(1) POLARIZATION

$\boxed{k = \frac{2\pi}{\lambda}}$ or $\boxed{k = \frac{\omega}{c}}$ w/ $\lambda = \frac{c}{f} \Rightarrow \frac{1}{\lambda} = \frac{f}{c}$

and $\boxed{\omega = 2\pi f}$

ANGULAR FREQUENCY of a WAVE VECTOR $\vec{k} = k \vec{u}_z$.

[1] POLARIZATION VECTOR:

$$\boxed{\vec{E} = \vec{E}_x u_x + \vec{E}_y u_y}$$

and $\begin{cases} \nabla \times E = \frac{\partial B}{\partial t} \\ \nabla \cdot E = \frac{\rho}{\varepsilon_0} \end{cases}$ } SAMPLE OF MAXWELL'S EQUATIONS

1) w/ $E_x = E_{0x} \cos(kz - \omega t)$

$E_y = E_{0y} \cos(kz - \omega t + \phi)$.

for $\frac{\partial^2 E_x}{\partial z^2} = -k^2 E_{0x} \cos(kz - \omega t) = -k^2 E_x$

$\frac{\partial^2 E_y}{\partial z^2} = -k^2 E_{0y} \cos(kz - \omega t) = -k^2 E_y$

a) $\frac{\partial^2 E_x}{\partial t^2} = -\omega^2 E_{0x}$ $\quad \frac{\partial^2 E_y}{\partial t^2} = -\omega^2 E_y$.

• HIGH SPEED BARRIER FIX

$\omega^2 = c^2 k^2 + \left(\frac{mc^2}{\hbar}\right)^2$ and

$\omega^2 = c^2\left(\frac{\omega^2}{c^2}\right) + \left(\frac{mc^2}{\hbar}\right)^2$

$= \omega^2 + \left(\frac{mc^2}{\hbar}\right)^2$

so $\left(\frac{mc}{\hbar}\right)^2 = 0$

and $\boxed{m = 0}$

$-k^2 E_x - k^2 E_y - \omega^2 E_x - \omega^2 E_y + m^2(E_x + E_y)$

b) hence $(\nabla^2 + \partial_t^2 + m^2)\vec{E} = -k^2 E_x - k^2 E_y + \omega^2(E_x + E_y) + m^2(E_x + E_y) \overset{?}{=} 0$

$= -(k^2(E_x + E_y) + \omega^2(E_x + E_y) + m^2 \ldots$ $\boxed{k^2 + \omega + m^2 = 0}$

$-(k^2 + \omega^2 + m^2) = 0 \rightarrow$

$k^2 = \frac{\omega^2}{c^2}$ so $-(k^2 + \omega^2 + m^2) = \frac{\omega^2}{c^2} + \omega^2 + m^2 = \omega^2\left(\frac{1}{c^2} + 1\right) + m^2 = 0$

$\frac{c^2 + 1}{c^2} + m^2 = 0$

$\frac{1}{c^2} + 1 + m^2 = \ldots$

c.) we: $\boxed{c^2\omega^2 = \omega^2 + c^2 m^2}$

$c^2 + 1 = (mc)^2$

$\boxed{c^2 = (mc)^2 - 1}$

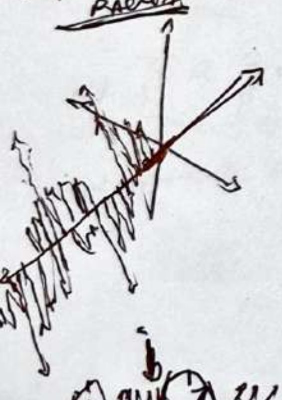
and: $\boxed{\omega^2 = \frac{c^2 m^2}{c^2 - 1}}$

and if $c^2 = 1$ then $\omega^2 = \frac{m^2}{0}$ can't be ZERO!

If $m^2 = 0$ then: $\omega^2 = 0$ and

in the field equation is only true IF MASS IS 0.

• POLARIZED BARRIER



2) $E_x = E_0 e^{i(kz - \omega t)} \rightarrow E = E_{0x} e^{i(kz - \omega t)} \vec{u}_x + E_{0y} e^{i(kz - \omega t + \phi)} \vec{u}_y$

$E_y = E_0 e^{i(kz - \omega t + \phi)}$

w/ $\boxed{\|E_0\| = (E_{0x}^2 + E_{0y}^2)^{\frac{1}{2}}}$.

3) hence $\vec{E} = E_0 e^{i(kz - \omega t)} \left[\frac{E_{0x}}{E_0} u_x + \frac{E_{0y}}{E_0} e^{i\phi} u_y\right]$

$$\boxed{\vec{\varepsilon} = \frac{E_{0x}}{E_0}\vec{u}_x + \frac{E_{0y}}{E_0} e^{i\phi}\vec{u}_y}$$

MALUS' LAW

thus $|\varepsilon| = \sqrt{\varepsilon^* \cdot \varepsilon} = \sqrt{\left(\frac{E_{0x}}{E_0}\right)^2 + \left(\frac{E_{0y}}{E_0}\right)^2} = 1$

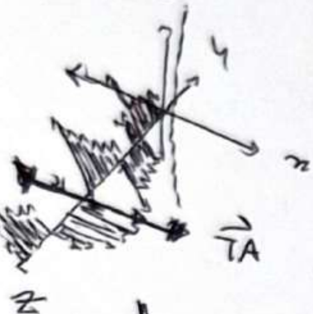
4) : (DETECTORS $\not\Rightarrow$ FIELDS) $\wedge$ (DETECTORS $\Rightarrow$ POWER FUNCTION).

for: $I = |E|^2 = E^* \cdot E$

and if $\varepsilon \cdot \varepsilon^* = 1$ $\quad I = E^* \cdot E = \left(E_0 e^{i(kz - \omega t)} \varepsilon^*\right)\left(E_0 e^{-i(kz - \omega t)} \varepsilon\right) = E_0^2$

POLARIZATION MECHANICS (2.3) (2.4)

(3) MODIFYING POLARIZATION



[i] LINEAR POLARIZATION:

$$\vec{E}_i = E_0 e^{i(kz-\omega t)}\vec{\varepsilon}$$

$$\vec{E}_T = (\vec{E}_i \cdot u_\theta)\,u_\theta = E_0 e^{i(kz-\omega t)}(\vec{\varepsilon}\cdot u_\theta)\,u_\theta$$

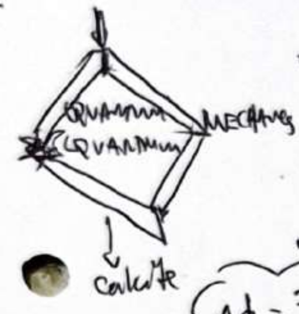
$$\boxed{E_T = E_0 e^{i(kz-\omega t)}(\varepsilon_0 \cdot u_\theta)\,u_\theta}$$ The output is linearly polarized on u_θ.

• Linearly polarized lights produces an angle ϕ wrt horizontal ($\varepsilon = u_\phi$)

∴ so off by: $\cos(\theta - \phi)$

[ii] SPLITTING BEAMS & [iii] WAVE PLATES

POLARIZING BEAM SPLITTER (PBS)



• Piece of a Birifringent material can be used as a PBS

• Polarizing Beam Splitter (PBS) can also use Thin Film coatings.

1) $\phi_f = \frac{n_f 2\pi l}{\lambda}$ f = FAST AXIS REFRACTION

$\phi_s = \frac{n_s 2\pi l}{\lambda}$ s = SLOW AXIS REFRACTION

$$\boxed{\Delta\phi = \phi_s - \phi_f = \frac{(n_s - n_f)2\pi l}{\lambda}}$$

$$\Delta\phi = 2\pi j + \frac{\pi}{2}$$

2) QUARTER WAVE LENGTH

$$\therefore \Delta\phi = \frac{2\pi}{4} = \frac{1\pi}{2} = \frac{\pi}{2}$$

$$\therefore \boxed{\Delta\phi_{QUARTER} = \frac{\pi}{2}}$$

• wave plate = optical element = phase shift in orthogonal frames.

(4) JONES VECTORS & MATRICES

[i] JONES VECTORS

$$\boxed{\begin{matrix}\varepsilon_H = u_x \\ \varepsilon_V = u_y\end{matrix}}$$ H = Horizontal $\rightarrow$ V = VERTICAL $\uparrow$

$$\text{so: } \varepsilon_{45} = \frac{1}{\sqrt{2}}u_x + \frac{1}{\sqrt{2}}u_y = \frac{1}{\sqrt{2}}(u_x + u_y) = \frac{1}{\sqrt{2}}(\varepsilon_H + \varepsilon_V)$$

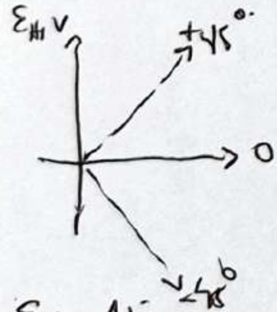
$$\text{so } \boxed{\varepsilon_{45} = \frac{1}{\sqrt{2}}(\varepsilon_H + \varepsilon_V)}$$

[ii] JONES MATRICES

GEOMETRY: 1) so $\varepsilon_H = \begin{bmatrix}1\\0\end{bmatrix}$ $\varepsilon_V = \begin{bmatrix}0\\1\end{bmatrix}$

$$\varepsilon_{+45} = \frac{1}{\sqrt{2}}\left(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

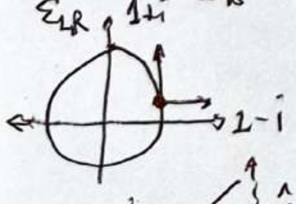
$$\varepsilon_{-45} = \frac{1}{\sqrt{2}}\left(\begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\right) = \frac{1}{\sqrt{2}} = \begin{bmatrix}1\\-1\end{bmatrix}$$



so: $$\boxed{\varepsilon_{+45} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \quad \varepsilon_{-45} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}}$$

$$\text{now: } \varepsilon_{+45}\varepsilon_{-45} = \frac{1}{(\sqrt{2})^2}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\begin{bmatrix}1\\1\end{bmatrix}\\-1\begin{bmatrix}1\\1\end{bmatrix}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\cdot1 + 1\cdot1\\1\cdot1 + -1\cdot1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}2-2\\2-2\end{bmatrix}$$

$$\begin{bmatrix}0\ \frac{1}{2}\\0\ \frac{1}{2}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \text{ so: } \boxed{\varepsilon_{+45}\varepsilon_{-45} = 0^\circ}$$



$$\text{so } \varepsilon_L = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \frac{1}{\sqrt{2}}(\varepsilon_H + i\varepsilon_V)$$

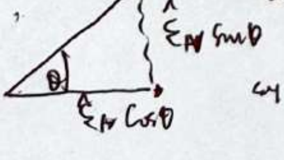
$$\varepsilon_R = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} = \frac{1}{\sqrt{2}}(\varepsilon_H - i\varepsilon_V)$$

$$\text{so: } \boxed{\begin{matrix}\varepsilon_L = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix} \\ \varepsilon_R = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}\end{matrix}}$$

$$-(i \cdot i) = -(-1) = 1$$

$$\text{and: } \varepsilon_L\varepsilon_R = \frac{1}{2}\begin{bmatrix}1\\i\end{bmatrix}\begin{bmatrix}1\\-i\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\begin{bmatrix}1\\i\end{bmatrix}\\i\begin{bmatrix}1\\i\end{bmatrix}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\cdot1 + i\cdot1\\1(i) + -i\cdot i\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1+i\\1-i\end{bmatrix}$$

ε_θ:



$$\text{so } \boxed{\varepsilon_\theta = \cos\theta \cdot \varepsilon_H + \sin\theta \cdot \varepsilon_V}$$

$$1 = \int_0^{\infty} |E(z,t)|^2 \, dz = E_0^2 \int_0^{\infty} \left(e^{i(kz-\omega t)}\right)^2 dz = E_0^2 \int e^{2i(kz-\omega t)} dz$$

$$= E_0^2 \frac{1}{2ik} e^{2i(kz-\omega t)} \Big|_0^{\infty} = E_0^2 \frac{1}{2ikz} e^{i2(k(\infty)-\omega t)} = E_0^2 \frac{1}{2ikz} e^{-i2\omega t}$$

NORMALIZATION FIX

$$\int_0^1 E(z,t) \, dz$$

$$= |E_0|^2 \int_0^{\infty} e^{-2\lambda z} \, dz$$

$$= |E_0|^2 \left[\frac{1}{2\lambda}\right]$$

$$\Rightarrow E_0 = \sqrt{2\lambda}$$

and: $\frac{d}{dt}\int_0^{\infty} |E(z,t)|^2 dz = 0$

$$|E|^2 = 2\lambda e^{-2\lambda z}$$

hence: $\boxed{E_0^2 = 2ike^{i2\omega t}}$ and $\boxed{E_0 = \sqrt{2ike^{2i\omega t}}}$

an $\omega = \frac{v}{r}$, $v = \frac{2\pi}{t}$

If $\omega = \frac{2\pi}{t}\frac{1}{r}$

The $\omega t = r 2\pi$.

w/ $e^{i2\omega t} = \cos(2\omega t) + i\sin(2\omega t)$

$= \cos(2\pi r) + i\sin(2\pi r)$

hence

$$E(z,t) = E_0 e^{i(kz-\omega t)} = \left(2ike^{i2\omega t}\right)^{\frac{1}{2}} e^{i(kz-\omega t)}$$

$U = 2ike^{i\omega t}$

$$\frac{\partial E}{\partial t} = \frac{\partial}{\partial t}\left(2ike^{i\omega t}\right)^{\frac{1}{2}} e^{i(kz-\omega t)} = \left(2ike^{i\omega t}\right)^{\frac{1}{2}} \frac{\partial}{\partial t} e^{-i\omega t} \cdot e^{ikz}$$

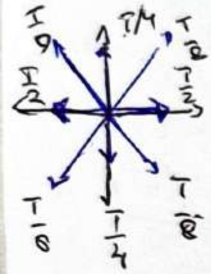
$\frac{dE}{dt} = \frac{dE}{dU}\frac{dU}{dt}$

$= U^{\frac{1}{2}} \frac{dU}{dt}$

So: $\boxed{E(z,t) = Ae^{-\lambda z} e^{i(kz-\omega t)}}$

Is more neutral.

THE WAVE FUNCTION IS STILL TIME DEPENDANT!



[ii] LINEAR POLARIZATION

$$\vec{\varepsilon} = \frac{E_{0x}}{E_0} u_x + \frac{E_{0y}}{E_0} u_y$$

an $\boxed{\theta = \tan^{-1}\left(\frac{E_{0y}}{E_{0x}}\right)}$

If AF The zero phase shift ($\phi = 0$) Is Assumed, Then The The change between $E_0(y,x)$ is perpendicular

CIRCULAR POLARIZATION

$\cos(kz - \omega t + \frac{\pi}{2})$

$$\downarrow$$

$\boxed{\frac{E_0}{\sqrt{2}} \sin(kz - \omega t)}$

[iii] CIRCULAR POLARIZATION

1) Consider $\phi = \frac{\pi}{2}$ and $(E_{0x}, E_{0y}) = \frac{E_0}{\sqrt{2}}$ then

$$\boxed{\vec{\varepsilon} = \frac{1}{\sqrt{2}} u_x + i u_y}$$

2) So: $E_x = \frac{E_0}{\sqrt{2}}\cos(-\omega t) = \frac{E_0}{\sqrt{2}}\cos(\omega t)$

$E_y = \frac{E_0}{\sqrt{2}}\sin(-\omega t) = \frac{E_0}{\sqrt{2}}\sin(\omega t)$

[iv] ELLIPTICAL POLARIZATION

and $\boxed{E_x = \frac{E_0}{\sqrt{2}}\cos(\omega t)}$ and $\boxed{E_y = \frac{E_0}{\sqrt{2}}\sin(\omega t)}$

Elliptical polarization Happens when for a vector field, $\boxed{\neg(\phi = 0) \land \neg(\phi = \frac{\pi}{2})}$ in its a special case

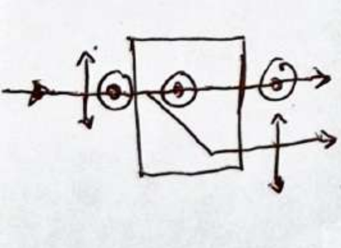
(2) BIREFRINGENCE

[i] BIREFRINGENT MATERIALS.

1) $k = \frac{\omega}{v} = \frac{n\omega}{c} = n\frac{2\pi}{\lambda}$

how $\boxed{k = \frac{\omega}{v}}$ and $\boxed{\frac{2\pi}{\lambda} = k}$ w/ $n = \frac{c}{v}$ so $v = \frac{c}{n}$

an $\frac{1}{v} =$



2) BIREFRINGENT Materials = Anisotropic = Refraction Index = Polarization

QUANTUM OPERATIONS (4.1) (4.2) (4.3) (4.5)

(1) BASICS

1)

$$|\psi_n\rangle = O_N|\psi_{N-1}\rangle = \hat{O}_N \ldots \hat{O}_2\hat{O}_1|\psi_i\rangle$$

so: $\boxed{|\psi_n\rangle = \prod_{i=1}^{N}(\hat{O}_N)\cdot|\psi_i\rangle}$ w/ $|\psi_n\rangle$ = quantum state, $\hat{O}_n$ = operator.

w/ $e^{\hat{O}} = \sum_n^\infty \frac{1}{n!}\hat{O}^n$ => Taylor series

2)

$$\langle A|F_P(\theta) = |+45\rangle$$
$$\langle A|F_P^\dagger(\theta) = \langle +45|$$

so: $\boxed{\langle H|F_P F_P^\dagger|H\rangle = \langle +45|+45\rangle = 1.}$ as unitary operators

2.1)

$$\hat{U}^\dagger\hat{U} = \hat{U}^\dagger\hat{U} = \hat{1}$$
$$\hat{U}|\psi_1\rangle = e^{i\phi}|\psi_2\rangle = [\cos(\phi) + i\sin(\phi)]|\psi_2\rangle$$
$$\hat{U}^\dagger\langle\psi_1| = e^{-i\phi}\langle\psi_2| = [\cos(\phi) - i\sin(\phi)]\langle\psi_2|$$

So: $\hat{U}|\psi_1\rangle\langle\psi_1|\hat{U}^\dagger = \langle\psi_1|\hat{U}\hat{U}^\dagger|\psi_1\rangle.$

$$\langle\psi_1|\hat{U}\hat{U}^\dagger|\psi_1\rangle = \langle\psi_2|e^{i\phi}e^{-i\phi}|\psi_2\rangle = \langle\psi_2|\psi_2\rangle e^{i\phi-i\phi}$$
$$= 1\cdot 1 = 1 \checkmark$$

so: $\boxed{\langle\psi_1|\hat{U}\hat{U}^\dagger|\psi_1\rangle = \langle\psi_2|e^{i\phi}e^{-i\phi}|\psi_2\rangle = 1.}$ unitary when

- $|\psi\rangle = \psi_1|HV_1\rangle + \psi_2|HV_2\rangle$
$= \sum_j \psi_j|HV_j\rangle$

st $\boxed{\psi_j = \langle HV_j|\psi\rangle}$

- $|\psi\rangle = \begin{bmatrix}\langle HV_1|\psi\rangle \\ \langle HV_2|\psi\rangle\end{bmatrix} = \begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix}_{HV}$

4.1) $\hat{P}_H|\psi\rangle = C_H|H\rangle$ -> probability of Hamiltonian state H. w/ $\hat{P}_H$ = PROJECTION TO H.

$$\hat{P}_V|\psi\rangle = C_V|V\rangle$$

so: $\boxed{|\psi\rangle = C_H|H\rangle + C_V|V\rangle}$ w/ $\boxed{C_H + C_V = 1}$

state(ψ) = Prob(H) × state(H) + Prob(V) × state(V)

- $\hat{P}_H = \hat{C}_H$

so: $\hat{P}_H|\psi\rangle = \hat{C}_H|\psi\rangle$

- since $C_H|H\rangle = C_H|\psi\rangle$

then $C_H|\psi\rangle = |H\rangle C_H$

so

but $\hat{P}_H|\psi\rangle = C_H|\psi\rangle = |H\rangle C_H = |H\rangle\langle H|\psi\rangle$

$$\hat{P}_H = |\psi\rangle\langle H|$$

so: $|\psi\rangle = C_H|H\rangle + C_V|V\rangle = |H\rangle C_H + |V\rangle C_V = \langle H|H\rangle|\psi\rangle + \langle V|V\rangle|\psi\rangle$
$= \langle H|\langle H|\psi\rangle + \langle V|\langle V|\psi\rangle = (\hat{P}_H + \hat{P}_V)\psi$

so: $\boxed{|\psi\rangle = \hat{P}_H + \hat{P}_V|\psi\rangle}$

$\boxed{\begin{matrix}|H\rangle = |HV_1\rangle \\ |V\rangle = |HV_2\rangle\end{matrix}}$

(2) MATRIX REPRESENTATION:

[1] THEORY

1) $\boxed{|\varphi\rangle = \hat{O}|\psi\rangle}$ $\boxed{|\varphi\rangle \triangleq \begin{bmatrix}\varphi_1 \\ \varphi_2\end{bmatrix}_{HV}}$

$$\varphi_i = \langle HV_i|\varphi\rangle = \langle HV_i|\hat{O}|\psi\rangle = \langle HV_i|\hat{O}\cdot\hat{1}|\psi\rangle$$

Lid $\bar{J}_H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\bar{J}_V = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $\bar{J}_\theta = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$.

$\boxed{\bar{J}_{\frac{\lambda}{4}\pm 45} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \mp i \\ \mp i & 1 \end{bmatrix}}$ $\boxed{\bar{J}_{\frac{\lambda}{2}\theta} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}}$

(1.1) Calculate $\mathbb{P}(S_y = \frac{\hbar}{2} \,|\, |x\rangle)$.

Ans:

$$\boxed{\mathbb{P}(S_y = \tfrac{\hbar}{2} \,|\, |+x\rangle) = |\langle +y|+x\rangle|^2}$$

1.) $|+y\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$ w/ $\frac{1}{\sqrt{2}}\left[|+z\rangle + i|-z\rangle\right] = \frac{1}{\sqrt{2}}\left[\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} i\right] = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$

$|+x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ wh/ $\frac{1}{\sqrt{2}}\left[|+z\rangle + |-z\rangle\right] = \frac{1}{\sqrt{2}}\left[\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right] = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

$\begin{bmatrix}a\\b\end{bmatrix}^\dagger = \begin{bmatrix}a\\-b\end{bmatrix}^T = (a \;\, -b]$

2.) $\langle +y| = |+y\rangle^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}^\dagger = \frac{1}{\sqrt{2}}[1 \;\, -i]$

3.) So: $\langle +y|+x\rangle = \left(\frac{1}{\sqrt{2}}[1 \;\, -i]\right) \times \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\right)$

$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}[1 \;\, -i]\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\cdot 1 + 1\cdot -i\\ -i\cdot 1 + 1\cdot 1\end{bmatrix} = \frac{1}{2}(1-i)$

4.) $|\langle +y|+x\rangle|^2 = \left(\frac{1}{2}(1-i)\right)^2 = \frac{1}{4}\left(1^2 + (-i)^2\right) = \frac{1}{4}(1+1) = \frac{1}{4}\cdot 2 = \boxed{\frac{1}{2}}$

So: $\boxed{\mathbb{P}(S_y = \tfrac{\hbar}{2} \,|\, |+x\rangle) = |\langle +y|+x\rangle|^2 = \tfrac{1}{2}}$

(1.2) Calculate $\mathbb{P}(S_x = \frac{\hbar}{2} \,|\, |-y\rangle)$

Ans:

$$\boxed{\mathbb{P}(S_x = -\tfrac{\hbar}{2} \,|\, |-y\rangle) = |\langle -x|-y\rangle|^2}$$

1.) $S_x = -\frac{\hbar}{2} \equiv |-x\rangle$ and: $\langle -x| = |-x\rangle^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}^\dagger$

S_x: $|-x\rangle \in x = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$ $= \frac{1}{\sqrt{2}}[1 \;\, -1]$.

2.) $\langle -x|-y\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\cdot\begin{bmatrix}1\\-i\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}^T$

$= \frac{1}{2}\cdot\begin{bmatrix}1\\-i\end{bmatrix}[1 \;\, -1] = \frac{1}{2}(1\cdot 1 + -i\cdot -1) = \frac{1}{2}(1+i)$

3.) $|\langle -x|-y\rangle|^2 = \left|\frac{1}{2}(1+i)\right|^2 = \frac{1}{4}(1^2 + i^2) = \frac{1}{4}(1^2 + 1^2) = \frac{1}{4}2 = \boxed{\frac{1}{2}}$

(3.) write $\hat{S}_z$ in terms of projection operator $|+z\rangle, |-z\rangle$:

Ans: $|+z\rangle = \frac{\hbar}{2}\begin{bmatrix}1\\0\end{bmatrix}$ $\langle +z| = \frac{\hbar}{2}[1 \;\, 0]$

$|-z\rangle = \frac{\hbar}{2}\begin{bmatrix}0\\1\end{bmatrix}$ $\langle -z| = \frac{\hbar}{2}[0 \;\, 1]$

So: $\hat{S}_z = \frac{\hbar}{2}\left\{|+z\rangle\langle +z| - |-z\rangle\langle -z|\right\} =$

QUANTUM MATRICES - HERMITES - BASIS (4.4) (4.5) (4.6)

1. MATRIX EXTRAS

[i] — 1 Continuation — $\boxed{\text{FIND } P_H \, |{+45}\rangle}$

EX 4.3 $\hat{P}_H |{+45}\rangle = |H\rangle\langle H| \left[\frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)\right] = \frac{1}{\sqrt{2}} |H\rangle$

w: $\hat{P}_H |{+45}\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{HV} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{HV} = \begin{bmatrix} 1\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}1\\1\end{bmatrix} \\ 0\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}1\\1\end{bmatrix} \end{bmatrix}_{HV} \frac{1}{\sqrt{2}}$

$= \begin{bmatrix} 1+0 \\ 0+0 \end{bmatrix}_{HV} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} |H\rangle$

w: $\boxed{\hat{P}_H |{+45}\rangle = \frac{1}{\sqrt{2}} |H\rangle}$

[ii] CORRESP. BETWEEN QM and CM.

[i] JONES MATRICES

NAME	#	POLARIZATION	POLARIZATION VECTOR	JONES
			$\vec{\varepsilon}_H$	$\begin{bmatrix}1\\0\end{bmatrix}$
Horizontal	1)	$\vec{\varepsilon}_H$	$\vec{\varepsilon}_V$	$\begin{bmatrix}1\\0\end{bmatrix}$
Vertical	2)	$\vec{\varepsilon}_V$	$\frac{1}{\sqrt{2}}(\varepsilon_H + \varepsilon_V)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$
+45° linear	3)	ε_{+45}	$\frac{1}{\sqrt{2}}(\varepsilon_H + \varepsilon_V)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$
-45° linear	4)	ε_{-45}	$\cos\theta\,\varepsilon_H + \sin\theta\,\varepsilon_V$	$\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$
Angular	5)	ε_θ	$\frac{1}{\sqrt{2}}(\varepsilon_H + i\varepsilon_V)$	
Left circular	6)	ε_L	$\frac{1}{\sqrt{2}}(\varepsilon_H + i\varepsilon_V)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$
Right circular	7)	ε_R		

JONES MATRIX

$\varepsilon_H = \begin{bmatrix}1\\0\end{bmatrix}$

⇓

$\bar{J}_{\frac{1}{2}\theta} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$

$\Rightarrow \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta \\ \sin 2\theta \end{bmatrix}$.

[iii] IMPLICATIONS:

1) Recall: $\hat{P}_H = \begin{bmatrix} \langle H|H\rangle & 0 \\ \langle V|H\rangle & 0 \end{bmatrix}$ is basically Jones matrix of horizontal polarizer.

2) Each OPTICAL ELEMENTS: Each QUANTUM OPERATOR.

3) ⑦ Why isn't the half wave plate physical implementation of the rotation operator? Because we have to get:

$$\boxed{\hat{R}_p(\theta)|\phi\rangle = |\phi + \theta\rangle}$$

2. CHANGING BASES

[i] THEORY

1) $\boxed{|\psi\rangle = \psi_H |H\rangle + \psi_V |V\rangle}$

2) $|\psi\rangle = \psi_{+45} |{+45}\rangle + \psi_{-45} |{-45}\rangle$

$|{+45}\rangle = \frac{1}{\sqrt{2}}[|H\rangle + |V\rangle]$

$|{-45}\rangle = \frac{1}{\sqrt{2}}[|H\rangle - |V\rangle]$

3) $\psi_{+45} = \langle{+45}| (\psi_H |H\rangle + \psi_V |V\rangle) = \psi_H \langle{+45}|H\rangle + \psi_V \langle{+45}|V\rangle$

$= \psi_H \langle{45}|H\rangle + \psi_V \langle{45}|V\rangle$

$\psi_{-45} = \langle{-45}| (\psi_H |H\rangle + \psi_V |V\rangle]$

4) $\langle{+45}|H\rangle = \frac{1}{\sqrt{2}}$ $\langle{+45}|V\rangle = \frac{1}{\sqrt{2}}$

$\langle{-45}|H\rangle = \frac{1}{\sqrt{2}}$ $\langle{-45}|V\rangle = -\frac{1}{\sqrt{2}}$

$$\boxed{\begin{aligned} \psi_{+45} &= \frac{1}{\sqrt{2}}(\psi_H + \psi_V) \\ \psi_{-45} &= \frac{1}{\sqrt{2}}(\psi_H - \psi_V) \end{aligned}}$$

PROBLEM 4

① Proof that unitary operator doesn't change the magnitude.

Ans: 1) Let $|\psi\rangle, |\psi'\rangle$ be vectors s.t:

$$\boxed{\hat{U}|\psi\rangle = |\psi'\rangle}$$

$\mathbb{I}$ = Identity operator

So we prove that: $\|\hat{U}\| = \||\psi\rangle\|$ w/ $\hat{U}^\dagger \hat{U} = \mathbb{I}$

2) So: $\||\psi'\rangle\| = \langle\psi'|\psi'\rangle = \langle U\psi|U\psi\rangle = \langle\psi|UU^\dagger|\psi\rangle$

$= \langle\psi|\mathbb{I}|\psi\rangle = \langle\psi|\psi\rangle$

So: $\boxed{\||\psi'\rangle\| = \langle\psi|\psi\rangle}$ The magnitude of vector ψ' is still the same unitary operator of $|\psi\rangle$ ∎

② Show that $\hat{P}_\psi^2 = \hat{P}_\psi$

Ans: 1) Recall that: $\hat{P}_\psi = \langle\psi|\psi\rangle$ so $\hat{P}_\psi^2 = |\langle\psi|\psi\rangle|^2 = |1|^2 = 1 = \langle\psi|\psi\rangle = \hat{P}_\psi$ ✓

2) So $\hat{P}_\psi^2 = |\langle\psi|\psi\rangle|^2 = \langle\psi|\psi\rangle\langle\psi|\psi\rangle = \psi\psi^* \cdot \psi\psi^* = \psi\cdot\psi \cdot \psi^*\psi^*$

$= \langle\psi|\psi\rangle\langle\psi^*|\psi^*\rangle = \langle\psi^*|\psi^*\rangle = \hat{P}_\psi$ ∎ so: $\boxed{\hat{P}_\psi^2 = |\langle\psi|\psi\rangle|^2 = \hat{P}_\psi}$ ∎

③ If $|\psi\rangle = c_H|H\rangle + c_V|V\rangle$ what's $\boxed{\langle\psi|\hat{P}_H?}$

$\langle\psi|\psi\rangle\hat{P}_H = [c_H|H\rangle + c_V|V\rangle]\hat{P}_H$

Ans: 1) $|\psi\rangle = c_H|H\rangle + c_V|V\rangle \Rightarrow$

$\hat{P}_H = c_H\hat{P}_H|H\rangle + c_V\hat{P}_H|V\rangle$

2) Recall that: $\hat{P}_H|\psi\rangle = \langle H|H\rangle|\psi\rangle$ so: $\hat{P}_H = \langle H|H\rangle$.

$\hat{P}_H = c_H\langle H|H\rangle|H\rangle + c_V\langle H|H\rangle|V\rangle$

$= c_H\langle H|H\rangle|H\rangle + c_V\langle H|V\rangle|H\rangle = |H\rangle[c_H\langle H|H\rangle + c_V\langle H|V\rangle]$

$= c_H|H\rangle + 0 = c_H|H\rangle$

3) So: $\hat{P}_H|\psi\rangle = c_H|\hat{H}\rangle \longrightarrow \langle\psi|\hat{P}_H = \hat{P}_H^\dagger|\psi\rangle^\dagger = \hat{c}_H^\dagger\langle\psi| = c_H^\dagger\langle H|$

and: $c_H^\dagger\langle H| = \langle H|c_H^\dagger = \langle H|\langle\psi|H\rangle = \langle\psi|\langle H|H\rangle$.

4) To proof this: $\langle\psi|H\rangle\langle H|\psi\rangle = \langle\psi|c_H^\dagger c_H|\psi\rangle = 1$ ∎ ✓

④ Verify the matrix representation of $\hat{R}_p(\theta)$ is unitary.

Ans: Unitary if $\hat{U}\hat{U}^\dagger = 1$. and we see that:

$$\hat{R}_p(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The prove that: $R_p^\dagger(\theta)\hat{R}_p(\theta) = \hat{1}$

So: $R_p^\dagger R_p = \cancel{\begin{bmatrix} \cos\theta^\dagger & \sin\theta^\dagger \\ \sin\theta^\dagger & \cos\theta^\dagger \end{bmatrix}}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$= \cos^2 + \sin^2$
$= \boxed{1}$ ✓

why: $\hat{R}_p(\theta)|H\rangle = \cos\theta|H\rangle + \sin\theta|V\rangle$

$\hat{R}_p(\theta)^\dagger\langle H| = \sin\theta\langle H| + \cos\theta\langle V|$

So: $\langle H|\hat{R}_p^\dagger R_p|H\rangle \equiv [\cos|H\rangle + \sin\theta|V\rangle][\cos\langle H| + \sin\theta\langle V|]$

$= \cos^2\langle H|H\rangle + 2\sin\theta\cos\theta\langle H|V\rangle + \sin^2\langle V|V\rangle$

$=$

PROBLEMS 6

(4) Show That:

$$S_y|+z\rangle = i\frac{\hbar}{2}|-z\rangle \text{ and } \hat{S}_y|-z\rangle = -i\frac{\hbar}{2}|+z\rangle$$

Ans: 1.) $\hat{S}_y = i\frac{\hbar}{2}\sigma_y = \frac{i\hbar}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

$|+z\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_z$

So: $\hat{S}_y|+z\rangle = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}|+z\rangle = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= \frac{\hbar}{2}\begin{bmatrix} (0)(1) + (-i)(0) \\ (i)(1) + (0)(0) \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} 0 \\ i \end{bmatrix} = i\frac{\hbar}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$= i\frac{\hbar}{2}|-z\rangle$ ✓

2) Similarly:

$S_y|-z\rangle = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ +i & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix} -i \\ 0 \end{bmatrix} = -i\frac{\hbar}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -i\frac{\hbar}{2}|z+\rangle$

So: $\boxed{\begin{array}{l} S_y|+z\rangle = i\frac{\hbar}{2}|-z\rangle \\ S_y|-z\rangle = -i\frac{\hbar}{2}|+z\rangle \end{array}}$

(9). The Relativistic gyro eqs. is $\gamma_{rel} = g\frac{q}{2m}$ while $\gamma_{class} = \frac{q}{2m}$

So: $\boxed{\gamma_{rel} = g\,\gamma_{class}}$ meaning its off by g. What is g?

Ans: g is the Anomalous magnetic moment of $\gamma_{classical}$ via the Dirac eqs

1.) Recall: $\boxed{(i\gamma^\mu\partial_\mu + m)\phi = 0}$ → Dirac eqs for Field Theory

Coupling: $\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$ and $\partial_\mu = D_\mu - iqA_\mu$

So: $(i\gamma^\mu\partial_\mu + m)\phi = (i\gamma^\mu(D_\mu - iqA_\mu) + m)\phi = 0$

Suppose $A^\mu = 0$ (no field interaction)

$(i\gamma^\mu(D_\mu - iqA_\mu) + m)\phi = (i\gamma^\mu D_\mu + q\gamma^\mu A_\mu + m)\phi = 0$

$\boxed{(i\gamma^\mu D_\mu + m)\phi = 0}$

So: $\boxed{\hbar(\sigma\cdot\sigma) = g}$

FOLDY WOUTHUYSEN

$\sim m + \sum_n \frac{x^n}{2m}$

$\sim \underbrace{m}_{0^{th}\text{ term}} + \underbrace{\frac{(p - qA)_r^2}{2m}}_{1^{st}\text{ term}} - \underbrace{\frac{q\hbar\,\sigma\cdot B}{m}}_{2^{nd}\text{ term}}$

2.) Recall: $V = \mu\cdot\vec{B}$ $\quad F = \left|\frac{\partial V}{\partial r}\right| = \mu\frac{\partial\vec{B}}{\partial r}$.

and: $\boxed{H = \frac{1}{2m}(p^2_{classical} - qA)^2 - \frac{q\hbar}{2m}\vec{\sigma}\cdot\vec{B}}$

So: $-\frac{q\hbar}{2m}\vec{\sigma}\cdot\vec{B} \Rightarrow \mu_s = \frac{q\hbar}{2m}\vec{\sigma}$

3.) $S_z = \frac{\hbar}{2}\vec{\sigma}$

$\mu_s = \frac{q}{m}\vec{S}$

So: $\frac{\mu_s}{\frac{q\hbar\vec{\sigma}}{2m}} = \frac{\mu_s}{\frac{q}{m}\vec{S}}$

$\frac{\hbar}{\vec{\sigma}\cdot\vec{S}} = \frac{2m}{m}$

$\frac{\hbar}{\sigma S} = 2.$

$\boxed{g = 2}$

PARTICLE INTERFERENCE

(1) APPLICATIONS OF 4th/5th ELEMENTS (6.4) (6.5)

I) EXPECTATION OF S_y:

$$\langle S_y\rangle = \langle +x|\hat{S}_y|+x\rangle$$
$$= \frac{1}{\sqrt{2}}(1\ 1)_z\left[\frac{\hbar}{2}\bar{\sigma}_y\right]\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}_z$$
$$= \frac{1}{\sqrt{2}}(1\ 1)_z\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}_z$$
$$= \frac{\hbar}{4}(1\ 1)_z\begin{pmatrix}-i\\ i\end{pmatrix}_z = \frac{\hbar}{4}(-i+i) = 0$$

So: $$\boxed{\langle S_y\rangle = \langle +x|\hat{S}_y|+x\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(1\ 1)\begin{pmatrix}1\\1\end{pmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} = 0}$$

II) COMMUTATOR OF S_x, S_y

1) $$[\hat{S}_x\ \hat{S}_y] = \hat{S}_x\hat{S}_y - \hat{S}_y\hat{S}_x$$
$$= \left(\frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}_z\frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\right) - \left(\frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}_z\frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}_z\right)$$
$$= \frac{\hbar^2}{4}\left\{\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix} - \begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix}\right\}$$
$$= \frac{\hbar^2}{4}\begin{Bmatrix}i & 0\\ i & -i\end{Bmatrix}_z = \frac{\hbar^2}{4}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = i\hbar\frac{\hbar}{2}\sigma_z.$$
$$= \frac{i\hbar^2}{2}\bar{\sigma}_z = i\hbar\hat{S}_z$$

So: $$\boxed{[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z}$$

2) Find cyclic permutations:

$$\boxed{\begin{aligned}[\hat{S}_y\ \hat{S}_z] &= i\hbar\hat{S}_x\\ [\hat{S}_z, \hat{S}_x] &= i\hbar\hat{S}_y\\ [\hat{S}_y, \hat{S}_x] &= -i\hbar\hat{S}_z\end{aligned}}$$ Angular momentum operator

III) The uncertainty principle:

$$\boxed{\Delta S_x\,\Delta S_y \geq \frac{\hbar}{2}|\langle S_z\rangle|}$$

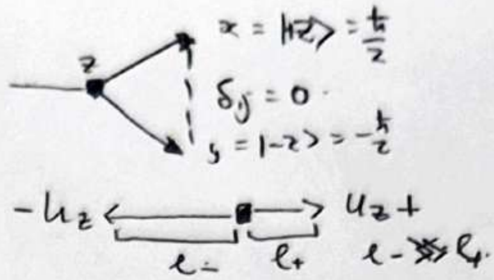
(2) PARTICLE INTERFERENCE

2) Problem:

$\langle +z | -z \rangle = 0$ s.t

$(+u_z)(-u_z) = -1$ s.t

So: $\boxed{\langle +z | -z \rangle \neq (+u_z)(-u_z) \quad 1 \neq 0}$



• Classical Intuition Fails!

[ii] The $\boxed{\hat{S}_z}$ operator

1.) $\hat{S}_z |+z\rangle = \frac{\hbar}{2} |+z\rangle$

$S_z |-z\rangle = -\frac{\hbar}{2} |-z\rangle$

2.) $|+z\rangle = \begin{pmatrix} R^+ \\ R^- \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

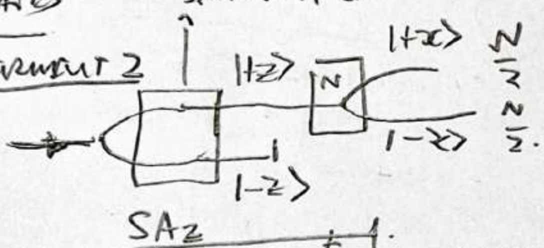
$|-z\rangle = \begin{pmatrix} R^+ \\ R^- \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

3.) $S_z = \frac{\hbar}{2} |+z\rangle\langle +z| = \frac{\hbar}{2} \sigma_z$

$$\boxed{S_z = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{\hbar}{2} \sigma_z}$$ Pauli spin matrix.

(3) More spin states

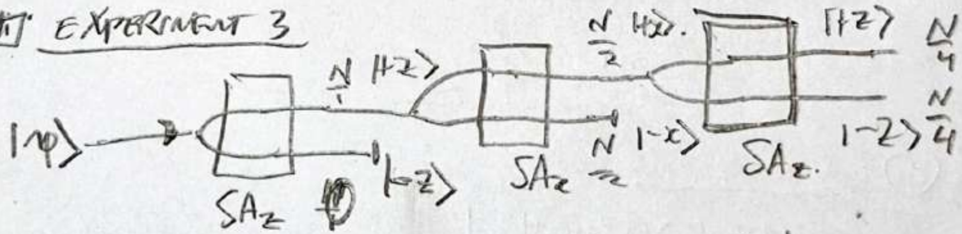
[i] Experiment 2



So: $\boxed{\begin{aligned} |x\rangle &= S_x \tfrac{\hbar}{2} \\ |-x\rangle &= -S_x \tfrac{\hbar}{2} \end{aligned}}$

Newton law = motion (orthogonals) are independent
= SA_x splits in $|x\rangle$, $|-x\rangle$ since $\exists |\varphi\rangle$.

[ii] Experiment 3



1.) So: $\mathbb{P}(S_z = \frac{\hbar}{2} \mid |+x\rangle) = |\langle +z | +x \rangle|^2 = \frac{1}{2}$

$\mathbb{P}(S_z = -\frac{\hbar}{2} \mid |+x\rangle) = |\langle -z | +x \rangle|^2 = \frac{1}{2}$

$A \sqrt{\sum x_i^2} = 1$

$A \sqrt{1 + 1^2} = 1$

$A \sqrt{2} = 1$

$\boxed{A = \frac{1}{\sqrt{2}}}$

2.) $$\boxed{\begin{aligned} |x\rangle &= \frac{1}{\sqrt{2}} \left(|+z\rangle + |-z\rangle \right) \\ |-x\rangle &= \frac{1}{\sqrt{2}} \left(|+z\rangle - |-z\rangle \right) \end{aligned}}$$

STERN GERLACH EXPERIMENT (6.1) (6.2) (6.3)

(1) TORQUE ON A MAGNETIC DIPOLE

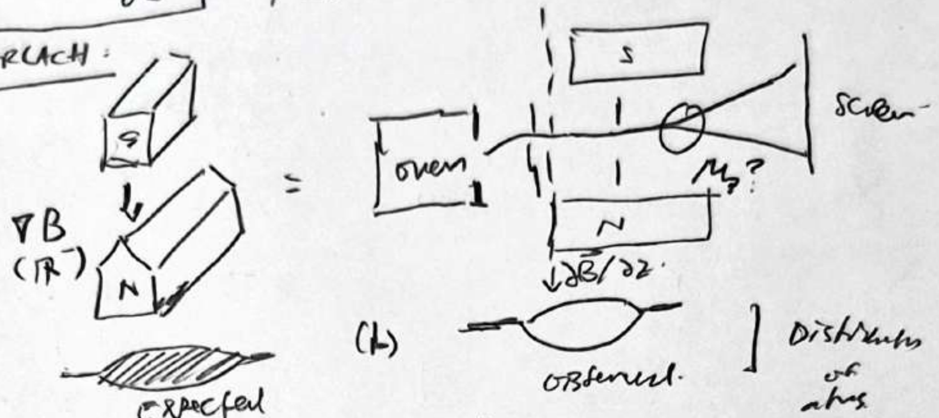
$$\boxed{V = \vec{\mu} \cdot \vec{B}}$$

1.) $\vec{F} = -\nabla V = \nabla(\mu \cdot \vec{B})$

2.) $\boxed{\vec{F} = \mu_z \frac{\partial B_z}{\partial z}}$ IF, the Dipole moment (μ) is so small, $F=0$ basically

- OTTO STERN
- WALTER GERLACH

[i] STERN GERLACH:



Gradient ($\nabla\vec{B}$) = Negative Gradient

Mass atom x Geometry = Deflection via μ_z.

- George Uhlenbeck
- Samuel Goudsmit

∴ Particle was actually spinning

[ii] SPIN:

1.) $\boxed{\vec{\mu} = \gamma \vec{S}}$ Gyromagnetic Ratio

- Uhlenbeck - Goudsmit = Spinning
- Spinning = Magnetic moments x Charged particle
- Uhlenbeck (problem) = Spinning >>> Spinning (normally)
- Spinning >>> Spinning (normally) = Dipole moment

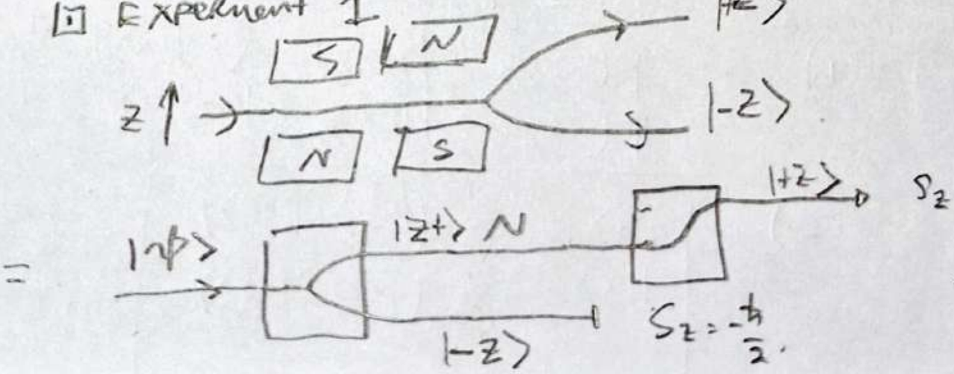
2.) $\boxed{\gamma_e = -1.76 \times 10^{11}\, s^{-1} T^{-1}}$

Recall: s^{-1} = seconds T = Tesla = $\frac{1N}{1A \cdot 1m}$.

3.) $\boxed{S_z = \pm \frac{1}{2}\hbar}$

(2) SPIN STATES

[i] EXPERIMENT 1



→ $\boxed{\langle +z | -z \rangle =}$

∴

(6) Use the matrix Representation of Rotation operator $\hat{R}_p(\theta)$ to write that $\boxed{\hat{R}_p(\theta)|\phi\rangle = |\phi+\theta\rangle}$. ϕ = state of photon?

Ans: so: $\hat{R}_p(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and let φ to be a cos. number.

and $\hat{R}_p(\theta)|\phi\rangle = \cos\theta|\phi\rangle + \sin\theta|\varphi\rangle$

$\hat{R}_T(\theta)|\phi\rangle = -\sin\theta|\cdot\phi\rangle + \cos\theta|\varphi\rangle$

$+ \sin\theta\langle H|H\rangle|\phi\rangle$

and $\hat{R}_\phi(\theta)|\phi\rangle\langle H|H\rangle = \cos\theta\langle H|H\rangle|\phi\rangle$

$\hat{R}_p(\theta)|\phi\rangle = \cos\theta|\phi\rangle$

$\hat{R}_p(\theta)|\phi\rangle\langle V|V\rangle = -\sin\theta\langle V|V\rangle|\phi\rangle + \cos\theta\langle V|H\rangle|\phi\rangle$

$= -\sin\theta|\phi\rangle$

so: $R(\theta)^{+} R_p(\theta)|\phi\rangle|\phi\rangle = (\cos^2\theta + \sin^2\theta)|\phi\rangle$ and $|\phi\rangle = [\phi$

$= \cos^2\theta|\phi\rangle + \sin^2\theta|\phi\rangle$. $|\theta\rangle = [$

$= \langle x(\theta)|x(\phi)\rangle$

also: $\cos\theta|\phi\rangle = x(\theta)|\phi\rangle =$

$|\varphi\rangle\ \hat{R}_p(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{matrix} \cos\theta\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} - \sin\theta\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} \\ \sin\theta\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} + \cos\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}\end{matrix}$

$= \begin{bmatrix} \cos(\theta+\phi_1) + \cos(\theta+\phi_2) & -\sin(\theta+\phi_1) - \sin(\theta+\phi_1) \\ \sin(\theta+\phi_1) + \sin(\theta+\phi_2) & \cos(\theta+\phi_2) - \cos(\theta+\phi_1) \end{bmatrix}$

$= \begin{bmatrix} \cos(\cdot) & -\sin(\cdot) \\ \sin(\cdot) & \cos(\cdot) \end{bmatrix}\begin{bmatrix} \phi_1+\theta_1 \\ \phi_2+\theta_2 \end{bmatrix} = \begin{bmatrix} \phi_1+\theta_1 \\ \phi_2+\theta_2 \end{bmatrix} = |\theta+\phi\rangle$

Ans ★ Let $|\phi\rangle = \cos\phi|H\rangle + \sin\phi|V\rangle = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$.

so: $R_p(\theta)|\phi\rangle = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$

$= \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi \end{bmatrix} = \begin{bmatrix} \cos\phi+\theta \\ \sin\phi+\theta \end{bmatrix} = |\phi+\theta\rangle$

so $\boxed{R_p(\theta)|\phi\rangle = |\phi+\theta\rangle}$

2) $\varphi_i = \langle HV_i | \hat{O} \left[\sum_j |HV_j\rangle\langle HV_j| \right] |\psi\rangle$

$= \sum_j \langle HV_i | \hat{O} | HV_j\rangle \langle HV_j | \psi\rangle$

$= \sum_j \langle HV_i | \hat{O} | HV_j\rangle \psi_j$

$$\boxed{\varphi_i = \sum_j \langle HV_i | \hat{O} | HV_j\rangle \psi_j}$$

3) $O_{ij} = \langle HV_i | \hat{O} | HV_j\rangle$

O_{ij}: $\boxed{\varphi_i = \sum_j O_{ij} \psi_j}$ $\quad O_{ij}\psi_j = \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix} \cdot \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}_{HV}$

$$\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}_{HV} = \begin{bmatrix} O_{11} & O_{12} \\ O_{12} & O_{22} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}_{HV}$$

[ii] EXAMPLES:

[1] 1) $\hat{P}_H \triangleq \begin{bmatrix} \langle HV_1 | \hat{P}_H | HV_1\rangle & \langle HV_1 | \hat{P}_H | HV_2\rangle \\ \langle HV_2 | \hat{P}_H | HV_1\rangle & \langle HV_2 | \hat{P}_H | HV_2\rangle \end{bmatrix}_{HV}$

$= \begin{bmatrix} \langle H | \hat{P}_H | H\rangle & \langle H | \hat{P}_H | V\rangle \\ \langle V | \hat{P}_H | H\rangle & \langle V | \hat{P}_H | V\rangle \end{bmatrix}$

so. $\hat{P}_H = \begin{bmatrix} \langle H | \hat{P}_H | H\rangle & \langle H | \hat{P}_H | V\rangle \\ \langle V | \hat{P}_H | H\rangle & \langle V | \hat{P}_H | V\rangle \end{bmatrix}$

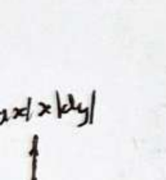
2) $\langle i | j\rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ $\quad$ ho. $\delta_{HV} = \begin{cases} \pm 1 & H = H;\ V = V \\ 0 & H \neq V,\ V \neq H \end{cases}$

hence: $\hat{P}_H = \begin{bmatrix} \hat{P}_H & 0 \\ 0 & \hat{P}_H \end{bmatrix}$ ST det ~~X~~

But Recall That: $\hat{P}_H |H\rangle = |H\rangle$ and $\hat{P}_H |V\rangle = 0$.

ho: $\hat{P}_H = \begin{bmatrix} \langle H|H\rangle & 0 \\ \langle V|H\rangle & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

So: $\boxed{\hat{P}_H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{HV}}$ $\rightarrow$



[2] POLARIZATION ROTATION OPERATOR:

1) $\hat{R}_p(\theta) | HV_1\rangle = \hat{R}_p(\theta) |H\rangle = \cos\theta |H\rangle + \sin\theta |V\rangle$

$\hat{R}_p(\theta) | HV_2\rangle = \hat{R}_p(\theta) |V\rangle = -\sin\theta |H\rangle + \cos\theta |V\rangle$.

$\langle H|H\rangle \cos\theta = \cos\theta$
$\langle H|V\rangle \sin\theta = 0$
$\langle V|H\rangle \cos\theta = 0$
$\langle V|V\rangle \sin\theta = \sin\theta$
etc
...

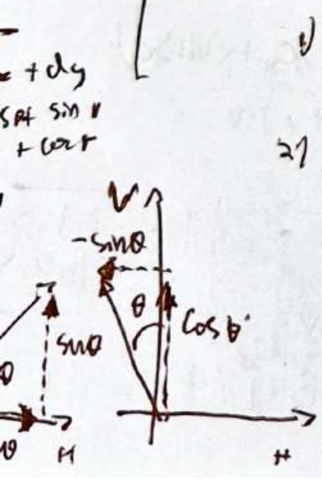
burs

+dy
sin v
+cos t

2) $\hat{R}_p(\theta) = \begin{bmatrix} \langle HV_1 | \hat{R}_p | HV_1\rangle & \langle HV_1 | \hat{R}_p(\theta) | HV_2\rangle \\ \langle HV_2 | \hat{R}_p | HV_1\rangle & \langle HV_2 | \hat{R}_p(\theta) | HV_2\rangle \end{bmatrix}_{HV}$

$= \begin{bmatrix} \langle H | [\cos\theta |H\rangle + \sin\theta |V\rangle] & \langle H | [-\sin\theta |H\rangle + \cos\theta |V\rangle] \\ \langle V | [\cos\theta |H\rangle + \sin\theta |V\rangle] & \langle V | [-\sin\theta |H\rangle + \cos\theta |V\rangle] \end{bmatrix}$

$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_{HV}$ $\quad$ ho: $\boxed{\hat{R}_p(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}$



PROBLEM 6

(13.) From the state $|-n\rangle$ which corresps ~~to~~ the

Ans: If: $\boxed{|+n\rangle = \cos\left(\frac{\theta}{2}\right)|+z\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|-z\rangle}$

$dx = \sin\frac{\theta}{2} + \cos\left(\frac{\theta}{2}\right)$

Then :an analogy here $ds = dx + dy$ :: $dy = -\cos\frac{\theta}{2} + \sin\left(\frac{\theta}{2}\right)$

It should be:

$$\boxed{|-n\rangle = -\sin\left(\frac{\theta}{2}\right)|+z\rangle + e^{i\phi}\cos\left(\frac{\theta}{2}\right)|-z\rangle}$$

(14.) Proof that $|-n\rangle$ is orthogonal to $+n\rangle$

Ans: 1.) $\langle -n| = |-n\rangle^{\dagger} = -\sin\left(\frac{\theta}{2}\right)|+z\rangle^{\dagger} + (e^{i\phi})^{\dagger}\cos\left(\frac{\theta}{2}\right)|-z\rangle^{\dagger}$

$= -\sin\left(\frac{\theta}{2}\right)\langle +z| + e^{-i\phi}\cos\left(\frac{\theta}{2}\right)\langle -z|$.

So: 2.) $\langle -n|+n\rangle = \left[-\sin\left(\frac{\theta}{2}\right)\langle +z| + e^{-i\phi}\cos\left(\frac{\theta}{2}\right)\langle -z|\right]\left[\cos\left(\frac{\theta}{2}\right)|+z\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|-z\rangle\right]$

$= -\sin\left(\frac{\theta}{2}\right)\underbrace{\langle +z|+z\rangle}_{1}\cos\left(\frac{\theta}{2}\right) + \underbrace{e^{-i\phi}e^{i\phi}}_{e^{-i\phi+i\phi} = e^{0} = 1}\cos\left(\frac{\theta}{2}\right)\underbrace{\langle -z|-z\rangle}_{1}\sin\left(\frac{\theta}{2}\right)$

$= -\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) = \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$

$= \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \boxed{0}$ ✓ **QED**

3.)

Problems 8

(1) Calculate the expected value of $\mathcal{P}^{s}_{HV}$, $\mathcal{P}^{i}_{HV}$, and $\mathcal{P}^{si}_{HV}$ for photons prepared in state $|V,L\rangle$

Ans. $|V,L\rangle = \frac{1}{\sqrt{2}}\left[|V\rangle_s \otimes |H\rangle_i + i|V\rangle_s \otimes |V\rangle_i\right]$

$\langle V,L| = \frac{1}{\sqrt{2}}\left[\langle V|_s \otimes \langle H|_i - i\langle V|_s \otimes \langle V|_i\right]$

so, $\langle V,L|V,L\rangle = \left(\frac{1}{\sqrt{2}}\right)^2\left[\langle V|V\rangle\langle H|H\rangle + i\langle V|V\rangle\langle H|V\rangle - i\langle V|V\rangle\langle V|H\rangle + (-i)(i)\langle V|V\rangle\langle V|V\rangle\right]$

$= \frac{1}{2}\left[1+0-0+1\right] = \frac{1}{2}2 = \boxed{1}$ ✓

and so: $\langle \mathcal{P}^{s}_{HV}\rangle = \langle V,L|\mathcal{P}^{s}_{HV}|V,L\rangle = \langle V,L|V,L\rangle = 1$

Similarly $\langle \mathcal{P}^{i}_{HV}\rangle = \langle V,L|\mathcal{P}^{i}_{HV}|V,L\rangle = \langle V,L|V,L\rangle = 1$

$\langle \mathcal{P}^{s}_{iHV}\rangle = \langle \mathcal{P}^{s}_{HV}\rangle\langle \mathcal{P}^{i}_{HV}\rangle = 1\times 1 = 1$

so: $\boxed{\langle \mathcal{P}^{s}\rangle = \langle \mathcal{P}^{i}_{HV}\rangle = \langle \mathcal{P}^{si}_{HV}\rangle = 1}$ → The operator is ~~[illegible]~~ since the:

- Projects only relevant subspace
- Projected component lies entirely on the basis you're projecting on.
- If the state is orthogonal then it's 0! $\langle V|L\rangle = 0$.!

(2) Calculate the expectation values of $\mathcal{P}^{s}_{HV}$, $\mathcal{P}^{i}_{HV}$, $\mathcal{P}^{si}_{HV}$ for photons prepared in state of:

$$\boxed{|\psi\rangle = \left(\frac{1}{3}\right)^{\frac{1}{2}}|H,+45\rangle + \left(\frac{2}{3}\right)^{\frac{1}{2}}|H,-45\rangle}$$

Ans. $\langle \mathcal{P}^{s}_{HV}\rangle = \langle\psi|\mathcal{P}^{s}_{HV}|\psi\rangle = \mathcal{P}^{s}_{HV}\langle\psi|\psi\rangle$? $\langle\psi|$...

since $|\psi\rangle = \left(\frac{1}{3}\right)^{\frac{1}{2}}|H,+45\rangle + \left(\frac{2}{3}\right)^{\frac{1}{2}}|H,-45\rangle = C_1|H,+45\rangle + C_2|H-45\rangle$

$= C_1|H\rangle\otimes|+45\rangle + C_2|H\rangle\otimes|-45\rangle$

so $\langle\psi| = |\psi\rangle^{+} = \left(\frac{1}{3}\right)^{\frac{1}{2}}|H,+45\rangle^{*} - \left(\frac{2}{3}\right)^{\frac{1}{2}}|H,-45\rangle.$

$= \left(\frac{1}{3}\right)^{\frac{1}{2}}\langle H,+45| - \left(\frac{2}{3}\right)^{\frac{1}{2}}\langle H,+45| = C_1\langle H,+45| - C_2\langle H-45|.$

$= C_1\langle H|\otimes\langle +45| - C_2\langle H|\otimes\langle -45|$

so $\langle\psi|\psi\rangle = \left[C_1|H,+45\rangle + C_2|H,-45\rangle\right] =$

① TIME EVOLUTION OPERATOR

BASICS:

- The Schrodinger Eq'n evolves in time w/t:

$$|\psi_0\rangle \longrightarrow |\psi(t)\rangle$$

- If some operator changes our state from initial time $|\psi(t_0)\rangle$ then:

$$\boxed{|\psi(t)\rangle = \hat{U}(t,t_0)\,|\psi(t_0)\rangle}$$

- and if $t_0 = 0$: $|\psi(t_0=0)\rangle = \hat{U}(t,0)\,|\psi(0)\rangle \equiv \hat{U}(t)\,|\psi(0)\rangle$

$$|\psi(t)\rangle\Big|_{t_0=0} = \hat{U}(t,t_0)\Big|_{t_0=0}$$

Given: $\boxed{|\psi(t)\rangle = \hat{U}(t)\cdot|\psi(0)\rangle}$

- Recall That: $\frac{\partial}{\partial t}|\psi(t)\rangle \equiv \frac{\partial \psi(t)}{\partial t} \rightarrow i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \mathcal{H}|\psi(t)\rangle$ and $\mathcal{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x})$.

and That: $\langle x|\psi(t)\rangle = \psi(x,t)$ and $\mathcal{H}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec{x})\psi$.

- Then: $\langle x|\psi(t)\rangle \pm \hat{U}\langle x|\psi(0)\rangle \Rightarrow \psi(x,t) = \hat{U}\psi(x,0) \equiv \hat{U}(t)\psi(x)$.

So: $\boxed{\psi(x,t) = \hat{U}(t)\,\hat{\psi}(x)} \Rightarrow$ The Returned for T.I.S.E

- Follows w/-out saying: $\langle\psi(0)|\psi(0)\rangle U(t)^{\dagger}U(t) = 1\cdot 1 = \boxed{1}$.

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|\,U(t)^{\dagger}U(t)\,|\psi(0)\rangle$$

- **THE PHILOSOPHY / RELEVANCE**: let a system evolve for infinitesimal time dt and letting $\hat{U}(t)$.
  - If we let a system evolve for infinitesimal time dt and letting $\hat{U}(t)$ to be The "TIME operator", Then:

$$\boxed{\hat{U}(dt) = \hat{1} - i\hat{G}_t\,dt} \quad \text{w/ } \hat{G}_t = \text{Generator for Infinitesimal Time Evolution}$$

  - This is also unitary!: $\hat{1} + i(\hat{G}_t + i\hat{G}_t^{\dagger})dt = \hat{1} - i(\hat{G}_t^{\dagger} - \hat{G}_t)dt$.

$$\hat{U}^{\dagger}(dt)\,\hat{U}(dt) = (\hat{1} + i\hat{G}_t\,dt)(\hat{1} - i\hat{G}_t\,dt)$$

$$\hat{=}\ 1 + (0)dt = \boxed{1} \quad \text{so That } \hat{G}_t^{\dagger} = \hat{G}_t$$

② THE SCHRODINGER EQS

[i] THE HAMILTONIAN:

- **PLANK'S ENERGY:**
  - $E = hf = \hbar\omega$ so: $\rightarrow E = \frac{f}{\omega} = \frac{\hbar}{h}$ and $\frac{f}{\omega} \propto \frac{1}{\sqrt{\mu_0\varepsilon_0}}$

  So $\boxed{E = \hbar\omega}$ w/ $\omega = \sqrt{\frac{k}{m}}$.

- **T.I.D.S.E**
  - We can write: $\hat{U}(dt) = 1 - \frac{i}{\hbar}\hat{H}\,dt \rightarrow \boxed{\frac{\partial}{\partial t}\hat{U} = -\frac{i}{\hbar}\hat{H}\hat{U}(t)}$

  w/ $\hat{U}(dt) - 1 = -\frac{i}{\hbar}\hat{H}$ and $\hat{U}(dt) - U(t) = -\frac{i}{\hbar}\hat{U}(t)$ and $\hat{U}(dt) \Rightarrow U(t) \equiv \hat{U}(\Delta t) - U(t) \equiv \frac{dU}{dt}$ ✓

- Continuing:

$$|\psi_0\rangle\left(\frac{\partial\hat{U}}{\partial t} = -\frac{i}{\hbar}\hat{H}\hat{U}(t)\right) \rightarrow -\frac{i}{\hbar}H\hat{U}(t)|\psi_0\rangle \rightarrow \frac{\partial}{\partial t}|\psi(t)\rangle = -\frac{i}{\hbar}H|\psi(t)\rangle$$

so: $\boxed{\frac{\partial}{\partial t}|\psi(t)\rangle = -\frac{i}{\hbar}H|\psi(t)\rangle}$

[ii] SOLUTIONS TO THE SCHRÖDINGER EQS

$$\boxed{\hat{U}(t) = e^{-iHt/\hbar}}$$

w/ $\frac{\partial \hat{U}}{\partial t} = -\frac{iH}{\hbar} e^{-iHt} = -\frac{iH}{\hbar}\hat{U}(t) = -\frac{i}{\hbar}\hat{H}\hat{U}(t)$ & so $\boxed{\frac{\partial \hat{U}}{\partial t} = -\frac{i}{\hbar}\hat{H}\hat{U}}$ ✓

■ ENERGIES & DISCRETE ENERGIES

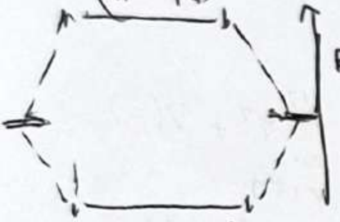
- have $\frac{\partial \hat{U}}{\partial t} \equiv \frac{\partial}{\partial t}|\psi(t)\rangle$ — This means that

$$\boxed{|\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|\psi(0)\rangle} \quad (9.13)$$

$|E_n\rangle = E_n.$

CHEMICAL BONDING

$\sigma(\psi_1 - \psi_2)$



$\sigma(\psi_1 + \psi_n)$

$P_1(\psi_1) : C_1 = \sqrt{\frac{1}{3}}$

$P_2(\psi_2) : C_2 = \sqrt{\frac{2}{3}}$

- The Hamiltonian is an Energy operator so: $\boxed{\hat{H}|E_n\rangle = E_n|E_n\rangle}$

$\hat{H}|E_n\rangle = E_n|E_n\rangle = E_n|E_n\rangle$ (9.14) so:

$\langle x|E_n\rangle = \sum_n C_n\psi_n \equiv C_1\psi_1 + C_2\psi_2 = \frac{1}{3}\psi_1 + (1-\frac{1}{3})\psi_2 \therefore \langle E_n\rangle$

- If the initial energy eigenstate $|\psi(0)\rangle = |E_n\rangle$ then

$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}}|E_n\rangle = \exp\left(-\frac{iE_nt}{\hbar}\right)|E_n\rangle \equiv \exp\left(-\frac{i\omega_nt}{\hbar}\right)|E_n\rangle$

so: $\boxed{|\psi(t)\rangle = \exp\left(-\frac{i\omega_nt}{\hbar}\right)|E_n\rangle}$ (9.15)

- Consequently: if we take $\langle x|$ to EQS (9.15), then

$\langle x|\psi(t)\rangle = \exp\left(-\frac{i\omega_nt}{\hbar}\right)\langle x|E_n\rangle$ let $a_n = \frac{i\omega_n}{\hbar}$

$\Rightarrow \psi(x,t) = \sum_n \exp(-a_nt)\psi_n = e^{-a_nt}[C_1\psi_1 + C_2\psi_2]$.

$\Rightarrow \therefore \boxed{\psi(x,t) \equiv e^{-a_nt}[C_1\psi_1 + C_2\psi_2]} \rightarrow |\psi(t)\rangle = e^{i\omega_nt}|E_n\rangle$ doesn't change in time! The overall phase changes but the state is $|E_n\rangle$

■ EIGENSTATE EXAMPLE

- we let: $|\psi(0)\rangle = C_1|E_1\rangle + C_2|E_2\rangle$. Then in later times:

$|\psi(t)\rangle = \exp\left(-\frac{iHt}{\hbar}\right)|\psi(0)\rangle = \exp(-a_nt)\left[C_1|E_1\rangle + C_2|E_2\rangle\right]$

$\rightarrow C_1|E_1\rangle e^{-a_1t} + C_2|E_2\rangle e^{-a_2t} \equiv e^{-i\omega_1t}\left[C_1|E_1\rangle + C_2e^{-i(\omega_2-\omega_1)t}|E_2\rangle\right]$

so: $\boxed{|\psi(t)\rangle = e^{-i\omega_1t}\left[C_1|E_1\rangle + C_2e^{-i(\omega_2-\omega_1)t}|E_n\rangle\right]}$

③ EXPECTATION VALUES

1.) $\langle H\rangle(t) = \langle\psi(t)|\hat{H}|\psi(t)\rangle = \langle\psi(0)|e^{-i\frac{\hat{H}t}{\hbar}}\cdot\hat{H}\cdot e^{+i\frac{\hat{H}t}{\hbar}}|\psi(0)\rangle$

$= \langle\psi(0)|\hat{H}|\psi(0)\rangle = \langle H\rangle(t=0)$ so: $\boxed{\langle H\rangle(t) = \langle H\rangle(0)}$

2) How about other observables? well:

$\frac{d}{dt}\langle A\rangle = \frac{d}{dt}\langle\psi(t)|\hat{A}|\psi(t)\rangle = \left[\frac{d}{dt}\langle\psi(t)|\right]\hat{A}|\psi(t)\rangle + \langle\psi(t)|\hat{A}\left[\frac{d}{dt}|\psi(t)\rangle\right]$ chain rule!

$= \frac{i}{\hbar}\langle\psi(t)|\hat{H}\hat{A}|\psi(t)\rangle - \frac{i}{\hbar}\langle\psi(t)|\hat{A}\hat{H}|\psi(t)\rangle$

$= \frac{i}{\hbar}\langle\psi(t)|[\hat{H},\hat{A}]|\psi(t)\rangle$

so: $\boxed{\frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}\langle\psi(t)|[\hat{H},\hat{A}]|\psi(t)\rangle}$

MAGNETIC FIELD SPIN

(4) SPIN $\frac{1}{2}$ IN A MAGNETIC FIELD

BASICS

- Energy of a Magnetic dipole w/ $\vec{\mu}$ moment in $\vec{B}$ magnetic field is

$$\boxed{H = -\vec{\mu}\cdot\vec{B}}$$

- Assume it points at z-direction w/ $(\vec{B} = B\,\vec{u}_z)$ and $(\vec{\mu} = \gamma\vec{S})$. Then

$$\boxed{H = -\gamma S_z \vec{B}}$$

so $\boxed{\hat{H} = -\hat{S}_z \Omega}$ w/ s^{-1} units

- So the Hamiltonian is:

$$\hat{H} = -\gamma \hat{S}_z \vec{B} = -\gamma \vec{B}\,\hat{S}_z = -\hat{S}_z \Omega$$

The LARMOR FREQUENCY

$\{S_z = \frac{\hbar}{2}\}$

INTRODUCING THE TIME OPERATOR

- At Time zero, say we have a state $|+z\rangle$. Then

$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}}|+z\rangle = \exp\left(-\frac{i\Omega\hat{S}_z}{\hbar}t\right)|+z\rangle = \exp\left(-i\Omega\left(\frac{\hbar}{2}\right)t\frac{1}{\hbar}\right)|+z\rangle$$

$$= \exp\left(-i\frac{\Omega}{2}\right)|+z\rangle \quad \text{so } \boxed{|\psi(t)\rangle = e^{-i\frac{\Omega}{2}}|+z\rangle}$$

- But this will be different if we start on $|+x\rangle$ s.t.:

$$|\psi(t)\rangle = \exp\left(-\frac{iH}{\hbar}t\right)|+x\rangle = \exp\left(-\frac{iH}{\hbar}t\right)\frac{1}{\sqrt{2}}\left[|+z\rangle + |-z\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[e^{+\frac{i\Omega S_z t}{\hbar}}|+z\rangle + e^{\frac{i\Omega S_z t}{\hbar}}|-z\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[e^{i\Omega\left(\frac{\hbar}{2}\right)\frac{t}{\hbar}}|+z\rangle + e^{-i\Omega\frac{\hbar}{2}t}|z\rangle\right]$$

$$= e^{\frac{i\Omega t}{2}}\frac{1}{\sqrt{2}}\left[|+z\rangle + e^{-i\Omega t}|-z\rangle\right] \quad \text{so } \boxed{|\psi(t)\rangle = e^{\frac{i\Omega t}{2}}\frac{1}{\sqrt{2}}\left[|+z\rangle + e^{-i\Omega t}|-z\rangle\right]}$$

- At $\left(t = \frac{\pi}{2\Omega}\right)$, the state is (EX):

$$\left|\psi\left(\frac{\pi}{2\Omega}\right)\right\rangle = e^{i\frac{\pi}{4}}\frac{1}{\sqrt{2}}\left[|+z\rangle + e^{-i\frac{\pi}{2}}|-z\rangle\right].$$

$$= \frac{1}{\sqrt{2}}\left(e^{i\frac{\pi}{4}}|+z\rangle + e^{-i\frac{\pi}{4}}e^{-i\frac{\pi}{2}}|-z\rangle\right) \Rightarrow$$

$$= \frac{1}{\sqrt{2}}\left(e^{i\frac{\pi}{4}}|+z\rangle + e^{-i\frac{3\pi}{4}}|-z\rangle\right)$$

$$= e^{i\frac{\pi}{4}}|-y\rangle \quad \text{w/ } \boxed{\frac{\pi}{2} := y}$$

$$\exp\left(-\left(i\frac{\pi}{4} + i\frac{\pi}{2}\right)\right) = \exp\left(-\left(i\left(\frac{\pi}{4} + \frac{\pi}{2}\right)\right)\right) = \exp\left(-\left(i\left(\frac{\pi + 2\pi}{2}\right)\right)\right) = \exp\left(-\left(i\left(\frac{3\pi}{2}\right)\right)\right) = \exp\left(-i(4.7124)\right) = e^{-i4.71234}$$

- Whilst at $\left(t = \frac{\pi}{\Omega}\right)$:

$$\left|\psi\left(t = \frac{\pi}{\Omega}\right)\right\rangle = e^{i\frac{\pi}{2}}\frac{1}{\sqrt{2}}\left[|+z\rangle + e^{i\pi}|-z\rangle\right]$$

$$= e^{i\frac{\pi}{2}}|x\rangle \qquad \boxed{\pi := x}$$

Since $\frac{\pi}{2} + \frac{\pi}{2} = \pi = x$.

$$\text{so } \boxed{\begin{aligned}\left|\psi\left(\tfrac{\pi}{2\Omega}\right)\right\rangle &= e^{i\frac{\pi}{4}}|-y\rangle \\ \left|\psi\left(\tfrac{\pi}{\Omega}\right)\right\rangle &= e^{i\frac{\pi}{2}}|-x\rangle\end{aligned}}$$

EXPECTED VALUE OF SPIN

- How do we confirm that the state is changing in time? Well one way to look at it is to take the expected value of $\vec{S}$ s.t.:

$$\boxed{\langle S\rangle(t) = \langle S_x\rangle(t)\,\vec{u}_x + \langle S_y\rangle(t)\,\vec{u}_y + \langle S_z\rangle(t)\,\vec{u}_z}$$

- The expectation value is thus for (x):

$$\langle S_x\rangle(t) = \langle\psi(t)|\hat{S}_x|\psi(t)\rangle = \left(e^{-\frac{i\Omega t}{2}}\frac{1}{\sqrt{2}}\left[\langle +z| + e^{i\Omega t}\langle -z|\right]\right)\hat{S}_x\left(e^{\frac{i\Omega t}{2}}\frac{1}{\sqrt{2}}\left[|z\rangle + e^{-i\Omega t}|-z\rangle\right]\right.$$

$$= \frac{\hbar}{4}\left(e^{-i\Omega t} + e^{i\Omega t}\right) = \frac{1}{2}\left[\langle +z| + e^{i\Omega t}\langle -z|\right]\left[\frac{\hbar}{2}|z\rangle + e^{-i\Omega t}\frac{\hbar}{2}|+z\rangle\right]$$

$$= \frac{1}{2}\left(\frac{\hbar}{2}\langle +z|-z\rangle + e^{\eta(t)}\langle +z|+z\rangle\frac{\hbar}{2} + \langle -z|-z\rangle\frac{\hbar}{2} + e^{-\eta(t)}\frac{\hbar}{2}\right)$$

$$= \frac{1}{2}\left(0 + e^{\eta t}\frac{\hbar}{2} + e^{-\eta t}\frac{\hbar}{2} + 0\right) = \frac{\hbar}{4}\left(e^{\eta t} + e^{-\eta t}\right)$$

$$= \boxed{\frac{\hbar}{2}\cos(\Omega t)}$$

- For y it must be $\langle S_y\rangle_t = -\frac{\hbar}{2}\sin(\Omega t)$:

w/ $\eta(t) = \eta_t = \Omega t$.

$$\langle S_y\rangle(t) = \langle \psi(t)|\hat{S}_y|\psi(t)\rangle$$

$$= \left[e^{-\frac{i\Omega t}{2}}\frac{1}{\sqrt{2}}\left[\langle +z| + e^{-i\Omega t}\langle -z|\right]\right]\hat{S}_y\left[e^{\frac{i\Omega t}{2}}\frac{1}{\sqrt{2}}\left[|+z\rangle + e^{-i\Omega t}|-z\rangle\right]\right].$$

$$= \frac{1}{2}\left[\left[\langle +z| + e^{i\Omega t}\langle -z|\right]\left[i\frac{\hbar}{2}|-z\rangle - ie^{-i\Omega t}\frac{\hbar}{2}|+z\rangle\right]\right]$$

$$= \frac{1}{2}\frac{\hbar}{2}\left[-ie^{-i\Omega t} + ie^{i\Omega t}\right]$$

$$= \frac{\hbar}{4}\left[2ie^{-i\Omega t} + ie^{\Omega t}\right] = \boxed{-\frac{\hbar}{2}\sin(\Omega t)}$$

$$\boxed{\begin{aligned}\langle S_x\rangle(t) &= \frac{\hbar}{2}(\cos(\Omega t))\\ \langle S_y\rangle(t) &= -\frac{\hbar}{2}(\sin(\Omega t))\end{aligned}}$$

Wave Eqs [illegible] (11.1) (11.2) (11.3)

① $\boxed{\hat{H}|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle}$ Schrodinger Eqs. - Compact form

① what can we do with this?

[i] Assumptions:

1) $\hat{H} = \frac{P^2}{2m} + V(x)$: $P^2 = (i\hbar)^2\frac{\partial^2}{\partial x} = -\hbar^2\nabla^2$. $P = i\hbar\nabla$

2) $\psi(x,t) = \langle x|\psi(t)\rangle$.

So: $\hat{H}\psi = \left[-\frac{\hbar\nabla^2}{2m} + V(x)\right]\psi(x,t) = \boxed{-\frac{\hbar}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}}$

Schrodinger Eqs - [illegible]

[ii] Time independent solutions:

1) $\Psi(x,t) = \psi(x)\,\phi(t)$

$|\psi(t)\rangle = e^{i\frac{Ht}{\hbar}}|\psi(0)\rangle$

[illegible] $\boxed{\Psi = \sum_n c_n\psi_n(x)\,\phi_n(t)}$

$c_n = \int_0^w \psi_n^*(x,0)\,\psi(x,0)\,dx$

$= \int_0^\infty |\psi|^2\,dx = 1$

2) $|\psi(0)\rangle = \sum_n c_n|\psi_n\rangle$

$\psi(x,0) = \langle x|\psi(0)\rangle = \sum_n c_n\langle x|\psi_n\rangle = \sum_n c_n\psi_n(x)$

So: $\boxed{|\psi(0)\rangle = \sum_n c_n\langle x|\psi_n\rangle = \sum_n c_n\psi_n(x)}$

$\boxed{E = \hbar w_n}$

3) $c_n = \langle\psi_n|\psi_0\rangle = \int_0^\infty\langle\psi_n|x\rangle\langle x|\psi_0\rangle\,dx = \int_0^\infty\psi_n^*\,\psi(x,0)\,dx$.

4) $|\psi(t)\rangle = \sum_n c_n e^{-i\frac{Ht}{\hbar}}|\psi_n\rangle = \sum_n c_n e^{-i\frac{E_n t}{\hbar}}|\psi_n\rangle = \sum_n e^{i\omega_n t}|\psi_n\rangle$

or: $\boxed{|\psi(t)\rangle = \sum_n e^{i\omega_n t}|\psi_n\rangle}$

5) $\Psi(x,t) = \langle x|\psi(t)\rangle \Rightarrow \boxed{\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}}$ $\boxed{11.8}$

[iii] Boundary conditions

• If uncertainty is finite [V(x) finite] ⇒ ∴ The integral is finite

are smooth functions

$$\boxed{\int_{x_0-\eta}^{x_0+\eta}\frac{\partial^2\psi}{\partial x^2} = \frac{d\psi_n}{dx}(x_0+\eta) - \frac{d\psi_n}{dx}(x_0-\eta) = \frac{2m}{\hbar}\int_{x_0-\eta}^{x_0+\eta}[V - E_n]\psi_n\,dx}$$

② Free particle

[i] $\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V_0\psi = E\psi \Rightarrow \frac{\partial^2\psi}{\partial x^2} = \frac{2m(E-V_0)\psi(x)}{\hbar}$

$k = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$

So: $\boxed{\psi(x) = Ae^{ikx} + Be^{-ikx}}$

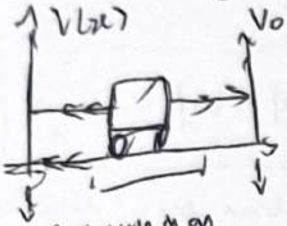
$\boxed{\Psi(x,t) = Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}}$ → Full solution.

[A] Implications:

1) $\psi(A,B)$: A is travelling positive direction $\xrightarrow{+}$
B is travelling negative direction $\xleftarrow{-}$

2.) The eigenstates are the same as $\boxed{k = \frac{p}{\hbar}}$ where h is a De-Broglie relation: $\boxed{A = \frac{1}{\sqrt{2\pi\hbar}}}$

$V = \begin{cases} 0 & \text{if not push} \\ \infty & \text{if pushed} \end{cases}$



A wagon in an infinite well. Enough of infinite time it always bounces but 0 if it's not moved

[3] POTENTIAL STEP

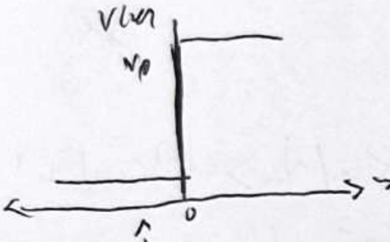
[i] Reflection / Absorption

1) $V(x) = \begin{cases} 0 \\ V_0 \end{cases}$

2.) $\psi(x) = A_1 e^{ik_1 x} + B_1 e^{-ik_1 x}$
w/ $\boxed{k_1 = \sqrt{\frac{2mE}{\hbar^2}}}$

3.) $\psi(x) = A_2 e^{ik_2 x} + B_2 e^{-ik_2 x}$
w/ $\boxed{k_2 = \sqrt{\frac{2m(E - V_0)}{\hbar^2}}}$ $\boxed{E \to E - V_0}$

[ii] Probability Flux

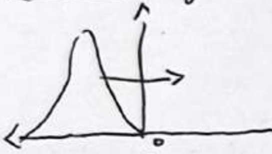


1.) $\frac{\partial |\psi(x,t)|^2}{\partial t} = \frac{\partial \psi^* \psi}{\partial t} = \psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t}$

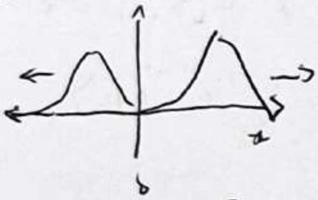
2.) So: $\boxed{\frac{\partial |\psi|^2}{\partial t} = \psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial t}}$

3.) Recall: $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} \longrightarrow \boxed{\frac{\partial \psi}{\partial t} = -\frac{\hbar}{i2m}\frac{\partial^2 \psi}{\partial x^2}}$

$i\hbar \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi^*}{\partial x^2} \longrightarrow \boxed{\frac{\partial \psi^*}{\partial t} = -\frac{\hbar}{i2m}\frac{\partial^2 \psi^*}{\partial x^2}}$



So we: $\psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial t} = \frac{\hbar}{i2m}\left[\psi^* \frac{\partial^2 \psi}{\partial x^2} - \psi \frac{\partial^2 \psi^*}{\partial x^2}\right]$.



2.) Let $\boxed{J_x(x,t) = \frac{\hbar}{i2m}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)}$.

Thus $\frac{\partial J}{\partial x} = \frac{\hbar}{i2m}\left[\cancel{\left(\frac{\partial \psi^*}{\partial x}\right)}\left(\frac{\partial \psi}{\partial x}\right) + \psi^* \frac{\partial^2}{\partial x^2}\psi - \cancel{\left(\frac{\partial \psi}{\partial x}\frac{\partial \psi^*}{\partial x}\right)} - \psi \frac{\partial^2 \psi}{\partial x^2}\psi^*\right]$

$= \frac{\hbar}{i2m}\left[\psi^* \nabla \psi - \psi \nabla \psi^*\right]$

So: $\boxed{\frac{\partial |\psi(\vec{r},t)|^2}{\partial t} = \frac{\partial J}{\partial x} = \nabla J(x)}$

Divergence Theorem to replace
· volume integral Ω
· closed surface S that bounds S.

3.) w/: $\iiint_\Omega \frac{\partial |\psi|^2}{\partial t} d^3\vec{r} = -\iiint \nabla \cdot \vec{J}\, d^3\vec{r}$.

$\frac{\partial}{\partial t}\iiint_\Omega |\psi(\vec{r},t)|^2 d^3\vec{r} = -\oiint \vec{J}(\vec{r},t)\, d\vec{s}$.

Thus $\iiint \frac{\partial |\psi|^2}{\partial t} d^3\vec{r} = -\iiint \nabla \cdot \vec{J}\, d^3\vec{r}$.

MOSFET, Extensions, Tunneling (11.3) (11.4)

① CASES of the flux eq.

recall: $\boxed{\nabla J(x,t) = \frac{\partial |\phi(x,t)|^2}{\partial t}}$

i) $E > V_0$.

1) recall that: $\begin{cases} k_1 = \sqrt{\frac{2mE}{\hbar^2}} \\ k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}} \end{cases}$

2) Assume $k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}} \Big|_{E > V_0}$ s.t $E - V_0 > 0$.

here $\boxed{k_2 \in \mathbb{R}^+ \ \forall t \ E - V_0 \ \forall E > V_0.}$

3.) $A_1 + B_1 = A_2$
$\boxed{ik_1A_1 - ik_1B_1 = ik_2A_2.}$ → Applied the boundary condition of $\frac{2m}{\hbar^2}\int_{x_0-\eta}^{x_0+\eta}[V(x) - E_n]\phi\, dx$

✓ PROBLEM 11.2

confirm eqs 11.44 are correct.

4.) $\cancel{ik_1}A_1 - \cancel{i}k_2A_2 = \cancel{i}k_1B_1$
$k_1A_1 - k_2A_2 = k_1B_1$
$k_1B_1 = k_1A_1 - k_2(A_1+B_1) = k_1A_1 - k_2A_1 - k_2B_1.$
$k_1B_1 = k_1A_1 - k_2A_1$
$k_1B_1 + k_2B_1 = k_1A_1 - k_2A_1$
$B_1(k_1+k_2) = A_1(k_1-k_2)$
$\boxed{B_1 = \frac{(k_1-k_2)}{(k_1+k_2)}A_1}$ (11.44a).

recall that:
$\phi_1 = A_1e^{ik_1x} + B_1e^{ik_1x}$
$\phi_2 = A_2e^{ik_2x} + B_2e^{ik_2x}$

5) $k_1B_1 - k_1A_1 = -k_2A_2$ $\quad B_1 = A_2 - A_1$
$k_1(A_2 - A_1) - k_1A_1 = -k_2A_2$
$k_1A_2 - k_1A_1 - k_1A_1 = -k_2A_2$
$k_1A_2 - k_1A_1 - k_1A_1 = +k_2A_2 - k_1A_2$
$+k_1A_1 - k_1A_1 = $
$2k_1A_1 = A_2(k_2+k_1) \Rightarrow \boxed{A_2 = \frac{2k_1}{k_2+k_1}A_1}$ (11.44B)

$B_1 = \frac{k_1-k_2}{k_1+k_2}A_1$
$= \frac{k_1-k_2}{k_1+k_2}\frac{k_2+k_1}{2k_1}A_2$
$= \frac{k_1-k_2}{2k_1}$

6) Summary:
- $A_1 = \frac{k_2+k_1}{2k_1}A_2$
- $A_2 = \frac{2k_1}{k_2+k_1}A_1$
- $B_1 = \frac{(k_1-k_2)}{k_1+k_2}A_1$
- $B_1 = \frac{k_1-k_2}{2k_1}A_2.$

7.) $J_x^{A_1}(x,t) = \frac{\hbar}{m}\mathrm{Im}\left(A_1^*e^{-i(k_1x-\omega_1t)} \cdot \frac{\partial}{\partial x}A_1e^{ik_1x-\omega_1t}\right)$

recall $i(k_1x-\omega t)$
$F(x,t) = e$
are $F \in (\mathbb{R}, \mathbb{C})$
but in this case $F \in \mathbb{R}$.

$= \frac{\hbar}{m}\mathrm{Im}\left(A_1^*e^{ik_1x-\omega t)} \, ik_1A_1e^{(ik_1x-\omega t)}\right)$

$\frac{\hbar k_1}{m} = \frac{p_1}{m} = v$ (11.46)
as $\boxed{p = mv}$

$= \frac{\hbar}{m}\mathrm{Im}(A_1^*\cancel{F}ik_1A_1\cancel{F}) = \frac{\hbar}{m}k_1A^*A_1 = \frac{\hbar k_1}{m}|A_1|^2.$
$= v_1|A_1|^2.$ $\quad$ so: $\boxed{J^{A_1} = v|A_1|^2}$ (11.45)

8.) $\boxed{R = \frac{J_x^{B_1}(x,t)}{J_x^{A_1}(x,t)} = \text{probability that the particle will be found to reflect}}$

R = Reflect
T = Transmit

w: $R = \left|\frac{J_x^{B_1}}{J_x^{A_1}}\right| = \frac{k_1 |B_1|^2}{v_1 |A_1|^2} = \frac{|B_1|^2}{|A_1|^2} = \frac{(k_1-k)^2}{(k_1+k_2)^2} = \frac{(\sqrt{E}-\sqrt{E-V_0})^2}{(\sqrt{E}+\sqrt{E-V_0})^2}$

5) $\boxed{T = 1-R} = 1 - \frac{(k_1-k_2)^2}{(k_1+k_2)^2} = \frac{(k_1+k_2)^2-(k_1-k_2)^2}{(k_1+k_2)^2}$

$= \frac{[k_1^2+2k_1k_2+k_2^2]-[k_1^2-2k_1k_2+k_2^2]}{(k_1+k_2)^2}$

$= k_1^2+2k_1k_2+k_2^2-k_1^2+2k_1k_2-k_2^2/(k_1+k_2)^2$

$= (k_1^2+k_2^2)-(k_1+k_2^2)+2k_1k_2+2k_1k_2/(k_1+k_2)^2$

$= 4k_1k_2/(k_1+k_2)^2 = \frac{4\sqrt{E(E-V_0)}}{(E+\sqrt{E-V_0})^2}$

w: $T = \frac{4\sqrt{E(E-V_0)}}{(E+\sqrt{E-V_0})^2}$, $R = \frac{(E-\sqrt{E-V_0})^2}{(\sqrt{E}+\sqrt{E-V_0})^2}$

$R=1$, $T=0$ if $E \approx V_0$

$R=0$, $T=1$ if $E > V_0$

- It transmits if there's "no V_0" since its wholly energy
- It Reflects if the energy apparently matches the potential

[ii] $E < V_0$

1) Then $k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar}}$ | $E<V_0$

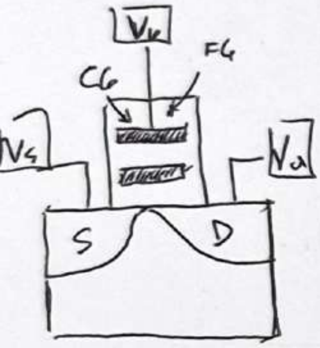
$= \sqrt{-\frac{2m(E-V_0)}{\hbar}} = i\sqrt{\frac{2m(E-V)}{\hbar}} = i\alpha$ w/ $\alpha \in \mathbb{R}^+$

2) w: $\boxed{\phi(x) = Ae^{\alpha x} + Be^{\alpha x}}$

3) then: $B_1 = \frac{ik_1+\alpha}{(ik_1-\alpha)}A_1$ $A_2 = \frac{2ik_1}{(ik_1-\alpha)}A$

4) $R = \left|\frac{J_x^{B_1}(x+t)}{J_x^{A_1}(x+t)}\right| = \frac{k|B_1|^2}{k_1|A_1|^2} = \left|\frac{(ik+\alpha)}{(ik-\alpha)}\right|^2 = 1.$

so clearly $T=0$



S = Source
D = Drain
CG = Control Gate
V = Volts

↓

MOSFET

Metal-oxide Field Effect Transistor

③ QUANTUM TUNNELING

So far: $\phi(x) = \begin{cases} A_1 e^{ikx} + B_1 e^{-ikx} & x<0 \\ A_2 e^{-\alpha x} + B_2 e^{\alpha x} & 0<x<L \\ A_3 e^{ikx} & x>L \end{cases}$

w/ $k = \sqrt{\frac{2mE}{\hbar^2}}$, $\alpha = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}$

[i] Implications
1) TUNNEL = Doesn't have Enough Energy to Go Through Barrier
2) FULL REFLECTION = $L \gg \alpha^{-1}$ so $P(R)=1$
3) IMAGINARY = $i\alpha$ means that the particle doesn't "propogate" through the Barrier in a Traditional sense.
4) SUPERLUMINAL

Angular Momentum & Commutation Rules (7.1) (7.2) (7.3)

① Commuting observables:

$$\boxed{[\hat{A},\hat{B}]=0}$$

This means: $\hat{A}\hat{B}-\hat{B}\hat{A} = \hat{A}^2 - A^2 = 0 = \hat{B}^2-\hat{B}^2 = \hat{A}\hat{B}-\hat{B}\cdot A$ $\forall \begin{vmatrix}\hat{A}_x & \hat{A}_y\\ \hat{B}_x & \hat{B}_y\end{vmatrix}$

1) For Commutation $[\hat{A},\hat{B}]=0$, there exists a complete set of states $|\lambda\rangle$ that are simultaneously eigenstates of both $\hat{A}$, $\hat{B}$ s.t.:

$$\boxed{\begin{aligned}\hat{A}|\lambda\rangle &= a_\lambda|\lambda\rangle\\ \hat{B}|\lambda\rangle &= b_\lambda|\lambda\rangle\end{aligned}}$$ $\forall [\hat{A},\hat{B}]=0\ \exists|\lambda\rangle$ s.t. $\begin{cases}\hat{A}|\lambda\rangle = a_\lambda|\lambda\rangle\\ \hat{B}|\lambda\rangle = b_\lambda|\lambda\rangle\end{cases}$ • Completeness of Eigenvalues

2) Let $|\lambda\rangle = |a,b\rangle$. Then:

$\hat{A}|\lambda\rangle = \hat{A}|a,b\rangle \rightarrow \langle\lambda|\hat{A}|\lambda\rangle = \langle a,b|\hat{A}|a,b\rangle$

$\hat{A}=\hat{A}$ and $a=\hat{A}$ so: $A|\lambda\rangle = a|a,b\rangle$

similarly $\hat{B}|\lambda\rangle = \hat{B}|a,b\rangle \rightarrow \langle\lambda|\hat{B}|\lambda\rangle = \langle a,b|\hat{B}|a,b\rangle$

$\langle a,b|\hat{B}B^\dagger|a,b\rangle = [\ ][\ ]\ \hat{B}\hat{B}^\dagger = 1.$

thus $\boxed{\begin{aligned}\hat{A}|a,b\rangle &= a|a,b\rangle\\ \hat{B}|a,b\rangle &= b|a,b\rangle\end{aligned}}$

Inversely: $\forall [\hat{A},\hat{B}]\neq 0,\ \nexists|\lambda\rangle$ with $\begin{cases}\hat{A}|\lambda\rangle \neq a_\lambda|\lambda\rangle\\ \hat{B}|\lambda\rangle \neq b_\lambda|\lambda\rangle\end{cases}$ • Incompleteness of Eigenvalues

② Angular Momentum Operators

[1] Total:

1) a) $\vec{S} = S_x\vec{u}_x + \hat{S}_y\vec{u}_y + \hat{S}_z\vec{u}_z \rightarrow$ Spin vector

b) $\boxed{\vec{J} = \vec{S}+\vec{L}} \rightarrow$ Total Angular Momentum.

c) $\vec{J} = \hat{J}_x\vec{u}_x + \hat{J}_y\vec{u}_y + \hat{J}_z\vec{u}_z \rightarrow$

2) $[\hat{J}_x,\hat{J}_y] = i\hbar J_z$

$[\hat{J}_y,J_z] = i\hbar J_x$

$[\hat{J}_z,J_x] = i\hbar J_y$

thus $\boxed{[J_i,J_j] = i\hbar J_k}$ w/ $i\neq j\neq k$.

Since $i=j$ then $[J_i,J_i]=0$.

• If $\vec{L} = \int(r\times\omega)\,d\vec{r} = 0$

then $\vec{J} = S+\vec{L} = S+0 = S$

or $\boxed{J=S}$ iff $L=0$.

3) $\hat{J}_z|+z\rangle = \frac{\hbar}{2}|+z\rangle$

$\hat{J}_z|-z\rangle = \frac{\hbar}{2}|-z\rangle$

[ii] The J^2 operator:

1) $\boxed{\hat{J}^2 = J\cdot J = \hat{J}_x^2+\hat{J}_y^2+\hat{J}_z^2}$ w/

$$\begin{bmatrix}J_x\\J_y\\J_z\end{bmatrix}\cdot\begin{bmatrix}J_x\\J_y\\J_z\end{bmatrix} = \begin{bmatrix}J_x\begin{bmatrix}J_x\\J_y\\J_z\end{bmatrix}\\ J_y\begin{bmatrix}J_x\\J_y\\J_z\end{bmatrix}\\ J_z\begin{bmatrix}J_x\\J_y\\J_z\end{bmatrix}\end{bmatrix} = \begin{bmatrix}J_x^2+J_xJ_y+J_xJ_z\\ J_xJ_y+J_y^2+J_yJ_z\\ J_zJ_z+J_zJ_y+J_z^2\end{bmatrix} = \begin{bmatrix}J_x^2+0+0\\0+J_y^2+0\\0+0+J_z^2\end{bmatrix} = \begin{bmatrix}J_x^2\\J_y^2\\J_z^2\end{bmatrix} \equiv J_x^2+J_y^2+J_z^2$$

2) $[\hat{A}\hat{B},\hat{C}] = \hat{A}\hat{B}(\hat{C}) - \hat{C}(\hat{A}\hat{B})$

$= \hat{A}(\hat{B}\hat{C}) - (\hat{A}\hat{C})\hat{B}$

$= \hat{A}(\hat{B}\hat{C}-\hat{C}\hat{B}) - \hat{B}(\hat{A}\hat{C}-\hat{C}\hat{A})$

$= \hat{A}[\hat{B},\hat{C}] + \hat{B}[\hat{A},\hat{C}]$

thus $\boxed{[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}]+\hat{B}[\hat{A},\hat{C}]}$

3) $\hat{J}^2, J_z = [(\hat{J}_x^2+J_y^2+J_z^2), J_z] = [J_x^2,\hat{J}_z] + [J_y^2,J_z] + \cancel{[J_z^2,J_z]} = 0.$

$= \hat{J}_x[\hat{J}_x,\hat{J}_z] + [\hat{J}_x,\hat{J}_z]\hat{J}_x + \hat{J}_y[\hat{J}_y,\hat{J}_z] + [\hat{J}_y,\hat{J}_z]\hat{J}_y \ldots$

so: $[\hat{J}^2,J_z] = [J_x^2,\hat{J}_z] + [J_y^2,J_z] = 0$

2) $\hat{J}^2 = J_x^2 + J_y^2 + J_z^2 = \frac{\hbar^2}{4}\{\hat{\sigma}_x^2 + \hat{\sigma}_y^2 + \hat{\sigma}_z^2\} = \frac{\hbar^2}{4}\left\{\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix} + \begin{bmatrix}0&-i\\i&0\end{bmatrix}\begin{bmatrix}0&-i\\i&0\end{bmatrix} + \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\right\}$

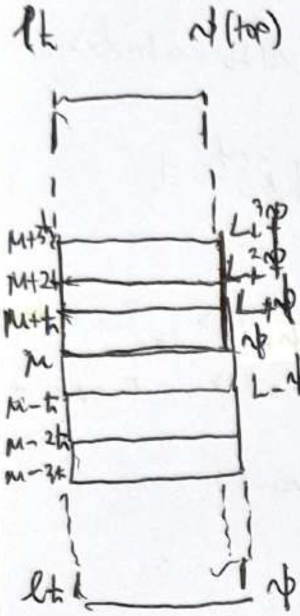
$= \frac{\hbar^2}{4}\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix} + \begin{bmatrix}1&0\\0&1\end{bmatrix} + \begin{bmatrix}1&0\\0&1\end{bmatrix}\right\} = \frac{3}{4}\hbar^2\begin{bmatrix}1&0\\0&1\end{bmatrix}$

hence $\boxed{\hat{J}^2 = J_x^2 + J_y^2 + J_z^2 = \frac{3}{4}\hbar^2\begin{bmatrix}1&0\\0&1\end{bmatrix}}$

~~$\sigma_x^2 = \begin{bmatrix}0&1\\1&0\end{bmatrix}^2 \quad \sigma_y^2 = \begin{bmatrix}0&-i\\i&0\end{bmatrix}^2 \quad \sigma_z^2 = \begin{bmatrix}1&0\\0&-1\end{bmatrix}^2$~~

③ EIGENVALUES and EIGENSTATES

LADDER



- Each Neighbors are separated via one unit ħ
- Eventually we're gonna see/reach a state for which the z-component exceeds the total.
- hence $\boxed{\exists\ \psi(top),\ L_+\psi(top) = 0}$

[i] BASICS:

1) $\boxed{\begin{aligned}\hat{J}^2|j,m_j\rangle &= j(j+1)\hbar^2|j,m_j\rangle\\ \hat{J}_z|j,m_j\rangle &= m_j\hbar|j,m_j\rangle\end{aligned}}$

Allowed values of j are $j = \frac{n}{2}$, $n \in \mathbb{Z}$ so: $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots$

2) allowed values of m_j are $m_j \in [-j, j] = -j, -j+1, -j+2, \ldots, j-2, j-1, j$

$= 0-j, 1-j, 2-j, \ldots, j-2, j-1, j-0.$

3) matrix formalism:

$\hat{J}^2 = \hbar^2 j(j+1) I = \hbar^2 2\cdot I = \hbar^2\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}$

$J_z = \hbar\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}$ hence $\boxed{J_z = \hbar\begin{bmatrix}1&&\\&0&\\&&-1\end{bmatrix} \quad J^2 = \hbar^2\begin{bmatrix}2&&\\&2&\\&&2\end{bmatrix}}$

4) Derivation?:

- $[L^2, L_z] = 0 \rightarrow \begin{cases}L^2\psi = \lambda\psi\\ L_z\psi = \mu\psi\end{cases}$ with ψ is a test function. $(\lambda, \mu) :=$ Eigenstates(L^2, L_z).

 and $\boxed{L_\pm = L_x \pm iL_y}$

- $[L_z, L_\pm] = [L_z, L_x] \pm i[L_z, L_y] = i\hbar L_y \pm i(-i\hbar L_x) = \pm\hbar(L_x \pm iL_y)$.

 $L_z(L_\pm\psi) = (L_zL_\pm - L_\pm L_z)\psi + L_\pm L_z\psi = \pm\hbar L_\pm\psi + L_\pm(\mu\psi)$

 $= (\mu \pm \hbar)L_\pm\psi$

- $\begin{cases}L_z\psi(top) = l\hbar\psi(top)\\ L^2\psi(top) = \lambda\psi(top)\end{cases} \Rightarrow L_\pm L_\mp = (L_x \pm iL_y)(L_x \mp iL_y) = L_x^2 + L_y^2$

 $= L_x^2 + L_y^2 \mp i(L_xL_y - L_yL_x)$

 $= L^2 - L_z^2 \mp i(i\hbar L_z)$.

$\rightarrow \boxed{L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z}$

hence $L^2\psi(top) = (L_-L_+ + L_z^2 + \hbar L_z)\psi(top) = (0 + \hbar^2l^2 + \hbar^2 l)\psi(top)$

$= \hbar^2 l(l+1)\psi(top)$

[ii] RAISING and LOWERING OPERATORS

1) $\hat{J}_\pm = J_x \pm iJ_y \rightarrow \begin{cases}\hat{J}_+|j,m_j\rangle = \hbar[j(j+1) - m_j(m_j+1)]^{\frac{1}{2}}|j,m_j+1\rangle\\ \hat{J}_-|j,m_j\rangle = \hbar[j(j+1) - m_j(m_j-1)]^{\frac{1}{2}}|j,m_j-1\rangle\end{cases}$

hence $\boxed{J_\pm|j,m_j\rangle = \hbar\sqrt{j(j+1) - m_j(m_j\pm1)}\times|j,m_j\pm1\rangle}$

2) If $m_j = j$ then $J_+|j,j\rangle = \hbar|j(j+1) - j(j+1)|^{\frac{1}{2}}|j,j\rangle = \hbar\cdot0\cdot0 = \boxed{0}$ so $\boxed{J_+|j,j\rangle = 0}$

[iii] SPINS:

1) $S^2|+z\rangle = \frac{3}{4}\hbar|+z\rangle = \frac{1}{2}(\frac{1}{2}+1)\hbar^2|+z\rangle$

$S^2|-z\rangle = \frac{3}{4}\hbar|-z\rangle = \frac{1}{2}(\frac{1}{2}+1)\hbar^2|-z\rangle$ and $\boxed{\begin{aligned}\hat{S}_z &= \frac{1}{2}\hbar|+z\rangle\\ \hat{S}_z &= -\frac{1}{2}\hbar|-z\rangle\end{aligned}}$

2) $|+z\rangle = |\frac{1}{2}, \frac{1}{2}\rangle$, $|-z\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle$ or $\boxed{\begin{matrix}z+ & z-\\ |++\rangle, & |+-\rangle\end{matrix}}$

Problem 2

(a) Find the column vector representing $\boxed{|\psi\rangle = \cos(\theta)|H\rangle + e^{i\phi}\sin(\theta)|V\rangle}$ Let $|\psi\rangle = 1$

Ans: 1) $\langle +45|H\rangle = \frac{1}{\sqrt{2}}[1\ 1]\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$

$\langle +45|V\rangle = \frac{1}{\sqrt{2}}[1\ 1]\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}$

So $\langle +45|\cos\theta|H\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}\cos\theta = \frac{1}{\sqrt{2}}\begin{bmatrix}\cos\theta & 0\\\cos\theta & 0\end{bmatrix}$

$\langle +45|e^{i\phi}\sin\theta|V\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}e^{i\phi}\sin\theta = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 0\\e^{i\phi}\sin\theta & e^{i\phi}\sin\theta\end{bmatrix}$

2) So $\langle +45|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}^T\begin{bmatrix}\cos\theta\\e^{i\phi}\sin\theta\end{bmatrix} = \frac{1}{\sqrt{2}}(\cos\theta + e^{i\phi}\sin\theta)$

Similarly: $\langle -45|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}^T\begin{bmatrix}\cos\theta\\e^{i\phi}\sin\theta\end{bmatrix} = \frac{1}{\sqrt{2}}(\cos\theta - e^{i\phi}\sin\theta)$

3) So: $\begin{bmatrix}\langle 45|\psi\rangle\\\langle -45|\psi\rangle\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}(\cos\theta + e^{i\phi}\sin\theta)\\\frac{1}{\sqrt{2}}(\cos\theta - e^{i\phi}\sin\theta)\end{bmatrix}$

So $\boxed{\langle \pm 45|\psi\rangle = \frac{1}{\sqrt{2}}\cos\theta \pm e^{i\phi}\sin\theta}$

(b) Work out the matrix representations of states as below:

(a) $\hat{P}_H = |H\rangle\langle H|$, $\hat{P}_V = |V\rangle\langle V|$ using $|L\rangle$, $|R\rangle$ states as basis. $\hat{P}_H\hat{P}_V = \hat{P}_V\hat{P}_H = 0$!

(b) Check that the relationships $\hat{P}_H^2 = \hat{P}_H$, $\hat{P}_V^2 = \hat{P}_V$ or

Ans.. [i] Answer 1:

If, $|R\rangle = \frac{1}{\sqrt{2}}(|H\rangle - i|V\rangle)$, $|L\rangle = \frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle)$.

1) Then: $|H\rangle = \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle)$; $\frac{i}{\sqrt{2}}(|R\rangle - |L\rangle)$.

So $|H\rangle = \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle) \to \hat{P}_H = \left(\frac{1}{\sqrt{2}}(|R\rangle + |L\rangle)\right)\left(\frac{1}{\sqrt{2}}(\langle R| + \langle L|)\right)$

$= \frac{1}{2}(\langle R|R\rangle + \langle L|R\rangle + \langle R|L\rangle + \langle L|L\rangle)$

$= \frac{1}{2}(1 + 0 + 0 + 1) = \frac{1}{2}(2) = \boxed{1}$

2) Similarly: $|V\rangle = \frac{1}{2}(\langle R|R\rangle - |R\rangle\langle L| - |L\rangle\langle R| + \langle L|L\rangle)$

$= \frac{1}{2}(1 - 0 - 0 + 1) = \frac{1}{2}(1) = \frac{2}{2} = \boxed{1}$

3) Now $\frac{1}{2}(\langle R|R\rangle + \langle L|R\rangle\langle R|L\rangle + \langle L|L\rangle)(\langle R|R\rangle - \langle L|R\rangle - \langle R|L\rangle + \langle L|L\rangle)$

$= \frac{1}{2}(1)(1)$

[ii] Answer 2

1) $\hat{P}_V = |V\rangle\langle V| = \frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$ $\hat{P}_H = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = |H\rangle\langle H|$

w/ $|V\rangle\langle V| = \begin{bmatrix} 1 \\ -1 \end{bmatrix} [1 \ -1] \frac{1}{2}$ $\quad |H\rangle\langle H| = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \ 1] \frac{1}{2}$

so $\boxed{\begin{matrix} \langle R|\hat{P}_H = \frac{1}{2}[1 \ 1] \\ \langle L|\hat{P}_H = \frac{1}{2}[1 \ 1] \end{matrix}}$ $\boxed{\begin{matrix} \langle R|\hat{P}_V \ \frac{1}{2}[1 \ -1] \\ \langle L|\hat{P}_V \ \frac{1}{2}[1 \ -1] \end{matrix}}$

3) to verify: $\hat{P}_H^2 = \frac{1}{2^2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^2 = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1[1 \ 1] + 1[1 \ 1] \\ 1[1 \ 1] + 1[1 \ 1] \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

so $\boxed{\hat{P}_H^2 = P_H}$

and $\frac{1}{4}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{4} & \frac{2}{4} \\ \frac{2}{4} & \frac{2}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = P_H$

4) Similarly: $\hat{P}_V^2 = \frac{1}{2^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix} 1\cdot 1 + (-1)(-1) & 1(-1) + (-1)(1) \\ (-1)(1) + (1)(-1) & (-1)(-1) + (1) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = P_V$ so $\boxed{\hat{P}_V^2 = P_V}$

5) Last but not least:

$P_H P_V = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\ 1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{bmatrix}$

so $\boxed{\begin{bmatrix} P_H & 0 \\ 0 & P_V \end{bmatrix} = \begin{bmatrix} P_H P_H & P_H P_V \\ P_H P_V & P_H P_H \end{bmatrix}}$ $= \begin{bmatrix} 1\cdot 1 + 1\cdot(-1) & 1\cdot(-1) + 1\cdot 1 \\ 1\cdot 1 + 1(-1) & 1\cdot(-1) + 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 1-1 & 1-1 \\ 1-1 & 1-1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$= \boxed{0}$ ✓ so $P_H P_V = 0$

Problems 4

① Show that: $[\hat{A}\hat{B}, C] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$ $\quad \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} + \hat{B}\hat{A}\hat{C} - \hat{B}\hat{C}\hat{A}$

Ans: 1) $[\hat{A}\hat{B}, C] = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{A}\hat{C} =$
$= \hat{A}(\hat{B}\hat{C} - \hat{C}\hat{B}) + \hat{B}(\hat{A}\hat{C} - \hat{B}\hat{A})$
$= \hat{A}[\hat{B}, \hat{C}] + \hat{B}[\hat{A}, \hat{C}]$

2) Rewrite:
$\hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B} = \hat{A}(\hat{B}\hat{C} - \hat{C}\hat{B}) + \hat{B}(\hat{A}\hat{C} - \hat{C}\hat{A})$
$= (\hat{A}\hat{B} - \hat{B}\hat{A})\hat{C} + (\hat{B}\hat{A} - \hat{A}\hat{B})\hat{C} = [A, B]\hat{C} + \hat{C}[\hat{B}, \hat{A}]$
$= ([\hat{A}, \hat{B}] + [\hat{B}, \hat{A}])\hat{C} = [\hat{A}\hat{B}, \hat{C}] \neq 0$ unless $\hat{C} = 0$. ✓

② Show that $\hat{J}^2$ is hermitian

Ans: we need show $\boxed{(\hat{J}^2)^\dagger = \hat{J}^2}$

Try 1) $(\hat{J}^2)^\dagger = (\hat{J}\cdot\hat{J})^\dagger = \hat{J}^\dagger\hat{J}^\dagger = \hat{J}\cdot\hat{J} = \hat{J}^2$

2) $(\hat{J}^2)^\dagger = (\hat{J}_x^2 + J_y^2 + J_z^2)^\dagger = \hat{J}_x^{2\dagger} + \hat{J}_y^{2\dagger} + \hat{J}_z^{2\dagger} = \hat{J}^2$.

So: Indeed $\boxed{\hat{J}^2 \text{ is Hermitian!}}$ $\longleftrightarrow$ $\boxed{A^\dagger = A}$

3) Also: $\langle\phi|\hat{A}|\psi\rangle = (\langle\psi|A|\phi\rangle)^*$

Also J^2 is Hermitian because it has a complete set of Eigenstates of real values.

③ Compute $[\hat{J}^2, \hat{J}_x]$ and $[\hat{J}^2, J_y]$

Ans: 1) $[\hat{J}^2, \hat{J}_x] = [(\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2), \hat{J}_x]$
$= \hat{J}_x^2\hat{J}_x + \hat{J}_y^2\hat{J}_x + \hat{J}_z^2\hat{J}_x = J_x \cdot J_x \cdot J_x \neq 0 + 0 = 0$

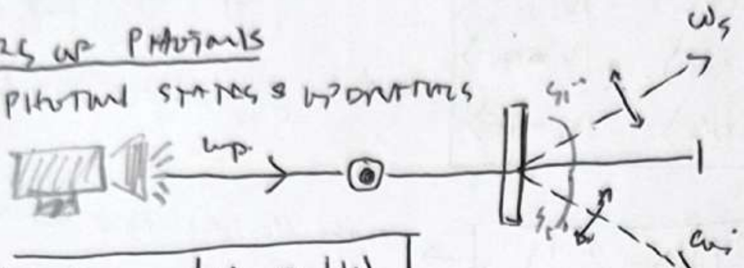
with $[\cancel{J_x^2}, J_x] + [J_y^2, \hat{J}_x] + [J_z^2, \hat{J}_x]$
$= \begin{cases} \hat{J}_y[J_y, J_x] + J_z[J_z J_x] = -i\hbar J_z J_y + i\hbar J_y J_z \\ J_y[J_x, J_y] + J_z[J_z J_x] = -i\hbar J_z J_y + i\hbar J_y J_z \end{cases}$
$= 0$ ✓

So: $\boxed{\begin{matrix}[\hat{J}^2, \hat{J}_x] = 0 \\ [\hat{J}^2, \hat{J}_y] = 0\end{matrix}}$

Two PARTICLE Systems
ENTANGLEMENT - PAIRS of PHOTONS (8.1)

(I) PAIRS of PHOTONS

[1] 2 PHOTON STATES & OPERATORS



SPONTANEOUS PARAMETRIC DOWNCONVERSION
- One Pump photon at angular Frequency $\langle \omega_p \xrightarrow{\Delta} (\omega_s, \omega_i) \rangle$ Inside a crystal.
- Polarization of the signal and idler photons are orthogonal of the pump.

1) $\boxed{|H,H\rangle = |H\rangle_s \otimes |H\rangle_i}$ HILBERT SPACE METHOD OF POLARIZATION

w/ $\otimes$ = Direct vector product That combines the 2 particle System in an Enlarged Hilbert space.

2.) $|H,H\rangle = |H\rangle_s |H\rangle_i = |HH\rangle$

and $|HH\rangle = (C_H^s|H\rangle_s + C_V^s|V\rangle_s)(C_H^i|H\rangle_i + C_V^i|V\rangle_i)$.

$= C_H^s C_H^i |H\rangle_s|H_i\rangle + 2C_H^s C_V^i |H\rangle_s|V\rangle_s + C_V^s C_V^i |V\rangle_s|V\rangle_i$

EIGENVALUE

$|\Psi\rangle = \sum_n C_n|\psi_0(x)\rangle$

$= \sum_n e^{if_n}|\psi_0(x)\rangle$

$= C_H^s C_H^i |H,H\rangle + C_V^s C_V^i |V,V\rangle + C_H^s C_V^i |H\rangle_s|V\rangle_i + C_V^s C_H^i |V\rangle_s|H\rangle_i$

$= C_H^s C_H^i |H,H\rangle + C_V^s C_V^i (V,V\rangle + C_H^s C_V^i |H,V\rangle + C_V^s C_H^i |V,H\rangle$

and since: $C_H^s C_H^i \equiv P_H(s,i)$. $C_V^s C_V^i \equiv P_V(s,i)$ $\begin{cases} C_H^s C_V^i = P_H^s(s)\, P_V^i(s) \\ C_V^s C_H^i = P_V(s)\, P_H(i). \end{cases}$

then: $\boxed{|HH\rangle = P_H(s,i)|H,H\rangle + P_V(s,i)|V,V\rangle + P_H^s(s)P_V(s)|H,V\rangle + P_V^s(s)P_H(i)|V,H\rangle}$

3.) when The Beam is $|+45,H\rangle$, then

$|+45,H\rangle = |+45\rangle_s \otimes |H\rangle_i = \frac{1}{\sqrt{2}}(|H\rangle_s + |V\rangle_s) \otimes |H\rangle_i = \frac{1}{\sqrt{2}}(|H\rangle_s \otimes |H\rangle_i + |V\rangle_s \otimes |H\rangle_i)$

$= \frac{1}{\sqrt{2}}(|H,H\rangle + |V,H\rangle)$

now: $\boxed{|+45,H\rangle = \frac{1}{\sqrt{2}}[|H,H\rangle + |V,H\rangle]}$

4) Suppose we Modify The polarization state of The Idler Beam.

$|+45,R\rangle = |+45\rangle_s \otimes |R\rangle_i = \frac{1}{\sqrt{2}}[|H\rangle_s + |V\rangle_s] \otimes \frac{1}{\sqrt{2}}[|H\rangle_i - i|V\rangle_i]$

$= \frac{1}{2}(|H\rangle_s|H\rangle_i - i|H\rangle_s|V\rangle_i + |V\rangle_s|H\rangle_i - i|V\rangle_s|V\rangle_i]$

$= \frac{1}{2}[|H,H\rangle - i|H,V\rangle + |V,H\rangle - i|V,V\rangle]$

5.) $\langle V,+45|R,H\rangle = ({}_s\langle V|\,{}_i\langle +45|)(|R\rangle_s|H_i\rangle) = {}_s\langle V|R\rangle_s\, {}_i\langle +45|H_i\rangle$

$= \left(\frac{-i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{-i}{2}$ w+ $\langle V| = \begin{bmatrix} -i \\ 0 \end{bmatrix}\frac{1}{\sqrt{2}}$ $|H\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \frac{1}{\sqrt{2}}$

6) Suppose That $(\wp_{HV}^s, \wp_{HV}^i)$ Be a photon state That acts on The (signal, idler) Resp. Then: $\boxed{[\hat{\wp}_{HV}^s, \hat{\wp}_{HV}^i] = 0}$. For Example: (?) Calculate The action operators such as $\hat{\wp}_{HV}^s$, $\hat{\wp}_{HV}^i$ and $\hat{\wp}_{HV}^{si}$ on $|V,+45\rangle$

Ans: a) $\hat{\wp}_{HV}^s|V,+45\rangle = (\hat{\wp}_{HV}^s|V\rangle_s) \otimes |+45\rangle_i = (-1)|V\rangle_s \otimes |+45\rangle_i = -|V,+45\rangle$.

b) $\hat{\wp}_{HV}^i(V,+45\rangle = |V\rangle_s \otimes (\hat{\wp}_{HV}^i|+45\rangle_i) = |V\rangle_s \otimes [\wp_{HV}^i \frac{1}{\sqrt{2}}[|H\rangle_i + |V\rangle_i]]$.

$= |V\rangle_s \otimes [\frac{1}{\sqrt{2}}[|H\rangle_i - |V\rangle_i]] = |V\rangle_s \otimes |-45\rangle_i = |V,-45\rangle$.

c) $\hat{\mathcal{P}}^{si}_{HV} |V, +45\rangle = \hat{\mathcal{P}}^{s}_{HV} \hat{\mathcal{P}}^{i}_{HV} |V, +45\rangle = (\hat{\mathcal{P}}^{s}_{HV} |V\rangle_s) \otimes (\hat{\mathcal{P}}^{i}_{HV} |+45\rangle_i)$

$= (-1) |V\rangle_s \otimes |-45\rangle_i = -|V, -45\rangle.$

Sum: $\boxed{\begin{aligned} \hat{\mathcal{P}}^{s}_{HV} |V, +45\rangle &= -|V, +45\rangle \\ \hat{\mathcal{P}}^{s}_{HV} |V, +45\rangle &= +|V, +45\rangle \\ \hat{\mathcal{P}}^{si}_{HV} |V, +45\rangle &= -|V, -45\rangle \end{aligned}}$

$(\hat{\mathcal{P}}^{s}_{HV}\ \hat{\mathcal{P}}^{i}_{HV}) = (-1, +1).$

wrt: $\hat{\mathcal{P}}^{i}_{HV} (|H\rangle_i + |V\rangle_i) = (|H\rangle_i + V_i\rangle$
$= |H\rangle_i - |V_i\rangle$ *

iii) PROBABILITIES

make measurements at observable $\hat{Q}$.

1) w/ $\boxed{\hat{Q} |\lambda_n\rangle = \lambda_n |\lambda_n\rangle}$

$P(\lambda_n \mid |\psi\rangle) = |\langle \lambda_n | \psi_n\rangle|^2 = \langle \psi | \lambda_n\rangle \langle \lambda_n | \psi\rangle = \langle \psi | \psi\rangle \langle \lambda_n | \lambda_n\rangle.$

$= \langle \psi | \hat{P}_{\lambda_n} | \psi\rangle = \langle \hat{P}_{\lambda_n}\rangle$

Sum: $\boxed{P(\lambda_n \mid |\psi\rangle) = \langle \hat{P}_{\lambda_n}\rangle}$

me $\langle a\rangle = \sum_i^N P(x) \delta a_i$
$= \alpha \pi - (1-\alpha) \lambda_i$

me $\langle \hat{P}_{\lambda_n}\rangle \equiv \alpha \lambda_1 + (1-\alpha) \lambda_2 =$
$= \alpha \lambda_1 + \lambda_2 - \alpha \lambda_2 = \alpha(\lambda_1 - \lambda_2) + \lambda_2.$

mm: $P(\lambda_n) = P(\lambda_n \mid |\psi\rangle)$

* EXAMPLE PROBLEMS

2) Calculate the probability that the signal photon will be measured to have vertical polarization and the idler photon will be measured to have horizontal polarization in a state $|R, +45\rangle$

Ans: a) If $\hat{P}(V_s, H_i)$ is a proj-operator, then:

$\hat{P}_{V_s, H_i} (\cdot) = |V, H\rangle\langle V, H| \to P(V_s, H_i) = \langle P_{V_s; H_i}\rangle = \langle V, H | \langle R, +45 | V, H\rangle$

b) so: $\langle P_{V_s, H_i}\rangle = \langle R, +45 | V, H\rangle \langle V, H | R, +45\rangle = |\langle R, +45 | V, H\rangle|^2$

$= |{}_s\langle R | V\rangle_s\ {}_i\langle +45 | H\rangle_i|^2 = \left|\frac{i}{\sqrt{2}} \frac{1}{\sqrt{2}}\right|^2 = \left|\frac{i^2}{(\sqrt{2})^2} \frac{1^2}{(\sqrt{2})^2}\right| = \left|\frac{-1}{2} \frac{1}{2}\right| = \left|-\frac{1}{4}\right|$

$= \frac{1}{4}$

So: $\boxed{\langle P_{V_s, H_i}\rangle = \frac{1}{4}}$ where w/ $\langle R|$

2c) Calculate the probability that the idler photon will be measured to have horizontal polarization on a system prepared in state $|R, +45\rangle$

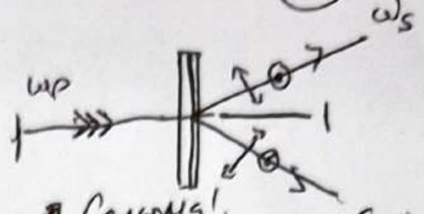
Ans: $P(H_i) = \langle \hat{P}_{H_i}\rangle = \langle R, +45 | |H\rangle_i\ {}_i\langle H | |R, +45\rangle = {}_s\langle R | R\rangle_s\ {}_i\langle +45 | H\rangle_i\ {}_i\langle H | +45\rangle_i$

$= (1) \left(\frac{1}{\sqrt{2}}\right) \left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$ so: $\boxed{P(H_i) = \frac{1}{2}}$

(d) Calculate the probability that the signal photon will be measured to have vertical polarization, given that the idler photon is measured to be horizontally polarized, for a system prepared in $|R, +45\rangle$.

Ans: $P(V_s | H_i) = \frac{P(V_s, H_i)}{P(H_i)} = \frac{1/4}{1/2} = \frac{1}{2}$ s.t. $\langle P_{V_s, H_i}\rangle \equiv P(V_s, H_i) = \frac{1}{4}$
$\langle P_{H_i}\rangle = P(H_i) = \frac{1}{2}.$

② ENTANGLED STATES:



1) $$\boxed{|\phi^+\rangle = \frac{1}{\sqrt{2}}\left[|H,H\rangle + |V,V\rangle\right]} \quad (8.19)$$

$$\Rightarrow \boxed{|\psi_A,\psi_B\rangle = |\psi_A\rangle_A \otimes |\psi_B\rangle_B} \quad (8.20)$$

w/ ψ_A = state particle A ψ_B = state particle B.

■ Crystals!:
- Ensure that when 2 crystals whose axes is rotated 90° wrt each other.
- Pump is polarized at 45°

2) For 2 photons prepared in state $|\phi^+\rangle$ of Eqs (8.19), determine the probabilities of obtaining (3 a.) The signal photon is polarized Horizontally b) The signal photon is measured Horizontally polarized given idler photon polarized.

Ans: a) $P(H_s)$ = Probability of signal photon polarized.

so: $$P(H_s) = \frac{1}{\sqrt{2}}\left[\langle H,H| + \langle V,V|\right]\left[|H\rangle_s\,{}_s\langle H|\right]\frac{1}{\sqrt{2}}\left[|H,H\rangle + |V,V\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[\langle H,H|H\rangle_s\,{}_s\langle H| + \langle V,V|H\rangle_s\,{}_s\langle H|\right]\frac{1}{\sqrt{2}}\left[|H,H\rangle + |V,V\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\Big[\langle H,H|H,H\rangle|H\rangle_s\,{}_s\langle H| + \langle H,H|H,H\rangle|V,V\rangle + \langle V,V|V,V\rangle|H\rangle_s\,{}_s\langle H|$$

$$= \frac{1}{2}\Big[\langle H,H|H,H\rangle|H\rangle_s\,{}_s\langle H| + \langle H,H|H,H\rangle|V,V\rangle + \langle V,V|V,V\rangle|H,H\rangle\Big]$$

$$= \frac{1}{2}\left[{}_s\langle H|H\rangle_s\,{}_i\langle H| + {}_s\langle V|H\rangle_s\,{}_i\langle V|\right]\left[{}_s\langle H|H\rangle_s|H\rangle_i + {}_s\langle H|V\rangle_s|V\rangle_i\right]$$

$$= \frac{1}{2}\,{}_i\langle H|H\rangle_i = \boxed{\frac{1}{2}} \quad \text{so: } \boxed{P(H_s) = \frac{1}{2}}$$

b) : we proceed to use $P(H_s|H_i) = \frac{P(H_s,H_i)}{P(H_i)}$ such that $P(H_i) = \frac{1}{2}$ so:

$$P(H_s,H_i) = \langle \hat{P}_{H_s,H_i}\rangle = \frac{1}{\sqrt{2}}\left[\langle H,H| + \langle V,V|\right]\left[|H,H\rangle\langle H,H|\right]\frac{1}{\sqrt{2}}\left[|H,H\rangle + |V,V\rangle\right].$$

$$= \frac{1}{2}\left[\langle H,H|H,H\rangle\langle H,H| + \langle V,V|H,H\rangle\langle H,H|\right]\left[|H,H\rangle + |V,V\rangle\right].$$

$$= \frac{1}{2}\Big[\langle H,H|H,H\rangle\langle H,H|H,H\rangle + \langle H,H|H,H\rangle\langle H,H|V,V\rangle + \langle V,V|H,H\rangle\langle H,H|H,H\rangle$$
$$+ \langle V,V|H,H\rangle\langle H,H|V,V\rangle\Big] = \frac{1}{2}\left[\langle H,H\rangle + \langle V,V|H,H\rangle\right]\left[\langle H,H|H,H\rangle + \langle H,H|\ldots\right.$$

$$= \frac{1}{2}[1+0][1+0] = \frac{1}{2}[1][1] = \frac{1}{2}[1] = \boxed{\frac{1}{2}}$$

$$\text{so: } P(H_s|H_i) = \frac{P(H_s,H_i)}{P(H_i)} = \frac{1/2}{1/2} = \frac{1}{2}\,\frac{2}{1} = \frac{2}{2} = \boxed{1} \quad \text{so: } \boxed{P(H_s|H_i) = 1}$$

c). Interpretation:
- The signal photon is measured to have Horizontal Polarization at Half pr
- The Results at The Idler are The Same : It

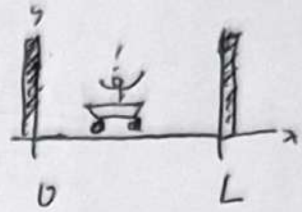
INFINITE SQUARE WELL (N.6)

QM = Discrete Energy levels
Classical = Continuum of Energies.

(1) INFINITE SQUARE WELL.

1] THEORY

1) $\boxed{V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{elsewhere} \end{cases}}$ (11.5a)



2) $A + B = 0$

$\boxed{B = -A}$ (11.62) $\Rightarrow$ Boundary conditions Require the wave function to be continuous on the edges of wall

s.t. $Ae^{ikx} + Be^{-ikx} = \psi(x)$

$\psi(0) = 0$

$Ae^{0} + Be^{0} = A + B = 0$

$\boxed{A = -B}$

3) $\psi(x) = Ae^{ikx} + Be^{-ikx}$ | $B = -A$

To Be Precise:

$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & \text{if } 0 < x < L \\ 0 & \text{elsewhere} \end{cases}$

a) $= Ae^{ikx} + (-Ae^{-ikx}) = Ae^{ikx} - Ae^{-ikx} = A(e^{ikx} - e^{-ikx})$

so.. $\boxed{\psi(x) = A(e^{ikx} - e^{-ikx})}$ (11.63)

b) Recall: $e^{ikx} = \cos(kx) + i\sin(kx)$
$e^{-ikx} = \cos(kx) - i\sin(kx)$

so: $e^{ikx} - e^{-ikx} = 0 + i\sin(kx) - (-i\sin(kx)) = i\sin kx + i\sin kx$
$= 2i\sin(kx)$

c) so: $\psi(x) = A2i\sin(kx) = C\sin(kx)$ where $\boxed{C = 2iA}$ so $\boxed{\psi = C\sin(kx)}$

4) Boundary conditions:

a) $0 < x < L$ so: $\psi(0) < \psi(x) < \psi(L)$
at $\psi(0) = 0$ so: $0 < \psi(x) < \psi(L)$
at $\psi(L) = C\sin(kx) = C\sin(kL)$.

so: $\boxed{\psi(L) = C\sin(kL) = C\sin(n\pi)}$

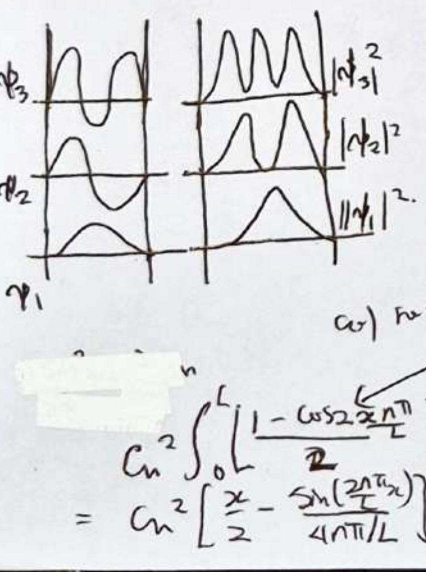
But:
$\boxed{\begin{matrix} \psi(L) = 0 \\ \text{and } C \neq 0 \\ \text{so } \sin(kL) = 0\,! \end{matrix}}$

Since $k = \frac{n\pi}{L}$
then $\boxed{kL = n\pi}$

1. $n \in \mathbb{Z}^+$ when $n > 0$.

so: $\boxed{k_n = \frac{n\pi}{L}}$

So



5) Recall That: $E = \frac{\hbar^2}{2m}k^2$ so: $E_n = \frac{\hbar^2(n\pi)^2}{2m(L)^2}$ where $p = \hbar k$, $p^2 = \hbar^2 k^2$

and $T = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$

so: $\boxed{E_n = \frac{\hbar^2 n^2 \pi^2}{2mL^2}}$

6) Normalization: $\int_{-\infty}^{\infty} |\psi(x)|^2 \, dx = 1$

a) so: $\int_{-\infty}^{\infty} |\psi|^2 \, dx = C_n^2 \int_0^L \sin^2\left(\frac{n\pi}{L}x\right) dx$.

Let $u = \sin\left(\frac{n\pi}{L}x\right)$, $x = \frac{n\pi}{L}x$
$du = \frac{n\pi}{L}\cos\left(\frac{n\pi}{L}x\right) dx$
$dx = \frac{L}{n\pi}\frac{du}{u}$

$C_n^2 \int_0^L \left[\frac{1 - \cos 2x\frac{n\pi}{L}}{2}\right] dx$

$= C_n^2 \left[\frac{x}{2} - \frac{\sin\left(\frac{2n\pi x}{L}\right)}{4n\pi/L}\right]_{x=0}^{L} = C_n^2 \frac{L}{2}$

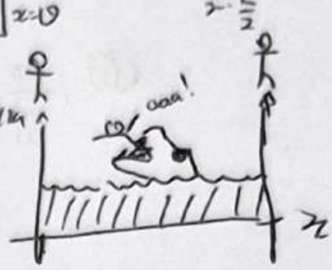
$\frac{\partial \sin^2 x}{\partial x} = \frac{\partial}{\partial x}\sin x \sin x$
$= \frac{\partial \sin x}{\partial x}\sin x + \frac{\partial \sin x}{\partial x}\sin x$
$= \cos x \sin x + \cos x \sin x$
$= 2\cos x \sin x$

$\int \sin^2 x \, dx$
$\int \sin x \sin x \, dx$

$\sin x\cos x - \int \cos x\cos x$ -
$\sin x\cos x - \cos x \sin x$ -

iv) and $C_n^2 \frac{L}{2} = 1$ then: $\boxed{C_n = \sqrt{\frac{2}{L}}}$

v) So: $$\boxed{\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) & 0 < x < L \\ & \text{Elsewhere} \end{cases}}$$



[ii] **EXAMPLE — The ILLEGAL wave function**

1) $$\psi(x,0) = \begin{cases} \sqrt{\frac{2}{L}} & 0 < x < \frac{L}{2} \\ 0 & \text{else} \end{cases}$$ So:

$\cos\left(\frac{n\pi}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$

So: $\cos\left(\frac{n\pi}{2}\right) = \sum_{n=0}^{\infty} (-1)^k \frac{\left(\frac{n\pi}{2}\right)^k}{(2k)!}$

$= \sum_{n=0}^{\infty} (-1)^k \frac{n^k \pi^k}{(2k)!} \frac{1}{2^k}$

2) $C_n = \int_0^{\infty} \psi_n^*(x)\, \Psi(x,0)\, dx$

3) $= \frac{2}{L} \int_0^{\frac{L}{2}} \sin\left(\frac{n\pi x}{L}\right) dx = \frac{-2}{n\pi} \cos\left(\frac{n\pi x}{L}\right)\Big|_0^{\frac{L}{2}}$

$= -\frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right]$

3) So: $$\boxed{C_n = \begin{cases} 2/n\pi & n \in (2k+1) \text{ odd} \\ 4/n\pi & n = 2, 6, 10 \\ 0 & n = 4, 8, 12 \end{cases}}$$

4) The wave func:

Energy → A) $\Psi(x,t) = \sum^{\infty} C_n \psi_n(x) e^{-i\omega t} = \frac{2}{\pi}\sqrt{\frac{2}{L}} \sum_{n=1,3,5}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right) e^{i\left(\frac{n^2\pi^2\hbar t}{2mL^2}\right)}$

$+ \frac{4}{\pi}\sqrt{\frac{2}{L}} \sum_{2,6,10}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right) e^{-i\frac{n^2\pi^2\hbar t}{2mL^2}}$

$(E_n,t) = |C_n|^2$

$\begin{cases} \frac{4}{(n\pi)^2} & \text{odd} \\ \frac{16}{(n\pi)^2} & 2, 6, 10 \\ 0 & 4, 8, 12 \end{cases}$

Probability → B) $\langle \psi_n | \psi(t) \rangle = \sum_n^{\infty} C_m e^{-i\omega t} \langle \psi_n | \psi_m \rangle = \sum_m^{\infty} C_m e^{-i\omega t} \delta_{nm}$

$= C_n e^{-i\omega t}$

∴ 5): $$\boxed{\begin{aligned} &\text{ENERGY}: \Psi(x,t) = \frac{2}{\pi}\sqrt{\frac{2}{L}} \sum_{1,3,5}^{\infty} \frac{1}{n} \sin(u) e^{iE_n} + \frac{4}{\pi}\sqrt{\frac{2}{L}} \sum_{2,6,10}^{\infty} \frac{1}{n} \sin(u) e^{-iE_n} \\ &\text{Probability}: \langle \psi_n | \psi(t) \rangle = C_n e^{-i\omega t} \end{aligned}}$$

② TIME EVOLUTION

[i] PREFACE —

Problems 1.4

(1.1) Use separation of variables to split PDE into 2 ODEs?!

Ans: 1) $\Psi = \psi(x)\phi(t)$

$\nabla^2\Psi = \frac{d^2\psi}{dx^2}\phi(t) = \psi''\phi$ $\qquad \frac{\partial\Psi}{\partial t} = \psi(x)\frac{d\phi}{dt} = \psi\dot{\phi}$

So: $\boxed{-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}}$

$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}\phi + V\psi\phi = i\hbar\psi\dot{\phi}$

2.) $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}\phi = i\hbar\psi\frac{d\phi}{dt} - V\psi\phi = i\hbar\psi\left[\frac{d\phi}{dt} - V\phi\right]$

$-\frac{\hbar^2}{2m}\nabla^2\psi - i\hbar\frac{\partial\phi}{\partial t}\frac{1}{\phi} + V = 0$

$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \frac{i\hbar d\phi}{dt}\frac{1}{\phi} + V = 0$

$\Rightarrow -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}\frac{\phi}{\psi} = i\hbar\frac{d\phi}{dt} - V\phi$ $\qquad$ Let

$\rightarrow -\frac{\hbar^2}{m}\frac{d^2\psi}{dx^2} = i\hbar\frac{d\phi}{dt}\frac{1}{\phi} - V$

3.) Let $\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}\frac{1}{\psi} = -a$ and $i\hbar\frac{1}{\phi}\frac{d\phi}{dt} - V = a$ so $-a + a = 0$.

w: a.) $\boxed{i\hbar\frac{d\phi}{dt}\frac{1}{\phi} - V = -a}$ $\rightarrow i\hbar\frac{d\phi}{dt}\frac{1}{\phi} = a + V$ $\qquad \boxed{E = a - V}$

$\rightarrow \frac{d\phi}{dt}\frac{1}{\phi} = \frac{-a+V}{i\hbar}$ $\rightarrow \int\frac{d\phi}{\phi} = \int\frac{-a+V}{i\hbar}dt$

$(E(t)\cdot\frac{\hbar^2}{2m})^{\frac{1}{2}}$

$(2m)^1(2m)^{-\frac{1}{2}}$

$(2m)^{1-\frac{1}{2}}$

$(2m)^{\frac{2}{2}-\frac{1}{2}}$

$= 2m^{\frac{1}{2}}$

$= \sqrt{2m}$

so $\phi(t) = e^{\frac{-Et}{i\hbar}} = e^{\frac{-iEt}{\hbar}}$

$(-1)^{\frac{1}{2}}\cdot\left(\frac{\hbar^2}{2m}\right)$

b.) $\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}\frac{1}{\psi} = +a}$ $\rightarrow \frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + a\psi = 0$

Solves with the charact: $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{0 \pm \sqrt{0 - \frac{\hbar^2}{2m}}}{2\frac{\hbar^2}{2m}} = \frac{\sqrt{-\frac{\hbar^2}{2m}a}}{2\frac{\hbar^2}{2m}}$

$= i\sqrt{\frac{\hbar^2}{2m}\left(\frac{2m}{\hbar^2}\right)^1} = i\frac{\hbar}{\sqrt{2m}}\frac{2m}{\hbar^2} = i\frac{2m}{\sqrt{2m}}\frac{1}{\hbar} = \frac{i\sqrt{2m}}{\hbar}$ $\qquad \boxed{e^{ikx}}$ ✓

$T(t) = \phi(t)$

$i\hbar\psi\frac{dT}{dt} = -\frac{\hbar^2}{2m}T\frac{d^2\psi}{dx^2} + V\psi T$

$i\hbar\psi\frac{dT}{dt}\frac{1}{T} = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi$

$\frac{dT}{dt} = \frac{iET(t)}{\hbar}$

$T(t) = e^{\frac{iET(t)}{\hbar}}$

$i\hbar\frac{\partial\psi}{\partial t} = \frac{i\hbar}{i\hbar}E\psi(x) = E\psi(x)$

So: $\psi(x) = e^{\frac{i\sqrt{2m}}{\hbar}x} = \cos\left(\sqrt{\frac{2m}{\hbar}}x\right) + i\sin\left(\sqrt{\frac{2m}{\hbar}}x\right)$

3.) So: $\boxed{\Psi(x,t) = e^{i(kx - Et)}}$ Typical solution!

4.) The PDE → ODE is:

$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x)}$

(13) varying (k-map).

Ans: 1) $v_1 |A_1|^2 = v_1 \left| \frac{(k_1+k_2)}{(k_1-k_2)} B_1 \right|^2 = v_1 \frac{(k_1+k_2)^2}{(k_1-k_2)^2} B_1^2$.

$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$

$k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$

and: $(k_1+k_2)^2 = k_1^2 + 2k_1k_2 + k_2^2$

$= \left(\sqrt{\frac{2mE}{\hbar^2}}\right)^2 + 2\sqrt{\frac{2mE}{\hbar^2}}\sqrt{\frac{2mE-V_0}{\hbar^2}} + \left(\sqrt{\frac{2m(E-V_0)}{\hbar^2}}\right)^2$

$= \frac{2mE}{\hbar^2} + \frac{2m(E-V_0)}{\hbar^2} + 2\left(\frac{2mE}{\hbar^2}\right)^{\frac{1}{2}}\left(\frac{2mE-V_0}{\hbar^2}\right)^{\frac{1}{2}}$

$= \frac{2m}{\hbar^2}\left(E+(E-V_0)\right) + 2\left(\frac{2mE}{\hbar^2}\cdot\frac{2m(E-V_0)}{\hbar^2}\right)^{\frac{1}{2}}$

$= \frac{2m\left(\sqrt{E}+\sqrt{E-V_0}\right)^2}{\hbar^2} + 2\left(\frac{2m}{\hbar^2}\sqrt{E(E-V_0)}\right) = k + 2k'$

$(xy)^{\frac{1}{2}} = x^{\frac{1}{2}}y^{\frac{1}{2}}$

$(16\cdot 4)^{\frac{1}{2}} = 16^{\frac{1}{2}}4^{\frac{1}{2}}$

$(64)^{\frac{1}{2}} = 4\cdot 2$

$64 = 4^2\cdot 2^2$

$64 = 16\cdot 4$

$64 = 64$ ✓

how: $k_1^2 + k_2^2 = k$

$k_1k_2 = k'$

It follows also that:

$(k_1-k_2)^2$

$k_1^2 - 2k_1k_2 + k_2^2$

$(k_1+k_2)^2 - 2k_1k_2$

$\frac{2m}{\hbar^2}\left\{(\sqrt{E}+\sqrt{E-V_0})^2 - \sqrt{E(E-V_0)}\right\}$

how: ...

2. ~~$2m(\ldots)$~~ $\rightarrow \frac{2m}{\hbar^2}\underbrace{\sqrt{E(E-V_0)}}_{k'}$ or $\underbrace{\frac{4mE-2mV_0}{\hbar^2}}_{k}$

so: $\frac{k_1k_2}{k_1^2+k_2^2} = \frac{k'}{k} = \frac{2m/\hbar^2\sqrt{E(E-V_0)}}{2m/\hbar^2(\sqrt{E}+\sqrt{E-V_0})^2} = \boxed{\frac{\sqrt{E(E-V_0)}}{(\sqrt{E}+\sqrt{E-V_0})^2}}$

3. how: $T = 1 - R = 4\frac{\left(2m/\hbar^2\sqrt{E(E-V_0)}\right)}{\left(2m/\hbar^2(\sqrt{E}+\sqrt{E-V_0})^2\right)} = \boxed{\frac{4\sqrt{(E-V_0)E}}{(\sqrt{E}+\sqrt{E-V_0})^2}}$ ✓

Problems N.

(13) Infinite square well potential:

$$\boxed{\Psi(x,0) = \begin{cases} \sqrt{\frac{2}{3L}} \sin\left(\frac{2\pi}{L}x\right) + i\sqrt{\frac{4}{3L}} \sin\left(\frac{3\pi}{L}x\right) & 0 < x < L \\ 0 & \text{else} \end{cases}}$$

(c) Probability $P(E_n,t)$?

Ans: $P(E_n,t) = |\langle \phi_n | \psi(t) \rangle|^2 = \sum_m C_m e^{-i\omega_m t} \langle \phi_n | \phi_m \rangle$

1st) $\langle \phi_n^* | \Psi(x,0) \rangle = \int_a^b \phi_n^*(x)\, \Psi(x,0)\, dx$

$$= \int_a^b \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \left\{ \sqrt{\frac{2}{3L}} \sin\left(\frac{2\pi}{L}x\right) + i\sqrt{\frac{4}{3L}} \sin\left(\frac{3\pi}{L}x\right) \right\} dx.$$

$$= \sqrt{\frac{2}{L}}\left(\sqrt{\frac{2}{3L}} + i\sqrt{\frac{4}{3L}}\right) \int_a^b \left[\sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{2\pi}{L}x\right) + i\sqrt{\frac{4}{3L}} \sin\left(\frac{3\pi}{L}x\right) \sin\left(\frac{n\pi L}{x}\right) \right] dx$$

$$= \int_0^L \frac{4}{3L^2} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{2\pi}{L}x\right) + i\frac{8}{3L^2} \sin\left(\frac{3\pi}{L}x\right) \sin\left(\frac{n\pi L}{x}\right) \Big\} dx.$$

Let $\alpha = \frac{n\pi x}{L}$, $\beta_1 = \frac{2\pi}{L}x$, $\beta_2 = \frac{3\pi}{L}x$.

$\bar{R} = \frac{4}{3L^2}$

$\bar{C} = i\frac{8}{3L^2}$

by trig: $\sin\alpha \sin\beta_1 = \frac{1}{2}\left[\cos(\alpha-\beta_1) - \cos(\alpha+\beta_1)\right] = \frac{1}{2}\cos\left(\frac{n\pi x}{L} - \frac{2\pi x}{L}\right) - \cos\left(\frac{n\pi}{L}x + \ldots\right)$

$\sin\alpha \sin\beta_2 = \frac{1}{2}\left[\cos(\alpha-\beta_2) - \cos(\alpha+\beta_2)\right] = \frac{1}{2}\cos\left(\frac{n\pi x}{L} - \frac{3\pi x}{L}\right) - \cos\left(\frac{n\pi x}{L} + \frac{3\pi}{L}\ldots\right)$

So: $\int_0^L \bar{R}\left[\frac{1}{2}\left[\cos\left(\frac{n\pi x}{L} - \frac{2\pi x}{L}\right) - \cos\left(\frac{n\pi x}{L} + \frac{2\pi x}{L}\right)\right]\right] + \bar{C}\left[\frac{1}{2}\left[\cos\left(\frac{n\pi x}{L} - \frac{3\pi x}{L}\right) - \cos\left(\frac{n\pi x}{L} + \frac{3\pi x}{L}\right)\right]\right]$

$$= \frac{1}{2}\int_0^L \left[\bar{R}\left[\cos\left(\frac{\pi(n-2)x}{L}\right) - \cos\left(\frac{\pi(n+2)x}{L}\right)\right] + \bar{C}\left[\cos\left(\frac{\pi(n-3)x}{L}\right) - \cos\left(\frac{\pi(n+3)x}{L}\right)\right]\right] dx.$$

$\boxed{\text{E}}$ Recall: $\int_0^L \cos\left(\frac{k\pi x}{L}\right) dx = \frac{1}{\frac{k\pi}{L}} \sin\left(\frac{k\pi x}{L}\right)\Big|_0^L = \frac{L}{k\pi}\sin\left(\frac{k\pi L}{L}\right) = \frac{L}{k\pi}\sin(k\pi)$

$\frac{d\sin(kx)}{dx}$

$\frac{dy}{dU}\frac{dU}{dx}$

$\frac{dy}{dU}(kx)\frac{d}{dx}(\sin(x))$

$k\cos(x)$

So: $\frac{1}{2}\left[\bar{R}\left\{\frac{L}{\pi(n-2)}\sin(\pi(n-2)) - \frac{L}{\pi(n+2)}\sin[\pi(n+2)]\right\} + \bar{C}\left\{\frac{L}{\pi(n-3)}\sin(\pi(n-3)) - \frac{L}{\pi(n-3)}\sin(\pi(n-3))\right\}\right]$

the form ex: let $(3,2) \equiv M$ $\quad \frac{1}{2}\frac{L}{\pi(n-M)}\sin(\pi(n-M)) \Rightarrow \begin{cases} 0 & \text{if } n \neq M \\ \frac{L}{2} & \text{if } n = M \end{cases}$

Conclusion:

$\boxed{C_1 + C_2 + C_3 = 1}$

2) So: $P(E_n,t) = \bar{R}\phi_1 + \bar{C}_2\phi_2 = \bar{R} + \bar{C}$

$C_1 = 0$; $\quad C_2 = \sqrt{\frac{2}{L}} \cdot \sqrt{\frac{2}{3L}}\frac{L}{2} = \sqrt{\frac{1}{3}}$ $\quad C_3 = \sqrt{\frac{2}{L}} \cdot \sqrt{\frac{4}{3L}} \cdot \frac{L}{2} = i\sqrt{\frac{2}{3}}$

So: $\begin{cases} P(E_1) = 0 \\ P(E_2) = \sqrt{\frac{1}{3}} \\ P(E_3) = i\sqrt{\frac{2}{3}} \end{cases}$ $\quad$ now: $\boxed{P(E_1) + P(E_3) = \sqrt{\frac{1}{3}} + i\sqrt{\frac{2}{3}}}$

(b) $\langle E(t)\rangle = |C_2|^2 E_2 + |C_3| E_3$

$$= \frac{1}{3}\cdot\frac{2^2\pi^2\hbar^2}{2mL^2} + \frac{2}{3}\cdot\frac{3^2\pi^2\hbar^2}{2mL^2} = \frac{4}{3}\frac{\pi^2\hbar^2}{2mL^2} + \frac{18}{3}\frac{\pi^2\hbar^2}{2mL^2}$$

$$= \left(\frac{4+18}{3}\right)\frac{\pi^2\hbar^2}{2mL^2} = \frac{22}{3}\frac{\pi^2\hbar^2}{2mL^2} = \frac{11}{3}\frac{\pi^2\hbar^2}{mL^2}$$

$$\therefore \boxed{\langle E\rangle = |C_2|^2 E_2 + |C_3| E_3 = \frac{11}{3}\frac{\pi^2\hbar^2}{mL^2}}$$

(c) $\langle x(t)\rangle = \langle x(0)\rangle = \frac{1}{m}\int \langle \psi(t)|\hat{x}|\psi(t)\rangle\, dt.$

$= \frac{1}{m}\int x|\psi(x,0)|^2\, dx$? So: $\boxed{\langle x\rangle = \frac{1}{m}\int x|\psi|^2\, dx}$

we see that $\int |\psi|^2 dx \triangleq \int \psi_n \Psi(x, t=0)\, dx = \frac{L}{2}$.

so: $\langle x\rangle = \int_0^L x\cos\left(\frac{k\pi x}{L}\right) = \frac{L^2}{k^2\pi^2}\cos(k\pi) - \cos(0)$

$$= \frac{L^3}{(k\pi)^2}\left((-1)^k - 1\right)$$

$$\Rightarrow \boxed{\int_0^L x\cos\left(\frac{k\pi x}{L}\right) = \begin{cases} 0 & \text{even} \\ \frac{-2L^2}{k^2\pi^2} & \text{odd}\end{cases}}$$

but probably $\langle x\rangle = c_1\dot{x}_1 + c_2 x_2 - \frac{1}{3}\frac{L}{2} + \frac{2}{3}\frac{L}{2} = \boxed{\frac{L}{2}}$ L.

(d) $\langle p\rangle = \int_0^L \psi^*\left(i\hbar\nabla\right)\psi'\, dx$

or $\langle p\rangle = C_2 P_2 + C_3 P_3$ and see that $p = m\frac{dx}{dt}$ so $\langle p\rangle = m\langle\dot{x}\rangle$

But we know that $\langle\dot{x}\rangle = 0$. So: $\boxed{\langle p\rangle = 0}$

SOLVING THE SCHRÖDINGER EQS DIRECTLY (12.A)

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2x^2\psi=E\psi(x)}\quad (12.A.1)$$

$\beta^2 = \frac{m\omega}{\hbar}$ (12.A.2).

① Let $\chi=\beta x$ s.t. $\beta=\left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}}$

1) Then, $\frac{d^2\psi(x)}{dx^2}=\frac{d}{dx}\left(\frac{d\psi}{d\chi}\frac{d\chi}{dx}\right)=\frac{d}{dx}\left(\frac{d\psi}{d\chi}\beta\right)=\frac{d^2\psi}{d\chi^2}\frac{d\chi}{dx}\beta=\boxed{\frac{d^2\psi}{d\chi^2}\beta^2}$

Since $\beta=\left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}}\rightarrow\beta^2=\frac{m\omega}{\hbar}$

2) $T=-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}=-\frac{\hbar^2}{2m}\frac{m\omega}{\hbar}\frac{d^2\psi}{d\chi^2}=-\frac{\hbar\omega}{2}\frac{d^2\psi}{d\chi^2}=-\frac{\hbar\omega}{2}\frac{\chi^2}{\beta^2}\frac{d^2\psi}{d\chi^2}$

3) $V=\frac{1}{2}m\omega^2x^2\psi=\frac{1}{2}m\omega^2\frac{\chi^2}{\beta^2}=\frac{1}{2}m\omega^2\frac{\chi^2}{\beta^2}\psi(\chi)$.

then $H=T+V\rightarrow\boxed{-\frac{\hbar\omega}{2}\frac{d^2\psi}{d\chi^2}+\frac{1}{2}m\omega^2\frac{\chi^2}{\beta^2}\psi(\chi)=E\psi(\chi)}$.

4) then $-\frac{\hbar\omega}{2}\frac{d^2\psi(\chi)}{d\chi^2}+\frac{1}{2}m\omega^2\chi^2\frac{1}{\frac{m\omega}{\hbar}}\psi(\chi)\leftrightarrow\frac{1}{2}m\omega^2\chi^2\frac{\hbar}{m\omega}=\frac{1}{2}\chi^2\hbar\omega$

thus $\boxed{-\frac{\hbar\omega}{2}\frac{d^2\psi(\chi)}{d\chi^2}+\frac{\hbar\omega}{2}\chi^2\psi(\chi)=E\psi(\chi)}$ (12.A.3)

② Let $\varepsilon=\frac{2E}{\hbar\omega}$ so,

1) $\frac{\hbar\omega}{2}\left[-D_\chi^2\psi(\chi)+\chi^2\psi(\chi)\right]=E\psi(\chi)$

$-D_\chi^2\psi(\chi)+\chi^2\psi(\chi)=\frac{2E}{\hbar\omega}\psi(\chi)=\varepsilon\psi(\chi)$.

so. $\boxed{\frac{d^2\psi}{d\chi^2}=(\chi^2-\varepsilon)\psi(\chi)}$ (12.A.5). (12.A.6).

2) so: $\boxed{\psi(\chi)=Ae^{-\frac{\chi^2}{2}}+Be^{-\frac{\chi^2}{2}}}$

$\psi(\chi)\xrightarrow[\chi\to\infty]{}Ae^{-\frac{\chi^2}{2}}$ so: $\psi(\chi)=h(\chi)e^{-\frac{\chi^2}{2}}$. $\left(\frac{dh(x)}{dx}\chi-h(\chi)\right)e^{-\chi^2}$

3) $\frac{d\psi(\chi)}{d\chi}=\frac{dh(\chi)}{d\chi}e^{-\frac{\chi^2}{2}}+h(\chi)(-\chi)e^{-\frac{\chi^2}{2}}$. $\chi(x)e^{-\frac{\chi^2}{2}}$

$=\left(\frac{dh(\chi)}{d\chi}-\chi h(\chi)\right)e^{-\frac{\chi^2}{2}}$ $+\left(\frac{dh(\chi)}{d\chi}-h(\chi)\right)\chi e^{-\frac{\chi^2}{2}}$.

$\frac{d^2\psi(\chi)}{d\chi^2}=\left(\frac{d^2h(\chi)}{d\chi^2}-\frac{d}{d\chi}(\chi h(\chi))\right)e^{-\frac{\chi^2}{2}}+\left(\frac{dh(\chi)}{d\chi}-h(x)\right)\chi e^{-\frac{\chi^2}{2}}$

$=\left(\frac{d^2h(\chi)}{d\chi^2}-\frac{d\chi}{d\chi}h(\chi)-\frac{dh(\chi)}{d\chi}\chi\right)e^{-\frac{\chi^2}{2}}+\left(\frac{dh(\chi)}{d\chi}-h(x)\right)\chi e^{-\frac{\chi^2}{2}}$

$=\left(\frac{d^2h(\chi)}{d\chi^2}-h(\chi)+\chi h(\chi)-2\chi\frac{dh(\chi)}{d\chi}\right)e^{-\frac{\chi^2}{2}}$

where since
$\frac{d}{dx}\left(X(\chi)e^{-\frac{\chi^2}{2}}\right)$
$\frac{dX}{d\chi}e^{-\chi^2}-X\chi e$

LHS
$e^{\frac{U}{2}}=e^{-\frac{\chi^2}{2}}$
$U=-\chi^2$
$\frac{\partial\phi}{\partial U}\frac{\partial U}{\partial\chi}$
$\frac{\partial}{\partial U}\left(e^{\frac{U}{2}}\right)\frac{\partial}{\partial\chi}(-\chi^2)$
$\frac{1}{2}e^{U}\cdot-2\chi$
$e^U\chi=e^{-\frac{\chi^2}{2}}\chi$

3.) then: $\frac{d^2\psi(x)}{dx^2} = (x^2-\varepsilon)\psi(x)$

$$\cancel{e^{-\frac{x^2}{2}}}\left(\frac{d^2h(x)}{dx^2} - h(x) - 2x\frac{dh(x)}{dx}\right) = (x^2-\varepsilon)\, h(x)\cancel{e^{-\frac{x^2}{2}}}.$$

$$\frac{d^2h(x)}{dx^2} + x^2 h(x) - h(x) - 2x\frac{dh(x)}{dx} = (x^2-\varepsilon)\, h(x)$$

$$\frac{d^2h(x)}{dx^2} - 2x\frac{dh(x)}{dx} + h(x)(x-1) = (x^2-\varepsilon)\, h(x)$$

then $\boxed{\frac{d^2h(x)}{dx^2} - 2x\frac{dh(x)}{dx} + (\varepsilon-1)\, h(x) = 0}$ (12.A.10).

③ SERIES SOLUTION: $\boxed{h(x) = \sum_j a_j x^j}$ (12.A.11) then. $\partial_x h = \sum_a j x^{j-1} a_j$

$\partial_{xx} h = \sum_j j(j-1) x^{j-2} a_j$

1) for: $\frac{d^2h(x)}{dx^2} - 2x\frac{dh(x)}{dx} + (\varepsilon-1)\, h(x) = 0$ we $\partial_x h \longrightarrow \sum_j (j+1) x^j a_{j+1}$

$\partial_{xx} h \longrightarrow \sum_j (j+2)(j-1) x^j a_{j+2}$

$$= \sum_j j(j-1) a_j x^{j-2} - 2x\sum_j j a_j x^{j-1} + (\varepsilon-1)\sum_j a_j x^0 = 0$$

$$= \sum_{j=0} (j+2)(j+1)\, a_{j+2}\cancel{x^j} + \sum_{0} \left[-2j + (\varepsilon-1)\right] a_j \cancel{x^j} = 0$$

then: $\sum_{j=0} (j+2)(j+1) a_{j+2} - (2j+\varepsilon-1) a_j = 0 \rightarrow \therefore a_{j+2}[(j+2)(j+1)] - a_j[-2j+\varepsilon-1]$

2) and: $a_{j+2} = \frac{-a_j\,[2j+\varepsilon-1]}{(j+2)(j+1)} = \frac{2j+1-\varepsilon}{(j+2)(j+1)}\, a_j$

then $\boxed{a_{j+2} = \frac{2j+1-\varepsilon}{(j+2)(j+1)}\, a_j}$ (12.A.16)

As long as this relationship is satisfied, the series $h(x)$ in (12.A.11), is a solution to the DE of (12.A.10).

3). $\frac{a_{j+2}}{a_j} \to \frac{2}{j}$ as $j \to \infty$, then: $\frac{2}{j} = \frac{2j+1-\varepsilon}{(j+2)(j+1)}$ | $\lim j \to \infty$.

and: $e^{x^2} = \sum_{n=0} \frac{1}{n!}(x^2)^n = \sum_{\substack{j=0\\ j=\text{even}}} b_j x^j$ letting $2n = j$

now $\boxed{\frac{b_{j+2}}{b_j} = \frac{\left(\frac{j}{2}\right)!}{\left(\frac{j}{2}+1\right)!}}$ and $\boxed{\frac{b_j}{b_j} \xrightarrow[j\to\infty]{} \frac{2}{j}}$

④ FINALE: at $h_n(x) = \sum_{j=0}^{n} a_j x^j$

ii) by $\psi_n(x) = h_n(x)\, e^{-\frac{x^2}{2}}$

& and $\boxed{\varepsilon_n = 2n+1}$ to $\boxed{E_n = \hbar\omega\left(n+\frac{1}{2}\right)}$

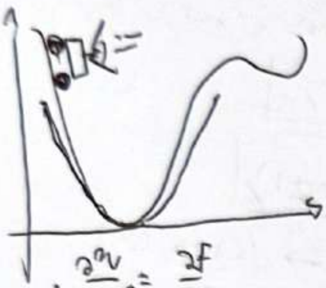
HARMONIC OSCILLATOR [1.21] [n.2]

① W.H.Y?

$$\boxed{V(x) = \sum_j \frac{d^k}{dx_0^k}\left(V(x_0)\right)[x-x_0]^k}$$

so $V(x) = V(x_0) + V'(x_0)(x-x_0) + \frac{1}{2}V''(x_0)\ldots$

V(x)



1) and: $\frac{\partial V}{\partial x} = F(x)$ $\quad \frac{\partial^2 V}{\partial x^2} = \frac{\partial F(x)}{\partial x}$ and $F(x) = \omega^2 x$

so: $\frac{\partial F}{\partial x} = \omega^2$, $F(x) = \omega^2 x$ ...

2) $V(x) = \sum_j D^k(V(x_0))[x-x_0]^k = -\frac{1}{2}\omega^2x^2 + \omega^2 x(x-x_0) + \omega^2(x-x_0)$. Let $x_0 = 0$ so

$= -\left(\frac{1}{2}\omega^2x^2 + \omega^2x^2 + \omega^2x_0 + \omega^2x + \omega^2x_0\right)$

$= -\frac{1}{2}\omega^2x^2 + \omega^2x^2 + \omega^2x$.

$= -\left(\frac{1}{2}\omega^2x^2 + \frac{2}{2}\omega^2x^2 + \omega^2x\right) = -\left(\frac{3}{2}\omega^2x^2 + x\omega^2\right)$

$= -\omega^2\left(\frac{3}{2}x^2 + x\right)$ so $V \sim \left(\frac{3}{2}\omega^2x^2 - \omega^2x\right)$.

3) $\boxed{V(x) = \frac{1}{2}k(x-x_0)^2}$ or $\boxed{V = \frac{1}{2}\omega^2x^2}$

- $\frac{\partial^2 V}{\partial x^2} = \frac{\partial F}{\partial x}$
- $\frac{\partial V}{\partial x} = -F(x)$
- $V = -\int F(x)\,dx$
- $V'' = \omega^2$
- $V' = -\omega^2x$
- $V = -\frac{1}{2}\omega^2x^2$

② CREATION, ANNIHILATION, & NUMBER OPERATORS.

$$\boxed{\hat H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2}$$ Hamiltonian w/ Harmonic oscillator

$= \frac{1}{2}\left(\frac{p^2}{m} + m\omega^2x^2\right) = \frac{1}{2}\left(\frac{p^2 + m^2\omega^2x^2}{m}\right)$

here 1) $\hat H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 = \frac{1}{2}\left(\frac{p^2}{m} + m\omega^2x^2\right)$

$= \frac{1}{2m}\left(p^2 + m^2\omega^2x^2\right) = \frac{1}{2}m\omega^2\left[x^2 + \frac{1}{(m\omega)^2}p^2\right]$

so: $\boxed{\hat H = \frac{1}{2}m\omega^2\left[x^2 + \frac{1}{(m\omega)^2}p^2\right]}$

2) $J_\pm = \hat J_x \pm iJ_y$?? $\hat a = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}}\left(x + \frac{i}{m\omega}\hat p\right)$, $\hat a^\dagger = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}}\left(x - \frac{i}{m\omega}\hat p\right)$.

Find such that $\boxed{\hat n \equiv a^\dagger a}$

3) we proceed: $\hat n = \frac{m\omega}{2\hbar}\left(\hat x - \frac{i}{m\omega}\hat p\right)\left(\hat x + \frac{i}{m\omega}\hat p\right)$.

$= \frac{m\omega}{2\hbar}\left[x^2 + \frac{1}{(m\omega)^2}p^2 + \frac{i}{m\omega}(\hat x\hat p - \hat p\hat x)\right] = \frac{m\omega}{2\hbar}\left[x^2 + \frac{p^2}{(m\omega)^2} + \frac{i[x,p]}{m\omega}\right]$

$= \frac{1}{\hbar\omega}\left[\frac{1}{2}m\omega^2\hat x^2 + \frac{p^2}{2m}\right] + \frac{i}{2\hbar}[\hat x, \hat p] = \frac{1}{\hbar\omega}\left(\hat H - \frac{1}{2}\right)$

and $H = \hbar\omega\left(\hat n + \frac{1}{2}\right) = \hbar\omega\left(\hat a^\dagger\hat a\right) + \frac{1}{2}$

so: $\boxed{\hat H = \hbar\omega\left(\hat a^\dagger\hat a + \frac{1}{2}\right)}$ and $[\hat a, \hat a^\dagger] = 1$

3) $[\hat n, \hat a] = [\hat a^\dagger\hat a, \hat a]$

$= \hat a^\dagger[\hat a, \hat a] + [\hat a^\dagger, \hat a]\hat a = 0 + [\hat a^\dagger, \hat a]\hat a = -(1)\hat a = -\hat a$

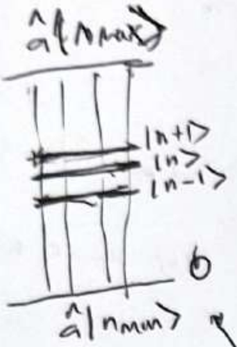
so $\boxed{[\hat n, \hat a] = -a}$

n) $[\hat{n}, \hat{a}] = \hat{n}\hat{a} - \hat{a}\hat{n} = -\hat{a}$
$\boxed{\hat{n}\hat{a} = \hat{a}\hat{n} - \hat{a}}$

$\hat{H}|\phi\rangle = \hat{E}|\phi\rangle$
$\hat{H}|n\rangle = \hat{E}|n\rangle$ so $\boxed{|n\rangle = |\phi\rangle}$

II EIGENVALUES & EIGENSTATES :

• The LADDER GRAPH



1) $\boxed{\begin{array}{l}\hat{H}|n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)|n\rangle \\ E_n|n\rangle = \hbar\omega\left(n+\frac{1}{2}\right)|n\rangle\end{array}}$

w/ : $\hat{n}|n\rangle = n|n\rangle$
: $\hbar\omega\left(n+\frac{1}{2}\right) \geq 0$
• $n \geq \frac{1}{2}$..

so: $\boxed{\hat{a}|n\rangle = |\phi\rangle}$ w/ : $\hat{a}$ = annihilation operator
: ϕ = particle wave state

2) $\hat{n}|\phi\rangle = \hat{n}\hat{a}|n\rangle = (\hat{a}\hat{n} - \hat{a})|n\rangle = (\hat{a}n - \hat{a})|n\rangle = (n-1)\hat{a}|n\rangle$
$= (n-1)|\phi\rangle$ so: $\boxed{\hat{n}|\phi\rangle = (n-1)|\phi\rangle}$

3) $\begin{cases}\hat{a}|n\rangle = C_-|n-1\rangle \\ \hat{a}^\dagger|n\rangle = C_+|n+1\rangle\end{cases}$ so $\boxed{\hat{a}^r|n\rangle = C_\pm|n\pm1\rangle}$

w/ $r = \begin{cases}\dagger \to :(C_\pm \to C_-) \\ 1 \to :(C_\pm \to C_+)\end{cases}$

4) $\hat{n}|n_{min}\rangle = n_{min}|n_{min}\rangle$
$a^\dagger\hat{a}|n_{min}\rangle = n_{min}|n_{min}\rangle$
$a^\dagger[0] = n_{min}|n_{min}\rangle \rightarrow \hat{a}|n_{min}\rangle = 0.$
$\boxed{0 = n_{min}}$

so: $E_n = \hbar\omega\left(n+\frac{1}{2}\right) \in [0, T]$. so $\boxed{E_n \in \mathbb{R}^+}$

• MINIMUM ENERGY FOR THE UNCERTAINTY PRINCIPLE

$\langle H\rangle = \frac{(\Delta p)^2}{2m} + \frac{1}{2}m\omega^2(\Delta x)^2$

$\rightarrow (\Delta p)^2 \geq \frac{\hbar^2}{4(\Delta x)^2}$

$\langle H\rangle \gtrsim \frac{1}{2m}\frac{\hbar^2}{4(\Delta x)^2} + \frac{1}{2}m\omega^2(\Delta x^2)$

⚠ CAUTION: $\boxed{E_0 \neq 0}$ since $E_0 = \frac{\hbar\omega}{2}$ ~~for~~ the minimal required !

so, $\Delta x\Delta p \geq \frac{\hbar}{2}$ since $0 \geq \frac{\hbar}{2}$! → ~~since~~ $\Delta x\,\Delta p = 0$ is not true !
⟷ since $\Delta x\,\Delta p = 0$ is $E_0 = 0$ (PARTICLE AT REST).

$\boxed{\hbar}$

HARMONIC OSCILLATORS - WAVE FUNCS & FOCK STATES

| 12.3 | n. 4 |

3. WAVE FUNCTION

GROUND STATE:

• The Ground state of a particle in $\boxed{\hat{a}|0\rangle = 0}$ with $\hat{a}|n\rangle = a|n+1\rangle\sqrt{0}$ $= a|0\rangle\sqrt{0}$ $= 0$.

• The ground state wave function has property $\boxed{\langle x|\hat{a}|0\rangle = 0}$ & GROUND STATE WAVE, $\psi_0(x)$

>> w/ $\langle x|\hat{a}|0\rangle = \langle x|\frac{\beta}{\sqrt{2}}\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right)|0\rangle = \langle x|0\rangle\frac{\beta}{\sqrt{2}}\left[\hat{x}+\frac{i}{m\omega}\hat{p}\right]$.

$= \psi_0(x)\frac{\beta}{\sqrt{2}}\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right) = \frac{\beta}{\sqrt{2}}\left(\psi_0\hat{x}+\frac{i}{m\omega}\hat{p}\psi_0\right) = \frac{\beta}{\sqrt{2}}\left(\psi_0\hat{x}+\frac{i}{m\omega}(-i\hbar\nabla\psi_0)\right)$

use EX-12.1 $= \frac{\beta}{\sqrt{2}}\left(\langle x\rangle_0+\frac{\hbar}{m\omega}\nabla\psi_0\right) = \frac{\beta}{\sqrt{2}}\left(\langle x\rangle_0+\frac{\hbar}{m\omega}\left(\frac{\partial x}{\partial x}\frac{\partial\psi}{\partial x}+x\frac{\partial\psi}{\partial x}-\frac{\partial(x\psi)}{\partial x}\right)\right)$

$= \frac{\beta}{\sqrt{2}}\left(\langle x\rangle_0+\frac{\hbar}{2m}\frac{\partial\langle x\rangle}{\partial x}\right) = \frac{\beta}{\sqrt{2}}\left(0+\frac{\hbar}{2m}\frac{\partial 0}{\partial x}\right) = \frac{\beta}{\sqrt{2}}(0) = \boxed{0}$

GAUSSIAN WAVE.

• The Differential eqs yields soln: $\boxed{\psi_0 = \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}}e^{-\frac{\beta^2x^2}{2}}} \iff \boxed{\frac{\partial\psi_0}{\partial x} = -\beta^2\psi_0(x)x}$

>> w/: $\frac{d\psi_0}{dx} = -\beta^2\psi_0 x$ * $\int\frac{d\psi}{\psi} = \int -\beta^2 x\,dx = \ln\psi_0 - \ln A = -\frac{1}{2}\beta^2x^2$

* $\ln\psi_0 = \ln A - \frac{1}{2}\beta^2x^2$ * $\boxed{\psi_0(x) = Ae^{-\frac{1}{2}\beta x^2}}$

>> Normalization: $\int|\psi|^2dx = A^2\int e^{-\beta^2x^2}dx = A^2\sqrt{\frac{\pi}{\beta^2}} = 1$

* eqs: $\psi_0(x) = \left(\frac{\beta}{\pi}\right)^{\frac{1}{4}}\exp\left(-\frac{1}{2}\beta^2x^2\right)$ * $A^2 = \sqrt{\frac{\beta^2}{\pi}}$ * $A^2 = \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{2}\frac{1}{2}} = \left(\frac{\beta}{\pi}\right)^{\frac{1}{4}}$

HERMITE SOLUTION:

• We previously used the fact that: $\boxed{\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle}$ & "HARMONIC" SERIES

$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$

$\frac{1}{\sqrt{n}}\hat{a}|n\rangle = |n-1\rangle$

>> w/:

* $\hat{a}^\dagger|n\rangle = \frac{\beta}{\sqrt{2}}\left(x-\frac{i}{m\omega}\hat{p}\right)$.

* $\frac{\beta}{\sqrt{2}}\left(x-\frac{i}{m\omega}\hat{p}\right)|n\rangle = \frac{\beta}{\sqrt{2}}\left(x|n\rangle-\frac{i}{m\omega}p|n\rangle\right)$.

• The same technique can be applied: $\boxed{\hat{a}^\dagger|n\rangle \equiv \psi_{n+1}(x)}$ & "APPLIED" WAVE

$= \langle x|\frac{\beta}{\sqrt{2}}\left(x|n\rangle-\frac{1}{m\omega}p|n\rangle\right) = \langle x|n+1\rangle\sqrt{n+1}$

$\frac{\beta}{\sqrt{2(n+1)}}\langle x|\left(x-\frac{1}{m\omega}p\right)|n\rangle = \frac{\beta}{\sqrt{2n+1}}\left(x\psi_n-\frac{1}{\beta^2}\frac{\partial\psi_n}{\partial x}\right) \equiv \psi_{n+1}(x)$

* $\frac{i\beta}{\sqrt{2}\sqrt{n+1}}\left(\langle x|x|n\rangle-\frac{1}{m\omega}\langle x|p|n\rangle\right) =$

• For Example, $n=1$ is: $\boxed{\psi_1(x) = \frac{\beta}{\sqrt{2}}\left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}}2xe^{-\frac{\beta x^2}{2}}}$ & 1st order polynom

>> w/ $\frac{\beta}{\sqrt{2}}\left(x\psi_0-\frac{1}{\beta^2}\frac{\partial\psi}{\partial x}\right)$

$= \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}}\frac{\beta}{\sqrt{2}}\left(xe^{-\beta^2x^2}-\frac{1}{\beta^2}\frac{\partial e^{-\beta x^2}}{\partial x}\right)$

$= \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}}\frac{\beta}{\sqrt{2}}\left(xe^{-\frac{\beta^2x^2}{2}}-\frac{1}{\beta^2}\left(\frac{-2x\beta^2}{2}\right)e^{-\frac{\beta^2x^2}{2}}\right)$

$= \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}}\frac{\beta}{\sqrt{2}}2xe^{-\beta x^2}$

• Thus, Born is the Hermite polynomial ~~Generalized wave func~~: $\boxed{\psi_n(x) = \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(\beta x)e^{-\frac{\beta^2x^2}{2}}}$ & HARMONIC OSCILLATOR WAVE.

• AND $\boxed{E_n = \hbar\omega\left(n+\frac{1}{2}\right) = \frac{1}{2}m\omega^2x_n^2}$

3) FOCK STATES & PHOTONS.

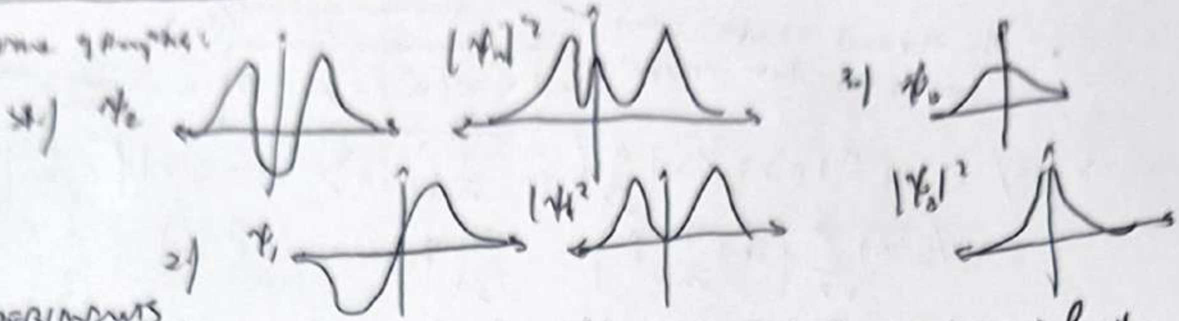
TIME DEPENDENT WAVE:

$$\boxed{\Psi(x,t) = \psi_n(x)\, e^{-\frac{iE_nt}{\hbar}}}$$

- It's normalizable:

$$|\Psi|^2 = \psi^* e^{\frac{iE_nt}{\hbar}}\, \psi\, e^{-\frac{iE_nt}{\hbar}} = \psi\psi^* = |\psi_n(x)|^2 \checkmark$$

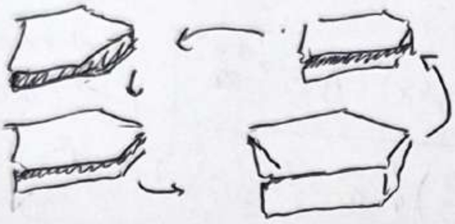
- Some graphics:



EXPERIMENTS:

- The Harmonic oscillator doesn't oscillate, since the $\boxed{\text{HAMILTONIAN-TIME-INDEPENDENT}}$
- The Hamiltonian is in a superposition of states corresp. to $\boxed{\text{2-ADJACENT-ENERGY-LEVELS}}$
- The separation of the energy levels becomes long. $\boxed{\Delta E = \hbar\omega}$ as n increases, as $\boxed{\frac{dE}{dn} = \hbar\omega}$
- But if the classical limit applies a large energy $\boxed{\lim_{n\to\infty} \Delta P_n = \text{CLASS}}$, no matter how large n gets.
- The $\boxed{\text{FOCK STATES}}$ are $\boxed{\text{NON-CLASSICAL}}$ such that $\boxed{E_n = n\hbar\omega}$ the particle grows but still won't be large.

:



$\omega \sim 6.175$ GHz

$$\boxed{\omega \sim \frac{1}{T}}$$

(5) COHERENT STATES

SERIES ANALYSIS:

- The Annihilation operator a:

$$\boxed{\hat{a}|\alpha\rangle = \alpha|\alpha\rangle}$$

- The series & coefficients of the Annihilation operator n: $\boxed{c_n = \frac{\alpha^n}{\sqrt{n}} c_0}$ $\boxed{|\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle}$

* coefficient w/ "DRIFT"   * Annihilation series

>> The coefficient is derived:

* $|\alpha\rangle\alpha = \hat{a}|\alpha\rangle$

* $\alpha\sum_n c_0 \alpha^n = \hat{a}\sum_n c_n \alpha^n$

* $\alpha\sum_n c_n\alpha^n = \hat{a}\sum_{n-1} c_n\alpha^{n-1} = \hat{a}\sum_{n+1} c_{n-1}\alpha^n$

* $\alpha c_n = \hat{a} c_{n-1} \rightsquigarrow \frac{\alpha}{\hat{a}} = \frac{c_{n-1}}{c_n} \rightsquigarrow \frac{c_n}{c_{n-1}} = \frac{\hat{a}}{\alpha}$

\# $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$

$$\boxed{|\alpha\rangle\hat{a} = \sum_{n=1} c_n\sqrt{n}|n-1\rangle}$$

* $\hat{a}|\alpha\rangle = \sum c_{n-1}\alpha|n-1\rangle$

* $\sqrt{n}c_n = c_{n-1}\alpha$ ← w/ $\langle\alpha|n\rangle$ $= \langle\alpha|\frac{1}{\sqrt{n}}\hat{a}|n\rangle$ $= \alpha\frac{1}{\sqrt{n}}\hat{a}|n\rangle$

* $\boxed{c_n = \frac{\alpha}{\sqrt{n}} c_{n-1}}$

\# Similar too: $\sigma_T = \frac{\sigma_0}{\sqrt{T}}$ ...

$\frac{\sigma_T\sqrt{T} - \sigma_T}{\sqrt{T}} = \Sigma_T$ Random walk w/ drift

* Thus: $|\alpha\rangle = c_0\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n}}|n\rangle$

- One of the fundamentals: $\boxed{\langle\alpha|\alpha\rangle = c_0^2 e^{|\alpha|^2} = 1}$ of orthogonality common

>> w/:

$$\langle\alpha|\alpha\rangle = \left(c_0\sum_n\frac{\alpha^n}{\sqrt{n!}}|n\rangle\right)\left(c_0\sum_{n'}\frac{\alpha^{*n}}{\sqrt{n'!}}|n'\rangle\right) = c_0^2\sum_{nn'}\frac{\alpha^n\alpha^{*'}}{\sqrt{n!}\sqrt{n'!}}\langle n|n'\rangle$$

$$= c_0^2\sum_n\frac{\alpha^n\alpha^{*n}}{n!}(1) = c_0^2\sum_n\frac{(\alpha\alpha^*)^n}{n!} = c_0^2 e^{\alpha^2}$$

- The normalization factor express: $\boxed{|\alpha\rangle = e^{-\frac{\alpha^2}{2}}\sum_{n=1}\frac{\alpha^n}{\sqrt{n}}|n\rangle}$ * NORMALIZED ANNIHILATION

>> w/ $\langle\alpha|\alpha\rangle = c_0^2 e^{\alpha^2} = 1$

* $c_0^2 = e^{-\alpha^2}$ * $c_0^{2\frac{1}{2}} = e^{-\alpha^2\frac{1}{2}}$ * $\boxed{c_0 = e^{-\frac{\alpha^2}{2}}}$

and $\{|\alpha=0\rangle = |n=0\rangle\}$

Since $|\alpha=0\rangle = |0\rangle = c_0\sum\frac{0^n}{\sqrt{n!}}|n\rangle$

$0 = 0|n\rangle = \boxed{0}$

STATISTICS & EXPECTATIONS:

- Dimensionless operator of space: $\boxed{\hat{x} = \frac{\beta}{\sqrt{2}}\xi = \frac{1}{2}(\hat{a} + a^\dagger)}$ (real)
- Dimensionless operator of momentum: $\boxed{\hat{p} = \frac{1}{\hbar\beta\sqrt{2}}\hat{p} = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)}$ (real)
- The creation & annihilation operators is linear: $\left\{\begin{array}{l}\hat{a} = (\hat{x} + i\hat{p})\\ \hat{a}^\dagger = (\hat{x} - i\hat{p})\end{array}\right.$ Dimensionless annihilation.

$\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$

$\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1$

$\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} = 2\hat{a}^\dagger\hat{a} + 1$

>> mean: $\langle x\rangle = \langle\alpha|\hat{x}|\alpha\rangle = \frac{1}{2}\langle\alpha|(a + a^\dagger)|\alpha\rangle = \frac{1}{2}(\alpha + \alpha^\dagger)\langle\alpha|\alpha\rangle = \mathrm{Re}(\alpha)$

>> std.dev: $\langle x^2\rangle = \langle\alpha|\hat{x}^2|\alpha\rangle = \frac{1}{4}\langle\alpha|(\hat{a} + \hat{a}^\dagger)(a + \hat{a}^\dagger)|\alpha\rangle = \frac{1}{4}\langle\alpha|\hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}|\alpha\rangle$

$= \frac{1}{4}($

EXERCISE 12

(1.) PROOF $\boxed{[a^\dagger \hat{a}] = 1}$

Ans: $[\hat{a}^\dagger \hat{a}] = \hat{a}^\dagger \hat{a} - \hat{a}^\dagger \hat{a} = \hat{n} - \hat{n}^\dagger = \left(\frac{1}{\hbar\omega}\hat{H} + \frac{1}{2}\right) - \left(\frac{1}{\hbar\omega}\hat{H} - \frac{1}{2}\right)$

$$= \underbrace{\frac{1}{\hbar\omega}\hat{H} + \frac{1}{2} - \frac{1}{\hbar\omega}\hat{H}}{} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = \boxed{1}$$

(2.) PROOF $\boxed{[\hat{n}, \hat{a}^\dagger] = \hat{a}^\dagger}$

Ans: $[\hat{n}, \hat{a}^\dagger] = \hat{n}\hat{a}^\dagger - \hat{a}^\dagger \hat{n} = \hat{a}\hat{a}^\dagger a^\dagger - \hat{a}^\dagger \hat{a}\hat{a}^\dagger = a^\dagger(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a})$

$$= \hat{a}^\dagger[\hat{a}, a^\dagger] = \hat{a}^\dagger(1) = \hat{a}^\dagger$$

(3) PROOF $\boxed{\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle}$

Ans:
- Basically ~~if~~ $n = |C_-|^2$. Then:
- So $a^\dagger|n\rangle = C_+|n+1\rangle$
$= \sqrt{n+1}\,|n+1\rangle$ QED

$n = |C_-|^2$

$n+1 = C_-^2 + 1$

$(n+1)^{\frac{1}{2}} = C_-^{2\frac{1}{2}} + 1^{\frac{1}{2}}$

$\sqrt{n+1} = C_- + 1$

$\sqrt{n+1} = C_+$

$C_- + 1 = C_+$

$C_- - C_+ = 1$

$\sqrt{n} - \sqrt{n+1} = 1$

$\sqrt{n}^2 - \sqrt{n+1}^2 = 1^2$

$n - n + 1 = 1$

$\boxed{1 = 1}$ ✓

$\boxed{\text{QED}}$

SCHRÖDINGER 3D - SEPARATION I (17.1) (17.2)

(1.) The 3D Schrödinger eqs

[1] BASICS: $\hat{H} = \frac{1}{2m}\hat{p}^2 + V(\vec{r}) = \frac{(p_x^2 + p_y^2 + p_z^2)}{2m} + V(\vec{r})$ $\quad H|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle$

1) Eq: $\boxed{\hat{H}\psi = \left(\frac{p^2(x,y,z)}{2m} + V(x,y,z)\right)\psi = i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t}}$

2) $\langle\vec{r}|H|\psi(t)\rangle = \langle r|i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle$

$= \langle\vec{r}\left|\frac{p^2(x,y,z)}{2m} + V\right|\psi(t)\rangle = \langle r|i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle$

$= \frac{(-i\hbar)^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V(\vec{r})\Big|\psi(\vec{r},t) = i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t}.$

3) hw: $\boxed{-\frac{\hbar^2}{2m}\nabla^2\Psi(\vec{r},t) + V\Psi(\vec{r},t) = i\hbar\frac{\partial}{\partial t}\Psi(\vec{r},t)}$

4) solutions & normalization:

a) $\Psi(\vec{r},t) = \sum_n c_n\psi_n(\vec{r})\cdot e^{-i\omega_n t}$ $\quad$ b) $\iiint_{\forall\, \text{space}} |\psi(\vec{r})|^2 = 1.$

SPHERICAL COORDINATES

[2] PDE Solutions:

$\psi(\vec{r}) = X(x)\,Y(y)\,Z(z) = XYZ$

w: $-\frac{\hbar^2}{2m}\left[YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2}\right] = E\,XYZ$

$-\frac{\hbar^2}{2m}\left[\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2}\right] = E \rightarrow$ $\quad -\frac{\hbar^2}{2m}\frac{1}{X}\frac{d^2X}{dx^2} = E_x$, $\quad -\frac{\hbar^2}{2m}\frac{1}{Y}\frac{d^2Y}{dy^2} = E_y$, $\quad -\frac{\hbar^2}{2m}\frac{1}{Z}\frac{d^2Z}{dz^2} = E_z.$

$-\frac{\hbar^2}{2m}\left[E_x + E_y + E_z\right] = E.$

d: $\boxed{E_{n_x} = \frac{n_x^2\pi^2\hbar^2}{2mL_x^2}} \rightarrow \boxed{X_{n_x}(x) = \begin{cases}\sqrt{\frac{2}{L_x}}\sin\left(\frac{n_x\pi}{L}x\right) & 0 < x < L.\\ 0\end{cases}}$

• Boundary conditions
• Free ends

here $E(n_x n_y n_z) = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right)$

h.w.: $\boxed{\psi_{n_x n_y n_z}(\vec{r}) = \sqrt{\frac{8}{L_x L_y L_z}}\sin\left(\frac{n_x\pi}{L_x}x\right)\sin\left(\frac{n_y\pi}{L_y}y\right)\sin\left(\frac{n_z\pi}{L_z}z\right)}$

(2.) CENTRAL POTENTIALS - PART I

1.) Assumptions:

a) $r = \sqrt{x^2+y^2+z^2}$ $\quad \theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$ $\quad$ b) $\boxed{\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta\end{aligned}}$

$\phi = \tan^{-1}\left(\frac{y}{x}\right)$ $\quad \theta = \cos^{-1}\left(\frac{z}{r}\right)$

2) $d^3r = r^2\sin\theta\cdot dr\cdot d\theta\, d\phi$ $\quad$ w: $\int d^3r = \int r^2\sin\theta\,|\psi(\vec{r})|^2\, dr\, d\theta\, d\phi = 1$

3) $\boxed{\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}}$ $\quad$ • More in the Appendix!

w/: $\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} \rightarrow \nabla\varphi = \frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y} + \frac{\partial\varphi}{\partial z} = \frac{\partial}{\partial r}$

ar: $\frac{\partial\varphi}{\partial x_i} = \frac{\partial\varphi}{\partial r}\frac{\partial r}{\partial x_i} + \frac{\partial\varphi}{\partial\theta}\frac{\partial\theta}{\partial x_i} + \frac{\partial\varphi}{\partial\phi}\frac{\partial\phi}{\partial x_i}$

1) SEPARATION OF VARIABLES

1) $-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \Rightarrow -\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] + V\psi = E\psi$

Let $\boxed{\psi(r,\theta,\phi) = R(r)\,Y(\theta,\phi)}$ or $RY = \psi$

thus $-\frac{\hbar^2}{2m}\left[\frac{Y(\theta,\phi)}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{R(r)}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{R}{r^2\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] + VRY = ERY$

2) $-\frac{2mr^2}{\hbar^2 RY}\left(-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi\right) = -\frac{2mr^2}{\hbar^2 \cancel{RY}}E\cancel{RY} \rightarrow +\frac{2mr^2\cancel{\hbar^2}}{\cancel{\hbar^2}RY\cancel{2m}}\nabla^2\psi + \frac{V\cancel{RY}}{\cancel{RY}}\frac{2mr^2}{\hbar^2} = -E\frac{2mr^2}{\hbar^2}$

or: $\boxed{\frac{r^2}{RY}\nabla^2\psi - \frac{2m}{\hbar^2}r^2\left[V(r) - E\right] = 0}$

3) thus $\frac{r^2}{RY}\left[\frac{Y}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{R}{r^2\sin\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] = \frac{\cancel{r^2}\cancel{Y}}{\cancel{r^2}R\cancel{Y}}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{\cancel{R}\cancel{r^2}}{\cancel{r^2}\cancel{R}Y}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{\partial^2 Y}{\partial\phi^2}\frac{\cancel{R}\cancel{r^2}}{\cancel{r^2}\sin^2\theta\cancel{R}Y}$

$= \frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{Y}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{Y\sin^2\theta}\frac{\partial Y}{\partial\phi}$

thus $\boxed{\frac{1}{R}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)\right] + \frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial Y}{\partial\phi}\right] - \frac{2m}{\hbar^2}r^2\left[V - E\right] = 0}$

4) Let: $\left\{\begin{array}{l} \frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2m}{\hbar^2}r^2(V-E) = l(l+1) \rightarrow \text{Radial part} \\ \frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial Y}{\partial\phi}\right] = -l(l+1) \rightarrow \text{Angular part} \end{array}\right.$

thus $\boxed{l(l+1) - l(l+1) = 0}$

ii) ANGULAR EQS:

1) $\sin\theta\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial Y}{\partial\theta}\right] + l(l+1)\sin^2\theta\, Y + \frac{\partial^2 Y}{\partial\phi^2} = 0$ and $\boxed{Y(\theta,\phi) = \Theta(\theta)\,\Phi(\phi)}$

thus: $\boxed{\frac{\sin\theta}{\Theta(\theta)}\frac{d}{d\theta}\sin\theta\frac{d\Theta}{d\theta} + l(l+1)\sin^2\theta\frac{1}{\Phi(\phi)}\frac{d^2\Phi}{d\phi^2} = 0}$

2) and: $\left(\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m_l^2\Phi\right) \wedge \left(\left(\frac{\sin\theta}{\Theta(\theta)}\frac{d}{d\theta}\sin\theta\frac{d\Theta}{d\theta}\right) + l(l+1)\sin^2\theta = m_l^2\right)$

iii) AZIMUTHAL EQS:

TYPICAL CHARACTERISTIC EQUATION

1) $\boxed{\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + m_l^2\Phi = 0} \rightarrow \frac{-b \pm\sqrt{b^2-4ac}}{2a} = \frac{0\pm\sqrt{0-4(1)(m_l^2)}}{2} = \frac{\pm\sqrt{-4m_l^2}}{2}$

$= \pm i\frac{2}{2}m_l = \boxed{\pm i m_l}$ $\quad \sqrt{-4m_l^2} = \sqrt{-4}\sqrt{m_l^2} = i\sqrt{4}\,m_l = i2m_l$

thus $\boxed{\Phi(\phi) = e^{im_l\phi}}$

and $\boxed{\Phi(\phi) = \Phi_{m_l}(\phi + 2\pi)}$ w/ $2\pi = 360° = 0°$

thus: $e^{im_l\phi} = e^{im_l(\phi+2\pi)} = e^{im_l\phi}e^{im_l 2\pi}$ and $e^{im_l 2\pi} = 1$.

2) is the orthonormality:

$$\boxed{\int_0^{2\pi}\Phi^*_{m_l}(\phi)\,\Phi_{m_l'}(\phi) = 2\pi\,\delta_{m_l m_l'}}$$

SCHRÖDINGER 3D - SEPARATION II

(2) CENTRAL POTENTIALS - PART II

[i] POLAR:

1) $\sin\theta \dfrac{\partial}{\partial\theta}\left[\sin\theta \dfrac{\partial\Theta}{\partial\theta}\right] + \left[l(l+1)\sin^2\theta - m_l^2\right]\Theta\,\theta = 0$

$$\boxed{\Theta_l^{m_l}(\theta) = P_l^{m_l}(\cos(\theta))}$$ in which $\forall\, 0 \le \theta \le \pi,\ l \in \mathbb{Z}^+$

$P_l^{m_l}(x) = (1-x)^{\frac{m}{2}} (D_x)^{m_l} P_l(x)$ → Rodrigues

$P_l(x) = \dfrac{1}{2^l l!}(D_x)^l (x^2-1)^l$. → Legendre.

2). $P_l^{m_l}(x) = (1-x)^{\frac{m}{2}} (D_x)^{m_l} P_l(x) = (1-x)^{\frac{m}{2}} (D_x)^{m_l}\left[\dfrac{1}{2^l l!}(D_x)^l (x^2-1)^l\right]$

$= \dfrac{(1-x^2)^{\frac{m}{2}}}{2^l l!}(D_x)^{m_l}(D_x)^l (x^2-1)^l = \dfrac{1}{2^l l!}(1-x^2)^{\frac{m_l}{2}}(D_x)^{l+m_l}(x^2-1)^l$

so $\boxed{P_l^{m}(x) = \dfrac{1}{2^l l!}(1-x^2)^{\frac{m_l}{2}}(D_x)^{l+m_l}(x^2-1)^l}$

3) It follows the negative value m_l as long as $(l+m_l) \ge 0$ (well def!)

in which. $\boxed{P_l^{-m_l}(x) = (-1)^{m_l}\dfrac{(l-m_l)!}{(l+m_l)!}P_l^{m_l}(x)}$

4) It's also orthogonal!

$$\int_{-1}^{1} P_l^{m}(x) P_{l'}^{m}(x) = \int_0^{\pi} P_l^{m_l}[\cos\theta]\sin\theta\, P_{l'}^{m_l}[\cos\theta] = \frac{2}{2l+1}\frac{(l+m_l)!}{(l-m_l)!}\delta_{ll'}$$

so $\boxed{\langle P_l^{m}(x) | P_l^{m}(x)\rangle = \dfrac{2}{2l+1}\dfrac{(l+m_l)!}{(l-m_l)!}\delta_{ll'}}$

[ii] SPHERICAL HARMONICS:

$$\boxed{Y_l^{m_l}(\theta,\phi) = (-1)^{m_l}\left[\frac{(2l+1)(l-m_l)!}{4\pi(l+m_l)!}\right]^{\frac{1}{2}} P_l^{m}[\cos\theta]\, e^{im_l\phi}}$$

and so: $\psi(r,\theta,\phi) = R(r)Y(\theta,\phi) \equiv R(r)Y_l^{m_l}(\theta,\phi)$

$= R(r)(-1)^{m_l}\left[\dfrac{(2l+1)(l-m_l)!}{4\pi(l+m_l)!}\right]^{\frac{1}{2}} P_l^{M}[\cos\theta]\, e^{im\phi}$

so $\boxed{\psi(r,\theta,\phi) = R(r)(-1)^{m_l}\left[\dfrac{(2l+1)}{4\pi}\dfrac{(l-m_l)!}{(l+m_l)!}\right]^{\frac{1}{2}} P_l^{M}[\cos\theta]\, e^{im\phi}}$

2) $Y_l^{-m_l} = (-1)^m Y_l^{m_l *}(\theta,\phi)$.

so $\displaystyle\int_0^{2\pi}\int_0^{\pi} Y_l^{m_l *}(\theta,\phi)\sin\theta\, Y_l^{m_l}(\theta,\phi) = \delta_{ll}\delta_{m_l m_l}$.

so $\boxed{\langle Y_l^{m_l} | Y_l^{m_l}\rangle = \delta_{ll}\delta_{m_l}}$

3) Common ones:

$Y_0^0 = \dfrac{1}{\sqrt{4\pi}}$ $\quad Y_1^0 = \sqrt{\dfrac{3}{4\pi}}\cos\theta$ $\quad Y_1^{\pm 1} = \mp\sqrt{\dfrac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$ $\quad Y_2^0 = \sqrt{\dfrac{5}{16\pi}}(3\cos^2\theta - 1)$

PROBLEMS 13

(1) Solve the harmonic oscillator of a 3D Schrödinger!

ANS: Let $r^2 = x^2 + y^2 + z^2$ and $\psi(r) = R(r)$

Then $\boxed{-\frac{\hbar^2}{2m}\nabla^2 R - \frac{1}{2}m\omega^2 r^2 = E}$ or $\boxed{\frac{\hbar^2}{2m}\nabla^2 R + \frac{1}{2}m\omega^2 r^2 - E = 0}$

1) Recall: $\nabla^2 R \equiv \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = \frac{d^2R}{dr^2}\frac{r^2}{r^2} + \frac{1}{r^2}\frac{dr^2}{dr}\frac{dR}{dr} = \frac{d^2R}{dr^2} + \frac{2r}{r^2}\frac{dR}{dr}$

$$= \frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}$$

2) So: $\frac{\hbar^2}{2m}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right) + \frac{1}{2}m\omega^2 r^2 R - ER = 0$

$$\frac{2m}{\hbar^2}\frac{\hbar^2}{2m}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right) + \frac{1}{\cancel{2}}\frac{m\omega^2 r^2 \cancel{2}m}{\hbar^2}R - \frac{2mE}{\hbar^2}R = 0$$

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \frac{m^2\omega^2 r^2}{\hbar^2}R - \frac{2mE}{\hbar^2}R = 0$$

Let $\frac{m\omega}{\hbar} = \alpha$ or $\frac{m^2\omega^2}{\hbar^2} = \alpha^2$

$\frac{2mE}{\hbar^2} = \varepsilon$

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \alpha^2 r^2 R - \varepsilon R = 0$$

GAUSSIAN DE.

GAUSSIAN APPROXIMATION

as $r \to \infty$,

$\frac{d^2R}{dr^2} + [\alpha r^2 - \varepsilon]R \approx 0$

$$\boxed{\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[\alpha r^2 - \varepsilon\right]R = 0}$$

3.) Let $R(r) \approx e^{-\frac{\alpha r^2}{2}}$ then $R(r) = e^{-\frac{\alpha r^2}{2}}u(r)$

so $\frac{dR}{dr} = -\alpha r\, u(r)\, e^{-\frac{\alpha r^2}{2}} + \frac{du}{dr}e^{-\frac{\alpha r^2}{2}} = \left(-\alpha r u(r) + \frac{du}{dr}\right)e^{-\frac{\alpha r^2}{2}}$

$$\frac{d^2R}{dr^2} = \left(-\alpha u(r) - \alpha r\frac{du}{dr} + \frac{d^2u}{dr^2}\right)e^{-\frac{\alpha r^2}{2}} + -\alpha r\left(-\alpha r u + \frac{du}{dr}\right)e^{-\frac{\alpha r^2}{2}}$$

$$= \left(-\alpha u + \alpha r\frac{du}{dr} - \alpha^2 r^2 u + \frac{du}{dr}\alpha r + \frac{d^2u}{dr^2}\right)e^{-\frac{\alpha r^2}{2}}$$

$$= \left(\frac{d^2u}{dr^2} + 2\alpha r\frac{du}{dr} + (\alpha^2 r^2 - \alpha)u\right)e^{-\frac{\alpha r^2}{2}}$$

So: $$\boxed{\frac{d^2u}{dr^2} + \left(\frac{2}{r} - 2\alpha r\right)\frac{du}{dr} + \left(\varepsilon - 3\alpha - \alpha^2 r^2\right)u = 0}$$

4.) Frobenius: Let $\sum_{n=0}^{?} a_n r^n = u(r)$

$$\sum_n^{?} n a_n r^{n-1}\ \text{...}\ \left(\frac{2}{r} - 2\alpha r\right)\sum_n^{?} n a_n r^{n-1} + \left(\varepsilon - 3\alpha - \alpha^2 r^2\right)\sum_n^{?} a_n r^n = 0$$

Then: $$\sum_n^{?}(n-1)n\, a_n r^{n-2} + \left(\frac{2}{r} - 2\alpha r\right)\sum_n^{?}(n+1)a_{n+1} r^{n} + \left(\varepsilon - 3\alpha - \alpha^2 r^2\right)\sum_n^{?} a_n r^n$$

$$\sum_n^{?}(n-1)(n+1)a_n r^n + \left(\frac{2}{r} - 2\alpha r\right)(n+1)a_{n+1} + \left(\varepsilon - 3\alpha - \alpha^2 r^2\right)a_n$$

$$a_{n+2}(n-1)(n+1) + \left(\frac{2}{r} - 2\alpha r\right)(n+1)a_{n+1} + \left(\varepsilon - 3\alpha - \alpha^2 r^2\right)a_n$$

... (skipping algebra)

$$\boxed{a_{n+2} = \frac{2\alpha(n+1) - \varepsilon + 3\alpha}{(n+2)(n+3)}a_n}$$

v) The solving after rearrange n, $\frac{\alpha}{}$n = 2N :

$\alpha n + 2 = 0 \rightarrow \varepsilon = 2\alpha(N+1) + 3\alpha \rightarrow \boxed{\varepsilon = 2\alpha(N+1)}$

since $\varepsilon = \frac{2mE}{\hbar^2}$, $\alpha = \frac{m\omega}{\hbar}$

so, $\frac{2mE}{\hbar^2} = 2 \cdot \frac{m\omega}{\hbar}(N+2) \longrightarrow \boxed{E = \hbar\omega(N+2)}$

so, $\boxed{E_n = \hbar\omega\left(n + \frac{3}{2}\right)}$

we since ~~$n = n_x + n_y + n_z$~~ here:

$\boxed{E_n = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right)}$

$\tan\theta = \frac{y}{x}$

$\theta = \tan^{-1} y/x$

(3) ~~Differentiate~~, 1st Derivative of the cylindrical coordinate (r, θ, ϕ).

as $\boxed{\begin{array}{l} x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{array}}$

Ans : Recall that :

$\varphi(x(r, \theta, \phi)) = r\sin\theta\cos\phi$ =.

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$

so, $\nabla_\partial \varphi(x) = \frac{\partial\varphi}{\partial r} + \frac{\partial\varphi}{\partial\theta} + \frac{\partial\varphi}{\partial\phi} = \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\phi}$

$= \sin\theta\cos\phi \frac{\partial\varphi}{\partial x}\Big|_r + r\cos\theta\cos\phi\frac{\partial\varphi}{\partial x}\Big|_\theta - r\sin\theta\sin\phi\frac{\partial\varphi}{\partial x}\Big|_\phi$

since $r =$

$\nabla_x = \sin\theta\cos\phi\frac{\partial}{\partial r} + r\cos\theta\cos\phi\frac{\partial}{\partial\theta} - r\sin\theta\sin\phi\frac{\partial}{\partial\phi}$

$\theta = \cos^{-1}\frac{z}{r}$

$r\cos\theta = \frac{z}{r}$

$r\cos\theta = z$

$r\cos\theta = r\cos\theta$

(i) Non-Degenerate Theory

Formulation from Hamiltonian

- Let $\hat{H}$ be the total Hamiltonian. Then

(14.1) $\boxed{\hat{H} = H_0 + \hat{H}_p}$ w/ • $\hat{H}_p$ = perturbed state; • H_0 = initial state.

- The exact energies are:

(14.2) $\boxed{\hat{H}_0 |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(0)}\rangle}$ w/ $X^{(0)}$ = 0th order unperturbed energies. $E_n^{(0)}$ = unique eigenstate. $|\psi_n^{(0)}\rangle$ = 2nd piece of $\hat{H}_p$.

- We modify it w/:

(14.3) $\boxed{H = \hat{H}_0 + \lambda \hat{H}_p}$ and $\lambda \in [0,1]$

- To find the eigenstates, we sum over power series in terms of λ^k so –

(14.4) $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$

and: $E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \cdots \doteq \sum_k \lambda^k E_n^{(k)}$.

$|\psi_n\rangle = |\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle + \lambda^2|\psi_n^{(2)}\rangle + \cdots \doteq \sum_k \lambda^k |\psi_n^{(k)}\rangle$.

∴ Thus: $\boxed{\begin{aligned} E_n &\sim \textstyle\sum_k \lambda^k E_n^{(k)} \\ |\psi_n\rangle &\sim \textstyle\sum_k \lambda^k |\psi_n^{(k)}\rangle \end{aligned}}$ (14.5) (14.6)

Nth order

- Continuing in substituting (14.4):

1.) $\hat{H}|\psi_n\rangle = \hat{H}\sum_k \lambda^k |\psi_n^{(k)}\rangle = (H_0 + \lambda H_p)\sum_k \lambda^k |\psi_n^{(k)}\rangle$.

$E_n|\psi_n\rangle = \left(\sum_k \lambda^k E_n^{(k)}\right)\left(\sum_k \lambda^k |\psi_n^{(k)}\rangle\right)$

2.) So;

$(\hat{H}_0 + \lambda\hat{H}_p)(|\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle + \ldots) = (E_n^{(0)} + \lambda E_n^{(1)} + \ldots)(|\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle)$

$\Rightarrow \hat{H}_0(|\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle) + \hat{H}_p(\lambda|\psi_n^{(0)}\rangle + \lambda^2|\psi_n^{(1)}\rangle + \ldots) = (E_n^{(0)} + \lambda E_n^{(1)} + \ldots)(|\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle$

$\Rightarrow \hat{H}_0(|\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle) + \hat{H}_p(\ldots) \ldots + \lambda^2(\hat{H}_0|\psi_n^{(2)}\rangle + \hat{H}_p|\psi_n^{(1)}\rangle)$

$\Rightarrow \hat{H}_0|\psi_n^{(0)}\rangle + \lambda(\hat{H}_0|\psi_n^{(1)}\rangle + \hat{H}_p|\psi_n^{(0)}\rangle) + \lambda^2(\hat{H}_0|\psi_n^{(2)}\rangle + \hat{H}_p|\psi_n^{(1)}\rangle)$

$= E_n^{(0)}|\psi_n^{(0)}\rangle + \lambda(E_n^{(0)}|\psi_n^{(1)}\rangle + E_n^{(1)}|\psi_n^{(0)}\rangle) + \lambda^2(E_n^{(0)}|\psi_n^{(2)}\rangle + E_n^{(1)}|\psi_n^{(1)}\rangle + E_n^{(2)}|\psi_n^{(0)}\rangle)$

or $\Rightarrow \sum_k^n \lambda^k H_0 + \sum_k \lambda^{k+1} H_p = \sum_k \lambda^{2k} \lambda |\psi_n^{(k)}\rangle$ w/ $\Lambda(k) = \sum_k \lambda^k$.

$= \Lambda(2k) E_n|\psi_n\rangle$.

$\equiv \Lambda(k) H_0 + \Lambda(k+1) H_p = \Lambda(2k) E_n|\psi_n\rangle$

i.e. $\boxed{\Lambda_k H_0 + \Lambda_{k+1} H_p = \Lambda_{2k} E_n|\psi_n\rangle}$

- 1st order term yields:

$\boxed{\hat{H}_0|\psi_n^{(1)}\rangle + \hat{H}_p|\psi_n^{(0)}\rangle = E_n^{(0)}|\psi_n^{(1)}\rangle + E_n^{(1)}|\psi_n^{(0)}\rangle}$. (14.9)

- 2nd order term yields:

$\boxed{\hat{H}_0|\psi_n^{(2)}\rangle + \hat{H}_p|\psi_n^{(1)}\rangle = E_n^{(0)}|\psi_n^{(2)}\rangle + E_n^{(1)}|\psi_n^{(1)}\rangle + E_n^{(2)}|\psi_n^{(0)}\rangle}$. (14.10)

[1] 1st Order Corrections

- Suppose we project $\langle\psi_n^{(0)}|$ into (14.9) set:

$\langle\psi_n^{(0)}|H_0|\psi_n^{(1)}\rangle + \langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(0)}\rangle = \langle\psi_n^{(0)}|E_n^{(0)}|\psi_n^{(1)}\rangle + \langle\psi_n^{(0)}|E_n^{(1)}|\psi_n^{(0)}\rangle$

$$\Rightarrow E_n^{(0)}\langle\psi_n^{(0)}|\psi_n^{(1)}\rangle + \langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(1)}\rangle = E_n^{(0)}\langle\psi_n^{(0)}|\psi_n^{(1)}\rangle + E_n^{(1)}$$

$$\Rightarrow E_n^{(0)}\langle\psi_n^{(0)}|\psi_n^{(1)}\rangle - E_n^{(0)}\langle\psi_n|\psi_n^{(1)}\rangle + \langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(0)}\rangle = E_n^{(1)}$$

$$\Rightarrow 0 + \langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(0)}\rangle = E_n^{(1)}$$

Thus: $\boxed{E_n^{(1)} = \langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(0)}\rangle}$

THE CORRECTION

- Let $E_n = E_n^{(0)} + E_n^{(1)}$ be the total energy s.t. we will employ this with the fact that: $\boxed{|\psi_n^{(1)}\rangle = \sum_n' C_{mn}^{(1)}|\psi_{mn}^{(0)}\rangle}$ (14.13)

- Our goal is to find eigenstates $|\psi_n\rangle$ of full Hamiltonian $\hat{H}$ s.t.
  1.) States are given ($\lambda = 1$) and contains $|\psi_n^{(0)}\rangle$. So no inclusion of $|\psi_n^{(0)}\rangle$.
  2.) By assuming that $\langle\psi_n^{(0)}|\psi_n^{(1)}\rangle = 0$, we can find the expression

* Thus it follows:

1.) $|\psi_n^{(1)}\rangle = \sum_n' C_{mn}^{(1)}|\psi_n^{(0)}\rangle$ 2.) $C_{mn}^{(1)} = \langle\psi_m^{(0)}|\psi_n^{(1)}\rangle$.

- Applying the Bra of $\psi_m^{(0)}$ we can derive:

$$\langle\psi_m^{(0)}|\hat{H}_0|\psi_n^{(1)}\rangle + \langle\psi_m^{(0)}|\hat{H}_p|\psi_n^{(0)}\rangle = \langle\psi_m^{(0)}|E_n^{(0)}|\psi_n^{(1)}\rangle + \langle\psi_m^{(0)}|E_n^{(1)}|\psi_n^{(0)}\rangle$$

$$\Rightarrow E_m^{(0)}\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle + \langle\psi_m^{(0)}|\hat{H}_p|\psi_n^{(0)}\rangle = E_n^{(0)}\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle$$

$$\Rightarrow E_m^{(0)} - E_n^{(0)} + \frac{\langle\psi_m^{(0)}|H_p|\psi_n^{(0)}\rangle}{\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle} = 0$$

$$\Rightarrow E_n^{(0)} - E_m^{(0)} = \frac{\langle\psi_m^{(0)}|H_p|\psi_n^{(0)}\rangle}{\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle} \Rightarrow \boxed{C_{mn} = \frac{\langle\psi_m^{(0)}|H_p|\psi_n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}}}$$ (14.18)

[II] 2nd ORDER CORRECTION

- Project:
  - Same as before, we project $\langle\psi_n^{(0)}|$ to (14.10) s.t.

$$\Rightarrow E_n^{(0)}\langle\psi_n^{(0)}|\psi_n^{(2)}\rangle + \langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(1)}\rangle = E_n^{(0)}\langle\psi_n^{(0)}|\psi_n^{(2)}\rangle + E_n\underbrace{\langle\psi_n^{(0)}|\psi_n^{(1)}\rangle}_{0} + E_n^{(2)}$$

$$\Rightarrow E_n^{(0)}\cancel{\langle\psi_n^{(0)}|\psi_n^{(2)}\rangle} + \langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(1)}\rangle = E_n^{(0)}\cancel{\langle\psi_n^{(0)}|\psi_n^{(2)}\rangle} + E_n^{(2)}$$

$$\boxed{E_n^{(2)} = \langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(1)}\rangle}$$ (14.27)

- Using (14.18) s.t. $(E_n^{(0)} - E_n^{(0)})C_{mn}$ is substituting on RHS of (14.27).

$$E_n^{(2)} = \langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(1)}\rangle = (E_n^{(0)} - E_m^{(0)})C_{mn} = (E_n^{(0)} - E_m^{(0)})\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle$$

$$= \sum_{m \neq n}' \frac{|\langle\psi_m^{(0)}|\hat{H}_p|\psi_n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

$\therefore$ So:

$$\boxed{\begin{gathered} E_n^{(1)} = \frac{\langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(0)}\rangle}{E_n^{(1)} - E_m^{(0)}} \text{ \# FIRST-ORDER.} \\ E_m^{(1)} = \frac{|\langle\psi_m^{(0)}|\hat{H}_p|\psi_n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}} \text{ \# 2nd ORDER} \end{gathered}}$$

Time Independent Perturbation
– Degenerate Theory & Relativistic Correction

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12. Degenerate Theory:

- **Philosophy**: { Degenerate = split in energy levels; Non degenerate = no split in energy }

- **Math**: • We assume that: $E_n^{(0)}$ is N-fold degenerate and have $|\psi_{nij}^{(0)}\rangle$ corresp. eigenstates w/ $j \in \mathbb{Z}^+$. The particular linear combination is:

$$\boxed{\hat{H}_0 \sum_{j=1}^{N} b_{nij} |\psi_{nij}^{(0)}\rangle = E_n^{(0)} \sum_{j=1}^{N} b_{nij} |\psi_{nij}^{(0)}\rangle} \quad (14.7m)$$

  • observe that: $|\psi_n^{(0)}\rangle \mapsto \sum_j^N b_{nij} |\psi_{nij}^{(0)}\rangle$ (14.28)

  similar how $f(x) \to \epsilon(x)$ with $\epsilon(x) = P_r(x) \approx \sum b_{nij} |\psi_{nij}^{(0)}\rangle$

  • we substitute this to eqs (14.4) ~
  to have: $\hat{H}_0 |\psi_n^{(1)}\rangle + \hat{H}_p |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(1)}\rangle + E_n^{(1)} |\psi_n^{(0)}\rangle$

$$\Rightarrow \langle\psi_{nii}^0| H_0 |\psi_n^1\rangle + \langle\psi_{nii}| H_p \sum_{j=1}^{N} b_{nij} |\psi_{nij}\rangle$$
$$= \langle\psi_{nii}^0| E_n^0 |\psi_n^{(1)}\rangle + \langle\psi_{nii}^{(0)}| E_n^{(1)} \sum_{j}^{N} b_{nij} |\psi_{nij}^{(0)}\rangle$$

  • The matrix elements of $\hat{H}_p$ in the N-dimensional subspace of the degenerate states to be: $\boxed{H_{pij} = \langle\psi_{nii}^0 | \hat{H}_p | \psi_{nij}^0\rangle}$ and $\boxed{\sum_{j=1}^{N} b_{nij} \langle\psi_{nii}^0|\hat{H}_p|\psi_{nij}^0\rangle = E_n^1 b_{nii}}$

  • using this we can we have:

$$\boxed{\sum_{j=1}^{N} H_{pij}\, b_{nij} = E_n^{(1)} b_{nii}}$$

- **Simplified**:
  • Basically: $H_0 + H_p = E_0 + E_1 \Rightarrow H_0|\psi\rangle + H_p|\psi\rangle = E_0|\psi\rangle + E_1|\psi\rangle$
  $\Rightarrow H_0 B_0 |\psi_0\rangle + H_p \underbrace{b_k |\psi_k\rangle}_{k>0} = E_0 b_0 |\psi_0\rangle + E_1 \underbrace{b_k |\psi_k\rangle}_{k>0}$
  • where: $\hat{H}_p \langle\psi_i|\psi_j\rangle = \hat{H}_{pij}$ so: $H_0 + H_{pij} b_k = E_0 + E_1 b_k |\psi_k\rangle$
  • or just: if $H_0, E_0 = 0 \Rightarrow \boxed{H_{pij} b_k = E_1 b_k |\psi_k}$

$\gg H_0 = E_n$

$\gg H_0 + H_p = E_n$

$\Rightarrow H_0 + \sum b H_p = \sum b E_n'$ $\langle\psi_i|b|\psi_j\rangle H_i = \langle\psi_i|b|\psi_j\rangle E$

$\Leftrightarrow \langle\psi_i|\hat{H}_0|\psi_j\rangle + \langle\psi_i|\hat{H}_p b|\psi_j\rangle = \langle\psi_i b E|\psi_j\rangle$

$\Rightarrow \sum_n b_n \langle\psi_i|H_p|\psi_j\rangle = E_n b_{nij}$

$= \sum_n H_{pij} b_{ij}^0 = E_n b_{ij}$

$\Rightarrow \hat{H}_p b_n = E_n b_n$

13. Relativistic Correction – Hydrogen

- **Fine structure of Hydrogen**:
  • The Einstein energy: $\underline{K + mc^2} = m^2c^4 - p^2c^2 + mc^2 = mc^2\left(1 + \frac{p^2c^2}{m^2c^2}\right)$
  since: $\boxed{K = mc^2\left|1 + \frac{p^2c^2}{mc^2}\right| - mc^2}$

  • There basically a binomial expansion of: $\text{Binom}\left(\frac{p^2}{2m}\frac{1}{c^2}\right) \approx H_R^k = \frac{p^2}{2m} + \frac{p^4}{8m^3c^2} + \cdots$

- The kinetic energy is:

$$\boxed{\hat{H}_R = -\frac{\hat{p}^4}{8m^3c^2} = -\frac{1}{2mc^2}\left(\frac{\hat{p}^2}{2m}\right)^2}$$

- Some caveats:
  - >> Hydrogen mass is literally: $\boxed{m = m_e}$
  - # since 1
  - >> so we'll use a $\boxed{\text{Non degenerate}}$ theory w/ $|n, l, m_e\rangle$ and fix n.
  - >> computing hydrogen requires computing $\langle n l m | \cdot | n l m \rangle$.
  - >> Avoiding degeneracy matrices requires us to use $[L_z, L^2]$ as roots

$$[p^2, \hat{L}] = [p_x^2, [p_x x]] = [p_x^2, i\hbar] \qquad p_x = i\hbar\nabla$$

$$= [-p_x \hbar, i\hbar]\phi = -\nabla^2\phi\, \hbar^2 i + \nabla^2\phi\, \hbar^2 i \qquad p_x^2 = -\hbar\nabla^2$$

$$= \boxed{0}\ V$$

- Defining:
  - >> $H_0 = \frac{p^2}{2m} + V(r) \Rightarrow \frac{p^2}{2m} = V(r) - H_0$ >> $H_R = -\frac{1}{2mc^2}(H_0 - V)(H - V)$
  - >> and: $\boxed{\frac{p^2}{2m} = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} - H_0}$ ★ do: $H_R = -\frac{1}{2mc^2}(H_0 - V)^2 = -\frac{1}{2mc^2}(H_0^2 - 2H_0V + V^2)$

- The perturbed state:

$$H_R = -\frac{1}{2mc^2}\left(H_0^2 - 2H_0V + V^2\right) = -\frac{1}{2mc^2}\left[\left(\frac{p^4}{4m} + \frac{p^2}{2m}\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} + \frac{e^4}{16\pi^2\varepsilon_0^2}\frac{1}{r^2}\right) + \left(2\frac{e^2}{2m}\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} + \frac{e^4}{16\pi^2\varepsilon_0^2}\frac{1}{r^2}\right) + \left(\frac{e^4}{16\pi^2\varepsilon_0^2}\frac{1}{r^2}\right)\right]$$

- This energy is expected as:

$$E_R^{(1)} = \langle n, l, m_l | H_R | n l m_l \rangle$$

$$= \left\langle n l m_l \right| -\frac{1}{2mc^2}\left(H_0^2 - 2H_0V + V^2\right)\left| n l m_l \right\rangle = -\frac{1}{2mc^2}\left[\langle n l m | H_0^2 | n l m \rangle - \langle n l m_l | 2H_0 V | n l m_l\rangle + \langle n l m_l | V^2 | n l m_l \rangle\right]$$

$$= -\frac{1}{2mc^2}\left[E^{0\,2} - 2E^0\frac{e^2}{4\pi\varepsilon_0}\langle n l m_l | \tfrac{1}{r} | n l m_l\rangle + \left(\frac{e^2}{4\pi\varepsilon_0}\right)^2 \langle n l m_l | \tfrac{1}{r^2} | n l m_l\rangle\right]$$

★ $\boxed{\left\langle \frac{1}{r}\right\rangle = \frac{1}{n^2 a_0}}$ ★ $\boxed{\left\langle \frac{1}{r^2}\right\rangle = \frac{1}{n^3 a_0^2 (l + \frac{1}{2})}}$

$$>> E_R^{(1)} = -\frac{1}{2mc^2}\left[-2E^0\frac{e^2}{4\pi\varepsilon_0}\left(\frac{1}{n^2 a_0}\right) + (E^0)^2 + \frac{e^4}{16\pi^2\varepsilon_0}\left(\frac{1}{n^3 a_0^2 (l + \frac{1}{2})}\right)\right]$$

$$= \frac{(E_n^{(0)})^2}{2mc^2}\left[\frac{4n}{(l + \frac{1}{2})} - 3\right]$$

- The fine structure and unperturbed energy:
  - >> $\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} \approx \frac{1}{137}$

- Our final equation is:

$$E_n^0 = \frac{m c^2 \alpha^4}{8n^4}\left[\frac{4n}{(l + \frac{1}{2})} - 3\right]$$

$$\boxed{E_R^{(1)} = -\frac{mc^2\alpha^4}{8n^4}\left(\frac{4n}{(l + \frac{1}{2})} - 3\right)}$$

1st PERTURBATION of EINSTEIN's energy

Time Independent Degeneracy Examples

(14.Ex)

(1) 1st Order Example - 14.1

Scenario: Suppose an electric field $\vec{E} = E\hat{x}$ is applied to a particle w/ charge q in an infinite potential well: an

$$\boxed{V(x) = \begin{cases} -qE\hat{x} & 0 < x < L \\ \infty & \text{elsewhere} \end{cases}}$$

(1) Determine the allowed energies!

Answer:

- The unperturbed states is basically:

$$\boxed{E_n^{(0)} = \frac{n^2 \pi^2 \hbar^2}{2mL^2}} \ \forall n \in \mathbb{Z}^+ \quad (14.21)$$

- The wave function on the other hand is:

$$\boxed{\psi_n^{(0)}(x) = \sqrt{\frac{2}{L}} \cdot \sin\left(\frac{n\pi x}{L}\right)} \text{ and (0) elsewhere } (14.22)$$

$\forall \psi_n \in 0 < x < L$

- The perturbed Hamiltonian is:

$$\boxed{H_p \equiv V(x) = -qE\hat{x}} \quad (14.23)$$

∴ The 1st order correction is:

$$E_n^{(1)} = \langle \psi_n^{(0)} | \hat{H}_p | \psi_n^{(0)} \rangle = \frac{2}{L}\int_0^L \sin^2\left(\frac{n\pi}{L}x\right) \cdot -qEx \, dx.$$

$$= \frac{2}{L}\int_0^L \left(\frac{1-\cos\left(\frac{2n\pi x}{L}\right)}{2}\right)(-qEx)\,dx = \frac{2}{L}\int_0^L -\frac{1}{2}qEx + \frac{1}{2}\cos\left(\frac{2n\pi}{L}\right)qEx \ldots$$

$$= -\frac{qEL}{2} \quad \text{so,} \quad \boxed{E_n = -\frac{qEL}{2}}$$

(2) 2nd Order Example.

Scenario: A uniform electric field of $\tilde{E} = \tilde{E}\hat{x}$ is applied to a particle w/ charge of q is situated in a harmonic oscillator potential. Determine the energies of the particle.

Answer: We apply: $\boxed{V(x) = \frac{1}{2}m\omega^2 x^2 - qEx}$ & Net potential of harmonic oscillator and cancels ~~forms~~

- The perturbation shifts: $E_n^0(x) \to E_n^1(x)$; $-\frac{q^2E^2}{2m\omega^2} \to -\frac{qE^2}{m\omega^2}$.
- The perturbation's solution form:

$$\boxed{\hat{H}_p(x) = -qE\hat{x}} \text{ w/ the eigenvalues: } \boxed{E_n^0 = \hbar\omega\left(n + \frac{1}{2}\right)}$$

- The matrix element:

$$\langle \psi_j^0 | \hat{H}_p | \psi_n^0 \rangle = -qE\langle j|\hat{x}|n\rangle = -qE\sqrt{\frac{\hbar}{2m\omega}} \cdot \langle j|\hat{a} + \hat{a}^\dagger|n\rangle$$

$$= -qE\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\langle j|n-1\rangle + \sqrt{n+1}\langle j|n+1\rangle\right)$$

$$= -qE\sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\cdot\delta_{j,n-1} + \sqrt{n+1}\cdot\delta_{j,n+1}\right].$$

$n = |c-|^2$

$\langle j|\hat{a} + a^\dagger|n\rangle$

$= \hat{a}\langle j|n\rangle + a^\dagger\langle j|n\rangle$

$= \langle j|(\hat{a}|n\rangle) + (a^\dagger|n\rangle)\langle j|$

$= \langle j|(c_-|n\rangle) + (c_+|n+1\rangle)\langle j|$

$= \sqrt{n}\langle j|n-1\rangle + \sqrt{n+1}\langle j|n-1\rangle$

- The 1st order correction:

$$\boxed{E_n^{0} = \langle \psi_n^{0} | \hat{H}_p | \psi_n^{0} \rangle = 0}$$

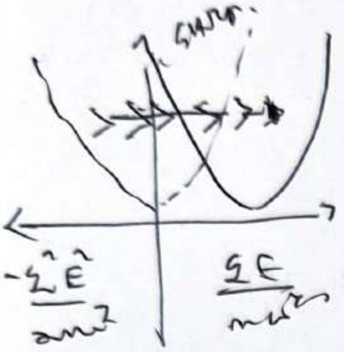
- So up to 1st order of correction:

$$\begin{cases} E_n^{(0)} = 0 \\ E_n^{(1)} = 0 \\ E_n^{(2)} = ? \end{cases}$$

- Therefore that:

$$E_n^{(2)} = \sum_{j \neq n} \frac{|\langle \psi_j^{(0)} | \hat{H}_p | \psi_n^{(0)} \rangle|^2}{E_n^{0} - E_j^{0}} = \sum_{j \neq n} \frac{1}{E_n^{0} - E_j^{0}} \left[qE \sqrt{\frac{\hbar}{2m\omega}} \left(\langle j | n-1 \rangle \sqrt{n} + \langle j | n+1 \rangle \sqrt{n+1} \right) \right]^2$$

$$= \frac{q^2 E \hbar}{2m\omega} \sum_{j \neq n} \frac{\left(\langle j | n-1 \rangle \sqrt{n} + \langle j | n+1 \rangle \sqrt{n+1} \right)^2}{E_n^{0} - E_j^{0}} = \frac{q^2 E \hbar}{2m\omega} \sum_{j \neq n} \left[\frac{\left(\sqrt{n}\, \delta_{j,n-1} + \sqrt{n+1}\, \delta_{j,n+1} \right)^2}{E_n^{0} - E_j^{0}} \right]$$



$$= \frac{q^2 E \hbar}{2m\omega} \sum_{j \neq n} \frac{1}{E_n^{0} - E_j^{0}} \sqrt{n}\, \delta_{j,n-1} \delta_{j,n-1} \sqrt{n} + \dots + \sqrt{n+1}\, \delta_{j,n+1} \delta_{j,n+1} \sqrt{n+1}$$

* we know this

$$= \frac{q^2 E \hbar^2}{2m\omega} \sum_{j \neq n} \frac{1}{E_n^{0} - E_j^{0}} \left((n) + (n+1) \right) = \frac{q^2 E \hbar^3}{2m\omega} \frac{n}{E_n^{0} - E_{n+1}^{0}} + \frac{n-1}{E_n^{0} - E_{n+1}^{0}}$$

$$= \frac{q^2 E \hbar^2}{2m\omega} \left[\frac{n}{\hbar\omega} - \frac{n+1}{\hbar\omega} \right] = -\frac{qE^2}{2m\omega^2}$$

$$\boxed{E_n^{(2)} = -\frac{qE^2}{2m\omega^2}}$$

2nd order correction is the model answer!

EXERCISES 14

① SIMPLE SYSTEM:

Calculate to 1st order an $H_p = V_0$ applied to $\hat{H}_0$. Calculate the energies to 2nd order and wave function

Ans. $E^1 = \langle \psi_i | V_0 | \psi_j \rangle = V_0 \langle \psi_i | \psi_j \rangle = V_0 \int |\psi|^2 dx = V_0$

$E^2 = \frac{|\langle \psi | V | \psi \rangle|^2}{\epsilon_i - \epsilon_j} = \frac{V_0^2}{\epsilon_i - \epsilon_j} \delta_{kn}^2 = 0$ & $|\psi_n^{(1)}\rangle = \sum_k \frac{\langle k | H | n \rangle |k\rangle}{\epsilon_n - \epsilon_k}$

$|\psi_n^{(1)}\rangle = 0$

② POTENTIAL WELL:

Infinite potential well extends to/ $x \in [0, L]$ and has a small bump in the middle. Assume perturbation:

$$\boxed{H_p = \begin{cases} V_0 & \frac{L}{2} - \frac{a}{2} < x < \frac{L}{2} + \frac{a}{2} \\ & \text{else} \end{cases}}$$

ⓐ Calculate the 1st order:

Ans: $E^1 = \langle \psi_j | V_0 | \psi_i \rangle = V_0 \langle \psi_j | \psi_i \rangle = V_0 \frac{2}{L} \int_{\frac{L}{2} - \frac{a}{2}}^{\frac{L}{2} + \frac{a}{2}} \sin^2\left(\frac{n \pi x}{L}\right) dx$

$$= V_0 \frac{2}{L} \int_{\frac{1}{2}(L-a)}^{\frac{1}{2}(L+a)} \left[\frac{1 - \cos(2 k_n x)}{2} dx \right] = \frac{2 V_0}{L} \left[\frac{a}{2} - \frac{1}{2} \int_{x_0}^{x_0 + a} \cos(2 k_n x) \right]$$

$k_n = \frac{n \pi}{L}$

$$= \frac{\cancel{2} V_0}{L} \frac{1}{\cancel{2}} \left[a - \frac{\sin(2 k_n (x_0 + a)) - \sin(2 k_n x_0)}{4 k_n} \right]$$

$$= \frac{V_0}{L} \left[\frac{a}{2} + (-1)^{n+1} \frac{\sin\left(n \pi \frac{a}{2}\right)}{2 k_n} \right]$$

$$\Rightarrow \boxed{E_n^{(1)} = V_0 \frac{a}{L} + (-1)^n V_0 \frac{\sin(n \pi a / L)}{n \pi}}$$

TIME INDEPENDENT PERTURBATION - SPIN ORBITS

3.7 FINE STRUCTURE - SPIN ORBIT ; $\boxed{14.3.2}$

$\left\{ 1\text{ volt} = \frac{1\text{ Joule}}{1\text{ coulomb}} \right.$

Coulomb: $[M^0 L^0 T^1 I^1]$

Joule $= [M L^2 T^{-2}]$

$I =$ coulombs $= \frac{1\text{ Joule}}{1\text{ volt}}$

GR principles: • The current loop produced by a magnetic field is:

$$\boxed{B = \frac{\mu_0 I}{2r} = \frac{I}{\varepsilon_0 c^2 2r}} \quad (6.8)$$

>> why start w/: $V = IR$

$$\left\{ \begin{array}{l} F = \nabla(\vec{\mu} \cdot \vec{B}) \quad \vec{\mu} = \frac{1}{\mu_0} \\ \mu_e = \gamma e S \\ \text{or } \mu_0 = \gamma_0 S \end{array} \right.$$

>> So: $V = IR \Rightarrow \frac{1}{\mu_0} \vec{B} = \frac{I}{2r} \Rightarrow \vec{B} = \frac{I}{2r}\mu_0 = \frac{1}{\varepsilon_0 c^2 2r}$

$\star : \left\{ \begin{array}{l} R V I = \frac{1}{2r} \\ \vec{\mu} = \frac{1}{\mu_0} \end{array} \right.$

$\left\{ \mu_0 = \frac{1}{c^2 \varepsilon_0} \right.$ since $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$, $c^2 = \frac{1}{\mu_0 \varepsilon_0}$

>> The effective current is: $I = \frac{e}{T}$, so:

$$B = \frac{I}{\varepsilon_0 c^2 2r} = \frac{\cancel{e}}{\varepsilon_0 c^2 2r}\frac{e}{T} = \frac{e}{\varepsilon_0 c^2 2r\left(\frac{2\pi r}{v}\right)} = \frac{e}{4\pi\varepsilon_0}\frac{v}{c^2 r^2}.$$

>> Gyromagnetic: $\boxed{\gamma_e = \frac{g_e}{2m} q_e = -\frac{g_e}{2m} e}$ by which: $g_e q_e \sim \tilde{g}_e \hat{q}_e = \hat{\zeta}$

or: $\frac{\hat{\zeta}^2}{2m} \propto \frac{p^2}{2m}$

>> If angular momentum is:

$$L = \vec{r} \times \vec{p} = m[\vec{r} \times \vec{v}] = mrv$$

$$\Rightarrow L = mrv \cdot \boxed{v = \frac{L}{mr}}$$

>> Then: $B = \frac{e}{4\pi\varepsilon_0}\frac{v}{c^2 r^2} = \frac{e}{4\pi\varepsilon_0}\frac{L}{mr}\frac{1}{c^2 r^2} = \frac{e}{4\pi\varepsilon_0}\frac{L}{mc^2 r^3}$

>> our dipole is: $\boxed{V = -\mu \cdot \vec{B} = \frac{g_e e^2}{8\pi\varepsilon_0 m^2 c^2 r^3} S \cdot \vec{L}}$

• why $B = \frac{e}{4\pi\varepsilon_0}\frac{L}{mc^2 r^3}$ $\quad \mu = -\frac{g_e e^2}{2m}$

So we: $\boxed{V = -\mu \cdot \vec{B} = \hat{\zeta}(S \cdot L)}$

$\zeta(r) = \frac{g_e e^2}{8\pi\varepsilon_0 c^2 r^3 m^2}$ why $\left\{ \zeta(1) = 1.38 \times 10^{20} \frac{\text{m}}{\text{kg}} \Big|_{r=1} \right\}$ $\quad B = \frac{S \cdot L}{\mu}$

$\hat{\zeta} = \zeta \cdot \frac{1}{r} = 1.38 \times 10^{20} \frac{\text{m}}{\text{kg}}\left(\frac{1}{r^3}\right)$

$r \to \infty$ $\hat{\zeta} = 0$ $\boxed{V =}$

• Turns out, the Hamiltonian in Reference frame is: due to

$$\boxed{\hat{H}_{SO} = \frac{(g_e - 1)}{8\pi\varepsilon_0 m^2 c^2 r^3}\hat{S}\cdot\hat{L} = \frac{e^2}{8\pi\varepsilon_0 m^2 c^2 r^3}\hat{S}\cdot\hat{L}}$$

SPIN TO ORBIT HAMILTONIAN

why $g_e \sim 2$.

>> $g_e \sim 2 \Rightarrow H = \frac{1}{2m}(p^2 - qA)^2 - \frac{g\hbar}{2m}\vec{\sigma}\cdot\vec{B}$

$= 0 - \frac{g\hbar}{2m}\vec{\sigma}\cdot\vec{B} = -\frac{g\hbar}{2m}\vec{\sigma}\cdot\vec{B}$

$= -\frac{g\hbar}{2m}\vec{\sigma}\frac{SL}{\mu}$

$\left\} \begin{array}{l} S_z = \frac{\hbar}{2}\vec{\sigma} \\ \mu_s = \frac{g}{m} S \end{array} \right.$

$\left\{ \begin{array}{l} \mu_s = \mu_S \\ \frac{g\hbar}{2m}\sigma = \frac{g}{m} \\ \frac{\hbar}{2}\frac{1}{\sigma S} = 1 \\ \boxed{\frac{\hbar}{\sigma S} = 2} \end{array} \right.$

8.) The subscript is used due to the spin orbit of the angular momentum

• The total angular is:

$$J^2 = \hat{J}\hat{J} = (S+L)(S+L) = \hat{S}^2 + \hat{L}^2 + 2S\cdot L$$

$\Rightarrow \frac{1}{2}(\hat{J}^2 - S^2 - L^2) = S \cdot L$

$\Rightarrow \hat{H}_{SO} = \frac{e^2}{16\pi\varepsilon_0 m^2 c^2 r^3}(\hat{J}^2 - S^2 \ldots$

HAMILTONIANS & [illegible]

- Practically: $E_{SO}^{(1)} = \langle \psi | \hat{H}_{SO} | \psi \rangle$
$$= \langle \psi | \frac{e^2}{16\pi\varepsilon_0 m^2c^2r^3}(J^2 - L^2 - S^2) | \psi \rangle.$$
$$= \langle \psi | \frac{e^2}{16\pi\varepsilon_0 m^2c^2r^3} j(j+1) - l(l+1) - s(s+1) | \psi \rangle$$
$$= \frac{e^2}{16\pi\varepsilon_0 m^2c^2} \left[j(j+1) - l(l+1) - s(s+1) \right] \langle \frac{1}{r^3} \rangle \quad \text{w/} \langle \frac{1}{r^3} \rangle = \langle \psi | \frac{1}{r^3} | \psi \rangle.$$

- Now expected value of integer ν in $\hat{r}$:
$$\langle \hat{r}^{\nu} \rangle = \left[n \frac{a_0}{2} \right]^{\nu} \frac{1}{2n(\beta+1)!} \sum_{k=0}^{\beta} (-1)^{\beta+k} \binom{\beta}{k} \binom{n+l-1+k}{\beta} \frac{1}{\binom{2l+\beta+1-k}{\beta+1}}.$$
w/ $\binom{x}{y} = \frac{x!}{y!(y!-x!)}$

>> So that: $$\langle \hat{r}^{-3} \rangle = \left[n \frac{a_0}{2} \right]^{-3} \frac{1}{2n(\beta+1)!} \sum_{k=0}^{\beta} (-1)^{\beta+k} \binom{\beta}{k} \binom{n+l-1+k}{\beta} \frac{1}{\binom{2l+\beta+1-k}{\beta+1}}$$

>> Practically this is: $$\langle \frac{1}{r^3} \rangle = \frac{1}{a_0^3 n^3 l(l+1)(l+\frac{1}{2})}$$

>> Doing algebra yields:
$$E_{SO}^{(1)} = \frac{e^2}{16\pi\varepsilon_0 m^2c^2} \left[j(j+1) - l(l+1) - s(s+1) \right] \langle \frac{1}{r^3} \rangle = \frac{e^2}{16\pi\varepsilon_0 m^2c^2} \left[j(j+1) - l(l+1) - s(s+1) \right] \times \frac{1}{a_0^3 n^3 l(l+1)(l+\frac{1}{2})}.$$
$$= \frac{e^2}{16\pi\varepsilon_0 m^2c^2} \frac{j(j+1) - l(l+1) - \frac{3}{4}}{n^3 l(l+1)(l+\frac{1}{2}) a_0^3 n^3} = \frac{mc^2\alpha^4}{4} \left[\frac{j(j+1) - l(l+1) - \frac{3}{4}}{n^3 l(l+1)(l+\frac{1}{2})} \right]$$

>> then: $$\boxed{E_{SO}^{(1)} = \frac{mc^2\alpha^4}{4} \left[\frac{j(j+1) - l(l+1) - \frac{3}{4}}{n^3 l(l+1)(l+\frac{1}{2})} \right]}$$

DARWIN TERM: • In Relativistic quantum mechanics, the particle can't just be localized smaller than: $\boxed{\lambda_c = \frac{\hbar}{mc}}$

- To Elaborate this, Expand V:
$$V(r+r') = V + \sum_{i=1}^{3} \left(\frac{\partial V}{\partial r_i} \right) r_i' + \frac{1}{2} \sum_{ij=1}^{3} \left(\frac{\partial^2 V}{\partial r_i \partial r_j} \right) r_i' r_j' + \ldots$$
$$= V(r) + \frac{1}{2} \left[\frac{1}{3} \lambda_c^2 \right] \nabla^2 V(r) + \ldots = V(r) + \frac{\hbar^2}{10 m^2c^2} \nabla^2 V(r) + \ldots$$

MAXWELL'S EQUATIONS
$$\nabla \cdot E = \nabla \left(\frac{q}{4\pi\varepsilon_0} \frac{1}{r} \right) = \nabla \frac{4\pi}{4\pi} \varepsilon_0 \rho$$
$$\nabla^2 \left[\frac{1}{r} \right]$$
$$\boxed{\nabla^2 \left(\frac{1}{r} \right) = 4\pi \delta^3(r)}$$

- Darwin Hamiltonian: $$\boxed{H_D = \frac{\hbar^2}{8m^2c^2} \nabla^2 V(r)}$$

- Coulomb potential is: $$H_D = \frac{\hbar^2}{8m^2c^2} \left[-\frac{e^2}{4\pi\varepsilon_0} \nabla^2 \left(\frac{1}{r} \right) \right] = \frac{\hbar^2 e^2}{8m^2c^2\varepsilon_0} \delta^3(r)$$
- Expected value is:
$$\boxed{E_D^{(1)} = \langle \hat{H}_D^{(1)} \rangle = \frac{\hbar^2 e^2}{8m^2c^2\varepsilon_0} |\psi_{nlm}(0)|^2}$$
w/ it's: $$E_D^{(1)} = \frac{mc^2\alpha^4}{2n^3}.$$

EXERCISE 14

(15) Verifying $E_{S_0} = \frac{mc^2\alpha^4}{2n^3}$ by using $L_j^k (x) = \frac{(j+k)!}{j!\,k!}$

Ans: We have: $E_{S_0} = \langle H_0 \rangle = \frac{\hbar^2 e^2}{8 m^3 c^2 \varepsilon_0} \cdot |\psi_{nlm}|^2$

>> with: $|\psi_{nlm}|^2 = R_{nl}(r) \cdot Y_{lml}(\theta, \phi)$

>> and $R_{nl} = \frac{2}{n^2}\sqrt{\frac{(n-l-1)!}{a_0^3 (n+l)!}} \left[\frac{2r}{na_0}\right]^l e^{-\frac{r}{na_0}} L^{2l+1}_{n-l-1}\left(\frac{2r}{na_0}\right)$.

• Letting: $R_{nl} \doteq R_{n0} = \frac{2}{n^2}\sqrt{\frac{(n-1)!}{a^3 l!}} \left[\frac{2r}{na_0}\right]^0 e^{-\frac{r}{na_0}} L^1_{n-1}\left(\frac{2r}{na_0}\right)$

$$= \left(\frac{2}{na_0}\right)^{\frac{3}{2}} \sqrt{\frac{(n-1)!}{2n\,n!}}\; e^{-\rho/2} L^1_{n-1}(\rho) \quad \text{with } \rho = \frac{2r}{na_0}.$$

=> where: $n! = n(n-1)!$ => $\sqrt{\frac{(n-1)!}{2n \cdot n!}} = \sqrt{\frac{1}{2n^2}} = \frac{1}{\sqrt{2}\,n}$.

=> have: $\boxed{R_{n0} = \frac{1}{\sqrt{2}\,n} e^{-\frac{\rho}{2}} L^1_{n-1}(\rho)}$

• Laguerre polynomial form :