INTRODUCTION TO $\mathbb{R}$ - ANALYSIS

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DIRCHLET DISCONTINUOUS (C.1) II.

① PROOF OF DISCONTINUITY & SEQUENCE

[I] SEQUENCE PROOF

1) Let P be the proposition that the negation is formulated as follows: ... $\Rightarrow$ ∴ Continuous$(f|c)$.

$P: \forall x_n \in A,\ f(x_n) \to f(c) \Rightarrow$ ∴ Continuous$(f|c)$

NEG(P): $\exists x_n \in A,\ \big(f(x_n) \to f(c)\big) \land$ Discontinuous$(f|c)$

so: prove $\boxed{\big(f(x_n) \to f(c)\big) \land \text{Discontinuous}(f|c)}$ 6/ is TRUE.

2.) $f(x_n) := f(x_{\delta_n})$ s.t $|x_{\delta_n} - c| < \frac{1}{n}$

so: $f(x_1 \cup \ldots \cup x_n) \longrightarrow f(c)$ since $f\left(\bigcup_{i=1}^{n} x_n\right) = f\left(\bigcup_{i=1}^{n} x_{\delta_n}\right)$

and $f\left(\bigcup_{i=1}^{n} x_{\delta_n}\right) := f(c) = \lim_{x \to c} f(x)$

so: $\boxed{f\left(\bigcup_{i=1}^{n} x_{\delta_n}\right) := f(c) = \lim_{x \to c} f(x)}$ is continuous

$f(x_n) \longrightarrow f(c)$ s.t it's a Continuous.

3.) If that's true then

so:

NEG(P) $:= \big(f(x_n) \to f(c)\big) \land$ Discontinuous$(f|c)$ 6/ true

$=$ Continuous$(f|c) \land$ Discontinuous$(f|c)$ 6/ true

$=$ Continuous$(f|c) \land \neg$ Continuous$(f|c)$.

$\boxed{\text{∴ CONTRADICTION! : Continuous}(f|c) \land \neg\,\text{Continuous}(f|c).}$

[II] DISCONTINUOUS PROOF

1) We just flip the signs / logics s.t:

$\boxed{\text{NEG}(P): \exists x_n \in A,\ f(x_n) \not\to f(c) \land \text{Continuous}(f|c)}$

2.) Then if $f(x_n) \not\to f(c)$. Then $f(x_1 \cup \ldots \cup x_n) \not\to f(c)$.

s.t $x_n := x_{\delta_n} \notin A$ since it doesn't map x from A to $\mathbb{R}$ w/ the condition $f: A \to \mathbb{R}$ is defined in C.

3.) Then $f(x_n) \to f(c)$ is DISCONTINUOUS s.t Discontinuous$(f|c)$.

so:

NEG(P): $\exists (x_n \in A),\ f(x_n) \not\to f(c) \land$ Continuous$(f|c)$ 6/ true

$=$ Discontinuous$(f|c) \land$ Continuous$(f|c)$

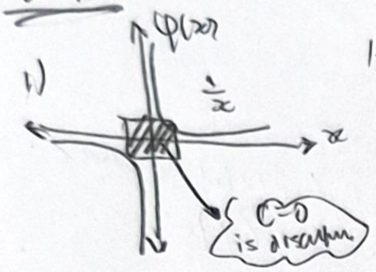
$= \neg$ Continuous$(f|c) \land$ Continuous$(f|c)$

$\boxed{\text{∴ CONTRADICTION! : }\neg\,\text{Continuous}(f|c) \land \text{Continuous}(f|c).}$

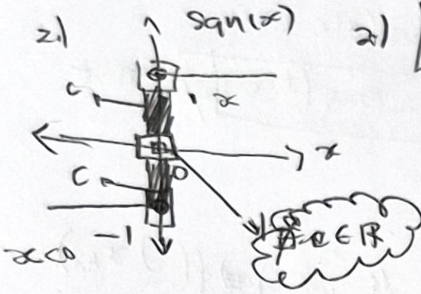
② EXAMPLES

GRAPHS:

1)



2)



[I] SIMPLE FUNCTIONS

1) $\boxed{\varphi(x) = \frac{1}{x}}$ isn't continuous s.t Discontinuous$(\varphi/0)$.

- observe $\varphi(x)\big|_{x\to 0} = \lim_{x\to 0} \varphi(x) = \lim_{x\to 0} \frac{1}{x} = \frac{1}{0} = \times$

- Then φ isn't at 0, so: $\boxed{\text{Discontinuous}(\varphi/0)}$ (A_2)

2) $\boxed{\text{sgn}(x)}$ isn't continuous at 0.

- observe. $\lim_{x\to 0} \text{sgn}(x) = \text{sgn}(0) = \begin{cases} +1 & x>0 \\ -1 & x<0 \\ 0 & x=0 \end{cases}$

So: $\boxed{\text{sgn}(0)=0}$ But also $\boxed{\text{sgn}(x) \notin \mathbb{R}}$ (B_2)

[II] DIRICHLET'S DISCONTINUOUS FUNCTION

1) Let $A := \mathbb{R}$. Then: $\boxed{f(x) := \begin{cases} 1 & x \text{ is rational} \\ 0 & x \text{ is irrational} \end{cases}}$

2) Let x_n be the sequence of irrational numbers that converges to c.
- Since $f(x_n) = 0$ $\forall n \in \mathbb{N}$, we have $\lim f(x_n) = 0$
- while $f(c) = 1$
- $\therefore$ and $\lim_{x\to c} f(x_n) \neq f(c)$ so it's <u>not continuous</u>!

K.J Thomae (1875)

3) Let $A := \{x \in \mathbb{R} \mid x > 0\}$. $\forall x > 0$, $h(x) =$ via $\frac{m}{n}$. Then define $\boxed{h\left(\frac{m}{n}\right) = \frac{1}{n}}$ $\Big\}$ Then $h(x)$ is <u>continuous</u> for every <u>irrational</u> number at A. But <u>discontinuous</u> for $\forall$ <u>rational</u>

(3) REMARKS:

[I] Limit point L

1) Sometimes $f: A \to \mathbb{R}$ isn't continuous at c.

2) If $\lim_{x\to L} f(x)$ $\exists$. Then $\boxed{F \overset{\text{def}}{=} A \cup \{c\} \to \mathbb{R}}$ s.t.:

$$\boxed{F(x) = \begin{cases} L & x=c \\ f(x) & x \in A \end{cases}}$$

[II] NO LIMIT AT C ?

1) If $g: A \to \mathbb{R}$ isn't continuous at C, Then $\nexists G: A \cup \{c\} \to \mathbb{R}$. s.t $\boxed{G(x) := \begin{cases} C & x=c \\ g(x) & x \in A \end{cases}}$

[III] EXAMPLE.

EXERCISE 5.1

UNIFORM CONTINUITY IN NEIGHBORHOOD

(a) Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$ be continuous $c \in A$.
Show that $\forall \varepsilon > 0, \left(\exists V_\delta(c), (x,y) \in A \cap V_\delta(c) \implies |f(x) - f(y)| < \varepsilon \right)$

Ans: [i] Using the triangle inequality:

1.) Let $\varepsilon > 0$. Since f is continuous then:
$\exists \delta > 0, \left[\forall x \in A, \text{ if } |x - c| < \delta \implies \therefore |f(x) - f(c)| < \varepsilon \right]$.

2.) Pick any $(x,y) \in A \cap V_\delta(c)$ s.t. it satisfies
$|x - c| < \frac{\varepsilon}{2}$ and $|y - c| < \frac{\varepsilon}{2}$. Then:
$$|f(x) - f(y)| = \underbrace{|f(x) - f(c) + f(c) - f(y)| \leq |f(x) - f(c)| + |f(c) - f(y)|}_{\text{Triangle Inequality}}$$

3.) By the triangle inequality this means:
$$|f(x) - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
$\therefore$ So: $\boxed{|f(x) - f(y)| < \varepsilon}$ QED.

[ii] Using contradiction

1.) We prove the negation that:
$\forall V_\delta(c), \left[(x,y) \in A \cap V_\delta(c) \right] \wedge \left[|f(x) - f(y)| \geq \varepsilon \right]$

2.) Recall: $\boxed{V_\delta(c) := \{ x \in \mathbb{R} \mid |x - c| < \varepsilon \}}$
Then for $\forall A \cap V_\delta(c)$, A is in $\mathbb{R}$ and A is continuous at c.

3.) So $(x,y) \in A \cap V_\delta(c)$ is continuous.

4.) But! $|f(x) - f(y)| = |f(x) - f(x)|_y| = 0 \geq \varepsilon$ so $\boxed{\varepsilon \leq 0}$
s.t. $\{ f(x) - f(x)|_y = 0 \} \iff |x - y| \leq \varepsilon$.

5.) Contradiction since the assumption $|x - y| < \varepsilon$ holds hence
the proposition must be true. QED

(b) Let $f: \mathbb{R} \to \mathbb{R}$ be continuous and let $S := \{ x \in \mathbb{R} \mid f(x) = 0 \}$ be
the "zero set". If $\left[(x_n \text{ is in } S) \wedge (x = \lim(x_n)) \right] \implies$ show that $x \in S$.

Ans: · Given $x_n \to x$ as $n \to \infty$
· Then: $f(x_n) \to f(x)$
· Since: $f(x) = 0$
· Then: $\cancel{f(x_n) = 0}$
· Thus $\boxed{(x_n, x) \in S}$ QED

(7) Let $f: \mathbb{R} \to \mathbb{R}$ be continuous at c and let $f(c) > 0$.
Show that $\exists V_\delta(c)$, $x \in V_\delta(c) \implies f(x) > 0$

ANS: [i] USING EPSILON:

1) let $\boxed{\varepsilon = \frac{f(c)}{2} > 0}$

2) so:

$\boxed{\exists \delta < 0, \ |x-c| \implies |f(x) - f(c)| < \frac{f(c)}{2}}$

3.) Do algebra:

$-\frac{f(c)}{2} < f(x) - f(c) < \frac{f(c)}{2}$

$-\frac{f(c)}{2} + f(c) < f(x) - f(c) + f(c) < \frac{f(c)}{2} + f(c)$.

$-\frac{f(c)}{2} < f(x) < \frac{f(c)}{2} \implies f(x) > \frac{f(c)}{2} > 0$

Thus $\boxed{2f(x) > f(c) > 0}$ QED

[ii] NEGATING:

1) NEG: $\forall V_\delta$, $x \in V_\delta \wedge f(x) \leq 0$.

Remark: so we proof $f(x) \leq 0$ is true given $x \in V_\delta$

2.) Iff $f(x) \leq 0$ Then $(f(x) = 0) \wedge (f(x) < 0)$ is True.

3.) But $f(c) > 0$ s.t. $\lim_{x \to c} f(x) > 0$ or $x > 0$ as $x \to c$

4) Thus $f(x) < 0$ is false. Hence its a contradiction.

5. Negate: $\nexists V_\delta, (x \in V_\delta) \wedge (f(x) \leq 0)$

$= \boxed{\exists V_\delta, \ x \in V_\delta \implies f(x) > 0}$

(A) DOUBLE FUNCTIONS

[i] PREFACE

1.) Let $A \subseteq \mathbb{R}$ and let (f, g) be fncs. Defined on $A \to \mathbb{R}$
s.t $f: A \to \mathbb{R}$ $g: A \to \mathbb{R}$

2.) These have properties of [sum, difference, product, multiple]:
a.) $f+g$ b.) $f-g$ c.) fg d. bf

3.) If $h: A \to \mathbb{R}$ s.t $h(x) \neq 0$ $\forall x \in A$, then $\frac{f}{h}$ is the [illegible]

[ii] CONTINUITY OF FUNCTION OPERATIONS

Assumptions
1.) Let $A \subseteq \mathbb{R}$, $(f, g): A \to \mathbb{R}$ s.t (f, g) are continuous.
- let $b \in \mathbb{R}$
- suppose $c \in A$ and continuous $(f, g \mid c)$.

2.) Then:
a.) $(f+g), (f-g), (fg), (bf)$ are continuous in c
b.) $\forall x \in A$, $(h: A \to \mathbb{R}) \wedge (h(x) \neq 0) \Rightarrow \therefore \frac{f}{h}$ is continuous in c

3.) PROOF:
a.) $f(c) = \lim_{x \to c} f(x)$ and $g(c) = \lim_{x \to c} f(c)$.
so: $(f+g)(c) = f(c) + g(c) = \boxed{\lim_{x \to c} (f(x) + g(c))}$ ∎

b.) Since $c \in A$, then $h(c) \neq 0$ s.t $\lim_{x \to c} h(x) \neq 0$,
so: $\frac{f}{h}(c) = \frac{f(c)}{h(c)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} h(x)} = \boxed{\lim_{x \to c} \frac{f(x)}{h(x)}}$ ∎

[iii] CONTINUITY OF DOMAIN

1.) Suppose the previous assumptions then:
a.) $(f+g), (f-g), (fg), (bf)$ is continuous on A
b.) $\forall x \in A$, $(h: A \to \mathbb{R}) \wedge (h(x) \neq 0) \Rightarrow \therefore \frac{f(x)}{h(x)}$ is continuous on A.

2.) REMARK:
a.) If $\varphi: A \to \mathbb{R}$ let $A_1 := \{x \in A \mid \varphi(x) \neq 0\}$.
b.) so: $\boxed{\left(\frac{f}{\varphi}\right)(x) := \frac{f(x)}{\varphi(x)} \text{ for } \forall x \in A_1}$

[iv] EXAMPLES

1) POLYNOMIALS.

Let $P(x)$ be a polynomial function s.t $P(x) = \sum_{n=0}^{N} a_n x^n$.

Then:

$$\lim_{x \to c} P(x) = \lim_{x \to c} \sum_{n=1}^{N} a_n x^n = \sum_{n=0}^{N} a_n c^n = P(c)$$

2). RATIONAL FUNCTIONS

a.) Let $P(x)$, $q(x)$ be polynomial fncs. on $\mathbb{R}$. Then are at most a finite number $\alpha_1 \dots \alpha_m$ of Real Roots of q

b.) If $x \notin \{\alpha_1, \dots, \alpha_m\}$ then $q(x) \neq 0$. s.t.

$r(x) := \frac{p(x)}{q(x)}$ $\forall x \notin \{\alpha_1 \dots \alpha_m\}$

$\Rightarrow r(x) = \frac{p(c)}{q(c)} = \lim_{x \to c} \left(\frac{p(x)}{q(x)} \right) = \lim_{x \to c} r(x)$.

3.) TRIG FUNCTIONS - SINE

a.) $\forall (x, y, z) \in \mathbb{R}$, $|\sin z| \leq |z|$, $|\cos z| \leq 1$.

$\sin x - \sin y = 2 \sin\left[\frac{1}{2}(x - y)\right] \cos\left[\frac{1}{2}(x+y)\right]$

b.) If $c \in \mathbb{R}$, Then:

$\sin(c) - \sin(c) \leq 2 \cdot \frac{1}{2} |x - c| = |x - c|$ s.t $\boxed{|x - c| < \varepsilon}$

$\therefore$ Ergo $\boxed{\sin(c) - \sin(c) \leq |x - c| \text{ is Continuous in } \mathbb{R}}$

4.) TRIG FUNCTIONS - COSINE w/ $|\sin z| \leq |z|$, $|\sin z| \leq 1$

Same as Before;

$\cos x - \cos y = -2 \sin\left[\frac{1}{2}(x+y)\right] \sin\left[\frac{1}{2}(x-y)\right]$

$c \in \mathbb{R} \leftrightarrow \cos x - \cos c = 2 \cdot \frac{1}{2} |c - x| = |x - c|$

for: Cos

5.) $\boxed{\cot x := \frac{\cos x}{\sin x}}$

(2.) COMPOSITION OF CONTINUOUS FUNCTIONS

[i] Let $(A, B) \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. be functions s.t $f(A) \subseteq B$. If f is Continuous at point $c \in A$ and g Continuous at $b = f(c) \in B$, Then $g \circ f : A \to \mathbb{R}$.

s.t: $\boxed{(f(c) \text{ on } A) \wedge (g(c) \text{ on } b \mid f(c) \in B = b) \Rightarrow g \circ f : A \to \mathbb{R}}$

[ii] Proof: (Next Page)

Composition of Continuous Functions (5.2)

① Proof:

1.) Let W be a ε-neighborhood of $g(b)$ s.t. $W_\varepsilon(g(b))$

2.) $\forall$ Continuous $(g|b)$, $(\exists V_\delta(b = f(c))$ $y \in B \cap V \Rightarrow g(y) \in W)$

3.) $\forall$ Continuous $(f|B)$, $(\exists U_\gamma(c)$, $x \in A \cap U \Rightarrow f(x) \in V)$

4.) Since $f(A) \subseteq B$, it follows that if $x \in A \cap U$, then $f(x) \in B \cap V$

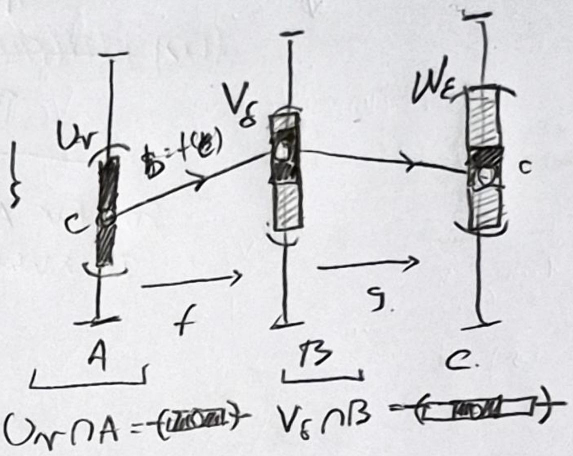
5.) So: $\boxed{g \circ f(x) = g(f(x)) \in W}$

6.) have $W_\varepsilon(g(b))$, then $g \circ f$ is continuous at C.

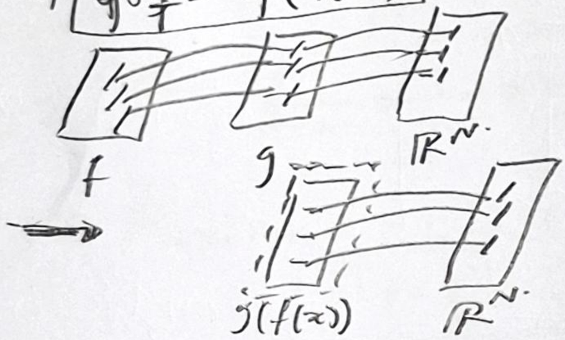
② ILLUSTRATION:

1) a) $\exists V_\delta(b = f(c))$, $y \in (B \cap V) \Rightarrow g(y) \in W$

b) $\exists U_\gamma(c)$, $x \in (A \cap V) \Rightarrow f(x) \in V$

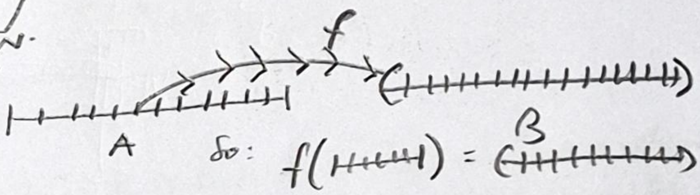


2.) $\boxed{g \circ f = g(f(\cdot))}$



So: $g \circ f(x)$ s.t. $(g, f) : (A, B) \longrightarrow \mathbb{R}$.

3.) $\boxed{f(A) \subseteq B}$:



③ EXAMPLES

[i] 1.) Let $\boxed{g_1(x) := |x|}$ for $\forall x \in \mathbb{R}$.

2.) Then: $\boxed{|g_1(x) - g_1(c)| \leq |x - c|}$ By the triangle inequality

3.) g_1 is continuous at $c \in \mathbb{R}$ and $\forall f : A \to \mathbb{R}$, $\boxed{g_1 \circ f = |f|}$

[ii] $\boxed{g_2(x) := \sqrt{x}}$ $\forall x \geq 0$

Then $g_2(x) \Rightarrow g_2(c)$ is continuous at $c \geq 0$.

1.) Let $f : A \to \mathbb{R}$.

2.) and $\boxed{g_2 \circ f = \sqrt{f}}$

[iii] $\boxed{g_2(x) := \sin x \quad f(x) = \frac{1}{x}}$

Then: $\boxed{g_2 \circ f = \sin\left(\frac{1}{x}\right)}$ and $\sin\left(\frac{1}{x}\right)$ is continuous at every point except $x = 0$.

(2) Following properties

1) $\sqrt{f}(x) := \sqrt{f(x)}$ $(\forall c \in A)$

a) f is continuous so: $\sqrt{f(c)}$

b) $\boxed{f(A) \Rightarrow \sqrt{f(A)}}$

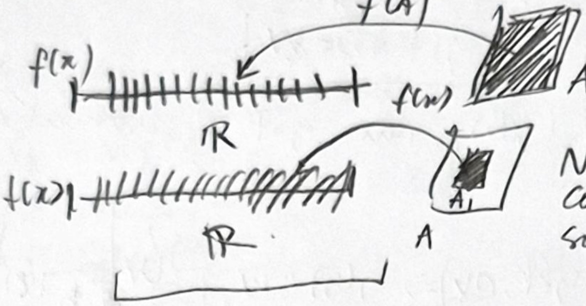
2) $|f(x)| := |f|(x)$ $(\forall c \in A)$

a) f is continuous so: $|f(c)|$

b) $\boxed{f(A) \Rightarrow |f(A)|}$

Iff f is continuous on A.

3)



Continuous?

Not continuous i.e. continuous except at $|A - A_1|$ so A_1 continuous?

So for A to be continuous it must be $\boxed{\text{POINTWISE}}$

RIEMANN INTEGRAL

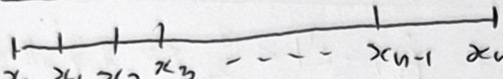
① PARTITIONS and TAGGED PARTITIONS

■ **PARTITIONS**

1) IF $I := [a, b]$ is a closed bounded interval $\in \mathbb{R}$, then
Then a partition I is finite, ordered set $\mathcal{P} := (x_0, x_1, \ldots x_{n-1}, x_n)$
of points in I st $a = x_0 < x_1 \cdots < x_n = b$

i.e.: $\boxed{\mathcal{P} = \{[x_{i-1}, x_i] \mid x_{i-1}, x_i \in \mathbb{R}\}}$ $\triangleq \begin{cases} I_1 [x_0, x_1] \\ I_2 [x_1, x_2] \\ I_3 [x_2, x_3] \end{cases}$

ILLUST



2) To be precise, the notation is $\mathcal{P} = \{[x_{i-1}, x_i]\}_{i=1}^{n}$ set

The NORM/mesh of $\mathcal{P}$ is:

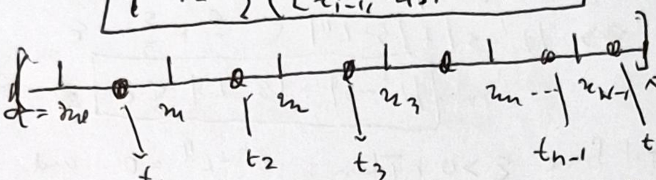
$$\boxed{\|\mathcal{P}\| = \max\{x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}\}}$$ → The NORM

■ THE RIEMANN SUM AS TAGS

3) IF a point t_i in between subintervals I_i, $\forall i \in \mathbb{Z}^+$, then
The points are called TAGS s.t.:

$$\boxed{\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}}$$

ILLUST:

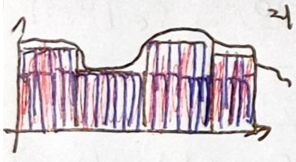


4) IF $\dot{\mathcal{P}}$ is the tagged interval of a partition, we define a Riemann sum as:

$$\boxed{S(f; \dot{\mathcal{P}}) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})}$$

② RIEMANN INTEGRAL

1. DEFINITION
 - a function $f: [a, b] \to \mathbb{R}$ is said to be Riemann Integrable on $[a, b]$ if there exists a number $L \in \mathbb{R}$ s.t. for every $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ s.t. any tagged partition of $[a, b]$ w/ $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$, $|S(f; \dot{\mathcal{P}}) - L| < \varepsilon \leftrightarrow \dot{\mathcal{P}} = \{[a, b], t_i\}$



2) i.e.: $\forall f: [a, b] \to \mathbb{R}$, $\exists L \in \mathbb{R} (\forall \varepsilon > 0 (\exists \delta_\varepsilon > 0,$
then: $\boxed{|S(f; \dot{\mathcal{P}}) - L| < \varepsilon \leftrightarrow \dot{\mathcal{P}} = ([a, b], t) \land S = \sum_i^n f(t_i) \Delta x_i}$

3) If this is true then: $\boxed{\mathcal{R}[a, b] := \text{set of all Riemann Integrable}}$

$\|\mathcal{P}\|$ = Large

$\|\mathcal{P}\|$ = small cuts

$\|\mathcal{P}\|$ = no more

For number L, L is the "limit" of the Riemann sums $S(f, \dot{\mathcal{P}})$ as $\|\dot{\mathcal{P}}\| \to 0$

then: $\boxed{L = \int_a^b f(x)\,dx}$ w/ x_1 $x_1 x_2 x_2 x_3 x_{11}$ ... x_3 → $t \to \infty$

⇒: Area $= \int_a^b f(x)\,dx$

a b

$\|\dot{\mathcal{P}}\| \to 0$, as $t \to$ (to generalize)

* some texts is also noted such
then $L = \int_a^b f_{only}(x)\,dx$
"leads to this"

(iv) UNIQUELY DETERMINED (Th 7.1.2)

Th: $\boxed{\text{If } f \in \mathcal{R}[a,b], \text{ then the value of the Integral is uniquely determined}}$

Proof:

1) Let (L', L'') be satisfactory of Definition and let $\varepsilon > 0$.
Then:
$$\forall (L', L'') \wedge (\varepsilon > 0), \exists \delta'_{\frac{\varepsilon}{2}} > 0 \left(\|\dot{\mathcal{P}}_1\| < \delta'_{\frac{\varepsilon}{2}} \implies |S(f; \dot{\mathcal{P}}_1) - L'| < \frac{\varepsilon}{2} \right)$$
Same: $\boxed{|S(f; \dot{\mathcal{P}}_1) - L'| < \frac{\varepsilon}{2} \iff \|\dot{\mathcal{P}}\| < \delta'_{\frac{\varepsilon}{2}} \quad \exists \delta_{\frac{\varepsilon}{2}} > 0}$

2) Also there exists $\delta''_{\frac{\varepsilon}{2}} > 0$, such:
$$\boxed{|S(f : \dot{\mathcal{P}}_1) - L'| < \frac{\varepsilon}{2} \iff \|\dot{\mathcal{P}}_2\| < \delta''_{\frac{\varepsilon}{2}}}$$

3) Let $\delta_\varepsilon := \min\{\delta'_{\frac{\varepsilon}{2}}, \delta''_{\frac{\varepsilon}{2}}\} > 0$ and $\dot{\mathcal{P}}$ be tagged partition / $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$.
Since both $\|\dot{\mathcal{P}}\| < \delta_{\varepsilon/2}$ and $\|\dot{\mathcal{P}}\| < \delta''_{\frac{\varepsilon}{2}}$ Then:
$\boxed{|S(f; \dot{\mathcal{P}}) - L'| < \frac{\varepsilon}{2}}$ and $\boxed{|S(f : \dot{\mathcal{P}}) - L''| < \frac{\varepsilon}{2}}$

4) Do the Algebra?
$$|L' - L''| = |L' - S(f; \dot{\mathcal{P}}) + S(f; \dot{\mathcal{P}}) - L''| = |L' - S| + |S - L''|.$$
$$\iff |L' - S| + |S - L''| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
So: $\boxed{|L' - S| + |S - L''| < \varepsilon}$ ∎ such

is satisfy

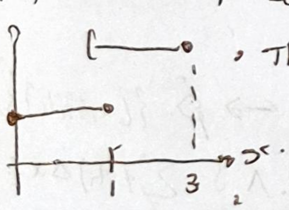
4.1 Since $\varepsilon > 0$, Then: $L' - L'' = 0$ and $0 - S + S - 0$
$-S + S = 0$ so $\boxed{0 < \varepsilon}$ ∎

(B) Some Examples

(1) Every case. $f \in [a,b]$ is in $\mathcal{R}[a,b]$.
• Let $f(x) := k$, $\forall x \in [a,b]$. if $\dot{\mathcal{P}} := \{([x_{i-1}, x_i]), t_i\}_{i=1}^{n}$ Then.
$$\boxed{S(f; \dot{\mathcal{P}}) = \sum_{i=1}^{n} k(x_i - x_{i-1}) = k(b-a)} \iff k\int_a^b dx = kx\Big|_a^b = k(b-a)$$

2) Let $q : [0,3] \to \mathbb{R}$, be defined via $\begin{cases} q_1(x) := 2 \ \forall\, 0 \le x \le 1. \\ q_2(x) := 3 \ \forall\, 1 \le x \le 3. \end{cases}$

$q(x)$



• The integral is $\begin{cases} \int_0^1 q_1(x)\,dx = 2\int_0^1 dx = 2x\Big|_0^1 = 2(1-0) = \boxed{2} \text{ and } S \in [0,3] \text{ is satisfy} \\ \int_1^3 q_2(x)\,dx = 3\int_3^1 dx = 3x\Big|_3^1 = 3(1-3) = 3 - 3 = \boxed{0} \text{ on } S \in [0,3] \end{cases}$

• an Illustration
$$S(q; \dot{\mathcal{P}}) = S(q; \dot{\mathcal{P}}_1) + S(q; \dot{\mathcal{P}}_2) \quad \text{to ensure } \boxed{|S(q; \dot{\mathcal{P}}) - 8| < \varepsilon}$$
$2(1-\delta) \le S(q; \dot{\mathcal{P}}_1) \le 2(1+\delta)$. why $\forall \|\dot{\mathcal{P}}\| < \delta$, if $u \in [0, 1-\delta] \wedge h \in [x_{i-1}, x_i]$
$3(2-\delta) \le S(q; \dot{\mathcal{P}}_2) \le 3(2+\delta)$. Then $x_{i-1} \le 1 - \delta$

$\dot{\mathcal{P}} := [0,3]$
$\|\dot{\mathcal{P}}\| := \delta$
$\begin{cases} \dot{\mathcal{P}}_1 := [0,1] & q(x) = 2 \\ \dot{\mathcal{P}}_2 := [1,3] & q(x) = 3 \end{cases}$

$$\to \boxed{8 - 5\delta \le S(q; \dot{\mathcal{P}}) = S(q; \dot{\mathcal{P}}) + S(q; \dot{\mathcal{P}}) \le 8 + 5\delta} \implies \boxed{|S(q; \dot{\mathcal{P}}) - 8| \le 5\delta}$$

3) Let $h(x) := x$, $\forall x \in [0,1]$. We show that $h \in \mathcal{R}[0,1]$
• TRICK: If $\{I_n\}_{i=1}^{n}$ set $[0,1] \ni I_n$, Then: $\boxed{q_i := \frac{1}{2}(x_{i-1} + x_i)}$ is a midpoint
and:
$$h(q_i)(x_i - x_{i-1}) = \frac{1}{2}(x_i + x_{i-1})(x_i - x_{i-1}) = \frac{1}{2}(x_i^2 - x_{i-1}^2)$$
So: $\boxed{S(h; \dot{Q}) = \frac{1}{2}(1^2 - 0^2) = \frac{1}{2}}$

$\int_0^1 x\,dx$
$= \frac{1}{2}x^2\Big|_0^1$
$= \frac{1}{2}(1-0) = \boxed{\frac{1}{2}}$

Exercise 7.1

(13) Suppose that $f:[a,b]\to\mathbb{R}$ and $f(x)=0$ except for a finite points $c_1 \dots c_n$ in $[a,b]$. Prove that $f\in\mathcal{R}[a,b]$ and $\int_a^b f = 0$.

Ans (Proof): If we say that $f(x)=0$ and $\mathcal{R}[a,b]\ni f$.

then $f(x)=\begin{cases} f(c) & \text{if } c_1\dots c_n \ni f \\ 0 & \text{if } c_1\dots c_n \not\ni f\end{cases}$ so: $\boxed{f(x)\in[0,f(c)]}$

2) and $f:[a,b]\to\mathbb{R}$ and $c_1\dots c_n\in a,b$ then $f(c)\in[a,b]$, $f(c)\in[a,b]$

so: since $\mathcal{R}(a,b)\ni f(a,b)$ then $f(c)$ must be in $\mathcal{R}$.

3) Then $f(x)=\begin{cases} f(c) \\ 0\end{cases} \to \int f\,dx = \begin{cases}\int_a^b f(c)\,dx = f(c)x\Big|_a^b = f_c(b-a)\Big|_0^1 = \boxed{f_c} \\ \int_a^b f\cdot dx = \boxed{0}\end{cases}$

so that! f is zero everywhere other than c, so it could either be zero or discontinuous. Since $\{c_1 \dots c_n\}$ has Lebesgue measure $\to 0$

(14) Suppose that $c\le d$ are points in $[a,b]$. If $\varphi:[a,b]\to\mathbb{R}$ satisfies $\varphi(x)=\alpha>0$ for $x\in[c,d]$ and $\varphi(x)=0$ elsewhere $[a,b]$, prove that $\varphi\in\mathcal{R}[a,b]$ and that $\int_a^b \varphi = \alpha(d-c)$! HINT: Given $\varepsilon>0$, let $\delta_\varepsilon := \frac{\varepsilon}{4\alpha}$ and show if $\|\dot{\mathcal{P}}\|<\delta_\varepsilon$ then we have $\alpha(d-c-2\delta_\varepsilon)\le S(\varphi;\dot{P})\le\alpha(d-c+2\delta_\varepsilon)$.

Ans:

1) Assumptions:
- $c\le d\in[a,d]$
- $\varphi:[a,b]\to\mathbb{R}\ni\varphi(x)=\alpha$
- $\alpha>0$
- $\{x\in[c,d]\ni\alpha$; $x\in[a,b]\ni\varphi$; $x\in$ else $=0$

(3)(1) $\int_a^b\varphi = \alpha(d-c)$, ?

(2) $\varphi\in\mathcal{R}[a,b]$

$\|4\delta_\varepsilon\| = \sqrt{4\delta_\varepsilon} = 2\sqrt{\delta_\varepsilon}\approx 2\delta_\varepsilon$

2) Given $\varepsilon>0$, let $\delta_\varepsilon=\frac{\varepsilon}{4\alpha}$. Then:

$|S(\dot{P}:\varphi)-L|<\varepsilon$ and $S(\dot{P}:\varphi)-L\to L-L=0$ for $\varepsilon>0$

and $\boxed{|S(\dot{\varphi}:\varphi)-L|<\delta_\varepsilon\cdot 4\alpha}$ and: $\delta_\varepsilon=\frac{\varepsilon}{4\alpha}>0$

so: $\delta_\varepsilon=\frac{\varepsilon}{4\alpha}\Big|_{\varepsilon>0}^{\alpha\neq 0}$

3) The norm $\|\dot{\varphi}\| = \max(\Delta x_N)<\delta_\varepsilon$

we have $\begin{cases}\min L(\varphi,P)=\alpha & \text{if in } [a,b], \text{ else } = 0 \\ \max U(\varphi,P)=\alpha & \text{max in } [a,b], \text{ else } = 0\end{cases}$

Infimum → Supremum

$L=\sum_{i=0}^{n-1}\inf_{t\in[x_i,x_{i+1}]} f(x)(x_{i+1}-x_i)$

$U=\sum_{i=0}^{n-1}\sup_{t\in[x_i,x_{i+1}]} f(x)(x_{i+1}-x_i)$

4) Define

$$\boxed{\begin{aligned} L(\varphi,P) &\ge \varphi(d-c-2\delta_\varepsilon) \\ U(\varphi,P) &\le \alpha(d-c+2\delta_\varepsilon)\end{aligned}}$$

5) Thus

$$\alpha(d-c-2\delta_\varepsilon)\le S\le\alpha(d-c+2\delta_\varepsilon)$$

$$\alpha(d-c-2\delta_\varepsilon)\le L\le\int_a^b\varphi\,dx\le U<\alpha(d-c+2\delta_\varepsilon)$$

then $U-L\le 4\alpha\delta_\varepsilon = \varepsilon$ so: $U-L\le\varepsilon$ and $\boxed{\varphi\in\mathcal{R}[a,b]}$

6) So: $\int_d^c\varphi(x)\,dx = \int_d^c\alpha\,dx = \alpha[c-d]$ so $\boxed{\int_b^a\varphi\,dx = \alpha[c,d]}$

(1) EXERCISES 7.12

Let $f : [a,b] \to \mathbb{R}$. Show that $f \notin \mathcal{R}[a,b]$ iff $\exists \varepsilon_0$ so such $\forall n \in \mathbb{N}$,
$\exists (\dot{P}_n, \dot{Q}_n)$ w/ $\|\dot{P}_n\| < \frac{1}{n}$ and $\|\dot{Q}_n\| < \frac{1}{n}$ such

$$\boxed{|S(f;\dot{P}_n) - S(f;\dot{Q}_n)| \geq \varepsilon_0}$$

Ans: 1) Assumptions:

- $\exists \varepsilon_0 > 0$ = arbitrary quantity
- $n \in \mathbb{N}$ = n is in the space of natural numbers
- $\|\dot{P}_n\| \leq \frac{1}{n}$ so: $\|\dot{P}_n\| - \frac{1}{n} < 0$
- $\|\dot{Q}_n\| < \frac{1}{n}$ $\|\dot{Q}_n\| - \frac{1}{n} < 0$
- $\begin{cases} S(f;\dot{P}_n) = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) \\ S(f;\dot{Q}_n) = \sum_{j=1}^{n} f(t_j)(x_j - x_{j-1}). \end{cases}$
- $\mathcal{R}[a,b] = \int_a^b f\,dx.$

2) Theorem:

$\forall f \in \mathcal{R}[a,b], \exists \varepsilon_0 > 0 \left(\forall n \in \mathbb{N} \left(\exists \dot{P}, \dot{Q} ; |S_{f\dot{P}} - S_{f\dot{Q}}| \geq \varepsilon_0 \right) \right).$

3) In the professional:

$$\left|\sum_i^n f(t_i)(x_i - x_{i-1}) - \sum_j^n f(t_j)(x_j - x_{j-1})\right| \geq \varepsilon_0.$$

The formula means:

$S(f;\dot{Q}_n$

ALGEBRAIC PROPERTIES - I

2.1

① ALGEbraic properties in $\mathbb{R}$:

w/ A = Addition
M = Multiplication
D = distributive

(A1) $a+b = b+a$ $\forall a, b \in \mathbb{R}$

(A2) $(a+b)+c = a+(b+c)$ $\forall a, b, c \in \mathbb{R}$

(A3) $\exists 0 \in \mathbb{R}$, $0+a=a$ & $a+0=a$ $\forall a \in \mathbb{R}$

(A4) $\forall a \in \mathbb{R}$, $\exists(-a \in \mathbb{R})$, $a+(-a)=0$ and $(-a)+a=0$

A

(M1) $a \cdot b = b \cdot a$ $\forall a, b \in \mathbb{R}$

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, $\forall a, b, c \in \mathbb{R}$

(M3) $\exists(1 \in \mathbb{R} \mid$ distinct from zero$)$, $1 \cdot a = a$. $a \cdot 1 = a$

(M4) $\forall (a \neq 0) \in \mathbb{R}$, $\exists\left(\frac{1}{a} \in \mathbb{R}\right)$, $a \cdot \left(\frac{1}{a}\right) = 1$ and $\frac{1}{a} \cdot a = 1$

M

(D) $a \cdot (b+c) = (a \cdot b) + (a \cdot c)$ and
$(b+c) \cdot a = (b \cdot a) + (c \cdot a)$.

② **Th 2.1.2 :**

(a) If $z, a \in \mathbb{R}$ w/ $z+a=a$, Then $z=0$

(b) If $(u$ and $b \neq 0) \in \mathbb{R}$ w/ $u \cdot b = b$, Then $u=1$

(c) If $a \in \mathbb{R}$, Then $a \cdot 0 = 0$

Proof: (a) Using (A3), (A4), (A2), hypothesis $z+a=a$ & (A4), we get:

$$z = z+0 = z+(a+(-a)) = z+a+(-a) = a+(-a) = \boxed{0}$$

(b) Using (M3), (M4), (M4), assume $u \cdot b = b$, & (M4),

$$u = \underbrace{u \cdot 1}_{(M3)} = u \cdot \underbrace{\left(b \cdot \frac{1}{b}\right)}_{(M4)} = u \cdot b \cdot \frac{1}{b} = u \cdot \frac{b}{b} = \frac{u \cdot b}{b} = \frac{1}{b} \cdot b = \boxed{1}$$

(c) follows from (a). s.t:

$$a + a \cdot 0 = a \cdot 1 + a \cdot 0 = a \cdot (1+0) = a \cdot (1) = \boxed{a}$$

③ **Th 2.13**

$\boxed{i}$ If ($a \neq 0$ and $b) \in \mathbb{R}$ are s.t $a \cdot b = 1$, Then $b = \frac{1}{a}$

$\boxed{ii}$ If $(a \cdot b) = 0$, Then either $a = 0, b = 0$

Proof $\boxed{i}$ (M3)(M4)(M2) & hypotheses$(a.b) = 1$ &(M3),

we have:

$$\boxed{b = 1 \cdot b = \left(\left(\frac{1}{a}\right) \cdot a\right) \cdot b = \frac{1}{a} \cdot a \cdot b = \frac{1}{a} \cdot 1 = \frac{1}{a}.}$$ ∎

$\boxed{ii}$ It suffices to assume $a \neq 0$ and Prove $b = 0$.

why? we multiply $a \cdot b$ by $\frac{1}{a}$ and apply (M2)(M4)(M3)

$$a \cdot b \cdot \frac{1}{a} = \underbrace{\left(\left(\frac{1}{a}\right) a\right)}_{(M2)} \cdot b = \underbrace{1 \cdot b}_{(M4)} = \underbrace{\boxed{b}}_{(M3)}$$

④ **Th 2.14**

$\nexists r \in \mathbb{Q}$, $r^2 = 2$

Proof:
- Suppose $(p, q) \in \mathbb{Z}$ and $\left(\frac{p}{q}\right)^2 = 2$.

$\boxed{i}$
- Assume $(p, q) \in \mathbb{Z}^+$ and have no common factors other Than 1

(why?) since: $\frac{p^2}{q^2} = 2 \rightarrow \boxed{p^2 = 2q^2} \rightarrow$ even.

- Implying p is even since if odd:

$p = 2n-1 \rightarrow p^2 = (2n-1)(2n-1) = 2(2n^2-2n+1)-1$ is odd

- Therefore, since (p, q) don't have Common factor 2, q must be an odd natural number.

$\boxed{ii}$
- Since p is even, Then $\boxed{p = 2m}$ for some $m \in \mathbb{N}$

hence $\boxed{4m^2 = 2q^2}$ so that $2m^2 = q^2$. Therefore q^2 is even.

- Since q is both even and odd, It's contradictory, ergo false ∎

ORDER Properties of $\mathbb{R}$

(1) $\exists\left(\mathbb{P} \text{ of } \mathbb{R} \mid \mathbb{P} \neq \emptyset\right)$ called the set of POSITIVE $\mathbb{R}$ NUMBERS

[i] If $a, b \in \mathbb{P}$, Then $a+b \in \mathbb{P}$

[ii] If $a, b \in \mathbb{P}$, Then $ab \in \mathbb{P}$

[iii] If a belongs to $\mathbb{R}$, Then exactly one of the following holds: (i) $a \in \mathbb{P}$, $a=0$, $-a \in \mathbb{P}$

* If $a \in \mathbb{P}$, then $a > 0$ is strictly positive

If $a \in \mathbb{P} \cup \{0\}$, we write $a \geq 0$ a is nonneg

If $-a \in \mathbb{P}$, we write $a < 0$.

(2) 2.1.6 Def: Let $a, b \in \mathbb{R}$. Then:

[i] $a-b \in \mathbb{P}$, we write $a > b$ or $b < a$.

[ii] $a-b \in \mathbb{P} \cup \{0\}$, we write $a \geq b$ and $b \leq a$.

- $(a > b,\ a = b,\ a < b)$
- (If both $a \leq b$, $b \leq a$, Then $a = b$.)
- $(a < b < c)$

EXERCISES SECTION 2.1

(1.) If $a, b \in \mathbb{R}$, Prove:

(a.) $a+b=0$ Then $b=-a$

(b.) $(-1)a=-a$

(c.) $-(-a)=a$

(d.) $(-1)(-1)=1$

ANS

(a.) $a+b = a+(-a) \xrightarrow{(A4)} = 0 = a+b$

$a+(-a) = a+b$

$a-a+(-a) = b$

$0+(-a) = \boxed{-a=b}$ ∎

(b.) $(-1)\cdot a = \underbrace{(-1)(1)(a)}_{(M3)} = \underbrace{(-1)(1)(a+0)}_{(A3)}$ ∎

$= (-1)(a+0) = \underbrace{(-1)\cdot a + (-1)\cdot 0}_{(D)} = -a+0 = \boxed{-a}$

(c.) $-(-a) = -\underbrace{((-1)\cdot a)}_{(M3)} = \underbrace{(-1)\cdot((-1)\cdot a)}_{\text{from (b.)}}$

$= (-1)(-a) = (-1)(-a+0) = \underbrace{(-1)(-a)+(-1)(0)}_{(D)}$ ∎

(d.) follows (LoL) $= \boxed{a}$

(3.) Solve each of the eqs by justyfying the appropriate property

(a.) $\boxed{2x+5=8}$

$2x+5-5 = 8-5$ → Subtraction Property of equality

$2x = 3$ → Division property of equality

$\frac{2x}{2} = \frac{3}{2}$

$\boxed{x = \frac{3}{2}}$ → PROOF: $2\left(\frac{3}{2}\right)+5 = 3+5 = \boxed{8}$

QED

(b.) $\boxed{x^2 = 2x}$

$x^2-2x = 2x-2x$ → Subtraction property

$x^2-2x=0$

$x(x-2) = \begin{cases} x=2 \\ x=0 \end{cases}$ →

(9.) Let $\boxed{K := \{ s + t\sqrt{2} \mid s, t \in \mathbb{Q} \}}$ show that

(a) If $x_1, x_2 \in K$, then $x_1 + x_2 \in K$ and $x_1 x_2 \in K$

(b) If $x \neq 0$ and $x \in K$, then $\frac{1}{x} \in K$

Proof. ~~Let~~ If $x_1, x_2 \in K$ Then $(x_1, x_2) := s + t\sqrt{2}$ $s, t, x_1, x_2 \in \mathbb{Q}$

(a) (i) Then $\begin{cases} s_1 + t_1\sqrt{2} = x_1 \\ s_2 + t_2\sqrt{2} = x_2 \end{cases}$ so: $x_1 + x_2 = (s_1 + t_1\sqrt{2}) + (s_2 + t_2\sqrt{2})$

$$\boxed{x_1 + x_2 = \sqrt{2}(t_1 + t_2) + (s_1 + s_2)}$$

since $t_1 + t_2 \in \mathbb{Q}$
$s_1 + t_2 \in \mathbb{Q}$

$\therefore x_1 + x_2 \in K$ ∎

(b) (ii) Similarly: $x_1 x_2 = (s_1 + t_1\sqrt{2})(s_2 + t_2\sqrt{2})$

$= s_1 s_2 + s_1 t_1\sqrt{2} + s_2 t_2\sqrt{2} + t_2 t_1 2.$

$= \cancel{s_1 s_2 +} \sqrt{2}(s_1 t_1 + s_2 t_2) + 2t_2 t_1$

$= \boxed{(s_1 s_2 + 2t_1 t_2) + \sqrt{2}(s_1 t_1 + s_2 t_2)}$

Since $s_1 s_2 + 2t_1 t_2 \in \mathbb{Q}$

(b) If $x \neq 0$ and $x \in K$ then $\frac{1}{x} \in K$

$s\sqrt{2}(+s t\sqrt{2})2$

If $x \neq 0$ Then $(\cancel{x} = s + t\sqrt{2}) \neq 0$

Proof: $x \neq 0$ Then

so: $\frac{1}{x} = \frac{1}{s + t\sqrt{2}} \Rightarrow$ $\frac{s - t\sqrt{2}}{s^2 - 2t^2}$ $s + t\sqrt{2}$

$\frac{1}{x} \cdot \frac{x}{x} = \frac{1}{s + t\sqrt{2}} \cdot \frac{s - t\sqrt{2}}{s - t\sqrt{2}} =$ $(s^2 + 2t^2) + (2st\sqrt{2})$

so: $\boxed{\frac{1}{x} = \frac{s - t\sqrt{2}}{s^2 - 2t^2}}$

Since $s^2 - 2t^2 \in \mathbb{Q}$ This Means that $\frac{s - t\sqrt{2}}{s^2 - 2t^2} \in \mathbb{R}$ and ergo $\frac{1}{x} \in \mathbb{Q}$ ∎

and $\frac{1}{x} \in K$

$\therefore K$ is an ordered field lying in $[\mathbb{Q}, \mathbb{R}]$

ABSOLUTE VALUE and The Real LINE (2.2)

1

$$\boxed{|a| := \begin{cases} a & \text{If } a>0 \\ 0 & \text{If } a=0 \\ -a & \text{If } a<0 \end{cases}}$$

Consequence

ex: $|5| = 5$ $\quad |-8| = 8$

$|a| \geq 0, \ \forall a \in \mathbb{R}$
$|a| = 0$, iff $a = 0$
$|-a| = |a|$.

2

$$\boxed{\begin{array}{l} 1)\ |ab| = |a||b| \ \forall a,b \in \mathbb{R} \\ 2)\ |a|^2 = a^2 \\ 3)\ \text{If } c \geq 0, \text{ Then } |a| \leq c \text{ iff } -c \leq a \leq c \\ 4)\ -|a| \leq a \leq |a| \ \forall a \in \mathbb{R}. \end{array}}$$

Theorem 2.2.2

i Proof:

If $(a \text{ or } b = 0) \Rightarrow |ab| = |a||b| = 0$

1.) If $(a>0, b>0) \Rightarrow ab > 0$ s.t $|ab| = ab = |a||b|$.

If $(a>0, b<0) \Rightarrow ab<0$ s.t $|ab| = -ab = |a||b|$.

etc.... ergo It satisfies $|ab| = |a||b| = 0$ for $\forall$ cases ∎

2.) $|a|^2 = |a||a| = a \cdot a = a^2 = |aa| = |a|^2$

$\forall a^2 \geq 0$, ∎

3.) Suppose $c \geq 0$ is true. Then $|a| \leq c \rightarrow |a| - c \leq 0$

$\Rightarrow -c \leq |a| \Rightarrow -|a| + c \leq 0$ $\quad \boxed{c \geq |a|}$ ≤ 0

$c - |a| \leq 0$

so $c \leq |a|$

So If $(|a| \leq c)$ we have $(a \leq c) \wedge (-a \leq c)$ ∎

If $(-c \leq a \leq c)$ we have $(a \leq c) \wedge (-a \leq c)$.

$\Rightarrow -c \leq |a| \geq c$

4.) Suppose $|a| = c$. Then $|a| \geq 0$.

Then $-|a| \leq 0$. So It follows that:

$\boxed{-|a| \leq a \leq |a|} \rightarrow$ Since for example
$-|5| \leq 5$ ✓ ∎
$5 \leq 5$

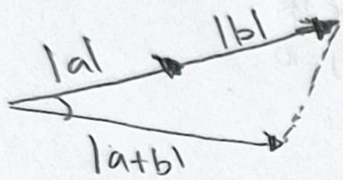
③ Triangle Inequality (2.2.3)

$$\boxed{\text{If } (a, b \in \mathbb{R}) \Rightarrow |a+b| \le |a| + |b|.}$$

Proof: • from 2.2.2 - 4) we have

$-|a| \le a \le |a|$

$-|b| \le b \le |b|$



so it follows: $\boxed{-(|a| + |b|) \le a+b \le |a| + |b|}$

• by 2.2.2 - 3) we have:

$$\boxed{|a+b| \le |a|+|b|}$$ ■

④ $\boxed{\begin{array}{l}\text{If } a, b \in \mathbb{R}, \text{ then:} \\ 1) \ ||a|-|b|| \le |a-b| \\ 2) \ |a-b| \le |a|+|b|\end{array}}$ Corollary 2.2.4.

Proof: 1.) start w/ $a = a-b+b$ and using the triangle Inequality.

[i] $|a| = |(a-b)+b| \le |a+b| + |b|$

$|a|-|b| \le |(a-b)+b| = |a-b|$

[ii] similarly $|b| \le |b-a+a| \le |b-a|+|a|$

$|a|-|b| \ge -|b-a| = -|a-b|$

here we get $\begin{array}{l}|a|-|b| \le |a-b| \\ |a|-|b| \ge -|a-b|\end{array} \Rightarrow \boxed{-|a-b| \le |a|-|b| \le |a-b|}$

Then by 2.2.2-3) it follows that

$$\boxed{||a|-|b|| \le |a-b|}$$ ■

2) We just replace b w/ $-b$ so that:

$|a-b| \le |a+b| \Rightarrow |a-b| \le |a| + |-b|$.

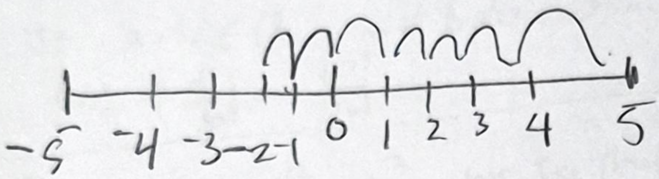
so: $\boxed{|a-b| \le |a|+|b|}$ ■

⑤ $\boxed{\text{If } (a_1 \cdots a_n \in \mathbb{R}) \rightarrow |a_1+a_2+\cdots a_n| \le |a_1|+|a_2|+\cdots}$

Corollary 2.2.5

REAL LINE (2.2)

$\mathbb{R}$



$|-2 + 3| =$

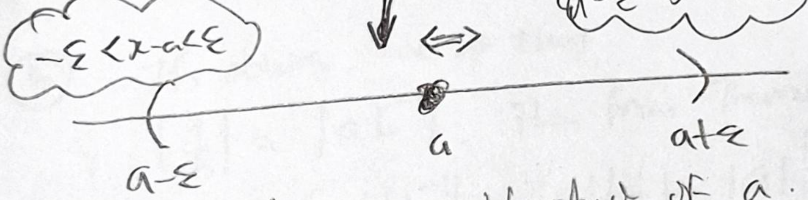
(1.) Let $(a \in \mathbb{R} \wedge \varepsilon > 0) \Rightarrow$ ε-neighborhood of a

is the set $V_\varepsilon(a) :- \{x \in \mathbb{R} \mid |x-a| < \varepsilon\}$.

so. $\boxed{V_\varepsilon(a) := \{x \in \mathbb{R} \mid |(x-a)| < \varepsilon\}}$

Neighborhood of a

$\Downarrow$ $\Leftrightarrow$

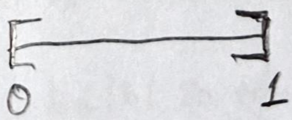


* An ε-neighbourhood of a.

(2) $\boxed{\text{Let } (a \in \mathbb{R}), \text{ If } (x \in V_\varepsilon(a), \forall\, \varepsilon > 0) \Rightarrow : x = a}$

[i] proof: x satisfies $|x-a| < \varepsilon$ $\forall\, \varepsilon > 0$. Then it follows from 2-1.9 that:

$\boxed{|x-a| = 0 \quad \text{hence } x = a}$



[ii] example:

$U := \{x \mid 0 \leq x < 1\}$

If $a \in U$. Then ε be a smaller of 2 numbers $\{a, 1-a\}$

Then $V_\varepsilon(a)$ is contained in U. by which $\forall\, x \in U$,

x has some neighborhood of it in U.

EXERCISE 2.3

(1) If $a, b \in \mathbb{R}$ and $b \neq 0$ show that (a) $|a| = \sqrt{a^2}$
(b) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$.

Ans (a) By $|a|^2 = a^2$, we see that: (Th. 2.2.2 (2))

$$\sqrt{a^2} = \sqrt{|a|^2} = |a|^{2\cdot\frac{1}{2}} = |a|$$

so $\boxed{\sqrt{a^2} = |a|}$

and since $|a| := \begin{cases} a \\ 0 \\ -a \end{cases}$ This means That $|a|$ satisfies

$\sqrt{a^2}$ s.t $|a|^2 = \begin{cases} a^2 \\ 0 \\ -a^2 - a \end{cases}$ so: $\sqrt{|a|^2} = \begin{cases} \sqrt{a^2} = a \\ \sqrt{0} = 0 \\ \sqrt{a^2} = a \end{cases}$

(b) It's obvious. we see that: Theorem 2.2.2 (i), we see that

$\left|\frac{a}{b}\right| = |ab^{-1}|$. Then from

$$|ab^{-1}| = |a||b^{-1}| = |a|\left|\frac{1}{b}\right| = |a|\frac{|1|}{|b|} = \boxed{\frac{|a|}{|b|}}$$

(2) If $a, b \in \mathbb{R}$, show that $|a+b| = |a| + |b|$ iff $ab \geq 0$

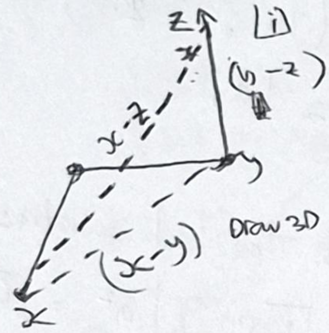
Ans: (i) We see that $|a+b| \leq |a|+|b|$ is satisfied here. But suppose its $-|a| \leq a \leq |a|$ and $-|b| \leq b\ |b|$. Then This can only occur $-|a| = |a|$ and $-|b| = |b|$ Then $|a+b| = |a|+|b|$ is fulfilled since if $(a, b) = 0$. so both are zero

(ii) suppose the case that $a \leq |a|$ and $b \leq |b|$ so that $a, b \geq 0$. Then $|a|\cdot|b| \geq 0$. and Thisnemf: $|a+b| = ||a| + |b|| = |a| + |b|$. This is true since $-|a| \leq |a|$ and $-|b| \leq |b|$ we set $a \leq |a|$ and $b \leq |b|$ s.t and $(-|a| = |a|, -|b| = |b|)$ iff $a \& b$ is 0

(iii)

(3.) If $x, y, z \in \mathbb{R}$, and $x \le z$, show that $x \le y \le z$ iff $|x-y| + |y-z| = |x-z|$ w/ Geometric Interpretation.

Proof : The triangle Inequality collorary $\boxed{|a-b| \le |a| + |b|}$ can be used by which:



i) $|x| + |y| + |y| + |z| = |x| + |z|$

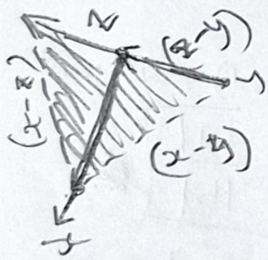
$|x| + 2|y| + |z| = |x| + |z|$

$|x| - |x| + 2|y| + |z| - |z| = 0$

$2|y| = 0 \longrightarrow$ for the case $|a-b| = |a| + |b|$

=

fold it 2D



ii) for $|x-y| \le |x| + |y|$
$|y-z| \le |y| + |z|$ $\Longrightarrow \boxed{z - x = |z - x|}$

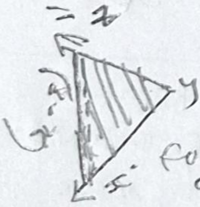
so: $y < x$ and $y > z$ is impossible since this means that

$(y-x) = |x-y| \longrightarrow (x-y)$
$(z-y) = |y-z| \longrightarrow (y-z)$ $\Longrightarrow (x-y) + (y-z) = |x-z|$

$(x-z) = |x-z|$

$\Longrightarrow \boxed{|z-x| \ne |x-z|}$ which is false.

= z



iii) OR: $y < x \le z$

so: $|x-y| = 0$ and $y = x$ which is a contradiction.

fold again as you get only 1 right angle triangle!

(4.) $\boxed{\text{Show that } |x-a| < \varepsilon \text{ iff } a - \varepsilon < x < a + \varepsilon.}$

THEOREMS IN REAL ANALYSIS

1.1.4 Th: If A, B, C are sets, Then (a) $A \setminus B \cup C = A \setminus B \cap A \setminus C$
(b) $A \setminus B \cap C = A \setminus B \cup A \setminus C$

1.3.2 Th: $\boxed{\text{Uniqueness}}$ If S is a finite set, Then number of elements in set S is a unique number in $\mathbb{N}$

1.3.3 Th: The set of Natural numbers is an Infinite set.

1.3.4 Th: (a) If A is a set w/ m-elements and B w/ n elements and If $A \cap B = \emptyset$, Then $A \cup B$ has $m+n$ elements

$A = \{a_i\}_{i=1}^{m}$
$B = \{b_j\}_{j=1}^{n}$

(b) If $A = \{x_i \mid m \in \mathbb{N}\}_{i=0}^{m}$ and $C \subseteq A$ w/ 1 element Then $A \setminus C$ Is a set w/ $m-1$ elements

(c) If C is an Infinite set and B is Finite Then $C \setminus B$ is Infinite.

1.3.5 Th: Suppose (S, T) are sets s.t $T \subseteq S$. Then:
(a) If S is a finite set, Then T is finite →
(b) If T is an Infinite set, Then S is an Infinite set. →

1.3.8 Th: The set $E := \{2n \mid n \in \mathbb{N}\}$ of even numbers is denumerable.
(a) Since the mapping $f: \mathbb{N} \to E$ defined $f(n) := 2n$ $\forall n \in \mathbb{N}$, Is a bijection of $\mathbb{N}$ + E.
Similarly, the set $O := \{2n-1 \mid n \in \mathbb{N}\}$ of odd Natural Numbers Is denumerable

(b) $\forall x \in \mathbb{Z}$, x is denumerable.

(c) $A = \{a_i\}_{i=1}^{n}$, $B = \{b_j\}_{j=1}^{m}$, $A \cup B$ Is denumerable.

1.3.8 Th: The set $\mathbb{N} \times \mathbb{N}$ Is denumerable.

1.3.9 Th: $\forall T \subseteq S$, (a) If S Is countable, Then T is countable
(b) If T Is uncountable, Then S is uncountable

1.3.10 Th: (a) S Is a countable set
(b) $\exists$ Surjection $(\mathbb{N} \to S)$
(c) $\exists$ Injection $(S \to \mathbb{N})$

1.3.11 Th: The set of $\mathbb{Q}$ of all rational numbers Is denumerable.

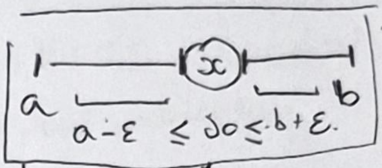
1.3.12 Th: If A_m is a countable set $\forall (m \in \mathbb{N})$, Then.

+$\nabla_\mu V^\mu$:

EXERCISE 2.2

NEW NEIGHBOR

(14) Let $\varepsilon > 0$, $\delta > 0$ and $a \in \mathbb{R}$. Show that $V_\varepsilon(a) \cap V_\delta(a)$ and $V_\varepsilon(a) \cup V_\delta(a)$ are γ-neighb. for $\forall$ appropriate values of γ.

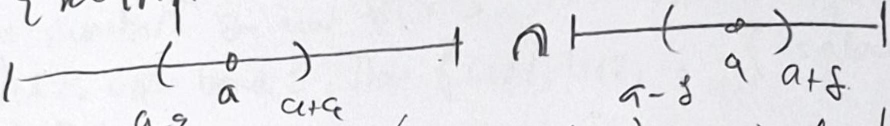


Ans: 1.) $V_\varepsilon(a) := \{ x \in \mathbb{R} \mid |x-a| < \varepsilon \}$

$V_\delta(a) := \{ x \in \mathbb{R} \mid |x-a| < \delta \}$

$V_\varepsilon \cap V_\delta := \{ x \in \mathbb{R} \mid |x-a| < \varepsilon \} \cap \{ x \in \mathbb{R} \mid |x-a| < \delta \}$

$V_\varepsilon \cup V_\delta := \{ x \in \mathbb{R} \mid |x-a| < \varepsilon \} \cup \{ x \in \mathbb{R} \mid |x-a| < \delta \}$



2.) basically: $(-\varepsilon < |x-a| < \varepsilon)$ or $(-\delta < |x-a| < \delta)$ so: a is contained in neighborhood $(V_\varepsilon, V_\delta)$ s.t. If $(x \in V_\varepsilon(a), \forall \varepsilon > 0)$, $x = a$. Thus:

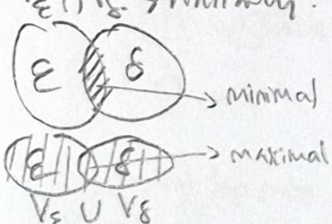
$\boxed{\forall (V_\varepsilon, V_\delta),\ V_\varepsilon(a) = (a-\varepsilon, a+\varepsilon)\ \ V_\delta(a) = (a-\delta, a+\delta)}$ w/ $\boxed{x=a}$

3.) Many; $V_\varepsilon \cap V_\delta = (a - \min(\varepsilon,\delta), a + \min(\varepsilon,\delta)) = V_\gamma(a)$

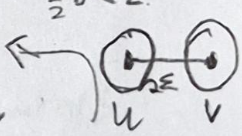
and $\gamma = \min(\varepsilon, \delta) > 0$. so $\boxed{V_\gamma(a)}$, $\forall \gamma = \max(\varepsilon,\delta)$

Similarly: $V_\varepsilon \cup V_\delta = (a - \max(\varepsilon,\delta), a + \max(\varepsilon,\delta)) = V_\gamma(a)$.

and $\gamma = \max(\varepsilon,\delta) > 0$, so: $\boxed{V_\gamma(a)}$ $\forall \gamma = \max(\varepsilon,\delta)$



$d(a,b) = 2\varepsilon$ s.t. $\frac{1}{2}a < \varepsilon$, $\frac{1}{2}b < \varepsilon$



(15) THE NULL NEIGHBOR: $\exists (V_\varepsilon(a), V_\varepsilon(b))$ s.t. $U \cap V = \emptyset$.

Let $a, b \in \mathbb{R}$ and $a \neq b$. Then

PROOF: If $V_\varepsilon(a), V_\varepsilon(b)$ conditions are satisfied Then $V_\varepsilon := (a+\varepsilon, a-\varepsilon)$ and $V_\varepsilon(b) := (b+\varepsilon, b-\varepsilon)$. Then Like The previous problem:

$V_\varepsilon \cap V_\varepsilon = ((a-\varepsilon, a+\varepsilon), \cap (b-\varepsilon, b+\varepsilon)) \Rightarrow a+\varepsilon = \frac{2a+b}{2} = a + \frac{b}{2}$, $b - \varepsilon = \frac{2b+a}{2} = b + \frac{a}{2}$

and: $a + \frac{b}{2} < b + \frac{a}{2} \rightarrow \boxed{U \cap V = \emptyset}$ This is basically a Hausdorff space

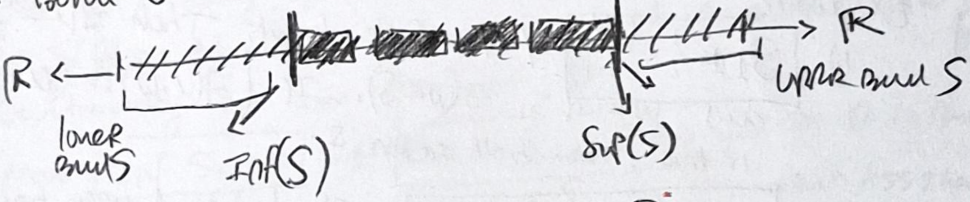
SUPREMA & INFIMA (2-3)

(1) Let S be a nonempty subset of $\mathbb{R}$.

1) S is bounded above iff $\exists u \in \mathbb{R}$, $s \leq u$; $\forall s \in S$ => :: upperbounds S
2) S is bounded below iff $\exists w \in \mathbb{R}$, $w \leq s$; $\forall s \in S$ => :: Lowerbounds S
3.) Let X be a set. X is bounded if (1.), 2.)) is true, and unbounded if one of them is false

Definition 2.3.1

[i] Example: $S := \{x \in \mathbb{R} \mid x < 2\}$ is bounded above; The number 2 and $\forall (x > 2)$ is an upper bound of S. s.t if u is upper bound S, Then $\{u+1, u+2, \ldots\}$ is also upper bound S.



[ii] Let S be a non empty set in $\mathbb{R}$. Then:

1) $\sup\{S\} := \{u \mid u$ is up-Bound(S) and if v is any upbound S, Then $u \leq v\}$
2) $\inf\{S\} := \{w \mid w$ is LowBound(S) and if t any lower S, $t \leq w\}$

Definition 2.3.2

[iii] Some properties:

(1.) (1) To show $u = \sup(S)$, show that 1.), 2.) of definition 2.3.2 holds

(2.) How to show? well there's Alternates:

1.) u is an upper bounds S
1') $s \leq u$, $\forall s \in S$

2) If v is any upper bound S Then $u \leq v$
2') If $z < u$, Then z isn't an upper bound of S
2'') If $z < u$, Then $\exists s_z \in S$ s.t $z < s_z$
2''') If $\varepsilon > 0$, Then $\exists s_\varepsilon \in S$ s.t $u - \varepsilon < s_\varepsilon$.

(2) LEMMA's 2.3.3 and 2.3.4

[i]
1) $u = \sup(S)$ $\forall S \in \mathbb{R}$ iff $\begin{cases} (1) & s \leq u \quad \forall s \in S \\ (2) & \text{if } v < u, \ \exists s' \in S \text{ s.t } v < s' \end{cases}$
2) $u = \sup(S)$ $\forall S \in \mathbb{R}$ iff $\forall \varepsilon > 0$, $\exists (s_\varepsilon \in S)$ s.t $u - \varepsilon < s_\varepsilon$)

ii PROOF (Lemma 2.3.4)

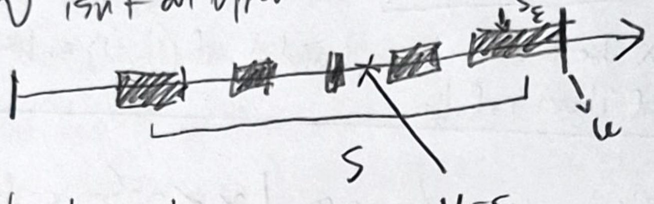
If ($u :\triangleq$ upper bound S that satisfies $v < u$) $\Rightarrow \varepsilon := u - v$. (Lemma 2.3.3)

If ($\varepsilon := u - v$) $\Rightarrow \varepsilon > 0$

If ($\varepsilon > 0$) $\Rightarrow (\exists s_\varepsilon \in S)$, $v = u - \varepsilon < s_\varepsilon$)

$\therefore$ Therefore v isn't an upper bound of S s.t $\boxed{u = \sup S}$ ✓ ∎

Illustration:



Remarks:

The converse is also true s.t: $u - \varepsilon$

$u = \sup S \quad \forall \varepsilon > 0$, and since $u - \varepsilon < s_\varepsilon$ Then $u - \varepsilon$ isn't

The upper bound of S, s.t $\boxed{s < u - \varepsilon}$. so $\boxed{u - \varepsilon < s_\varepsilon}$

iii Examples:

1.) $\boxed{S_1 \triangleq \{\cdot\}_{i=1}^{N}}$ w/ $N :=$ finite elements. Then $\exists (u \in S)$, s.t $u = \sup S_1$ and $w = \inf S_1$ $\exists (w \in S)$. If $(\exists (u, w) := (u = \sup S) \land (w = \inf S))$ is true, Then both are in S.

2.) $\boxed{S_2 := \{x \mid 0 \leq x \leq 1\}}$. Clearly has 1 upper bound which is 1. easy to proof. If $(v < 1) \Rightarrow \exists (s' \in S_2)$, $v < s'$. Therefore v isn't an upper bound of S_2 and v is an arbitrary number $v < 1$. So. $S_2 = 1$.

3.) $\boxed{S_3 := \{x \mid 0 < x < 1\}}$ also has 1 as an upper bound w/ (2.)'s Argument

② Completeness in $\mathbb{R}$ – Definition 2.3.6

$$\boxed{\begin{array}{l} \text{if } (\bar{S} := \{-s \mid s \in S\}) \Rightarrow \exists (u := \sup \bar{S}) \in \mathbb{R}. \\ \text{if } (S := \{s \mid s \in S\}) \Rightarrow \exists (-u := \inf S) \in \mathbb{R} \end{array}}$$

EXERCISE 2.3

1. Let $\boxed{S_1 := \{ x \in \mathbb{R} \mid x \geq 0 \}}$. Show That In detail S_1 has lower bounds but no upper bounds. Show that $\inf S_1 = 0$

ANS: Let u be an upper bound. Then $v < u$ is satisfied. we want to

[i] show that $u \notin S$ s.t $\nexists (u := \sup(S))$. For $u := \sup(S)$ be valid it must at least have 1 upper bound.

[ii] If a set has 1 up-bound, Then it has many (2.31 definition). and The sequence s.t $(u+1, u+2, \ldots) := \sup(S)$. and

$\{u+n \mid n \in \mathbb{R}\} \equiv \{u+x \mid x \in \mathbb{R}\}_x$ an as $x \to \infty$, so $\nexists (u := \sup(S))$ ■

There's Infinitely upperbounds, s.t it "explodes".

ANS 2: Assume M having upperbound S_1 for $\boxed{x \leq M \ \forall (x \geq 0)}$. But

① for $\forall x \in M$, we can choose $x = M+1$, since $M+1 > 0$, Contradicting M is an upperbound. ■

[ii] Now $\boxed{\inf S_1 = 0}$ s.t $\boxed{\forall x \in S_1,\ x \geq 0}$ and so 0 is lowerbound S. and $\forall \varepsilon > 0$, $0 + \varepsilon = \varepsilon$, $\varepsilon \in S_1$, so no number less than 0 can be The $\inf(S_1)$. ■

2. Let $\boxed{S_2 := \{ x \in \mathbb{R} \mid x > 0 \}}$ Does S_2 have lower bounds but no upper bounds? does $\inf(S_2)$ exist? Does $\sup(S_2)$? Proofe it!

ANS: Let M be an upperbound. Then $x < M$ s.t $\forall x > 0$. $\forall x \in M$, $M > -1$ and $(0 > -1)$ meaning $\nexists \sup$

Then $x = M+1$ so $M+1 > 0$ and

Let N be a lowerbound. Then $x \geq N$. So $\forall x > 0$, and $\forall x \in N$, we can choose: $x \in (0, N)$ s.t $\exists (x \in S_2)$. But This means that $N > 0$ meaning that $N = 0$ isn't the lower bound. Thus $\boxed{\exists \inf(S_2),\ \inf S_2 = 0}$

3. Let $\boxed{S_3 = \{ \frac{1}{n} \mid n \in \mathbb{N} \}}$. Show that $\sup S_3 = 1$ and $\inf S_3 \geq 0$.

ANS: Let M be an upperbound in S_3. So choose M s.t $n \in M$ and M is The upper bound of $\frac{1}{n}$. So $n = M+1$ s.t $\frac{1}{n} = \frac{1}{M+1} > 1$

We know $M \neq -1$ or $M+1 \neq 0$. So $M+0 = 1$ is true but ii If L is the lower bound choose (n, L) s.t n is the "cap" while L is the lowest. we see that $L \neq 0$ since $\frac{1}{n} \to \frac{1}{0} X$.

So: $L \geq 0$. Suppose the case $L < 0$. Then it's also true for n.

But since $n \in \mathbb{N}$, Then it's false (we can't have $-\frac{1}{0.2}, \frac{1}{0.3}$ since it's'nt Integer)

So: $\boxed{L \geq 0}$.

iii Thus for M: $\frac{1}{M+1} > 0$ and $\frac{1}{0+1} = \frac{1}{1} = \boxed{1}$

and $1 > u$.

Ans II - Limits : We wanna show that: $\lim_{n \to \infty} \frac{1}{n} = 0$ and $\forall n \in \mathbb{N}, \frac{1}{n} \leq 1$.

i for $\lim_{n \to \infty} a_n = 0$, $\lim_{n \to \infty} \frac{1}{n} = \left\{ \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \frac{1}{\infty} \right\} = 0$.

and $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t $\forall n \geq N$, $\left| \frac{1}{n} - 0 \right| < \varepsilon$

and $\lim_{n \to \infty} \left| \frac{1}{n} - 0 \right| < \lim_{n \to \infty} \varepsilon = \lim_{n \to \infty} \left| \frac{1}{n} \right| < \lim_{n \to \infty} \varepsilon$

$= \boxed{0 < \varepsilon \text{ (as } n \to \infty)}$ which is true!

ii since $0 < \varepsilon$, as $\lim_{n \to 1} \frac{1}{n} : \left\{ \frac{1}{3}, \frac{1}{2.0}, \frac{1}{1.9}, \frac{1}{1.8}, \frac{1}{1.7}, \cdots \frac{1}{1} \right\}$

Then $\frac{1}{n} = \frac{1}{1} = \boxed{1}$ and $\boxed{0 < 1}$ So we say:

$\therefore \boxed{\begin{array}{l} Inf(S_3) = 0 \\ Sup(S_3) = 1 \end{array}}$ ■

7. Let $S \subseteq \mathbb{R}$ be nonempty. Show that $u \in \mathbb{R}$ is an upper bound of S iff the conditions $\boxed{(t \in \mathbb{R} \wedge t > u) \Rightarrow t \notin S}$

Ans: Since $t > u$, Then if u is the upper bound of S, Then $t > v$, s.t only if $t \in S$ and $t := Upb(S)$

It's a contradiction, since $t > v$, Then v is the upper bound in S. ⊠

But since $t \notin S$,

$$\frac{P \wedge Q}{T} \quad \frac{R}{T} \quad \frac{\square}{T} \quad \text{s.t} \quad \left.\begin{array}{l} P = t \in S \text{ is true} \\ Q = t > v \text{ is true} \end{array}\right\} \text{True } \square$$

Remark:

$\quad T \quad F \quad F \qquad R = t \notin S$ is false

If $t \in \mathbb{R}$ and $t \in S$,

Then $t \leq u$ since v is the upper bound but $t > v$! contradiction

Then the statement is false

LIMIT THEOREMS (4.2)

① Let $A \subseteq \mathbb{R}$, let $f: A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A. We say f is bounded on neighborhood c if there $\exists V_\delta(c)$ s.t. $f(x) \leq M$ $\forall$ $A \cap V_\delta(c) \ni x$.

so: $\boxed{\exists (V_\delta(c) \text{ w/ } M > 0) \text{ s.t. } |f(x)| \leq M \ \forall (x \in A \cap V_\delta)}$

Definition 4.2.1

i Theorem 4.22:

$\boxed{\text{If } A \subseteq \mathbb{R} \text{ and } f: A \to \mathbb{R} \text{ has a limit } c \in \mathbb{R} \text{ then } f \text{ is bounded on } V_{(\cdot)}(c)}$

Proof: If $\left(L := \lim_{x \to c} f(x)\right) \implies \forall \varepsilon = 1, \left(\exists \delta > 0, 0 < |x - c| < \delta\right)$

1.) If $\left(\forall \varepsilon > 1, \left(\exists \delta > 0, 0 < |x-c| < \delta\right)\right) \implies |f(x) - L| < 1$.

If $\left(|f(x) - L| < 1\right) \implies \boxed{|f(x)| - |L| \leq |f(x) - L| < 1}$

If $\left(x \in \left(A \cap V_\delta(c)\right)\right), x \neq c$, then $|f(x)| \leq |L| + 1$, we take

2) Therefore, $M = |L| + 1$ while if $c \in A$, we take $M := \sup\{|f(c)|, L+1\}$.

3.) It follows: If $(x \in A \cap V_\delta(c)) \implies f(x) \leq M$. Thus $\boxed{f \text{ is bounded in } V_\delta(c)}$

ii Remark: Basically if $\lim_{x \to c} f(x)$, then $\exists \delta < 0$, $0 < |x - c| < \delta$ and $\boxed{f(x) - L < 1}$ since $\varepsilon = 1$. So: $\lim_{x \to c} f(x) = f(c)$, and: $f(x) - f(c) < 1$ and

② Let $A \subseteq \mathbb{R}$ and let $f + g$ be functions defined on $(A, \mathbb{R})$. We define some properties:

i

1) $(f+g)(x) = f(x) + g(x)$ 2.) $(f-g)(x) = f(x) + g(x)$

3.) $(f \cdot g)(x) = f(x) g(x)$ 4.) $(b f)(x) = b \cdot f(x)$ $\forall x \in A$, w/ $b = \text{cons.}$

5.) $\left(\frac{f}{h}\right)(x) = \frac{f(x)}{h(x)}$ $\forall x \in A$.

Definition 4.2.3.

ii Theorem 4.24: Let $c \in \mathbb{R}$, be a cluster point of A. Let $b \in \mathbb{R}$, then we can define some properties.

1.) $\lim_{x\to c}(f+g) = L+M$ 2.) $\lim_{x\to c} fg = LM$ 3.) $\lim_{x\to c}(f-g) = L-M$

4.) $\lim(bf) = bL$ w/ $\begin{cases}\lim_{x\to c} f(x) = L \\ \lim_{x\to c} g(x) = M\end{cases}$

iii Examples:

1.) $\lim_{x\to c} \frac{1}{x} = \frac{1}{\lim_{x\to c} x} = \boxed{\frac{1}{c}}$

2.) $\lim_{x\to 2}\left(\frac{x^3-4}{x^2-1}\right) = \frac{\lim_{x\to 2}(x^3-4)}{\lim_{x\to 2}(x^2+1)} = \frac{2^3-4}{2^2+1} = \frac{8-4}{4+1} = \boxed{\frac{4}{5}}$

x^2-4:
$=2^2-4=0$

$3(2)-6$
$6-6=0$

3.) $\lim_{x\to 2}(x^2+1)(x^3-4) = \lim_{x\to 2}(x^2+1)\cdot\lim_{x\to 2}(x^3-4)$
$= (2^2+1)(2^3-4) = (4+1)(8-4) = 5\cdot 4 = \boxed{20}$

4.) $\lim_{x\to 2}\frac{x^2-4}{3x-6} = \frac{(x+2)\cancel{(x-2)}}{3\cancel{(x-2)}}\cdot\lim_{x\to 2} = \lim_{x\to 2}\frac{x+2}{3} = \boxed{\frac{4}{3}}$

② Let $A\subseteq\mathbb{R}$ w/ $f_1 \ldots f_n$ functions on $A\to\mathbb{R}$ and c a cluster.

If $\forall L_k$ $L_k := \lim_{x\to c} f_k$ $\forall k\in\mathbb{Z}^n$

i Then: $L_1+L_2+L_3+\ldots L_n = \lim_{x\to c}(f_1+f_2+\ldots+f_n)$

$L_1\cdot L_2\cdot L_3\cdot\ldots\cdot L_n = \lim_{x\to c}(f_1\cdot f_2\cdot f_3\cdot\ldots)$

Thus: $\boxed{\sum_n L_n = \lim_{x\to c}\sum_n f_n(x)}$; $\boxed{\prod_n L_n = \lim_{x\to c}\prod_n f_n(x)}$

ii In particular: $\boxed{L^n = \lim_{x\to c}(f(x))^n}$

EXERCISE 4.2

(12) Let $f: \mathbb{R} \to \mathbb{R}$ such that $f(x+y) = f(x) + f(y)$ $\forall x, y \in \mathbb{R}$.
Assume $\lim_{x \to 0} f = L$ exists. Proof That $L = 0$ and Then Proof That f has a limit at every point $c \in \mathbb{R}$.

* Hint: [1^{st} note $f(2x) = f(x) + f(x) = 2f(x)$ $\forall x \in \mathbb{R}$. Also $f(x) = f(x-c) + f(c)$].

Ans : Since $\lim_{x \to 0} f(x) = L$ exists, we continue to proof That.

indeed $\lim_{x \to 0} f(x) = f(0) = 0$.

CONTINUOUS FUNCTIONS (5.1) I

(E) **DISCONTINUITY**

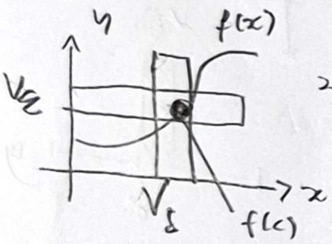
[i] Let $A \subseteq \mathbb{R}$ and let $f: A \to \mathbb{R}$ and $c \in A$.
f is continuous at c if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. if $|x-c| < \delta \Rightarrow |f(x) - f(c)|$

So: $\boxed{\text{Continuous}(f) := \forall \varepsilon > 0, [(\exists \delta > 0), |x-c| < \delta \rightarrow \therefore [f_x - f_c] < \varepsilon]}$

[ii] Proof: Let P be the proposition that:

1.) $P: \forall \exists \delta > 0, [|x-c|] < \delta \Rightarrow \therefore f(x) - f(c) < \varepsilon$

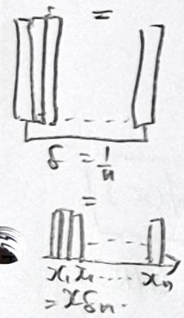
Then $NEG(P) = \{|x-c| < \delta\} \wedge \{|f(x) - f(c)| \geq \varepsilon\}$ wh/ $NEG(P)$ = The Negation of P.



2.) · Let $\delta_n = \frac{1}{n}$ s.t we divide delta n's into n slices of chunks
· Then $x_n := x\delta_n$ (see figure left).

3.) Then: $x_n(\delta_n) \triangleq \boxed{|x_{\delta_n} - c| < \frac{1}{n}}$

OR: $\boxed{x_{\delta_n} - c - \frac{1}{n} < 0}$



4.) observe that $f(x - c - \frac{1}{n}) = f(x) - f(c) - f(\frac{1}{n})$.

Then $f(|x-c-\frac{1}{n}|) = |f(x) - f(c) f(\frac{1}{n})| < 0$.

5.) Then $\{|f(x) - f(c) - f(\frac{1}{n})| := J\} < 0$

and $\{f(x) - f(c) - f(\frac{1}{n}) : J\} \geq \varepsilon$

$\therefore$ So: $\varepsilon < J < 0$ and $\varepsilon < 0$. But $\varepsilon > 0$ so

This contradicts the initial claim that:

$\boxed{\forall \varepsilon > 0, P}$ But we get $\boxed{\forall \varepsilon > 0 \text{ and } P \text{ shows } \varepsilon < 0}$

A contradiction!

[iii] **EXAMPLES:**

1.) $f(x) = x^2$ is Continuous $(\mathbb{R})$.
$\Rightarrow$ If $c \in \mathbb{R}$, Then $\lim_{x \to c} f = c^2$ The $\lim_{x \to \infty} = \infty^2 = \boxed{\infty}$. ✓

2.) $g(x) = x$ is Continuous $(\mathbb{R})$
$\Rightarrow$ If $c \in \mathbb{R}$, Then $\lim_{x \to c} f(x) = \lim_{x \to c} x = c$ So $\boxed{f(c) = c}$ ✓

(2) NEIGHBORHOOD

[i] A function $f: A \to \mathbb{R}$ is continuous at point $c \in A$ iff. given any ε-Neighb.

1.) $V_\varepsilon(f(c))$, $\exists\,\delta$-neighborhood $V_\delta(c)$ of c s.t. if x is any point in $V_\delta \cap A$, Then $f(x)$ belongs to $V_\varepsilon(f(c))$ that is:

$$\boxed{f(A \cap V_\delta(c)) \subseteq V_\varepsilon(f(c))}$$

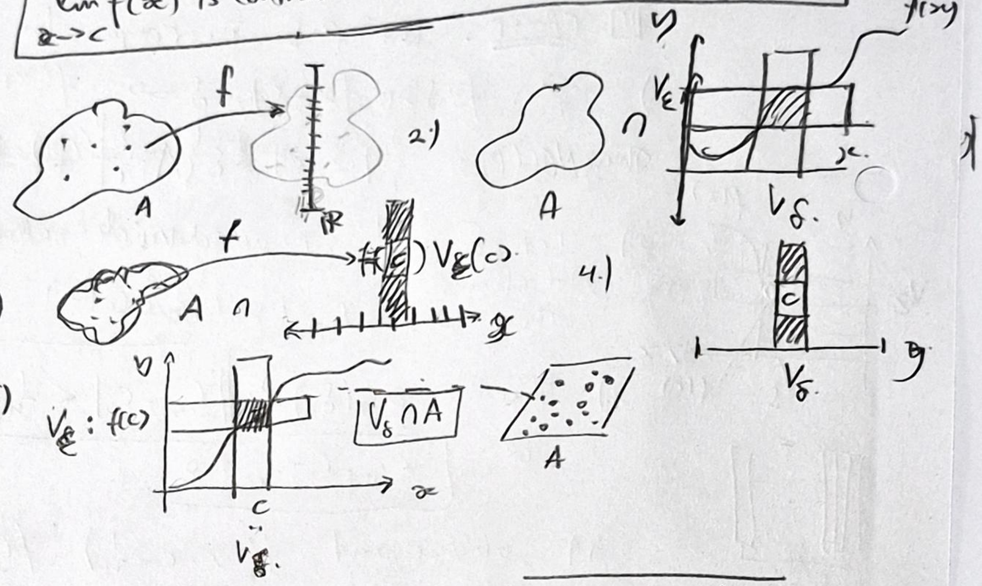
2.) $\text{Continuous}(c|A) \iff f(A \cap V_\delta(c)) \subseteq V_\varepsilon(f(c))$

$\text{Continuous}(c|A) = \text{Continuous}(f: A \to \mathbb{R}\ (c)\ |\ \text{as } x \to c)$

So: $\boxed{\lim_{x \to c} f(x) \text{ is continuous} \iff f(A \cap V_\delta(c)) \subseteq V_\varepsilon(f(c))}$

[ii] VISUAL:

1.) 2.) 3.) 4.) 5.)



[iii] REMARK:

1) If $c \in A$, Then continuous(f): $\boxed{f(c) = \lim_{x \to c} f(x)}$

2.) If c is a cluster point of A, then

a.) f must be defined at c b.) limit f at $c := \exists\,\mathbb{R}$.

c.) These 2 values must be equal.

$\therefore$ So: $\boxed{\text{As } x \to c, \text{ iff}; (f \text{ is at } c) \wedge (f(c) \in \mathbb{R}) \wedge (f(c) = \lim_{x \to c} f(x))}$

otherwise $f(x)$ isn't continuous.

3.)

(3) SEQUENTIAL CRITERION & DISCONTINUITY CRITERION.

proof is on next page

[i] Sequential:

1) Function $f: A \to \mathbb{R}$ is continuous at point $c \in A$ iff $\forall x_n \in A$, $f(x_n) \to c$.

So: $\boxed{\forall x_n \in A \,|\, A \subseteq \mathbb{R},\ (f(x_n) \to f(c)) \Rightarrow \therefore \text{continuous}(f|c)}$

[ii] Discontinuity

2.) $\boxed{\forall x_n \in A \,|\, A \subseteq c,\ f(x_n) \not\to f(c) \Rightarrow \therefore \text{Discontinuous}(f|c).}$