{t-2} + x_{t-1}) + (1-\lambda)\frac{1}{2}(m_{t-1} + m_t)]$

$= \{m_{t-1} + u_t\} - [\lambda\{p_{t-1}\} + (1-\lambda)\frac{1}{2}(m_{t-1} + \{m_{t-1} - u_t\})]$

$= m_{t-1} + u_t - \lambda p_{t-1} + (1-\lambda) m_{t-1} + (1-\lambda)\frac{1}{2} u_t = \lambda(m_{t-1} - p_{t-1}) + \frac{1+\lambda}{2} u_t$

$= \lambda y_{t-1} + \frac{1+\lambda}{2} u_t$ $\therefore$ SO $\boxed{y_t = \lambda y_{t-1} + \frac{1+\lambda}{2} u_t}$ ⑥

TAYLOR MODEL

- MODEL / ASSUMPTIONS:

1) $m_t = m_{t-1} + u_t$

2) $x_t = \frac{1}{2}(p_t^* + E_t p_{t-1}^*) - \frac{1}{2}\{[\phi m_t + (1-\phi)p_t] + [\phi E_t m_{t-1} + (1-\phi) E_t p_{t+1}]$

$$\therefore \text{SO } x_t = \frac{1}{2}\Big\{\underbrace{[\phi m_t + (1-\phi)p_t]}_{\text{CURRENT ADJUSTMENTS}} + \underbrace{[\phi E_t m_{t-1} + (1-\phi) E_t p_{t+1}]}_{\text{EXPECTED ADJUSTMENTS}}\Big\}$$

3) $x_t = \phi m_t + \frac{1}{4}(1-\phi)(x_{t-1} + 2x_t + E_t x_{t+1})$

4) $\boxed{x_t = A(x_{t-1} + E x_{t+1}) + (1-2A) m_t}$ ①, $A \equiv \frac{1}{2}\frac{1-\phi}{1+\phi}$

- UNDETERMINED COEFFICIENTS

1.) $\boxed{x_t = \mu + \lambda x_{t-1} + v m_t}$ w/ $\left\{\begin{matrix}\mu \\ \lambda \\ v\end{matrix}\right\}$ SOLUTIONS TO the MODEL

2) $\boxed{m_t = \mu + \lambda m_t + V m_t.}$ ② w/ PRICE SETTERS' PRICE IS ADJUSTED w/ EQUILIBRIUM FLEXIBLE PRICES to y.

3.) $x_t = \lambda x_{t-1} + (1-\lambda) m_t$

4) $E_t x_{t+1} = \lambda[\lambda x_{t-1} + (1-\lambda) m_t] + (1-\lambda) m_t$

$= \lambda^2 x_{t-1} + (1-\lambda^2) m_t$ $\therefore$ SO $\boxed{E_t x_{t+1} = \lambda^2 x_{t-1} + (1-\lambda^2) m_t}$

5.) ③ $\boxed{x_t = A[x_{t-1} + E x_{t+1}] + (1-2A) m_t}$ ← sub w/ 4

$\Rightarrow x_t = A\left[x_{t-1} + \{\lambda^2 x_{t-1} + (1-\lambda^2) m_t\}\right] + (1-2A) m_t)$

$= A x_{t-1} + A\lambda^2 x_{t-1} + A(1-\lambda^2) m_t + (1-2A) m_t$

$= (A + A\lambda^2) x_{t-1} + [A(1-\lambda^2) + (1-2A) m_t]$

$\therefore$ SO: $\boxed{x_t = A + A\lambda^2 x_{t-1} + [A(1-\lambda^2) + (1-2A) m_t]}$ ④

6.) $\boxed{A + A\lambda^2 = \lambda}$; $A + A\lambda^2 = \lambda \Rightarrow A(1-\lambda^2) + (1-2A) = 1-\lambda$

$\Rightarrow A - A\lambda^2 + 1 - 2A \Rightarrow -A\lambda^2 - A + 1 - 1 - \lambda = \boxed{-A\lambda^2 - A - \lambda}$

$\Rightarrow \boxed{\lambda = \frac{1 \pm \sqrt{1-4A^2}}{2A}}$ ⑤ $\Rightarrow \boxed{\left\{\begin{matrix}\lambda_1 = \frac{1-\sqrt{\phi}}{1+\sqrt{\phi}} \\ \lambda_2 = \frac{1+\sqrt{\phi}}{1-\sqrt{\phi}}\end{matrix}\right.}$

7) $y_t = m_t - \frac{1}{2}\{[\lambda x_{t-2} + (1-\lambda) m_{t-1}] + [\lambda x_{t+1} + (1-\lambda) m_t].\}$

a) $m_t = m_{t-1} + u_t$

b) $\frac{x_{t-1} + x_{t-2}}{2} = p_t$

$\Rightarrow y_t = m_t - \frac{1}{2}\{[\lambda x_{t-2} + (1-\lambda) m_t] + [\lambda x_{t-1} + (1-\lambda) m_t]\}$

$= m_t - [\lambda \frac{1}{2}(x_{t-2} + x_{t-1}) + (1-\lambda)\frac{1}{2}(m_{t-1} + m_t)]$

$= \{m_{t-1} + u_t\} - [\lambda\{p_{t-1}\} + (1-\lambda)\frac{1}{2}(m_{t-1} + \{m_{t-1} - u_t\})]$

$= m_{t-1} + u_t - \lambda p_{t-1} + (1-\lambda) m_{t-1} + (1-\lambda)\frac{1}{2} u_t = \lambda(m_{t-1} - p_{t-1}) + \frac{1+\lambda}{2} u_t$

$= \lambda y_{t-1} + \frac{1+\lambda}{2} u_t$ $\therefore$ SO $\boxed{y_t = \lambda y_{t-1} + \frac{1+\lambda}{2} u_t}$ ⑥

• IMPLICATIONS

1) $$y_t = \lambda y_{t-1} + \frac{1+\lambda}{2} u_t$$

1.) SHOCKS ↔ AGGREGATE DEMAND HAVE LONG LASTING EFFECTS ON OUTPUT

⇒ if $\lambda_1 > 0$ and $\phi < 1$, Then aggregate demand in the economy will have effects on output

⇒ where did this EFFECT come from? from PRICE SETTER'S reluctance to allow variations in prices I.e.: $\boxed{P_t^* = \phi m_t + (1-\phi) P_t}$

2) RESPONSE OF PRICE LEVEL IS THE FLIP SIDE OF OUTPUT.

⇒ The price level Rises by $[1-(1+\lambda)/2]\, u^0$ in the initial Period

⇒ The fraction $(1-\lambda)$ Rises after $(1-(\frac{1+\lambda}{2}))\, u^0$ in after period

3.) SHOCKS STILL HAVE EFFECTS EVEN THOUGH FIRMS CHANGE PRICES

since not all firms adjust, $y \uparrow = \uparrow [1 \pm \frac{\lambda}{2}] u^0$ w/ u = shocks.

• LAG OPERATORS

1) $$x_t = A(x_{t-1} + E_t x_{t+1}) + \frac{1-2A}{2} m_t + \frac{1-2A}{2} E_t m_{t+1}$$

2) a.) $\boxed{L z_t = z_{t-1}}$ c.) $E_t m_{t+1} = m_t$

b.) $L E_t x_{t+1} = E_t x_t = x_t$ d.) $E_t x_{t+1} = L^{-1} x_t$

LAG OPERATORS

3) $$x_t = A(x_{t-1} + E_t x_{t+1}) + \frac{1-2A}{2} m_t + \frac{1-2A}{2} E_t m_{t+1}$$

$$\Rightarrow x_t = A(\{L x_t\} + \{L^{-1} x_t\}) + \frac{1-2A}{2} m_t + \frac{1-2A}{2}\{L^{-1} m_t\}$$

$$= A L x_t + A \frac{1}{L} x_t + \frac{1-2A}{2}\left(m_t + \frac{1}{L} m_t\right)$$

$$= (AL + AL^{-1}) x_t + \frac{1-2A}{2}(1 + L^{-1}) m_t$$

$$\Rightarrow x_t - (AL + AL^{-1}) x_t = \frac{1-2A}{2}(1+L^{-1}) m_t \Rightarrow \boxed{(I - AL - AL^{-1}) x_t = \frac{1-2A}{2}(1+L^{-1}) m_t}$$

Identity

I : $I z_t = z_t$

4.) $$(I - AL + AL^{-1}) x_t = \frac{1-2A}{2}(1+L^{-1}) m_t \Rightarrow (I - \lambda L^{-1})(I - \lambda L)\left(\frac{1}{A}\right) x_t = \frac{1}{A}\frac{1-2A}{2}(1+L^{-1}) m_t$$

$$\Rightarrow (I - \lambda L^{-1})(1 - \lambda L) x_t = \frac{\lambda}{A}\frac{1-2A}{A}(I + L^{-1}) m_t$$

$$\Rightarrow (I - \lambda L) x_t = \frac{\lambda}{A}\frac{1-2A}{A}\left[\sum_j (1+\lambda) L^{-j}\right] m_t$$

w/ $\sum_j (1+\lambda) L^{-j} = [I + (1+\lambda) L^{-1} + (1+\lambda) L^{-2} + \dots]$

$$\Rightarrow x_t = \frac{\lambda}{A}\frac{1-2A}{A}\left\{\frac{\sum_j (1+\lambda) L^{-j}}{\sum_j (1-\lambda) L^{-j}}\right\} m_t \approx \frac{\lambda}{A}\frac{1-2A}{2}\left(1 + \left\{\frac{1+\lambda}{1-\lambda}\right\}\right) + \lambda x_{t-1}$$

$$\therefore \text{so}: \boxed{x_t = \frac{\lambda}{A}\frac{1-2A}{2}\left(1 + \frac{1+\lambda}{1-\lambda}\right) + \lambda x_{t-1}}$$

5.) $$E_t x_{t+k} - A E_t x_{t+k-1} - A E_t x_{t+k+1} = \frac{1-2A}{2}\left[E_t m_{t+k} + E_t m_{t+k+1}\right]$$

CHAPTER 4 – SUMMARY

BUILDING BLOCKS + EQUATIONS

(1.) $\sum_{t}^{\infty} \beta^t \left[U(C_t) - V(L_t) \right]$ w/ $U(C_t) = \frac{C_t^{1-\theta}}{1-\theta} \Rightarrow \frac{dU}{dC_t} = C_t^{-\theta}$, $V(L_t) = \frac{1}{\gamma} B_t^{\gamma} \Rightarrow \frac{dV}{dL_t} = B_t^{\gamma-1}$ # HOUSEHOLD

(2.) $R_t = \upsilon_{it} \left[\left(\frac{P_{it}}{P_t}\right)^{1-\eta} - \left(\frac{W_t}{P_t}\right)\left(\frac{P_{it}}{P_t}\right)^{-\eta} \right]$

(3.) $A = E_1 \left[\sum_{t=0}^{\infty} q_t \lambda_t Y_t P^{\eta-1} F(P_{it}, P^*) \right]$ } # FIRM BEHAVIOR

(4.) $R_i = \sum_{t=0}^{\infty} \tilde{\omega}_t E[P_t^*]$ w/ a.) $E[P_t^*] = E[\phi M_t + (1-\phi) P_t^2]$, b.) $\tilde{\omega}_t = \frac{q_t}{\sum_{\tau=0}^{\infty} q_\tau}$ # PRICES DYNAMICS

(5.) $P_t^1 = \frac{1-\phi}{1+\phi} P_t^2 + E_t M_t \frac{2\phi}{1+\phi}$, $P_t^2 = E_{t-2} M_t$ $\Rightarrow$ $\begin{cases} P_t = E_{t-2} M_t + \frac{\phi}{1+\phi} E_{t-1} M_t - E_{t-2} M_t \\ y_t = \frac{1}{1+\phi} E_{t+1} M_t - E_{t-2} M_t + M_t - E_{t-1} M_t \end{cases}$

ADJUSTMENT AND FIRES

(6.) $y_t = \frac{1}{1+\phi} \left[E_{t-1} M_t - E_{t-2} M_t \right] + \left[M_t - E_{t-1} M_t \right]$

(7.) $\begin{cases} x_t = \mu + \lambda x_{t-1} + V M_t \\ M_t = \mu + \lambda M_{t-} + V M_t \end{cases} \Rightarrow x_t = A\left[x_{t-1} - E x_{t+1}\right] + 1-2A\, M_t$

$y_t = \lambda y_{t-1} + \frac{1+\lambda}{2} u_t$ $\Rightarrow$ AGREGATE DEMAND IS EFFECTED by SHOCKS EVEN Though FIRMS CHANGE PRICES

(8.) $\sum_{j} (1-\lambda L) x_t = \frac{\lambda}{A} \frac{1-2A}{A} \left[\sum_{j} (1+\lambda) L^{-j} \right] M_t$

$\Rightarrow x_t = \frac{\lambda}{A} \frac{1-2A}{2} \left(1 + \frac{1+\lambda}{1-\lambda} \right) + \lambda x_{t-1}$ $\Rightarrow$ LAG OPERATOR DERIVATIVE OF y_t.

The Permanent Income Hypothesis

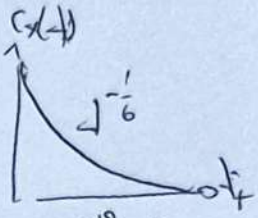
(1) PIH & Consumption under certainty.

PIH - Permanent

$$\boxed{A_0 + \sum_{t=1}^{T} Y_t \geq \sum_{t=1}^{T} C_t}$$ Budget constraint.

- We want to:

$$\boxed{\max_{C_t} \sum_{t=0}^{T} u(C_t) \quad s.t \quad A_0 + \sum_{t=1}^{T} y_t - \sum_{t=1}^{T} C_t \geq 0.}$$



- Thus the Lagrangian is:

$$\boxed{\mathcal{L} = \sum_{t=1}^{T} u(C_t) - \lambda\left[A_0 + \sum_{t=1}^{T}(y_t - C_t)\right]}$$

- The FOC is:

$$\frac{\partial \mathcal{L}}{\partial C_t} = \sum_{t=1}^{T} u'(C_t) - \lambda = 0 \Rightarrow \sum_{t=1}^{T} u'(C_t) = \lambda$$

$$C_t^{-\theta} = \lambda$$

$$\boxed{C_t = \lambda^{-\frac{1}{\theta}}}$$

Rate of Decay w/ Elasticity

$$\mathcal{L} = \sum_{t=1}^{T} u(C_t) - A_0 \sum_{t=1}^{T} u'(C_t) - \sum_{t=0}^{T} y_t u'(C_t) - C_t u'(C_t) = 0$$

- Since marginal level of consumption is cons. The $C_1 = \dots = C_T$ such that

$C_1 = C_2 \rightarrow C_1 + C_2 = 2C_2$

The constraint is actually cons. Thus

$$\sum_{t=1}^{T} u(C_t) = A_0 \sum_{t=1}^{T} u' + \sum_{t=0}^{T} y_t u' + C_t u' =$$

$$u(C_1) + u(C_2) = A_0\left(u'(C_1) + u'(C_2)\right) + \left(u'(C_1) + u(C_2)\right)(y_1 + u_2) + C_1 u_1' + C_2 u_2'$$

$$2u(C_2) = A_0\left(2u'(C_2)\right) + 2u'(C_2)2y_2 + 2C_2 u_2'$$

$$Tu(C_T) = A_0 u_T' + u_T' y_T + C_T u_T' = (A_0 + y_T + C_T) u_T'$$

$$T\frac{u_T}{u_T'} = A_0 + y_T + C_T \rightarrow T\frac{u_T}{C_T u_T'} = \frac{A_0 + y_T}{C_T} \rightarrow T C_T = A_0 + y_T$$

$$\text{Thus: } C_T = \frac{1}{T}\left(A_0 + \sum_{t=1}^{T} E(y_t)\right) \rightarrow \therefore \boxed{C_t = \frac{1}{T}\left[A_0 + \sum_{t=0}^{T} E(y_t)\right]}$$

$$T = A_0 + y_T + C_T \Rightarrow \frac{T}{u_T'} = \frac{A_0 + y_T + C_T}{u_T'} \Rightarrow \frac{T}{u_T'} = \frac{A_0 + y_T + C_T}{u_T'}$$

- **Implications**

C_t = Permanent Income

$\frac{1}{T}[A_0 + Y_t]$ = Transitory Income

C_{t+1} = Used $\frac{1}{T}(A_0 + Y_t)$

C_{t+2} = Used $\frac{1}{T} A_0 + Y_{t+1}$

- Consumption is determined by Permanent Income.
- Temporary Tax cuts may have little effect on consumption since ~~Z Rise but~~ $\frac{Z}{T}$ only Rise.
- Thus: $\boxed{S_t = Y_t - C_t = \left(Y_t + \frac{1}{T}\sum_{\tau=1}^{T} Y_\tau\right) - \frac{1}{T} A_0}$

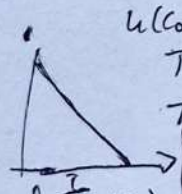
Remark:

$$u(C_0) + \dots u(C_T) = A_0 + \sum_{t=1}^{T} E(y_t) - TC_t$$

$$Tu(C_0) + TC_t = A_0 + \sum_{t=1}^{T} E(y_t)$$

$$T(0 + C_t) = A_0 + \sum_{t=1}^{T} E(y_t)$$

$$\boxed{C_t = \frac{1}{T}\left[\sum E(y_t) + A_0\right]}$$



$$\lim_{T \to \infty} \sum_{t=0}^{T} u(C_t) = \sum_{t=0}^{\infty} u(C_t) = 0 \text{ thus}$$

$$U = \int_0^{\infty} u(C_t)\, dt = 0$$

$$u(C_1) + u(C_2) = 2u(C_0)$$

$$u(C_1) + \dots + u(C_T) = Tu(C_0)$$

since $u(0) = 0$,

(4) INTEREST RATE SAVING

[i] INTEREST RATE & CONSUMPTION GROWTH

MATH

- For Non-Zero interest, the Budget constraint is the PV of Lifetime consumption doesn't exceed initial wealth + income:

$$\boxed{\sum_{t=1}^{T} \frac{1}{(1+r)^t} C_t \leq A_0 + \sum_{t=1}^{T} \frac{1}{(1+r)^t} Y_t}$$

- If we allow non-zero interest, then we'll use a Non zero Discount Rate.

Let:

$$\boxed{U = \sum_{t=1}^{T} \frac{1}{(1+\rho)^t} \frac{C_t^{1-\theta}}{1-\theta}}$$

- If we have some Decrease in Consumption at t and increase at $t+1$,

Let:

$$\boxed{\frac{1}{(1+\rho)^t} C_t^{-\theta} = (1+r)\frac{1}{(1+\rho)^{t+1}} C_{t+1}^{-\theta}}$$

- Doing the Algebra:

$$\frac{1}{(1+\rho)^t}\frac{1}{C_t^{\theta}} = \frac{(1+r)}{(1+\rho)^{t+1}}\frac{1}{C_{t+1}^{\theta}} \rightarrow \frac{C_{t+1}^{\theta}}{C_t^{\theta}} = \frac{(1+r)(1+\rho)^t}{(1+\rho)^{t+1}} = (1+r)(1+\rho)^{t-t-1}$$

$$= \frac{(1+r)}{(1+\rho)} \rightarrow \left(\frac{C_{t+1}}{C_t}\right)^{\theta} = \frac{(1+r)}{(1+r)} \rightarrow \frac{C_{t+1}}{C_t} = \left(\frac{1+r}{1+r}\right)^{\frac{1}{\theta}}$$

Let:

$$\boxed{\frac{C_{t+1}}{C_t} = \left(\frac{1+r}{1+\rho}\right)^{\frac{1}{\theta}}}$$

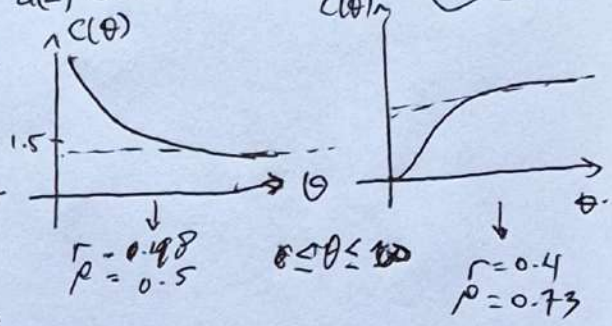
or if $C_t = L^{1} C_{t+1}$ Then

$$\frac{1}{a(L)}\frac{C_{t+1}}{C_{t+1}} = \frac{1}{a(L)} = \frac{(1+r)}{(1+\rho)}^{\frac{1}{\theta}}$$

$$a(L)\frac{(1+r)}{(1+\rho)}^{\frac{1}{\theta}} = 1$$

THEORY:

- We allow the possibility that Real interest and discount rates aren't equal ($\Delta(\rho \neq r)$).
- If $r > \rho \rightarrow$ Rise in consumption
  If $r < \rho \rightarrow$ Fall in consumption
- Campbell, Mankiw, Hall (1980s) Found that θ is High thus, Intertemporal Elasticity substitution is Low

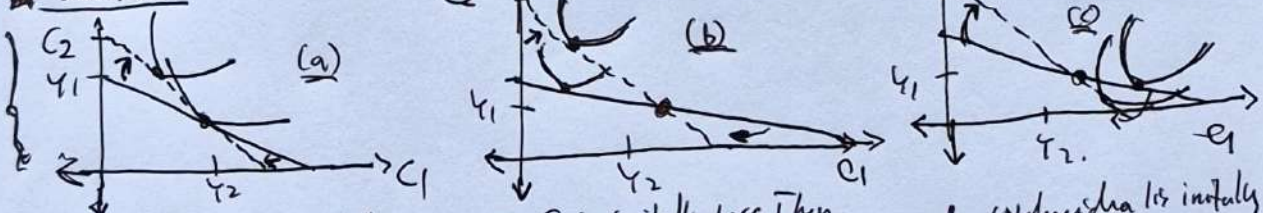


[ii] INTERTEMPORAL SAVING IN 2 PERIOD CASE

GRAPHS

Assumptions

- (C_1, C_2) goes (Y_1, Y_2)
- Individual can choose to consume higher income each period
- Slope is $-(1+r)$



- Saving (Y_1, Y_2) is initially zero
- (r) has No Income effect, The Individual's initial consumption
- 1st Period consumption falls and Saving rises

- C_1 is initially Less Than Y_1. Thus Saving is positive
- The Individual can afford more than his/her initial bundle
- Saving Doesn't change

- Individual is initially Borrowing.

∴ So

(6) Beyond the Permanent Income Hypothesis

(i) Buffer Stock Saving

- **High Consumption in Growing Countries of Income**
  - Individuals in Countries w/ Income growth is high will Consume More
  - Consumption growth in slower ones have lower

So: $\boxed{\text{Growing Econ} = \text{Growing Consump} \quad \text{Slowing Econ} = \text{Lower Consump}}$

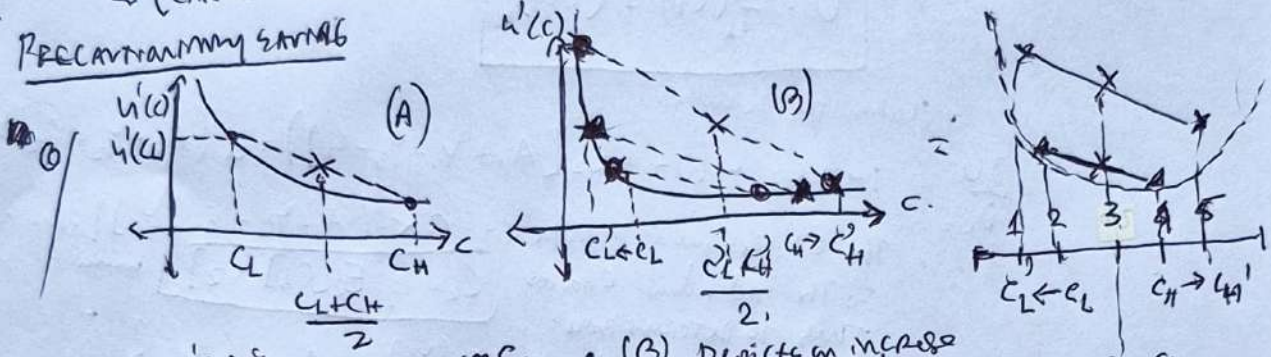
\# Managers and Professionals typically have earning profiles that rise steeply until middle age (Carroll & Summers (1991))

- **Failure of PIH**
  - PIH predicts that there's no relation between expected growth of income & consumption
  - Consumption is determined by real interest and discount rate
  - So no time varying income

$\therefore$ PIH $\doteq \boxed{E_t\left[\frac{\partial C}{\partial t}\right] \neq E_t\left[\frac{\partial y_t}{\partial t}\right]}$

- **Buffer Stock Savings**

(ii) Precautionary Saving



- $u'(c)$ is decreasing in c.
- If $p(c_L) = p(c_H) = \frac{1}{2}$, Then the Expected Marginal utility is just $\frac{c_L + c_H}{2}$
- $u'''(c)$ is positive
- (B) depicts an increase in uncertainty
- Increase in uncertainty raises the expected marginal utility for a given expected value fnctn

So: $c_L - c_L' = 2 - 1 = 1$

$c_H - c_H' = 4 - 5 = -1$

$\frac{c_L + c_H}{2} = \frac{2+4}{2} = \frac{6}{2} = 3$

So: $\frac{\Delta c_L}{\Delta c_H} = \frac{c_L - c_L'}{c_H - c_H'} = -\frac{1}{1} = \boxed{-1}$

So: -1 offsets for consumption since

Mathematics

- Suppose we Formulate:

$$\boxed{u'(c_t) = E_t\left[u'(c_t)\right]}$$

- Let $u(c_t) = \frac{c_t^{1-\theta}}{1-\theta} \rightarrow u' = c_t^{-\theta}$ and:

$E[u'] = \rho_1 u_1' + \rho_2 u_2' = \rho u_1' + (1-\rho) u_2'$ and $\rho = \frac{1}{2}$

$= \frac{1}{2} u_1' + \frac{1}{2} u_2' = \frac{u_1' + u_2'}{2}$

So: $\boxed{E[u'] = \frac{u_1' + u_2'}{2}}$ w/ if $u_1' \rightarrow c_t = c_t^{-\theta}$, $u_2' \rightarrow c_H = c_{t+1}^{-\theta}$ Then $\boxed{E[c_t] = \frac{c_H + c_L}{2}}$

- Recall that:

$$\boxed{E[r^i] = \rho + \theta E[g^c] + \theta \operatorname{cov}(r^i, g^c) - \frac{1}{2}\theta(\theta+1)\operatorname{var}(g^c)}$$

- Then if $r = \rho$ Then:

$\rho \approx \rho + \theta E[g^c] + \frac{1}{2}(\theta+1)\theta \operatorname{var}(g^c) \rightarrow \boxed{E[g^c] \sim \frac{1}{2}(\theta+1)\operatorname{var}(g^c)}$

- If $\rho_t^i = E_t\left[\sum_{k=1}^{\infty} \frac{1}{(1+\rho)^k} \frac{u'(c_{t+k})}{u'(c_t)} \rho_{t+k}^i\right]$
- Then: $E_t[1+r_{t+1}^i] = \frac{1}{E_t[u'(c_{t+1})]}\left[(1+\rho)u'(c_t) + \operatorname{cov}(1+r_{t+1}^i, c_{t+1})\right]$
- The EPP: $c_t^{-\theta} = \frac{-1}{1+\rho} E_t\left[1+r_{t+1}^i\right] c_{t+1}^{-\theta} \rightarrow$

Implications

- The analysis suggest precautionary saving raises expected consumption growth:
- Example, if CRRA is 4 and STDDEV is 0.1 then $\frac{1}{2}(4+1)(0.1)^2 = 25\%$
- Households are "usually" impatient so they have steep paths of expected income

III LIQUIDITY CONSTRAINTS:

- **LIQUIDITY CAN RAISE SAVINGS**
  - 2 ways liquidity raises savings:
    1.) BINDING LIQUIDITY = LESS CONSUMPTION
    2.) FUTURE BINDING = LESS CONSUMPTION NOW
  - to: $\boxed{\text{FLOW OF INCOME + EXPECTED FLOW OF INC}}$

- **ASSUMPTIONS**
  - Constraint takes place only the individual can't Borrow at all
  - utility fnc is quadratic

- **TOTAL UTILITY + MATH**
  - If A_t is asset at t, then $(C_3 = A_2 + Y_3)$, w/ $(C_3 = A_1 + Y_2 + Y_3 - C_2)$.

Therefore:

$$\boxed{U = \left(C_2 - \tfrac{1}{2}aC_2^2\right) + E_2\left[(A_1 + Y_2 + Y_3 - C_2)\right] - \tfrac{1}{2}a\left[(A_1 + Y_2 + Y_3 - C_2)^2\right]}$$

w/ $C_2 - \frac{1}{2}aC_2^2 = \frac{a^0 C_2^1}{1!} + \frac{1}{2!}a^1 C_2^2 = \frac{a^{k-1}C_2^k}{k!} = f(C)$

$C_3 \Leftrightarrow \frac{1}{1!}E_2^1[C_3] + \frac{1}{2!}\sigma^2[C_3]$ as $E^2[C_3] = C_3^2$

so $\boxed{U = f_k(C_2) + \langle C_3 \rangle_k}$

- Taking the derivative:

$$\frac{\partial U}{\partial C_2} = 1 - aC_2 - \left(1 - aE_2[A_1 + Y_2 + Y_3 - C_2]\right) = a\left(A_1 + Y_2 + E_2[Y_3] - 2C_2\right)$$

- The expression is positive set $C_2 < \frac{1}{2}(A_1 + Y_2 + E_2[Y_3])$ and negative.
  So the individual chooses $\boxed{C_2 = \frac{1}{2}E_2[C_3 - C_2]}$ when
  Thus to determine BIND:

$$\boxed{C_2 = \min\left\{\frac{A_1 + Y_2 + E_2[Y_3]}{2} \; ; \; A_1 + Y_2\right\}}$$

$$\left\{\begin{array}{l} \text{BIND} = \frac{1}{2}E_2[(C_3 - C_3)] \\ \text{BIND} = A_1 + Y_2 \end{array}\right\}$$

Thus liquidity effects future consumption.

IV CREDIT LIMITS & BORROWING – GROSS & SOULES (2002)

- **STATISTICAL FORMULATION:**
  - Examining Household credit cards non Repayment

$$\boxed{\Delta B_{it} = \beta_0 \Delta L_{it} + \beta_1 \Delta L_{i,t-1} + \dots + \beta_{12}\Delta L_{i,t-3} + a'X_{it} + \varepsilon_{it}}$$

w/ ΔB_{it} = INTEREST-BEARING CC HOUSEHOLD i at $L\beta_{it}^j = \sum_{t=0}^{T=12} \beta_t^j$
ΔL_{it} = CREDIT LIMIT AT HOUSEHOLD i at $L^j\beta_t = \beta_{t-j}$

- The Theory:
  1) ABSENCE of Constraints, LENDER'S willingness to lend doesn't Effect Consumption
  2) when credit

- **STATISTICAL RESULTS:**

$\boxed{\sum_{i=1}^{12} \beta_i = 0.12} \rightarrow$ $\sum_i \beta_i = 0.011$
s.e $= 0.018$
t-stat $= \frac{\beta}{se} = \frac{0.011}{0.018} = 6.6$

- So an increase in Credit Borrow w/ 11% is due to \$1 increase in Credit Limit
- For households w/ High utilization Rates

PERMANENT INCOME HYPOTHESIS

(1)

$$\boxed{U = \sum_{t=1}^{T} u(C_t)}$$

$u'(\cdot) > 0$
$u''(\cdot) < 0$

Individual's consumption

$$\frac{\partial U}{\partial C_t} = \sum_{t=1}^{T} \frac{\partial u}{\partial C_t}\frac{\partial C_t}{\partial t} = C_t^{-\theta}\frac{\partial C_t}{\partial t} = \frac{\dot{C}_t}{C_t^{\theta}}$$

$$\frac{\partial^2 u}{\partial C_t^2} = \sum_{t=1}^{T} \frac{\partial}{\partial C_t}\frac{\dot{C}_t}{C_t^{\theta}} =$$

$$\boxed{\sum_{t=1}^{T} C_t \leq A_0 + \sum_{t=1}^{T} Y_t}$$

→ Budget constraint

$$A_0 + \sum_{t=1}^{T} Y_t - \sum_{t=1}^{T} = 0$$

$$L = \sum_{t=1}^{T} u(C_t) + \lambda\left[\sum_{t=1}^{T} Y_t - \sum_{t=1}^{T} C_t + A_0\right]$$

$$\frac{\partial L}{\partial C_t} = \sum_{t=1}^{T} u'(C_t) - \lambda = 0$$

$$\lambda = u'(C_t)$$

$$\sum_{t=1}^{T} u(C_t) - \left[u'(C_t)\sum_{t=1}^{T} Y_t - u'(C_t)\sum_{t=1}^{T} C_t + u(C_t)A_0\right] = 0$$

$$C_t T - \left[\sum_{t=1}^{T} Y_t + A_0\right] = 0$$

$$C_t T = \sum_{t=1}^{T} Y_t + A_0$$

$$\boxed{C_t = \frac{1}{T}\left[A_0 + \sum_{t=1}^{T} Y_t\right]}$$

→ Permanent Income Hypothesis

(2) $$\boxed{S_t = Y_t - C_t = \left(Y_t - \frac{1}{T}\sum_{\tau=1}^{T} Y_\tau\right) - \frac{1}{T}A_0}$$

- PERMANENT INCOME : $\frac{1}{T}\left(A_0 + \sum_{\tau=1}^{T} Y_\tau\right)$
- TRANSITORY : $Y_t - \frac{1}{T}\sum_{\tau=1}^{T} Y_\tau - \frac{1}{T}A_0$
- SUPPOSE I GET AN INCREASE SALARY BY Z. Z RAISES MY CURRENT INCOME AND RAISES PERMANENT INCOME by $\frac{Z}{T}$.

(3) $$\boxed{\text{SAVINGS} = \text{FUTURE CONSUMPTION}}$$

(1) ¢ : POOR PEOPLE SAVE SMALLER ON THEIR INCOMES THAN THE WEALTHY SINCE POOR people pay $\frac{1}{3}$ FOR STAFFLIVING

℞ : $\begin{cases}\text{Poor}\\ \text{wealthy}\end{cases}$ ⇒ Trouble std today ≠ Trouble std. tomorrow.

(2) ¢ : A's CONSUMPTION COMPARISON TO B's CONSUMP = A's ↑ RAISING LUXURY CONSUMPTION Relative to B

℞ : SAVINGS = Future consumption.
saving ↓ = ↓ Future consumption.
⇒ ∴ Falling Further behind than B's C_t.

$$\boxed{C_A(t) \neq C_B(t)}$$

w/ $S_t^A = Y_t^A - C_t^A \Rightarrow \frac{\Delta S_{t+1}^A}{S_t^A} = \frac{\Delta Y_{t+1}^A}{Y_t} - \frac{\Delta C_{t+1}^A}{C_t}$

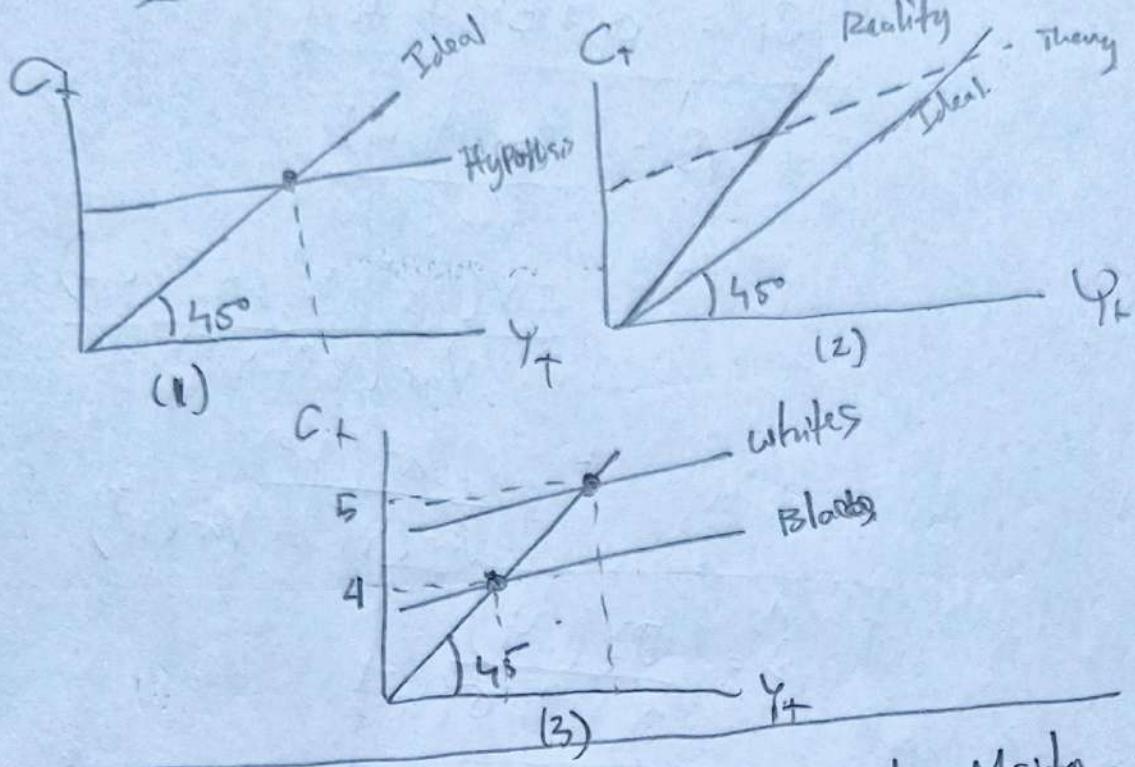
$S_t^B = Y_t^B - C_t^B \Rightarrow \frac{\Delta S_{t+1}^B}{S_t^B} = \frac{\Delta Y_{t+1}^B}{Y_t} - \frac{\Delta C_{t+1}^B}{C_t}$

Also? why / $\boxed{A_0^A \neq A_0^B}$

UNDERSTANDING ESTIMATED CONSUMPTION FUNCTIONS - EMPIRICAL APP

8/12

(1)



> " The Amount of Aggregate Consumption Mainly Depends on the Amount of Aggregate Income. "

↳ Keynes ⇔ Not consistent w/ Data

(a) Keyne's Hypothesis

(b) Reality ($C_t \perp Y_t$)

(c) Cross example ($\beta_W > \beta_B$ w/ slope β_W / β_B)

(2)

$$Y_{it} = Y_{it}^{P} + Y_{it}^{T} \rightarrow$$ 1st. Current Income of Individual (i)

$$N(\mu = 0, \sigma = 1)$$

$$\rho(Y_{it}^{P}, Y_{it}^{T}) = 0$$

$$U_i = \sum_{t=1}^{T} u[C_{it} - e_{it}] \rightarrow$$ 2nd. Utility of individual i w/ $\mu = 0$.

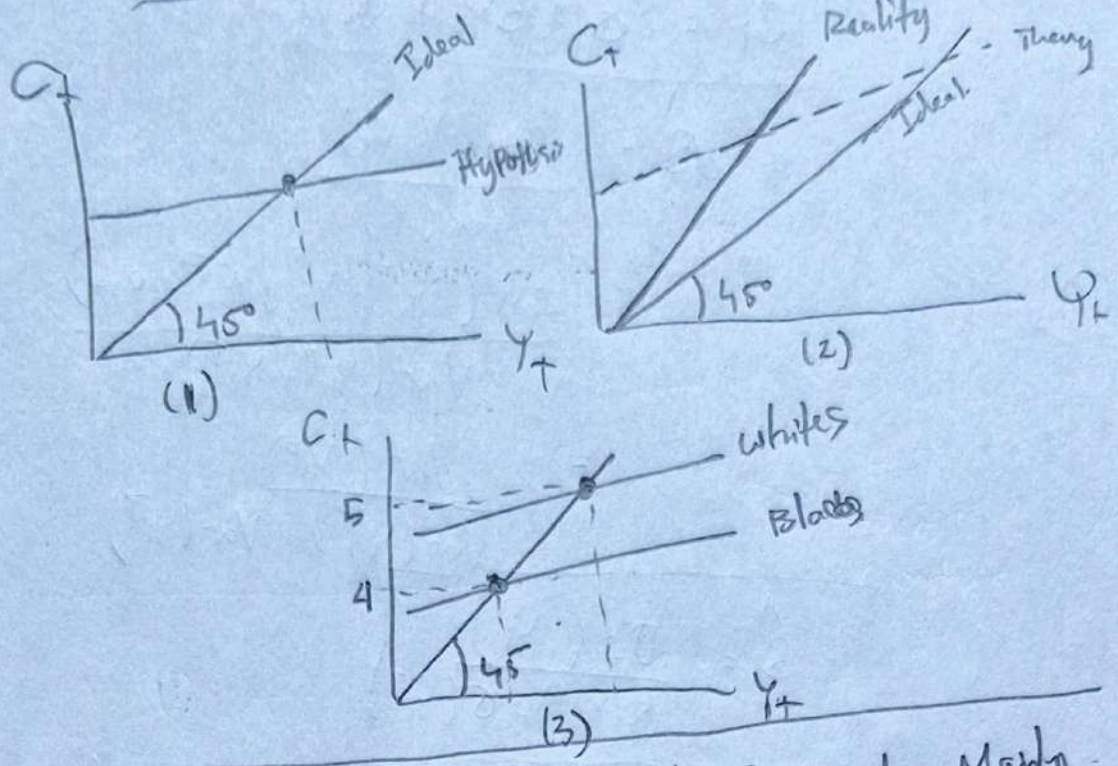
↳ • Medical Emergency
• Wedding of a Child } Fluctuations in Time

↳ (?) CONSUMPTION = CONSTANT

UNDERSTANDING ESTIMATED CONSUMPTION FUNCTIONS - EMPIRICAL APP

8/12

(1)



> " The Amount of Aggregate Consumption Mainly Depends on the Amount of Aggregate Income "

↳ Keynes $\overset{\Delta}{\Leftrightarrow}$ Not consistent w/ Data

(a) Keyne's Hypothesis
(b) Reality ($C_t \perp Y_t$)
(c) Cross example ($\beta_W > \beta_B$ w/ slope $\beta_{W/B}$)

(2)

$$Y_{it} = Y_{it}^{P} + Y_{it}^{T}$$ → 1st: Current Income of Individual (i)

$$N(\mu = 0, \sigma = 1)$$
$$\rho(Y_{it}^{P}, Y_{it}^{T}) = 0$$

$$U_i = \sum_{t=1}^{T} u\left[C_{it} - e_{it} \right]$$ → 2nd: Utility of individual i w/ $\mu = 0$.

↳ • Medical Emergency
• Wedding of a Child } Fluctuations in time

↳ (?) CONSUMPTION = CONSTANT

(3) $C_{it} = Y_{ii}^{P} + e_{it} \longrightarrow$ Transitory Consumption

$$\boxed{C_i = \alpha + \beta Y_i + U_i}$$

$$\hat{\beta} = \frac{Cov(Y_i, C_i)}{Var(Y_i)}$$

$$= \frac{Cov(Y_i^P + Y_i^T ; Y_i^P + e_{it})}{Var(Y_i^P + Y_i^T)}$$

$$= \frac{Var(Y_i^P)}{Var(Y_i^P) + Var(Y_i^T)}$$

$$\boxed{\hat{\beta} = \frac{\sigma(Y_i^P)}{\sigma(Y_i^P) + \sigma(Y_i^T)}}$$

w/ $Cov(Y_i^P + Y_i^T ; Y_i^P + e_{it})$

$\triangleq [Cov(Y_i^P) + Cov(Y_i^T)][Cov(Y_i^P) + Cov(C_i)]$

$\triangleq Cov(Y_i^P)\, Cov(Y_i^P)$

$\triangleq Cov(Y_i^P)^2 = \boxed{\sigma^2(Y_i^P)}$

wh/ $\underline{Cov(X, X) = Var(X)}$.

$$\alpha = \bar{C} + \hat{\beta}\bar{Y} = (1 - \hat{\beta})\bar{Y}^P$$

↳ univariate Regression

(4) $U_i = \sum_{t=1}^{T} U(C_t - e_{it})$

$C_i = \alpha + \beta Y_i + U_i$

(1) Individual w/ higher Current Income will consume more than the other to extent that her transitory Income is higher [ie. $\uparrow Y = \uparrow C_t \Leftrightarrow Y^T \uparrow$]

(2) Blacks & White Consumption is via their own Average Income $((1-\beta)\bar{Y}^P)$ Rather than Culture

ie $\boxed{\text{AVERAGE INCOME} \gg \text{CULTURE, TASTES}}$

RANDOM WALK HYPOTHESIS

(1) $$E[U] = E\left[\sum_{t=1}^{T}\left[C_t - \frac{a}{2}C_t^2\right]\right]$$

$$1 - aC_1 = E_1[1 - aC_t]$$

$$\cancel{1} - aC_1 = \cancel{E(1)} - E[aC_t] = \cancel{E(1)} - E(C_t)a$$

$$-aC_1 = \cancel{a}E_1[C_t]$$

So: $$\boxed{C_1 = E_1[C_t]}$$

$$\sum_{t=1}^{T} E_1[C_t] = A_0 + \sum_{t=1}^{T} E_1[Y_t]$$

$$\boxed{C_1 = \frac{1}{T}\left(A_0 + \sum_{t=1}^{T} E_1[Y_t]\right)}$$

$$C_t = E_{t-1}[C_t] + e_t = C_{t-1} + e_t$$

$$\boxed{C_t = C_{t-1} + e_t}$$ → Random walk / Time series regression.

(2) $$C_2 = \frac{1}{T-1}\left[A_1 + \sum_{t=2}^{T} E_2[Y_t]\right]$$

$$= \frac{1}{T-1}\left[A_0 + Y_1 - C_1 + \sum_{t=2}^{T} E_2[Y_t]\right]$$

$$= \frac{1}{T-1}\left[\left(A_0 + Y_1 - (C_1) + \sum_{t=2}^{T} E_1[Y_t]\right) + \left(\sum_{t=2}^{T} E_2[Y_t] - \sum_{t=2}^{T} E_1[Y_t]\right)\right]$$

$$= \frac{1}{T-1}\left[TC_1 - C_1 + \sum_{t=2}^{T} E_2[Y_t] - \sum_{t=2}^{T} E_1[Y_t]\right]$$

$$\boxed{C_2 = C_1 + \frac{1}{T-1}\left(\sum_{t=2}^{T} E_2[Y_t] - \sum_{t=2}^{T} E_1[Y_t]\right)}$$

So: $C_1 = \frac{1}{T}\left(A_0 + \sum_{t=1}^{T} E_1[Y_t]\right)$

$$C_2 = C_1 + \frac{1}{T-1}\left(\sum_{t=2}^{T} E_2[Y_t] - \sum_{t=1}^{T} E_1[Y_t]\right).$$

(3) $u'(C_1) = E_1[u'(C_2)] \longrightarrow$

$u'(C_1) = u'(E_1[C_2]) \longrightarrow$ Quadratic utility = Linear

$$\boxed{C_1 = E_1[C_2]}$$

w/ $E_1[1 - aC_2] = 1 - aE[C_2]$.

TWO TESTS OF RANDOM WALK

MANKIW ← HALL

8.3.1A

RANDOM WALK = UNPREDICTABILITY ~~WRONG~~ OF CONSUMPTION.

(1). $\max E \sum_{s=0}^{T-t} \frac{1}{1+r} U(k_{t+s})$

$$\text{s.t} \sum_{s=0}^{T-t} \frac{1}{(1+r)^s}\left[k_{t+s} - (1-\delta)k_{t+s-1} + w_{t+s}\right] - A_t = 0$$

$$L = E\sum_{s=0}^{T-t} \frac{1}{(1+r)} U(k_{t+s}) - \lambda\left[\sum_{s=0}^{T-t} \frac{1}{(1+r)^s}\left[k_{t+s} - (1-\delta)k_{t+s-1} + w_{t+s}\right] - A_t\right]$$

$$\frac{\partial L}{\partial k_{t+s}} = \frac{1}{(1+r)^t} U'(k_{t+s}) - \lambda \frac{1}{(1+r)^s}(1-(1-\delta)) = 0$$

$$\frac{\partial L}{\partial \lambda} = \sum_{s=0}^{T-t} \frac{1}{(1+r)^s}\left[k_{t+s} - (1-\delta)\cdot k_{t+s-1} - w_{t+s}\right] - A_t = 0$$

(1) $$\frac{1}{(1+r)^t} U'(k_{t+s}) = \lambda \frac{1}{(1+r)^s}(1-(1-\delta))$$

$$\lambda = \frac{(1+r)^s}{(1+r)^t} \frac{U'(k_{t+s})}{(1-(1-\delta))} = \frac{(1+r)^s}{(1+r)^t} \frac{U'(k_{t+s})}{-\delta}$$

$$\phi = \frac{1+r}{\delta+r} \qquad = \frac{(1+r)^s}{(1+r)^t \delta} U'(k_{t+s})$$

(2) $$A_t = \sum_{s=0}^{T-t} \frac{1}{(1+r)^s}\left[k_{t+s} - (1-\delta)k_{t+s-1} - w_{t+s}\right]$$

$$\boxed{\sum_{s=0}^{T-1} \frac{1}{(1+r)^s} \phi w_{t+s} = \phi A_t + (1-\delta)\phi k_{t-1} - \frac{(1-\delta)\phi k_T}{(1+r)^{T+1}}}$$

(2+) $E_t U'(K_{t+1}) = \frac{1+r}{1+r} U'(K_t)$

$$K_{t+1} = a_0 + a_1 K_t + u_{t+1}$$

$$K_{t+1} = (1-\delta) K_t + C_{t+1}$$

$$(1-\delta) K_t + C_{t-1} = a_0 + a_1 K_t + u_{t+1}$$

$$\boxed{C_{t-1} = a_0 + a_1 K_t - (1-\delta) K_t + u_{t+1}}$$

OR: $\boxed{C_{t-1} = \delta a_0 + a_1 C_t + u_{t+1} - (1-\delta) u_t}$

(1) No Info from K_t can be used to Predict K_{t+1} by which $V(K_t, K_{t+j})$ $\forall t$, Is a $\boxed{\text{RANDOM WALK}}$

(2). Const Quadratic Utility equals to the Representation of an AR(1):

$$K_{t+1} = a_0 + a_1 K_t + U_{t+1}$$

(3)

x_i	(1)	(2)	(3)
		β_i	
Cons.			
C_t	2.65	-4.37	3.48
C_{t-1}	1.002	1.015	1.044
C_{t-2}			0.096
C_{t-3}			-0.086
δ		1.038	
		SE	
C_t	5.04	3.49	5.08
C_{t-1}	0.011	0.011	0.102
C_{t-2}			0.147
C_{t-3}			0.147
δ		0.032	
se	14.8	14.9	14.8
DW	1.89		1.99

$F = 0.8[illegible]$

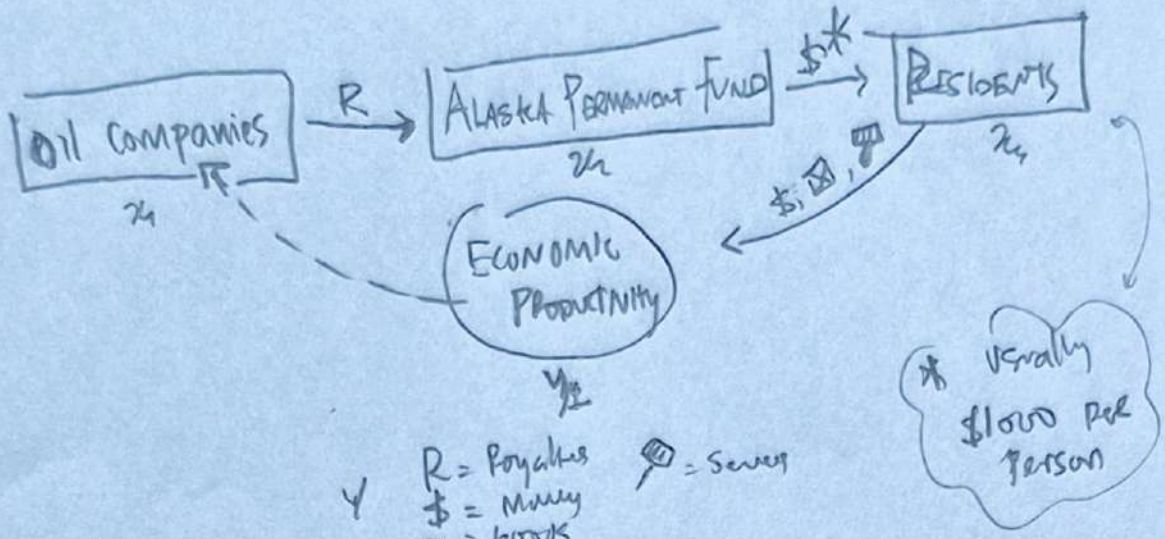
$F^* = 2.70$

(1) Nominal Interest Rates are $\boxed{\text{Predictors}}$ of Consumer Expenditure

(2) Stock Prices $\boxed{\text{Cannot}}$ Predict Consumer Expenditure

HSIEH'S TEST (8.3.2)

(1)



Y R = Royalties 🔨 = Services
$ = Money
⊠ = Goods

(?) PIH Predicts NO INCREASE IN CONSUMPTION since the Arrival of Payments Doesn't Provide Info on PERMANENT INCOME

(?) H_0: CONSUMPTION OF ALASKA >> CONSUMPTION OTHER STATES ?
(1) ALASKA'S WEATHER
(2) ALASKA PERMANENT FUND.

(2)

$$\boxed{\ln\left(\frac{e_{hy}^{Q_4}}{C_{hy}^{Q_3}}\right) = \alpha_0 + \alpha_1\left(\frac{PFD_y \text{ Family size}_{hy}}{Y_{hy}}\right) + \alpha_2 X_{hy} + e_{hy}}$$

X_{hy} = Control Variables.

$$\ln \hat{C}_{FOOD} = 0.0237 + 0.0002\left(\frac{PFD}{Y}\right) + 0.0049 X_{hy}$$
$$(0.0324) \qquad (0.0303)$$

$$\ln \hat{C}_{Entertainment} = 0.0594 + 0.0674\left(\frac{PFD}{Y}\right) + 0.0674 X_{hy}$$
$$(0.0758) \qquad (0.0783)$$

$$\boxed{u'(C_t)\, P^i_t = E_t\left[\sum_{k=1}^{\infty} \frac{1}{(1+\rho)^k}\, u'(C_{t+k})\, D^i_{t+k}\right]} \quad \forall i$$

w/ $\frac{1}{(1+\rho)^k} D^i_{t+k} \triangleq D^i_{t+1}, D^i_{t+2} \ldots$ = PAYOFF STREAMS OR DIVIDEND

P^i_t = PRICE OF AN ASSET.

$$u'(C_t) = \frac{1}{1+\rho} E_t\left[(1+r^i_{t+1})\, u'(C_{t+1})\right] \quad \forall i$$

$$r^i_{t+1} = \frac{D^i_{t+1}}{P^i_t} - 1.$$

$$u'(C_t) = \frac{1}{1+\rho} E_t\left[(1+r^i_{t+1})\right] E\left[u'(C_{t+1})\right]$$

$$= \frac{1}{1+\rho}\left\{E_t\left[1+r^i_{t+1}\right] E\left[u'(C_{t+1})\right] + \mathrm{cov}\left\{1+r^i_{t+1}, u'(C_{t+1})\right\}\right.$$

$$= \frac{1}{1+\rho}\left\{E_t\left[1+r^i_{t+1}\right] E_t\left[u'(C_{t+1})\right] - a\,\mathrm{cov}_t\left(1+r^i_{t+1}, C_{t+1}\right)\right\}$$

w/ $u(C) \triangleq \sum_{c=1}^{n} \frac{c^n}{n!} a^{n-1} \triangleq c a^{1-1} - \frac{1}{2} c^2 a^{2-1}$

$= c - \frac{1}{2} a c^2$

$\partial_c u(C) = 1 - ac$

HENCE:

$$\boxed{u'(C_t) = \frac{1}{1+\rho}\left[E_t\left[1+r_t\right] E_t\left[u'(C_{t+1})\right] - a\,\mathrm{cov}\left(1+r_{t+1}, C_{t+1}\right)\right]}$$

ASSUME $i=1$.

INVESTMENT and the COST of CAPITAL

(1) $\boxed{\Pi(K, X_1, X_2, \ldots X_n) - r_K K}$ = net profit

qR: $R(K, X_1 \ldots X_n) - r_K K$ = Rental price

$X(\cdot)$ = costs & externalities

$R(\cdot)$ = Accounts

$R_{KK} < 0$

$R_K > 0$

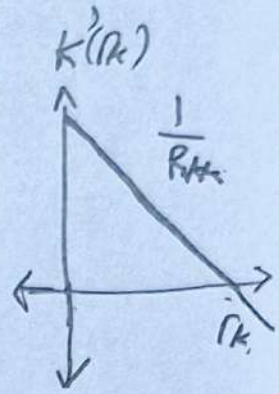
[1] We want maximize.

$$\frac{\partial R(K, X_1 \ldots X_n)}{\partial K} = r_K$$

$$\frac{\partial^2 R(K, X_1 \ldots X_n)}{\partial K^2} = 1$$

$$\frac{\partial R}{\partial K} = r_K \Rightarrow \frac{\partial R}{\partial r_K} = \frac{\partial R}{\partial K}\frac{\partial K}{\partial r_K} = R'_K \frac{\partial K}{\partial r_K} = 1$$

and $R'_K \triangleq R^*_{KK} \frac{\partial K}{\partial r_K} = 1 \Rightarrow \boxed{\frac{\partial K}{\partial r_K} = \frac{1}{R_{KK}}}$



[ii] USER'S COST OF CAPITAL:

Let $r_K(t) = r(t)P_K(t) + \delta P_K(t) - \dot{P}_K(t)$

$$= \left(r(t) + \delta \frac{P_K(t)}{P_K(t)} - \frac{\dot{P}_K(t)}{P_K(t)}\right) P_K(t)$$

So: $\boxed{P_K(t) = (r(t) + \delta - \dot{P}_K / P_K) P_K(t)}$

wh:

$r_t P_K$ = Real cost per unit time w/ r = Interest

δ = depreciation

$\dot{P}$ = "Variable" cost

- If we add taxes:

$$r_K(t) \Big|_{\partial T} \rightarrow \left[r(t) + \delta - \frac{\dot{P}_K(t)}{P_K(t)}\right](1 - f T) P_K$$

wh

T = Marginal corporate Income tax

f = fraction of Capital goods

So: $\boxed{r_K(t) = \left(r_t + \delta - \frac{\dot{P}_K(t)}{P_K(t)}\right)(1 - fT) P_K}$ $\forall$ fraction f for Capital goods $P_K(t)$

[iii] Difficultres

1.) Failings of the model?

a.) Exogenous like Interest rates of The Fed $\Rightarrow \downarrow r_K$.

Since $\dot{K}_t = sK_t - \delta K_t$, a discret $\Delta K_t \propto$ Infinite Investment Rate

b.) Expectations and Investment Demand $\Rightarrow$ Expectations of Demand and costs makes The model to be able Marginal Revenue

2.) Internal Adjustment Cost $\uparrow$ = $\uparrow$ Direct costs changing Capital stock

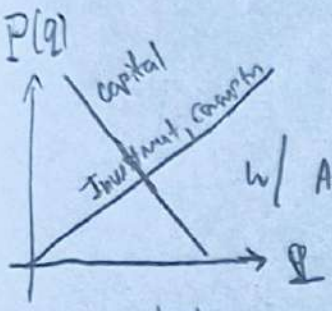
EX: New capital Installment, Training Workers

MODEL OF INVESTMENT W/ ADJ. COST

(9.2) (9.3)

(A.)

$$\boxed{\Pi = \int_0^\infty e^{-rt}\left[\pi(k_t)x_t - I_t - C(I_t)\right]dt}$$



S(q): Investment
D(q): Capital

w/ Π: net profit, π: Profit as capital times, x_t: stock, I: Investment, C: Costs

w/ Assumptions: 1.) (N)idential firms in an Economy
2.) Neglecting Installments and agreement of capital stocks
3.) $\dot{x}_t(t) = I(t)$ 4.) $C(0)=0,\ C'(0)=0,\ C'(\cdot)>0$.

[1] Discrete time version:

1) $$\boxed{\tilde{\Pi} = \sum_{t=0}^{\infty}\frac{1}{(1+r)^t}\left[\pi(k_t)x_t - I_t - C(I_t)\right]}$$

The firm before has $x_t = x_{t-1} + I_t\ \forall t$, so it must project its Revenue of $\tilde{\Pi}$ w/ the constraint. The

$$\mathcal{L} = \tilde{\Pi} - \lambda\hat{x}_t = \sum_{t=0}^{\infty}\left\{\frac{1}{(1+r)^t}\left[\pi_k x_t - I_t - C_{I_t}\right] - \lambda_t\left(x_{t-1} + I_t - x_t\right)\right\}$$

$$\mathcal{L} \rightarrow \boxed{\mathcal{L}' = \sum_{t=0}^{\infty}\frac{1}{(1+r)^t}\left[\pi x_t - I_t - C(I_t) + q_t\left(x_{t-1} + I_t - x_t\right)\right]}$$

w/: $$\frac{\partial \mathcal{L}'}{\partial I_t} = \left[-1 - C'(I_t) + q_t\right]\frac{1}{(1+r)^t} = 0$$

$$\Leftrightarrow \boxed{q_t = 1 + C'(I_t)}$$

unit capital cost = Fixed price + Marginal adj costs.

2) $$\frac{\partial \mathcal{L}'}{\partial x_t} = \frac{1}{(1+r)^t}\cdot\left(\pi(k_t) - q_t\right) = 0$$

$$\frac{\partial \mathcal{L}'}{\partial x_{t+1}} = \frac{1}{(1+r)^{t+1}}\cdot q_{t+1} = 0 \quad \text{So: } +$$

$$\frac{1}{(1+r)^t}\left(\pi(k_t) - q_t\right) + \frac{1}{(1+r)^{t+1}}q_{t+1} = 0$$

$$\frac{(1+r)^{t+1}}{(1+r)^t}\left[\pi(k_t) - q_t\right] + \frac{(1+r)^{t+1}}{(1+r)^{t+1}}q_{t+1} = 0$$

$$(1+r)\left[\pi(k_t) - q_t\right] + q_{t+1} = 0$$

$$(1+r)\pi(k_t) + q_{t+1} = (1+r)q_t$$

$$\boxed{(1+r)\pi(k_t) = (1+r)q_t - q_{t+1}} \Rightarrow$$

Marginal Revenue Product of Capital = opportunity cost of Capital

[a.]

$$\boxed{\pi(k_t) = \frac{1}{1+r}\left(rq_t - \Delta q_t\right)}$$

s.t. $\Delta q_t = q_{t+1} - q_t$

$q_t = q_{t+1} - \Delta q_t$

[b.]

$$\boxed{q_t = \pi(k_t) + \frac{1}{1+r}q_{t+1}}$$

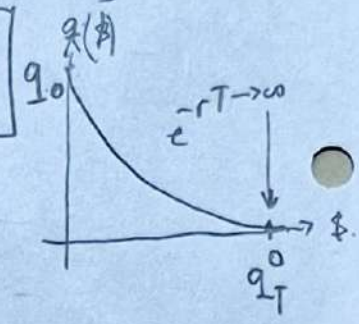
ii Profit Maximizing Behavior (MB)

1.) $$\boxed{MB = \lim_{T\to\infty}\left[\sum_{t=0}^{T-1}\frac{1}{(1+r)^t}\pi(k_t)\right]}$$ w/ MB = Maximizing Behavior

so:
$$q_0 = \pi(k_0) + \frac{1}{1+r}q_1$$
$$q = \pi(k_0) + \frac{1}{1+r}\left[\pi(k_1) + \frac{1}{(1+r)}q_2\right]$$
$$= \pi(k_0 + \frac{1}{1+r}\left[\pi(k_1) + \frac{1}{1+r}\left[\pi(k_2) + \frac{1}{(1+r)}q_3\right]\right]$$
$$= \cdots \lim_{T\to\infty}\left\{\left[\sum_{t=0}^{T-1}\frac{1}{(1+r)^t}\pi(k_t)\right] + \frac{1}{(1+r)^T}q_T\right\}$$

now:
$$\lim_{T\to\infty}\frac{1}{(1+r)^T}q_T = \frac{1}{(1+r)^T}q_t \sim \boxed{e^{-rT=\infty}q_t = 0}$$

so:
$$\boxed{\lim_{T\to\infty}\frac{1}{(1+r)^T}q_T = 0}$$

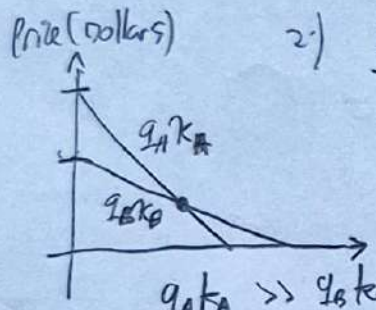


(?) What if it fails? well
Raising Investment $(t=0)$ = marginal cost q_0
Holding Additional Capital forever.
$\therefore$ Firm not maximizing profit

2.) Alternative:
$$\boxed{\lim_{T\to\infty}\frac{1}{(1+r)^T}q_T K_T = 0} \text{ s.t } q_t K_t e^{rT\to\infty} = 0$$

can't be optimal by holding capital.

TRANSVERSALITY CONDITION



$q_A k_A >> q_B k_B$

- since A has a higher Interest s.t Capital depreciates less faster w/ higher Interest
- but if we suppose a Budget $B(\cdot)$, Then $\lim_{T\to\infty} e^{rT}B(\cdot) = 0$ since a higher discount minimizes the Budget as $T\to\infty$.

iii Continuous Case: Current value Hamiltonian w/ $q_t = \mu(\cdot)$

$$\boxed{H(x_t, I_t) = \pi\{k_t\}\cdot x_t - I_t - C(I_t) + q_t I_t}$$
w/ $q(t) = 1 + C'(I_t)$

$$\frac{\partial H}{\partial k_t} = \pi(k_t)$$
$$= rq - \frac{\partial q}{\partial t}$$

so: $$\boxed{\pi(k_t) = rq - \frac{\partial q}{\partial t}}$$

H:
y - control = I_t
x = state = x
$\dot{x}$ = transition
λ = costate = q

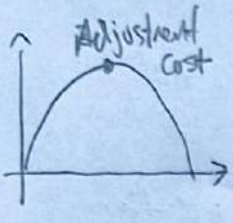
so: $H = f + \mu g \Rightarrow f = \pi(k_t)x_t - I_t - C(I_t)$
$g = I_t$

so: $\partial H / \partial I_t = -1 - C'(I_t) + q_t$ so: $\boxed{q_t = 1 + C'(I_t)}$

TOBIN'S q (9.4) (9.5)

(1) 1) $$\boxed{q(t) = \int_{\tau=t}^{\infty} e^{-r(\tau-t)}\,\tilde{\pi}(k_\tau)\,d\tau}$$ w/ q = Market value of unit capital

q-High = Increase capital stock
q-Low = Reduce capital stock



2.) $$\boxed{\text{TOBIN'S } q = \frac{\text{Market value unit capital}}{\text{Purchase price capital}}}$$

3) $$\frac{\partial q}{\partial RC} = \text{Marginal } q = \frac{\partial q}{\partial MV}\frac{\partial MV}{\partial RC} = \frac{1}{PP}\cdot MV'(\cdot) = \frac{MV'(\cdot)}{PP}$$

and $\boxed{\text{Marginal } q \lesseqgtr \text{Average } q}$

4.)

[i] ANALYZING MODEL:

1) $$\boxed{\dot{k}(t) = f(q(t))}$$ $f(1)=0,\ f'(0)>0$

w/ $C'(I) \perp I$, $f(q) \perp q$

an: $f(q) \equiv NC^{-1}(q-1) = N\frac{1}{C'(q-1)}$

2.) $$\boxed{\frac{dq(t)}{dt} = rq(t) - \pi(k(t))}$$ w/ if $\frac{dk}{dt}$ is $\begin{cases}\text{positive as exceeds } 1\\ \text{negative } q<1\end{cases}$

Solve using Laplace: $\mathcal{L}\{\dot{q}\} = \mathcal{L}\{rq\} - \mathcal{L}\{\pi(k_t)\}$

a.) an'. $\mathcal{L}\{\frac{dq}{dt}\} = sQ(s) - q_0$
$\mathcal{L}\{rq\} = rQ(s)$

Convolution identity: So: $sQ(s) - q_0 = rQ(s) - \mathcal{L}\{\pi\}$

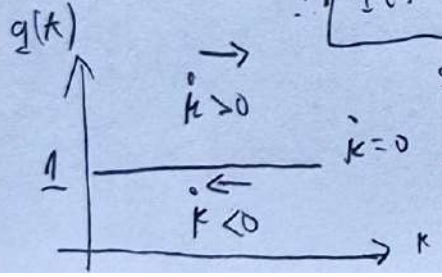
$\Rightarrow sQ(s) - rQ(s) = q_0 + \mathcal{L}(\pi) \Rightarrow \boxed{Q(s) = \frac{q_0 + \mathcal{L}(\pi)}{s-r}}$

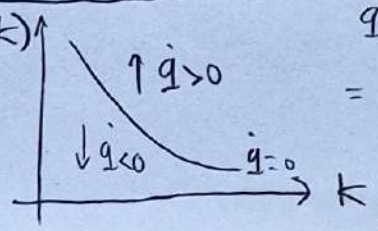
b.) Then: $q(t) = q_0 e^{rt} - \mathcal{L}^{-1}\left\{\frac{\mathcal{L}\{\pi(k_t)\}}{s-r}\right\}$

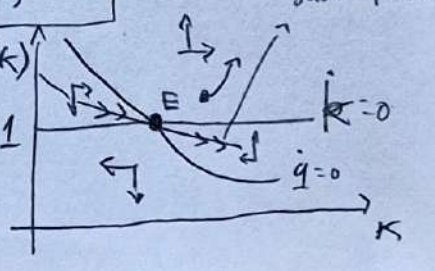
$$q(t) = q_0 e^{rt} - \int_0^t e^{r(t-\tau)}\pi(k_\tau)\,d\tau$$

[ii] GRAPH:

$$\boxed{q(t) = e^{rt}\left(q_0 - \int_0^t e^{-\tau}\pi(k_\tau)\right)d\tau}$$

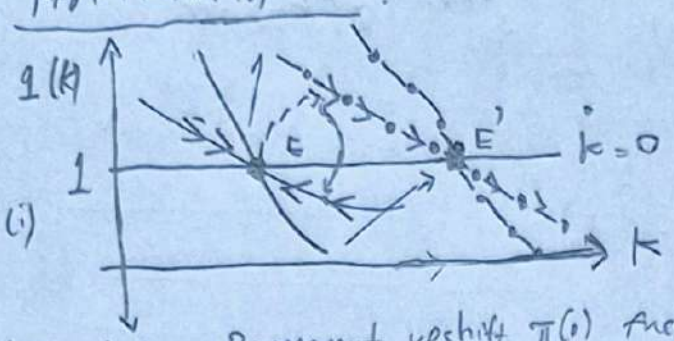


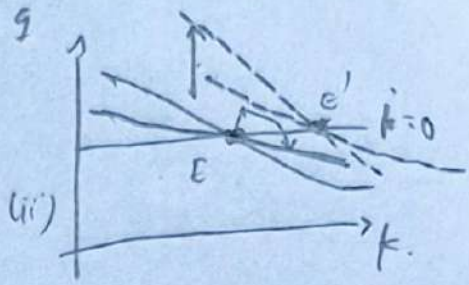




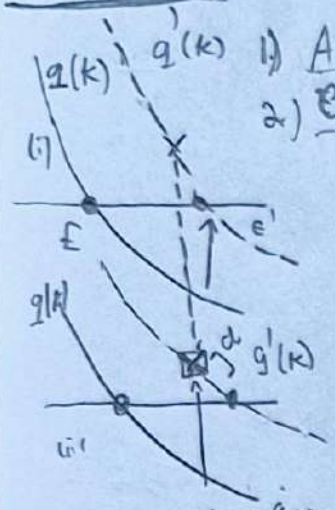
② IMPLICATIONS

[i] Profitability





Zooming [i]



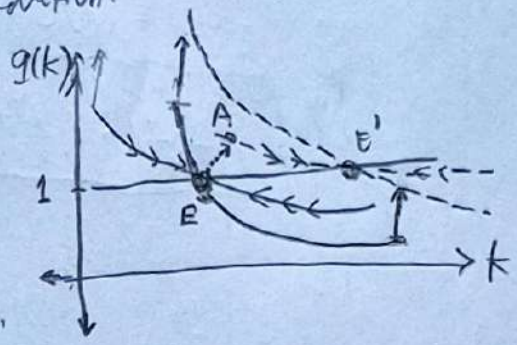
*(i) Permanent (d=0)
(ii) Temporary (d>0)

1) Assumption = Permanent upshift $\pi(\cdot)$ function.

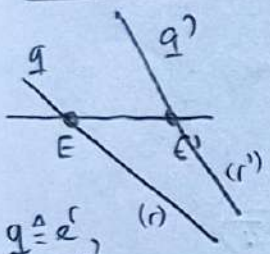
2) Consequence =
- Shift $\pi(\cdot)$ = Shift Tech = Shift product
- No Anticipated jumps in q
- At (T), (k, q) must be on the saddle path.

3) Effect =
- Firms increase capital stocks
- Increase capital stocks = $\Delta \dot{q}$ Declines.
- Since q determines Investment, Investment Responses less.
- Fall in firm Production price due to higher Production.

[ii] INTEREST RATES



Zooming [ii]



$q \triangleq e^r$
$q' \triangleq e^{r'}$
$e^r > e^{r'}$

1) Assumption:
- Interests = shifts profit
- shifts profit = Investment.

2) Consequence:
- q jumps to a new saddle path
- ~~Higher~~ Interests = Temporary Boom in Investment
  Investment BOOM = permanent Higher Capital Stock.

3) EFFECT:
- Long Term < Short Term = short term fall
- short term fall = No Value in Long Term Bonds!

[iii] TAX CREDITS

Problems 8

(1.) BASIC FIRM MECHANICS

- Firm uses a Cobb-Douglas Combination of Capital / Labour or $\boxed{Y = K^{\alpha} L^{1-\alpha}}$, $0 < \alpha < 1$.
- Output Firm price is fixed in the short run, thus it takes both price of its product p and quantity Y as given.
- Input markets are competitive, thus firms take W wage and r Rental.

(a.) Firm's choice of L given p, Y, w, K?

Ans: $\min_L \{ C = wL + rK \}$ s.t $\{ Y = K^{\alpha} L^{1-\alpha} \}$

So: $Y = K^{\alpha} L^{1-\alpha} \rightarrow \frac{Y}{K^{\alpha}} = L^{1-\alpha} \rightarrow L^{\frac{1-\alpha}{1-\alpha}} = \left(\frac{Y}{K^{\alpha}}\right)^{\frac{1}{1-\alpha}}$

and: $\boxed{L = \frac{Y^{\frac{1}{1-\alpha}}}{K^{\frac{\alpha}{1-\alpha}}}}$; $\boxed{C = \left(\frac{Y}{K^{\alpha}}\right)^{\frac{1}{1-\alpha}} w + rK}$

(b.) Profit Function of p, Y, w, K?

Ans: $\boxed{\pi(p, Y, w, K) = R(p, Y, w, K) - C(p, Y, w, K).}$

So: $R = pY$ and $C = wL + rK$

So: $\boxed{\pi = pY - wL - rK}$

(c.) Solve 1st order condition for K:

Ans: i) $\max_K \{ \pi = pY - w\left(\frac{Y}{K^{\alpha}}\right)^{\frac{1}{1-\alpha}} + rK \}$

$-\alpha - 1 - \alpha$
$= -1 - \alpha - \alpha$
$= -1 - 2\alpha$

and: $\frac{\partial \pi}{\partial K} = -wY^{\frac{1}{1-\alpha}} \frac{\partial}{\partial K}\left(K^{-\frac{\alpha}{1-\alpha}}\right) + r = 0$

$= -wY^{\frac{1}{1-\alpha}} - \alpha \frac{1}{1-\alpha} K^{-\frac{\alpha}{1-\alpha} - 1} = -wY^{\frac{1}{1-\alpha}} \frac{\alpha}{1-\alpha} K^{\frac{-\alpha-1+\alpha}{1-\alpha}} = -wY^{\frac{1}{1-\alpha}} \frac{\alpha}{1-\alpha} K^{-1} = r$

$y(K) = \left(\frac{Y}{K^{\alpha}}\right)^{\frac{1}{1-\alpha}} = Y^{\frac{1}{1-\alpha}} K^{-\frac{\alpha}{1-\alpha}}$

and: $\boxed{r = -wY^{\frac{1}{1-\alpha}} \times \frac{\alpha}{1-\alpha} K^{\frac{-1}{1-\alpha}}.}$

- Higher wages = Labour expensive
  So: Capital is substitute
- Capital more expensive = $r\uparrow$
  So: Capital less
- $Y\uparrow \rightarrow K\uparrow$ = higher output

ii) Suppose $\alpha = \frac{1}{3}$:

$r = -w^{\frac{1}{1-\frac{1}{3}}} \times \frac{\frac{1}{3}}{1-\frac{1}{3}} K^{-\frac{1}{1-\frac{1}{3}}} = -wY^{\frac{3}{2}} \cdot \frac{1}{3} \cdot \frac{3}{2} K^{-\frac{3}{2}}$

$= -wY^{\frac{3}{2}} \cdot \frac{1}{2} K^{-\frac{3}{2}} = -\frac{1}{2} w Y^{\frac{3}{2}} K^{-\frac{3}{2}} = -\frac{1}{2} w \left(\frac{Y}{K}\right)^{\frac{3}{2}}$

So $\boxed{r\left(\alpha = \frac{1}{3}\right) = \frac{1}{2} w \left(\frac{Y}{K}\right)^{\frac{3}{2}}}$

(3) STRAIGHT LINE DEPRECIATION

- Corporations in US are allowed to subtract depreciation allowances from taxable income.
- Depreciation = Purchase of capital. Purch. capital at t an deduct $D(s)$ at the purch. price from taxable income $t+s$.
- Depreciation takes form $D(s) = \frac{1}{T}$ $s \in (0, T)$ $s > T$: T = tax life

(a) Assume straight line, and marginal corporate tax is τ at T w/ interest i, ask: How much does purchasing a unit capital at price P_K reduce the present value of firm's corporate tax liabilities as a function of T, τ, i, P_K?

Ans: 1.) Firm buys capital P_K s.t. $\frac{1}{T}$ depreciation so $\tau = \tau\left(\frac{P_K}{T}\right)$.

2.) Firm when buying capital, buys w/ present value: $X = \int e^{-rt} k \, dt$.

So: $$X = \int_{s=t}^{s=t+T} e^{i(s-t)} \tau\left(\frac{P}{T}\right) ds.$$

3.) $$X = \tau \int_{s=t}^{s=t+T} e^{i(s-t)} ds = \tau \int_{s=t}^{s=t+T} e^{is-it} ds = \tau \left. \frac{e^{is-it}}{i} \right|_{t}^{t+T}$$

$$= \tau\left\{\frac{e^{i(t+T)-it}}{i} - \frac{e^{it-it}}{i}\right\} = \tau\left\{\frac{e^{it+iT-it}}{i} - \frac{1}{i}\right\} = \tau\left\{\frac{e^{iT}-1}{i}\right\}$$

So: $$= \boxed{X = \tau\left(\frac{P_K}{T}\right)\left(\frac{e^{iT}-1}{i}\right)} \rightarrow \boxed{P_K^{AT} = P_K\left[1 - \left(\frac{\tau}{T}\right)\left(\frac{1-e^{iT}}{i}\right)\right]}$$

(b) Suppose that $i = r + \pi$, and π increases w/ no Δr. How does this effect the after price.

Tax shield

Ans: 1.) $X = \tau\left(\frac{P_K}{T}\right) \frac{\exp((r+\pi)T) - 1}{i}$ let $\pi \rightarrow \pi + \Delta\pi$. Then:

2.) $$X_{\Delta\pi} = \tau\left(\frac{P_K}{T}\right) \frac{e^{(r+\pi+\Delta\pi)T} - 1}{i}$$

So: $$P_K^{AT}\Big|_{\Delta\pi} = P_K\left(1 - \frac{\tau}{T}\left\{\frac{1 - e^{(r+\pi+\Delta\pi)T}}{i}\right\}\right)$$

So Inflation increases the user's $\boxed{\text{Cost of Capital}}$ such that the change in nominal interest rate increases the discount factor of the firm's discount!

Meaning that $c(P_K^{AT}) < c\left(P_K^{AT}\Big|_{\Delta\pi}\right)$ s.t.

$$(r+\pi)\cancel{T} < (r+\pi+\Delta\pi)\cancel{T}$$

$$\cancel{r+\pi} + \Delta\pi > \cancel{r+\pi} \rightarrow \boxed{\Delta\pi > 0} \checkmark$$

So: $$\boxed{\pi_{nf}\uparrow = \text{PV Tax shield} \downarrow = P_K\uparrow}$$

Exercise 9

TRANSVERSALITY CONDITION

(4) TRANSVERSALITY CONDITION

- Individual wants to maximize $G : \int_0^\infty e^{-\rho t}\left[-\frac{1}{2}aG(t)^2\right] dt$ at $a>0$, $\rho>0$
- Here, $G(t)$ is the amount of quarterly the investment costs, we can allow $G(t) \in [-\infty, \infty]$.
- Individual's creation or savings will affect the stock of capital T, evolving at $T(0)=0$, $\dot{T}(t)=G(t)$.

(a) Show that utility max path is $G_t = 0 \ \forall t$.

Ans: 1) Obviously if $G=0$, then if

$$X = \int_0^\infty e^{-\rho t}\left[-\frac{1}{2}aG^2\right] dt$$

then $X=0$. and if $\dot{T}(t) = G(t)$. Then, Euler-Lagrange eq

$$\boxed{L = ae^{-\rho t}\dot{T}}$$

1) $$\boxed{\begin{aligned} X &= \int_0^\infty e^{-\rho t}\left[-\frac{1}{2}\dot{T}(t)^2\right] dt : \\ \delta X &= \int_0^\infty L(T, \dot{T}, t) = 0. \end{aligned}}$$

2) $$\frac{\partial L}{\partial T} = \frac{d}{dt}\frac{\partial L}{\partial \dot{T}} \Rightarrow \frac{\partial L}{\partial \dot{T}} = 0, \quad \frac{\partial L}{\partial \dot{T}} = -\int e^{-\rho t}\dot{T}(t)\, dt$$

so: $$\frac{d}{dt}\frac{\partial L}{\partial \dot{T}} = \frac{d}{dt}\left\{-\int e^{-\rho t}\frac{dT}{dt}\right\} dt = -e^{-\rho t}\frac{dT}{dt}$$ so:

3) $$\frac{d}{dt}\frac{\partial L}{\partial \dot{T}} - \frac{\partial L}{\partial T} = -e^{-\rho t}\frac{dT}{dt} - 0 = 0 \quad \text{s.t.} \quad -e^{-\rho t}\frac{dT}{dt} = 0$$

since $\dot{T} = G(t) \Rightarrow -e^{\rho t}G(t) = 0 \Rightarrow \therefore \boxed{G(t) = 0}$ ■

(b) more formal:

1) $$\frac{d}{dt}\left(-aTe^{-\rho t}\right) = -a\ddot{T}(t)e^{-\rho t} + a\rho\dot{T}e^{-\rho t} = 0$$

so: $$\cancel{a\rho\dot{T}e^{-\rho t}} = \cancel{a\ddot{T}e^{-\rho t}}$$

so: $$\boxed{\ddot{T} = \rho\dot{T}}$$

$$A(C) = -\frac{1}{2}aC^2$$

2) $$\ddot{T} - \rho\dot{T} = \dot{G}(t) - \rho G(t) = 0 \Rightarrow \frac{dG}{dt} = \rho G(t) \Rightarrow G(t) = Ce^{\rho t}.$$

3) $$-\frac{1}{2}aG(t)^2 \Rightarrow -\frac{1}{2}aC^2e^{-\rho t}e^{2\rho t} = -\frac{1}{2}aC^2e^{\rho t} = 0. \text{ Since } \boxed{\frac{\partial A(C)}{\partial t} = 0}$$ ■

(b) Use Hamiltonian approach

$$\boxed{H(x, \dot{x}, y, \mu) = f(x, \dot{x}, y) + \mu g(\dot{x}, x, y)}$$

1) Set: $\mu(t) = Le^{-\rho t}$ and $G(t) := C$ (constant), $T(t) =$ state. So:

$$\boxed{\mathcal{H}_c = -\frac{a}{2}G + \mu G}$$

2) $\frac{\partial H_c}{\partial \sigma} = -a\sigma(t) + \mu$. so $\boxed{\mu = +a\sigma(t)}$

3.) $\dot{\mu} = -\rho\mu - \frac{\partial H_c}{\partial T} = \underbrace{\rho\mu(t)}_{\forall T \in \mathbb{R}} \Rightarrow \mu(t)\rho = \dot{\mu}$ $\boxed{\mu = \mu(0)e^{\rho t}}$

4). $\dot{T} = \sigma = \frac{\mu}{a} = \frac{\mu_0 e^{\rho t}}{a}$

$\frac{dT}{dt} = \frac{\mu_0 e^{\rho t}}{a} \Rightarrow T(t) = \int_0^t \frac{\mu_0}{a} e^{\rho t} = \frac{1}{a\rho}\mu_0 e^{\rho t} - 1$

so: $\boxed{T(t) = \frac{1}{a\rho}\mu_0 e^{\rho t} - 1}$

5.) The Transversality condition:

$\boxed{\lim_{t\to\infty} e^{-\rho t}\mu(t)T(t) = 0}$

so: $\lim_{t\to\infty} e^{-\rho t}\mu T = \lim_{t\to\infty} e^{-\rho t}(\mu_0 e^{\rho t})\times\left(\frac{1}{a\rho}\mu_0 e^{\rho t} - 1\right)$

$= \mu_0^2 e^{-\rho t} e^{\rho t} e^{\rho t}\frac{1}{a\rho} - \mu_0 e^{\rho t} = \mu_0^2 e^{\rho t}\frac{1}{a\rho} - \mu_0 e^{\rho t}$

$\Rightarrow: \left(\mu_0^2\frac{1}{a\rho} - \mu_0\right)e^{\rho t} = 0$

only zero iff $\mu_0 = 0$. otherwise ∞!

so: $\boxed{\sigma(t) = 0}$!

EXERCISE 8.9

(9.) Housing MARKET MODEL:

- Let H = housing stock, I = Investment Rate, P_H = Real price Housing, R = rent
- Assume I is Increasing in P_H, so $\boxed{I = I(P_H)}$ w/ $I'(\cdot) > 0$ ...
- $\boxed{\dot{H} = I - \delta H}$
- Assume a decreasing Func. H: $R = R(H)$, $R'(\cdot) < 0$
- Assume Income + Capital gains = Returns $\Rightarrow r = \frac{(R + \dot{P}_H)}{P_H}$

(a.) Sketch (H, P_H), $\dot{H} = 0$ in a phase diagram.

(b.) Dynamics of (H, P_H)? Sketch saddle path

(c.) Suppose Market is initially in long run equilibrium and that it Becomes known that $\exists$ (Permanent Increase in r in the T)

(d.) What happens to (H, P_H) at the time of the news.

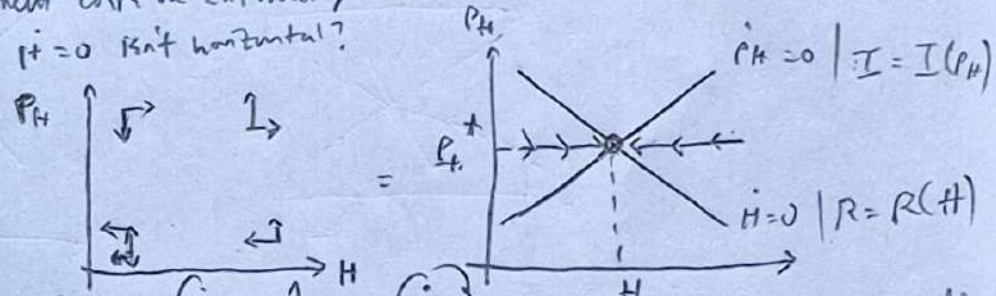
- What happens to H, P_H, I, R?

(e.) Adjustment Costs are external / Internal?

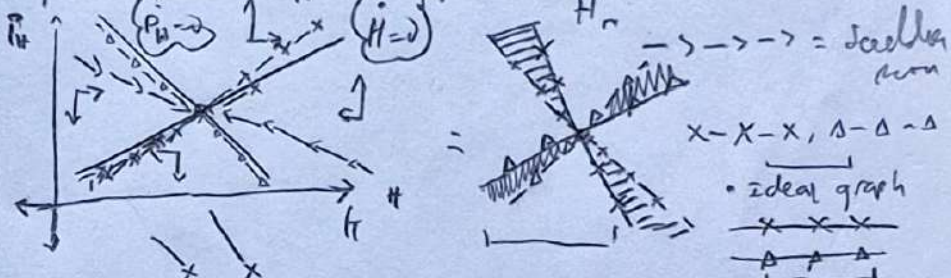
(f.) Why $\dot{P}_H = 0$ isn't horizontal?

Ans: $\boxed{1}$

1)

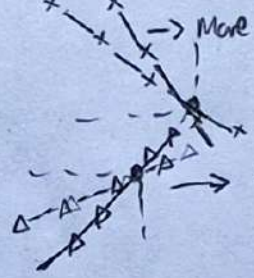


2)



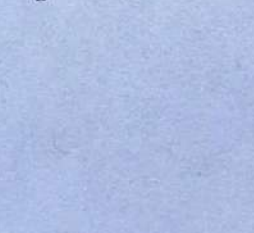
a) (X–X–X) → (XXX)
The graph reaches a Saddle part such that it "Reaches it's Stable equilibrium".

a)



→ More steeper =

b)



→ More steeper = supply Manage. Approach

MODEL FOR PERFECT MARKETS & ASSUMPTIONS

(10.1) (10.2)

$$\boxed{V = \sum_j^{\infty} \beta^j (c_{t+j})}$$

$u(c_t)$ $\beta h(c_t)$ $= \beta^j u(c_{t+j})$

(1) MODEL FOR PERFECT MARKETS:

[i] Assumptions & EQUILIBRIUM:

1) $\boxed{V = U(c_1) + \beta U(c_2)}$ $\beta > 0$ $U'(w) > 0$, $U''(w) < 0$

a) $\begin{cases} \text{There are } N \text{ possible investment projects} \\ \text{There are } S \text{ - possible states of the world} \\ \text{All present 2 output cases for period 1 Input} \end{cases}$

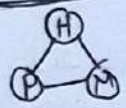
$\exists N, I_1, \dots I_{n_2}$
$\exists S, S_1(I_{n_1}) \dots S_j(I_{n_2})$
$S_2(I_{n_1}) \dots S_i(I_{n_2})$
$\forall C_2, E[C_2 | C_1] = C_1$

b) If Quantity K_i of Revenues 1 is derivative to project i
It produces $R_{is} K_i$ in period 2 state S. $(R_{is} \geq 0)$

c) $\sum_{s=1}^{S} \pi_s = 1$ set $K \in \mathbb{R}^+$

d. $\boxed{\text{ARROW - DEBREU} = [\{K_i\}, \{q_s\}, \{C_2^s\}, C_1]^T}$

- WALRASIAN AUCTIONEER
- Produces an transform Commodities into Bundles
- Market: $\begin{bmatrix} p_1^M \\ \vdots \\ p_n^M \end{bmatrix}$ only choose Market price p_i^M vector
- (?) Market has no utility or PROFIT. So → Markets Decide to p_i^M to Ready WALRASIAN AUCTIONEER
- ARROW-DEBREU



- Households Posses Endowments or Interest
- Endowments are sold in Market st
  - Working Hours
  - Land use
  - Tons of Corn
- Households can be Thought as JSC
- Profit Producer is divided on Households: $\frac{1}{h}$

2) Maximizing Budget Constraint

$$\boxed{e_1 + \sum_{s=1}^{S} q_s e_2^s = E} \quad \frac{\partial E}{\partial e_1} = 1 \quad \frac{\partial E}{\partial e_2^s} = \sum_{s=1}^{S} q_s$$

b. $\nabla \cdot E = \frac{\partial E}{\partial e_1} \hat{i} + \frac{\partial E}{\partial C_2^s} \hat{j} = 1 + \sum_{s=1}^{S} q_s \rightarrow \begin{bmatrix} 1 \\ q_s \end{bmatrix}$ is a Consumption Gradient Vector

3) $\frac{1}{\beta} \frac{U'(C_1)}{U''(C_2^s)} = \frac{1}{q_s} \pi_s \Rightarrow \boxed{q_s = \frac{U'(C_2^s)}{U''(C_1)} \beta \pi_s}$. equilibrium of a price of a claim an unit in state S = probability state occurs

4) $\sum_{s=1}^{S} q_s R_{is} = \begin{cases} =1 & \text{if } K_i > 0 \\ \leq 1 & \text{if } K_i = 0 \end{cases} \forall i$

wh/ payoff is the revenue from selling a state contingent contract is $\sum_{s=1}^{S} q_s R_{is}$.

5) $\boxed{C_1 + \sum_{i=1}^{n} K_i = E}$ MARKET Clearing (1) $\boxed{\sum_{i=1}^{n} K_i R_s = C_2^s}$ market Clear 2.

→ $u(c_i) = \frac{c_i^{1-\gamma}}{1-\gamma}$ Follow CRRA type Preferences.

[ii] Discussion:

1) Household Perspective = Partial Equilibrium as Consumption (Ricardian Agents)

2) Riskfree Assets:
- Debt : Riskless Debt pays $\sum_{s=1}^{S} 1 \cdot q_s = \sum_{s=1}^{S} q_s$
- Insurance : Insurance always costs $q_s = \begin{cases} 1 \\ 0 \end{cases}$.

wh/ $\boxed{\text{Riskfree Assets} = \sum_{s=1}^{S} q_s x_s}$

3) Financial Assets ∈ Economy ⟹ Walrasian outcome = Arrow-Debreu Allocation
Arrow-Debreu Allocation = Equity Markets = trade on Risk assets.

(2) AGENCY COSTS AND THE FINANCIAL ACCELERATOR

[i] INTERMEDIATION

1) Firm Information >> Intermediate Info = $\boxed{\Omega_F >> \Omega_I}$
2) Informational Mediator = Banks, Mutual Funds, Bond Rating Agencies

3.) But these Institutions also don't convey that specific information of a client

Ex: Ω_{cc} (Banks | Depositors): The Bank doesn't have that much information on the Colvents Balance sheet Dispersions.

4.) Asymmetric Information ($\Omega_i \neq \Omega_j$) = Agency problems.
Agency problems = Distortion of Investment provides

Lem. $\boxed{(\Omega_i \neq \Omega_j) \longrightarrow \therefore \text{Most efficient project?}}$

II Assumptions:

1) Basic Notations.

- W = Entrepreneur wealth
- $1-W$ = Entrepreneur "Demand"
- r = Rate of Return Expectation
- r = Rate of Investment.

Then: $\boxed{\forall (W, 1-W) \xrightarrow{\Omega, \Pi} r \gg 1+r}$

{ For all Entrepreneur's wealths and Demand, The Entrepreneur's Information (Ω) and Investment Seeking (Π) must Bring $r \gg 1+r$ }

2.) The Entrepreneurs (Π) = Invest at Risk Free ~~Face~~ Rate
Invest at Risk Free Rate = Undertakes Project if $r \wedge E_t(\beta^j x_{t+j}) \ggg (1+r) W$

Lem: $\boxed{W_{\Pi,\Omega} : E_t(y_t - r' x_{t+j}) \ggg (1+r_t) W(y_t, x_t)}$

{ The wealth of an Entrepreneurs via Investment Seeking and Info (Π, Ω) is Done iff Expectation of Net output and Payment, Exceeds interest costs }

3.) ⚠ $\boxed{\text{KEY ASSUMPTION}}$ = ENTREPENEURS ARE BETTER INFORMED

III Equilibrium under Symmetric Information.

$\boxed{(1-W)(1+r)} \xrightarrow{\text{by}} \boxed{\frac{(1-W)(1+r)}{r}} \xrightarrow{\text{by}}$

Expected payments Providing of Entre.

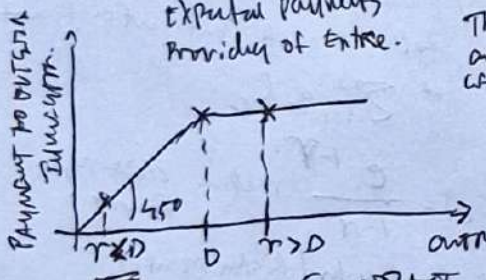
There's many contracts. an Example is the Fraction of payments to mean (3)

$(1-W)(1+r) = r$
$(1-W)(1+r) - r$
$(1-W) + (1-W)r - r$
$W(1+r) - (1+W) - r$ - an let $\beta^{-1} = (1+r)$
$\boxed{W\beta - \beta - r = 0}$

{ ALGEBRA }

Given $r \gg 1+r = \beta^{-1}$ By Assumption
Then: if $W(1)^{-1} - (1)^{-1} = 2 = W - 1 - 2$
$= W - 2 = 0$
$= \boxed{W = 2}$

Then will take the Invs



optimal contract is
∴ The minimization of ~~Fraction~~ of rec.
That the Investors carry out whilst planning contract Investor the Required Rate of return.

IV Form of Contract

1) $r > D$ = Pays D
$r < D$ =

2) $\boxed{\min_{\Omega} \Omega(r_t, Q_t, W_t) = (1-W_t) Q_t - r_t}$ by: $\frac{d r_t}{dt} = 0$

$-\frac{dW_t}{dt} Q_t - \frac{dQ_t}{dt} + \frac{dQ_t}{dt} W_t = \frac{dr_t}{dt}$

Lem. $\frac{d\Omega}{dt} = \left(\frac{d(1)}{dt} - \frac{dW_t}{dt}\right) Q_t + \frac{dQ_t}{dt}(1-W_t) - \frac{dr_t}{dt}$

$= -(\Delta W_t Q_t - \Delta Q_t - \Delta Q_t W_t - \Delta r_t) = -W_t Q_t - Q_t W_t + Q_t = r_t$

$Q_t(1 - 2W_t) = r_t \rightarrow \boxed{Q_t = \frac{r_t}{1-2W_t}}$ by if $r = 2$, $Q = 1$ $\rightarrow 1 - 2W_t = 2$, $2W_t = -1$, $\boxed{W_t = \frac{1}{2}}$

by: $W_t = \frac{1}{2}$, $1 - W_t = \frac{1}{2}$ cut $\boxed{W_t + (1-W_t) = 1}$ as $r \gg Q$.

If $Q_t \ll r$
then $Q_t - r < 0$

$\boxed{Q_t - r < 0 < \frac{r_t - r_t(1-2W_t)}{1-2W_t}}$

3) So The minimization is $\boxed{Q - r < 0 < \frac{r - r(1-2W)}{1-2W}}$ ∴ If The Net Return is Bigger than wealth Transfer

ENTREPRENEUR'S EQUILIBRIUM

(1) ANALYSIS COSTS & FINANCIAL ACCELERATOR

EQUILIBRIUM w/ D

PROBLEMATIC:

If we have:

$$\boxed{\Omega = (1-W_t)Q_t - r_t}$$ w/ W_t = wages, Q_t = output and r_t = return

$$-W_tQ_t + 2W_tQ_t - r_t = -W_tQ_t - r_t$$

2.) and if $\max_t \Omega = (1-W_t)Q_t - r_t \rightarrow -(\Delta W_tQ_t + \Delta Q_tW_t) - \Delta r_t = \Delta\Omega$.

$$-\Delta r_t - \Delta Q_t + \Delta W_tQ_t + \Delta Q_tW_t = \frac{d(1-W_t)}{dt}Q_t + \frac{dQ_t}{dt}(1-W_t) - r_t$$

$$= -\frac{dW_t}{dt}Q_t + \frac{dQ_t}{dt} - \frac{dQ_t}{dt}W_t - \frac{dr_t}{dt} \overset{\Delta}{=} -(\Delta W_tQ_t) + \Delta Q_t - \Delta Q_tW_t - \Delta r_t$$

$$= -\Delta W_tQ_t + \Delta Q_t(1-W_t) - \Delta r_t = 0.$$

$$\Delta Q_t(1-W_t) = \Delta r_t + \Delta W_tQ_t \quad \text{and:} \quad \Delta Q_t(1-W_t) - \Delta W_tQ_t = \Delta r_t$$

$$(1-W_t) = \frac{\Delta r_t + \Delta W_tQ_t}{\Delta Q_t}$$

$$\text{or} \quad (1-W_t) = \frac{dr_t + dW_tQ_t}{dQ} = \frac{dr}{dQ} + \frac{dWQ}{dQ} = \frac{dr}{dQ} + \frac{dW}{dQ}Q.$$] HEURISTIC

$$\text{Thus:} \quad \boxed{1 - W_t = \frac{dr}{dQ} + \frac{dW}{dQ}Q}$$

NOT ~~PROBLEMATIC~~, HEURISTIC, ABSURD

REVISION:

1.) Suppose that $D < 2r$. Then output y_T is $y_T \lessgtr D$. If $y_T > D$, then the investor doesn't pay costs s.t. it uniformly distributed in $[0, 2r]$

$$y_1(D) = \frac{y_T}{\underbrace{b-a}_{\text{uniform distribution}}} = \frac{y_T}{2r-D} \rightarrow y_T\left(1 - \frac{D}{2r}\right) = y_T\left(\frac{2r-D}{2r}\right)$$

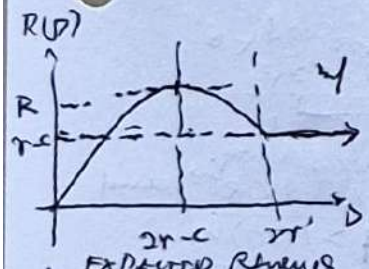
$$\therefore \boxed{y_1 = \frac{2r-D}{2r}y_T}$$

2.) On the other hand if output is $D \geq 2r$ then the need for insurance $\boxed{\frac{1}{2}D - C \geq 0}$

w/ Executive cost c, s.t.

$$y_2(D) = \frac{D}{2r}E(c) = \frac{D}{2r}\left(P_1y_T + P_2C\right) = \frac{D}{2r}\left(\frac{1}{2}y_T - C\right)$$

$$\therefore \boxed{y_2(D) = \frac{D}{2r}\left(\frac{1}{2}y_T - C\right)} \text{ or } \frac{D}{2r}\left(\frac{y_T - 2C}{2}\right) \text{ w/ } r.$$

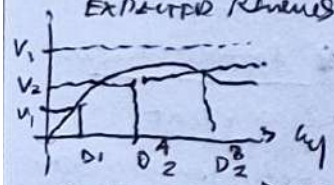


$\Rightarrow$ The total payouts is $\boxed{R = y_1 + y_2}$ $\forall$ $D \leq 2r$ s.t.

$$R(D) = y_1 + y_2 = \frac{D}{2r}\left(\frac{1}{2}y_T - C\right) + \frac{2r-D}{2r}y_T \text{ and if } y_T \equiv D \text{ then}$$

$$= \frac{D}{2r}\left(\frac{1}{2}y_T - C\right) + \frac{2r-D}{2r}D \quad \text{and } R(D)\big|_{D>2r} = r - C, \text{ so:}$$

$$\boxed{R(D) = \begin{cases} \frac{2r-D}{2r}D + \frac{D}{2r}\left(\frac{D}{2} - C\right) & \text{if } D \leq 2r \\ r - C & \text{if } D > 2r. \end{cases}}$$



$$\text{w/ } \frac{dR}{dD} = 1 - \left[\frac{C}{2r}\right] - \left[\frac{D}{2r}\right] = 0 \text{ s.t. } 1 = \frac{C}{2r} + \frac{D}{2r} = \frac{C+D}{2r}$$

$$2r = C + D \rightarrow \boxed{D = 2r - C} \rightarrow$$

- Dividends are Rising until $D = 2r - C$ then decreases
- When the investors Receive compensat. The net amount receive is also less than $D = 2r - C$

$$\text{So } R^{max}(D = 2r - C) = \left(\frac{2r-C}{2r}\right)^2 r \equiv R^{max}$$

$$\text{Thus } \boxed{\left(\frac{2r-C}{2r}r\right) = R^{max}} \rightarrow$$ Thus the Maximum Expected Renewal equal Expected output is when Costs zero.

and further decreases to no $r - C$

CREDIT RATIONING

- The Required Net Revenue is $(1+r)(1+w)$.
- If the Required $(R_\Pi = V_1 > r-c) = 0$, then $\exists D$, $D := E[R_\Pi]$.
- If the Revenue Exceed R^{MAX} w/ $(R_\Pi = V_3)$, then $\nexists D$, $D := E(R_\Pi)$

Thus this where they credit ration.

So:
$$\boxed{\begin{array}{l} R = V_1 \longrightarrow \exists D(D = E(R)) : \text{use Reserve cash} \\ R = V_3 \longrightarrow \nexists D(D = E(R)) : \text{use credit rationing} \end{array}}$$

IN BETWEEN

$(R^{MAX} \wedge r-c) := R_2 = V_2$:

- In between R^{MAX} yields either (D_A^2, D_B^2) but $R(D) = V_2$.
- If an Investor is making a loan to an Entrepreneur w/ D_B^2 requirement, Then other INVESTORS can make loan w/ more favorable terms.
- Competition drives $(D \longrightarrow D^A)$ such that

$R(D) = (1+r)(1+w)$; and

$$\boxed{D^* = 2r - c - \sqrt{(2r-c)^2 - 4r(1+r)(1+w)}} \text{ for } \boxed{(1+r)(1+w) \le R^{max}}$$

VI EQUILIBRIUM INVESTMENT

OBTAIN FINANCING

(?) When does he/she take the project?

- They want to obtain some financing at some interest

MATH SETUP

- Entrepreneur obtains $(1+r)W$ when investing in ASSETS (safe)
- If take, then $E(D) - r$.
- If financing, then $(1+r)(1-w) - E(C)$.

$\therefore$ So: $\boxed{\begin{array}{l} E(D) - r = \text{No financing} \\ (1+r)(1+w) - E(C) = \text{Yes financing} \end{array}}$

THE MATH II:

If the investor's verification cost

$$A = \frac{D^*}{2r} c = \frac{1}{2r} c \left[2r - c - \sqrt{(2r-c)^2 - 4r(1+r)(1-w)} \right] = c\left[\frac{2r-c}{2r} - \sqrt{\left(\frac{2r-c}{2r}\right)^2 - \frac{(1+r)(1+w)}{r}}\right]$$

So: $$\boxed{A = \frac{D^* c}{2r} = \left[\frac{2r-c}{2r} - \sqrt{\left(\frac{2r-c}{2r}\right)^2 - \frac{(1+r)(1+w)}{r}}\right] c.}$$

WILLINGNESS

- Investors will undertake the project if a loan is available
- If W is very small, the investors will take it even if R is for interest.

1) If we differentiate prev. w/ $A = A(c, r, W, r)$ w/ $\underbrace{\frac{\partial A}{\partial c} > 0, \frac{\partial A}{\partial r} > 0}_{\text{cost \& return are high}}$, $\underbrace{\frac{\partial A}{\partial W} < 0 \quad \frac{\partial A}{\partial r} < 0}_{\text{but wealth and payout is decreasing}}$

2) Expected payments is $\boxed{(1-w)(1+r) + A(c, r, W, r)}$

w/ Then at $(1+r)(1-w) \le R^{max}$ and

$$\boxed{r - (1+r)(1+W) - A > (1+r)W}$$

VII DISCUSSION

MORAL HAZARD and ASSYMETRIC RISK

- Suppose there's assymetric information on how much risk the entrepreneur is willing to take.
- Then the investor bears the cost of poor outcomes
- Assymetry of information = Moral Hazard and can decrease both income and

$\therefore$ So $\boxed{\text{MORAL HAZARD} = \ominus \text{ INVESTOR} + \ominus \text{ ENTRE} \text{ if ASSYMETRIC INFORMATION}}$

AGENCY COST MECHANICS

- Under plausible assumptions:

1) $\frac{\partial A}{\partial W} < 0 \rightarrow \frac{\partial A}{\partial c} > 0$.

2) $\frac{\partial A}{\partial r} > 0 \rightarrow \frac{\partial A}{\partial r} < 0$

- So an increase in assymetric information at Ω_1 increases c by which W declines

ADVERSE SELECTION

- Suppose that entrepreneurs are heterogeneous in risk.
- Suppose that risk isn't publicly observable
- Then there is agency costs outside of financing
- There are ~~agency costs~~ on to $\boxed{\text{ADVERSE SELECTION}}$

THE FINANCIAL ACCELARATOR.

② AGENCY COSTS & FINANCIAL ACCELARATOR

VIII FINANCIAL ACCELARATOR

IMPERFECTIONS OF MARKET

- Investment = $E(O\Omega) = E(M_t O_t)$.
- $E(M_t O_t) \doteq \begin{cases} \Omega(\cdot) & \text{iff Perfect} \\ 1 & \text{iff Imperfect} \end{cases}$
- Lower w/ $\frac{\partial A}{\partial w} > 0 \rightarrow \frac{\partial \Omega}{\partial I} > 0$.

EXCESS REVENUES FOR INVESTMENT

- Resources = Investment
- Investment = Excess(Revenue - Expense).
- $Z(Rev - Exp) \neq D(\pi) + (H, \vec{z})$
- ie: $\boxed{I = Z(\pi) + X(H, \vec{n})}$

w/ I = Investment, Z_π = Excess profits, $X(H, \vec{n})$ = Resources to purchase Houses + durable goods

AGENCY COSTS (Bernanke & Gertler (1989))

- Agency Costs associated w/ an Investment also Rises
- If $\Delta C \gg \Delta I$, then output D also falls.
- Application: Tech shocks to real investment

VALUE OF COLLATERAL

- Financial Market Imperfections = Shocks current resources = Value of collateral Fall?
- 2 Reasons why Asset values fall:
  1.) Current $E(O) = E_{t+1}(O)$
  2.) Falls in collateral amount

$E_t(O) = E_{t+1}(O)$

- Declines in output are somewhere persistent $\lim_{t \to \infty} (E_t - E_{t+1}) O = 0$
- Expected future marginal products or services flows from assets fall
- Means less value today

so: $(E_t - E_{t+1}) O \rightarrow \frac{\partial}{\partial q} E_t(X_t(q)) \gtrless 0$
$= \frac{\partial}{\partial q} E_{t+1}(X_t(q)) < 0$

FALL OF COLATERAL VALUE

- Productive Assets are held long 2 sides of this
  1) CREDIT CONSTRAINED | 2) SUBCONSTRAINED
- When a fall in economy wide output raises Agency costs.
  = Limit Asset purchase of unconstrained

∴ $\boxed{\text{FALL COLATERAL} = \text{Limit Asset purchase}}$

① THE FINANCIAL ACCELARATOR

- Diminishing marginal products. Creates the CREDIT conts / ¬CONS firm fall.
- Baseline = firm (Cr (CONS or ¬CONS)) Borrows at safe interest
- $\boxed{\text{Financial Accelarator} = \text{imperfecting magnify cost}}$

XI OTHER IMPLICATIONS ① RIGIDITY ② INTEREST

OUTPUT MAGNIFICATION via INTEREST CHANGES

- Suppose an increase in safe interest rate = Monetary policy in resp. to sticky prices
- Agency costs rise Not only the fall in output but

Δ Interest = Δ Agency cost

- ie: $\boxed{\frac{\partial Y}{\partial i} < 0 = \frac{\partial Y}{\partial e} \lessgtr 0}$
- ie if $A_r > 0 = A_w < 0$, then $R(O) \lessgtr 0$ is high chance

∴ $\boxed{A_r > 0 = A_w < 0 = (P(C^i) > \frac{1}{2}) > \frac{1}{2}}$

REAL EFFECTS OF STICKS LARGER AS INCREASE RIGIDITY

- Recall that firms' incentives to change prices are important determinant of real rigidity
- Therefore result rigidity by when smaller responsiveness of profit maximizing firms (see prices) reduces incentives. = ↑ shocks

∴ $\boxed{\text{STICK PRICES (+)} = \text{Shocks (+)}}$

MANY VARIABLES THAT don't EFFECT INVESTMENT when PERFECT / ¬PERFECT

- Entrepreneurs wealth via (r, w) when Heterogeneous, if the market is PERFECT then it's predictive
- But $\forall (r, w)$, if ¬PERFECT the project rate $\boxed{r}$!

② why?:

$r_{perfect}$ = Fundamental or not depends only on (r)

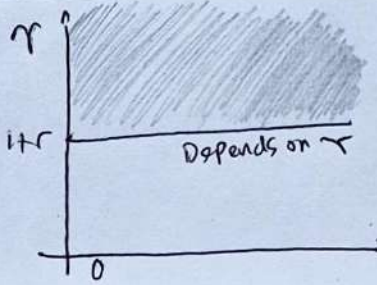
$r_{\neg perfect}$ = Distorted or (r, w) room

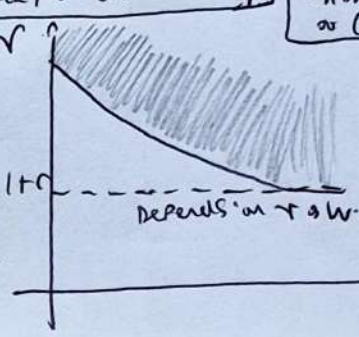
AVERAGE TAX and IDIOSYNCRATIC RISK

- Suppose taxes are added to the model
- Then the average rate effects investment

① How do they effect investment

→ ∴ The ability to use internal funds + idiosyncratic.





Disruption: The financial system does effect aggregate output?

(3) EMPIRICAL APPLICATION: CASH FLOW & INVESTMENT

[i] ECONOMETRIC FORMULATION & RESULTS

Cash flow = R+ - E+ - T+.

1) $I_i = \beta_0 + \beta D_i^{HCF} + \varepsilon_i$

$$\boxed{I_i = \beta_0 + \beta_1 D_i^{HCF} + \beta_2 D_i^{LD} + \beta_3 D_i^{HCF} D^{LD} + \varepsilon_i}$$

to be logged in $I = \kappa D_{HCF}^{\beta_1} D_{LD}^{\beta_2} (D_{HCF} D_{LD})^{\beta_3} e$

$= \kappa D_{HCF}^{\beta_1+\beta_3} D_{LD}^{\beta_2+\beta_3}$

w/ $\ell(\cdot) = \log(\cdot) \equiv \ln(\cdot)$.

$\beta_1+\beta_3$ = Diff. Low Dividends

β_3 = Cash impact cash flow on invest

here:

$$\boxed{\ell(I) = \beta_0 + (\beta_1+\beta_3)\ell(D_{HCF}) + (\beta_2+\beta_3)\ell(D_{LD})} + \varepsilon_i \Rightarrow I = \beta_0 + \beta_1 D^{HCF} D^{LD} + \beta_2 D^{HCF} D^{HD} + \varepsilon_i$$

2) Data: • Time: 1970 - 1984
• N = 422 Relatively Large US Firms w/ $\frac{D^L}{\pi} < 10\%$, $\frac{D^H}{\pi} > 20\%$

3) RESULTS:

	β_i	SE	t-stat
$D^{LD} D^{HCF}$	0.230	0.0010	12.1
$D^{HD} D^{HCF}$	0.461	0.021	12.1

• So Lower Dividends has 23 Cents more cash flow
Higher Dividend has 46 Cents more cash flows.

[ii] CONSEQUENCES

YOUNG & LESS ESTABLISH

- Low dividend Firms Differ Really greatly Than Highones
- LD Firms tend more younger and less Established.
- They also have DIFFERENT ACCESSES to External Financing

1 OIL COMPANIES & VARIATIONS (Lamont (1997))

- Variations are also studied concerning this approach
- Oil Stocks:
- Scenario: Non oil subsidiary collapse of the 1986
- solution: Internal finance is cheaper than External Fund

→ : $\boxed{\text{Internal} \downarrow = \text{Subsid. Inv.} \downarrow}$

2 PENSION EFFECTS w INTERNAL FUNDS (Rauh (2006))

- Credit availability was important firm behavior in 2008 crisis.
- $\boxed{\text{Funding Pensions cheaper} = \text{INTERNAL FUNDS}}$

GENERAL EFFICIENCY WAGE MODEL

(i) RAMIFICATIONS:

1) Higher wage = Worker Quality ↑ = Productive ↑

market clearing wage

2) Higher wage = Worker's effort ↑

* Walrasian = Indifferent on jobs losing?

3) Higher wage = Attract higher ability

4) Higher wage = Loyalty or workers

[1] Assumptions

1) $\pi = Y - wL$

2) $Y = F(eL)$

$\frac{\partial Y}{\partial L} > 0, \quad \frac{\partial^2 Y}{\partial L^2} < 0.$

3) $\boxed{e = e(w)}$ → e = effort.

(ii) Analysis:

1) $\boxed{\max_{L,w} \{ F(e(w) \cdot L) - wL \}}$

let $U = e(w) \cdot L$ so $F(e(w)L) = F(U)$

$\frac{\partial F}{\partial L} : \frac{\partial F}{\partial U}\frac{\partial U}{\partial L} = F'(U)\, e(w) - w = 0$

a) $\frac{\partial F}{\partial w} : \frac{\partial F}{\partial U}\frac{\partial U}{\partial w} = F'(U)\, L\, e'(w) - L = 0$

$\boxed{F'(U)\, e(w) - w = F'(U)\, e'(w) - L}$

so:

b) $F'(U)\, e(w) - w - F'(U)\, e'(w) - 1 = 0$ Remember: $F'(U) = \frac{w}{e(w)}$

so: $\frac{w}{e(w)} e(w) - w - \frac{w}{e(w)} e'(w) - 1 = 0$

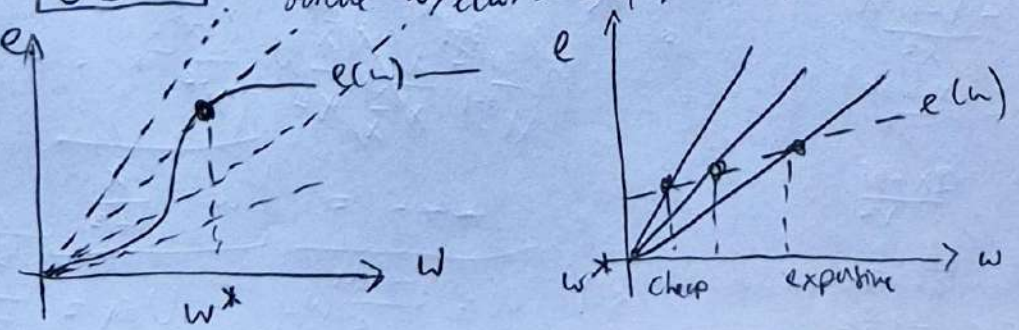
$w - w - \frac{w}{e(w)} e'(w) - 1 = -\frac{w\, e'(w)}{e(w)} - 1$

or: $w \frac{\Delta e}{\Delta w} \frac{1}{e} = 1$

$\boxed{\frac{w}{e}\frac{\Delta e}{\Delta w} = 1}$ → obtain $w/e(w)$ = cost per unit effective Labour

so: $\boxed{w \cdot \frac{e'(w)}{e(w)} = 1}$ → at the optimum, the elasticity of effort w/ resp to wage is 1

2)



[iii] Implications.

1.) Efficiency = unemployment

2.) ↑ Demand = $w^* = e(\cdot)$ = No Rate to adjust.

3.) Labour Demand = Large movements in unemployment

4.)

[2] General Version

w_a =

1.) $\boxed{e = e(w, w_a, u)}$

let $A = w, w_a, u$

So: $\frac{\partial e}{\partial A} > 0$, $\frac{\partial^2 e}{\partial A^2} < 0$, $\frac{\partial^3 e}{\partial A^3} > 0$

$\downarrow e_1$ $\downarrow e_2$ $\downarrow e_3$

$$F'(e(A), L) = \frac{w}{e(A)}$$

$$\frac{w\, e_1(A)}{e(A)} = 1$$

and $\boxed{EQ: \ w = w_a}$

2.) $e = \begin{cases} \left(\frac{w-x}{x}\right)^{\beta} & \text{if } w > x \\ 0 & \text{else} \end{cases}$

$b > 0$, normal

$b = 1$, x = wage paid at other firms x employed

$b \not> 1$, workers put less weight

So: $\boxed{e = \left(\frac{w-x}{x}\right)^{\beta} \ \forall w > x}$ x = labour market measures

here: $x = (1 - bu) w_a$

3.) $\boxed{\frac{1}{w} e_1 = e_1}$

$$e_1 = \frac{\partial e}{\partial w} = \frac{\partial e}{\partial J}\frac{\partial J}{\partial w} \quad \frac{\partial e}{\partial J} = \frac{\partial}{\partial J} J^{\beta} \quad \frac{\partial}{\partial w}\left(\frac{w}{x} - \frac{x}{x}\right)$$

$$\frac{\partial}{\partial x}(w - x) = \frac{\partial w}{\partial x} - \frac{\partial x}{\partial x} = 0 - 1$$

$f' = -1$

$f = w - x$

$g = x$

$$\frac{d}{dx}\frac{f}{g} = \frac{f'g - fg'}{g^2} = \frac{-1x - w - x}{x^2} = \frac{-x - w - x}{x^2} = -\left(\frac{x + w + x}{x^2}\right) = \frac{2x + w}{x^2} = \frac{2x}{x^2} + \frac{w}{x^2} = \frac{2}{x} + \frac{w}{x^2}$$

$$\frac{\partial}{\partial x} - \frac{w}{x^2}$$

$$= \beta J^{\beta-1}\left(\frac{\partial w}{\partial w}\frac{1}{x} - x\right)$$

So: $\boxed{e_1 = w\beta\left[\frac{w-x}{x}\right]^{\beta-1}\frac{1}{x} = \frac{1}{w}\left(\frac{w-x}{x}\right)^{\beta}}$

4.) $\beta\left[\frac{w-x}{x}\right]^{\beta-1}\frac{1}{x} \cdot w\left[\frac{w-x}{x}\right]^{-\beta} = 1$

$$\beta\left(\frac{w^{\beta-1}}{x^{\beta-1}} - \frac{x^{\beta-1}}{x^{\beta-1}}\right)\frac{1}{x^2} w \cdot \left(\frac{w^{-\beta}}{x^{-\beta}} - 1\right) = \beta\frac{w^{\beta-1+1}}{x^{\beta-1+1}} - \frac{w}{x}$$

$$\text{---} = \boxed{w = \frac{x}{1-\beta}}$$

$(1-\beta)w_a = (1+bu)w_a$

$1-\beta = 1+bu$

$\boxed{u = \frac{\beta}{b}}$

5.) $\boxed{w = \frac{1-bu}{1-\beta} w_a}$ $\boxed{U_{EQ} = \frac{\beta}{b}}$

THE SHAPIRO-STIGLITZ MODEL

(11.2)

(1) Shapiro-Stiglitz: Limited Monitoring = Extent Effort

(i) Assumptions:

1.) $U = \int_0^\infty e^{-\rho t} u(t)\, dt$ → utility PV.

2.) $u(t) = \begin{cases} w(t) - e(t) & \text{iff Employed} \\ 0 & \text{iff unemployed} \end{cases}$

3.) $\boxed{P(t) = e^{-b(t-t_0)}}$ Poisson process

probability worker is still employed at time (t).

w/ $P(t+T) = e^{-b(t+T-t_0)}$

a.) $P(t) = e^{-b(t-t_0)}$

$\frac{P(t+T)}{P(t)} = e^{-b(t+T-t_0)+b(t-t_0)} = e^{-b(t-t+t_0-t_0+T)} = e^{bT}$

So: $\boxed{\frac{P(t+T)}{P(t)} = e^{bT}}$ Recall Poisson: $P(x) = \frac{e^{-\lambda t}}{\lambda!}$

b.) Hazard rate: $\boxed{\frac{\partial P(t)}{\partial t} = -b P(t)}$ → Hazard Rate

probability of an employed worker ceases employment at dt w/ t → $b\,dt$ as $t \to 0$.

(3)

4.) $\boxed{\pi(t) = F[\bar{e}L(t)] - w_t[L_t + S_t]}$

w/ $S(t)$ = Shirkers

Firm's profit = production function$|_L$ − wages × (L + shirkers)

w/: $F' > 0$, $F'' < 0$.

(ii) Values of E, U, S:

1.) $V_E(\Delta t) = \int_{t=0}^{\Delta t} e^{-bt} e^{-\rho t}(w-\bar{e})\,dt + e^{-\rho\Delta t}\left[e^{-b\Delta t} V_E(\Delta t) + (1-e^{-b\Delta t}) V_U(\Delta t)\right]$

s.t.: $V_E(\Delta t), V_U(\Delta t)$ = Values of Employment, Values of unemployment Resp.

2.) $I = \int_{t_0}^{\Delta t} e^{-(b+\rho)t}(w-\bar{e})\,dt = -\frac{1}{b-\rho} e^{-(b+\rho)t}(w-\bar{e})\Big|_0^{\Delta t}$

$= -\frac{1}{b-\rho} e^{-(b+\rho)\Delta t}(w-\bar{e}) + \frac{1}{b-\rho} e^{-(b+\rho)(0)}(w-\bar{e})$

$= \frac{1}{b+\rho}(w-\bar{e})\left[-e^{-(b+\rho)\Delta t} + 1\right] = \frac{1}{b+\rho}(w-\bar{e})\left[1 - e^{-(b+\rho)\Delta t}\right]$

→) $V_E(\Delta t) = I + e^{-(\rho+b)\Delta t} V_E(\Delta t) + e^{-\rho\Delta t}(1-e^{-b\Delta t}) V_U(\Delta t)$

$= \frac{1}{b+\rho}(w-\bar{e}) e^{-(\rho+b)\Delta t} V_E + \left(e^{-\rho\Delta t} - e^{-(\rho+b)\Delta t}\right) V_U$

$= \frac{1}{b+\rho}(w-\bar{e}) e^{-(\rho+b)\Delta t} - e^{-(\rho+b)\Delta t} V_U + e^{-(\rho+b)\Delta t} V_E + e^{-\rho\Delta t}$

$= e^{-(\rho+b)\Delta t}\left[\frac{1}{\rho+b}(w-\bar{e}) - V_U + V_E\right] + e^{-\rho\Delta t} V_U$

$\doteq \frac{1}{\rho+b}\left[w-\bar{e}\right] - \cancel{V_U} + V_E + \cancel{V_U} = \frac{1}{\rho+b}\left((w-\bar{e}) + bV_U\right)$

So: $$\boxed{V_E = \frac{1}{\rho + b}\left[(w - \bar{e}) + bV_U\right]} \quad (11.29)$$

$w - \bar{e}$: Dividend Rate when employed
bV_U = No Dividends when unemployed
∴ ASSET PRICED BY RISK NEUTRAL INVESTORS

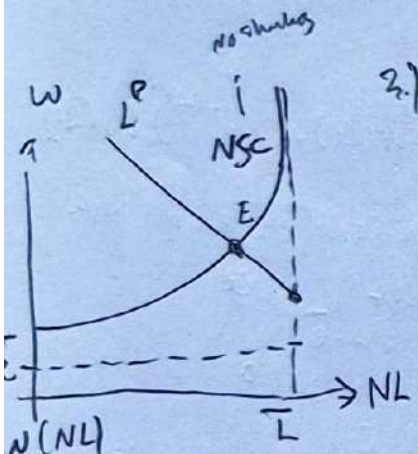
3.) $\rho V_E(\rho + b) = (w - \bar{e}) + bV_U$
$V_E\rho + V_E b = (w - \bar{e}) + bV_U$
$V_E\rho = (w - \bar{e}) + bV_U - bV_E$
$$\boxed{\rho V_E = (w - \bar{e}) - b(V_E - V_U)} \quad (11.28)$$

4). If worker is strategy, Then "dividend" is w/ two ones
The Expected Capital loss is $(b+q)(V_S - V_U)$ so:
$$\boxed{\begin{aligned} \rho V_S &= w - (b+q)(V_S - V_U) \\ \rho V_U &= a(V_E - V_U) \end{aligned}} \quad \begin{aligned}(11.29)\\(11.30)\end{aligned}$$

2. No-Shirking Condition

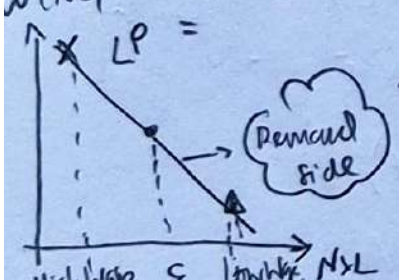
1) $\boxed{\Sigma : V_E = V_S}$ Firms must pay enough so that (11.31) $V_E \geq V_S$.

2) $(w - \bar{e}) - b(V_E - V_U) = w - (b+q)(V_E - V_U)$ (11.32)
$$\boxed{V_E - V_U = \frac{\bar{e}}{q}} \quad (11.33)$$



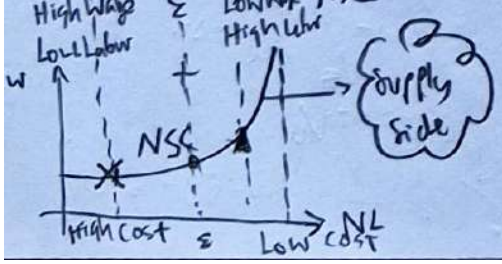
3.) So: $\rho(V_E - V_U) = (w - \bar{e}) - (a+b)(V_E - V_U)$
$\rho(V_E - V_U) + \bar{e} + (a+b)(V_E - V_U) = w$
$\rho\frac{\bar{e}}{q} + \bar{e} + (a+b)\frac{\bar{e}}{q} = \frac{\rho\bar{e}}{q} + \bar{e} + \frac{a\bar{e}}{q} + b\frac{\bar{e}}{q}$
$= \frac{\bar{e}}{q}(\rho + a + b) + \bar{e}$ so: $\boxed{W = \frac{\bar{e}}{q}(\rho + a + b)}$

So wage is affected by:
a) Rate of Finding jobs (a)
b) Rate of job breakup (b)
c) Discounted rate (ρ)



4.) $\boxed{a = \frac{NLb}{L - NL}}$ w/ $\frac{NLb}{L - NL} = \frac{\text{Firms} \times \text{Labour} \times \text{Breakup}}{\text{Average Employment} - \text{Firms} \times \text{Labor}}$

So: $W = \bar{e} + \frac{\bar{e}}{q}(\rho + a + b) = \frac{\bar{e}}{q}\left(\rho + \frac{NLb}{L - NL} + b\right) + \bar{e}$



So: $W_{\text{No shirking}} = \bar{e} + \rho t$

SEARCH & MATCH MODELS (II.4)

(I) Assumptions:

[i] QUALITATIVE

1) Employed Worker: • Exogenous = $y(t)$ → Production
• Endogenous = $w(t)$ →

2) Unemployed worker: Cons. Income: $b > 0$ (Exogenous)

3.) Workers are Risk Neutral so:

$$\boxed{X_U = \begin{cases} w(t) & \text{iff Employed} \\ b & \text{iff unemployed} \end{cases}}$$

4.) Vacancy has cost C w/ $C > 0$.
* If Vacant / Filled, it has C as Exogenous

5.) $$\boxed{\text{Profits} = \begin{cases} y - w(t) - C & \leftrightarrow \text{filled job} \\ b + c & \leftrightarrow \text{vacant} \end{cases}}$$

6.) • Fewer jobs + ∃ unemployed = job creation ($+\pi$)
More jobs + ∃ unemployed = job threshold ($-\pi$)

[ii] QUANTITATIVE:

1) $\boxed{M(t) = M(U_t, V_t)}$, $M_U > 0$, $M_V > 0$
meetings in t is by unemployed, filled (V)

2) $$\boxed{\frac{dE(t)}{dt} = M(U_t, V_t) - \lambda E_t}$$

so that: $\frac{dE(t)}{dt} + \lambda E(t) = 0 \rightarrow \frac{dE(t)}{E(t)} = -\lambda\, dt \rightarrow \boxed{E(t) = E_0 e^{-\lambda t}}$
Poisson Process

3.) $$\boxed{\exists \phi (0 \le \phi \le 1);\ [\phi := L, (1-\phi) := N]}$$
Nash Bargaining.

(II) MATCHING FUNCTION.

[i] 1) $M(U_t, V_t) = U_t M(1, V_t / U_t)$
$= U(t)\, m(\theta_t)$

so: $\boxed{M(U_t V_t) = U_t m(\theta_t)}$ w/ $m(\theta) = M(1, \theta)$ is a job finding rate

2) $a(t) = \frac{M(U_t, V_t)}{U_t} = \frac{m(\theta_t) U_t}{U_t} = m(\theta_t)$

so: $\boxed{a(t) = m(\theta_t)}$ Probability of finding job = job rate

3.) $\alpha(t) = \frac{M}{V} = \frac{U_t m(\theta_t)}{V(t)} = \frac{m(\theta_t)}{\theta(t)}$

so: $\boxed{\alpha(t) = \frac{m(\theta_t)}{\theta_t}}$

4.) $\boxed{m(\theta) = k\theta^{\gamma}}$ → Cobb-Douglas

Terms:
• Employed's
Exo + Endo
• Cons Income on unemployed
• Vacancy has cost c
• Profits = $\begin{cases} y - w - c \\ b + c \end{cases}$

Eqs:
• $M = U \cdot M(1, \frac{V}{U}) = U \cdot m(\theta)$
• $a = m(\theta)$
• $\alpha = \frac{m(\theta)}{\theta}$

Equilibrium Conditions

1.) $$\boxed{rV_E(t) = w(t) + \frac{\partial V_E}{\partial t} - \lambda\left[V_E(t) - V_U(t)\right]}$$

Return of capital = Dividend + Capital gain − Capital loss

rV $\quad w(t)$ $\quad \dot{V}_E$ $\quad \lambda EV$

2.) $$rV_U(t) = b + \frac{\partial V_U(t)}{\partial t} + \alpha(t)\left[V_E(t) - V_U(t)\right]$$

$$rV_F(t) = \left[y - w(t) - c\right] + \frac{\partial V_F}{\partial t} - \lambda\left[V_F(t) - V_V(t)\right]$$

$$rV_V(t) = -c + \frac{\partial V_V(t)}{\partial t} + \alpha(t)\left[V_F(t) - V_V(t)\right]$$

1. $\underbrace{\dot{V}_F - \lambda\left[V_F - V_V\right]}_{\text{Net Capital}}$

2. $\underbrace{\alpha(t)\left[V_F - V_V\right] + \dot{V}_V}_{\text{Total Capital via Rates of Filling}}$

So:

Net Rate of Unemployed I = Cons + Capital gain + Prob of capital loss

Net Rate of Filled = Profit + Capital gain − Prob of Capital loss

Rate of Unemployed = −Cost + Rate of Vacancy × Capital loss

*Recall:
- Capital gain ($\dot{V}_E$) = The number employed
- Rate Unemployed I = Risk of Cost/Exogs + Risk of Unemployed
- Rate Unemployed II = Risk of Cons + Getting a Jobs
- Rate of Fillings = Job Filled profit + Net Capital

3.) $$\frac{\partial E(t)}{\partial t} = U_t^{1-\tau} V_t^{\tau} - \lambda E(t)$$

4.) $$\boxed{V_E(t) - V_U(t) = \frac{\phi}{1-\phi}\left[V_F - V_V\right]} \quad V_V = 0 \ \forall t.$$

SHAPIRO STIGLITZ & SEARCH MODELS

(1) Assumptions: $\boxed{U = \int e^{-\rho t}\,dt}$ $\boxed{P(t) = e^{-bt+t_0}}$ $\boxed{\frac{P(t+\tau)}{P(t)} = e^{b\tau}}$

Production function $\pi(t) = F(\bar{e}L(t)) - w_t(L_t + S_t)$

1) Firm

$$V_E(\Delta t) = \int_{t=0}^{\Delta t} e^{-bt} e^{-\rho t}(w-\bar{e})\,dt + e^{-\rho\Delta t}\left[e^{-b\Delta t}V_E(\Delta t) + (1-e^{-\Delta t})V_U(\Delta t)\right]$$

$$= \int_{t_0}^{\Delta t} u(bt)\,u(\rho t)[w-\bar{e}] + u(\rho\Delta t)\left[u(b\Delta t)V_E + (1-e^{-\Delta t})V_U\right]$$

$$= \int_{t_0}^{\Delta t} U(b,\rho)[w-\bar{e}]\,dt + U(b,\rho|\Delta t)V_E + [1-e^{-\Delta t}]V_U$$

So: $\boxed{V_E(\Delta t) = \int_{t_0}^{\Delta t} U(b,\rho)[w-\bar{e}]\,dt + U(b,\rho)\Delta t)V_E + [1-e^{-\Delta t}]V_U}$

2) So

$$I = \int_{t=0}^{\Delta t} e^{-(b+\rho)t}\,dt = \int_{t_0}^{\Delta t} e^{-(b+\rho)\,dt}\,dt = \frac{1}{-b-\rho}e^{-(b+\rho)\,dt}\Big|_0^{\Delta t}$$

$$= \frac{1}{-b+\rho}e^{-(b+\rho)\Delta t} + \frac{1}{b+\rho}e^{-(b+\rho)0} = \frac{1}{b+\rho} + \frac{1}{b+\rho}e^{-(b+\rho)\Delta t}$$

$$= \frac{1}{b+\rho}\left(1-e^{-(b+\rho)\Delta t}\right)$$ So: $\boxed{I(b,\rho)\cdot(w-\bar{e}) = \frac{1}{b+\rho}\left(1-e^{-(b+\rho)\Delta t}\right)(w-\bar{e})}$

2) Meaning:

$$V_E(\Delta t) = \frac{1}{b+\rho}\left(1-e^{-(b+\rho)\Delta t}\right)(w-\bar{e}) + U(b,\rho|\Delta t)V_E + [1-e^{-\Delta t}]V_U.$$

Inflation & Money Growth

(1)

$$\boxed{L(i,Y) = \frac{M}{P}}$$

w/ $L(i,Y)$ = Real Balances Demand w/ $\frac{\partial L}{\partial i} < 0$, $\frac{\partial L}{\partial Y} > 0$ i = Nominal Interest

so: $P = \frac{M}{L(i,Y)}$ $\quad \frac{\partial P}{\partial i} = \frac{\partial L^{-1}(i,Y)}{\partial i} M$

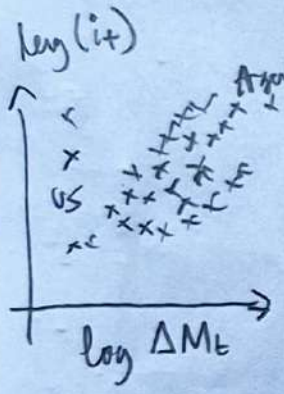
$$= \frac{\partial L^{-1}(i,Y)}{n(i)} \frac{n(i)}{i} M = \frac{1}{\frac{\partial L}{\partial n}\frac{\partial n}{\partial i}} M$$

$$\boxed{\frac{\partial P}{\partial Y} = \frac{M}{\partial L/\partial Y} = \frac{\Delta Y}{\Delta L} M}$$

$$= \frac{M}{L'(n)\,\alpha i^{\alpha-1}} = \frac{M \cdot i}{L'(n)} \quad \text{w/ } n(i) = i^{\alpha}, \ \alpha i^{\alpha-1}$$

[1] Inflation & Money growth:

1). Expected-inflation Component of $n(i)$ Reflects Inflation itself and the observed in real Interest is limited



" { Money demand ≙ doesn't fall crossy
Money supply ≙ huge variations (Large negative = deflation, Inflation Positive = Hyper inflation)

2) (Goldfeld & Sichel, 1990), $\alpha = -0.2$

so: $\log n(i_t) = \alpha_0 + \alpha_1 \log i_t$ s.t $\beta_0 + \log M - (\log \delta L + (\alpha-1)\log i_t) + \log i_t$

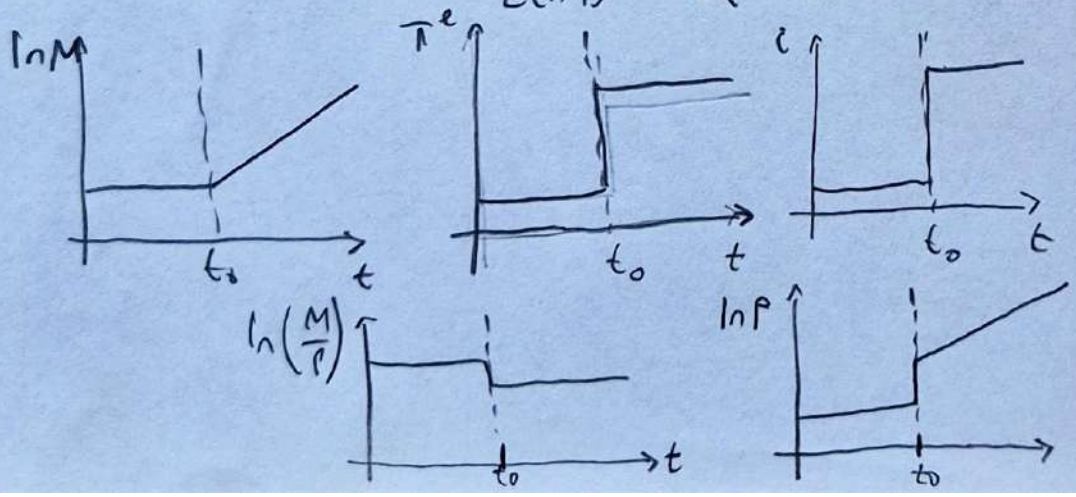
so: $\boxed{\log P = -\log \alpha + \log M_t + \log i_t - \log \delta L + (\alpha-1)\log i_t}$ ↘ Quickie Model

$\alpha = -0.2$ for i_t

(2)

$$\boxed{i \equiv r + \pi^e, \ r \equiv i - \pi^e}$$ Fisher Identity:

This means: $P = \frac{M}{L(i,Y)} = \frac{M}{L(\bar{r}+\pi^e, \bar{Y})}$



GOVERNMENT BUDGETS & FISCAL POLICY – CONSTRAINT

[17]

(1) BUDGET CONSTRAINT:

BASIC:

- The basic formulation of a budget constraint is:

$$\boxed{\sum_{t=0}^{T} \beta^t G_t \leq \sum_{t=0}^{T} \beta^t T(t) - D_0}$$

where $\beta = e^{-\delta(s)}$, so $\beta^t \sim \int_0^t e^{-\delta(s)}\,ds$, or $\beta^t = e^{-\delta t}$

↳ in discrete time

- Now in continuous time:

$$\boxed{\int_0^\infty e^{-R(t)} G(t)\,dt \leq -P_0 - \int_0^\infty e^{-R(t)} T(t)}$$

↳ continuous time

- The No Ponzi-condition then:

$$\boxed{\lim_{t \to \infty} e^{-R(t)} D(t) \leq 0}$$

↳ No Ponzi

where $R(t) = \int_{\tau=0}^{t} r(\tau)\,d\tau$.

- Debt changes (stock of real debt) then:

$$\boxed{\frac{\partial D_t}{\partial t} = [G(t) - T(t)] + r_t D_t}$$

↳ stock of real debt

- with $\boxed{\Pi_t = G_t - T_t}$ primary surplus (this is called primary surplus)

- usually they will aim at: $\sum_t \beta^t (G_t - T_t) = \sum_t \beta^t \Pi_t \Rightarrow \boxed{\sum_t \beta^t \Pi_t \geq P_0}$

↳ primary surplus consistent to repay the debt

MEASUREMENTS:

- The problem in measurement is that:

[1.] **INFLATION:** ≫ Particular inflation is now:

$$\frac{\partial D_t}{\partial t} = \Pi_t + r_t D_t \Rightarrow \frac{\partial P_t}{\partial t} \rho_t = \rho_t \Pi_t + \rho_t r_t D_t$$

$m_t v = m_{t-1} - m_{t+1}$

$m_t v^m = \rho_t y_t \rightarrow \frac{\partial B_t}{\partial t} = \rho_t \Pi_t + \rho_t r_t D_t \rightarrow \frac{\partial B_t}{\partial t} = \rho_t \{ \Pi_t + (\pi_t - i_t) D_t \}$

↳ iff depend on real rate (the wrong form)

$\boxed{\frac{\Delta m_t}{y_t} = \rho_t}$ ↳ Milton Friedman style inflation

$\boxed{\frac{\partial B_t}{\partial t} = \rho_t \{ \Pi_t + (r_t + \pi_t) D_t \}}$ ↳ Debt as the price level (the correct form)

≫ As long as the stock of debt is positive, higher inflation raises the conventional measure of deficit s.t.:

$$\boxed{\frac{\dot{B}_t}{\rho_t} = \dot{D}_t + \pi_t D_t}$$

≫ In time series.

≫ Another possibility is to assume that $\pi_t D_t = 0$, so:

$$\frac{E_t(B_{t+1})}{\rho_t} = E_t(\rho_{t+1}) + \pi_t D_t$$

$$\frac{E(\rho_{t+1})}{\rho_t} = 0 \Rightarrow -\pi_t D_t = E_t(D_{t+1})$$

$$\boxed{\frac{\dot{B}_t}{\rho_t} = \dot{D}_t + \pi_t D_t}$$

↳ If inflation or the inflation growth w.r.t. government debt is 0, then the expected future inflation will be equal to

$$\frac{1}{\rho_t} \frac{\partial B_t}{\partial t} = \frac{\partial D_t}{\partial t} \Rightarrow \boxed{\frac{B_t}{\rho_t} = D_t = \sum_j \beta^j S_{t+j}}$$

Rescale factors – intertemporal constraint

$$\pi \pi_t = \frac{\partial D_t}{\partial t} \frac{1}{D_t} \Rightarrow -\sum_{t=1}^{T} \pi(t) \leq -\ln D_t$$

$$\boxed{D_t = e^{\int_t^T \pi(\tau)\,d\tau}}$$

$$\boxed{\frac{D_t + E_t}{\rho_t} = \sum_j \beta^j A_{t+j}}$$

ie: (Liabilities + equities now = future assets) (consolidated form)

[2.] **SALE OF AN ASSET:**

- The sale of an asset: ⊕ Revenue & Deficit
  ⊖ opportunity profit from asset
- If affects the **DEFICIT** but not the **BUDGET CONSTRAINT**

[3.] **UNFUNDED LIABILITY**

- Unfunded Liability = Incur expenses in the future

PONZI GAMES:

- Take a look again at the equations: here

$$\int e^{-R_t} G_t\,dt \leq -P_0 - \int_{t_0}^{\infty} e^{-R_t} T(t)\,dt$$

$$\sum_t \beta^t [G_t - T_t]$$

or: $\boxed{\sum_t \beta^t \Pi_t \leq -D_0}$

- our equations dictate that if $(\beta^t G_t > \beta^t T(t))$, then basically

$$\boxed{D_t : \beta^t G_t - \beta^t T(t) > 0.}$$

DEBT IS STRICTLY POSITIVE

» if there's a finite number of agents, $\beta^t K_t$ is strictly positive [AFTER-TAX-INCOME]

» If there's many agents then this doesn't apply.

- The Diamond model consumption is

$$\boxed{\frac{C_{2t+1}}{C_{1t}} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}}}$$

vs DIAMOND consumption model.

» observe that [IMPOVERISHES SALES BADLY] to [DISSAVE IN FUTURE]

- This means that the [PRESENT VALUE OF PRIVATE SECTOR MONEY] will be EXCEEDING to [PRESENT VALUE OF PUBLIC SECTOR]

- The ponzi scheme is run such that (in account):
- The private sector can't run this due to the risk of [CAUGHT IN FRAUD] in, the process of EITER AUDIT.

- Taking a look again at the Diamond model

$$\boxed{S(r) = \frac{q_r^{\frac{1-\theta}{\theta}}}{q_\rho^{\frac{1}{\theta}} + q_r^{\frac{1-\theta}{\theta}}}}$$

DIAMOND SAVING MODEL

» OR if $\theta = 1$, $S(r) = \frac{q_r^{\frac{1-1}{\theta}}}{q_\rho^{\frac{1}{1}} + q_r^{\frac{1-1}{1}}} = \frac{1}{q_\rho + 1}$

» Say $\rho = 0.3$, Thus

$$\frac{1}{2+0.3} = \frac{1}{2.3} = \boxed{0.43} = \frac{1}{(1+\rho)+1} = \frac{1}{2+\rho} = \frac{1}{2} + \frac{1}{\rho}$$

SAVING 43% RATE

[ASSET, INITIAL OR INCOME]

$\theta = 1$
$\rho = 0.3$

$A_0 + E(y_t)$

$A_0 + A_t + E(y_t)$

S_t $\xrightarrow{\left(\frac{1+\rho}{1+r}\right)^{\frac{1}{\theta}} \frac{C_{2t+1}}{C_{1t}}}$ C_t

$t=0$ [SAVING at $t=0$] $t=1$ [CONSUMPTION at $t=1$]

etc.

MONEY (x) ... 500
DEBT (B) ... 500
INTEREST EARNED (4) ... 50
MONEY (x) ... 50

MONEY (x) ... 50
DEBT ... 50
INTEREST EXP ... 50
MONEY ... 50

∴ this means that

$$\boxed{X_t - q_t(i_t) B_{t-1} = B_t.}$$

$q_t B_t \to \infty$,
$T_t \to \infty$.
EROSION OF PUBLIC SAVING (S_t)
∴ and consumption decreases ($C_{2t+1} \to -\infty$)

and as $q_{i_t} B_{t-1} \to \infty$, $B_t \to (-\infty)$
since the real value of debt is decreasing in value due to interest expense