QUANTUM MECHANICS :
THEORY & EXPERIMENT

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(M) POLARIZATION

$$\boxed{k = \frac{2\pi}{\lambda}} \quad \text{or} \quad \boxed{k = \frac{\omega}{c}} \quad \text{w/} \quad \lambda = \frac{c}{f} \Rightarrow \frac{1}{f} = \frac{f}{c}$$

$$\text{an} \quad \boxed{\omega = 2\pi f}$$

ANGULAR FREQUENCY of a WAVE VECTOR $\vec{k} = k\vec{u}_z$.

[1] POLARIZATION VECTOR:

$$\boxed{\vec{E} = \vec{E}_x \vec{u}_x + \vec{E}_y u_y}$$

$$\text{and} \begin{cases} \nabla \times E = \frac{\partial B}{\partial t} \\ \nabla \cdot E = \frac{\rho}{\varepsilon_0} \end{cases}$$ SUMMARY of MAXWELL'S EQUATIONS

1) w/ $E_x = E_{0x} \cos(kz - \omega t)$

$E_y = E_{0y} \cos(kz - \omega t + \phi)$.

hence $\frac{\partial^2 E_x}{\partial z^2} = -k^2 E_{0x} \cos(kz - \omega t) = -k^2 E_x$

$\frac{\partial^2 E_y}{\partial z^2} = -k^2 E_{0y} \cos(kz - \omega t) = -k^2 E_y$.

a) $\frac{\partial^2 E_x}{\partial t^2} = -\omega^2 E_{0x}$ $\quad \frac{\partial^2 E_y}{\partial t^2} = -\omega^2 E_y$.

hence b) $(\nabla^2 + \partial_{tt}^2 + m^2)\vec{E} = -k^2 E_x - k^2 E_y - \omega^2 E_x - \omega^2 E_y + m^2(E_x + E_y) = 0$

$= -(k^2(E_x + E_y) + \omega^2(E_x + E_y) + m^2(E_x + E_y)) = 0$

$= -(k^2 + \omega^2 + m^2) = 0 \rightarrow \boxed{k^2 + \omega + m^2 = 0}$

- HIGH SPEED PARTICLE FIX

$\omega^2 = c^2 k^2 + \left(\frac{mc^2}{\hbar}\right)^2$ and $k^2 = \frac{\omega^2}{c^2}$ then $\frac{\omega^2}{c^2} + \omega^2 + m^2 = \omega^2\left(\frac{1}{c^2} + 1\right) + m^2 = 0$

$\omega^2 = c^2\left(\frac{\omega^2}{c^2}\right) + \left(\frac{mc^2}{\hbar}\right)^2$ $\quad \frac{1}{c^2} + 1 + m^2 = \frac{c^2 + 1}{c^2} + m^2 = 0$

$= \omega^2 + \left(\frac{mc^2}{\hbar}\right)^2$ $\quad c^2 + 1 = (mc)^2$

c.) we: $\boxed{c^2\omega^2 = \omega^2 + c^2 m^2}$ $\quad \boxed{c^2 = (mc)^2 - 1}$

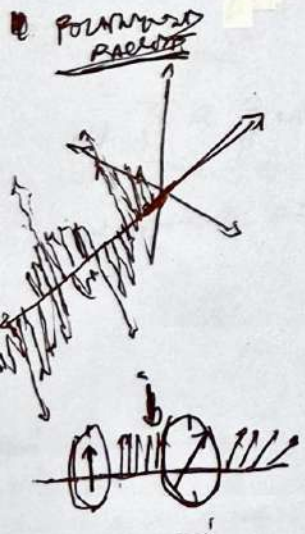
and: $\boxed{\omega^2 = \frac{c^2 m^2}{c^2 - 1}}$

$\lim \left(\frac{mc^2}{\hbar}\right)^2 = 0$ $\quad$ and if $c^2 = 1$ then $\omega^2 = \frac{m^2}{0}$ can't be ZERO!

and $\boxed{m = 0}$

If $m^2 = 0$ Then: $\omega^2 = 0$ and the FIELD EQUATION is and the PARTICLE IS 0.

- POLARIZED PACKAGE



2) $E_x = E_0 e^{i(kz - \omega t)}$

$E_y = E_0 e^{i(kz - \omega t + \phi)} \rightarrow E = E_{0x} e^{(kz - \omega t)} \vec{u}_x + E_{0y} e^{i(kz - \omega t + \phi)} \vec{u}_y$

w/ $\boxed{\|E_0\| = (E_{0x}^2 + E_{0y}^2)^{\frac{1}{2}}}$

3) $\vec{E} = E_0 e^{i(kz - \omega t)} \left[\frac{E_{0x}}{E_0} u_x + \frac{E_{0y}}{E_0} e^{i\phi} u_y\right]$

$$\boxed{\vec{\varepsilon} = \frac{E_{0x}}{E_0}\vec{u}_x + \frac{E_{0y}}{E_0} e^{i\phi} \vec{u}_y}$$

$|\varepsilon| = \sqrt{\varepsilon^* \cdot \varepsilon} = \sqrt{\left(\frac{E_{0x}}{E_0}\right)^2 + \left(\frac{E_{0y}}{E_0}\right)^2} = 1$

MALUS' LAW

$\Rightarrow$ (DETECTORS $\Rightarrow$ FIELDS) $\wedge$ (DETECTORS $\Rightarrow$ PARTICLE COUNTS).

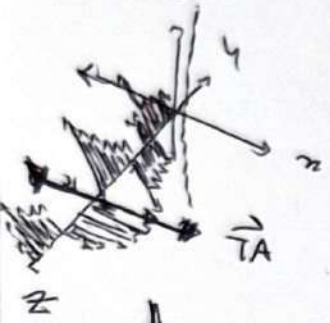
hence: $I = |E|^2 = E^* \cdot E = \left(E_0 e^{i(kz - \omega t)} \varepsilon^*\right)\left(E_0 e^{-i(kz - \omega t)} \varepsilon\right) = E_0^2$

and if $\varepsilon \cdot \varepsilon^* = 1$ $\quad I = E^* \cdot E = \left(E_0 e^{i(kz - \omega t)} \varepsilon^*\right)\left(E_0 e^{-i(kz - \omega t)} \varepsilon\right) = E_0^2$

POLARIZATION MECHANICS (2.3) (2.4)

(3) MODIFYING POLARIZATION

[i] LINEAR POLARIZATION:



$$\vec{E}_i = E_0 e^{i(kz-\omega t)}\,\vec{\varepsilon}$$

$$\vec{E}_t = (\vec{E}_i \cdot u_\theta)\,\vec{u}_\theta = E_0 e^{i(kz-\omega t)}(\vec{\varepsilon}\cdot u_\theta)\,u_\theta$$

$$\boxed{E_t = E_0 e^{i(kz-\omega t)}(\varepsilon_0 \cdot u_\theta)\,u_\theta}$$

The output is linearly polarized on u_θ.

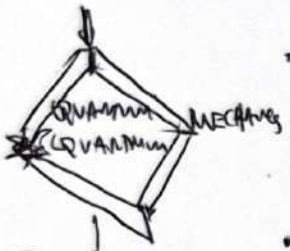
- Linearly polarized light passes an angle ϕ w/ horizontal ($\varepsilon = u_\phi$)

$\therefore$ Do off by a factor: $\cos(\theta-\phi)$

[ii] SPLITTING BEAMS & [iii] WAVE PLATES

POLARIZING BEAM SPLITTER (PBS)

- Prism of a birefringent material can be used as a PBS
- Polarizing Beam Splitter (PBS) can also use thin film coatings.



$$\Delta\phi = 2\pi j + \frac{\pi}{2}$$

1) $\phi_f = \dfrac{n_f\, 2\pi l}{\lambda}$ $\quad f$ = FAST AXIS REFRACTION

$\phi_s = \dfrac{n_s\, 2\pi l}{\lambda}$ $\quad s$ = SLOW AXIS REFRACTION

$$\boxed{\Delta\phi = \phi_s - \phi_f = \frac{(n_s - n_f)\,2\pi l}{\lambda}}$$

2) QUARTER WAVE LENGTH

$\therefore \Delta\phi = \dfrac{2\pi}{4} = \dfrac{1\pi}{2} = \dfrac{\pi}{2}$ $\quad \therefore \boxed{\Delta\phi_{QUARTER} = \dfrac{\pi}{2}}$

- Wave plate = optical element = phase shift in orthogonal frames.

(4) JONES VECTORS & MATRICES

[i] JONES VECTORS

$$\boxed{\begin{aligned}\varepsilon_H &= u_x \\ \varepsilon_V &= u_y\end{aligned}}$$

H = Horizontal $\rightarrow$

V = Vertical $\uparrow$

here 1) $\varepsilon_{45} = \dfrac{1}{\sqrt{2}}u_x + \dfrac{1}{\sqrt{2}}u_y = \dfrac{1}{\sqrt{2}}(u_x + u_y) = \dfrac{1}{\sqrt{2}}(\varepsilon_H + \varepsilon_V)$

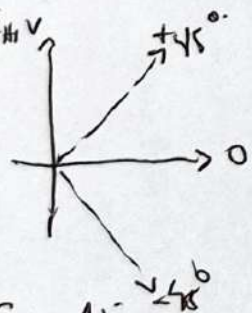
here $\boxed{\varepsilon_{45} = \dfrac{1}{\sqrt{2}}(\varepsilon_H + \varepsilon_V)}$

[ii] JONES MATRICES

CLIFFORD: 1) and $\varepsilon_H = \begin{bmatrix}1\\0\end{bmatrix}$ $\quad \varepsilon_V = \begin{bmatrix}0\\1\end{bmatrix}$

$$\varepsilon_{+45} = \frac{1}{\sqrt{2}}\left(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right) = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

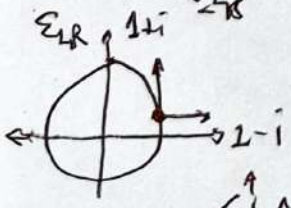
$$\varepsilon_{-45} = \frac{1}{\sqrt{2}}\left(\begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\right) = \frac{1}{\sqrt{2}} = \begin{bmatrix}1\\-1\end{bmatrix}$$



here $\boxed{\varepsilon_{+45} = \dfrac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} \quad \varepsilon_{-45} = \dfrac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}}$

now: $\varepsilon_{+45}\,\varepsilon_{-45} = \dfrac{1}{(\sqrt{2})^2}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \dfrac{1}{2}\begin{bmatrix}1\begin{bmatrix}1\\1\end{bmatrix}\\-1\begin{bmatrix}1\\1\end{bmatrix}\end{bmatrix} = \dfrac{1}{2}\begin{bmatrix}1\cdot1 + 1\cdot-1\\1\cdot1 + -1\cdot1\end{bmatrix} = \dfrac{1}{2}\begin{bmatrix}2-2\\2-2\end{bmatrix}$

$\begin{bmatrix}0 & \frac{1}{2}\\0 & \frac{1}{2}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ so: $\boxed{\varepsilon_{+45}\,\varepsilon_{-45} = 0}$



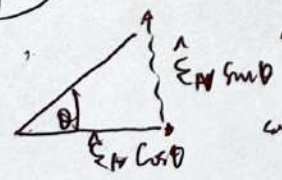
and $\varepsilon_L = \dfrac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix} = \dfrac{1}{\sqrt{2}}(\varepsilon_H + i\varepsilon_V)$

$\varepsilon_R = \dfrac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} = \dfrac{1}{\sqrt{2}}(\varepsilon_H - i\varepsilon_V)$

so: $\boxed{\begin{aligned}\varepsilon_L &= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}\\ \varepsilon_R &= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}\end{aligned}}$ $\quad -(i\cdot i) = -(-1) = 1$

and: $\varepsilon_L\,\varepsilon_R = \dfrac{1}{2}\begin{bmatrix}1\\i\end{bmatrix}\begin{bmatrix}1\\-i\end{bmatrix} = \dfrac{1}{2}\begin{bmatrix}1\begin{bmatrix}1\\-i\end{bmatrix}\\i\begin{bmatrix}1\\-i\end{bmatrix}\end{bmatrix} = \dfrac{1}{2}\begin{bmatrix}1\cdot1 + i\cdot1\\1\cdot(-i) + -i\cdot-i\end{bmatrix} = \dfrac{1}{2}\begin{bmatrix}1+i\\-1-i\end{bmatrix}$

ε_θ:



and $\boxed{\varepsilon_\theta = \cos\theta \cdot \varepsilon_H + \sin\theta \cdot \varepsilon_V}$

$$1 = \int_0^\infty |E(z,t)|^2\, dz = E_0^2 \int_0^\infty \left(e^{i(kz-\omega t)}\right)^2 dz = E_0^2 \int_0^\infty e^{2i(kz-\omega t)}\, dz$$

$$= E_0^2 \frac{1}{2ik} e^{2i(kz-\omega t)}\Big|_0^\infty = E_0^2 \frac{1}{2ik z} e^{i2(k(\infty-\omega t)} = E_0^2 \frac{1}{2ikz} e^{-i2\omega t}$$

Normalization Fix

$$\int_0^\infty |E(z,t)|\, dz$$

$$= |E_0|^2 \int_0^\infty e^{-2\lambda z}\, dz$$

$$= |E_0|^2 \left[\frac{1}{2\lambda}\right]$$

$$\text{or } E_0 = \sqrt{2\lambda}$$

$$\text{and: } \frac{\partial}{\partial t}\int_0^\infty |E(z,t)|^2\, dz = 0$$

Similarly $|E|^2 = 2\lambda e^{-2\lambda z}$ ✓

$$\text{hence: } \boxed{E_0^2 = 2ike^{i2\omega t}} \quad \text{and: } \boxed{E_0 = \sqrt{2ike^{2i\omega t}}}$$

$$\text{as } \omega = \frac{v}{r}, \quad v = \frac{\lambda c}{t}$$

$$\text{If } \omega = \frac{\lambda c}{t}\frac{1}{r}$$

$$\text{Then } \omega t = r\lambda c.$$

$$\text{w/ } e^{i2\omega t} = \cos(2\omega t) + i\sin(2\omega t)$$

$$= \cos(2\lambda c r) + i\sin(2\lambda c r)$$

$$\text{hence } E(z,t) = E_0 e^{i(kz-\omega t)} = \left(2ike^{i2\omega t}\right)^{\frac{1}{2}} e^{i(kz-\omega t)}$$

$$\frac{\partial E}{\partial t} = \frac{\partial}{\partial t}\left(2ike^{i\omega t}\right)^{\frac{1}{2}} e^{i(kz-\omega t)} = \left(2ike^{i\omega t}\right)^{\frac{1}{2}} \frac{\partial}{\partial t} e^{-i\omega t} \cdot e^{ikz}$$

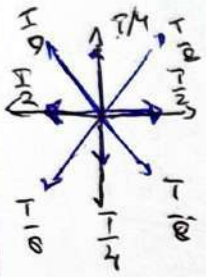
$$U = 2ike^{i\omega t}$$

$$\frac{dE}{dt} = \frac{dE}{dU}\frac{dU}{dt} = U^{\frac{1}{2}} \frac{dU}{dt}$$

$$\text{So: } \boxed{E(z,t) = Ae^{-\lambda z} e^{i(kz-\omega t)}}$$

To make sensible.

THE WAVE FUNCTION IS STILL TIME DEPENDENT!



(iii) LINEAR POLARIZATION

$$\vec{E} = \underbrace{E_{0x}}_{E_x} u_x + \underbrace{E_{0y}}_{E_y} u_y$$

$$\text{or } \boxed{\theta = \tan^{-1}\left(\frac{E_{0y}}{E_{0x}}\right)}$$

If the zero phase shift ($\phi = 0$) is assumed, then the the charge between $E_0(y,x)$ is proportional

LINEAR PUMPING

$$\cos\left(kz - \omega t + \frac{\pi}{2}\right)$$

$$\downarrow$$

$$\boxed{\frac{E_0}{\sqrt{2}} \sin(kz - \omega t)}$$

(iv) CIRCULAR POLARIZATION

Consider $\phi = \frac{\pi}{2}$ and $(E_{0x}, E_{0y}) = \frac{E_0}{\sqrt{2}}$. then

$$\boxed{\vec{E} = \frac{1}{\sqrt{2}} u_x + i\vec{u}_y}$$

$$\text{so: } E_x = \frac{E_0}{\sqrt{2}} \cos(-\omega t) = \frac{E_0}{\sqrt{2}} \cos(\omega t)$$

$$E_y = \frac{E_0}{\sqrt{2}} \sin(-\omega t) = \frac{E_0}{\sqrt{2}} \sin(\omega t)$$

(v) ELIPTICAL POLARIZATION

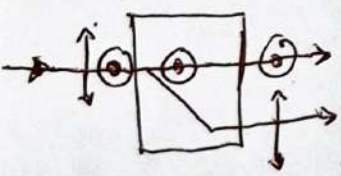
$$\text{or } \boxed{E_x = \frac{E_0}{\sqrt{2}} \cos(\omega t)} \text{ and } \boxed{E_y = \frac{E_0}{\sqrt{2}} \sin(\omega t)}$$

Elliptical polarization happens when for a vector field, $\boxed{\neg(\phi = 0) \wedge \neg(\phi = \frac{\pi}{2})}$ in its a spiral curve

(2) BIREFRÈGENCE

[1] BIREFRINGENT MATERIALS:

1) $k = \frac{\omega}{v} = \frac{n\omega}{c} = \frac{n2\pi}{\lambda}$ hence $\boxed{k = \frac{\omega}{v}}$ and $\boxed{\frac{2\pi n}{\lambda} = k}$ w/ $n = \frac{c}{v}$ & $v = \frac{c}{n}$ or $\frac{1}{v} =$



2) BIFRENGENT MATERIALS = Anisotropic = Refractive Index := polarized

Quantum operations

(4.1) (1.7) (2-3) (1.5)

(1) BASICS

1)

$$|\psi_n\rangle = O_N|\psi_{N-1}\rangle = \hat{O}_N \cdots \hat{O}_2\hat{O}_1|\psi_i\rangle$$

for: $\boxed{|\psi_n\rangle = \prod_{i=1}^{N}(\hat{O}_N)\cdot|\psi_i\rangle}$ w/ $|\psi_n\rangle$ = quantum state, $\hat{O}_n$ = operator.

why $e^{\hat{O}} = \sum_n \frac{1}{n!}\hat{O}^n$ => Taylor series

2)

$$\langle A|F_P(\theta) = |+45\rangle$$
$$\langle A|F_P^{\dagger}(\theta) = \langle +45|$$

for: $\boxed{\langle H|F_P F_P^{\dagger}|H\rangle = \langle +45|+45\rangle = 1.}$ as unitary operators

3.)

$$\hat{U}^{\dagger}\hat{U} = \hat{U}^{\dagger}\hat{U} = \hat{I}$$
$$\hat{U}|\psi_1\rangle = e^{i\phi}|\psi_2\rangle = [\cos(\phi) + i\sin(\phi)]|\psi_2\rangle$$
$$\hat{U}^{\dagger}\langle\psi_1| = e^{-i\phi}\langle\psi_2| = [\cos(\phi) - i\sin(\phi)]\langle\psi_2|$$

- $|\psi\rangle = \psi_1|HV_1\rangle + \psi_2|HV_2\rangle = \sum_j \psi_j|HV_j\rangle$

$\boxed{\psi_j = \langle HV_j|\psi\rangle}$

- $|\psi\rangle = \begin{bmatrix}\langle HV_1|\psi\rangle \\ \langle HV_2|\psi\rangle\end{bmatrix} = \begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix}_{HV}$

for: $\hat{U}|\psi_1\rangle\langle\psi_1|\hat{U}^{\dagger} = \langle\psi_1|\hat{U}\hat{U}^{\dagger}|\psi_1\rangle.$

$$\langle\psi_1|\hat{U}\hat{U}^{\dagger}|\psi_1\rangle = \langle\psi_2|e^{i\phi}e^{-i\phi}|\psi_2\rangle = \langle\psi_2|\psi_2\rangle e^{i\phi - i\phi}$$
$$= 1 \cdot 1 = 1 \checkmark$$

so: $\boxed{\langle\psi_1|\hat{U}\hat{U}^{\dagger}|\psi_1\rangle = \langle\psi_2|e^{i\phi}e^{-i\phi}|\psi_2\rangle = 1.}$ unitary wins

4.) $\hat{P}_H|\psi\rangle = C_H|H\rangle$ -> Probability of Hamiltonian state H.

why $\hat{P}_H$ = PROJECTION TO H

$$\hat{P}_V|\psi\rangle = C_V|V\rangle$$

for: $\boxed{|\psi\rangle = C_H|H\rangle + C_V|V\rangle}$ w/ $\boxed{C_H + C_V = 1}$

state(ψ) = Prob(H) × state(H) + Prob(V) × state(V)

- $\hat{P}_H = \hat{C}_H$

for: $\hat{P}_H|\psi\rangle = \hat{C}_H|\psi\rangle$

" since $C_H|H\rangle = C_H|\psi\rangle$ then $C_H|\psi\rangle = |H\rangle C_H$

for: $\hat{P}_H|\psi\rangle = C_H|\psi\rangle = |H\rangle C_H = |H\rangle\langle H|\psi\rangle$

$$\hat{P}_H = |\psi\rangle\langle\psi|$$

so: $|\psi\rangle = C_H|H\rangle + C_V|V\rangle = |H\rangle C_H + |V\rangle C_V = \langle H|H\rangle|\psi\rangle + \langle V|V\rangle|\psi\rangle$
$= \langle H|\langle H|\psi\rangle + \langle V|\langle V|\psi\rangle = (\hat{P}_H + \hat{P}_V)\psi$

for: $\boxed{|\psi\rangle = \hat{P}_H + \hat{P}_V|\psi\rangle}$

(2) MATRIX REPRESENTATION

1)

THEORY

$\boxed{|\psi\rangle = \hat{O}|\psi\rangle}$ $\boxed{|\psi\rangle \triangleq \begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix}_{HV}}$ $\boxed{\begin{aligned}|H\rangle &= |HV_1\rangle \\ |V\rangle &= |HV_2\rangle\end{aligned}}$

$$\psi_i = \langle HV_i|\psi\rangle = \langle HV_i|\hat{O}|\psi\rangle = \langle HV_i|\hat{O}\cdot\hat{I}|\psi\rangle$$

hol $\bar{J}_H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\bar{J}_V = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $\bar{J}_\theta = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$.

$$\boxed{\bar{J}_{\frac{\lambda}{4}\pm 45^\circ} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \mp i \\ \mp i & 1 \end{bmatrix}}$$

$$\boxed{\bar{J}_{\frac{\lambda}{2}\theta} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}}$$

(1) Calculate $\mathbb{P}(S_y = \frac{\hbar}{2} \mid |x\rangle)$.

Ans:

$$\boxed{\mathbb{P}(S_y = \tfrac{\hbar}{2} \mid |x\rangle) = |\langle +y|+x\rangle|^2}$$

1) $|+y\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$ w/ $\frac{1}{\sqrt{2}}\left[|+z\rangle + i|-z\rangle\right] = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} i = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$

$|+x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$ w/ $\frac{1}{\sqrt{2}}\left[|+z\rangle + |-z\rangle\right] = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$

$\begin{bmatrix}a\\b\end{bmatrix}^\dagger = \begin{bmatrix}a\\-b\end{bmatrix}^T$
$= (a \; -b]$

2) $\langle +y| = |+y\rangle^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\end{bmatrix}$

3.) So: $\langle +y|+x\rangle = \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\end{bmatrix}\right) \times \left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}\right)$

$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\cdot 1 + 1\cdot -i\\ -i\cdot 1 + 1\cdot 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1-i\end{bmatrix}$

4.) $|\langle +y|+x\rangle|^2 = \left(\frac{1}{2}(1-i)\right)^2 = \frac{1}{4}(1^2+i^2) = \frac{1}{4}(1+1) = \frac{1}{4}\cdot 2 = \boxed{\frac{1}{2}}$

So: $\boxed{\mathbb{P}(S_y = \tfrac{\hbar}{2} \mid |x+\rangle) = |\langle +y|+x\rangle|^2 = \frac{1}{2}}$

(2) Calculate $\mathbb{P}(S_x = \frac{\hbar}{2} \mid |-y\rangle)$:

Ans:

$$\boxed{\mathbb{P}(S_x = -\tfrac{\hbar}{2} \mid |-y\rangle) = |\langle -x|-y\rangle|^2}$$

1) $S_x = -\frac{\hbar}{2} \equiv |-x\rangle$

S_x: $|-x\rangle \; S_x = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$ w/: $\langle -x| = |-x\rangle^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}^\dagger$

$= \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\end{bmatrix}$.

2.) $\langle -x|-y\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}} \cdot \begin{bmatrix}1\\-i\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}^T$

$= \frac{1}{2} \cdot \begin{bmatrix}1\\-i\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix} = \frac{1}{2}\left(1\cdot 1 + i\cdot 1\right) = \frac{1}{2}(1+i)$

3.) $|\langle -x|-y\rangle|^2 = \left|\frac{1}{2}(1+i)\right|^2 = \frac{1}{4}(1^2+i^2) = \frac{1}{4}(1^2+1^2) = \frac{1}{4}\cdot 2 = \boxed{\frac{1}{2}}$

(3) write $\hat{S}_z$ in terms of projection operator $|+z\rangle, |-z\rangle$:

Ans:

$|+z\rangle = \frac{\hbar}{2}\begin{bmatrix}1\\0\end{bmatrix}$ $\langle +z| = \frac{\hbar}{2}\begin{bmatrix}1 & 0\end{bmatrix}$

$|-z\rangle = \frac{\hbar}{2}\begin{bmatrix}0\\1\end{bmatrix}$ $\langle -z| = \frac{\hbar}{2}\begin{bmatrix}0 & 1\end{bmatrix}$

So: $\boxed{S_z = \frac{\hbar}{2}\left\{|+z\rangle\langle +z| - |-z\rangle\langle -z|\right\} =}$

QUANTUM MATRICES - HERMITIAN - PAULI

(4.4) (4.5) (4.6)

(1.) MATRIX EXTRACTION

[i] — 1 Continuation — $\boxed{\text{FIND } P_H |+45\rangle}$

EX 4.3

$$\hat{P}_H |+45\rangle = |H\rangle\langle H| \left[\frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)\right] = \frac{1}{\sqrt{2}}|H\rangle$$

$$\text{or: } \hat{P}_H |+45\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{HV} \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{HV} = \begin{bmatrix} 1\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}1\\1\end{bmatrix} \\ 0\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}1\\1\end{bmatrix} \end{bmatrix}_{HV} \frac{1}{\sqrt{2}}$$

$$= \begin{bmatrix} 1+0 \\ 0+0 \end{bmatrix}_{HV} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}|H\rangle$$

$$\text{or: } \boxed{\hat{P}_H |+45\rangle = \frac{1}{\sqrt{2}}|H\rangle}$$

[ii] CORRESP. BETWEEN Q & CLASS CM.

[i] JONES MATRICES

NAME		POLARIZATION	POLARIZATION VECTOR $\vec{\varepsilon}_H$	JONES
• Horizontal	1)	$\vec{\varepsilon}_H$	$\vec{\varepsilon}_V$	$\begin{bmatrix}1\\0\end{bmatrix}$
• Vertical	2)	$\vec{\varepsilon}_V$	$\frac{1}{\sqrt{2}}(\varepsilon_H + \varepsilon_V)$	$\begin{bmatrix}1\\0\end{bmatrix}$
• +45° linear	3.)	$\vec{\varepsilon}_{+45}$	$\frac{1}{\sqrt{2}}(\varepsilon_H + \varepsilon_V)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$
• −45° linear	4.)	ε_{-45}	$\frac{1}{\sqrt{2}}(\varepsilon_H - \varepsilon_V)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$
• Another	5.)	ε_θ	$\cos\theta\,\varepsilon_H + \sin\theta\,\varepsilon_V$	$\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$
• Left circular	6.)	ε_L	$\frac{1}{\sqrt{2}}(\varepsilon_H + i\varepsilon_V)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$
• Right circular	7.)	ε_R	$\frac{1}{\sqrt{2}}(\varepsilon_H + i\varepsilon_V)$	

JONES MATRIX

$$\varepsilon_H = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\Downarrow$$

$$\bar{J}_{\frac{1}{2}\theta} = \begin{bmatrix}\cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}\cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\cos 2\theta \\ \sin 2\theta\end{bmatrix}.$$

[ii] IMPLICATIONS:

1) Certainly: $\hat{P}_H = \begin{bmatrix}\langle H|H\rangle & 0 \\ \langle V|H\rangle & 0\end{bmatrix}$ is physically realizable matrix of HORIZONTAL polarizer.

2.) Each OPTICAL ELEMENTS: Each QUANTUM OPERATOR.

3.) (?) Why isn't the half wave plate physical implementation of the rotation operator? Here we have to get:

$$\boxed{\hat{R}_p(\theta)|\phi\rangle = |\phi + \theta\rangle}$$

(2) CHANGING BASES

[i] THEORY

1) $$\boxed{|\psi\rangle = \psi_H|H\rangle + \psi_V|V\rangle}$$

2.) $$|\psi\rangle = \psi_{+45}|+45\rangle + \psi_{-45}|-45\rangle$$

$$|+45\rangle = \frac{1}{\sqrt{2}}[|H\rangle + |V\rangle]$$

$$|-45\rangle = \frac{1}{\sqrt{2}}[|H\rangle - |V\rangle]$$

3.) $$\psi_{+45} = \langle +45|(\psi_H|H\rangle + \psi_V|V\rangle) = \psi_H\langle +45|H\rangle + \psi_V\langle +45|V\rangle$$

$$\psi_{-45} = \langle -45|(\psi_H|H\rangle + \psi_V|V\rangle) = \psi_H\langle -45|H\rangle + \psi_V\langle -45|V\rangle$$

4.) $$\langle +45|H\rangle = \frac{1}{\sqrt{2}} \qquad \langle +45|V\rangle = \frac{1}{\sqrt{2}}$$

$$\langle -45|H\rangle = \frac{1}{\sqrt{2}} \qquad \langle -45|V\rangle = -\frac{1}{\sqrt{2}}$$

$$\boxed{\begin{aligned}\psi_{+45} &= \frac{1}{\sqrt{2}}(\psi_H + \psi_V) \\ \psi_{-45} &= \frac{1}{\sqrt{2}}(\psi_H - \psi_V)\end{aligned}}$$

PROBLEM 4

(1) Prove that unitary operators do not change the normal.

Ans: 1) Let $|\psi\rangle, |\psi'\rangle$ be vectors s.t:

$$\boxed{\hat{U}|\psi\rangle = |\psi'\rangle}$$

For the proof that: $\|\hat{U}\| = \||\psi\rangle\|$ w/ $\hat{U}^\dagger \hat{U} = \mathbb{I}$ $\mathbb{I}$ = Identity operator

2) So: $\||\psi'\rangle\| = \langle\psi'|\psi'\rangle = \langle U\psi|U\psi\rangle = \langle\psi|UU^\dagger|\psi\rangle$

$= \langle\psi|\mathbb{I}|\psi\rangle = \langle\psi|\psi\rangle$

So: $\boxed{\||\psi'\rangle\| = \langle\psi|\psi\rangle}$ The magnitude of vector $|\psi'\rangle$ is still the same making operator of $|\psi\rangle$ ■

(2) Show that $\hat{P}_\psi^2 = \hat{P}_\psi$

Ans: 1) Recall that: $\hat{P}_\psi = \langle\psi|\psi\rangle$ or $\hat{P}_\psi^2 = |\langle\psi|\psi\rangle|^2 = |1|^2 = 1 = \langle\psi|\psi\rangle = \hat{P}_\psi$ ✓

2) So $\hat{P}_\psi^2 = |\langle\psi|\psi\rangle|^2 = \langle\psi|\psi\rangle\langle\psi|\psi\rangle = \psi\psi^*\cdot\psi\psi^* = \underline{\psi\cdot\psi\cdot\psi^*\psi^*}$

$= \langle\psi|\psi\rangle\langle\psi^*|\psi^*\rangle = \langle\psi^*|\psi^*\rangle = \hat{P}_\psi$ ■ so: $\boxed{\hat{P}_\psi^2 = |\langle\psi|\psi\rangle|^2 = \hat{P}_\psi}$ ■

(3) If $|\psi\rangle = c_H|H\rangle + c_V|V\rangle$ what's $\boxed{\langle\psi|\hat{P}_H?}$

Ans: 1) $|\psi\rangle = c_H|H\rangle + c_V|V\rangle \Rightarrow \langle\psi|\psi\rangle\hat{P}_H = [c_H|H\rangle + c_V|V\rangle]\hat{P}_H$

$\hat{P}_H = c_H\hat{P}_H|H\rangle + c_V\hat{P}_H|V\rangle$

2) Recall that: $\hat{P}_H|\psi\rangle = \langle H|H\rangle|\psi\rangle$ so: $\hat{P}_H = \langle H|H\rangle$.

$\hat{P}_H = c_H\langle H|H\rangle|H\rangle + c_V\langle H|H\rangle|V\rangle$

$= c_H\langle H|H\rangle|H\rangle + c_V\langle H|V\rangle|H\rangle = |H\rangle[c_H\langle H|H\rangle + c_V\langle H|V\rangle]$

$= c_H|H\rangle\cdot 1 + 0 = c_H|H\rangle$

3) So: $\hat{P}_H|\psi\rangle = c_H|\hat{H}\rangle \longrightarrow \langle\psi|\hat{P}_H = \hat{P}_H^\dagger|\psi\rangle^\dagger = \hat{P}_H^\dagger\langle\psi| = c_H^\dagger\langle H|$

and: $c_H^\dagger\langle H| = \langle H|c_H^\dagger = \langle H|\langle\psi|H\rangle = \langle\psi|\langle H|H\rangle$.

4.) to prove this: $\langle\psi|H\rangle\langle H|\psi\rangle = \langle\psi|c_H^\dagger c_H|\psi\rangle = 1$ ■ ✓

(4) Verify the matrix representation of $\hat{R}_\theta(\theta)$ is unitary.

Ans: Verify that $\hat{U}\hat{U}^\dagger = 1$. and the see that:

$$\hat{R}_\theta(H) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The verify that: $R_\theta^\dagger(\theta)\,\hat{R}_\theta(\theta) = \hat{1}$

So: $\cancel{R_\theta^\dagger R_\theta = \begin{bmatrix} \cos\theta^\dagger & -\sin\theta^\dagger \\ \sin\theta^\dagger & \cos\theta^\dagger \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$

$= \cos^2 + \sin^2$

$= \boxed{1}$ ✓

w/: $\hat{R}_\theta(\theta)|H\rangle = \cos\theta|H\rangle + \sin\theta|V\rangle$

$\hat{R}_\theta(\theta)^\dagger\langle H| = \sin\theta\langle H| + \cos\theta\langle V|$

So: $\langle H|\hat{R}_\theta^\dagger R_\theta|H\rangle = [\cos|H\rangle + \sin\theta|V\rangle][\cos\langle H| + \sin\theta\langle V|]$

$= \cos^2\langle H|H\rangle + 2\sin\theta\cos\theta\langle H|V\rangle + \sin^2\langle V|V\rangle$

Problems 6

(4) Show that:

$$S_y|+z\rangle = i\frac{\hbar}{2}|-z\rangle \text{ and } \hat{S}_y|z\rangle = -i\frac{\hbar}{2}|+z\rangle$$

Ans: 1.) $\hat{S}_y = i\frac{\hbar}{2}\sigma_y = i\frac{\hbar}{2}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$

$|+z\rangle = \begin{bmatrix}1\\0\end{bmatrix}_z$

So: $\hat{S}_y|+z\rangle = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}|z\rangle = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$

$= \frac{\hbar}{2}\begin{bmatrix}(0)(1)+(-i)(0)\\ (i)(1)+(0)(0)\end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix}0\\ i\end{bmatrix} = i\frac{\hbar}{2}\begin{bmatrix}0\\1\end{bmatrix}$

$= i\frac{\hbar}{2}|-z\rangle$ ✓

2) Similarly:

$S_y|+z\rangle = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ +i & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix}-i\\0\end{bmatrix} = -i\frac{\hbar}{2}\begin{bmatrix}1\\0\end{bmatrix} = -i\frac{\hbar}{2}|z+\rangle$

So: $\boxed{\begin{aligned} S_y|+z\rangle &= i\frac{\hbar}{2}|-z\rangle \\ S_y|-z\rangle &= -i\frac{\hbar}{2}|+z\rangle \end{aligned}}$

(9). The Relativistic gyro eqs. is $\gamma_{rel} = g\frac{q}{2m}$ while $\gamma_{class} = \frac{q}{2m}$

So: $\boxed{\gamma_{rel} = g\,\gamma_{class}}$ Weeny its off by g. What is g?

Ans: g is the Anomalous magnetic moment of $\gamma_{classical}$ via the Dirac eqs

1.) Recall: $\boxed{(i\gamma^\mu\partial_\mu + m)\psi = 0}$ → Dirac eqs for Field Theory

Comply: $\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$ and $\partial_\mu = D_\mu - iqA_\mu$

So: $(i\gamma^\mu\partial_\mu + m)\psi = (i\gamma^\mu(D_\mu - iqA_\mu) + m)\psi = 0$

Suppose $A^\mu = 0$ (no field interaction)

$(i\gamma^\mu(D_\mu - iqA_\mu) + m)\psi = (i\gamma^\mu D_\mu + qA^\mu A_\mu + m)\psi = 0$

$\boxed{(i\gamma^\mu D_\mu + m)\psi = 0}$

So: $\hbar(\sigma\cdot\partial) = g$

FOLDY WOUTHUSEN

$\sim m + \sum_n \frac{x^n}{2m}$

$\sim m + \frac{(p^2 - qA)^2}{2m} - \frac{q\hbar}{m}\sigma\cdot B$

0th term, 1st term, 2nd term

2.) Recall: $V = \mu\cdot\vec{B}$ $\quad F = \left|\frac{\partial V}{\partial r}\right| = \mu\frac{\partial \vec{B}}{\partial r}$.

also $\boxed{H = \frac{1}{2m}(p^2 - qA)^2 - \frac{q\hbar}{2m}\vec{\sigma}\cdot\vec{B}}$

So: $-\frac{q\hbar}{2m}\vec{\sigma}\cdot\vec{B} \Rightarrow \mu_s = \frac{q\hbar}{2m}\vec{\sigma}$

3.) $S_z = \frac{\hbar}{2}\vec{\sigma}$

$\mu_s = \frac{q}{m}\vec{S}$

So: $\mu_s = \mu_s$

$\frac{\hbar\vec{\sigma}}{2m} = \frac{q}{m}\vec{S}$

$\frac{\hbar}{\vec{\sigma}\cdot\vec{S}} = \frac{2m}{m}$

$\frac{\hbar}{\hbar/2} = 2$.

$\boxed{g = 2}$

PARTICLE INTERFERENCE

(1) APPLICATIONS OF 4th/5th ELEMENTS: (6.4) (6.5)

[i] EXPECTATION OF S_y:

$$\langle S_y \rangle = \langle +x | \hat{S}_y | +x \rangle$$

$$= \frac{1}{\sqrt{2}}(1\ 1)_z \left[\frac{\hbar^2}{2} \bar{\sigma}_y \right] \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_z$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix}_z \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}_z$$

$$= \frac{\hbar}{4} \begin{pmatrix} 1 & 1 \end{pmatrix}_z \begin{pmatrix} -i \\ i \end{pmatrix}_z = \frac{\hbar}{4}(-i+i) = 0$$

So: $$\boxed{\langle S_y \rangle = \langle +x | \hat{S}_y | +x \rangle = \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} (1\ 1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = 0}$$

[ii] COMMUTATOR OF S_x, S_y

1) $$[\hat{S}_x \hat{S}_y] = \hat{S}_x \hat{S}_y - \hat{S}_y \hat{S}_x$$

$$= \left(\frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_z \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \right) - \left(\frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}_z \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_z \right)$$

$$= \frac{\hbar^2}{4} \left\{ \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} - \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \right\}$$

$$= \frac{\hbar^2}{4} \left\{ \begin{matrix} i & 0 \\ i & -i \end{matrix} \right\}_z = \frac{\hbar^2}{4} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = i\hbar \frac{\hbar}{2} \sigma_z.$$

$$= \frac{i\hbar^2}{2} \bar{\sigma}_z = i\hbar \hat{S}_z$$

So: $$\boxed{[\hat{S}_x, \hat{S}_y] = i\hbar \hat{S}_z}$$

2) Force cyclic permutations:

$$\boxed{\begin{aligned} [\hat{S}_y \hat{S}_z] &= i\hbar \hat{S}_x \\ [\hat{S}_z, \hat{S}_x] &= i\hbar \hat{S}_y \\ [\hat{S}_y, \hat{S}_z] &= -i\hbar \hat{S}_z \end{aligned}}$$

Angular momentum operator

[iii] The uncertainty principle:

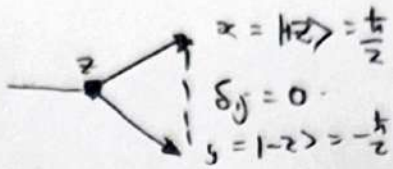
$$\boxed{\Delta S_x \Delta S_y \geq \frac{\hbar}{2} \left| \langle \hat{S}_z \rangle \right|}$$

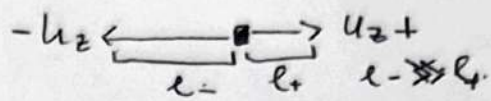
(2) PARTICLE INTERFERENCE

2) PROBLEM

$\langle +z | -z \rangle = 0$ s.t

$\langle +u_z \rangle \langle -u_z \rangle = -1$ s.t





So: $\boxed{\begin{array}{c} \langle +z | -z \rangle \neq (+u_z)(-u_z) \\ 1 \neq 0 \end{array}}$

- Classical Intuition Fails!

III The $\boxed{\hat{S}_z}$ operator

1.) $\hat{S}_z |+z\rangle = \frac{\hbar}{2} |+z\rangle$

$S_z |-z\rangle = -\frac{\hbar}{2} |-z\rangle$

2.) $|+z\rangle = \begin{pmatrix} R^+ \\ R^- \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

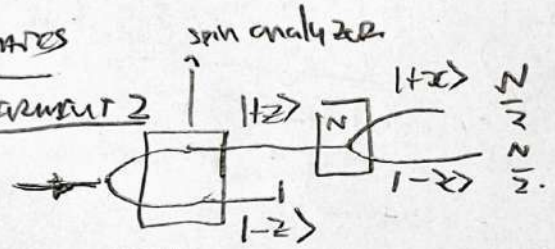
$|-z\rangle = \begin{pmatrix} R^+ \\ R^- \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

3.) $S_z = \frac{\hbar}{2} |+z\rangle\langle -z| = \frac{\hbar}{2} \bar{\sigma}_z$

$\boxed{S_z = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{\hbar}{2} \bar{\sigma}_z}$ Pauli spin matrix.

(3) MORE SPIN STATES

I EXPERIMENT 2

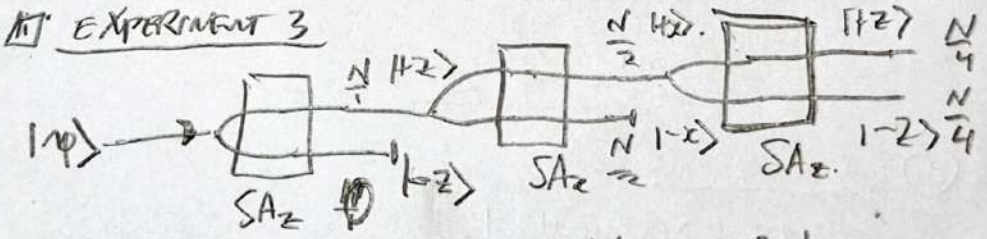


So: $\boxed{\begin{array}{l} |x\rangle = S_x \frac{\hbar}{2} \\ |-x\rangle = -S_x \frac{\hbar}{2} \end{array}}$

Newtonian = motion (orthogonals) are independent.

= SA_x splits in $|x\rangle, |-x\rangle$ since $\exists |\psi\rangle$.

II EXPERIMENT 3



1.) So: $\mathbb{P}(S_z = \frac{\hbar}{2} \mid |+x\rangle) = |\langle +z | +x \rangle|^2 = \frac{1}{2}$

$\mathbb{P}(S_z = -\frac{\hbar}{2} \mid |+x\rangle) = |\langle -z | +x \rangle|^2 = \frac{1}{2}$.

$A \sqrt{\sum x_i^2} = 1$

$A \sqrt{|1|^2} = 1$

$A\sqrt{2} = 1$.

$\boxed{A = \frac{1}{\sqrt{2}}}$

2.) $\boxed{\begin{array}{l} |x\rangle = \frac{1}{\sqrt{2}} (|+z\rangle + |-z\rangle) \\ |-x\rangle = \frac{1}{\sqrt{2}} (|+z\rangle - |-z\rangle) \end{array}}$

STERN GERLACH EXPERIMENT (6.1) (6.2) (6.3)

(1) FORCE ON A MAGNETIC DIPOLE

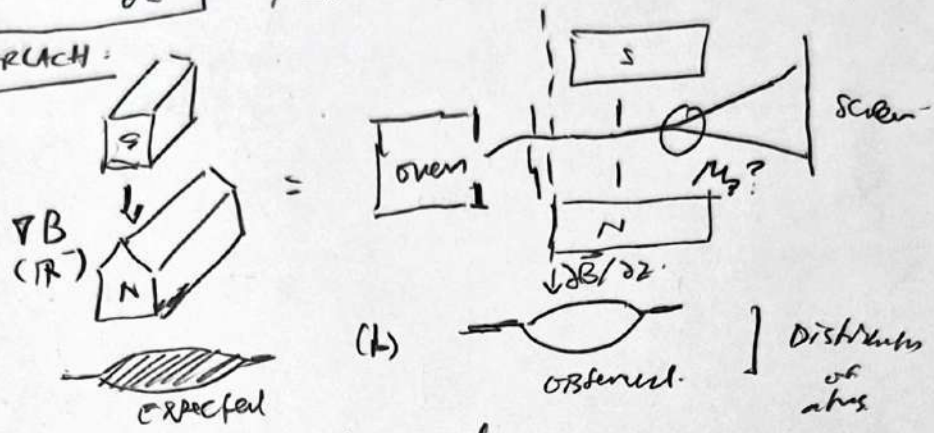
$$\boxed{V = \vec{\mu} \cdot \vec{B}}$$

1) $\vec{F} = -\nabla V = \nabla(\mu \cdot \vec{B})$

2) $\boxed{\vec{F} = \mu_z \frac{\partial B_z}{\partial z}}$ IF, the Dipole moment (μ) is so small, $F = 0$ basically

- OTTO STERN
- WALTER GERLACH

[i] STERN GERLACH:



Gradient ($\vec{\nabla B}$) = negative Gradient

Mass atom × Geometry = Deflection via $/\hbar z$.

- George Uhlenbeck
- Samuel Goudsmit

∴ Particle was actually spinning

[ii] SPIN:

1) $\boxed{\vec{\mu} = \gamma \vec{S}}$ Gyromagnetic Ratio

- Uhlenbeck - Goudsmit = Spinning
- Spinning = Magnetic moments × Charged particle
- Uhlenbeck (problem) = Spinning >>> Spinning (normally)
- Spinning >>> Spinning (normally) = Dipole moment

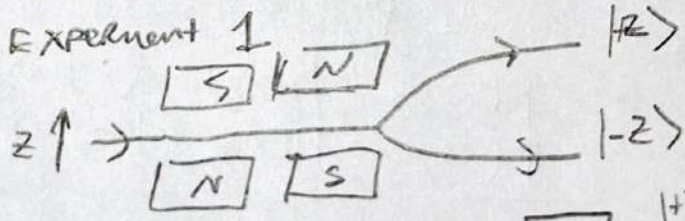
2) $\boxed{\gamma_e = -1.76 \times 10^{11}\, s^{-1} T^{-1}}$

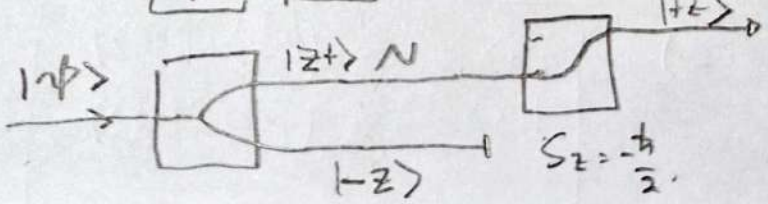
Recall: s^{-1} = Seconds T = Tesla = $\frac{1N}{1A \cdot 1m}$.

3) $\boxed{S_z = \pm \frac{1}{2}\hbar}$

(2) SPIN STATES

[i] EXPERIMENT 1





→ $\boxed{\langle +z | -z \rangle = 0}$

∴

(b) Use the matrix Representation of Rotation operator $\hat{R}_p(\theta)$ to write that $\boxed{\hat{R}_p(\theta)|\phi\rangle = |\phi+\theta\rangle}$ ϕ = state of photon?

Ans: so: $\hat{R}_p(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and let φ be a cons. number.

and $\hat{R}_p(\theta)|\phi\rangle = \cos\theta|\phi\rangle + \sin\theta|\varphi\rangle$

$\hat{R}_p(\theta)|\varphi\rangle = -\sin\theta|\phi\rangle + \cos\theta|\varphi\rangle$

and $\hat{R}_p(\theta)|\phi\rangle\langle H|H\rangle = \cos\theta\langle H|H\rangle|\phi\rangle + \sin\theta\langle H|V\rangle|\phi\rangle$

$\hat{R}_p(\theta)|\phi\rangle = \cos\theta|\phi\rangle$

$\hat{R}_p(\theta)|\phi\rangle\langle V|V\rangle = -\sin\theta\langle V|V\rangle|\phi\rangle + \cos\theta\langle V|H\rangle|\phi\rangle$

$= -\sin\theta|\phi\rangle$

so: $\cancel{R_p(\theta)^{-1}R_p(\theta)|\phi\rangle|\phi\rangle = (\cos^2\theta + \sin^2\theta)|\phi\rangle = \cos^2\theta|\phi\rangle + \sin^2\theta|\phi\rangle}$

and $|\phi\rangle = \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}$, $|\theta\rangle = \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}$

now: $\cos\theta|\phi\rangle = x(\theta)|\phi\rangle = \langle x(\theta)|x(\phi)\rangle$

$|\phi\rangle\hat{R}_p(\theta) = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \cos\theta\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} - \sin\theta\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}$

$\sin\theta\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} + \cos\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}$

$= \begin{bmatrix}\cos(\theta+\phi_1) + \cos(\theta+\phi_2) & -\sin(\theta+\phi_1) - \sin(\theta+\phi_1) \\ \sin(\theta+\phi_1) + \sin(\theta+\phi_2) & \cos(\theta+\phi_1) - \cos(\theta+\phi_1)\end{bmatrix}$

$= \begin{bmatrix}\cos(\cdot) & -\sin(\cdot) \\ \sin(\cdot) & \cos(\cdot)\end{bmatrix}\begin{bmatrix}\phi_1+\theta_1 \\ \phi_2+\theta_2\end{bmatrix} = \begin{bmatrix}\phi_1+\theta_1 \\ \phi_2+\theta_2\end{bmatrix} = |\theta+\phi\rangle$

Ans ★ $|\phi\rangle = \cos\phi|H\rangle + \sin\phi|V\rangle = \begin{bmatrix}\cos\phi \\ \sin\phi\end{bmatrix}$.

so: $R_p(\theta)|\phi\rangle = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\phi \\ \sin\phi\end{bmatrix}$

$= \begin{bmatrix}\cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi\end{bmatrix} = \begin{bmatrix}\cos\phi+\theta \\ \sin\phi+\theta\end{bmatrix} = |\phi+\theta\rangle$

so: $\boxed{R_p(\theta)|\phi\rangle = |\phi+\theta\rangle}$

2) $\varphi_i = \langle HV_i | \hat{O} \left[\sum_j |HV_j\rangle\langle HV_j| \right] | \psi \rangle$

$= \sum_j \langle HV_i | \hat{O} | HV_j \rangle \langle HV_j | \psi \rangle$

$= \sum_j \langle HV_i | \hat{O} | HV_j \rangle \psi_j$

$$\boxed{\varphi_i = \sum_j \langle HV_i | \hat{O} | HV_j \rangle \psi_j}$$

2) $O_{ij} = \langle HV_i | \hat{O} | HV_j \rangle$

O_{ij}: $\boxed{\varphi_i = \sum_j O_{ij} \psi_j}$ $\quad O_{ij}\psi_j = \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix} \cdot \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}_{HV}$

$$\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}_{HV} = \begin{bmatrix} O_{11} & O_{12} \\ O_{12} & O_{22} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}_{HV}$$

[iv] EXAMPLES:

[1] 1) $\hat{P}_H \triangleq \begin{bmatrix} \langle HV_1 | \hat{P}_H | HV_1 \rangle & \langle HV_1 | \hat{P}_H | HV_2 \rangle \\ \langle HV_2 | \hat{P}_H | HV_1 \rangle & \langle HV_2 | \hat{P}_H | HV_2 \rangle \end{bmatrix}_{HV}$

$= \begin{bmatrix} \langle H | \hat{P}_H | H \rangle & \langle H | \hat{P}_H | V \rangle \\ \langle V | \hat{P}_H | H \rangle & \langle V | \hat{P}_H | V \rangle \end{bmatrix}$

$\text{or. } \hat{P}_H = \begin{bmatrix} \langle H | \hat{P}_H | H \rangle & \langle H | \hat{P}_H | V \rangle \\ \langle V | \hat{P}_H | H \rangle & \langle V | \hat{P}_H | V \rangle \end{bmatrix}$

2) $\langle i | j \rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ $\quad$ $\text{no. } \delta_{HV} = \begin{cases} \pm 1 & H = H;\ V = V \\ 0 & H \neq V,\ V \neq H. \end{cases}$

$\text{hence: } \hat{P}_H = \begin{bmatrix} \hat{P}_H & 0 \\ 0 & \hat{P}_H \end{bmatrix}$ $\quad$ SIT det. ✗

But Recall That: $\hat{P}_H |H\rangle = |H\rangle$ and $\hat{P}_H |V\rangle = 0$.

$\text{so: } \hat{P}_H = \begin{bmatrix} \langle H | H \rangle & 0 \\ \langle V | H \rangle & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

So: $\boxed{\hat{P}_H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{HV}}$ $\rightarrow$

[2] POLARIZATION ROTATION OPERATOR..

1) $\hat{R}_p(\theta) |HV_1\rangle = \hat{R}_p(\theta) |H\rangle = \cos\theta |H\rangle + \sin\theta |V\rangle$

$\hat{R}_p(\theta) |HV_2\rangle = \hat{R}_p(\theta) |V\rangle = -\sin\theta |H\rangle + \cos\theta |V\rangle$.

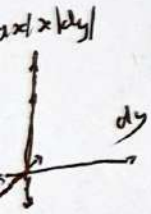
- $\langle H | H \rangle \cos\theta = \cos\theta$
- $\langle H | V \rangle \sin\theta = 0$
- $\langle V | H \rangle \cos\theta = 0$
- $\langle V | V \rangle \sin\theta = \sin\theta$

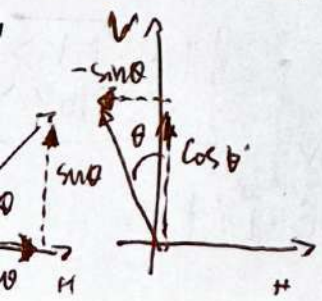
etc ....

2) $\hat{R}_p(\theta) = \begin{bmatrix} \langle HV_1 | R_p | HV_1 \rangle & \langle HV_1 | \hat{R}_p(\theta) | HV_2 \rangle \\ \langle HV_2 | R_p | HV_1 \rangle & \langle HV_2 | \hat{R}_p(\theta) | HV_2 \rangle \end{bmatrix}_{HV}$

$= \begin{bmatrix} \langle H | [\cos\theta |H\rangle + \sin\theta |V\rangle] & \langle H | [-\sin\theta |H\rangle + \cos\theta |V\rangle] \\ \langle V | [\cos\theta |H\rangle + \sin\theta |V\rangle] & \langle V | [-\sin\theta |H\rangle + \cos\theta |V\rangle] \end{bmatrix}$

$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_{HV}$ $\quad$ $\text{so: } \boxed{\hat{R}_p(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}$





PROBLEM 6

(13.) Find the state $|-n\rangle$ which corresp. to $|+n\rangle$

Ans: If: $\boxed{|+n\rangle = \cos\left(\frac{\theta}{2}\right)|+z\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|-z\rangle}$

Then an analogy like $ds = dx + dy :=$ $dx = \sin\frac{\theta}{2} + \cos\left(\frac{\theta}{2}\right)$, $dy = -\cos\frac{\theta}{2} + \sin\left(\frac{\theta}{2}\right)$

It should be:

$$\boxed{|-n\rangle = -\sin\left(\frac{\theta}{2}\right)|+z\rangle + e^{i\phi}\cos\left(\frac{\theta}{2}\right)|-z\rangle}$$

(14.) Proof that $|-n\rangle$ is orthogonal to $|+n\rangle$

Ans: 1) $\langle -n| = |-n\rangle^\dagger = -\sin\left(\frac{\theta}{2}\right)|+z\rangle^\dagger + (e^{i\phi})^\dagger\cos\left(\frac{\theta}{2}\right)|-z\rangle^\dagger$

$= -\sin\left(\frac{\theta}{2}\right)\langle +z| + e^{-i\phi}\cos\left(\frac{\theta}{2}\right)\langle -z|$.

So: 2) $\langle -n|+n\rangle = \left[-\sin\left(\frac{\theta}{2}\right)\langle +z| + e^{-i\phi}\cos\left(\frac{\theta}{2}\right)\langle -z|\right]\left[\cos\left(\frac{\theta}{2}\right)|+z\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|-z\rangle\right]$

$= -\sin\left(\frac{\theta}{2}\right)\underbrace{\langle +z|+z\rangle}_{1}\cos\left(\frac{\theta}{2}\right) + \underbrace{e^{-i\phi}e^{i\phi}}_{e^{-i\phi+i\phi}=e^0=1}\cos\left(\frac{\theta}{2}\right)\underbrace{\langle -z|-z\rangle}_{1}\sin\left(\frac{\theta}{2}\right)$

$= -\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) = \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\cos$

$= \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \boxed{0}$ ✓ QED

3)

Problems 8

(1) Calculate the Expected value of $\mathcal{P}^s_{HV}$, $\mathcal{P}^i_{HV}$, and $\mathcal{P}^{si}_{HV}$ for photons prepared in state $|V,L\rangle$

Ans. $|V,L\rangle = \frac{1}{\sqrt{2}}\left[|V\rangle_s \otimes |H\rangle_i + i|V\rangle_s \otimes |V\rangle_i\right]$

$\langle V,L| = \frac{1}{\sqrt{2}}\left[\langle V|_s \otimes \langle H|_i - i\langle V|_s \otimes \langle V|_i\right]$

so: $\langle V,L|V,L\rangle = \left(\frac{1}{\sqrt{2}}\right)^2\left[\langle V|V\rangle\langle H|H\rangle + i\langle V|V\rangle\langle H|V\rangle - i\langle V|V\rangle\langle V|H\rangle + (-i)(i)\langle V|V\rangle\langle V|V\rangle\right]$

$= \frac{1}{2}\left[1+0-0+1\right] = \frac{1}{2}2 = \boxed{1}$ ✓

and so: $\langle \mathcal{P}^s_{HV}\rangle = \langle V,L|\mathcal{P}^s_{HV}|V,L\rangle = \langle V,L|V,L\rangle = 1$

Similarly $\langle \mathcal{P}^i_{HV}\rangle = \langle V,L|\mathcal{P}^i_{HV}|V,L\rangle = \langle V,L|V,L\rangle = 1$

$\langle \mathcal{P}^{si}_{HV}\rangle = \langle \mathcal{P}^s_{HV}\rangle\langle \mathcal{P}^i_{HV}\rangle = 1\times 1 = 1$

so: $\boxed{\langle \mathcal{P}^s\rangle = \langle \mathcal{P}^i\rangle = \langle \mathcal{P}^{si}_{HV}\rangle = 1}$ → The operator is ~~[illegible]~~ since the:

- projects only relevant subspace
- Projected component was entirely on the basis you're projecting on.
- If the state is orthogonal then it's 0! $\langle V|L\rangle = 0$.!

(2) Calculate the Expectation values of $\mathcal{P}^s_{HV}$, $\mathcal{P}^i_{HV}$, $\mathcal{P}^{si}_{HV}$ for photons prepared in state of

$$\boxed{|\psi\rangle = \left(\frac{1}{3}\right)^{\frac{1}{2}}|H,+45\rangle + \left(\frac{2}{3}\right)^{\frac{1}{2}}|H,-45\rangle}$$

Ans. $\langle \mathcal{P}^s_{HV}\rangle = \langle\psi|\mathcal{P}^s_{HV}|\psi\rangle = \mathcal{P}^s_{HV}\langle\psi|\psi\rangle$? $\langle\psi|$?

Since $|\psi\rangle = \left(\frac{1}{3}\right)^{\frac{1}{2}}|H,+45\rangle + \left(\frac{2}{3}\right)^{\frac{1}{2}}|H,-45\rangle = c_1|H,+45\rangle + c_2|H,-45\rangle$

$= c_1|H\rangle\otimes|+45\rangle + c_2|H\rangle\otimes|-45\rangle$

now $\langle\psi| = |\psi\rangle^\dagger = \left(\frac{1}{3}\right)^{\frac{1}{2}}|H,+45\rangle^* - \left(\frac{2}{3}\right)^{\frac{1}{2}}|H,-45\rangle$

$= \left(\frac{1}{3}\right)^{\frac{1}{2}}\langle H,+45| - \left(\frac{2}{3}\right)^{\frac{1}{2}}\langle H,+45| = c_1\langle H,+45| - c_2\langle H,-45|$

$= c_1\langle H|\otimes\langle +45| - c_2\langle H|\otimes\langle -45|$

now $\langle\psi|\psi\rangle = \left[c_1|H,+45\rangle + c_2|H,-45\rangle\right] =$

BASICS of TIME OPERATORS

9.1 | 9.2 | 9.3

① TIME EVOLUTION OPERATOR

- **BASICS:**
  - The schrodinger eqn points us the way:

$$|\psi\rangle \longrightarrow |\psi(t)\rangle$$

  - If some operator changes one state from initial time $|\psi(t_0)\rangle$ then:

$$\boxed{|\psi(t)\rangle = \hat{U}(t,t_0)\,|\psi(t_0)\rangle}$$

  - and if $t_0 = 0$

$$|\psi(t)\rangle\Big|_{t_0=0} = \hat{U}(t,t_0)\Big|_{t_0=0} |\psi(t_0=0)\rangle = \hat{U}(t,0)\,|\psi(0)\rangle \equiv \hat{U}(t)\,|\psi(0)\rangle$$

Since: $\boxed{|\psi(t)\rangle = \hat{U}(t)\cdot|\psi(0)\rangle}$

  - Recall that:

$$\frac{\partial}{\partial t}|\psi(t)\rangle \equiv \frac{\partial\psi(t)}{\partial t} \rightarrow i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \mathcal{H}|\psi(t)\rangle \text{ and } \mathcal{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x})$$

  in that: $\langle x|\psi(t)\rangle = \psi(x,t)$ and $\mathcal{H}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec{x})\psi$

  - Then:

$$\langle x|\psi(t)\rangle = \hat{U}\langle x|\psi(0)\rangle \rightarrow \psi(x,t) = \hat{U}\psi(x,0) \equiv \hat{U}(t)\psi(x)$$

so: $\boxed{\psi(x,t) = \hat{U}(t)\,\hat{\psi}(x)}$ $\Rightarrow$ The Returned for T.I.S.E

  - Follows w/-out saying:

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|U(t)^\dagger U(t)|\psi(0)\rangle = \langle\psi(0)|\psi(0)\rangle\, U(t)^\dagger U(t) = 1\cdot 1 = \boxed{1}$$

- **THE PHILOSOPHY / RELEVANCE**
  - If we let a system evolve for infinitesimal time dt and letting $\hat{U}(t)$ to be the "TIME OPERATOR", then:

$$\boxed{\hat{U}(dt) = \hat{1} - i\hat{G}_t\,dt}$$

w/ $\hat{G}_t$ = Generator for Infinitesimal Time Evolution

  - This is also unitary!

$$\hat{U}^\dagger(dt)\,\hat{U}(dt) = (\hat{1} + i\hat{G}_t^\dagger dt)(\hat{1} - i\hat{G}_t\,dt) \approx \hat{1} - i(\hat{G}_t + i\hat{G}_t^\dagger)\,dt = \hat{1} - i(\hat{G}_t^\dagger - \hat{G}_t)\,dt$$

$$\triangleq 1 + (0)dt = \boxed{1}$$ so that $\hat{G}_t^\dagger = \hat{G}_t$

② THE SCHRODINGER EQS

[i] THE HAMILTONIAN:

- **PLANK'S ENERGY:**
  - $E = hf = \hbar\omega$ said $\rightarrow$ $E = \frac{1}{\omega} = \frac{\hbar}{h}$ and $\frac{1}{\omega} \propto \frac{1}{\sqrt{\mu_0\varepsilon_0}}$

in $\boxed{E = \hbar\omega}$ w/ $\omega = \sqrt{\frac{k}{m}}$

- **TDSE**
  - we can write:

$$\hat{U}(dt) = 1 - \frac{i}{\hbar}\hat{H}\,dt \rightarrow \boxed{\frac{\partial}{\partial t}\hat{U} = -\frac{i}{\hbar}\hat{H}\hat{U}(t)}$$

w/ $\hat{U}(dt) - 1 = -\frac{i}{\hbar}\hat{H}$ and $\hat{U}(dt) - U(t) = -\frac{i}{\hbar}\hat{U}(t)$ and $\hat{U}(dt) - U(t) \equiv \hat{U}(\Delta t) - U(t) \equiv \frac{dU}{dt}$ ✓

  - Continuing:

$$|\psi_0\rangle\left(\frac{\partial\hat{U}}{\partial t} = -\frac{i}{\hbar}\hat{H}\hat{U}(t)\right) \rightarrow -\frac{i}{\hbar}H\hat{U}(t)|\psi_0\rangle \rightarrow \frac{\partial}{\partial t}|\psi(t)\rangle = -\frac{i}{\hbar}H|\psi(t)\rangle$$

so: $\boxed{\frac{\partial}{\partial t}|\psi(t)\rangle = -\frac{i}{\hbar}|\psi(t)\rangle}$

[ii] SOLUTIONS TO THE SCHRÖDINGER EQS

$$\boxed{\hat{U}(t) = e^{-iHt/\hbar}}$$

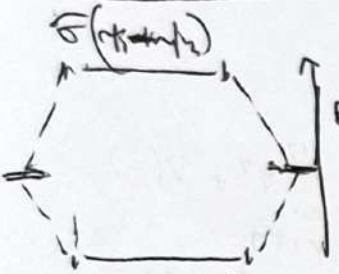
note/ $\frac{\partial \hat{U}}{\partial t} = -\frac{iH}{\hbar} e^{-iHt} = -\frac{iH}{\hbar}\hat{U}(t) = -\frac{i}{\hbar}\hat{H}\hat{U}(t)$ & so $\boxed{\frac{\partial \hat{U}}{\partial t} = -\frac{i}{\hbar}\hat{H}\hat{U}}$ ✓

- **ENERGIES & DISCRETE ENERGIES**
  - Since $\frac{\partial \hat{U}}{\partial t} \equiv \frac{\partial}{\partial t}|\psi(t)\rangle$. This means that:

$$\boxed{|\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|\psi(0)\rangle} \quad (9.13)$$

$|E_n\rangle = E_n$.

CHEMICAL BONDING



$P_1(\psi_1) = C_1 = \sqrt{\frac{1}{3}}$
$P_2(\psi_2) = C_2 = \sqrt{\frac{2}{3}}$

- The Hamiltonian is an Energy operator so: $\boxed{\hat{H}|E_n\rangle = E_n|E_n\rangle}$

$\hat{H}|E_n\rangle = E_n|E_n\rangle$ (9.14) so:

$\langle x|E_n\rangle = \sum_n C_n\psi_n = C_1\psi_1 + C_2\psi_2 = \frac{1}{2}\psi_1 + (1-\varepsilon)\frac{1}{2}\psi_2 = \langle E_n\rangle$, $n=2$

- If the initial Energy eigenstate $|\psi(0)\rangle = |E_n\rangle$ then

$$|\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|E_n\rangle = \exp\left(-\frac{iE_nt}{\hbar}\right)|E_n\rangle \equiv \exp\left(-\frac{i\omega_nt}{\hbar}\right)|E_n\rangle$$

so: $$\boxed{|\psi(t)\rangle = \exp\left(-\frac{i\omega_nt}{\hbar}\right)|E_n\rangle} \quad (9.15)$$

- Consequently if we take $\langle x|$ to EQS (9.15). then

$\langle x|\psi(t)\rangle = \exp\left(-\frac{i\omega_nt}{\hbar}\right)\langle x|E_n\rangle$ let $a_n = \frac{i\omega_n}{\hbar}$.

$\Rightarrow \psi(x,t) = \sum_n \exp(-a_nt)\psi_n = e^{-a_nt}[C_1\psi_1 + C_2\psi_2]$.

$\Rightarrow$: $\boxed{\psi(x,t) \equiv e^{-a_nt}[C_1\psi_1 + C_2\psi_2]}$ $\rightarrow$ $|\psi(t)\rangle = e^{i\omega_nt}|E_n\rangle$ doesn't change in time! The overall phase changes but the state is $|E_n\rangle$

- **EIGENSTATE EXAMPLE**
  - we let: $|\psi(0)\rangle = C_1|E_1\rangle + C_2|E_2\rangle$. Then in Later times:

$$|\psi(t)\rangle = \exp\left(-\frac{i\hat{H}t}{\hbar}\right)|\psi(0)\rangle = \exp(-a_nt)\left[C_1|E_1\rangle + C_2|E_2\rangle\right]$$

$$\rightarrow C_1|E_1\rangle e^{-a_1t} + C_2|E_2\rangle e^{-a_1t} \equiv e^{-i\omega_2t}\left[C_1|E_1\rangle + C_2e^{-i(\omega_2-\omega_1)t}|E_2\rangle\right]$$

so: $$\boxed{|\psi(t)\rangle = e^{-i\omega_2t}\left[C_1|E_1\rangle + C_2e^{-i(\omega_2-\omega_1)t}|E_n\rangle\right]}$$

(3) EXPECTATIONAL VALUES

1.) $\langle H\rangle(t) = \langle\psi(t)|\hat{H}|\psi(t)\rangle = \langle\psi(0)|e^{-\frac{i\hat{H}t}{\hbar}}\cdot\hat{H}\cdot e^{+\frac{i\hat{H}t}{\hbar}}|\psi(0)\rangle$

$= \langle\psi(0)|\hat{H}|\psi(0)\rangle = \langle H\rangle(t=0)$ so: $\boxed{\langle H\rangle = (H) = \langle H\rangle(0)}$

2) How about other observables? well:

$$\frac{d}{dt}\langle A\rangle = \frac{d}{dt}\langle\psi(t)|\hat{A}|\psi(t)\rangle = \left[\frac{d}{dt}\langle\psi(t)|\right]\hat{A}|\psi(t)\rangle + \langle\psi(t)|\hat{A}\left[\frac{d}{dt}|\psi(t)\rangle\right]$$

Chain rule!

$$= \frac{i}{\hbar}\langle\psi(t)|\hat{H}\hat{A}|\psi(t)\rangle - \frac{i}{\hbar}\langle\psi(t)|\hat{A}\hat{H}|\psi(t)\rangle$$

$$= \frac{i}{\hbar}\langle\psi(t)|[\hat{H},\hat{A}]|\psi(t)\rangle$$

so: $$\boxed{\frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}\langle\psi(t)|[\hat{H},\hat{A}]|\psi(t)\rangle}$$

MAGNETIC FIELD SPIN

4.4

(4) SPIN $\frac{1}{2}$ IN A MAGNETIC FIELD

BASICS

- Energy of a Magnetic dipole of $\vec{\mu}$ moment in $\vec{B}$ magnetic field is:

$$\boxed{H = -\vec{\mu}\cdot\vec{B}}$$

- Assume it points at z-direction of $(\vec{B} = B\vec{k}_z)$ and $(\vec{\mu} = \gamma\vec{S})$. Then

$$\boxed{H = -\gamma S_z \vec{B}}$$

so $\boxed{\hat{H} = -S_z\Omega}$ w/ s^{-1} units

The LARMOR FREQUENCY

- So the Hamiltonian is:

$$\hat{H} = -\gamma\hat{S}_z\vec{B} = -\gamma\vec{B}\hat{S}_z = -\hat{S}_z\Omega$$

INTERMEDIATE TIME OPERATOR

$\left\{S_z = \frac{\hbar}{2}\right\}$

- At time zero, say we have a state $|+z\rangle$. Then:

$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}}|+z\rangle = \exp\left(-\frac{i\Omega\hat{S}_z t}{\hbar}\right)|+z\rangle = \exp\left(-i\Omega\left(\frac{\hbar}{2}\right)t\frac{1}{\hbar}\right)|+z\rangle$$

$$= \exp\left(-i\frac{\Omega}{2}\right)|+z\rangle \quad \text{so: } \boxed{|\psi(t)\rangle = e^{-i\frac{\Omega}{2}}|+z\rangle}$$

- But this will be different if we start on $|+x\rangle$ s.t.:

$$|\psi(t)\rangle = \exp\left(-\frac{iHt}{\hbar}\right)|+x\rangle = \exp\left(-\frac{iHt}{\hbar}\right)\frac{1}{\sqrt{2}}\left[|+z\rangle + |-z\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[e^{+i\Omega\frac{S_z t}{\hbar}}|+z\rangle + e^{i\Omega\frac{S_z t}{\hbar}}|-z\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[e^{i\Omega\left(\frac{\hbar}{2}\right)\frac{t}{\hbar}}|+z\rangle + e^{-i\Omega\frac{\hbar t}{2}}|-z\rangle\right]$$

$$= e^{i\frac{\Omega t}{2}}\frac{1}{\sqrt{2}}\left[|+z\rangle + e^{-i\Omega t}|-z\rangle\right] \quad \text{so: } \boxed{|\psi(t)\rangle = e^{i\frac{\Omega t}{2}}\frac{1}{\sqrt{2}}\left[|+z\rangle + e^{-i\Omega t}|-z\rangle\right]}$$

- At $\left(t = \frac{\pi}{2\Omega}\right)$, the state is (EX):

$$|\psi\left(\tfrac{\pi}{2\Omega}\right)\rangle = e^{i\frac{\pi}{4}}\frac{1}{\sqrt{2}}\left[|+z\rangle + e^{-i\frac{\pi}{2}}|-z\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left(e^{i\frac{\pi}{4}}|+z\rangle + e^{-i\frac{\pi}{4}}e^{-i\frac{\pi}{2}}|-z\rangle\right) \rightarrow$$

$$= \frac{1}{\sqrt{2}}\left(e^{i\frac{\pi}{4}}|+z\rangle + e^{-i\frac{3\pi}{4}}|-z\rangle\right)$$

$$= e^{i\frac{\pi}{4}}|-y\rangle \quad \text{w/ } \boxed{\frac{\pi}{2} := y}$$

$$= \exp\left(-\left(i\frac{\pi}{4} + i\frac{\pi}{2}\right)\right)$$

$$= \exp\left(-\left(i\left(\frac{\pi}{4} + \frac{\pi}{2}\right)\right)\right)$$

$$= \exp\left(-\left(i\left(\frac{\pi + 2\pi}{2}\right)\right)\right)$$

$$= \exp\left(-\left(i\left(\frac{3\pi}{2}\right)\right)\right)$$

$$= \exp\left(-i\left(4.71239\right)\right)$$

$$= e^{-i4.71239}y \checkmark$$

- While at $\left(t = \frac{\pi}{\Omega}\right)$:

$$|\psi\left(t = \tfrac{\pi}{\Omega}\right)\rangle = e^{i\frac{\pi}{2}}\frac{1}{\sqrt{2}}\left[|+z\rangle + e^{i\pi}|-z\rangle\right]$$

$$= e^{i\frac{\pi}{2}}|x\rangle \quad \boxed{\pi := x}$$

true $\frac{\pi}{2} + \frac{\pi}{2} = \pi = x$.

so: $\boxed{|\psi\left(\tfrac{\pi}{2\Omega}\right)\rangle = e^{i\frac{\pi}{4}}|-y\rangle \quad |\psi\left(\tfrac{\pi}{\Omega}\right)\rangle = e^{i\frac{\pi}{2}}|-x\rangle}$

EXPECTED VALUE OF SPIN

- How do we confirm that the state is changing in time? Well one way to look at it is to take the expected value of S s.t.:

$$\boxed{\langle S\rangle(t) = \langle S_x\rangle(t)\vec{u}_x + \langle S_y\rangle(t)\vec{u}_y + \langle S_z\rangle(t)\vec{u}_z}$$

- The expectation value is thus for (x):

$$\langle S_x\rangle(t) = \langle\psi(t)|\hat{S}_x|\psi(t)\rangle = \left(e^{-i\frac{\Omega t}{2}}\frac{1}{\sqrt{2}}\left[\langle +z| + e^{i\Omega t}\langle -z|\right]\right)\hat{S}_x\left(e^{i\frac{\Omega t}{2}}\frac{1}{\sqrt{2}}\left[|z\rangle + e^{-i\Omega t}|-z\rangle\right]\right)$$

$$= \frac{\hbar}{4}\left(e^{-i\Omega t} + e^{i\Omega t}\right) = \frac{1}{2}\left[\langle +z| + e^{i\Omega t}\langle -z|\right]\left[\frac{\hbar}{2}|-z\rangle + e^{-i\Omega t}\frac{\hbar}{2}|+z\rangle\right]$$

$$= \boxed{\frac{\hbar}{2}\cos(\Omega t)}$$

$$= \frac{1}{2}\left(\frac{\hbar}{2}\langle +z|-z\rangle + e^{n(t)}\langle +z|+z\rangle\frac{\hbar}{2} + \langle -z|-z\rangle\frac{\hbar}{2} + e^{-n(t)}\frac{\hbar}{2}\right)$$

$$= \frac{1}{2}\left(0 + e^{n+\frac{\hbar}{2}} + e^{-n+\frac{\hbar}{2}} + 0\right) = \frac{\hbar}{4}\left(e^{n_t} + e^{-n_t}\right) \checkmark$$

- For y it must be $\langle S_y\rangle_t = -\frac{\hbar}{2}\sin(\Omega t)$:

w/ $n(t) = n_t = \Omega t$.

$$\langle S_y\rangle(t) = \langle \psi(t)|\hat{S}_y|\psi(t)\rangle$$

$$= \left[e^{-i\frac{\Omega t}{2}}\frac{1}{\sqrt{2}}\left[\langle +z| + e^{-i\Omega t}\langle -z|\right]\right]\hat{S}_y\left[e^{i\frac{\Omega t}{2}}\frac{1}{\sqrt{2}}\left[|+z\rangle + e^{-i\Omega t}|-z\rangle\right]\right]$$

$$= \frac{1}{2}\left[\left[\langle +z| + e^{i\Omega t}\langle -z|\right]\left[i\frac{\hbar}{2}|-z\rangle - ie^{-i\Omega t}\frac{\hbar}{2}|+z\rangle\right]\right]$$

$$= \frac{1}{2}\frac{\hbar}{2}\left[-ie^{-i\Omega t} + ie^{i\Omega t}\right]$$

$$= \frac{\hbar}{4}\left[2ie^{-i\Omega t} + ie^{i\Omega t}\right] = \boxed{-\frac{\hbar}{2}\sin(\Omega t)}$$

$$\boxed{\begin{aligned}\langle S_x\rangle(t) &= \frac{\hbar}{2}(\cos(\Omega t))\\ \langle S_y\rangle(t) &= -\frac{\hbar}{2}(\sin(\Omega t))\end{aligned}}$$

WAVE EQS MEASUREMENTS (11.1) (11.2) (11.3)

① $\boxed{\hat{H}|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle}$ Schrodinger EQS. - Complete Rev

② what can we do with this?

[i] Assumptions:

1) $\hat{H} = \frac{p^2}{2m} + V(\hat{x})$; $p^2 = (i\hbar)^2 \frac{\partial^2}{\partial x} = -\hbar^2 \nabla^2$. $p = i\hbar\nabla$

2) $\psi(x,t) = \langle x|\psi(t)\rangle$.

so: $\hat{H}\psi = \left[-\frac{\hbar \nabla^2}{2m} + V(x)\right]\psi(x,t) = \boxed{-\frac{\hbar}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}}$

Schrodinger EQS - 1d form.

[ii] TIME INDEPENDENT SOLUTIONS.

1) $\Psi(x,t) = \psi(x)\,\phi(t)$

$|\psi(t)\rangle = e^{i\frac{Ht}{\hbar}}|\psi(0)\rangle$

Stationary $\boxed{\Psi = \sum_n c_n \psi_n(x)\,\phi_n(t)}$

$c_n = \int_0^w \psi_n^*(x,0)\,\psi(x,0)\,dx = \int_0^\infty |\psi|^2\,dx = 1$

2) $|\psi(0)\rangle = \sum_n c_n|\psi_n\rangle$

$\psi(x,0) = \langle x|\psi(0)\rangle = \sum_n c_n\langle x|\psi_n\rangle = \sum_n c_n\psi_n(x)$

so: $\boxed{|\psi(0)\rangle = \sum_n c_n\langle x|\psi_n\rangle = \sum_n c_n\psi_n(x)}$

$\boxed{E = \hbar\omega_n}$

3) $c_n = \langle\psi_n|\psi_0\rangle = \int_0^\infty \langle\psi_n|x\rangle\langle x|\psi_0\rangle\,dx = \int_0^\infty \psi_n^*\psi(x,0)\,dx$.

4) $|\psi(t)\rangle = \sum_n c_n e^{-i\frac{Ht}{\hbar}}|\psi_n\rangle = \sum_n c_n e^{-i\frac{E_n t}{\hbar}}|\psi_n\rangle = \sum_n e^{i\omega_n t}|\psi_n\rangle$

so: $\boxed{|\psi(t)\rangle = \sum_n e^{i\omega_n t}|\psi_n\rangle}$

5) $\Psi(x,t) = \langle x|\psi(t)\rangle \Rightarrow \boxed{\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}}$ $\boxed{N.8}$

[iii] BOUNDARY CONDITIONS

are continuous functions

• If discontinuity is finite [V(x) finite] $\Rightarrow \therefore$ The Integral is finite

$$\boxed{\int_{x_0-\eta}^{x_0+\eta} \frac{\partial^2\psi}{\partial x^2} = \frac{\partial\psi_n}{\partial x}(x_0+\eta) - \frac{\partial\psi_n}{\partial x}(x_0-\eta) = \frac{2m}{\hbar}\int_{x_0-\eta}^{x_0+\eta}[V-E_n]\psi_n\,dx}$$

② FREE PARTICLE

$k = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$ [i] $\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V_0\psi = E\psi \Rightarrow \frac{\partial^2\psi}{\partial x^2} = \frac{2m(E-V_0)}{\hbar}\psi(x)$

so: $\boxed{\psi(x) = Ae^{ikx} + Be^{-ikx}}$

so $\boxed{\Psi(x,t) = Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}}$ $\Rightarrow$ Full solution.

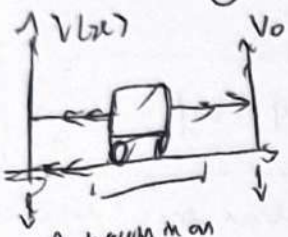
(A) Interpretations:

1) $\psi(A,B)$: A is travelling positive direction $\xrightarrow{+}$
B is travelling negative direction $\xleftarrow{-}$

2.) The Exponentiates are the case of $\boxed{k = \frac{p}{\hbar}}$ where h is a De-Broglie Relation: $\boxed{A = \frac{1}{\sqrt{2\pi\hbar}}}$

$V = \begin{cases} 0 & \text{If not Push} \\ \infty & \text{If Pushed} \end{cases}$

(2) POTENTIAL STEP



A Wagon in an Infinite well. Energy is Infinite since It always Bounces But 0 If it's not pushed

(1) REGIMES / Assumptions

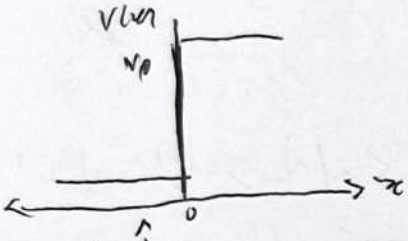
1.) $V(x) = \begin{cases} 0 \\ V_0 \end{cases}$

2.) $\psi(x) = A_1 e^{ik_1x} + B_1 e^{-ik_1x}$ where $\boxed{k_1 = \sqrt{\frac{2mE}{\hbar^2}}}$

3.) $\psi(x) = A_2 e^{ik_2x} + B_2 e^{-ik_2x}$ where $\boxed{k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}}$

$\boxed{E \to E - V_0}$

(2) PROBABILITY FLUX

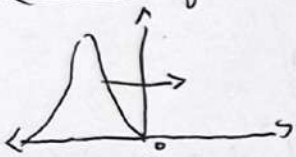


1.) $\frac{\partial|\psi(x,t)|^2}{\partial t} = \frac{\partial \psi^*\psi}{\partial t} = \psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}$

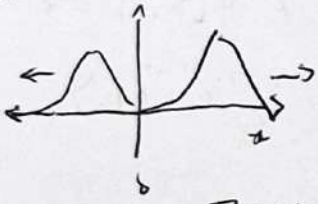
a.) So: $\boxed{\frac{\partial|\psi|^2}{\partial t} = \psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}}$

a) Recall: $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} \longrightarrow \boxed{\frac{\partial\psi}{\partial t} = -\frac{\hbar}{i2m}\frac{\partial^2\psi}{\partial x^2}}$

$i\hbar\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{\partial x^2} \longrightarrow \boxed{\frac{\partial\psi^*}{\partial t} = -\frac{\hbar}{i2m}\frac{\partial^2\psi^*}{\partial x^2}}$



So the: $\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t} = \frac{\hbar}{i2m}\left[\psi^*\frac{\partial^2\psi}{\partial x^2} - \psi\frac{\partial^2\psi^*}{\partial x^2}\right]$.



2.) Let $\boxed{J_x(x,t) = \frac{\hbar}{i2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right)}$.

Thus: $\frac{\partial J}{\partial x} = \frac{\hbar}{i2m}\left[\cancel{\left(\frac{\partial\psi^*}{\partial x}\right)\left(\frac{\partial\psi}{\partial x}\right)} + \psi^*\frac{\partial^2}{\partial x^2}\psi - \cancel{\left(\frac{\partial\psi}{\partial x}\frac{\partial\psi^*}{\partial x}\right)} - \psi\frac{\partial^2\psi}{\partial x^2}\psi^*\right]$

$= \frac{\hbar}{i2m}\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right]$

So: $\boxed{\frac{\partial|\psi(\vec{r},t)|^2}{\partial t} = \frac{\partial J}{\partial x} = \nabla J(x)}$

Divergence Theorem to Replace
- volume Integral Ω
- closed surface S that Bounds S.

Recall: $\iiint_\Omega \frac{\partial|\psi|^2}{\partial t}\, d^3\vec{r} = -\iiint \nabla\cdot\vec{J}\, d^3\vec{r}$

$\frac{\partial}{\partial t}\iiint_\Omega |\psi(\vec{r},t)|^2\, d^3\vec{r} = -\iint \vec{J}(r,t)\, d\vec{S}$

Thus: $\iiint \frac{\partial|\psi|^2}{\partial t}\, d^3\vec{r} = -\iiint \nabla\cdot\vec{J}\, d^3\vec{r}$.

MOSFET, Extensions, Tunneling (11.3) (11.4)

① Cases at the plane [illegible]

recall: $\boxed{\nabla J(x,t) = \frac{\partial |\psi(x,t)|^2}{\partial t}}$

[1] $E > V_0$

1) recall that: $\begin{cases} k_1 = \sqrt{\frac{2mE}{\hbar^2}} \\ k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}} \end{cases}$

2) assume $k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}\Big|_{E \geq V_0}$ s.t $E - V_0 > 0$.

hence $\boxed{k_2 \in \mathbb{R}^+ \text{ if } E - V_0 \ \forall E > V_0.}$

3.) $A_1 + B_1 = A_2$

$\boxed{ik_1A_1 - ik_1B_1 = ik_2A_2.}$ → Applied the boundary condition on. $\frac{2m}{\hbar^2}\int_{x_0-\eta}^{x_0+\eta}[V(x) - E_n]\phi\, dx$

4.) $\cancel{ik_1}A_1 - \cancel{ik_2}A_2 = \cancel{ik_1}B_1$

$k_1A_1 - k_2A_2 = k_1B_1$

$k_1B_1 = k_1A_1 - k_2(A_1+B_1) = k_1A_1 - k_2A_1 - k_2B_1.$

$k_1B_1 + k_2B_1 = k_1A_1 - k_2A_1$

$B_1(k_1+k_2) = A_1(k_1-k_2)$

$\boxed{B_1 = \frac{(k_1-k_2)}{(k_1+k_2)}A_1}$ (11.4a).

✓ PROBLEM 11.2

confirm EQS 11.4 and recall

recall that:

$\phi_1 = A_1e^{ik_1x} + B_1e^{ik_1x}$

$\phi_2 = A_2e^{ik_2x} + B_2e^{ik_2x}$

5) $k_1B_1 = k_1A_1 = -k_2A_2$ $\quad B_1 = A_2 - A_1$

$k_1(A_2 - A_1) - k_1A_1 = -k_2A_2$

$k_1A_2 - k_1A_1 - k_1A_1 = -k_2A_2$

$+k_1A_1 - k_1A_1 = +k_2A_2 - k_1A_2$

$2k_1A_1 = A_2(k_2+k_1)$ ⇒ $\boxed{A_2 = \frac{2k_1}{k_2+k_1}A_1}$ (11.4b)

$B_1 = \frac{k_1 - k_2}{k_1+k_2}A_1$

$= \frac{k_1-k_2}{\cancel{k_1+k_2}}\frac{\cancel{k_2+k_1}}{2k_1}A_2$

$= \frac{k_1-k_2}{2k_1}$

6) Summary:

- $A_1 = \frac{k_2+k_1}{2k_1}A_2$
- $A_2 = \frac{2k_1}{k_2+k_1}A_1$
- $B_1 = \frac{(k_1-k_2)}{k_1+k_2}A_1$
- $B_1 = \frac{k_1-k_2}{2k_1}A_2.$

7) $J_x^{A_1}(x,t) = \frac{\hbar}{m}\,\mathrm{Im}\left(A_1^*e^{-i(k_1x-\omega_1t)} \cdot \frac{\partial}{\partial x}A_1e^{ik_1x-\omega_1t}\right).$

recall $i(k_1x - \omega t)$

$\cancel{J(x,t)} = \cancel{B}$

$= \frac{\hbar}{m}\mathrm{Im}\left(A_1^*e^{ik_1x-\omega_1t)}\ ik_1A_1e^{ik_1x-\omega_1t}\right)$

$= \frac{\hbar}{m}\mathrm{Im}\left(A_1^*\cancel{e}\,ik_1A_1\cancel{e}\right) = \frac{\hbar}{m}k_1A_1^*A_1 = \frac{\hbar k_1}{m}|A_1|^2.$

$= v_1|A_1|^2.$ do: $\boxed{J^{A_1} = v|A|^2}$ (11.45)

uh/ $\frac{\hbar k_1}{m} = \frac{p_1}{m} = v$ (11.46)

or $\boxed{p = mv}$

or $F \in (\mathbb{R}, \mathbb{C})$ but in this case $F \in \mathbb{R}$.

4.) $\boxed{R = \frac{J_x^{B_1}(x,t)}{J_x^{A_1}(x,t)}}$ = probability that the particle will be found to reflect

$\boxed{R = \text{Reflection Coefficient}}$

w: $R = \left|\frac{J_x^{B_1}}{J_x^{A_1}}\right| = \frac{v_1|B_1|^2}{v_1|A_1|^2} = \frac{|B_1|^2}{|A_1|^2} = \frac{(k_1-k)^2}{(k_1+k)^2} = \frac{(\sqrt{E}-\sqrt{E-V_0})^2}{(\sqrt{E}+\sqrt{E-V_0})^2}$

5) $\boxed{T = 1-R} = 1 - \frac{(k_1-k_2)^2}{(k_1+k_2)^2} = \frac{(k_1+k_2)^2-(k_1-k_2)^2}{(k_1+k_2)^2}$

$= \frac{[k_1^2+2k_1k_2+k_2^2]-[k_1^2-2k_1k_2+k_2^2]}{(k_1+k_2)^2}$

$= k_1^2+2k_1k_2+k_2^2-k_1^2+2k_1k_2-k_2^2 / (k_1+k_2)^2$

$= (k_1^2+k_2^2)-(k_1+k_2)^2+2k_1k_2+2k_1k_2 / (k_1+k_2)^2$

$= 4k_1k_2/(k_1+k_2)^2 = \frac{4\sqrt{E(E-V_0)}}{(E+\sqrt{E-V_0})^2}$

w: $\boxed{T = \frac{4\sqrt{E(E-V_0)}}{(E+\sqrt{E-V_0})^2} \quad R = \frac{(E-\sqrt{E-V_0})^2}{(\sqrt{E}+\sqrt{E-V_0})^2}}$

$R = 1$, $T = 0$ if $E < V_0$

$R = 0$, $T = 1$ if $E > V_0$

- It transmits if there's "no V_0" since its wholly overlapping
- It reflects if the energy differential mismatches the potential

[ii] $E < V_0$

1) Then $k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar}}$

$= \sqrt{\frac{-2m(E-V_0)}{\hbar}} = i\sqrt{\frac{2m(E-V_0)}{\hbar}} = i\alpha$ w/ $\alpha \in \mathbb{R}^+$

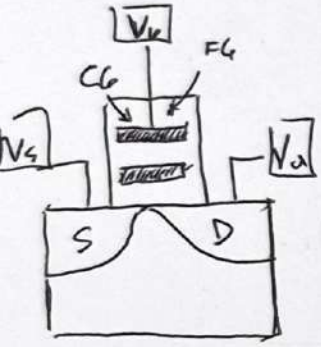
$E < V_0$

2) w: $\boxed{\phi(x) = Ae^{\alpha x} + Be^{-\alpha x}}$

3) Then: $B_1 = \frac{ik_1+\alpha}{(ik_1-\alpha)}A_1 \quad A_2 = \frac{2ik_1}{(ik_1-\alpha)}A$

4) $R = \left|\frac{J_x^{B_1}(x_{\neq})}{J_x^{A_1}(x_{\neq})}\right| = \frac{k|B_1|^2}{k_1|A_1|^2} = \left|\frac{(ik+\alpha)}{(ik-\alpha)}\right|^2 = 1.$

so clearly $T = 0$



S = Source
D = Drain
CG = Control Gate
V = Volts

↓

MOSFET

metal-oxide field effect transistor

② Quantum Tunneling

So far: $\boxed{\phi(x) = \begin{cases} A_1e^{ikx} + B_1e^{-ikx} & x<0 \\ A_2e^{-\alpha x} + B_2e^{\alpha x} & 0<x<L \\ A_3e^{ikx} & x>L \end{cases}}$

w/ $\boxed{k = \sqrt{\frac{2mE}{\hbar^2}}}$

$\boxed{\alpha = \sqrt{\frac{2m(V_0-E)}{\hbar^2}}}$

[i] Implications

1) Tunnel = Doesn't have enough energy to go through barrier

2) Full Reflection = $L \gg \alpha^{-1}$ so $P(R) = 1$

3) Imaginary = $i\alpha$ means that the particle doesn't "propagate" through the barrier in a traditional sense.

4) Superluminal

Angular Momentum & Commutation Basics (7.1) (7.2) (7.3)

(1) Commutable observables:

$$\boxed{[\hat{A},\hat{B}]=0}$$

That means: $\hat{A}B-\hat{B}A = \hat{A}^2-A^2=0 = \hat{B}^2-\hat{B}^2 = \hat{A}\hat{B}-\hat{B}-A \quad \forall \begin{vmatrix}\hat{A}_x & \hat{A}_y\\ \hat{B}_x & \hat{B}_y\end{vmatrix}$

1) For commutation $[\hat{A},\hat{B}]=0$, there exists a complete set of states $|\lambda\rangle$ that are simultaneously eigenstates of both $\hat{A}$, $\hat{B}$ such:

$$\boxed{\begin{aligned}\hat{A}|\lambda\rangle &= a_\lambda|\lambda\rangle\\ \hat{B}|\lambda\rangle &= b_\lambda|\lambda\rangle\end{aligned}} \quad \forall [\hat{A},\hat{B}]=0 \;\exists |\lambda\rangle \text{ s.t. } \left\{\begin{aligned}\hat{A}|\lambda\rangle &= a_\lambda|\lambda\rangle\\ \hat{B}|\lambda\rangle &= b_\lambda|\lambda\rangle\end{aligned}\right\} \bullet \text{ Completeness of eigenvalues}$$

2) Let $|\lambda\rangle = |a,b\rangle$. Then:

$\hat{A}|\lambda\rangle = \hat{A}|a,b\rangle \rightarrow \langle\lambda|\hat{A}|\lambda\rangle = \langle a,b|\hat{A}|a,b\rangle$

$\hat{A}=\hat{A}$ and $a=\hat{A}$ so: $A|\lambda\rangle = a|a,b\rangle$

similarly $\hat{B}|\lambda\rangle = \hat{B}|a,b\rangle \rightarrow \langle\lambda|\hat{B}|\lambda\rangle = \langle a,b|\hat{B}|a,b\rangle$

$\langle a,b|\hat{B}B^{\dagger}|a,b\rangle = [\;][\;]\hat{B}\hat{B}^{\dagger} = 1$

thus $\boxed{\begin{aligned}\hat{A}|a,b\rangle &= a|a,b\rangle\\ \hat{B}|a,b\rangle &= b|a,b\rangle\end{aligned}}$

Invariance: $\forall [\hat{A},B]\neq 0$, $\nexists |\lambda\rangle$ s.t. $\left\{\begin{aligned}\hat{A}|\lambda\rangle &\neq a_\lambda|\lambda\rangle\\ \hat{B}|\lambda\rangle &\neq b_\lambda|\lambda\rangle\end{aligned}\right\} \bullet$ Incompleteness of eigenvalues

(2) Angular momentum operators

[i] Total:

a) $\vec{S} = S_x\vec{u}_x + \hat{S}_y\vec{u}_y + \hat{S}_z\vec{u}_z \rightarrow$ Spin vector

b) $\boxed{\vec{J} = \vec{S}+\vec{L}} \rightarrow$ Total Angular Momentum.

c) $\vec{J} = \hat{J}_x\vec{u}_x + \hat{J}_y\vec{u}_y + \hat{J}_z\vec{u}_z \rightarrow$

2) $[\hat{J}_x,\hat{J}_y] = i\hbar J_z$

$[\hat{J}_y,J_z] = i\hbar J_x$

$[\hat{J}_z,J_x] = i\hbar J_y$

thus $\boxed{[J_i,J_j] = i\hbar J_k}$ w/ $i\neq j\neq k$.

Since $i=j$ then $[J_i,J_i]=0$.

If $\vec{L} = \int(r\times\omega)\,d\vec{r} = 0$ then $\vec{J} = S+\vec{L} = S+0 = S$ so $\boxed{J=S}$ iff $L=0$.

3) $\hat{J}_z|+z\rangle = \frac{\hbar}{2}|+z\rangle$

$\hat{J}_z|-z\rangle = \frac{\hbar}{2}|-z\rangle$

[ii] The J^2 operator:

1) $\boxed{\hat{J}^2 = J\cdot J = \hat{J}_x^2+\hat{J}_y^2+\hat{J}_z^2}$ with

$$\begin{bmatrix}J_x\\J_y\\J_z\end{bmatrix}\cdot\begin{bmatrix}J_x\\J_y\\J_z\end{bmatrix} = \begin{bmatrix}J_x\begin{bmatrix}J_x\\J_y\\J_z\end{bmatrix}\\ J_y\begin{bmatrix}J_x\\J_y\\J_z\end{bmatrix}\\ J_z\begin{bmatrix}J_x\\J_y\\J_z\end{bmatrix}\end{bmatrix} = \begin{bmatrix}J_x^2+J_xJ_y+J_xJ_z\\ J_xJ_y+J_y^2+J_yJ_z\\ J_zJ_x+J_zJ_y+J_z^2\end{bmatrix} = \begin{bmatrix}J_x^2+0+0\\ 0+J_y^2+0\\ 0+0+J_z^2\end{bmatrix} = \begin{bmatrix}J_x^2\\J_y^2\\J_z^2\end{bmatrix} \equiv J_x^2+J_y^2+J_z^2$$

2) $[\hat{A}\hat{B},\hat{C}] = \hat{A}\hat{B}(\hat{C}) - \hat{C}(\hat{A}\hat{B})$

$= \hat{A}(\hat{B}\hat{C}) - (\hat{A}\hat{C})\hat{B}$

$= \hat{A}(\hat{B}\hat{C}-\hat{C}\hat{B}) - \hat{B}(\hat{A}\hat{C}-\hat{C}\hat{A})$

$= \hat{A}[\hat{B},\hat{C}] + \hat{B}[\hat{A},C]$

thus $\boxed{[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}]+\hat{B}[\hat{A},\hat{C}]}$

3) $\hat{J}^2, J_z = [(J_x^2+J_y^2+J_z^2), J_z] = [J_x^2,J_z]+[J_y^2,J_z]+\cancel{[J_z^2,J_z]}=0$

$= \hat{J}_x[\hat{J}_x,\hat{J}_z] + [J_x,\hat{J}_z]\hat{J}_x + \hat{J}_y[\hat{J}_y\hat{J}_z] + [J_y,\hat{J}_z]\hat{J}_y + \ldots$

$= 0$

so: $\boxed{[\hat{J}^2,J_z] = [J_x^2,\hat{J}_z]+[J_y^2,J_z] = 0}$

2) $\hat{J}^2 = J_x^2 + J_y^2 + J_z^2 = \frac{\hbar^2}{4}\{\sigma_x^2 + \sigma_y^2 + \sigma_z^2\} = \frac{\hbar^2}{4}\left\{\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix} + \begin{bmatrix}0&-i\\i&0\end{bmatrix}\begin{bmatrix}0&-i\\i&0\end{bmatrix} + \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix}\right\}$

$= \frac{\hbar^2}{4}\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix} + \begin{bmatrix}1&0\\0&1\end{bmatrix} + \begin{bmatrix}1&0\\0&1\end{bmatrix}\right\} = \frac{3}{4}\hbar^2\begin{bmatrix}1&0\\0&1\end{bmatrix}$

hence $\boxed{\hat{J}^2 = J_x^2 + J_y^2 + J_z^2 = \frac{3}{4}\hbar^2\begin{bmatrix}1&0\\0&1\end{bmatrix}}$

Note: $\sigma_x^2 = \begin{bmatrix}0&1\\1&0\end{bmatrix}^2$ $\sigma_y^2 = \begin{bmatrix}0&-i\\i&0\end{bmatrix}^2$ $\sigma_z^2 = \begin{bmatrix}1&0\\0&-1\end{bmatrix}^2$.

(2) Eigenvalues and Eigenstates

LADDER

[i] BASICS:

1) $\boxed{\begin{aligned}\hat{J}^2|j,m_j\rangle &= j(j+1)\hbar|j,m_j\rangle\\ \hat{J}_z|j,m_j\rangle &= m_j\hbar|j,m_j\rangle\end{aligned}}$

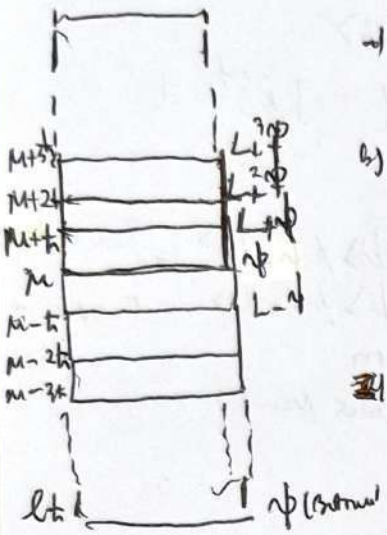
Allowed values of j are $j = \frac{n}{2}$, $n \in \mathbb{Z}$ so: $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots$

and allowed values of m_j are $m_j \in [-j, j] = -j, -j+1, -j+2, \ldots j-2, j-1, j$.
$= 0-j, 1-j, 2-j, \ldots j-2, j-1, j-0$.

2) matrix representation:

$\hat{J}^2 = \hbar^2 j(j+1) I = \hbar^2 2 \cdot I = \hbar^2 \begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}$

$J_z = \hbar\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}$ hence $\boxed{J_z = \hbar\begin{bmatrix}1&&\\&0&\\&&-1\end{bmatrix} \quad J^2 = \hbar^2\begin{bmatrix}2&&\\&2&\\&&2\end{bmatrix}}$



- Each Neighbors are separated via one unit $\hbar$.
- Eventually we're gonna feel reach a state for which the z-component exceeds the total.
- hence $\boxed{\exists\, \psi(top),\ L_+\psi(top) = 0}$

3) Derivation?:

- $[L^2, L] = 0 \rightarrow \begin{cases} L^2\psi = \lambda\psi \\ L_z\psi = \mu\psi \end{cases}$ w/ ψ is a test function
$(\lambda, \mu) :=$ Eigenstates (L^2, L).

on $\boxed{L_\pm = L_x \pm iL_y}$

- $[L_z, L_\pm] = [L_z, L_x] \pm i[L_z, L_y] = i\hbar L_y \pm i(-i\hbar L_x) = \pm\hbar(L_x \pm iL_y)$.

$L_z(L_+\psi) = (L_z L_+ - L_+ L_z)\psi + L_+ L_z\psi = \pm\hbar L_+\psi + L_+(\mu\psi)$
$= (\mu \pm \hbar)L_+\psi$

- $\begin{cases} L_z\psi(top) = \ell\hbar\psi(top) \\ L^2\psi(top) = \lambda\psi(top) \end{cases} \rightarrow L_\pm L_\mp = (L_x + iL_y)(L_x \mp iL_y) = L_x^2 + L_y^2$
$= L_x^2 + L_y^2 \mp i(L_xL_y - L_yL_x)$
$= L^2 - L_z^2 \mp i(\pm\hbar L_z)$.

$\rightarrow \boxed{L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z}$

hence $L^2\psi(top) = (L_-L_+ + L_z^2 + \hbar L_z)\psi(top) = (0 + \hbar^2\ell^2 + \hbar^2\ell)\psi(top)$
$= \hbar^2\ell(\ell+1)\psi(top)$

[ii] RAISING and LOWERING OPERATORS:

1) $\hat{J}_\pm = J_x + iJ_y \rightarrow \begin{cases} \hat{J}_+|j,m_j\rangle = \hbar[j(j+1) - m_j(m_j+1)]^{\frac{1}{2}}|j,m_j+1\rangle \\ \hat{J}_-|j,m_j\rangle = \hbar[j(j+1) - m_j(m_j-1)]^{\frac{1}{2}}|j,m_j-1\rangle \end{cases}$

hence $\boxed{J_\pm|j,m_j\rangle = \hbar\sqrt{j(j+1) - m_j(m_j \pm 1)}\,|j,m_j\pm1\rangle}$

2) If $m_j = j$ then $J_+|j,j\rangle = \hbar[j(j+1) - j(j+1)]^{\frac{1}{2}}|j,j\rangle = \hbar\, 0\, 0 = \boxed{0}$ $\rightarrow \boxed{J_+|j,j\rangle = 0}$

[iii] SPINS:

1) $S^2|+z\rangle = \frac{3}{4}\hbar|+z\rangle = \frac{1}{2}(\frac{1}{2}+1)\hbar^2|+z\rangle$
$S^2|-z\rangle = \frac{3}{4}\hbar|-z\rangle = \frac{1}{2}(\frac{1}{2}+1)\hbar^2|-z\rangle$ and $\boxed{\begin{aligned}\hat{S}_z &= \frac{1}{2}\hbar|+z\rangle\\ \hat{S}_z &= -\frac{1}{2}\hbar|-z\rangle\end{aligned}}$

2) $|+z\rangle = |\frac{1}{2},\frac{1}{2}\rangle$, $|-z\rangle = |\frac{1}{2},-\frac{1}{2}\rangle$ $\Rightarrow$ $\boxed{\begin{matrix}z+ & z-\\ |++\rangle, & |+-\rangle\end{matrix}}$

Problem 9

(a) Prove the Column vector representing $\boxed{|\psi\rangle = \cos(\theta)|H\rangle + e^{i\phi}\sin(\theta)|V\rangle}$ let $|\psi\rangle = \psi$

Ans: 1) $\langle +45|H\rangle = \frac{1}{\sqrt{2}}[1\ \ 1]\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$

$\langle +45|V\rangle = \frac{1}{\sqrt{2}}[1\ 1]\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}$

so $\langle +45|\cos\theta|H\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}\cos\theta = \frac{1}{\sqrt{2}}\begin{bmatrix}\cos\theta & 0\\\cos\theta & 0\end{bmatrix}$

$\langle +45|e^{i\phi}\sin\theta|V\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}e^{i\phi}\sin\theta = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 0\\e^{i\phi}\sin\theta & e^{i\phi}\sin\theta\end{bmatrix}$

2) so $\langle +45|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}^T\begin{bmatrix}\cos\theta\\e^{i\phi}\sin\theta\end{bmatrix} = \frac{1}{\sqrt{2}}(\cos\theta + e^{i\phi}\sin\theta)$

Similarly: $\langle -45|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}^T\begin{bmatrix}\cos\theta\\e^{i\phi}\sin\theta\end{bmatrix} = \frac{1}{\sqrt{2}}(\cos\theta - e^{i\phi}\sin\theta)$

3) so: $\begin{bmatrix}\langle +45|\psi\rangle\\\langle -45|\psi\rangle\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}(\cos\theta + e^{i\phi}\sin\theta)\\\frac{1}{\sqrt{2}}(\cos\theta - e^{i\phi}\sin\theta)\end{bmatrix}$

so: $\boxed{\langle \pm 45|\psi\rangle = \frac{1}{\sqrt{2}}\cos\theta \pm e^{i\phi}\sin\theta}$

(b) Work out the matrix representation of $\hat{P}_H = |H\rangle\langle H|$, $\hat{P}_V = |V\rangle\langle V|$ using $|L\rangle$ $|R\rangle$ states as Basis;

(a) check that the relationships $\hat{P}_H^2 = \hat{P}_H$, $\hat{P}_V^2 = \hat{P}_V$ and $\hat{P}_H\hat{P}_V = \hat{P}_V\hat{P}_H = 0$.!

Ans.. Answer 1:

if, $|R\rangle = \frac{1}{\sqrt{2}}(|H\rangle - i|V\rangle)$; $|L\rangle = \frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle)$.

then: $|H\rangle = \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle)$; $\frac{i}{\sqrt{2}}(|R\rangle - |L\rangle)$.

so $|H\rangle = \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle) \rightarrow \hat{P}_H = \left(\frac{1}{\sqrt{2}}(|R\rangle + |L\rangle)\right)\left(\frac{1}{\sqrt{2}}(\langle R| + \langle L|)\right)$

$= \frac{1}{2}(\langle R|R\rangle + \langle L|R\rangle + \langle R|L\rangle + \langle L|L\rangle)$

$= \frac{1}{2}(1 + 0 + 0 + 1) = \frac{1}{2}(2) = \boxed{1}$ ✓

2) Similarly: $|V\rangle = \frac{1}{2}(\langle R|R\rangle - |R\rangle\langle L| - |L\rangle\langle R| + \langle L|L\rangle)$

$= \frac{1}{2}(1 - 0 - 0 + 1) = \frac{1}{2}(1) = \frac{2}{2} = \boxed{1}$ ✓

2 now $\frac{1}{2}(\langle R|R\rangle + \langle L|R\rangle\langle R|L\rangle + \langle L|L\rangle)(\langle R|R\rangle - \langle L|R\rangle - \langle R|L\rangle + \langle L|L\rangle)$

$= \frac{1}{2}(1)(1)$

Answer 2

1) $\hat{P}_V = |V\rangle\langle V| = \frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$ $\hat{P}_H = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = |H\rangle\langle H|$

w/ $|V\rangle\langle V| = \begin{bmatrix}1\\-1\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix}\frac{1}{2}$ $|H\rangle\langle H| = \begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}\frac{1}{2}$

we $\boxed{\begin{aligned}\langle R|\hat{P}_H &= \tfrac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix}\\ \langle L|\hat{P}_H &= \tfrac{1}{2}\begin{bmatrix}1 & 1\end{bmatrix}\end{aligned}}$ $\boxed{\begin{aligned}\langle R|\hat{P}_V &= \tfrac{1}{2}\begin{bmatrix}1 & -1\end{bmatrix}\\ \langle L|\hat{P}_V &= \tfrac{1}{2}\begin{bmatrix}1 & -1\end{bmatrix}\end{aligned}}$

3) to verify: $\hat{P}_H^2 = \frac{1}{2^2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}^2 = \frac{1}{4}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}$

$$= \begin{bmatrix}1\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} & 1\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\\ 1\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} & 1\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}2 & 2\\2 & 2\end{bmatrix}$$

for $\boxed{\hat{P}_H^2 = P_H}$ we $\frac{1}{4}\begin{bmatrix}2 & 2\\2 & 2\end{bmatrix} = \begin{bmatrix}\frac{2}{4} & \frac{2}{4}\\ \frac{2}{4} & \frac{2}{4}\end{bmatrix} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = P_H$

4) Similarly: $\hat{P}_V^2 = \frac{1}{2^2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$

$$= \frac{1}{4}\begin{bmatrix}1\cdot 1 + (-1)(-1) & 1(-1) + (-1)(1)\\ (-1)(1) + (1)(-1) & (-1)(-1) + (1)\end{bmatrix} = \frac{1}{4}\begin{bmatrix}2 & -2\\-2 & 2\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = P_V \quad \text{we } \boxed{\hat{P}_V^2 = P_V}$$

5) Last but not least:

$$P_H P_V = \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} = \begin{bmatrix}1\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} & 1\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\\ 1\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix} & 1\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\end{bmatrix}$$

for $\boxed{\begin{bmatrix}P_H & 0\\0 & P_H\end{bmatrix} = \begin{bmatrix}P_H P_H & P_H P_V\\ P_H P_V & P_H P_H\end{bmatrix}}$

$$= \begin{bmatrix}1\cdot 1 + 1\cdot(-1) & 1\cdot(-1) + 1\cdot 1\\ 1\cdot 1 + 1(-1) & 1\cdot(-1) + 1\cdot 1\end{bmatrix} = \begin{bmatrix}1-1 & -1+1\\ 1-1 & -1+1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

$= \boxed{0}$ ✓ we $P_H P_V = 0$

Problem 4

(1) Show that: $[\hat{A}\hat{B},C] = \hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B}$

$\hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}B + \hat{B}\hat{A}\hat{C} - \hat{B}\hat{C}\hat{A}$

Ans: 1) $[\hat{A}\hat{B},C] = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{A}C = \hat{A}(\hat{B}\hat{C} - \hat{C}\hat{B}) + \hat{B}(\hat{A}\hat{C} - \hat{C}\hat{A})$

$= \hat{A}[\hat{B},\hat{C}] + \hat{B}[\hat{A},\hat{C}]$

2) Reverse:

$\hat{A}[\hat{B},\hat{C}] + [\hat{A},\hat{C}]\hat{B} = \hat{A}(\hat{B}\hat{C} - \hat{C}\hat{B}) + \hat{B}(\hat{A}\hat{C} - \hat{C}\hat{A})$

$= (\hat{A}\hat{B} - \hat{B}\hat{A})\hat{C} + (\hat{B}\hat{A} - \hat{A}\hat{B})\hat{C} = [A,B]\hat{C} + \hat{C}[\hat{B},\hat{A}]$

$= ([\hat{A},\hat{B}] + [\hat{B},\hat{A}])\hat{C} = [\hat{A}\hat{B},\hat{C}] \neq 0$ unless $\hat{C} = 0$. ✓

(2) Show that $\hat{J}^2$ is hermitian

Ans: we wanna show $\boxed{(J^2\psi) = \hat{J}^2}$

Try 1) $(\hat{J}^2)^\dagger = (\hat{J}\cdot\hat{J})^\dagger = \hat{J}^\dagger\hat{J}^\dagger = \hat{J}\cdot\hat{J} = \hat{J}^2$

2) $(\hat{J}^2)^\dagger = (\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2)^\dagger = \hat{J}_x^{2\dagger} + \hat{J}_y^{2\dagger} + \hat{J}_z^{2\dagger} = \hat{J}^2$.

So, Indeed $\boxed{\hat{J}^2 \text{ is Hermitian!}}$

3) Now: $\langle\psi|\hat{A}|\psi\rangle = (\langle\psi|A|\psi\rangle)^* \longleftrightarrow \boxed{A^\dagger = A}$

for J^2 is Hermitian because it has a complete set of Eigenstates w/ real values.

(3) Compute $[\hat{J}^2, \hat{J}_x]$ and $[\hat{J}^2, J_y]$

Ans: 1) $[\hat{J}^2, \hat{J}_x] = [(\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2), \hat{J}_x]$

$= \hat{J}_x^2\hat{J}_x + \hat{J}_y^2\hat{J}_x + \hat{J}_z^2\hat{J}_x = J_x\cdot J_x\cdot J_x \neq 0 + 0 = 0$

$[\cancel{J_x^3, J_x}] + [J_y^2, \hat{J}_x] + [J_z^2\hat{J}_x]$

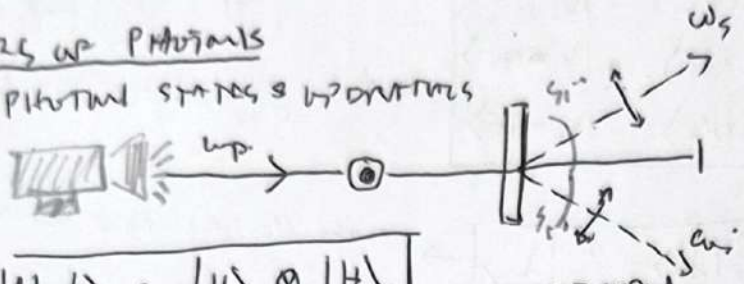
$= \begin{cases} \hat{J}_y[J_y, J_x] + J_z[J_z J_x] = \\ J_y[J_x, J_y] + J_z[J_z J_x] = \end{cases} \begin{cases} -i\hbar J_z J_y + i\hbar J_y J_z \\ -i\hbar J_z J_y + i\hbar J_y J_z \end{cases} +$

$= 0$ ✓

So: $\boxed{\begin{matrix}[\hat{J}^2, \hat{J}_x] = 0 \\ [\hat{J}^2, \hat{J}_y] = 0\end{matrix}}$

(1) Pairs of Photons

[1] 2 Photon States & Properties



Spontaneous Parametric Down Conversion

- One Pump photon at one wave frequency $\langle \omega_p \xrightarrow{\Delta} (\omega_s, \omega_i) \rangle$ Include a crystal.
- Polarization of the signal and idler photons are orthogonal of the pump.

1) $\boxed{|H,H\rangle = |H\rangle_s \otimes |H\rangle_i}$ HILBERT SPACE METHOD OF POLARIZATION

w/ $\otimes$ = Direct vector product That combines the 2 particle system in an Enlarged Hilbert space.

2.) $|H,H\rangle = |H\rangle_s |H\rangle_i = |HH\rangle$

$$\text{and } |HH\rangle = \left(C_H^s|H\rangle_s + C_V^s|V\rangle_s\right)\left(C_H^i|H\rangle_i + C_V^i|V\rangle_i\right).$$

$$= C_H^s C_H^i |H\rangle_s|H_i\rangle + 2C_H^s C_V^i |H\rangle_s|V\rangle_s + C_V^s C_V^i |V\rangle_H|V\rangle_i$$

$$= C_H^s C_H^i |H,H\rangle + C_V^s C_V^i |V,V\rangle + C_H^s C_V^i |H\rangle_s|V\rangle_i + C_V^s C_H^i |V\rangle_s|H\rangle_i$$

$$= C_H^s C_H^i |H,H\rangle + C_V^s C_V^i (V,V\rangle + C_H^s C_V^i |H,V\rangle + C_V^s C_H^i |V,H\rangle$$

EIGENVALUE

$|\psi\rangle = \sum_n C_n |\psi_0(n)\rangle$

$= \sum_n e^{iE_n t} |\psi_0(n)\rangle$

and since: $C_H^s C_H^i \equiv P_H(s,i)$. $C_V^s C_V^i \equiv P_V(s,i)$ $\begin{cases} C_H^s C_V^i = P_H^s(s)\, P_V^i(s) \\ C_V^s C_H^i = P_V(s)\, P_H(i). \end{cases}$

from: $\boxed{|HH\rangle = P_H(s,i)|H,H\rangle + P_V(s,i)|V,V\rangle + P_H^s(s)P_V(s)|H,V\rangle + P_V^s(s)P_H(i)|V,H\rangle}$

3.) here The Beam is $|+45,H\rangle_i$ then:

$$|+45,H\rangle = |+45\rangle_s \otimes |H\rangle_i = \frac{1}{\sqrt{2}}\left(|H\rangle_s + |V\rangle_s\right) \otimes |H\rangle_i = \frac{1}{\sqrt{2}}\left(|H\rangle_s \otimes |H\rangle_i + |V\rangle_s \otimes |H\rangle_i\right)$$

$$= \frac{1}{\sqrt{2}}\left(|H,H\rangle + |V,H\rangle\right)$$

now: $\boxed{|+45,H\rangle = \frac{1}{\sqrt{2}}\left[|H,H\rangle + |V,H\rangle\right]}$

4) Suppose we Modify The polarization state of the Idler Beam.

$$|+45,R\rangle = |+45\rangle_s \otimes |R\rangle_i = \frac{1}{\sqrt{2}}\left[|H\rangle_s + |V\rangle_s\right] \otimes \frac{1}{\sqrt{2}}\left[|H\rangle_i - i|V\rangle_i\right]$$

$$= \frac{1}{2}\left[|H\rangle_s|H\rangle_i - i|H\rangle_s|V\rangle_i + |V\rangle_s|H\rangle_i - i|V\rangle_s|V\rangle_i\right]$$

$$= \frac{1}{2}\left[|H,H\rangle - i|H,V\rangle + |V,H\rangle - i|V,V\rangle\right]$$

5.) $\langle V,+45|R,H\rangle = \left({}_s\langle V|\,{}_i\langle +45|\right)\left(|R\rangle_s|H_i\rangle\right) = {}_s\langle V|R\rangle_s\, {}_i\langle +45|H_i\rangle$

$$= \left(\frac{-i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{-i}{2} \quad \text{but } \langle V| = \begin{bmatrix} -i \\ 0 \end{bmatrix}\frac{1}{\sqrt{2}} \quad |H\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\frac{1}{\sqrt{2}}$$

6) Suppose That: $(\mathcal{P}^s_{HV}, \mathcal{P}^i_{HV})$ be a photon state That acts on The (signal, idler) Resp. Then: $\boxed{\left[\hat{\mathcal{P}}^s_{HV}, \hat{\mathcal{P}}^i_{HV}\right] = 0}$. For example: ? Calculate The action operators such as $\hat{\mathcal{P}}^s_{HV}$, $\hat{\mathcal{P}}^i_{HV}$, and $\hat{\mathcal{P}}^{si}_{HV}$ on $|V,+45\rangle$

Ans: a) $\hat{\mathcal{P}}^s_{HV}(V,+45) = \left(\hat{\mathcal{P}}^s_{HV}|V\rangle_s\right) \otimes |+45\rangle_i = (-1)|V\rangle_s \otimes |+45\rangle_i = -|V,+45\rangle$.

b) $\mathcal{P}^i_{HV}(V,+45) = |V\rangle_s \otimes \left(\hat{\mathcal{P}}^i_{HV}|+45\rangle_i\right) = |V\rangle_s \otimes \left[\mathcal{P}^i_{HV}\frac{1}{\sqrt{2}}\left[|H\rangle_i + |V\rangle_i\right]\right]$.

$$= |V\rangle_s \otimes \left[\frac{1}{\sqrt{2}}\left[|H\rangle_i - |V\rangle_i\right]\right] = |V\rangle_s \otimes |-45\rangle_i = |V,45\rangle.$$

c) $\hat{\mathcal{D}}^{si}_{HV}|V,+45\rangle = \hat{\mathcal{D}}^{s}_{HV}\,\hat{\mathcal{D}}^{i}_{HV}|V,+45\rangle = (\hat{\mathcal{D}}^{s}_{HV}|V\rangle_s)\otimes(\hat{\mathcal{D}}^{i}_{HV}|+45\rangle_i)$

$= (-1)|V\rangle_s \otimes |-45\rangle_i = -|V,-45\rangle.$

Ans:
$$\boxed{\begin{aligned}\hat{\mathcal{D}}^{s}_{HV}|V,+45\rangle &= -|V,+45\rangle\\ \hat{\mathcal{D}}^{s}_{HV}|V,+45\rangle &= +|V,-45\rangle\\ \hat{\mathcal{D}}^{si}_{HV}|V,+45\rangle &= -|V,-45\rangle\end{aligned}}$$

$(\mathcal{D}^{s}_{HV}\ \mathcal{D}^{i}_{HV}) = (-1,+1).$

and: $\mathcal{D}^{i}_{HV}(|H\rangle_i + |V\rangle_i) = (|H\rangle_i + V_i\rangle)^*$
$= |H\rangle_i - |V_i\rangle$

iii) PROBABILITIES

$\boxed{\hat{Q}|\lambda_n\rangle = \lambda_n|\lambda_n\rangle} \rightarrow$ Make Measurements on observable Q.

1) with $P(\lambda_n|\psi\rangle) = |\langle\lambda_n|\psi\rangle|^2 = \langle\psi|\lambda_n\rangle\langle\lambda_n|\psi\rangle = \langle\psi|\psi\rangle\langle\lambda_n|\lambda_n\rangle.$

$= \langle\psi|\hat{P}_{\lambda_n}|\psi\rangle = \langle\hat{P}_{\lambda_n}\rangle$

Ans. $\boxed{P(\lambda_n|\psi\rangle) = \langle\hat{P}_{\lambda_n}\rangle}$ and $\langle\hat{P}_{\lambda_n}\rangle \equiv \alpha\lambda_1 + (1-\alpha)\lambda_2 =$
$= \alpha\lambda_1 + \lambda_2 - \alpha\lambda_2 = \alpha(\lambda_1-\lambda_2) + \lambda_2.$

see $\langle a\rangle = \sum_i^N p(a)\,a_i \sim \alpha x - (1-\alpha)x_i$

and: $p(\lambda_n) = p(\lambda_n|\psi\rangle)$

* EXAMPLE PROBLEM

2) Calculate the probability that the signal photon will be measured to have vertical polarization and the idler photon will be measured to have horizontal polarization in a state $|R,+45\rangle$

Ans: a) If $\hat{\rho}(V_s,H_i)$ is a proj-operator:
$\hat{P}_{V_s,H_i}(\cdot) = |V,H\rangle\langle V,H| \Rightarrow p(V_s,H_i) = \langle P_{V_s;H_i}\rangle = \langle V,H|\langle R,+45|V,H\rangle$

b) so: $\langle P_{V_s,H_i}\rangle = \langle R,+45|V,H\rangle\langle V,H|R,+45\rangle = |\langle R,+45|V,H\rangle|^2$
$= |{}_s\langle R|V\rangle_s\ {}_i\langle +45|H\rangle_i|^2 = \left|\frac{i}{\sqrt{2}}\frac{1}{\sqrt{2}}\right|^2 = \left|\frac{i^2}{(\sqrt{2})^2}\frac{1^2}{(\sqrt{2})^2}\right| = \left|-\frac{1}{2}\frac{1}{2}\right| = \left|-\frac{1}{4}\right|$
$= \frac{1}{4}$

so: $\boxed{\langle P_{V_s,H_i}\rangle = \frac{1}{4}}$ ~~state~~ with $\langle R|$

2c) Calculate the probability that the idler photon will be measured to have horizontal polarization on a system prepared in state $|R,+45\rangle$

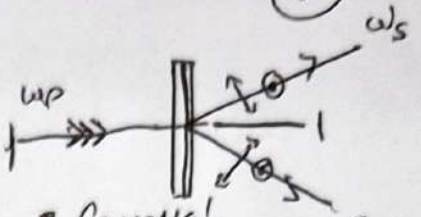
Ans: $p(H_i) = \langle\hat{P}_{H_i}\rangle = \langle R,+45|\,|H\rangle_i\langle H|\,|R,+45\rangle = {}_s\langle R|R\rangle_s\ {}_i\langle +45|H\rangle_i\ {}_i\langle H|+45\rangle_i$
$= (1)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$ so: $\boxed{P(H_i) = \frac{1}{2}}$

d) Calculate the probability that the signal photon will be measured to have vertical polarization, given that the the idler photon is measured to be horizontally polarized, for a system prepared in $|R,+45\rangle$.

Ans: $p(V_s|H_i) = \frac{p(V_s,H_i)}{p(H_i)} = \frac{1/4}{1/2} = \frac{1}{2}$ with $\langle P_{V_s,H_i}\rangle \equiv P(V_s,H_i) = \frac{1}{4}$
$\langle P_{H_i}\rangle = p(H_i) = \frac{1}{2}.$

② ENTANGLED STATES:

1) $$\boxed{|\phi^+\rangle = \frac{1}{\sqrt{2}}\left[|H,H\rangle + |V,V\rangle\right]} \quad (8.19)$$



2) $$\boxed{|\psi_A, \psi_B\rangle = |\psi_A\rangle_A \otimes |\psi_B\rangle_B} \quad (8.20)$$

w/ ψ_A = State particle A ψ_B = State particle B

■ CRYSTALS!:

- A source that uses 2 crystals whose Axes is Rotated 90° w.r.t each other.
- Pump is polarized at 45°

3) For 2 photons prepared in state $|\phi^+\rangle$ of Eqs (8.19), Determine the probabilities of obtaining: a.) The signal photon is polarized Horizontally
b) The signal photon is measured Horizontally polarized given idler photon

Ans: a) $P(H_s)$ = Probability of signal Photon polarized.

so: $$P(H_s) = \frac{1}{\sqrt{2}}\left[\langle H,H| + \langle V,V|\right]\left[|H\rangle_s{}_s\langle H|\right]\frac{1}{\sqrt{2}}\left[|H,H\rangle + |V,V\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[\langle H,H|H\rangle_s{}_s\langle H| + \langle V,V|H\rangle_s{}_s\langle H|\right]\frac{1}{\sqrt{2}}\left[|H,H\rangle + |V,V\rangle\right]$$

$$= \frac{1}{2}\left[\langle H,H|H,H\rangle|H\rangle_s{}_s\langle H| + \langle H,H|H,H\rangle|V,V\rangle + \langle V,V|V,V\rangle|H\rangle_s{}_s\langle H| + \langle V,V|V,V\rangle|H,H\rangle\right]$$

$$= \frac{1}{2}\left[{}_s\langle H|H\rangle_s\,{}_i\langle H| + {}_s\langle V|H\rangle_s\,{}_i\langle V|\right]\left[{}_s\langle H|H\rangle_s|H\rangle_i + {}_s\langle H|V\rangle_s|V\rangle_i\right]$$

$$= \frac{1}{2}\,{}_i\langle H|H\rangle_i = \boxed{\frac{1}{2}} \quad \text{so: } \boxed{P(H_s) = \frac{1}{2}}$$

b): We proceed to use $P(H_s|H_i) = \frac{P(H_s, H_i)}{P(H_i)}$ such that $P(H_i) = \frac{1}{2}$ so:

$$P(H_s, H_i) = \langle \hat{P}_{H_s,H_i}\rangle = \frac{1}{\sqrt{2}}\left[\langle H,H| + \langle V,V|\right]\left[|H,H\rangle\langle H,H|\right]\frac{1}{\sqrt{2}}\left[|H,H\rangle + |V,V\rangle\right]$$

$$= \frac{1}{2}\left[\langle H,H|H,H\rangle\langle H,H| + \langle V,V|H,H\rangle\langle H,H|\right]\left[|H,H\rangle + |V,V\rangle\right]$$

$$= \frac{1}{2}\left[\langle H,H|H,H\rangle\langle H,H|H,H\rangle + \langle H,H|H,H\rangle\langle H,H|V,V\rangle + \langle V,V|H,H\rangle\langle H,H|H,H\rangle + \langle V,V|H,H\rangle\langle H,H|V,V\rangle\right] = \frac{1}{2}\left[\langle H,H\rangle + \langle V,V|H,H\rangle\right]\left[\langle H,H|H,H\rangle + \langle H,H|\right]$$

$$= \frac{1}{2}\left[1+0\right]\left[1+0\right] = \frac{1}{2}[1][1] = \frac{1}{2}[1] = \boxed{\frac{1}{2}}$$

$$\text{so: } P(H_s|H_i) = \frac{P(H_s,H_i)}{P(H_i)} = \frac{1/2}{1/2} = \frac{1}{2}\cdot\frac{2}{1} = \frac{2}{2} = \boxed{1} \quad \text{so: } \boxed{P(H_s|H_i) = 1}$$

c). Interpretation:
- The signal photon is Measured to have Horizontal Polarization at Half Pr
- The Results of the Idler are the Same: It

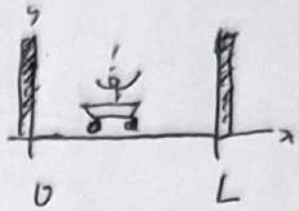
INFINITE SQUARE WELL (11.5)

QM = Discrete Energy Levels
Classical = Continuum of Energies.

(1) INFINITE SQUARE WELL.

[1] THEORY

1) $$\boxed{V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{elsewhere} \end{cases}} \quad (11.54)$$



2a) $A + B = 0$

$$\boxed{B = -A} \quad (11.62)$$

$\Rightarrow$ Boundary conditions require the wave function to be continuous on the edges of well

s.t. $Ae^{ikx} + Be^{-ikx} = \psi(x)$

$\psi(0) = 0$

$Ae^0 + Be^0 = A + B = 0$

$\boxed{A = -B}$ ✓

To be precise:

$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & \text{if } 0 < x < L \\ 0 & \text{elsewhere} \end{cases}$$

3a) $\psi(x) = Ae^{ikx} + Be^{-ikx}$ | $B = -A$

a.i) $= Ae^{ikx} + (-Ae^{-ikx}) = Ae^{ikx} - Ae^{-ikx} = A(e^{ikx} - e^{-ikx})$

so $\boxed{\psi(x) = A(e^{ikx} - e^{-ikx})}$ (11.63)

b.i) Recall: $e^{ikx} = \cos(kx) + i\sin(kx)$
$e^{-ikx} = \cos(kx) - i\sin(kx)$ −

c.i) $e^{ikx} - e^{-ikx} = 0 + i\sin(kx) - (-i\sin(kx)) = i\sin kx + i\sin kx$
$= 2i\sin(kx)$

c.ii) so: $\psi(x) = A2i\sin(kx) = C\sin(kx)$ so $\boxed{\psi = C\sin(kx)}$

where $\boxed{C = 2iA}$

But:
$\boxed{\psi(L) = 0 \text{ and } C \neq 0 \text{ so } \sin(kL) = 0!}$

4) Boundary conditions:

1. $n \in \mathbb{Z}^+$ where $n > 0$.

a.i) $0 < x < L$ so: $\psi(0) < \psi(x) < \psi(L)$

a.ii) s.t. $\psi(0) = 0$ so: $0 < \psi(x) < \psi(L)$

b.i) $\psi(L) = C\sin(kx) = C\sin(kL)$. since $k = \frac{n\pi}{L}$

then $\boxed{kL = n\pi}$

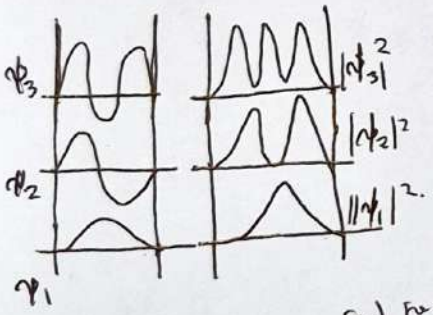
c.i) $\boxed{\psi(L) = C\sin(kL) = C\sin(n\pi)}$

so: $\boxed{k_n = \frac{n\pi}{L}}$

5.i) Recall that: $E = \frac{\hbar^2}{2m}k^2$ so: $E_n = \frac{\hbar^2(n\pi)^2}{2m(L)^2}$. where $p = \hbar k$, $p^2 = \hbar^2 k^2$

and $T = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$

c.ii) $\boxed{E_n = \frac{\hbar^2 n^2 \pi^2}{2mL^2}}$



6a). Normalization: $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$

c.i) so: $\int_{-\infty}^{\infty} |\psi|^2 dx = C_n^2 \int_0^L \sin^2\left(\frac{n\pi}{L}x\right) dx$

Let $u = \sin\left(\frac{n\pi}{L}x\right)$ $\quad x = \frac{n\pi}{L}x$

$du = \frac{n\pi}{L}\cos\left(\frac{n\pi}{L}x\right) dx$

$dx = \frac{L}{n\pi}\frac{du}{u}$

$\frac{\partial \sin^2 x}{\partial x} = \frac{\partial}{\partial x}\sin x \sin x$

$\int \sin x \sin x \, dx = \frac{\partial \sin x}{\partial x}\sin x + \frac{\partial \sin x}{\partial x}\sin x$

$= \cos x \sin x + \cos x \sin x$

$= 2\cos x \sin x$

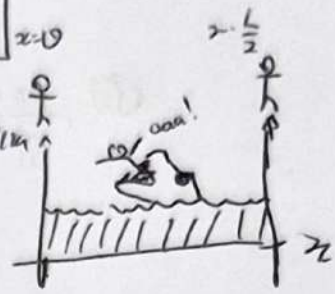
$\sin x \cos x - \cos x \cos x$
$\sin x \cos x - \cos x \sin x$

$$C_n^2 \int_0^L \left[\frac{1 - \cos\frac{2x n\pi}{L}}{2}\right] dx$$

$$= C_n^2 \left[\frac{x}{2} - \frac{\sin\left(\frac{2n\pi x}{L}\right)}{4n\pi/L}\right]_{x=0}^{L} = C_n^2 \frac{L}{2}$$

b) Since $C_n^2 \frac{L}{2} = 1$ then: $\boxed{C_n = \sqrt{\frac{2}{L}}}$

c) So: $\boxed{\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) & 0 < x < L \\ 0 & \text{elsewhere} \end{cases}}$



[iv] Example: The ILLEGAL wave function

1) $\psi(x,0) = \begin{cases} \sqrt{\frac{2}{L}} & 0 < x < \frac{L}{2} \\ 0 & \text{else} \end{cases}$

So:

$\cos\left(\frac{n\pi}{2}\right) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$

So:

$\cos\left(\frac{n\pi}{2}\right) = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{n\pi}{2}\right)^k}{(2k)!}$

$= \sum_{k=0}^{\infty} (-1)^k \frac{n^k \pi^k}{(2k)!\, 2^k}$

2) $C_n = \int_0^{\infty} \psi_n^*(x) \Psi(x,0)\, dx$

$= \frac{2}{L} \int_0^{\frac{L}{2}} \sin\left(\frac{n\pi x}{L}\right) dx = \frac{-2}{n\pi} \cos\left(\frac{n\pi x}{L}\right)\Big|_0^{\frac{L}{2}}$

$= -\frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right]$

3) So: $\boxed{C_n = \begin{cases} 2/n\pi & n \in (2k+1) \text{ odd} \\ 4/n\pi & n = 2, 6, 10 \\ 0 & n = 4, 8, 12 \end{cases}}$

4) The wave functions:

Energy → A) $\Psi(x,t) = \sum_n C_n \psi_n(x) e^{-i\omega_n t} = \frac{2}{\pi}\sqrt{\frac{2}{L}} \sum_{n=1,3,5} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right) e^{i\left(\frac{n^2\pi^2\hbar}{2mL^2}\right)}$

$+ \frac{4}{\pi}\sqrt{\frac{2}{L}} \sum_{2,6,10} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right) e^{-i\frac{n^2\pi^2\hbar}{2mL^2}}$

$(E_n t) = |C_n|^2$

$\begin{cases} \frac{4}{(n\pi)^2} & \text{odd} \\ \frac{16}{(n\pi)^2} & 2,6,10 \\ 0 & 4,8,12 \end{cases}$

Probability → B) $\langle \psi_n | \psi(t) \rangle = \sum_n C_n e^{-i\omega_n t} \langle \psi_n | \psi_n \rangle = \sum_n C_n e^{-i\omega_n t} \delta_{nn}$

$= C_n e^{-i\omega_n t}$

$\therefore$ So: ENERGY: $\Psi(x,t) = \frac{2}{\pi}\sqrt{\frac{2}{L}} \sum_{1,3,5} \frac{1}{n} \sin(n) e^{iE_n} + \frac{4}{\pi}\sqrt{\frac{2}{L}} \sum_{2,6,10} \frac{1}{n} \sin(n) e^{-iE_n}$

Probability: $\langle \psi_n | \psi(t) \rangle = C_n e^{-i\omega_n t}$

② TIME EVOLUTION

[i] PREFACE –

Problem #4

(4.) Use separation of variables to split PDE into 2 ODEs?

Ans: 1) $\Psi = \psi(x)\,\phi(t)$

$\nabla^2\Psi = \frac{\partial^2\psi}{\partial x^2}\phi(t) = \psi''\phi$ $\qquad \frac{\partial\Psi}{\partial t} = \psi(x)\frac{\partial\phi}{\partial t} = \psi\dot{\phi}$

So: $\boxed{-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}}$

$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}\phi + V\psi\phi = i\hbar\psi\dot{\phi}$

2.) $-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}\phi = i\hbar\psi\frac{\partial\phi}{\partial t} - V\psi\phi = i\hbar\psi\left[\frac{\partial\phi}{\partial t} - V\phi\right]$

$-\frac{\hbar^2}{2m}\nabla^2\psi - i\hbar\frac{\partial\phi}{\partial t}\frac{1}{\phi} + V = 0$

$\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} - \frac{i\hbar\partial\phi}{\partial t}\frac{1}{\phi} + V = 0$ $\Rightarrow -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}\frac{\phi}{\psi} = i\hbar\frac{\partial\phi}{\partial t} - V\phi$ Let

$\rightarrow -\frac{\hbar^2}{m}\frac{\partial^2\psi}{\partial x^2} = i\hbar\frac{\partial\phi}{\partial t}\frac{1}{\phi} - V$

3.) Let $\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}\frac{1}{\psi} = -a$ and $i\hbar\frac{1}{\phi}\frac{\partial\phi}{\partial t} = a$ so $a + a = 0$.

w: a.) $\boxed{i\hbar\frac{\partial\phi}{\partial t}\frac{1}{\phi} - V = -a}$ $\rightarrow i\hbar\frac{\partial\phi}{\partial t}\frac{1}{\phi} = a + V$ $\qquad \boxed{E = a - V}$

$\rightarrow \frac{\partial\phi}{\partial t}\frac{1}{\phi} = \frac{-a+V}{i\hbar} \rightarrow \int\frac{d\phi}{\phi} = \int\frac{-a+V}{i\hbar}\,dt$

$\therefore \phi(t) = e^{\frac{-Et}{i\hbar}} = e^{\frac{-iEt}{\hbar}}$

$(-1)\cdot\frac{\hbar^2}{2m})^{\frac{1}{2}}$

$(-1)^{\frac{1}{2}}\cdot\left(\frac{\hbar^2}{2m}\right)$

$(2m)^{1}(2m)^{-\frac{1}{2}}$

$(2m)^{1-\frac{1}{2}}$

$(2m)^{\frac{2}{2}-\frac{1}{2}}$

$= 2m^{\frac{1}{2}}$

$= \sqrt{2m}$

b.) $\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}\frac{1}{\psi} = +a} \rightarrow \frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + a\psi = 0$

Solve Quadratic Check: $\frac{-b^2 \pm \sqrt{b^2 - 4ac}}{2a} = \frac{0 \pm \sqrt{0 - \frac{\hbar^2 a}{2m}}}{2\frac{\hbar^2}{2m}} = \frac{\sqrt{-\frac{\hbar^2}{2m}a}}{2\frac{\hbar^2}{2m}}$

$= i\sqrt{\frac{\hbar^2}{2m}\left(\frac{2m}{\hbar^2}\right)} = i\frac{\hbar}{\sqrt{2m}}\frac{2m}{\hbar^2} = i\frac{2m}{\sqrt{2m}}\frac{1}{\hbar} = \frac{i\sqrt{2m}}{\hbar}$

$T(t) = \phi(t)$

$i\hbar\psi\frac{\partial T}{\partial t} = -\frac{\hbar^2}{2m}T\frac{\partial^2\psi}{\partial x^2} + V\psi T$

$i\hbar\psi\frac{\partial T}{\partial t}\frac{1}{T} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi$

$\frac{\partial T}{\partial t} = \frac{iE}{\hbar}T(t)$

$T(t) = e^{\frac{iE}{\hbar}T(t)}$

$i\hbar\frac{\partial\psi}{\partial t} = \frac{i\hbar}{i\hbar}E\psi(x)$

$= E\psi(x)$

So: $\psi(x) = e^{i\sqrt{\frac{2m}{\hbar}}x} = \cos\left(\sqrt{\frac{2m}{\hbar}}x\right) + i\sin\left(\sqrt{\frac{2m}{\hbar}}x\right) = \boxed{e^{ikx}}$ ✓

3.) So: $\boxed{\Psi(x,t) = e^{i(kx - Et)}}$ Theoretical Solution!

4.) The PDE $\rightarrow$ SODE is:

$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi(x)}$

(15) Varying (Kompl).

Ans: 1) $v_1|A_1|^2 = v_1\left|\frac{(k_1+k_2)}{(k_1-k_2)}B_1\right|^2 = v_1\frac{(k_1+k_2)^2}{(k_1-k_2)^2}B_1^2$

$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$

$k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$

$(k_1+k_2)^2 = k_1^2 + 2k_1k_2 + k_2^2$

$= \left(\sqrt{\frac{2mE}{\hbar^2}}\right)^2 + 2\sqrt{\frac{2mE}{\hbar^2}}\sqrt{\frac{2mE-V_0}{\hbar^2}} + \left(\sqrt{\frac{2m(E-V_0)}{\hbar^2}}\right)^2$

$= \frac{2mE}{\hbar^2} + \frac{2m(E-V_0)}{\hbar^2} + 2\left(\frac{2mE}{\hbar^2}\right)^{\frac{1}{2}}\left(\frac{2mE-V_0}{\hbar^2}\right)^{\frac{1}{2}}$

$= \frac{2m}{\hbar^2}\left(E+(E-V_0)\right) + 2\left(\frac{2mE}{\hbar^2}\cdot\frac{2m(E-V_0)}{\hbar^2}\right)^{\frac{1}{2}}$

$= \frac{2m\left(\sqrt{E}+\sqrt{E-V_0}\right)^2}{\hbar^2} + 2\left(\frac{2m}{\hbar^2}\sqrt{E(E-V_0)}\right) = k + 2k'$

$bxy^{\frac{1}{2}} = x^{\frac{1}{2}}y^{\frac{1}{2}}$

$(16\cdot 4)^{\frac{1}{2}} = 16^{\frac{1}{2}}4^{\frac{1}{2}}$

$(64)^{\frac{1}{2}} = 4\cdot 2$

$64 = 4^2\cdot 2^2$

$64 = 16\cdot 4$

$64 = 64$ ✓

hv: $k_1^2 + k_2^2 = k$

$k_1k_2 = k'$

It follows also that:

$(k_1-k_2)^2$

$k_1^2 - 2k_1k_2 + k_2^2$

$(k_1+k_2)^2 - 2k_1k_2$

$\frac{2m}{\hbar^2}\left\{\left(\sqrt{E}+\sqrt{E-V_0}\right)^2 - \sqrt{E(E-V_0)}\right)$

2.) $\frac{2m}{\hbar^2}\sqrt{E(E-V_0)}$ or $\frac{4mE-2mV_0}{\hbar^2}$

or: $\frac{k_1k_2}{k_1^2+k_2^2} = \frac{k'}{k} = \frac{2m/\hbar^2\sqrt{E(E-V_0)}}{2m/\hbar^2\left(\sqrt{E}+\sqrt{E-V_0}\right)^2} = \boxed{\frac{\sqrt{E(E-V_0)}}{\left(\sqrt{E}+\sqrt{E-V_0}\right)^2}}$

3. hv: $T = 1 - R = 4\left(\frac{2m/\hbar^2\sqrt{E(E-V_0)}}{2m/\hbar^2\left(\sqrt{E}+\sqrt{E-V_0}\right)^2}\right) = \boxed{\frac{4\sqrt{(E-V_0)E}}{\left(\sqrt{E}+\sqrt{E-V_0}\right)^2}}$ ✓

Problems N.

(13) Infinite square well potential:

$$\Psi(x,0) = \begin{cases} \sqrt{\frac{2}{3L}} \sin\left(\frac{2\pi}{L}x\right) + i\sqrt{\frac{4}{3L}} \sin\left(\frac{3\pi}{L}x\right) & 0 < x < L \\ 0 & \text{else} \end{cases}$$

(a) Determine $P(E_n, t)$?

Ans: $P(E_n, t) = |\langle \psi_n | \Psi(x,t) \rangle|^2 = \sum_m c_m e^{-i\omega_m t} \langle \psi_n | \psi_m \rangle$

1-a) $\langle \psi_n^* | \Psi(x,0) \rangle = \int_a^b \psi_n^*(x)\, \Psi(x,0)\, dx$

$$c_n = \int_a^b \left\{ \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \right\} \cdot \left\{ \sqrt{\frac{2}{3L}} \sin\left(\frac{2\pi}{L}x\right) + i\sqrt{\frac{4}{3L}} \sin\left(\frac{3\pi}{L}x\right) \right\} dx.$$

$$= \sqrt{\frac{2}{L}}\left(\sqrt{\frac{2}{3L}} + i\sqrt{\frac{4}{3L}}\right) \int_a^b \left[\sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{2\pi}{L}x\right) + i\sqrt{\frac{4}{3L}} \sin\left(\frac{3\pi}{L}x\right) \sin\left(\frac{n\pi L}{x}\right) \right] dx$$

$$= \int_0^L \frac{4}{3L^2} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{2\pi}{L}x\right) + i\frac{8}{3L^2} \sin\left(\frac{3\pi}{L}x\right) \sin\left(\frac{n\pi L}{x}\right) \Big\} dx -$$

제 하 Let $\alpha = \frac{n\pi x}{L}$, $\beta_1 = \frac{2\pi}{L}x$, $\beta_2 = \frac{3\pi}{L}x$.

$\bar{R} = \frac{4}{3L^2}$

$\bar{C} = i\frac{8}{3L^2}$

by eq: $\sin\alpha \sin\beta_1 = \frac{1}{2}[\cos(\alpha-\beta_1) - \cos(\alpha+\beta_1)] = \frac{1}{2}\cos\left(\frac{n\pi x}{L} - \frac{2\pi x}{L}\right) - \cos\left(\frac{n\pi}{L} + \frac{2\pi x}{L}\right)$

$\sin\alpha \sin\beta_2 = \frac{1}{2}[\cos(\alpha-\beta_2) - \cos(\alpha+\beta_2)] = \frac{1}{2}\cos\left(\frac{n\pi x}{L} - \frac{3\pi x}{L}\right) - \cos\left(\frac{n\pi x}{L} + \frac{3\pi x}{L}\right)$

So: $$\int_0^L \bar{R}\left[\frac{1}{2}\left[\cos\left(\frac{n\pi x}{L} - \frac{2\pi x}{L}\right) - \cos\left(\frac{n\pi x}{L} + \frac{2\pi x}{L}\right)\right]\right] + \bar{C}\left[\frac{1}{2}\left[\cos\left(\frac{n\pi x}{L} - \frac{3\pi x}{L}\right) - \cos\left(\frac{n\pi x}{L} + \frac{3\pi x}{L}\right)\right]\right] dx.$$

$$= \frac{1}{2}\int_0^L \left[\bar{R}\left[\cos\left(\frac{\pi(n-2)}{L}x\right) - \cos\left(\frac{\pi(n+2)}{L}x\right)\right] + \bar{C}\left[\cos\left(\frac{\pi(n-3)}{L}\right) - \cos\left(\frac{\pi(n+3)}{L}\right)\right]\right] dx.$$

$\frac{d\sin(kx)}{dx}$

$\frac{dy}{du}\frac{du}{dx}$

$\frac{dy}{du}(kx)\frac{d}{dx}(\sin(x))$

$k\cos(kx)$

$\boxed{E}$ Recall: $\int_0^L \cos\left(\frac{k\pi x}{L}\right) dx = \frac{1}{\frac{k\pi}{L}} \sin\left(\frac{k\pi x}{L}\right)\Big|_0^L = \frac{L}{k\pi}\sin\left(\frac{k\pi L}{L}\right) = \frac{L}{k\pi}\sin(k\pi)$

So $$\frac{1}{2}\left[\bar{R}\left\{\frac{L}{\pi(n-2)}\sin(\pi(n-2)) - \frac{L}{\pi(n+2)}\sin(\pi(n+2))\right\} + \bar{C}\left\{\frac{L}{\pi(n-3)}\sin(\pi(n-3)) - \frac{L}{\pi(n-3)}\sin(\pi(n-3))\right\}\right]$$

Define: $\frac{1}{2}\frac{L}{\pi(n-m)}\sin(\pi(n-m)) \Rightarrow \begin{cases} 0 & \text{if } n \neq m \\ \frac{L}{2} & \text{if } n = m \end{cases}$

Let $(3,2) \equiv m$

Conclusion:

$\boxed{C_1 + C_2 + C_3 = 1}$

2) So: $P(E_n, t) = \bar{\bar{R}}\psi_1 + \bar{\bar{R}}\psi_2 = \bar{\bar{R}} + \bar{\bar{C}}$

$C_1 = 0$; $C_2 = \sqrt{\frac{2}{L}} \cdot \sqrt{\frac{2}{3L}} \frac{L}{2} = \sqrt{\frac{1}{3}}$

$C_3 = \sqrt{\frac{2}{L}} \cdot \sqrt{\frac{4}{3L}} \frac{L}{2} = i\sqrt{\frac{2}{3}}$

So: $\begin{cases} P(E_1) = 0 \\ P(E_2) = \sqrt{\frac{1}{3}} \\ P(E_3) = i\sqrt{\frac{2}{3}} \end{cases}$

$\boxed{P(E_2) + P(E_3) = \sqrt{\frac{1}{3}} + i\sqrt{\frac{2}{3}}}$

(b) $\langle E(t)\rangle = |c_2|^2 E_2 + |c_3| E_3$

$$= \frac{1}{3}\cdot\frac{2^2\pi^2\hbar^2}{2mL^2} + \frac{2}{3}\cdot\frac{3^2\pi^2\hbar^2}{2mL^2} = \frac{4}{3}\frac{\pi^2\hbar^2}{2mL^2} + \frac{18}{3}\frac{\pi^2\hbar^2}{2mL^2}$$

$$= \left(\frac{4+18}{3}\right)\frac{\pi^2\hbar^2}{2mL^2} = \frac{22}{3}\frac{\pi^2\hbar^2}{2mL^2} = \frac{11}{3}\frac{\pi^2\hbar^2}{mL^2}$$

$$\therefore \boxed{\langle E\rangle = |c_2|^2 E_2 + |c_3| E_3 = \frac{11}{3}\frac{\pi^2\hbar^2}{mL^2}}$$

(c) $\langle x(t)\rangle = \langle x(0)\rangle = \frac{1}{m}\int \langle \psi(t)|\hat{x}|\psi(t)\rangle\, dx.$

$= \frac{1}{m}\int x|\psi(x,0)|^2\, dx$? for: $\boxed{\langle x\rangle = \frac{1}{m}\int x|\psi|^2 dx}$

we see that $\int |\psi|^2 dx \triangleq \int \phi_n \phi(x,t=0)\, dx = \frac{L}{2}$

so: $\langle x\rangle = \int_0^L x\cos\left(\frac{k\pi x}{L}\right) = \frac{L^2}{k^2\pi^2}\cos(k\pi) - \cos(0)$

$$= \frac{L^2}{(k\pi)^2}\left((-1)^k - 1\right)$$

$$\Rightarrow \boxed{\int_0^L x\cos\left(\frac{k\pi x}{L}\right) = \begin{cases} 0 & \text{even} \\ \frac{-2L^2}{k^2\pi^2} & \text{odd} \end{cases}}$$

but particular $\langle x\rangle = c_1 x_1 + c_2 x_2 = \frac{1}{3}\frac{L}{2} + \frac{2}{3}\frac{L}{2} = \boxed{\frac{L}{2}}$ L.

(d) $\langle p\rangle = \int_0^L \psi^*(-i\hbar\nabla)\psi\, dx$

or $\langle p\rangle = c_2 p_2 + c_3 p_3$ and see that $p = m\frac{dx}{dt}$ so $\langle p\rangle = m\langle \dot{x}\rangle$

But we know that $\langle \dot{x}\rangle = 0$. so: $\boxed{\langle p\rangle = 0}$

Solving the SCHRODINGER EQS directly (12.A)

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi = E\psi(x)} \quad (12.A.1)$$

$$\beta = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}} \quad \beta^2 = \frac{m\omega}{\hbar} \quad (12.A.2).$$

(1) answer: let $\chi = \beta x$ so

1) then, $\frac{d^2\psi(x)}{dx^2} = \frac{d}{dx}\left(\frac{d\psi}{d\chi}\frac{d\chi}{dx}\right) = \frac{d}{dx}\left(\frac{d\psi}{d\chi}\beta\right) = \frac{d^2\psi}{d\chi^2}\frac{d\chi}{dx}\beta = \boxed{\frac{d^2\psi}{d\chi^2}\beta^2}$

see $\beta = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}} \rightarrow \beta^2 = \frac{m\omega}{\hbar}$

2) $T = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = -\frac{\hbar^2}{2m}\frac{m\omega}{\hbar}x^2\frac{d^2\psi}{d\chi^2} = -\frac{\hbar\omega}{2}\frac{d^2\psi}{d\chi^2} = -\frac{\hbar\omega}{2}\frac{d^2\psi}{d\chi^2}$

3) $V = \frac{1}{2}m\omega^2 x^2\psi = \frac{1}{2}m\omega^2\frac{\chi^2}{\beta^2} = \frac{1}{2}m\omega^2\frac{\chi^2}{\beta^2}\psi(\chi).$

see $H = T + V \rightarrow \boxed{-\frac{\hbar\omega}{2}\frac{d^2\psi}{d\chi^2} + \frac{1}{2}m\omega^2\frac{\chi^2}{\beta^2}\psi(\chi) = E\psi(\chi).}$

4) so: $-\frac{\hbar\omega}{2}\frac{d^2\psi(\chi)}{d\chi^2} + \frac{1}{2}m\omega^2\chi^2\frac{1}{\frac{m\omega}{\hbar}}\psi(\chi) \leftrightarrow \frac{1}{2}m\omega^2\chi^2\frac{\hbar}{m\omega} = \frac{1}{2}\chi^2\hbar\omega$

see $\boxed{-\frac{\hbar\omega}{2}\frac{d^2\psi(\chi)}{d\chi^2} + \frac{\hbar\omega^2}{2}\chi^2\psi(\chi) = E\psi(\chi)}$ (12.A.3)

(2) let $\varepsilon = \frac{2E}{\hbar\omega}$. So:

1) $\frac{\hbar\omega}{2}\left[-D_\chi^2\psi(\chi) + \chi^2\psi(\chi)\right] = E\psi(\chi)$

$-D_\chi^2\psi(\chi) + \chi^2\psi(\chi) = \frac{2E}{\hbar\omega}\psi(\chi) = \varepsilon\psi(\chi).$

so: $\boxed{\frac{d^2\psi}{d\chi^2} = (\chi^2 - \varepsilon)\psi(\chi).}$ (12.A.5). (12.A.6)

2) so: $\boxed{\psi(\chi) = Ae^{-\frac{\chi^2}{2}} + Be^{-\frac{\chi^2}{2}}}$

$\psi(\chi) \xrightarrow[\chi\to\infty]{} Ae^{-\frac{\chi^2}{2}}$ so: $\psi(\chi) = h(\chi)e^{-\frac{\chi^2}{2}}$. $\left(\frac{dh(\chi)}{d\chi}\chi - h(\chi)\right)e^{-\chi^2}$

check note

$\frac{d}{d\chi}\left(\chi(\chi)e^{-\frac{\chi^2}{2}}\right)$

$\frac{d^2\chi}{d\chi}e^{-\chi^2} - \chi\chi e^{-}$

3) $\frac{d\psi(\chi)}{d\chi} = \frac{dh(\chi)}{d\chi}e^{-\frac{\chi^2}{2}} + h(\chi)(-\chi)e^{-\frac{\chi^2}{2}}$

$= \left(\frac{dh(\chi)}{d\chi} - \chi h(\chi)\right)e^{-\frac{\chi^2}{2}} = \chi(\chi)e^{-\frac{\chi^2}{2}}$

$\frac{d^2\psi(\chi)}{d\chi^2} = \left(\frac{d^2h(\chi)}{d\chi^2} - \frac{d}{d\chi}(\chi h(\chi))\right)e^{-\frac{\chi^2}{2}} + \left(\frac{dh(\chi)}{d\chi} - h(\chi)\right)\chi e^{-\frac{\chi^2}{2}}$

$= \left(\frac{d^2h(\chi)}{d\chi^2} - \frac{d\chi}{d\chi}h(\chi) - \frac{dh(\chi)}{d\chi}\chi\right)e^{-\frac{\chi^2}{2}} + \left(\frac{dh(\chi)}{d\chi} - h(\chi)\right)\chi e^{-\frac{\chi^2}{2}}$

$= \left(\frac{d^2h(\chi)}{d\chi^2} - h(\chi) - 2\chi\frac{dh(\chi)}{d\chi}\right)e^{-\frac{\chi^2}{2}}$

Lab

$e^{\frac{U}{2}} = e^{-\frac{\chi^2}{2}}$

$\frac{\partial\psi}{\partial U}\frac{\partial U}{\partial\chi}$ $U = -\chi^2$

$\frac{\partial}{\partial U}\left(e^{\frac{U}{2}}\right)\frac{\partial}{\partial\chi}(-\chi^2)$

$\frac{1}{2}e^{U} - 2\chi$

$e^{U}\chi = e^{-\frac{\chi^2}{2}}\chi$

3.) do: $\frac{d^2\psi(x)}{dx^2} = (x^2-\varepsilon)\psi(x)$

$$\cancel{e^{-\frac{x^2}{2}}}\left(\frac{d^2h(x)}{dx^2} - h(x) - 2x\frac{dh(x)}{dx}\right) = (x^2-\varepsilon)\, h(x)\, \cancel{e^{-\frac{x^2}{2}}}.$$

$$\frac{d^2h(x)}{dx^2} + x^2h(x) - h(x) - 2x\frac{dh(x)}{dx} = (x^2-\varepsilon)\, h(x)$$

$$\frac{d^2h(x)}{dx^2} - 2x\frac{dh(x)}{dx} + h(x)(x-1) = (x^2-\varepsilon)\, h(x)$$

then $\boxed{\frac{d^2h(x)}{dx^2} - 2x\frac{dh(x)}{dx} + (\varepsilon-1)\, h(x) = 0}$ (12.A.10).

③ SERIES SOLUTION: $\boxed{h(x) = \sum_j a_j x^j}$ (12.A.11) then: $\partial_x h = \sum_j j x^{j-1} a_j$ $j=2$.

$\partial_{xx} h = \sum_j j(j-1) x^{j-2} a_j$

1) for: $\frac{d^2h(x)}{dx^2} - 2x\frac{dh(x)}{dx} + (\varepsilon-1)\, h(x) = 0$ we $\partial_x h \to \sum_j (j+1) x^j a_{j+1}$

$\partial_{xx} h \to \sum_j (j+2)(j-1) x^j a_{j+2}$

$$= \sum_j j(j-1) a_j x^{j-2} - 2x\sum_j j a_j x^{j-1} + (\varepsilon-1)\sum_j a_j x^j = 0$$

$$= \sum_{j=0} (j+2)(j+1)\, a_{j+2}\, \cancel{x^j} + \sum_{j=0} \left[-2j + (\varepsilon-1)\right] a_j \cancel{x^j} = 0$$

do: $\sum_{j=0} (j+2)(j+1) a_{j+2} - (2j + \varepsilon - 1) a_j = 0 \rightarrow \therefore a_{j+2}[(j+2)(j+1)] - a_j[-2j+\varepsilon-1]$

2) we: $a_{j+2} = \frac{-a_j[-2j+\varepsilon-1]}{(j+2)(j+1)} = \frac{2j+1-\varepsilon}{(j+2)(j+1)} a_j$

then $\boxed{a_{j+2} = \frac{2j+1-\varepsilon}{(j+2)(j+1)} a_j}$ (12.A.16)

As long as this relationship is satisfied, the series $h(x)$ in (12.A.11), is a solution to the DE of (12.A.1).

3). $\frac{a_{j+2}}{a_j} \to \frac{2}{j}$ as $j \to \infty$, do: $\frac{2}{j} = \frac{2j+1-\varepsilon}{(j+2)(j+1)}$ | $\lim j \to \infty$.

we: $e^{x^2} = \sum_{n=0} \frac{1}{n!}(x^2)^n = \sum_{j=0} b_j x^j$ where $2n = j$

$j = \text{even}$

now $\boxed{\frac{b_{j+2}}{b_j} = \frac{(\frac{j}{2})!}{(\frac{j}{2}+1)!}}$ then $\boxed{\frac{b_j}{b_j} \xrightarrow[j\to\infty]{} \frac{2}{j}}$

④ FINALE: so $h_n(x) = \sum_{j=0}^{n} a_j x^j$

if $\psi_n(x) = h_n(x)\, e^{-\frac{x^2}{2}}$

& ones $\boxed{\varepsilon_n = 2n+1}$ do: $\boxed{E_n = \hbar\omega\left(n+\frac{1}{2}\right)}$

HARMONIC OSCILLATOR

[121] [12.2]

① W.H.Y?

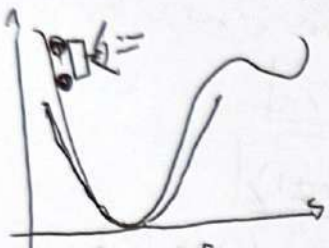
$$\boxed{V(x) = \sum_j \frac{d^k}{dx_0^k}\left(V(x_0)\right)[x-x_0]^k}$$

1) so $V(x) = V(x_0) + V'(x_0)(x-x_0) + \frac{1}{2}V''(x_0)\ldots$

and: $\frac{\partial V}{\partial x} = F(x)$ $\quad \frac{\partial^2 V}{\partial x^2} = \frac{\partial F(x)}{\partial x}$ $\quad$ ie $F(x) = \omega^2 x$

so: $\frac{\partial F}{\partial x} = \omega^2$, $F(x) = \omega^2 x$ so.

$V(x)$



- $\frac{\partial^2 V}{\partial x^2} = \frac{\partial F}{\partial x}$
- $\frac{\partial V}{\partial x} = -F(x)$
- $V = -\int F(x)\,dx$

- $V'' = \omega^2$
- $V' = -\omega^2 x$
- $V = -\frac{1}{2}\omega^2 x^2$

so $V(x) = \sum_k D^k(V(x_0))[x-x_0]^k = -\frac{1}{2}\omega^2 x^2 + \omega^2 x(x-x_0) - \omega^2(x-x_0)$.

$= -\left(\frac{1}{2}\omega^2 x^2 + \omega^2 x^2 + \omega^2 x_0 + \omega^2 x + \omega^2 x_0\right)$ let $x_0 = 0$ so.

so $= -\frac{1}{2}\omega^2 x^2 - \omega^2 x^2 - \omega^2 x$.

$= -\left(\frac{1}{2}\omega^2 x^2 + \frac{2}{2}\omega^2 x^2 + \omega^2 x\right) = -\left(\frac{3}{2}\omega^2 x^2 + \omega^2 x\right)$

$= -\omega^2\left(\frac{3}{2}x^2 + x\right)$ so: $V \sim \left(\frac{3}{2}\omega^2 x^2 - \omega^2 x\right)$.

so) $\boxed{V(x) = \frac{1}{2}k(x-x_0)^2}$ ie $\boxed{V = \frac{1}{2}\omega^2 x^2}$

② CREATION, ANNIHILATION, & NUMBER OPERATOR.

$$\boxed{\hat{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2}$$ Hamiltonian w/ Harmonic oscillator

here if $\hat{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}\left(\frac{p^2}{m} + m\omega^2 x^2\right) = \frac{1}{2}\left(\frac{p^2 + m^2\omega^2 x^2}{m}\right)$

$= \frac{1}{2m}\left(p^2 + m^2\omega^2 x^2\right) = \frac{1}{2}m\omega^2\left[x^2 + \frac{1}{(m\omega)^2}p^2\right]$.

so: $\boxed{\hat{H} = \frac{1}{2}m\omega^2\left[x^2 + \frac{1}{(m\omega)^2}p^2\right]}$

so) $J_\pm = \hat{J}_x \pm i\hat{J}_y$? $\begin{cases} \hat{a} = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}}\left(x + \frac{i}{m\omega}\hat{p}\right) \\ \hat{a}^\dagger = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}}\left(x - \frac{i}{m\omega}\hat{p}\right) \end{cases}$

then we put $\boxed{\hat{n} \equiv a^\dagger a}$

so we proceed: $\hat{n} = \frac{m\omega}{2\hbar}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$.

$= \frac{m\omega}{2\hbar}\left[x^2 + \frac{1}{(m\omega)^2}\hat{p}^2 + \frac{i}{m\omega}(\hat{x}\hat{p} - \hat{p}\hat{x})\right] = \frac{m\omega}{2\hbar}\left[x^2 + \frac{p^2}{(m\omega)^2} + \frac{i[x,p]}{m\omega}\right]$

$= \frac{1}{\hbar\omega}\left[\frac{1}{2}m\omega^2 x^2 + \frac{p^2}{2m}\right] + \frac{i}{2\hbar}[\hat{x},\hat{p}] = \frac{1}{\hbar\omega}\left(\hat{H} - \frac{1}{2}\right)$

ie $H = \hbar\omega\left(\hat{n} + \frac{1}{2}\right) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$

so: $\boxed{\hat{H} = \hbar\omega\left(a^\dagger\hat{a} + \frac{1}{2}\right)}$ and $[\hat{a}, \hat{a}^\dagger] = 1$

3) $[\hat{n}, \hat{a}] = [\hat{a}^\dagger\hat{a}, \hat{a}]$

$= \hat{a}^\dagger[\hat{a}, \hat{a}] + [\hat{a}^\dagger, \hat{a}]\hat{a} = 0 + [\hat{a}^\dagger, \hat{a}]\hat{a} = -(1)\hat{a} = -\hat{a}$

so $\boxed{[\hat{n}, \hat{a}] = -a}$

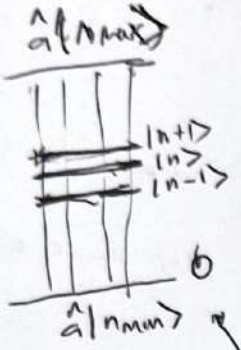
iii) $[\hat{n}, \hat{a}] = \hat{n}\hat{a} - \hat{a}\hat{n} = -\hat{a}$

$$\boxed{\hat{n}\hat{a} = \hat{a}\hat{n} - \hat{a}}$$

$\hat{H}|\psi\rangle = E|\psi\rangle$, $\hat{H}|n\rangle = \hat{E}|n\rangle$ $\Rightarrow$ $\boxed{|n\rangle \equiv |\psi\rangle}$

[1] EIGENVALUES & EIGENVECTORS

• The HARMONIC OSCILLATOR



1) $$\boxed{\begin{aligned} \hat{H}|n\rangle &= \hbar\omega\left(n+\tfrac{1}{2}\right)|n\rangle \\ E_n|n\rangle &= \hbar\omega\left(n+\tfrac{1}{2}\right)|n\rangle \end{aligned}}$$

why/ • $\hat{n}|n\rangle = n|n\rangle$
• $\hbar\omega\left(n+\frac{1}{2}\right) \geq 0$
• $n \geq \frac{1}{2}$..

for: $\boxed{\hat{a}|n\rangle = |\psi\rangle}$ why/ • $\hat{a}$ = annihilation operator
• ψ = particle wave state

2) $\hat{n}|\psi\rangle = \hat{n}\hat{a}|n\rangle = (\hat{a}\hat{n} - \hat{a})|n\rangle = (\hat{a}n - \hat{a})|n\rangle = (n-1)\hat{a}|n\rangle$
$= (n-1)|\psi\rangle$ for: $\boxed{\hat{n}|\psi\rangle = (n-1)|\psi\rangle}$

3) $$\begin{cases} \hat{a}|n\rangle = C_-|n-1\rangle \\ \hat{a}^\dagger|n\rangle = C_+|n+1\rangle \end{cases}$$ for: $\boxed{\hat{a}^\gamma|n\rangle = C_\pm|n\pm1\rangle}$

why/ $\gamma = \begin{cases} \dagger \rightarrow \therefore (C_\pm \rightarrow C_-) \\ 1 \rightarrow \therefore (C_\pm \rightarrow C_+) \end{cases}$

4) $\hat{n}|n_{min}\rangle = n_{min}|n_{min}\rangle$
$a^\dagger \hat{a}|n_{min}\rangle = n_{min}|n_{min}\rangle$
$a^\dagger[0] = n_{min}|n_{min}\rangle$ $\rightarrow$ $\hat{a}|n_{min}\rangle = 0.$
$$\boxed{0 = n_{min}}$$

MINIMUM ENERGY FOR THE UNCERTAINTY PRINCIPLE

$\langle H\rangle = \frac{(\Delta p)^2}{2m} + \frac{1}{2}m\omega^2(\Delta x)^2$
$\rightarrow (\Delta p)^2 \geq \frac{\hbar^2}{4(\Delta x)^2}$
$\langle H\rangle \gtrsim \frac{1}{2m}\frac{\hbar^2}{4(\Delta x)^2} + \frac{1}{2}m\omega^2(\Delta x^2)$

for: $E_n = \hbar\omega\left(n+\frac{1}{2}\right) \in [0, \infty]$. so $\boxed{E_n \in \mathbb{R}^+}$

⚠ CAUTION: $\boxed{E_0 \neq 0}$ since $E_0 = \frac{\hbar\omega}{2}$ is the minimal energy!

for, $\Delta x \Delta p \geq \frac{\hbar}{2}$ $\rightarrow$ $\Delta x \Delta p = 0$ is not true!
since $0 \geq \frac{\hbar}{2}$! $\leftrightarrow$ since $\Delta x \Delta p = 0$ is $E_0 = 0$ (PARTICLE AT REST).

$\boxed{\text{ii}}$

HARMONIC OSCILLATORS - WAVE FUNCS & FOCK STATES

[2.3] [2.4]

3. WAVE FUNCTION

GROUND STATE:

- The ground state of a particle is

$$\boxed{\hat{a}|0\rangle = 0}$$

why $\hat{a}|n\rangle = a|n+1\rangle\sqrt{0}$
$= a|0\rangle\sqrt{0}$
$= 0$.

- The ground state wave function has property: $\boxed{\langle x|\hat{a}|0\rangle = 0}$ $\psi_0(x)$ ★ GROUND STATE WAVE

>> WK/ $\langle x|\hat{a}|0\rangle = \langle x|\frac{\beta}{\sqrt{2}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)|0\rangle = \langle x|0\rangle \frac{\beta}{\sqrt{2}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$

$$\psi_0(x)\frac{\beta}{\sqrt{2}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right) = \frac{\beta}{\sqrt{2}}\left(\psi_0\hat{x} + \frac{i}{m\omega}\hat{p}\psi_0\right) = \frac{\beta}{\sqrt{2}}\left(\psi_0\hat{x} + \frac{i}{m\omega}\left(-i\hbar\nabla\psi_0\right)\right)$$

use EX 12.1

$$= \frac{\beta}{\sqrt{2}}\left(\langle x\rangle_0 + \frac{\hbar}{m\omega}\nabla\psi_0\right) = \frac{\beta}{\sqrt{2}}\left(\langle x\rangle_0 + \frac{\hbar}{m\omega}\left(\frac{\partial x}{\partial x}\frac{\partial\psi}{\partial x} + x\frac{\partial\psi}{\partial x} - \frac{\partial}{\partial x}(x\psi)\right)\right)$$

$$= \frac{\beta}{\sqrt{2}}\left(\langle x\rangle_0 + \frac{\hbar}{2m}\frac{\partial\langle x\rangle}{\partial x}\right) = \frac{\beta}{\sqrt{2}}\left(0 + \frac{\hbar}{2m}\frac{\partial 0}{\partial x}\right) = \frac{\beta}{\sqrt{2}}(0) = \boxed{0}$$

- The differential eqs gives solution: $\boxed{\psi_0 = \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\beta^2 x^2}{2}}}$ GAUSSIAN WAVE. $\Longleftrightarrow$ $\boxed{\frac{\partial\psi_0}{\partial x} = -\beta^2\psi_0(x)x}$

>> WK/: $\frac{\partial\psi_0}{\partial x} = -\beta^2\psi_0 x$

★ $\int\frac{\partial\psi}{\psi} = \int -\beta^2 x\,dx = \ln\psi_0 - \ln A = -\frac{1}{2}\beta^2x^2$

★ $\ln\psi_0 = \ln A - \frac{1}{2}\beta^2x^2$ ★ $\boxed{\psi_0(x) = A e^{-\frac{1}{2}\beta^2x^2}}$

>> Normalization: $\int|\psi|^2dx = A^2\int e^{-\beta^2x^2}dx = A^2\sqrt{\frac{\pi}{\beta^2}} = 1$

★ $A^2 = \sqrt{\frac{\beta^2}{\pi}}$ ★ $A^2 = \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{2}\cdot\frac{1}{2}} = \left(\frac{\beta}{\pi}\right)^{\frac{1}{4}}$

★ ergo: $\psi_0(x) = \left(\frac{\beta}{\pi}\right)^{\frac{1}{4}}\exp\left(-\frac{1}{2}\beta^2x^2\right)$

HERMITE SOLUTION:

- We previously used the fact that: $\boxed{\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle}$ ★ "HARMONIC" SERIES

$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

$\frac{1}{\sqrt{n}}\hat{a}|n\rangle = |n-1\rangle$

- The same technique can be applied: $\boxed{\hat{a}^\dagger|n\rangle \equiv \psi_{n+1}(x)}$ ★ "PROPER" WAVE

>> WK/:

★ $\hat{a}^\dagger|n\rangle = \frac{\beta}{\sqrt{2}}\left(x - \frac{i}{m\omega}\hat{p}\right)$

★ $\frac{\beta}{\sqrt{2}}\left(x - \frac{i}{m\omega}\hat{p}\right)|n\rangle = \frac{\beta}{\sqrt{2}}\left(x|n\rangle - \frac{i}{m\omega}p|n\rangle\right)$

$= \langle x|\frac{\beta}{\sqrt{2}}\left[x|n\rangle - \frac{1}{m\omega}p|n\rangle\right] = \langle x|n+1\rangle\sqrt{n+1}$

★ $\frac{\beta}{\sqrt{2}\sqrt{n+1}}\left(\langle x|x|n\rangle - \frac{1}{m\omega}\langle x|p|n\rangle\right) = \frac{\beta}{\sqrt{2(n+1)}}\langle x|\left(x - \frac{1}{m\omega}p\right)|n\rangle = \frac{\beta}{\sqrt{2n+1}}\left(x\psi_n - \frac{1}{\beta^2}\frac{\partial\psi_n}{\partial x}\right) \equiv \psi_{n+1}(x)$

- For example, $n=1$ is:

$$\boxed{\psi_1(x) = \frac{\beta}{\sqrt{2}}\left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}} 2x e^{-\frac{\beta^2x^2}{2}}}$$ ★ 1st HERMITE POLYNOMIAL

>> WK/ $\frac{\beta}{\sqrt{2}}\left(x\psi_0 - \frac{1}{\beta^2}\frac{\partial\psi}{\partial x}\right)$

$= \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}}\frac{\beta}{\sqrt{2}}\left(xe^{-\beta^2x^2} - \frac{1}{\beta^2}\frac{\partial e^{-\beta^2x^2}}{\partial x}\right)$

$= \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}}\frac{\beta}{\sqrt{2}}\left(xe^{-\frac{\beta^2x^2}{2}} - \frac{1}{\beta^2}\left(\frac{-2x\beta^2}{2}\right)e^{-\frac{\beta^2x^2}{2}}\right)$

$= \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}}\frac{\beta}{\sqrt{2}}2xe^{-\beta x^2}$

- Thus, βx is the Hermite polynomial. General wave func:

$$\boxed{\psi_n(x) = \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(\beta x)e^{-\frac{\beta^2x^2}{2}}}$$

★ HARMONIC OSCILLATOR.

- AND $\boxed{E_n = \hbar\omega\left(n + \frac{1}{2}\right) = \frac{1}{2}m\omega^2x_n^2}$

3) FOCK STATES & PHOTONS:

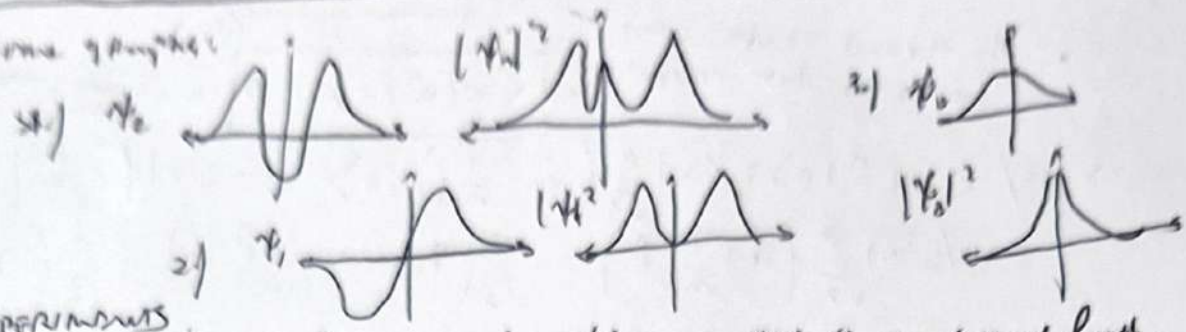
THE DEPARTING WAVE:

$$\boxed{\Psi(x,t) = \psi_n(x)\, e^{-\frac{iE_n t}{\hbar}}}$$

- It's normalizable:

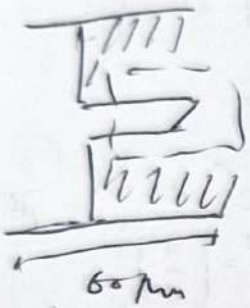
$$|\Psi|^2 = \psi^* e^{-\frac{iE_n t}{\hbar}} \psi^* e^{\frac{iE_n t}{\hbar}} = \psi\psi^* = |\psi_n(x)|^2 \checkmark$$

- Some graphs:



EXPERIMENTS:

- The Harmonic oscillator doesn't oscillate since the $\boxed{\text{HAMILTONIAN-TIME-INDEPENDENT}}$
- The Hamiltonian is in a superposition of states corresp. to $\boxed{\text{2-ADJACENT-ENERGY-LEVELS}}$
- The separation of the energy levels is constant. $\boxed{\Delta E = \hbar\omega}$ as n increases, as $\boxed{\frac{dE}{dn} = \hbar\omega}$
- But if the classical limit applies, large energies & $\boxed{\lim_{n\to\infty} \Delta E_n = \text{CLASS}}$, No matter how large n $n \to \infty$.
- The $\boxed{\text{FOCK STATES}}$ are $\boxed{\text{NON-CLASSICAL}}$ since that $\boxed{E_n = n\hbar\omega}$ the particle grows and grows and can't or cause



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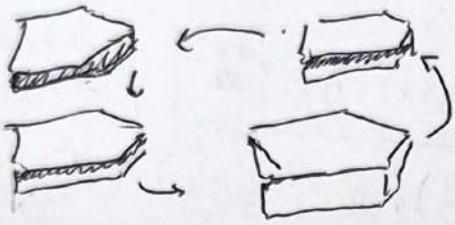


$\omega \sim 6.175$ GHz

$\boxed{\omega = \frac{1}{\tau}}$

(5) COHERENT STATES

SERIES ANALYSIS:

- The Annihilation operator $\hat{a}$:

$$\boxed{\hat{a}|\alpha\rangle = \alpha|\alpha\rangle}$$

- The series is conditioned by the Annihilation operator n:

$$\boxed{C_n = \frac{\alpha^n}{\sqrt{n}} C_0} \qquad \boxed{|\alpha\rangle = C_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle}$$

* COEFFICIENT OF "DROP" * ANNIHILATOR SERIES

>> The conditioning is denoted:

* $|\alpha\rangle\alpha = \hat{a}|\alpha\rangle$
* $\alpha \sum_n C_n \alpha^n = \hat{a} \sum_n C_n \alpha^n$
* $\alpha \sum_n C_n \alpha^n = \hat{a} \sum_{n-1} C_n \alpha^{n-1} = \hat{a} \sum_{n=1} C_{n-1} \alpha^n$
* $\alpha C_n = \hat{a} C_{n-1} \Rightarrow \frac{\alpha}{\hat{a}} = \frac{C_{n-1}}{C_n} \Rightarrow \frac{C_n}{C_{n-1}} = \frac{\hat{a}}{\alpha}$

$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$

$$\boxed{|\alpha\rangle\hat{a} = \sum_{n=1} C_n \sqrt{n} |n-1\rangle}$$

* $\hat{a}|\alpha\rangle = \sum C_{n-1} \alpha^n |n-1\rangle$
* $\sqrt{n} C_n = C_{n-1}\alpha$ ← w/ $\langle\alpha|n-1\rangle = \langle\alpha|\frac{1}{\sqrt{n}}\hat{a}|n\rangle = \alpha\frac{1}{\sqrt{n}}\hat{a}|n\rangle$
* $\boxed{C_n = \frac{\alpha}{\sqrt{n}} C_{n-1}}$

Similar to: $\sigma_T = \frac{\sigma_0}{\sqrt{T}}$ w/ $\frac{\sigma_T\sqrt{T} - \sigma_T}{\sqrt{T}} = \varepsilon_T$ Random walk by [illegible]

* Thus:

$$|\alpha\rangle = C_0 \sum_{n=1}^{\infty} \frac{\alpha^n}{\sqrt{n}} |n\rangle$$

- One of the fundamentals:

$$\boxed{\langle\alpha|\alpha\rangle = C_0^2 e^{|\alpha|^2} = 1}$$ & orthogonality condition

>> w/:

$$\langle\alpha|\alpha\rangle = \left(C_0 \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle\right)\left(C_0 \sum_{n'} \frac{\alpha^{*n}}{\sqrt{n'!}}|n'\rangle\right) = C_0^2 \sum_{nn'} \frac{\alpha^n \alpha^{*n'}}{\sqrt{n!}\sqrt{n'!}}\langle n|n'\rangle$$

$$= C_0^2 \sum_n \frac{\alpha^n \alpha^{*n}}{n!}(1) = C_0^2 \sum_n \frac{(\alpha\alpha^*)^n}{n!} = C_0^2 e^{\alpha^2}$$

- The normalization factor requires:

$$\boxed{|\alpha\rangle = e^{-\frac{\alpha^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n}} |n\rangle}$$ * NORMALIZED ANNIHILATOR

>> w/ $\langle\alpha|\alpha\rangle = C_0^2 e^{\alpha^2} = 1$

* $C_0^2 = e^{-\alpha^2}$ * $C_0^{2\frac{1}{2}} = e^{-\alpha^2 \cdot \frac{1}{2}}$ * $\boxed{C_0 = e^{-\frac{\alpha^2}{2}}}$ and $\{|\alpha=0\rangle = |n=0\rangle\}$

since $|\alpha=0\rangle = |0\rangle = C_0 \sum_n \frac{0^n}{\sqrt{n!}}|n\rangle$

$0 = 0|n\rangle = \boxed{0}$

STATISTICS & EXPECTATIONS:

- Dimensionless operator of space: $\boxed{\hat{x} = \frac{\beta}{\sqrt{2}}\hat{x} = \frac{1}{2}(\hat{a} + a^\dagger)}$ (real)
- Dimensionless operator of momentum: $\boxed{\hat{p} = \frac{1}{\hbar\beta\sqrt{2}}\hat{p} = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)}$ (real)
- The creation & annihilation operators are linear:

$$\boxed{\begin{cases} \hat{a} = (\hat{x} + i\hat{p}) \\ \hat{a}^\dagger = (\hat{x} - i\hat{p}) \end{cases}}$$ Dimensionless annihilation.

$\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$
$\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1$
$\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} = 2\hat{a}^\dagger\hat{a} + 1$

>> MEAN: $\langle\hat{x}\rangle = \langle\alpha|\hat{x}|\alpha\rangle = \frac{1}{2}\langle\alpha|(a + a^\dagger)|\alpha\rangle = \frac{1}{2}(\alpha + \alpha^*)\langle\alpha|\alpha\rangle = Re(\alpha)$

>> STD. DEV: $\langle\hat{x}^2\rangle = \langle\alpha|\hat{x}^2|\alpha\rangle = \frac{1}{4}\langle\alpha|(\hat{a} + \hat{a}^\dagger)(a + \hat{a}^\dagger)|\alpha\rangle = \frac{1}{4}\langle\alpha|\hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}|\alpha\rangle$

$= \frac{1}{4}($

EXERCISE 12

① PROOF $\boxed{[a^\dagger \hat{a}] = 1}$

Ans: $[\hat{a}^\dagger \hat{a}] = \hat{a}^\dagger \hat{a} - \hat{a}^\dagger \hat{a} = \hat{n} - \hat{n}^\dagger = \left(\frac{1}{\hbar\omega}\hat{H} + \frac{1}{2}\right) - \left(\frac{1}{\hbar\omega}\hat{H} + \frac{1}{2}\right)$

$= \frac{1}{\hbar\omega}\hat{H} + \frac{1}{2} - \frac{1}{\hbar\omega}\hat{H} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = \boxed{1}$

② PROOF $\boxed{[\hat{n}, \hat{a}^\dagger] = \hat{a}^\dagger}$

Ans: $[\hat{n}, \hat{a}^\dagger] = \hat{n}\hat{a}^\dagger - \hat{a}^\dagger \hat{n} = \hat{a}\hat{a}^\dagger a^\dagger - \hat{a}^\dagger \hat{a}\hat{a}^\dagger = a^\dagger(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger \hat{a})$

$= \hat{a}^\dagger[\hat{a}, a^\dagger] = \hat{a}^\dagger(1) = \hat{a}^\dagger$

③ PROOF $\boxed{\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle}$

Ans: . Basically if $n = |C_-|^2$. Then:

. So $a^\dagger |n\rangle = C_+ |n+1\rangle$

$= \sqrt{n+1}\,|n+1\rangle$ QED

$n = |C_-|^2$

$n+1 = C_-^2 + 1$

$(n+1)^{\frac{1}{2}} = C_-^{2\frac{1}{2}} + 1^{\frac{1}{2}}$

$\sqrt{n+1} = C_- + 1$

$\sqrt{n+1} = C_+$

$C_- + 1 = C_+$

$C_- - C_+ = 1$

$\sqrt{n} - \sqrt{n+1} = 1$

$\sqrt{n}^2 - \sqrt{n+1}^2 = 1^2$

$n - n + 1 = 1$

$\boxed{1 = 1}$

QED

SCHRÖDINGER 3D – INTERPRETATIONS I (17.1) (17.2)

(1.) The 3D Schrödinger eqs

[1] BASICS:

$$\hat{H} = \frac{-1}{2m}\hat{P}^2 + V(\vec{r}) = \frac{\hat{P}_x^2 + \hat{P}_y^2 + \hat{P}_z^2}{2m} + V(\vec{r})$$

$$H|\phi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\phi(\vec{r})\rangle$$

1) Cor:

$$\boxed{\hat{H}\phi = \left(\frac{P^2(x,y,z)}{2m} + V(x,y,z)\right)\phi = i\hbar\frac{\partial\phi(\vec{r},t)}{\partial t}}$$

2) $\langle\vec{r}|H|\phi(t)\rangle = \langle r|i\hbar\frac{\partial}{\partial t}|\phi(t)\rangle$

$$= \langle\vec{r}|\frac{P^2(x,y,z)}{2m} + V|\phi(t)\rangle = \langle r|i\hbar\frac{\partial}{\partial t}|\phi(t)\rangle$$

$$= \frac{(-i\hbar)^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V(\vec{r})\,|\phi(\vec{r},t)| = i\hbar\frac{\partial\phi(\vec{r},t)}{\partial t}.$$

3) hence:

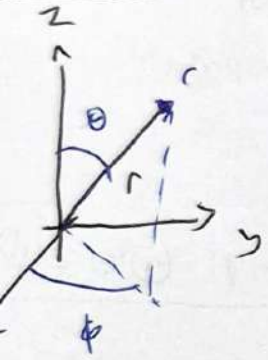
$$\boxed{-\frac{\hbar^2}{2m}\nabla^2\Phi(\vec{r},t) + V\Phi(\vec{r},t) = i\hbar\frac{\partial}{\partial t}\Phi(\vec{r},t)}$$

4) solutions & normalization:

a) $\Phi(\vec{r},t) = \sum_n C_n\phi_n(\vec{r})\cdot e^{-i\omega_n t}$ b) $\iiint_{\forall\phi\,\text{no}}|\phi(\vec{r})|^2 = 1.$

[2] PDE Solutions:

SPHERICAL COORDINATES



$$\phi(\vec{r}) = X(x)\,Y(y)\,Z(z) = XYZ$$

$$\text{w: } -\frac{\hbar^2}{2m}\left[YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2}\right] = E\,XYZ.$$

$$-\frac{\hbar^2}{2m}\left[\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2}\right] = E \rightarrow$$

$$-\frac{\hbar^2}{2m}\frac{1}{X}\frac{d^2X}{dx^2} = E_x$$

$$-\frac{\hbar^2}{2m}\frac{1}{Y}\frac{d^2X}{dy^2} = E_y$$

$$-\frac{\hbar^2}{2m}\frac{1}{Z}\frac{d^2Z}{dz^2} = E_z.$$

$$-\frac{\hbar^2}{2m}\left[E_x + E_y + E_z\right] = E.$$

$$\text{dr: } \boxed{E_{n_x} = \frac{n_x^2\pi^2\hbar^2}{2mL_x^2}} \rightarrow \boxed{X_{n_x}(x) = \begin{cases}\sqrt{\frac{2}{L_x}}\sin\left(\frac{n_x\pi}{L}x\right) & 0 < x < L\\ 0\end{cases}}$$

• Boundary conditions
• Fixed ends

$$\text{Since } E(n_x,n_y,n_z) = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right)$$

$$\text{h.w: } \boxed{\phi_{n_xn_yn_z}(\vec{r}) = \sqrt{\frac{8}{L_xL_yL_z}}\sin\left(\frac{n_x\pi}{L_x}x\right)\sin\left(\frac{n_y\pi}{L_y}y\right)\sin\left(\frac{n_z\pi}{L_z}z\right)}$$

(2.) Central Potentials – Part I

1) Assumptions:

a) $r = \sqrt{x^2+y^2+z^2}$ $\quad \theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$ $\quad \phi = \tan^{-1}\left(\frac{y}{x}\right)$ $\quad \theta = \cos^{-1}\left(\frac{z}{r}\right)$

b)
$$\boxed{\begin{aligned} x &= r\sin\theta\cos\phi\\ y &= r\sin\theta\sin\phi\\ z &= r\cos\theta\end{aligned}}$$

2) $d^3r = r^2\sin\theta\cdot dr\cdot d\theta\, d\phi$ $\quad$ hw: $\int d^3r = \int r^2\sin\theta\,|\phi(\vec{r})|^2\,dr\,d\theta\,d\phi = 1$

3)
$$\boxed{\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}}$$

Make an in Appendix!

hw/: $\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} \rightarrow \nabla_i = \frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y} + \frac{\partial\varphi}{\partial z} = \frac{\partial}{\partial r}$

$$\text{cw: } \frac{\partial\varphi}{\partial x_i} = \frac{\partial\varphi}{\partial r}\frac{\partial r}{\partial x_i} + \frac{\partial\varphi}{\partial\theta}\frac{\partial\theta}{\partial x_i} + \frac{\partial\varphi}{\partial\phi}\frac{\partial\phi}{\partial x_i}$$

i) SEPARATION OF VARIABLES

1) $-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \Rightarrow -\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] + V\psi = E\psi$

Let $\boxed{\psi(r,\theta,\phi) = R(r)\,Y(\theta,\phi)}$ or $RY = \psi$

hence $-\frac{\hbar^2}{2m}\left[\frac{Y(\theta,\phi)}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{R(r)}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{R}{r^2\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] + VRY = ERY.$

2) $-\frac{2mr^2}{\hbar^2 RY}\left(\frac{\hbar^2}{2m}\nabla^2\psi + V\psi\right) = -\frac{2mr^2}{\hbar^2 RY}ERY \Rightarrow +\frac{2mr^2}{\hbar^2 RY}\frac{\hbar^2}{2m}\nabla^2\psi + \frac{V\psi}{RY}\frac{2mr^2}{\hbar^2} = -E\frac{2mr^2}{\hbar^2}$

or: $\boxed{\frac{r^2}{RY}\nabla^2\psi - \frac{2m}{\hbar^2}r^2\left[V(r) - E\right] = 0}$

3) hence: $\frac{r^2}{RY}\left[\frac{Y}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{R}{r^2\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] = \frac{r^2 Y}{RY}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{R}{r^2 RY}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{\partial^2 Y}{\partial\phi^2}\frac{R}{r^2\sin^2\theta\, RY}$

$= \frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{Y}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}$

hence: $\boxed{\frac{1}{R}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)\right] + \frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] - \frac{2m}{\hbar^2}r^2\left[V - E\right] = 0}$

i.e. Let:
$$\begin{cases} \frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2m}{\hbar^2}r^2(V-E) = l(l+1) \rightarrow \text{Radial part} \\ \frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] = -l(l+1) \rightarrow \text{Angular part} \end{cases}$$

hence: $\boxed{l(l+1) - l(l+1) = 0}$

ii) ANGULAR EQS:

1) $\sin\theta\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial Y}{\partial\theta}\right] + l(l+1)\sin^2\theta\, Y + \frac{\partial^2 Y}{\partial\phi^2} = 0$ and $\boxed{Y(\theta,\phi) = \Theta(\theta)\,\Phi(\phi)}$

hence: $\boxed{\frac{\sin\theta}{\Theta(\theta)}\frac{d}{d\theta}\sin\theta\frac{d\Theta}{d\theta} + l(l+1)\sin^2\theta\frac{1}{\Phi(\phi)}\frac{d^2\Phi}{d\phi^2} = 0}$

2) and: $\left(\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m_l^2\right) \wedge \left(\left(\frac{\sin\theta}{\Theta(\theta)}\frac{d}{d\theta}\sin\theta\frac{d\Theta}{d\theta}\right) + l(l+1)\sin^2\theta = m_l^2\right)$

iii) AZIMUTHAL EQS:

1) $\boxed{\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + m_l^2\Phi = 0} \rightarrow \frac{-b \pm\sqrt{b^2-4ac}}{2a} = \frac{0\pm\sqrt{0-4(1)(m_l^2)}}{2} = \frac{\pm\sqrt{-4m_l^2}}{2}$

THEIRCAL CHARACTERISTIC EQUATION

$= \pm i\frac{2}{2}m_l = \boxed{\pm i m_l}$ $\quad \sqrt{-4m_l^2} = \sqrt{-4}\sqrt{m_l^2} = i\sqrt{4}\,m_l = i2m_l$

hence: $\boxed{\Phi(\phi) = e^{im_l\phi}}$

and $\boxed{\Phi(\phi) = \Phi_{m_l}(\phi + 2\pi)}$ w/ $2\pi = 360° = 0°$

hence: $e^{im_l\phi} = e^{im_l(\phi+2\pi)} = e^{im_l\phi}e^{im_l 2\pi}$ and $e^{im_l 2\pi} = 1$.

2) so the orthonormality:

$$\boxed{\int_0^{2\pi}\Phi^*_{m_l}(\phi)\,\Phi_{m_l'}(\phi) = 2\pi\delta_{m_l m_l'}}$$

(2) CENTRAL POTENTIALS - PART II

[i] POLAR:

1) $\sin\theta \frac{\partial}{\partial\theta}\left[\sin\theta \frac{\partial \Theta}{\partial\theta}\right] + \left[l(l+1)\sin^2\theta - m_l^2\right]\Theta = 0$

$$\boxed{\Theta_l^{m_l}(\theta) = P_l^{m_l}(\cos\theta)}$$ in which $\forall \theta \leq \theta \leq \pi$, $l \in \mathbb{Z}^+$

$P_l^{m_l}(x) = (1-x)^{\frac{m}{2}}(D_x)^{m_l} P_l(x)$ → Rodrigues

$P_l(x) = \frac{1}{2^l l!}(D_x)^l (x^2-1)^l$ → Legendre.

2). $P_l^{m_l}(x) = (1-x)^{\frac{m}{2}}(D_x)^{m_l} P_l(x) = (1-x)^{\frac{m}{2}}(D_x)^{m_l}\left[\frac{1}{2^l l!}(D_x)^l (x^2-1)^l\right]$

$= \frac{(1-x^2)^{\frac{m}{2}}}{2^l l!}(D_x)^{m_l}(D_x)^l (x^2-1)^l = \frac{1}{2^l l!}(1-x^2)^{\frac{m_l}{2}}(D_x)^{l+m_l}(x^2-1)^l$

$\therefore$ $$\boxed{P_l^{m_l}(x) = \frac{1}{2^l l!}(1-x^2)^{\frac{m_l}{2}}(D_x)^{l+m_l}(x^2-1)^l}$$

$\therefore$ It follows for negative value m_l as long as $(l+m_l) \geq 0$ (well defined!)

By which: $$\boxed{P_l^{-m_l}(x) = (-1)^{m_l}\frac{(l-m_l)!}{(l+m_l)!}P_l^{m_l}(x)}$$

2) It's also orthogonal!

$$\int_{-1}^{1} P_l^{m}(\cos\theta)P_{l'}^{m}(x) = \int_0^{\pi} P_l^{m_l}[\cos\theta]\sin\theta\, P_{l'}^{m_l}[\cos\theta] = \frac{2}{2l+1}\frac{(l+m_l)!}{(l-m_l)!}\delta_{ll'}$$

$\therefore$ $$\boxed{\langle P_l^{m_l}(x) | P_l^{m}(x)\rangle = \frac{2}{2l+1}\frac{(l+m_l)!}{(l-m_l)!}\delta_{ll'}}$$

[ii] SPHERICAL HARMONICS:

$$\boxed{Y_l^{m_l}(\theta,\phi) = (-1)^{m_l}\left[\frac{(2l+1)(l-m_l)!}{4\pi(l+m_l)!}\right]^{\frac{1}{2}} P_l^{m_l}[\cos\theta]\, e^{im_l\phi}}$$

and so: $\psi(r,\theta,\phi) = R(r)Y(\theta,\phi) \equiv R(r)Y_l^{m_l}(\theta,\phi)$

$= R(r)(-1)^{m_l}\left[\frac{(2l+1)(l-m_l)!}{4\pi(l+m_l)!}\right]^{\frac{1}{2}} P_l^{m}[\cos\theta]\, e^{im_l\phi}$

$\therefore$ $$\boxed{\psi(r,\theta,\phi) = R(r)(-1)^{m_l}\left[\frac{(2l+1)}{4\pi}\frac{(l-m_l)!}{(l+m_l)!}\right]^{\frac{1}{2}} P_l^{m}[\cos\theta]\, e^{im_l\phi}}$$

2) $Y_l^{-m_l} = (-1)^m Y_l^{m_l *}(\theta,\phi)$.

$\therefore$ $\int_0^{2\pi}\int_0^{\pi} Y_l^{m_l *}(\theta,\phi)\sin\theta\, Y_l^{m_l}(\theta,\phi) = \delta_{ll}\delta_{mm}$. $\therefore$ $$\boxed{\langle Y_l^{m_l} | Y_l^{m_l}\rangle = \delta_{ll}\delta_{mm}}$$

4) Common ones:

$Y_0^0 = \frac{1}{\sqrt{4\pi}}$ $\quad Y_l^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$ $\quad Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$ $\quad Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$

PROBLEMS 13

(4) Solve the harmonic oscillator in a 3D schrödinger!

Ans: Let $r^2 = x^2+y^2+z^2$ and $\psi(r) = R(r)$

Then: $\boxed{-\frac{\hbar^2}{2m}\nabla^2 R - \frac{1}{2}m\omega^2 r^2 = E}$ or $\boxed{\frac{\hbar^2}{2m}\nabla^2 R + \frac{1}{2}m\omega^2 r^2 - E = 0}$

1) Recall: $\nabla^2 R \equiv \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{d^2R}{dr^2}\frac{r^2}{r^2} + \frac{1}{r^2}\frac{dr^2}{dr}\frac{dR}{dr} = \frac{d^2R}{dr^2} + \frac{2r}{r^2}\frac{dR}{dr}$

$= \frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}$

2) So: $\frac{\hbar^2}{2m}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right) + \frac{1}{2}m\omega^2 r^2 R - ER = 0$

$\frac{2m}{\hbar^2}\frac{\hbar^2}{2m}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right) + \frac{1}{2}\frac{m\omega^2 r^2 2m}{\hbar^2}R - \frac{2mE}{\hbar^2}R = 0$

$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \frac{m^2\omega^2 r^2}{\hbar^2}R - \frac{2mE}{\hbar^2}R = 0$ Let: $\frac{m\omega}{\hbar^2} = \alpha$ so $\frac{m^2\omega^2}{\hbar^2} = \alpha^2$, $\frac{2mE}{\hbar^2} = \varepsilon$

$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \alpha^2 r^2 R - \varepsilon R = 0$

$\boxed{\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[\alpha r^2 - \varepsilon\right]R = 0}$ GAUSSIAN DE.

GAUSSIAN APPROXIMATION

as $r \to \infty$: $\frac{d^2R}{dr^2} + [\alpha r^2 - \varepsilon]R = 0$

3.) Let $R(r) = e^{-\frac{\alpha r^2}{2}}$ Then $R(r) = e^{-\frac{\alpha r^2}{2}}u(r)$

so $\frac{dR}{dr} = -\alpha r\, u(r)\, e^{-\frac{\alpha r^2}{2}} + \frac{du}{dr}e^{-\frac{\alpha r^2}{2}} = \left(-\alpha r u(r) + \frac{du}{dr}\right)e^{-\frac{\alpha r^2}{2}}$

$\frac{d^2R}{dr^2} = \left(-\alpha u(r) + \alpha r\frac{du}{dr} + \frac{d^2u}{dr^2}\right)e^{-\frac{\alpha r^2}{2}} + -\alpha r\left(-\alpha r u + \frac{du}{dr}\right)e^{-\frac{\alpha r^2}{2}}$

$= \left(-\alpha u + \alpha r\frac{du}{dr} - \alpha^2 r^2 u + \frac{du}{dr}\alpha r + \frac{d^2u}{dr^2}\right)e^{-\frac{\alpha r^2}{2}}$

$= \left(\frac{d^2u}{dr^2} + 2\alpha r\frac{du}{dr} + (\alpha^2 r^2 - \alpha)u\right)e^{-\frac{\alpha r^2}{2}}$

So: $\boxed{\frac{d^2u}{dr^2} + \left(\frac{2}{r} - 2\alpha r\right)\frac{du}{dr} + (\varepsilon - 3\alpha - \alpha^2 r^2)u = 0}$

4) Sol: Let $\sum_{n=0}^{\infty} a_n r^n = u(r)$

Then: $\sum_n (n-1)n\, a_{n-1} r^{n-2} + \left(\frac{2}{r} - 2\alpha r\right)\sum_n n a_n r^{n-1} + (\varepsilon - 3\alpha - \alpha^2 r^2)\sum_n a_n r^n = 0$

$\sum_n (n-1)(n+1)a_{n+1} r^n + \left(\frac{2}{r} - 2\alpha r\right)\sum_n (n+1)a_{n+1} r^n + (\varepsilon - 3\alpha - \alpha^2 r^2)\sum_n a_n r^n = 0$

$a_{n+2}(n-1)(n+1) + \left(\frac{2}{r} - 2\alpha r\right)(n+1)a_{n+1} + (\varepsilon - 3\alpha - \alpha^2 r^2)a_n = 0$

... (skipping algebra)

$\boxed{a_{n+2} = \frac{2\alpha(n+1)(-n\varepsilon + 3\alpha)}{(n+2)(n+3)}a_n}$

v.) The series must terminate, so, $\forall n \geq N$:

$\alpha_{N+2} = 0 \Rightarrow \varepsilon = 2\alpha(N+1) + 2\alpha \Rightarrow \boxed{\varepsilon = 2\alpha(N+1)}$

where: $\varepsilon = \frac{2mE}{\hbar^2}$, $\alpha = \frac{m\omega}{\hbar}$

so: $\frac{2mE}{\hbar^2} = 2 \cdot \frac{m\omega}{\hbar}(N+2) \Rightarrow \boxed{E = \hbar\omega(N+2)}$

so: $\boxed{E_n = \hbar\omega\left(n + \frac{3}{2}\right)}$

Or ~~since $n = n_x + n_y + n_z$ then~~:

$$\boxed{E_n = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right)}$$

$\tan\theta = \frac{y}{x}$

$\theta = \tan^{-1}\frac{y}{x}$

(3) Determine the 1st derivative of the cylindrical coordinate (r, θ, ϕ).

$\boxed{\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}}$

Ans: Recall that:

$\varphi(x(r, \theta, \phi)) = r\sin\theta\cos\phi$.

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$

so: $\nabla_x \varphi(x) = \frac{\partial\varphi}{\partial r} + \frac{\partial\varphi}{\partial\theta} + \frac{\partial\varphi}{\partial\phi} = \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\phi}$

$= \sin\theta\cos\phi \left.\frac{\partial\varphi}{\partial x}\right|_r - r\cos\theta\cos\phi \left.\frac{\partial\varphi}{\partial x}\right|_\theta - r\sin\theta\sin\phi \left.\frac{\partial\varphi}{\partial x}\right|_\phi$

$\nabla_x = \sin\theta\cos\phi\frac{\partial}{\partial r} + r\cos\theta\cos\phi\frac{\partial}{\partial\theta} - r\sin\theta\sin\phi\frac{\partial}{\partial\phi}$

where $r =$

$\theta = \cos^{-1}\frac{z}{r}$

$r\cos\theta = \frac{z}{r}$

$r\cos\theta = z$

$r\cos\theta = r\cos\theta$

(1) NON-DEGENERATE THEORY

FORMULATION OF THE HAMILTONIAN

- Let $\hat{H}$ be the total Hamiltonian. Then

(14.1) $\boxed{\hat{H} = H_0 + \hat{H}_p}$ w/: $\hat{H}_p$ = perturbed state, H_0 = initial state.

- The exact energies are:

(14.2) $\boxed{\hat{H}_0 |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(0)}\rangle}$ w/: $X^{(0)}$ = 0th order unperturbed energies. $E_n^{(0)}$ = unique eigenstate. $|\psi_n\rangle$ = 2nd piece of $\hat{H}_p$.

- The modified w/:

(14.3) $\boxed{H = \hat{H}_0 + \lambda \hat{H}_p}$ and $\lambda \in [0,1]$

- To derive the eigenstates, we sum over power series in terms of λ^k so:

(14.4): $\hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$

and: $E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \dots = \sum_k \lambda^k E_n^{(k)}$

$|\psi_n\rangle = |\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle + \lambda^2|\psi_n^{(2)}\rangle + \dots = \sum_k \lambda^k |\psi_n^{(k)}\rangle$.

$\therefore$ Thus: $\boxed{\begin{aligned} E_n &\sim \sum_k \lambda^k E_n^{(k)} \\ |\psi_n\rangle &\sim \sum_k \lambda^k |\psi_n^{(k)}\rangle \end{aligned}}$ (14.5) (14.6)

N^{TH} ORDER

- Continuing in substituting (14.4):

1.) $\hat{H}|\psi_n\rangle = \hat{H}\sum_k \lambda^k |\psi_n^{(k)}\rangle = (H_0 + \lambda H_p)\sum_k \lambda^k |\psi_n^{(k)}\rangle$.

$E_n|\psi_n\rangle = \left(\sum_k \lambda^k E_n^{(k)}\right)\left(\sum_k \lambda^k |\psi_n^{(k)}\rangle\right)$

2.) So:

$(\hat{H}_0 + \lambda \hat{H}_p)(|\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle + \dots) = (E_n^{(0)} + \lambda E_n^{(1)} + \dots)(|\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle)$

$\Rightarrow \hat{H}_0(|\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle) + \hat{H}_p(\lambda|\psi_n^{(0)}\rangle + \lambda^2|\psi_n^{(1)}\rangle + \dots) = (E_n^{(0)} + \lambda E_n^{(1)} + \dots)(|\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle$

$\Rightarrow \hat{H}_0|\psi_n^{(0)}\rangle + \lambda(\hat{H}_0|\psi_n^{(1)}\rangle + \hat{H}_p|\psi_n^{(0)}\rangle) + \lambda^2(\hat{H}_0|\psi_n^{(2)}\rangle + \hat{H}_p|\psi_n^{(1)}\rangle)$

$= E_n^{(0)}|\psi_n^{(0)}\rangle + \lambda(E_n^{(0)}|\psi_n^{(1)}\rangle + E_n^{(1)}|\psi_n^{(0)}\rangle) + \lambda^2(E_n^{(0)}|\psi_n^{(2)}\rangle + E_n^{(1)}|\psi_n^{(1)}\rangle + E_n^{(2)}|\psi$

or $\Rightarrow \sum_k \lambda^k H_0 + \sum_k \lambda^{k+1} H_p = \sum_k \lambda^{2k} \lambda |\psi_n^{(k)}\rangle$ w/ $\Lambda(k) = \sum_k \lambda^k$.

$\equiv \Lambda(k) H_0 + \Lambda(k+1) H_p = \Lambda(2k) E_n|\psi_n\rangle$.

w: $\boxed{\Lambda_k H_0 + \Lambda_{k+1} H_p = \Lambda_{2k} E_n |\psi_n\rangle}$

- 1st order term yields:

$\boxed{\hat{H}_0|\psi_n^{(1)}\rangle + \hat{H}_p|\psi_n^{(0)}\rangle = E_n^{(0)}|\psi_n^{(1)}\rangle + E_n^{(1)}|\psi_n^{(0)}\rangle}$. (14.9)

- 2nd order term yields:

$\boxed{\hat{H}_0|\psi_n^{(2)}\rangle + \hat{H}_p|\psi_n^{(1)}\rangle = E_n^{(0)}|\psi_n^{(2)}\rangle + E_n^{(1)}|\psi_n^{(1)}\rangle + E_n^{(2)}|\psi_n^{(0)}\rangle}$ (14

[1] 1st ORDER CORRECTIONS

- Suppose we project $\langle\psi_n^{(0)}|$ into (14.9) s.t.

$\langle\psi_n^{(0)}|H_0|\psi_n^{(1)}\rangle + \langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(0)}\rangle = \langle\psi_n^{(0)}|E_n^{(0)}|\psi_n^{(1)}\rangle + \langle\psi_n^{(0)}|E_n^{(1)}|\psi_n^{(0)}\rangle$

$$\Rightarrow E_n^{(0)}\langle\psi_n^{(0)}|\psi_n^{(1)}\rangle + \langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(0)}\rangle = E_n^{(0)}\langle\psi_n^{(0)}|\psi_n^{(1)}\rangle + E_n^{(1)}$$

$$\Rightarrow E_n^{(0)}\langle\psi_n^{(0)}|\psi_n^{(1)}\rangle - E_n^{(0)}\langle\psi_n^{(0)}|\psi_n^{(1)}\rangle + \langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(0)}\rangle = E_n^{(1)}$$

$$\Rightarrow 0 + \langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(0)}\rangle = E_n^{(1)}$$

Thus: $\boxed{E_n^{(1)} = \langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(0)}\rangle}$

- THE CORRECTION
  - Let $E_n = E_n^{(0)} + E_n^{(1)}$ be the total Energy s.t we will employ jointly with the fact that:

$$\boxed{|\psi_n^{(1)}\rangle = \sum_m C_{mn}^{(1)}|\psi_m^{(0)}\rangle} \quad (14.13)$$

  - Our goal is to find Eigenstates $|\psi_n\rangle$ of Full Hamiltonian $\hat{H}$ w/:
    1.) States are norm ($\lambda = 1$) and contains $|\psi_n^{(0)}\rangle$. to No Inclusion of $|\psi_n^{(0)}\rangle$.
    2.) By assuming that $\langle\psi_n^{(0)}|\psi_n^{(1)}\rangle = 0$, we can find the expansion
  - Thus it follows:

1.) $|\psi_n^{(1)}\rangle = \sum_m C_{mn}^{(1)}|\psi_m^{(0)}\rangle$ 2.) $C_{mn}^{(1)} = \langle\psi_m^{(0)}|\psi_n^{(1)}\rangle$.

  - Applying the Bra of $\langle\psi_m^{(0)}|$ we can have:

$$\langle\psi_m^{(0)}|\hat{H}_0|\psi_n^{(1)}\rangle + \langle\psi_m^{(0)}|\hat{H}_p|\psi_n^{(0)}\rangle = \langle\psi_m^{(0)}|E_n^{(0)}|\psi_n^{(1)}\rangle + \langle\psi_m^{(0)}|E_n^{(1)}|\psi_n^{(0)}\rangle$$

$$\Rightarrow E_m^{(0)}\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle + \langle\psi_m^{(0)}|\hat{H}_p|\psi_n^{(0)}\rangle = E_n^{(0)}\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle$$

$$\Rightarrow E_m^{(0)} - E_n^{(0)} + \frac{\langle\psi_m^{(0)}|H_p|H_n^{(0)}\rangle}{\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle} = 0$$

$$\Rightarrow E_n^{(0)} - E_m^{(0)} = \frac{\langle\psi_m^{(0)}|H_0|H_n^{(0)}\rangle}{\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle} \Rightarrow \boxed{C_{mn} = \frac{\langle\psi_m^{(0)}|H_0|H_n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}}} \quad (14.18)$$

2nd ORDER CORRECTION

- Project:
  - Same as before, we project $\langle\psi_n^{(0)}|$ to (14.10) s.t:

$$\Rightarrow E_n^{(0)}\langle\psi_n^{(0)}|\psi_n^{(2)}\rangle + \langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(1)}\rangle = E_n^{(0)}\langle\psi_n^{(0)}|\psi_n^{(2)}\rangle + E_n\underbrace{\langle\psi_n^{(0)}|\psi_n^{(1)}\rangle}_{0} + E_n^{(2)}$$

$$\Rightarrow \cancel{E_n^{(0)}\langle\psi_n^{(0)}|\psi_n^{(2)}\rangle} + \langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(1)}\rangle = \cancel{E_n^{(0)}\langle\psi_n^{(0)}|\psi_n^{(2)}\rangle} + E_n^{(2)}$$

$$\boxed{E_n^{(2)} = \langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(1)}\rangle} \quad (14.27)$$

  - using (14.18) s.t $(E_n^{(0)} - E_m^{(0)})C_{mn}$ is substitution on RHS of (14.27).

$$E_n^{(2)} = \langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(1)}\rangle = (E_n^{(0)} - E_m^{(0)})C_{mn} = (E_n^{(0)} - E_m^{(0)})\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle$$

$$= \sum_{m \neq n} \frac{|\langle\psi_m^{(0)}|\hat{H}_p|\psi_n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$$

$$\boxed{\begin{aligned} E_n^{(1)} &= \frac{\langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}} \quad \text{\# FIRST-ORDER.} \\ E_m^{(1)} &= \frac{|\langle\psi_m^{(0)}|\hat{H}_p|\psi_n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}} \quad \text{\# 2nd ORDER} \end{aligned}}$$

$\therefore$ SV:

TIME INDEPENDENT PERTURBATION
— DEGENERATE THEORY & RELATIVISTIC CORRECTIONS

12. DEGENERATE THEORY:

- **Philosophy:** $\begin{cases} \text{Degenerate} = \text{SPLIT of ENERGY LEVELS} \\ \text{Non Degenerate} = \text{NO SPLIT in ENERGY} \end{cases}$

- **MATH:** • We assume that: $E_n^{(0)}$ is N-fold degenerate and have $|\psi_{nij}^{(0)}\rangle$ corresp. Eigenspaces w/ $j \in \mathbb{Z}^+$. The particular Linear Combination is:

$$\boxed{\hat{H}_0 \sum_{i=1}^{N} b_{nij} |\psi_{nij}^{(0)}\rangle = E_n^{(0)} \sum_{i=1}^{N} b_{nij} |\psi_{nij}^{(0)}\rangle} \quad (14.3m)$$

• observe that: $|\alpha_n^{(0)}\rangle \mapsto \sum_{m}^{N} b_{nij} |\psi_{nij}^{(0)}\rangle$ · (14.3n)

• we substitute this to EQS (14.4)

For time:

$$\hat{H}_0 |\psi_n^{(1)}\rangle + \hat{H}_p |\psi_n^{(0)}\rangle = E_n^{(0)} |\psi_n^{(1)}\rangle + E_n^{(1)} |\psi_n^{(0)}\rangle$$

$$\Rightarrow \langle \psi_{nii}^{0} | H_0 | \psi_n^{1} \rangle + \langle \psi_{nii}^{0} | H_p \sum_{j=1}^{N} b_{nij}^{(0)} |\psi_{nij}\rangle = \langle \psi_{nii}^{0} | E_n^{0} | \psi_n^{(1)} \rangle + \langle \psi_{nii}^{(0)} | E_n^{(1)} \sum_{j=1}^{N} b_{nij} |\psi_{nij}^{(0)}\rangle$$

Under hence: $\rho(x) \to \epsilon(x)$ with $\epsilon(x) = \rho_r(x) \approx$ or $:= b_{nij} |\psi_{nij}\rangle$

• The matrix elements of $\hat{H}_p$ in the N-Dimensional subspace of the Degenerate States to be:

$$\boxed{H_{pij} = \langle \psi_{nii}^{0} | \hat{H}_p | \psi_{nij}^{0} \rangle} \quad \text{or} \quad \boxed{\sum_{j=1}^{n} b_{nij} \langle \psi_{nii}^{0} | \hat{H}_p | \psi_{nij}^{0} \rangle = E_n^{1} b_{nii}}$$

• using this we can write:

$$\boxed{\sum_{j=1}^{N} H_{pij} b_{nij} = E_n^{(1)} b_{nii}}$$

- **SIMPLIFIED:**

• Basically: $H_0 + H_p = E_0 + E_1 \Rightarrow H_0 |\psi\rangle + H_p |\psi\rangle = E_0 |\psi\rangle + E_1 |\psi\rangle$

$\Rightarrow H_0 b_0 |\psi_0\rangle + H_p \underbrace{b_k}_{k>0} |\psi_k\rangle = E_0 b_0 |\psi_0\rangle + E_1 b_k |\psi_k\rangle$

• using: $\hat{H}_p \langle \psi_i | \psi_j \rangle = H_{pij}$ in: $H_0 + H_{pij} b_k = E_0 + E_1 b_k |\psi_k\rangle$

• or just if $H_0, E_0 = 0 \Rightarrow \boxed{H_{pij} b_k = E_1 b_k |\psi_k\rangle}$

$\Rightarrow H_0 = E_n$

$\Rightarrow H_0 + H_p = E_n$

$\Rightarrow H_0 + \sum_n b H_p = \sum_n b E_n$

$\Rightarrow \langle \psi_i | H_0 | \psi_j \rangle + \langle \psi_i | H_p b | \psi_j \rangle = \langle \psi_i | b E | \psi_j \rangle$

$\langle \psi_i | b | \psi_j \rangle H_p = \langle \psi_i | b | \psi_j \rangle E$

$\Rightarrow \sum_n b_n \langle \psi_i | H_p | \psi_j \rangle = E_n^{(1)} b_{nij}$

$= \sum_n H_{pij} b_{ij} = E_n b_{ij}$

$\Rightarrow \hat{H}_p b_n = E_n b_n$

13. RELATIVISTIC CORRECTIONS — HYDROGEN

- **FINE STRUCTURE OF HYDROGEN:**

• The Einstein energy: $K + mc^2 = m^2c^4 - p^2c^2 + mc^2 = mc^2\left(1 + \frac{p^2c^2}{m^2c^2}\right)^{1/2}$

Hence: $\boxed{K = mc^2\left(1 + \frac{p^2c^2}{mc^2}\right) - mc^2}$

• This is basically a binomial expansion of: $\text{Binom}\left(\frac{p^2}{2m}\frac{1}{c^2}\right) \approx H_R^k = \frac{p^2}{2m} + \frac{p^4}{8m^3c^2} + \cdots$

• The kinetic energy is:

$$\boxed{\hat{H}_R = -\frac{\hat{p}^4}{8m^3c^2} = -\frac{1}{2mc^2}\left(\frac{\hat{E}^2}{2m}\right)^2}$$

• Some comments:

>> Hydrogen mass is literally: $\boxed{m_H = m_e}$

since 1

>> So we'll use a $\boxed{\text{non degenerate}}$ theory w/ $|n, l, m_l\rangle$ as fix n.

>> computing Hydrogen requires computing $\langle n|m|\cdot|n|m\rangle$.

>> Avoiding degeneracy matrices required us to use (L_z, L^2) as roots

Sol: $[p^2, \hat{L}] = [p_x^2, [p_x x]] = [p_x^2, i\hbar]$ $\quad p_x = i\hbar\nabla$

$= [-p_x^2\hbar, i\hbar]\phi = -\nabla^2\phi\hbar^2 i + \nabla^2\phi\hbar^2 i$ $\quad p_x^2 = -\hbar\nabla^2$

$= \boxed{0}\, V$

• Defining >> $H_0 = \frac{p^2}{2m} + V(r)$ => $\frac{p^2}{2m} = V(r) - H_0$ >> $H_R = -\frac{1}{2mc^2}(H_0 - V)(H - V)$

>> so: $\boxed{\frac{p^2}{2m} = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} - H_0}$ ★ or: $H_R = -\frac{1}{2mc^2}(H_0 - V)^2$

• The perturbed state:

$$H_R = -\frac{1}{2mc^2}\left(H_0^2 - 2H_0V + V^2\right) = -\frac{1}{2mc^2}\left[\left(\frac{p^4}{4m} + \frac{p^2}{2m}\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} + \frac{e^4}{16\pi^2\varepsilon_0^2}\frac{1}{r^2}\right) + \left(2\frac{p^2}{2m}\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} + \frac{e^4}{16\pi^2\varepsilon_0^2}\frac{1}{r^2}\right) + \left(\frac{e^4}{16\pi^2\varepsilon_0^2}\frac{1}{r^2}\right)\right]$$

$= -\frac{1}{2mc^2}\left(H_0^2 - 2H_0V + V^2\right)$

• This energy is expected as:

$$\Rightarrow E_R^{(1)} = \langle n, l, m_l | H_R | n\, l\, m_l\rangle$$

$$= \langle n|m_l| -\frac{1}{2mc^2}\left(H_0^2 - 2H_0V + V^2\right)|n|m_l\rangle = -\frac{1}{2mc^2}\left[\langle n|m|H_0^2|n|m\rangle - \langle n|m_l|2H_0V|n|m_l\rangle + \langle n|m_l|V^2|n|m_l\rangle\right]$$

$$= -\frac{1}{2mc^2}\left[E^{0\,2} - 2E^0\frac{e^2}{4\pi\varepsilon_0}\langle n|m_l|\frac{1}{r}|n|m_l\rangle + \left(\frac{e^2}{4\pi\varepsilon_0}\right)^2\langle n|m_l|\frac{1}{r^2}|n|m_l\rangle\right]$$

★ $\boxed{\left\langle\frac{1}{r}\right\rangle = \frac{1}{n^2a_0}}$ ★ $\boxed{\left\langle\frac{1}{r^2}\right\rangle = \frac{1}{n^3a_0^2(l + \frac{1}{2})}}$

$$\Rightarrow E_R^{(1)} = -\frac{1}{2mc^2}\left(-2E^0\frac{e^2}{4\pi\varepsilon_0}\left(\frac{1}{n^2a_0}\right) + (E^0)^2 + \frac{e^4}{16\pi^2\varepsilon_0}\left(\frac{1}{n^3a_0^2(l+\frac{1}{2})}\right)\right)$$

$$= \frac{(E_n^{(0)})^2}{2mc^2}\left[\frac{4n}{(l+\frac{1}{2})} - 3\right]$$

• The fine structure and unperturbed energy:

>> $\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \approx \frac{1}{137}$

=> $E_n^0 = \frac{m_e c^2\alpha^4}{8n^4}\left[\frac{4n}{(l+\frac{1}{2})} - 3\right]$

• Our final equation:

$$\boxed{E_R^{(1)} = -\frac{mc^2\alpha^4}{8n^4}\left(\frac{4n}{(l+\frac{1}{2})} - 3\right)}$$

1st PERTURBATION of EINSTEIN'S energy

Time-Independent Perturbation Theory Examples

14.Ex

(1) 1st Order Example - 14.1

Scenario:

Suppose an electric field $\vec{E} = \vec{E}\hat{x}$ is applied to a particle w/ charge q in an infinite potential well:

$$\boxed{V(x) = \begin{cases} -qEx & 0 < x < L \\ \infty & \text{Elsewhere} \end{cases}}$$

(a) Determine the allowed energies!

Answer:

- The unperturbed states is basically:

$$\boxed{E_n^{(0)} = \frac{n^2 \pi^2 \hbar^2}{2mL^2}} \quad \forall n \in \mathbb{Z}^+ \qquad (14.21)$$

- The wave function on the other hand is:

$$\boxed{\psi_n^{(0)}(x) = \sqrt{\frac{2}{L}} \cdot \sin\left(\frac{n\pi}{L}x\right)} \text{ and } (0) \text{ Elsewhere} \qquad (14.22)$$

$\forall \psi_n \in 0 < x < L$

- The perturbed Hamiltonian is:

$$\boxed{\hat{H}_p \equiv V(x) = -qE\hat{x}} \qquad (14.23)$$

∴ The 1st wave correction is:

$$E_n^{(1)} = \langle \psi_n^{(0)} | \hat{H}_p | \psi_n^{(0)} \rangle = \frac{2}{L}\int_0^L \sin^2\left(\frac{n\pi}{L}x\right) \cdot -qEx\, dx$$

$$= \frac{2}{L}\int_0^L \left(\frac{1-\cos\left(\frac{2n\pi x}{L}\right)}{2}\right)(-qEx)\, dx = \frac{2}{L}\int_0^L -\frac{1}{2}qEx + \frac{1}{2}\cos\left(\frac{2n\pi}{L}\right)qEx\, dx$$

$$= -\frac{qEL}{2} \quad \text{so,} \quad \boxed{E_n = -\frac{qEL}{2}}$$

(2) 2nd Order Example

Scenario:

A uniform electric field of $\vec{E} = \vec{E}\hat{x}$ is applied to a particle w/ charge of q is situated in a harmonic oscillator potential. Determine the energies of the particle.

Answer: We apply: $\boxed{V(x) = \frac{1}{2}m\omega^2 x^2 - qEx}$ ⇒ NET POTENTIAL OF HARMONIC OSCILLATOR and CONSTANT FORCE

- The perturbation changes: $E_n^0(\omega) \to E_n^0(\omega')$: $-\frac{q^2E^2}{2m\omega^2} \to -\frac{qE^2}{m\omega^3}$.
- The perturbation's solution is:

$$\boxed{\hat{H}_p(x) = -qE\hat{x}} \text{ w/the eigenstates: } \boxed{E_n^0 = \hbar\omega\left(n + \frac{1}{2}\right)}$$

- The matrix element:

$$\langle \psi_j^{(0)} | \hat{H}_p | \psi_n^{(0)} \rangle = -qE\langle j | \hat{x} | n \rangle = -qE\sqrt{\frac{\hbar}{2m\omega}} \cdot \langle j | \hat{a} + \hat{a}^\dagger | n \rangle$$

$$= -qE\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\langle j | n-1 \rangle + \sqrt{n+1}\langle j | n+1 \rangle\right)$$

$$= -qE\sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\cdot\delta_{j,n-1} + \sqrt{n+1}\cdot\delta_{j,n+1}\right]$$

$n = |c_-|^2$

$\langle j | \hat{a} + a^\dagger | n \rangle$

$= \hat{a}\langle j | n \rangle + a^\dagger \langle j | n \rangle$

$= \langle j | (\hat{a} | n \rangle) + (a^\dagger | n \rangle)\langle j |$

$= \langle j | (c_- | n \rangle) + (c_+ | n+1 \rangle)\langle j |$

$= \sqrt{n}\langle j | n-1 \rangle + \sqrt{n+1}\langle j | n-1 \rangle$

- The 1st order correction is:

$$\boxed{E_n^{0} = \langle \psi_n^{0} | \hat{H}_p | \psi_n^{0} \rangle = 0}$$

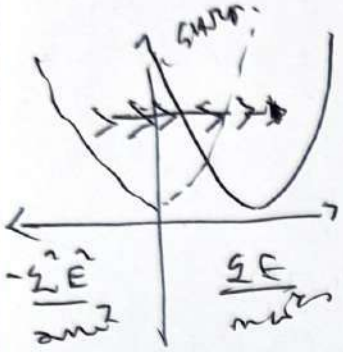
- So up to 1st order of correction:

$$\begin{cases} E_n^{(0)} = 0 \\ E_n^{(1)} = 0 \\ E_n^{(2)} = ? \end{cases}$$

- Turns out that:

$$E_n^{(2)} = \sum_{j \neq n} \frac{K \langle \psi_j^{(0)} | \hat{H}_p | \psi_n^{(0)} \rangle^2}{E_n^{0} - E_j^{0}} = \sum_{j \neq n} \frac{1}{E_n^{0} - E_j^{0}} \left[qE\sqrt{\frac{\hbar}{2m\omega}} \langle j | n-1 \rangle \sqrt{n} + \langle j | n+1 \rangle \sqrt{n+1} \right]^2$$

$$= \frac{q^2 E \hbar}{2m\omega} \sum_{j \neq n} \frac{\left(\langle j | n-1 \rangle \sqrt{n} + \langle j | n+1 \rangle \sqrt{n+1} \right)^2}{E_n^{0} - E_j^{0}} = \frac{q^2 E \hbar}{2m\omega} \sum_{j \neq n} \left[\frac{\left(\sqrt{n}\, \delta_{j,n-1} + \sqrt{n+1}\, \delta_{j,n+1} \right)^2}{E_n^{0} - E_j^{0}} \right]$$



$$= \frac{q^2 E \hbar}{2m\omega} \sum_{j \neq n} \frac{1}{E_n^{0} - E_j^{0}} \sqrt{n}\, \delta_{j,n-1}\, \delta_{j,n-1} \sqrt{n} + \dots + \sqrt{n+1}\, \delta_{j,n+1}\, \delta_{j,n+1} \sqrt{n+1}$$

*we know this

$$= \frac{q^2 E \hbar^2}{2m\omega} \sum_{j \neq n} \frac{1}{E_n^{0} - E_j^{0}} \left((n) + (n+1) \right) = \frac{q^2 E \hbar^3}{2m\omega} \left(\frac{n}{E_n^{0} - E_{n+1}^{0}} + \frac{n-1}{E_n^{0} - E_{n-1}^{0}} \right)$$

$$= \frac{q^2 E^2 \hbar}{2m\omega} \left[\frac{n}{\hbar\omega} - \frac{n+1}{\hbar\omega} \right] = -\frac{qE^2}{2m\omega^2}.$$

$$\boxed{E_n^{(2)} = -\frac{qE^2}{2m\omega^2}}$$

2nd order correction is the ground state.

EXERCISES 14

① **Simple system:**

Calculate to 1st order an $H_p = V_0$ applied to $\hat{H}_0$. Calculate the energies to 2nd order and wave functions

Ans. $E^1 = \langle \psi_i | V_0 | \psi_j \rangle = V_0 \langle \psi_i | \psi_j \rangle = V_0 \int |\psi|^2 dr = V_0$

$E^2 = \dfrac{|\langle \psi | V | \psi \rangle|^2}{\epsilon_i - \epsilon_j} = \dfrac{V_0^2}{\epsilon_i - \epsilon_j} \delta_{kn}^2 = 0$ & $|\psi_n^{(1)}\rangle = \sum_k \dfrac{|k|H|n\rangle|k\rangle}{\epsilon_n - \epsilon_k}$

$|\psi_n^{(2)}\rangle = 0$

② **Potential well:**

Infinite potential well extends to/ $x \in [0, L]$ and has a small bump in the middle. Assume perturbation:

$$H_p = \begin{cases} V_0 & \frac{L}{2} - \frac{a}{2} < x < \frac{L}{2} + \frac{a}{2} \\ 0 & \text{else} \end{cases}$$

(?) Calculate to 1st order:

Ans. $E^1 = \langle \psi_j | V_0 | \psi_j \rangle = V_0 \langle \psi_j | \psi_j \rangle = V_0 \dfrac{2}{L} \displaystyle\int_{\frac{L}{2}-\frac{a}{2}}^{\frac{L}{2}+\frac{a}{2}} \sin^2\left(\frac{n\pi x}{L}\right) dx$

$= V_0 \dfrac{2}{L} \displaystyle\int_{\frac{1}{2}(L-a)}^{\frac{1}{2}(L+a)} \left[\underbrace{\frac{1 - \cos(2k_n x)}{2}} dx \right] = \dfrac{2V_0}{L} \left[\dfrac{a}{2} - \dfrac{1}{2} \displaystyle\int_{x_0}^{x_0+a} \cos(2k_n x) \right]$

$k_n = \dfrac{n\pi}{L}$

$= \dfrac{\cancel{2}V_0}{L} \dfrac{1}{\cancel{2}} \left[a - \dfrac{\sin(2k_n(x_0+a)) - \sin(2k_n x_0)}{4k_n} \right]$

$= \dfrac{V_0}{L} \left[\dfrac{a}{2} + (-1)^{n+1} \dfrac{\sin\left(n\pi \frac{a}{2}\right)}{2k_n} \right]$

$$\therefore \boxed{E_n^{(1)} = V_0 \frac{a}{L} + (-1)^n V_0 \frac{\sin(n\pi a/L)}{n\pi}}$$

TIME INDEPENDENT PERTURBATION - SPIN ORBITS

(3.2) FINE STRUCTURE - SPIN ORBIT

14.3.2

(1 volt = $\frac{1 \text{ joule}}{1 \text{ coulomb}}$)

Coulomb = $[M^0 L^0 T^1 I^1]$

Joule = $[M L^2 T^{-2}]$

1 = coulomb = $\frac{1 \text{ joule}}{1 \text{ volt}}$

BG principles: • The current loop produced by a magnetic field is:

$$\boxed{B = \frac{\mu_0 I}{2r} = \frac{I}{\varepsilon_0 c^2 2r}} \quad (6.8)$$

>> why start with:

$$\begin{cases} V = IR \\ F = \nabla(\vec{\mu}\cdot\vec{B}) \quad \vec{\mu} = \frac{1}{\mu_0} \\ \mu_e = \gamma_e S \\ \text{or } \mu_0 = \gamma_0 S \end{cases}$$

>> So: $V = IR$

$$\Rightarrow \frac{1}{\mu_0}\vec{B} = \frac{I}{2r} \Rightarrow \vec{B} = \frac{I}{2r}\mu_0 = \frac{I}{\varepsilon_0 c^2 2r}$$

$$\left\{ \mu_0 = \frac{1}{c_2 \varepsilon_0} \right.$$

since $c^2 = \frac{1}{\sqrt{\mu_0 \varepsilon}}$, $c^2 = \frac{1}{\mu_0 \varepsilon}$

$$\star: \begin{cases} |R V| = \frac{1}{2r} \\ \vec{\mu} = \frac{1}{\mu_0} \end{cases}$$

>> The effective current is: $I = \frac{e}{T}$, So:

$$B = \frac{I}{\varepsilon_0 c^2 2r} = \frac{e}{\varepsilon_0 c^2 2r}\frac{v}{T} = \frac{e}{\varepsilon_0 c^2 2r\left(\frac{2\pi r}{v}\right)} = \frac{e}{4\pi\varepsilon_0}\frac{v}{c^2 r^2}$$

>> Gyromagnetic: $\boxed{\gamma_e = \frac{g_e}{2m} q_e = \frac{-g_e}{2m} e}$ by which: $g_e q_e \sim \hat{g}_e \hat{q}_e = \hat{\gamma}$

or: $\frac{\hat{\gamma}}{2m} \star \frac{p^2}{2m}$

>> Angular momentum is:

$$L = \vec{r}\times\vec{p} = m[\vec{r}\times\vec{v}] = mrv$$

$$\Rightarrow L = mrv = \boxed{v = \frac{L}{mr}}$$

>> Then:

$$B = \frac{e}{4\pi\varepsilon_0}\frac{v}{c^2 r^2} = \frac{e}{4\pi\varepsilon_0}\frac{L}{mr}\frac{1}{c^2 r^2} = \frac{e}{4\pi\varepsilon_0}\frac{L}{mc^2 r^3}$$

>> our Hamiltonian is:

$$\boxed{V = -\mu\cdot\vec{B} = \frac{g_e e^2}{8\pi\varepsilon_0 m^2 c^2 r^3} S\cdot L}$$

• why $B = \frac{e}{4\pi\varepsilon_0}\frac{L}{mc^2 r^3}$ $\quad \mu = -\frac{g_e e^2}{2m}$

So w:

$$\boxed{V = -\mu\cdot\vec{B} = \hat{\xi}(S\cdot L)}$$

$$\xi(r) = \frac{g_e e^2}{8\pi\varepsilon_0 c^2 r^3 m^2}$$

why $\left\{ \xi(1) = 1.38\times10^{20}\frac{M}{kg}\Big|_{r=1} \right.$

$B = \frac{S\cdot L}{\mu}$

• Turns out, the Hamiltonian in Reference frame is:

$$\boxed{\hat{H}_{SO} = \frac{(g_e - 1)}{8\pi\varepsilon_0 m^2 c^2 r^3}\hat{S}\cdot\hat{L} = \frac{e^2}{8\pi\varepsilon_0 m^2 c^2 r^3}\hat{S}\cdot\hat{L}}$$

SPIN & ORBIT HAMILTONIAN

$\hat{\xi} = \xi\cdot\frac{1}{r} = 1.38\times10^{20}\frac{M}{kg}\left(\frac{1}{r^3}\right)$

$r \to \infty \; \hat{\xi} = 0 \; \boxed{V =}$

>> $g_e \sim 2 \Leftrightarrow H = \frac{1}{2m}(p^2 - qA)^2 - \frac{q\hbar}{2m}\vec{\sigma}\cdot\vec{B}$

$$= 0 - \frac{q\hbar}{2m}\vec{\sigma}\cdot\vec{B} = \frac{q\hbar}{2m}\vec{\sigma}\cdot\vec{B}$$

$$= -\frac{q\hbar}{2m}\vec{\sigma}\frac{SL}{\mu}$$

why $g_e \sim 2$

$$\left\{ \begin{array}{l} S_z = \frac{\hbar}{2}\vec{\sigma} \\ \mu_S = \frac{q}{m} S \end{array} \right.$$

$$\left\{ \begin{array}{l} \mu_S = \mu_S \\ \frac{q\hbar}{2m}\sigma = \frac{q}{m} \\ \frac{\hbar}{2}\frac{1}{\sigma S} = 1 \\ \boxed{\frac{\hbar}{\sigma S} = 2} \end{array} \right.$$

q) The subscript is used due to the spin orbit of the angular momentum

• The total angular is:

$$J^2 = \hat{J}\cdot\hat{J} = (S+L)(S+L) = \hat{S}^2 + \hat{L}^2 + 2S\cdot L$$

$$\Rightarrow \frac{1}{2}(J^2 - S^2 - L^2) = S\cdot L$$

$$\Rightarrow \boxed{\hat{H}_{SO} = \frac{e^2}{16\pi\varepsilon_0 m^2 c^2 r^3}(J^2 - S^2 -}$$

HAMILTONIAN is a DERIVING

- Perturbation: $E_{SO}^{(1)} = \langle \hat{H}_{SO} | \psi \rangle$

$$= \langle \psi | \frac{e^2}{16\pi\varepsilon_0 m^2 c^2 r^3} (J^2 - L^2 - S^2) | \psi \rangle$$

$$= \langle \psi | \frac{e^2}{16\pi\varepsilon_0 m^2 c^2 r^3} j(j+1) - l(l+1) - s(s+1) | \psi \rangle$$

$$= \frac{e^2}{16\pi\varepsilon_0 m^2 c^2} \left[j(j+1) - l(l+1) - s(s+1) \right] \left\langle \frac{1}{r^3} \right\rangle \quad \text{w/ } \left\langle \frac{1}{r^3} \right\rangle = \langle \psi | \frac{1}{r^3} | \psi \rangle.$$

- Now expected value of integer ν is:

$$\langle \hat{r}^{\nu} \rangle = \left[n \frac{a_0}{2} \right]^{\nu} \frac{1}{2n(\beta+1)!} \sum_{k=0}^{\beta} (-1)^{\beta+k} \binom{\beta}{k} \binom{n+l-1+k}{\beta} \frac{1}{\binom{2l+\beta+1-k}{\beta+1}}.$$

$$\text{w/ } \binom{x}{y} = \frac{x!}{y!(y!-x!)}$$

$\Rightarrow$ So that: $$\langle \hat{r}^{-3} \rangle = \left[n \frac{a_0}{2} \right]^{-3} \frac{1}{2n(\beta+1)!} \sum_{k=0}^{\beta} (-1)^{\beta+k} \binom{\beta}{k} \binom{n-l-1+k}{\beta} \frac{1}{\binom{2l+\beta+1-k}{\beta+1}}$$

$\Rightarrow$ Particularly this is: $$\left\langle \frac{1}{r^3} \right\rangle = \frac{1}{a_0^3 n^3 l(l+1)(l+\frac{1}{2})}$$

$\Rightarrow$ Doing algebra yields:

$$E_{SO}^{(1)} = \frac{e^2}{16\pi\varepsilon_0 m^2 c^2} \left[j(j+1) - l(l+1) - s(s+1) \right] \left\langle \frac{1}{r^3} \right\rangle = \frac{e^2}{16\pi\varepsilon_0 m^2 c^2} \left[j(j+1) - l(l+1) - s(s+1) \right] \times \frac{1}{a_0^3 n^3 l(l+1)(l+\frac{1}{2})}$$

$$= \frac{e^2}{16\pi\varepsilon_0 m^2 c^2} \frac{j(j+1) - l(l+1) - \frac{3}{4}}{n^3 l(l+1)(l+\frac{1}{2}) a_0^3 n^3} = \frac{mc^2\alpha^4}{4} \left[\frac{j(j+1) - l(l+1) - \frac{3}{4}}{n^3 l(l+1)(l+\frac{1}{2})} \right]$$

$\Rightarrow$ Thus: $$\boxed{E_{SO}^{(1)} = \frac{mc^2\alpha^4}{4} \left[\frac{j(j+1) - l(l+1) - \frac{3}{4}}{n^3 l(l+1)(l+\frac{1}{2})} \right]}$$

DARWIN TERM:

- In Relativistic quantum mechanics, the particle can't just be localized smaller than: $\boxed{\lambda_c = \frac{\hbar}{mc}}$

- To Elaborate this, Expand V:

$$V(r+r') = V + \sum_{i=1}^{3} \left(\frac{\partial V}{\partial r_i} \right) r_i' + \frac{1}{2} \sum_{i,j=1}^{3} \left(\frac{\partial^2 V}{\partial r_i \partial r_j} \right) r_i' r_j' + \cdots$$

$$= V(r) + \frac{1}{2} \left[\frac{1}{3} \lambda_c^2 \right] \nabla^2 V(r) + \cdots = V(r) + \frac{\hbar^2}{10 m^2 c^2} \nabla^2 V(r) + \cdots$$

- Darwinian Hamiltonian: $$\boxed{H_D = \frac{\hbar^2}{8m^2c^2} \nabla^2 V(r)}$$

MAXWELL'S EQUATION

$$\nabla \cdot E = \nabla \left(\frac{q}{4\pi\varepsilon_0} \frac{1}{r} \right) = \nabla \frac{4\pi\varepsilon_0 \rho}{4\pi} = \nabla \cdot \left[\frac{1}{r} \right]$$

$$\Rightarrow \boxed{\nabla^2 \left(\frac{1}{r} \right) = 4\pi\delta^3(r)}$$

- Coulomb potential is: $$H_D = \frac{\hbar^2}{8m^2c^2} \left[-\frac{e^2}{4\pi\varepsilon_0} \nabla^2 \left(\frac{1}{r} \right) \right] = \frac{\hbar^2 e^2}{8m^2c^2\varepsilon_0} \delta^3(r)$$

- Expected value is:

$$\boxed{E_D^{(1)} = \langle \hat{H}_D^{(1)} \rangle = \frac{\hbar^2 e^2}{8m^2c^2\varepsilon_0} |\psi_{nlm}(0)|^2}$$

with it's: $$E_D^{(1)} = \frac{mc^2\alpha^4}{2n^3}$$

EXERCISE 14

(15) Verifying $E_{S0} = \frac{mc^2\alpha^4}{2n^3}$ by using $L_j^k(x) = \frac{(j+k)!}{j!\,k!}$

Ans: We have: $E_{S0} = \langle H_0 \rangle = \frac{\hbar^2 e^2}{8 m^2 c^2 \varepsilon_0} \cdot |\psi_{nlm}|^2$

>> also: $|\psi_{nlm}|^2 = R_{nl}(r) \cdot Y_{lm}(\theta, \phi)$

>> and $R_{nl} = \frac{2}{n^2}\sqrt{\frac{(n-l-1)!}{a_0^3(n+l)!}}\left[\frac{2r}{na_0}\right]^l e^{-\frac{r}{na_0}} L_{n-l-1}^{2l+1}\left(\frac{2r}{na_0}\right)$.

- Letting: $R_{nl} \doteq R_{n0} = \frac{2}{n^2}\sqrt{\frac{(n-1)!}{a^3 l!}}\left[\frac{2r}{na_0}\right]^a e^{-\frac{r}{na_0}} L_{n-1}^{l}\left(\frac{2r}{na_0}\right)$

$$= \left(\frac{2}{na_0}\right)^{\frac{3}{2}} \sqrt{\frac{(n-1)!}{2n\,n!}}\; e^{-\rho/2} L_{n-1}^{1}(\rho) \quad \text{with } \rho = \frac{2r}{na_0}.$$

=> where: $n! = n(n-1)!$ => $\sqrt{\frac{(n-1)!}{2n \cdot n!}} = \sqrt{\frac{1}{2n^2}} = \frac{1}{\sqrt{2n}}$.

=>> hence: $\boxed{R_{n0} = \frac{1}{\sqrt{2n}} e^{-\frac{\rho}{2}} L_{n-1}^{1}(\rho)}$

- Legendre polynomial form: