QUANTUM MECHANICS:

THEORY & EXPERIMENT

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(1) POLARIZATION

$\boxed{k = \frac{2\pi}{\lambda}}$ or $\boxed{k = \frac{\omega}{c}}$ w/ $\lambda = \frac{c}{f} \Rightarrow \frac{1}{\lambda} = \frac{f}{c}$

and $\boxed{\omega = 2\pi f}$

ANGULAR FREQUENCY of a WAVE VECTOR $\vec{k} = k \vec{u}_z$.

[1] POLARIZATION VECTOR:

$$\boxed{\vec{E} = \vec{E}_x \vec{u}_x + \vec{E}_y u_y}$$

and $\left\{ \begin{array}{l} \nabla \times E = \frac{\partial B}{\partial t} \\ \nabla \cdot E = \frac{\rho}{\varepsilon_0} \end{array} \right\}$ SAMPLE OF 2 MAXWELL'S EQUATIONS

1) w/ $E_x = E_{0x} \cos(kz - \omega t)$

$E_y = E_{0y} \cos(kz - \omega t + \phi)$.

then $\frac{\partial E_x}{\partial z^2} = -k^2 E_{0x} \cos(kz - \omega t) = -k^2 E_x$

$\frac{\partial E_y}{\partial z^2} = -k^2 E_{0y} \cos(kz - \omega t) = -k^2 E_y$.

a) $\frac{\partial^2 E_x}{\partial t^2} = -\omega^2 E_{0x}$ $\quad \frac{\partial^2 E_y}{\partial t^2} = -\omega^2 E_y$.

$-k^2 E_x - k^2 E_y - \omega^2 E_x - \omega^2 E_y + m^2(E_x + E_y)$

b) then $(\nabla^2 + \partial t^2 + m^2)\vec{E} = -k^2 E_x - k^2 E_y + \omega^2 (E_x + E_y) + m^2 (E_x + E_y) = 0$

$= -k^2(E_x + E_y) + \ldots$ $\quad \boxed{k^2 + \omega + m^2 = 0}$

$-(k^2 + \omega^2 + m^2) = 0 \rightarrow$

HIGH SPEED PARTICLE FIX

$\omega^2 = c^2 k^2 + \left(\frac{mc^2}{\hbar}\right)^2$ and $k^2 = \frac{\omega^2}{c^2}$ then $\frac{\omega^2}{c^2} + \omega^2 + m^2 = \omega^2\left(\frac{1}{c^2} + 1\right) + m^2 = 0$

$\omega^2 = c^2\left(\frac{\omega^2}{c^2}\right) + \left(\frac{mc^2}{\hbar}\right)^2$ $\quad \frac{1}{c^2} + 1 + m^2 = \frac{c^2 + 1}{c^2} + m^2 = 0$

$= \omega^2 + \left(\frac{mc^2}{\hbar}\right)^2$

c.) we: $\boxed{c^2 \omega^2 = \omega^2 + c^2 m^2}$ $\quad \boxed{\frac{c^2 + 1}{c^2} = (mc)^2 - 1}$

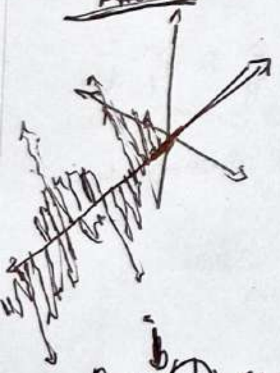
and: $\boxed{\omega^2 = \frac{c^2 m^2}{c^2 - 1}}$ $\quad$ and if $c^2 = 1$ then $\omega^2 = \frac{m^2}{0}$ can't be ZERO!

then $\left(\frac{mc}{\hbar}\right)^2 = 0$ and $\boxed{m = 0}$

If $m^2 = 0$ then: $\omega^2 = 0$

in the field equation is only true if MASS is 0.

POLARIZED RADIATION



2) $E_x = E_0 e^{i(kz - \omega t)} \rightarrow E = E_{0x} e^{i(kz - \omega t)} \vec{u}_x + E_{0y} e^{i(kz - \omega t + \phi)} \vec{u}_y$

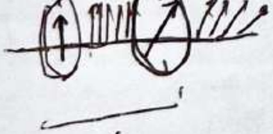
$E_y = E_0 e^{i(kz - \omega t + \phi)}$

w/ $\boxed{\|E_0\| = (E_{0x}^2 + E_{0y}^2)^{\frac{1}{2}}}$.

3) and $\vec{E} = E_0 e^{i(kz - \omega t)} \boxed{\frac{E_{0x}}{E_0} u_x + \frac{E_{0y}}{E_0} e^{i\phi} u_y}$

$$\boxed{\vec{\varepsilon} = \frac{E_{0x}}{E_0} \vec{u}_x + \frac{E_{0y}}{E_0} e^{i\phi} \vec{u}_y}$$

then $|\varepsilon| = \sqrt{\varepsilon^* \cdot \varepsilon} = \sqrt{\left(\frac{E_{0x}}{E_0}\right)^2 + \left(\frac{E_{0y}}{E_0}\right)^2} = 1$



MALUS' LAW

4) : (DETECTORS $\neq$ FIELDS) $\wedge$ (DETECTORS $=$ POWER INTENSITY).

then: $I = |E|^2 = E^* \cdot E$ $\left(E_0 e^{i(kz - \omega t)} \varepsilon^*\right)\left(E_0 e^{-i(kz - \omega t)} \varepsilon\right) = E_0^2$

and if $\varepsilon \cdot \varepsilon^* = 1$ $\quad I = E^* \cdot E = \left(E_0 e^{i(kz - \omega t)} \varepsilon^*\right)\left(E_0 e^{-i(kz - \omega t)} \varepsilon\right)$

$$1 = \int_0^{\infty} |E(z,t)|^2 \, dz = E_0^2 \int_0^{\infty} \left(e^{i(kz-\omega t)}\right)^2 dz = E_0^2 \int e^{2i(kz-\omega t)} dz$$

Normalization Fix

$$= E_0^2 \frac{1}{2ik} e^{2i(kz-\omega t)} \Big|_0^{\infty} = E_0^2 \frac{1}{2ikz} e^{i2(k(\infty)-\omega t)} = E_0^2 \frac{1}{2ik} e^{-i2\omega t}$$

$$\int \left| E(z,t) \right| dz$$

$$= |E_0|^2 \int_0^{\infty} e^{-2\lambda z} dz$$

$$= |E_0|^2 \left[\frac{1}{2\lambda}\right]$$

$$\therefore E_0 = \sqrt{2\lambda}$$

$$\frac{d}{dt}\int_0^{\infty} |E(z,t)|^2 dz = 0$$

$$|E|^2 = 2\lambda e^{-2\lambda z}$$

$$\boxed{E_0^2 = 2ike^{i2\omega t}} \quad \boxed{E_0 = \sqrt{2ike^{2i\omega t}}}$$

$$\omega = \frac{v}{r}, \quad v = \frac{2\pi}{t}$$

w/ $e^{i2\omega t} = \cos(2\omega t) + i\sin(2\omega t)$
$= \cos(2\theta r) + i\sin(2\theta r)$

If $\omega = \frac{2\pi}{t}\frac{1}{r}$, Then $\omega t = r2\pi$.

$$E(z,t) = E_0 e^{i(kz-\omega t)} = \left(2ike^{i2\omega t}\right)^{\frac{1}{2}} e^{i(kz-\omega t)}$$

$$\frac{\partial E}{\partial t} = \frac{\partial}{\partial t}\left(2ike^{i\omega t}\right)^{\frac{1}{2}} e^{i(kz-\omega t)} = \left(2ike^{i\omega t}\right)^{\frac{1}{2}} \frac{\partial}{\partial t} e^{-i\omega t} \cdot e^{ikz}$$

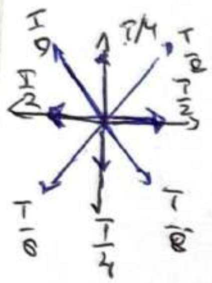
$U = 2ike^{i\omega t}$

$$\frac{\partial E}{\partial t} = \frac{\partial E}{\partial U}\frac{\partial U}{\partial t} = U^{\frac{1}{2}} \frac{\partial U}{\partial t}$$

So: $\boxed{E(z,t) = Ae^{-\lambda z} e^{i(kz-\omega t)}}$

To make normalized.

THE WAVE FUNCTION IS STILL TIME DEPENDENT!



[ii] LINEAR POLARIZATION

$$\vec{\varepsilon} = \frac{E_{0x}}{E_0} u_x + \frac{E_{0y}}{E_0} u_y$$

and $\boxed{\theta = \tan^{-1}\left(\frac{E_{0y}}{E_{0x}}\right)}$

CIRCULAR POLARIZATION — If the zero phase shift ($\phi = 0$) is assumed, Then The change between $E_0(y,z)$ is perpendicular

$$\cos(kz-\omega t+\tfrac{\pi}{2})$$

$$\downarrow$$

$$\boxed{\frac{E_0}{\sqrt{2}}\sin(kz-\omega t)}$$

[iii] CIRCULAR POLARIZATION

1) Consider $\phi = \frac{\pi}{2}$ and $(E_{0x}, E_{0y}) = \frac{E_0}{\sqrt{2}}$. then

$$\boxed{\vec{\varepsilon} = \frac{1}{\sqrt{2}} u_x + i u_y}$$

2) here: $E_x = \frac{E_0}{\sqrt{2}}\cos(-\omega t) = \frac{E_0}{\sqrt{2}}\cos(\omega t)$

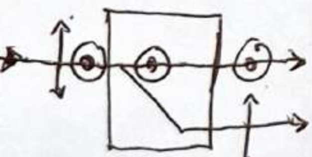
$E_y = \frac{E_0}{\sqrt{2}}\sin(-\omega t) = \frac{E_0}{\sqrt{2}}\sin(\omega t)$

[iv] ELLIPTICAL POLARIZATION and $\boxed{E_x = \frac{E_0}{\sqrt{2}}\cos(\omega t)}$ and $\boxed{E_y = \frac{E_0}{\sqrt{2}}\sin(\omega t)}$

Elliptical polarization happens when for a vector field, $\boxed{\neg(\phi=0) \wedge \neg(\phi=\frac{\pi}{2})}$ in its a special case

(2) BIREFRINGENCE

[i] BIREFRINGENT MATERIALS.



1) $k = \frac{\omega}{v} = \frac{n\omega}{c} = \frac{n2\pi}{\lambda}$ here $\boxed{k = \frac{\omega}{v}}$ and $\boxed{\frac{2\pi n}{\lambda} = k}$ w/ $n = \frac{c}{v}$ so $v = \frac{c}{n}$ and $\frac{1}{v} =$

2) BIREFRINGENT Materials = Anisotropic = Refraction Index := Polarization

POLARIZATION MECHANICS (2.3) (2.4)

(3) MODIFYING POLARIZATION

[1] LINEAR POLARIZATION:

$$\vec{E}_i = E_0 e^{i(kz-\omega t)}\, \vec{\varepsilon}$$

$$\vec{E}_t = (\vec{E}_i \cdot u_\theta)\, \bar{u}_\theta = E_0 e^{i(kz-\omega t)} (\vec{\varepsilon}\cdot u_\theta)\, u_\theta$$

$$\boxed{E_t = E_0 e^{i(kz-\omega t)} (\varepsilon \cdot u_\theta) u_\theta}$$ The output is linearly polarized on u_θ.

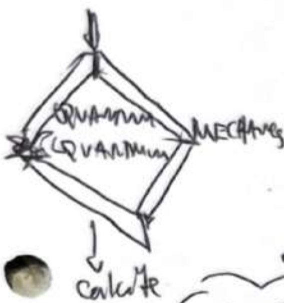
- Linearly polarized lights produces an angle φ w/ horizontal ($\varepsilon = u_y$)

: So off by ψ: $\cos(\theta - \psi)$

[ii] SPLITTING BEAMS & [iii] WAVE PLATES

POLARIZING BEAM SPLITTER (PBS)

- Prisms of a Birefringent material can be used as a PBS
- Polarizing Beam Splitter (PBS) can also use thin film coatings.



1) $\phi_f = \dfrac{n_f 2\pi l}{\lambda}$ f = FAST AXIS REFRACTION

$\phi_s = \dfrac{n_s 2\pi l}{\lambda}$ s = SLOW AXIS REFRACTION

$$\boxed{\Delta\phi = \phi_s - \phi_f = \frac{(n_s - n_f) 2\pi l}{\lambda}}$$

$$\Delta\phi = 2\pi j + \frac{\pi}{2}$$

2) QUARTER WAVE LENGTH

$: \Delta\phi = \dfrac{2\pi}{4} = \dfrac{1\pi}{2} = \dfrac{\pi}{2}$ $\quad$ so $\boxed{\Delta\phi_{QUARTER} = \dfrac{\pi}{2}}$

- wave plate = optical element = phase shift in orthogonal frames.

(4) JONES VECTORS & MATRICES

[i] JONES VECTORS:

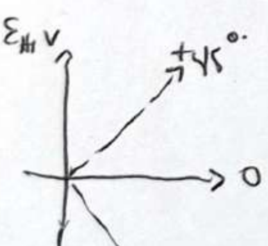
$$\boxed{\begin{matrix}\varepsilon_H = u_x \\ \varepsilon_V = u_y\end{matrix}}$$ H = Horizontal $\rightarrow$, V = VERTICAL $\uparrow$

Ex: 1) $\varepsilon_{45} = \dfrac{1}{\sqrt2} u_x + \dfrac{1}{\sqrt2} u_y = \dfrac{1}{\sqrt2}(u_y + u_x) = \dfrac{1}{\sqrt2}(\varepsilon_H + \varepsilon_V)$

then $\boxed{\varepsilon_{45} = \dfrac{1}{\sqrt2}(\varepsilon_H + \varepsilon_V)}$

[ii] JONES MATRICES

Geometry: 1)



a) $\varepsilon_H = \begin{bmatrix}1\\0\end{bmatrix}$ $\quad \varepsilon_V = \begin{bmatrix}0\\1\end{bmatrix}$

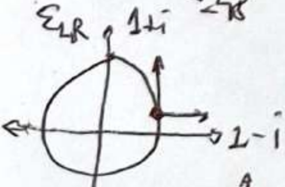
$\varepsilon_{+45} = \dfrac{1}{\sqrt2}\left(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right) = \dfrac{1}{\sqrt2}\begin{bmatrix}1\\1\end{bmatrix}$

$\varepsilon_{-45} = \dfrac{1}{\sqrt2}\left(\begin{bmatrix}1\\0\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}\right) = \dfrac{1}{\sqrt2} = \begin{bmatrix}1\\-1\end{bmatrix}$

so then $\boxed{\varepsilon_{+45} = \dfrac{1}{\sqrt2}\begin{bmatrix}1\\1\end{bmatrix} \quad \varepsilon_{-45} = \dfrac{1}{\sqrt2}\begin{bmatrix}1\\-1\end{bmatrix}}$

Now $\varepsilon_{+45}\varepsilon_{-45} = \left(\dfrac{1}{\sqrt2}\right)^2 \begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \dfrac{1}{2}\begin{bmatrix}1\begin{bmatrix}1\\1\end{bmatrix}\\-1\begin{bmatrix}1\\1\end{bmatrix}\end{bmatrix} = \dfrac{1}{2}\begin{bmatrix}1\cdot1 + 1\cdot-1\\ 1\cdot1 + -1\cdot1\end{bmatrix} = \dfrac{1}{2}\begin{bmatrix}2-2\\2-2\end{bmatrix}$

$= \begin{bmatrix}0 & \frac12\\ 0 & \frac12\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ so: $\boxed{\varepsilon_{+45}\varepsilon_{-45} = 0^0}$



b) $\varepsilon_L = \dfrac{1}{\sqrt2}\begin{pmatrix}1\\i\end{pmatrix} = \dfrac{1}{\sqrt2}(\varepsilon_H + i\varepsilon_V)$

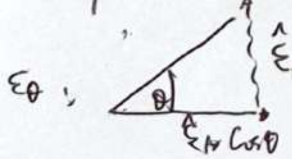
$\varepsilon_R = \dfrac{1}{\sqrt2}\begin{pmatrix}1\\-i\end{pmatrix} = \dfrac{1}{\sqrt2}(\varepsilon_H - i\varepsilon_V)$

so: $\boxed{\begin{matrix}\varepsilon_L = \dfrac{1}{\sqrt2}\begin{bmatrix}1\\i\end{bmatrix} \\ \varepsilon_R = \dfrac{1}{\sqrt2}\begin{bmatrix}1\\-i\end{bmatrix}\end{matrix}}$

$-(i\cdot i) = -(-1) = 1$

and: $\varepsilon_L\varepsilon_R = \dfrac{1}{2}\begin{bmatrix}1\\i\end{bmatrix}\begin{bmatrix}1\\-i\end{bmatrix} = \dfrac{1}{2}\begin{bmatrix}1\begin{bmatrix}1\\-i\end{bmatrix}\\ i\begin{bmatrix}1\\-i\end{bmatrix}\end{bmatrix} = \dfrac{1}{2}\begin{bmatrix}1\cdot1 + i\cdot1\\ 1(-i) + -i\cdot i\end{bmatrix} = \dfrac{1}{2}\begin{bmatrix}1+i\\-1-i\end{bmatrix}$

ε_θ:



so $\boxed{\varepsilon_\theta = \cos\theta \cdot \varepsilon_H + \sin\theta \cdot \varepsilon_V}$

and $\bar{J}_H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\bar{J}_V = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $\bar{J}_\theta = \begin{bmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$.

$$\boxed{\bar{J}_{\frac{\lambda}{4}\pm 45^\circ} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \mp i \\ \mp i & 1 \end{bmatrix}} \qquad \boxed{\bar{J}_{\frac{\lambda}{2}\theta} = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}}$$

QUANTUM OPERATORS

(4.1) (4.2) (4.3) (4.4)

1) BASICS

1)

$$|\psi_N\rangle = \hat{O}_N|\psi_{N-1}\rangle = \hat{O}_N \ldots \hat{O}_2\hat{O}_1|\psi_i\rangle$$

then $\boxed{|\psi_N\rangle = \prod_{i=1}^{N}(\hat{O}_N)\cdot|\psi_i\rangle}$ w/ $|\psi_N\rangle$ = quantum state, $\hat{O}_N$ = operator.

w/ $e^{\hat{O}} = \sum_n^{\infty} \frac{1}{n!}\hat{O}^n$ => Taylor series

2.)

$$\langle A|F_P(\theta) = \langle +45|$$
$$\langle A|F_P^{\dagger}(\theta) = \langle +45|$$

then $\boxed{\langle H|F_P F_P^{\dagger}|H\rangle = \langle +45|+45\rangle = 1.}$ => unitary operators

3.)

$$\hat{U}^{\dagger}\hat{U} = \hat{U}^{\dagger}\hat{U} = \hat{1}$$
$$\hat{U}|\psi_1\rangle = e^{i\phi}|\psi_2\rangle = [\cos(\phi) + i\sin(\phi)]|\psi_2\rangle$$
$$\hat{U}^{\dagger}\langle\psi_1| = e^{-i\phi}\langle\psi_2| = [\cos(\phi) - i\sin(\phi)]\langle\psi_2|$$

For: $\hat{U}|\psi_1\rangle\langle\psi_1|\hat{U}^{\dagger} = \langle\psi_1|\hat{U}\hat{U}^{\dagger}|\psi_1\rangle$.

$$\langle\psi_1|\hat{U}\hat{U}^{\dagger}|\psi_1\rangle = \langle\psi_2|e^{i\phi}e^{-i\phi}|\psi_2\rangle = \langle\psi_2|\psi_2\rangle e^{i\phi-i\phi}$$
$$= 1\times 1 = 1 \checkmark$$

then $\boxed{\langle\psi_1|\hat{U}\hat{U}^{\dagger}|\psi_1\rangle = \langle\psi_2|e^{i\phi}e^{-i\phi}|\psi_2\rangle = 1.}$ unitary operator

- $|\psi\rangle = \psi_1|HV_1\rangle + \psi_2|HV_2\rangle = \sum_j \psi_j|HV_j\rangle$

  st $\boxed{\psi_j = \langle HV_j|\psi\rangle}$

- $|\psi\rangle = \begin{bmatrix}\langle HV_1|\psi\rangle \\ \langle HV_2|\psi\rangle\end{bmatrix} = \begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix}_{HV}$

4.) $\hat{P}_H|\psi\rangle = C_H|H\rangle$ -> Probability of Hamiltonian state H. w/ $\hat{P}_H$ = PROJECTION TO H

$$\hat{P}_V|\psi\rangle = C_V|V\rangle$$

so, $\boxed{|\psi\rangle = C_H|H\rangle + C_V|V\rangle}$ w/ $\boxed{C_H + C_V = 1}$

state (ψ) = Prob(H) x state(H) + Prob(V) x state(V)

- $\hat{P}_H = \hat{C}_H$
- $\hat{P}_H|\psi\rangle = \hat{C}_H|\psi\rangle$

  Since $C_H|H\rangle = C_H|\psi\rangle$

  Then $C_H|\psi\rangle = |H\rangle C_H$

w/ $\hat{P}_H|\psi\rangle = C_H|\psi\rangle = |H\rangle C_H = |H\rangle\langle H|\psi\rangle$

$$\hat{P}\psi = |\psi\rangle\langle\psi|$$

then: $|\psi\rangle = C_H|H\rangle + C_V|V\rangle = |H\rangle C_H + |V\rangle C_V = \langle H|H\rangle|\psi\rangle + \langle V|V\rangle|\psi\rangle$

$$= \langle H|\langle H|\psi\rangle + \langle V|\langle V|\psi\rangle = (\hat{P}_H + \hat{P}_V)\psi$$

then, $\boxed{|\psi\rangle = \hat{P}_H + \hat{P}_V|\psi\rangle}$

2) MATRIX REPRESENTATION

[1] THEORY

1.) $\boxed{|\psi\rangle = \hat{O}|\psi\rangle}$ $\boxed{|\psi\rangle \triangleq \begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix}_{HV}}$ $\boxed{\begin{aligned}|H\rangle &= |HV_1\rangle \\ |V\rangle &= |HV_2\rangle\end{aligned}}$

$$\psi_i = \langle HV_i|\psi\rangle = \langle HV_i|\hat{O}|\psi\rangle = \langle HV_i|\hat{O}\cdot\hat{1}|\psi\rangle$$

3) $\varphi_i = \langle HV_i | \hat{O} \left[\sum_j |HV_j\rangle\langle HV_j| \right] |\psi\rangle$

$= \sum_j \langle HV_i | \hat{O} | HV_j\rangle \langle HV_j | \psi\rangle$

$= \sum_j \langle HV_i | \hat{O} | HV_j\rangle \psi_j$

$$\boxed{\varphi_i = \sum_j \langle HV_i | \hat{O} | HV_j\rangle \psi_j}$$

4) $O_{ij} = \langle HV_i | \hat{O} | HV_j\rangle$

O_{ij}: $\boxed{\varphi_i = \sum_j O_{ij} \psi_j}$ $\quad O_{ij}\psi_j = \begin{bmatrix} O_{11} & O_{12} \\ O_{21} & O_{22} \end{bmatrix} \cdot \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}_{HV}$

$$\begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}_{HV} = \begin{bmatrix} O_{11} & O_{12} \\ O_{12} & O_{22} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}_{HV}$$

[iv] EXAMPLES:

[1] 1) $\hat{P}_H \triangleq \begin{bmatrix} \langle HV_1 | \hat{P}_H | HV_1\rangle & \langle HV_1 | \hat{P}_H | HV_2\rangle \\ \langle HV_2 | \hat{P}_H | HV_1\rangle & \langle HV_2 | \hat{P}_H | HV_2\rangle \end{bmatrix}_{HV}$

$= \begin{bmatrix} \langle H | \hat{P}_H | H\rangle & \langle H | \hat{P}_H | V\rangle \\ \langle V | \hat{P}_H | H\rangle & \langle V | \hat{P}_H | V\rangle \end{bmatrix}$

so. $\hat{P}_H = \begin{bmatrix} \langle H | \hat{P}_H | H\rangle & \langle H | \hat{P}_H | V\rangle \\ \langle V | \hat{P}_H | H\rangle & \langle V | \hat{P}_H | V\rangle \end{bmatrix}$

2) $\langle i | j\rangle = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ $\quad$ here. $\delta_{HV} = \begin{cases} 1 & H = H;\ V = V \\ 0 & H \neq V,\ V \neq H. \end{cases}$

hence: $\hat{P}_H = \begin{bmatrix} \hat{P}_H & 0 \\ 0 & \hat{P}_H \end{bmatrix}$ ST det ✗

But recall that: $\hat{P}_H |H\rangle = |H\rangle$ and $\hat{P}_H |V\rangle = 0$.

we: $\hat{P}_H = \begin{bmatrix} \langle H | H\rangle & 0 \\ \langle V | H\rangle & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

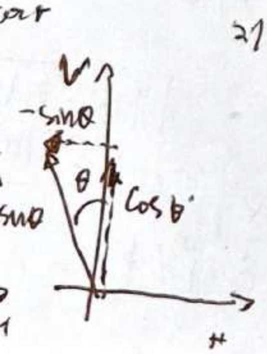
so: $\boxed{\hat{P}_H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{HV}}$ $\rightarrow$

[2] POLARIZATION ROTATION OPERATOR:

1) $\hat{R}_p(\theta) |HV_1\rangle = \hat{R}_p(\theta) |H\rangle = \cos\theta |H\rangle + \sin\theta |V\rangle$

$\hat{R}_p(\theta) |HV_2\rangle = \hat{R}_p(\theta) |V\rangle = -\sin\theta |H\rangle + \cos\theta |V\rangle$.

$\langle H | H\rangle \cos\theta = \cos\theta$
$\langle H | V\rangle \sin\theta = 0$
$\langle V | H\rangle \cos\theta = 0$
$\langle V | V\rangle \sin\theta = \sin\theta$
etc ......



2) $\hat{R}_p(\theta) = \begin{bmatrix} \langle HV_1 | \hat{R}_p | HV_1\rangle & \langle HV_1 | \hat{R}_p(\theta) | HV_2\rangle \\ \langle HV_2 | \hat{R}_p | HV_1\rangle & \langle HV_2 | \hat{R}_p(\theta) | HV_2\rangle \end{bmatrix}_{HV}$

$= \begin{bmatrix} \langle H | [\cos\theta |H\rangle + \sin\theta |V\rangle] & \langle H | [-\sin\theta |H\rangle + \cos\theta |V\rangle] \\ \langle V | [\cos\theta |H\rangle + \sin\theta |V\rangle] & \langle V | [-\sin\theta |H\rangle + \cos\theta |V\rangle] \end{bmatrix}$

$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_{HV}$ $\quad$ so: $\boxed{\hat{R}_p(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}$

QUANTUM MATRICES - HERMITES - BASIS

(4.4) (4.5) (4.6)

(1.) MATRIX EXTENSIONS

[i] — 1 Continuation — $\boxed{\text{FIND } P_H |+45\rangle}$

EX 4.3

$$\hat{P}_H |+45\rangle = |H\rangle\langle H| \left[\frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)\right] = \frac{1}{\sqrt{2}}|H\rangle$$

$$\text{w: } \hat{P}_H |+45\rangle = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}_{HV} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{HV} = \begin{bmatrix} 1\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}1\\1\end{bmatrix} \\ 0\begin{bmatrix}1\\1\end{bmatrix} + 0\begin{bmatrix}1\\1\end{bmatrix} \end{bmatrix}_{HV} \frac{1}{\sqrt{2}}$$

$$= \begin{bmatrix} 1+0 \\ 0+0 \end{bmatrix}_{HV} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}|H\rangle$$

$$\text{w: } \boxed{\hat{P}_H |+45\rangle = \frac{1}{\sqrt{2}}|H\rangle}$$

[ii] CORRESP. BETWEEN QM & CM.

[i] JONES MATRICES

NAME	#	POLARIZATION	POLARIZATION VECTOR	JONES
Horizontal	1)	$\vec{\varepsilon}_H$	$\vec{\varepsilon}_H$	$\begin{bmatrix}1\\0\end{bmatrix}$
Vertical	2.)	$\vec{\varepsilon}_V$	$\vec{\varepsilon}_V$	$\begin{bmatrix}1\\0\end{bmatrix}$
+45° Linear	3.)	$\vec{\varepsilon}_{+45}$	$\frac{1}{\sqrt{2}}(\varepsilon_H + \varepsilon_V)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$
-45° Linear	4.)	ε_{-45}	$\frac{1}{\sqrt{2}}(\varepsilon_H + \varepsilon_V)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$
Angle	5.)	ε_θ	$\cos\theta\,\varepsilon_H + \sin\theta\,\varepsilon_V$	$\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$
Left Circular	6.)	ε_L	$\frac{1}{\sqrt{2}}(\varepsilon_H + i\varepsilon_V)$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$
Right Circular	7.)	ε_R	$\frac{1}{\sqrt{2}}(\varepsilon_H + i\varepsilon_V)$	

JONES MATRIX

$$\varepsilon_H = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\Downarrow$$

$$\bar{J}_{\frac{\lambda}{2}\theta} = \begin{bmatrix}\cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}\cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta\end{bmatrix} = \begin{bmatrix}\cos 2\theta \\ \sin 2\theta\end{bmatrix}.$$

[iii] IMPLICATIONS:

1) See that: $\hat{P}_H = \begin{bmatrix}\langle H|H\rangle & 0 \\ \langle V|H\rangle & 0\end{bmatrix}$ is Basically Jones Matrix of HORIZONTAL polarizer.

2.) Each OPTICAL ELEMENTS: Each QUANTUM OPERATOR - plate physical Implementation.

3.) ⑦ Why can't the Half wave plate physical Implementation of the rotation operator? Since we have & get:

$$\boxed{\hat{R}_p(\theta)|\phi\rangle = |\phi + \theta\rangle}$$

(2) CHANGING BASES

[i] THINGS

1)
$$\boxed{|\psi\rangle = \psi_H|H\rangle + \psi_V|V\rangle}$$

2.)
$$|\psi\rangle = \psi_{+45}|+45\rangle + \psi_{-45}|-45\rangle$$

$$|+45\rangle = \frac{1}{\sqrt{2}}\left[|H\rangle + |V\rangle\right]$$

$$|-45\rangle = \frac{1}{\sqrt{2}}\left[|H\rangle - |V\rangle\right]$$

3.)
$$\psi_{+45} = \langle +45|\left(\psi_H|H\rangle + \psi_V|V\rangle\right) = \psi_H\langle +45|H\rangle + \psi_V\langle +45|V\rangle$$

$$\psi_{-45} = \langle -45|\left(\psi_H|H\rangle + \psi_V|V\rangle\right] = \psi_H\langle -45|H\rangle + \psi_V\langle -45|V\rangle$$

4.)
$$\langle +45|H\rangle = \frac{1}{\sqrt{2}} \quad \langle +45|V\rangle = \frac{1}{\sqrt{2}}$$

$$\langle -45|H\rangle = \frac{1}{\sqrt{2}} \quad \langle -45|V\rangle = -\frac{1}{\sqrt{2}}$$

$$\boxed{\begin{aligned}\psi_{+45} &= \frac{1}{\sqrt{2}}(\psi_H + \psi_V) \\ \psi_{-45} &= \frac{1}{\sqrt{2}}(\psi_H - \psi_V)\end{aligned}}$$

PROBLEM 4

① Proof that unitary operators do not change the magnitude.

Ans: 1) Let $|\psi\rangle, |\psi'\rangle$ be vectors s.t.:

$$\boxed{\hat{U}|\psi\rangle = |\psi'\rangle}$$

So we proof that: $\|\hat{U}\| = \| |\psi\rangle \|$ w/ $\hat{U}^{\dagger}\hat{U} = \mathbb{I}$ $\mathbb{I}$ = Identity operator

2) So: $\| |\psi'\rangle \| = \langle\psi'|\psi'\rangle = \langle U\psi | U\psi\rangle = \langle\psi| UU^{\dagger}|\psi\rangle$

$= \langle\psi|\mathbb{I}|\psi\rangle = \langle\psi|\psi\rangle$

So: $\boxed{\| |\psi'\rangle \| = \langle\psi|\psi\rangle}$ The magnitude of vector ψ' is still the same unitary operator of $|\psi\rangle$ ∎

② Show that $\hat{P}_{\psi}^{2} = \hat{P}_{\psi}$

Ans: 1) Recall that: $\hat{P}_{\psi} = \langle\psi|\psi\rangle$ so $\hat{P}^2 = |\langle\psi|\psi\rangle|^2 = |1|^2 = 1$
$= \langle\psi|\psi\rangle = \hat{P}_{\psi}$ ✓

2) So $\hat{P}_{\psi}^{2} = |\langle\psi|\psi\rangle|^2 = \langle\psi|\psi\rangle\langle\psi|\psi\rangle = \psi\psi^{*}\cdot\psi\psi^{*} = \psi\cdot\psi\cdot\psi^{*}\psi^{*}$

$= \langle\psi|\psi\rangle\langle\psi^{*}|\psi^{*}\rangle = \langle\psi^{*}|\psi^{*}\rangle = \hat{P}_{\psi}$ ∎   so: $\boxed{\hat{P}_{\psi}^{2} = |\langle\psi|\psi\rangle|^2 = \hat{P}_{\psi}}$

③ If $|\psi\rangle = C_H|H\rangle + C_V|V\rangle$ what's $\boxed{\langle\psi|\hat{P}_H ?}$

Ans: 1) $|\psi\rangle = C_H|H\rangle + C_V|V\rangle \Rightarrow \langle\psi|\psi\rangle\hat{P}_H = [C_H|H\rangle + C_V|V\rangle]\hat{P}_H$

$\hat{P}_H = C_H\hat{P}_H|H\rangle + C_V\hat{P}_H|V\rangle$

2) Recall that: $\hat{P}_H |\psi\rangle = \langle H|H\rangle|\psi\rangle$ so: $\hat{P}_H = \langle H|H\rangle$.

$\hat{P}_H = C_H\langle H|H\rangle|H\rangle + C_V\langle H|H\rangle|V\rangle$

$= C_H\langle H|H\rangle|H\rangle + C_V\langle H|V\rangle|H\rangle = |H\rangle[C_H\langle H|H\rangle + C_V\langle H|V\rangle]$

$= C_H|H\rangle + 0 = C_H|H\rangle$

3) So: $\hat{P}_H|\psi\rangle = C_H|\hat{H}\rangle \longrightarrow \langle\psi|\hat{P}_H = \hat{P}_H^{\dagger}|\psi\rangle^{\dagger} = \hat{C}_H^{\dagger}\langle\psi| = C_H^{\dagger}\langle H|$

and: $C_H^{\dagger}\langle H| = \langle H|C_H^{\dagger} = \langle H|\langle\psi|H\rangle = \langle\psi|\langle H|H\rangle$.

4) To proof this: $\langle\psi|H\rangle\langle H|\psi\rangle = \langle\psi|C_H^{\dagger}C_H|\psi\rangle = 1$ ∎ ✓

④ Verify the matrix representation of $\hat{R}_p(\theta)$ is unitary.

Ans: Unitary if $\hat{U}\hat{V}^{\dagger} = 1$. and we see that:

$$\hat{R}_p(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

The need that: $R_p^{\dagger}(\theta)\,\hat{R}_p(\theta) = \hat{1}$

So:

$$R_p^{\dagger}R_p = \begin{bmatrix} \cancel{\cos\theta^{\dagger}} & \cancel{\sin\theta^{\dagger}} \\ \cancel{\sin\theta^{\dagger}} & \cancel{\cos\theta^{\dagger}} \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where: $\hat{R}_p(\theta)|H\rangle = \cos\theta|H\rangle + \sin\theta|V\rangle$

$\hat{R}_p(\theta)^{\dagger}\langle H| = \sin\theta\langle H| + \cos\theta\langle V|$ $= \cos^2 + \sin^2 = \boxed{1}$ ✓

So: $\langle H|\hat{R}_p^{\dagger}R_p|H\rangle = [\cos|H\rangle + \sin\theta|V\rangle][\cos\langle H| + \sin\theta\langle V|]$

$= \cos^2\langle H|H\rangle + 2\sin\theta\cos\theta\langle H|V\rangle + \sin^2\langle V|V\rangle$

① Use the matrix Representation of Rotation operator $\hat{R}_p(\theta)$ to write that $\boxed{\hat{R}_p(\theta)\,|\phi\rangle = |\phi+\theta\rangle.}$ ϕ = state of photon

Ans:

so: $\hat{R}_p(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and let φ to be a cons number.

and $\hat{R}_p(\theta)\,|\phi\rangle = \cos\theta\,|\phi\rangle + \sin\theta\,|\varphi\rangle$

$\hat{R}_p(\theta)\,|\varphi\rangle = -\sin\theta\,|\phi\rangle + \cos\theta\,|\varphi\rangle$

and $\hat{R}_p(\theta)\,|\phi\rangle\langle H|H\rangle = \cos\theta\,\langle H|H\rangle|\phi\rangle + \sin\theta\,\langle H|H\rangle\,|\phi\rangle$

$\hat{R}_p(\theta)|\phi\rangle = \cos\theta\,|\phi\rangle$

and $\hat{R}_p(\theta)\,|\phi\rangle\langle V|V\rangle = -\sin\theta\,\langle V|V\rangle|\phi\rangle + \cos\theta\,\langle V|H\rangle\,|\phi\rangle$

$= -\sin\theta\,|\phi\rangle$

so: $R_p(\theta)^{\dagger} R_p(\theta)\,|\phi\rangle|\phi\rangle = (\cos^2\theta + \sin^2\theta)\,|\phi\rangle$ and $|\phi\rangle =$

$= \cos^2\theta\,|\phi\rangle + \sin^2\theta\,|\phi\rangle.$ $\quad |\theta\rangle =$

where: $\cos\theta\,|\phi\rangle = x(\theta)\,|\phi\rangle = \langle x(\theta)\,|\,x(\phi)\rangle$

$|\phi\rangle\,\hat{R}_p(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{matrix} \cos\theta \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} - \sin\theta \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \\ \sin\theta \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} + \cos \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \end{matrix}$

$= \begin{bmatrix} \cos(\theta+\phi_1) + \cos(\theta+\phi_2) & -\sin(\theta+\phi_1) - \sin(\theta+\phi_1) \\ \sin(\theta+\phi_1) + \sin(\theta+\phi_2) & \cos(\theta+\phi_2) - \cos(\theta+\phi_1) \end{bmatrix}$

$= \begin{bmatrix} \cos(\cdot) & -\sin(\cdot) \\ \sin(\cdot) & \cos(\cdot) \end{bmatrix}\begin{bmatrix} \phi_1+\theta_1 \\ \phi_2+\theta_2 \end{bmatrix} = \begin{bmatrix} \phi_1+\theta_1 \\ \phi_2+\theta_2 \end{bmatrix} = |\theta+\phi\rangle$

Ans ★

But $|\phi\rangle = \cos\phi\,|H\rangle + \sin\phi\,|V\rangle = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}.$

so: $R_p(\theta)\,|\phi\rangle = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$

$= \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi \end{bmatrix} = \begin{bmatrix} \cos\phi+\theta \\ \sin\phi+\theta \end{bmatrix} = |\phi+\theta\rangle$

so $\boxed{R_p(\theta)\,|\phi\rangle = |\phi+\theta\rangle}$

STERN GERLACH EXPERIMENT (6.1) (6.2) (6.3)

(1) FORCE ON A MAGNETIC POLE

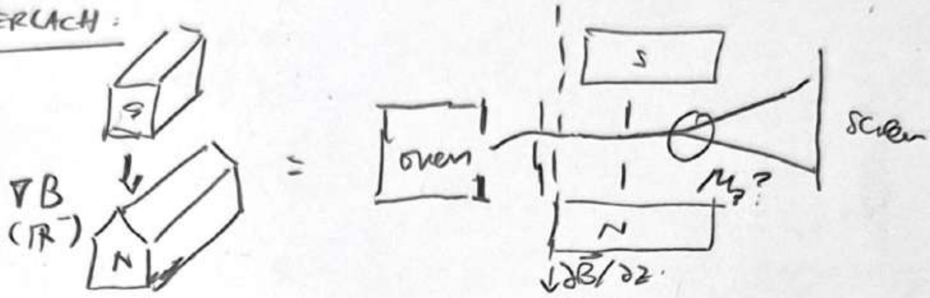
$$\boxed{V = \vec{\mu} \cdot \vec{B}}$$

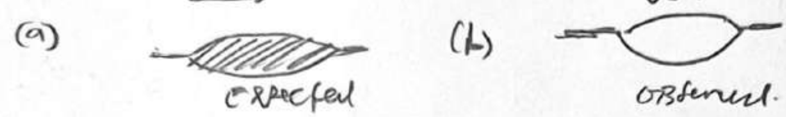
1.) $\vec{F} = -\nabla V = \nabla(\mu \cdot \vec{B})$

2.) $\boxed{\vec{F} = \mu_z \frac{\partial B_z}{\partial z}}$ IF, the dipole moment (μ) is so small, $F = 0$ basically

- OTTO STERN
- WALTER GERLACH

[i] STERN GERLACH:





Distribution of atoms

Gradient ($\nabla\vec{B}$) = Negative Spectrum

Mass atom × Geometry = Deflection via μ_z.

- George Uhlenbeck
- Samuel Goudsmit

∴ Particle was actually spinning

[ii] SPIN:

1.) $\boxed{\vec{\mu} = \gamma \vec{S}}$ Gyromagnetic Ratio

- Uhlenbeck - Goudsmit = Spinning
- Spinning = Magnetic moments × Charged particle
- Uhlenbeck (problem) = Spinning >>> Spinning (Normally)
- Spinning >>> Spinning (Normally) = Dipole moment

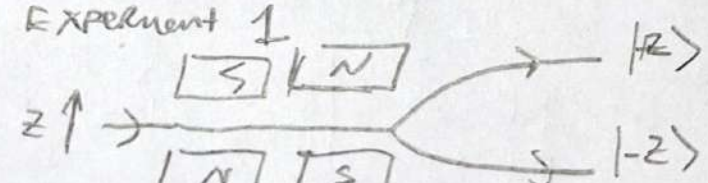
2.) $\boxed{\gamma_e = -1.76 \times 10^{11}\, s^{-1} T^{-1}}$

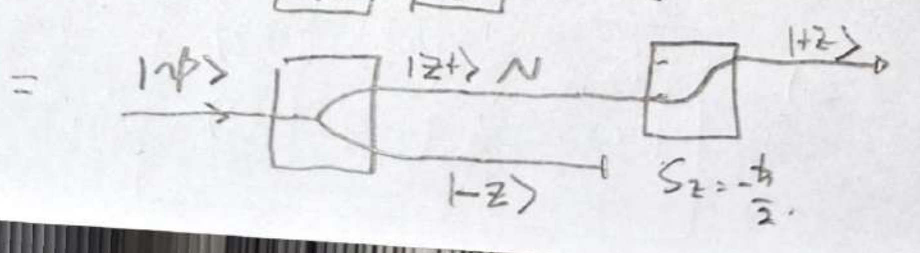
Recall: s^{-1} = Seconds T = Tesla = $\frac{1N}{1A \cdot 1m}$

3.) $\boxed{S_z = \pm \frac{1}{2}\hbar}$

(2) SPIN STATES

[i] EXPERIMENT 1





$S_z = \frac{\hbar}{2}$

$S_z = -\frac{\hbar}{2}$

→ $\langle +z | -z \rangle = 0$

∴

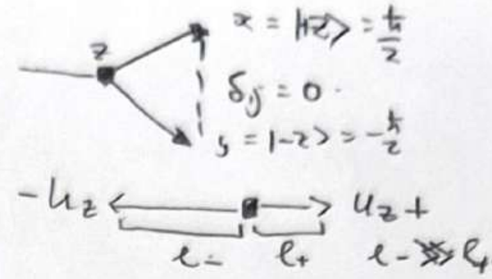
2) Problem:

$\langle +z | -z \rangle = 0$ s.t

$(+u_z)(-u_z) = -1$

s.t

so: $\boxed{\langle +z|-z\rangle \neq (+u_z)(-u_z) \quad 1 \neq 0}$



- Classical Intuition fails!

[iii] The $\boxed{\hat{S}_z}$ operator

1.) $\hat{S}_z |+z\rangle = \frac{\hbar}{2} |+z\rangle$

$S_z |-z\rangle = -\frac{\hbar}{2} |-z\rangle$

2.) $|+z\rangle = \begin{pmatrix} R^+ \\ R^- \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

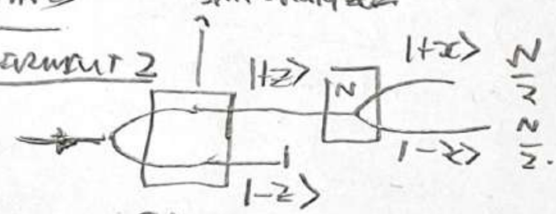
$|-z\rangle = \begin{pmatrix} R^+ \\ R^- \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

3.) $S_z = \frac{\hbar}{2} |+z\rangle\langle +z| = \frac{\hbar}{2} \bar{\sigma}_z$

$\boxed{S_z = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \frac{\hbar}{2} \bar{\sigma}_z}$ Pauli spin matrix.

(3.) MORE SPIN STATES

(i) EXPERIMENT 2

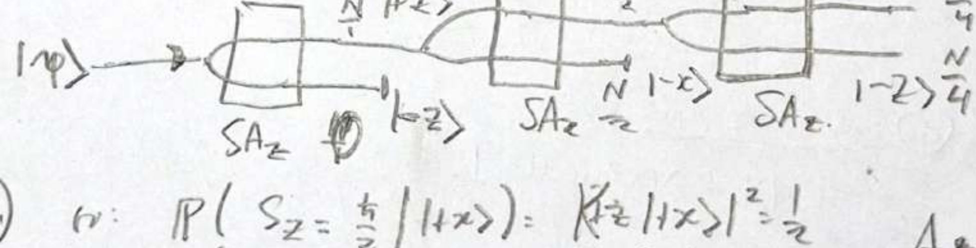


so: $\boxed{\begin{aligned} |x\rangle &= S_x \frac{\hbar}{2} \\ |-x\rangle &= -S_x \frac{\hbar}{2} \end{aligned}}$

Newton law = Motion (orthogonals) are Independent.

= SA_x splits in $|x\rangle, |-x\rangle$ since $\exists |\varphi\rangle$.

[ii] EXPERIMENT 3



1.) So: $\mathbb{P}(S_z = \frac{\hbar}{2} \,|\, |+x\rangle) = |\langle +z|+x\rangle|^2 = \frac{1}{2}$

$\mathbb{P}(S_z = -\frac{\hbar}{2} \,|\, |+x\rangle) = |\langle -z|+x\rangle|^2 = \frac{1}{2}$

$A\sqrt{\sum_i x_i^2} = 1$

$A\sqrt{1+1^2} = 1$

$A\sqrt{2} = 1$

$\boxed{A = \frac{1}{\sqrt{2}}}$

2.) $\boxed{\begin{aligned} |x\rangle &= \frac{1}{\sqrt{2}}\left(|+z\rangle + |-z\rangle\right) \\ |-x\rangle &= \frac{1}{\sqrt{2}}\left(|+z\rangle - |-z\rangle\right) \end{aligned}}$

PARTICLE INTERFERENCE

(1) APPLICATIONS OF $4^{th}/5^{th}$ EXPERIMENTS (6.4) (6.5)

[I] EXPECTATION OF S_y:

$$\langle S_y\rangle = \langle +x|\hat{S}_y|+x\rangle$$
$$= \frac{1}{\sqrt{2}}(1\ 1)_z\left[\frac{\hbar}{2}\bar{\sigma}_y\right]\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}_z$$
$$= \frac{1}{\sqrt{2}}(1\ 1)_z\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}_z$$
$$= \frac{\hbar}{4}(1\ 1)_z\begin{pmatrix}-i\\ i\end{pmatrix}_z = \frac{\hbar}{4}(-i+i) = 0$$

So: $$\boxed{\langle S_y\rangle = \langle +x|\hat{S}_y|+x\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}(1\ 1)\begin{pmatrix}1\\1\end{pmatrix}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix} = 0}$$

[II] COMMUTATION OF S_x, S_y

1) $$[\hat{S}_x\ \hat{S}_y] = \hat{S}_x\hat{S}_y - \hat{S}_y\hat{S}_x$$
$$= \left(\frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}_z\frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\right) - \left(\frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}_z\frac{\hbar}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}_z\right)$$
$$= \frac{\hbar^2}{4}\left\{\begin{bmatrix}i & 0\\ 0 & -i\end{bmatrix} - \begin{bmatrix}-i & 0\\ 0 & i\end{bmatrix}\right\}$$
$$= \frac{\hbar^2}{4}\begin{Bmatrix}i & 0\\ i & -i\end{Bmatrix}_z = \frac{\hbar^2}{4}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = i\hbar\frac{\hbar}{2}\sigma_z$$
$$= \frac{i\hbar^2}{2}\bar{\sigma}_z = i\hbar\hat{S}_z$$

So: $$\boxed{[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z}$$

2) Find cyclic permutations:

$$\boxed{\begin{aligned}[\hat{S}_y\ \hat{S}_z] &= i\hbar\hat{S}_x\\ [\hat{S}_z, \hat{S}_x] &= i\hbar\hat{S}_y\\ [\hat{S}_y, \hat{S}_z] &= -i\hbar\hat{S}_z\end{aligned}}$$

Angular momentum operator

[III] The uncertainty principle:

$$\boxed{\Delta S_x\,\Delta S_y \geq \frac{\hbar}{2}|\langle\hat{S}_z\rangle|}$$

(2) PARTICLE INTERFERENCE

Problems 6

(4) Show that:

$$S_y|+z\rangle = i\frac{\hbar}{2}|-z\rangle \text{ and } \hat{S}_y|z\rangle = -i\frac{\hbar}{2}|+z\rangle$$

Ans: 1.) $\hat{S}_y = i\frac{\hbar}{2}\sigma_y = \frac{i\hbar}{2}\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$

$|+z\rangle = \begin{bmatrix}1\\0\end{bmatrix}_z$

so: $\hat{S}_y|+z\rangle = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}|z\rangle = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$

$= \frac{\hbar}{2}\begin{bmatrix}(0)(1) + (-i)(0)\\ (i)(1) + (0)(0)\end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix}0\\i\end{bmatrix} = i\frac{\hbar}{2}\begin{bmatrix}0\\1\end{bmatrix}$

$= i\frac{\hbar}{2}|-z\rangle$ ✓

2) Similarly:

$S_y|+z\rangle = \frac{\hbar}{2}\begin{bmatrix}0 & -i\\ +i & 0\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} = \frac{\hbar}{2}\begin{bmatrix}-i\\0\end{bmatrix} = -i\frac{\hbar}{2}\begin{bmatrix}1\\0\end{bmatrix} = -\frac{i\hbar}{2}|z+\rangle$

so: $\boxed{\begin{aligned}S_y|+z\rangle &= i\frac{\hbar}{2}|-z\rangle\\ S_y|-z\rangle &= -i\frac{\hbar}{2}|+z\rangle\end{aligned}}$

(9). The relativistic gyro eqs. is $\gamma_{rel} = g\frac{q}{2m}$ while $\gamma_{class} = \frac{q}{2m}$

so: $\boxed{\gamma_{rel} = g\,\gamma_{class}}$ meaning its off by g. What is g?

Ans: g is the Anomalous magnetic moment of $\gamma_{classical}$ via the Dirac eqs

1.) Recall: $\boxed{(i\gamma^\mu\partial_\mu + m)\psi = 0}$ → Dirac eqs from Field theory

coupling: $\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$ and $\partial_\mu = D_\mu - iqA_\mu$

so: $(i\gamma^\mu\partial_\mu + m)\psi = (i\gamma^\mu(D_\mu - iqA_\mu) + m)\psi = 0$

Suppose $A^\mu = 0$ (no field interaction)

$(i\gamma^\mu(D_\mu - iqA_\mu) + m)\psi = (i\gamma^\mu D_\mu + q\gamma^\mu A_\mu + m)\psi = 0$

$\boxed{(i\gamma^\mu D_\mu + m)\psi = 0}$ so: $\boxed{\hbar(\vec{\sigma}\cdot\vec{D}) = g}$

Foldy Wouthuysen

$\sim m + \sum_m \frac{x^m}{2m}$

$\sim \underbrace{m}_{0^{th}\ term} + \underbrace{\frac{(p^2 - qA)^2_r}{2m}}_{1^{st}\ term} \underbrace{- \frac{1}{m}\frac{q\hbar}{}\sigma\cdot B}_{2^{nd}\ term}$

2.) Recall: $V = \mu\cdot\vec{B}$ $F = \left|\frac{\partial V}{\partial r}\right| = \mu\frac{\partial\vec{B}}{\partial r}$.

also: $\boxed{H = \frac{1}{2m}(p^2_{classical} - qA)^2 - \frac{q\hbar}{2m}\vec{\sigma}\cdot\vec{B}}$

so: $-\frac{q\hbar}{2m}\vec{\sigma}\cdot\vec{B} \Rightarrow \mu_s = \frac{q\hbar}{2m}\vec{\sigma}$

3.) $S_z = \frac{\hbar}{2}\vec{\sigma}$ so: $\mu_s = \mu_s$ $\frac{\hbar}{\vec{\sigma}\cdot\vec{S}} = \frac{2m}{m}$

$\mu_s = \frac{q}{m}\vec{S}$ $\frac{q\hbar\vec{\sigma}}{2m} = \frac{q}{m}\vec{S}$ $\frac{\hbar}{\sigma S} = 2$.

$\boxed{g = 2}$

(1.1) Calculate $\mathbb{P}(S_y = \frac{\hbar}{2} \mid |x\rangle)$.

Ans:

$$\boxed{\mathbb{P}(S_y = \tfrac{\hbar}{2} \mid |+x\rangle) = |\langle +y|+x\rangle|^2}$$

$\begin{bmatrix} a \\ b \end{bmatrix}^\dagger = \begin{bmatrix} a \\ -b \end{bmatrix}^T = (a \ -b]$

1.) $|+y\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$ w/ $\frac{1}{\sqrt{2}}\left[|+z\rangle + i|-z\rangle\right] = \frac{1}{\sqrt{2}}\left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} i\right] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$

$|+x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ w/ $\frac{1}{\sqrt{2}}\left[|+z\rangle + |-z\rangle\right] = \frac{1}{\sqrt{2}}\left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

2.) $\langle +y| = |+y\rangle^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}^\dagger = \frac{1}{\sqrt{2}}[1 \ -i]$

3.) So: $\langle +y|+x\rangle = \left(\frac{1}{\sqrt{2}}[1 \ -i]\right) \times \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$

$= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}[1 \ -i]\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1\cdot 1 + 1\cdot i \\ -i\cdot 1 + 1\cdot 1 \end{bmatrix} = \frac{1}{2}[1-i]$

4.) $|\langle +y|+x\rangle|^2 = \left(\frac{1}{2}(1-i)\right)^2 = \frac{1}{4}\left(1^2 + i^2\right) = \frac{1}{4}(1+1) = \frac{1}{4}\cdot 2 = \boxed{\frac{1}{2}}$

So: $\boxed{\mathbb{P}(S_y = \tfrac{\hbar}{2} \mid |+x\rangle) = |\langle +y|+x\rangle|^2 = \tfrac{1}{2}}$

(1.2) Calculate $\mathbb{P}(S_x = \frac{\hbar}{2} \mid |-y\rangle)$:

Ans:

$$\boxed{\mathbb{P}(S_x = -\tfrac{\hbar}{2} \mid |-y\rangle) = |\langle -x|-y\rangle|^2}$$

1) $S_x = -\frac{\hbar}{2} \equiv |-x\rangle$

S_x: $|-x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and: $\langle -x| = |-x\rangle^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}^\dagger$

$= \frac{1}{\sqrt{2}}[1 \ -1]$.

2.) $\langle -x|-y\rangle = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\cdot\begin{bmatrix} 1 \\ -i \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}^T$

$= \frac{1}{2}\cdot\begin{bmatrix} 1 \\ -i \end{bmatrix}[1 \ -1] = \frac{1}{2}(1\cdot 1 + i) = \frac{1}{2}(1+i)$

3.) $|\langle -x|-y\rangle|^2 = \left|\frac{1}{2}(1+i)\right|^2 = \frac{1}{4}(1^2 + i^2) = \frac{1}{4}(1^2+1^2) = \frac{1}{4}\cdot 2 = \boxed{\frac{1}{2}}$

(3.) Write $\hat{S}_z$ in terms of Projection operator $|+z\rangle, |-z\rangle$:

Ans:

$|+z\rangle = \frac{\hbar}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\langle +z| = \frac{\hbar}{2}[1 \ 0]$

$|-z\rangle = \frac{\hbar}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\langle -z| = \frac{\hbar}{2}[0 \ 1]$

So: $\boxed{\hat{S}_z = \frac{\hbar}{2}\left\{|+z\rangle\langle +z| - |-z\rangle\langle -z|\right\} =}$

s.t: $\hat{S}_z = \frac{\hbar}{2}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}[1 \ 0] - \begin{bmatrix} 0 \\ 1 \end{bmatrix}[0 \ 1]\right\} = \frac{\hbar}{2}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

So: $\boxed{\hat{S}_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}} = \boxed{\hat{S}_z = \frac{\hbar}{2}\hat{\sigma}_z}$

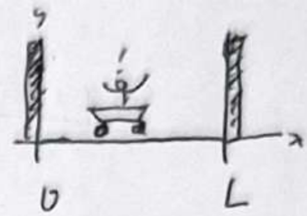
INFINITE SQUARE WELL (11.5)

QM = Discrete Energy levels
Classical = Continuum of Energies.

1. INFINITE SQUARE WELL.

1] THEORY

1] $$\boxed{V(x) = \begin{cases} 0 & 0 < x < L \\ \infty & \text{elsewhere} \end{cases}}$$ (11.5a)



2] $A + B = 0$

$\boxed{B = -A}$ (11.62) ⇒ Boundary conditions Require the wave function to be continuous on the edges of well

s.t. $Ae^{ikx} + Be^{-ikx} = \psi(x)$
$\psi(0) = 0$
$Ae^{0} + Be^{0} = A + B = 0$
$\boxed{A = -B}$ ✓

To Be Flexible:
$$\psi(x) = \begin{cases} Ae^{ikx} + Be^{-ikx} & \text{if } 0 < x < L \\ 0 & \text{elsewhere.} \end{cases}$$

3] $\psi(x) = Ae^{ikx} + Be^{-ikx}$ | $B = -A$

a] $= Ae^{ikx} + (-Ae^{-ikx}) = Ae^{ikx} - Ae^{-ikx} = A(e^{ikx} - e^{-ikx})$

so: $\boxed{\psi(x) = A(e^{ikx} - e^{-ikx})}$ (11.63)

b] Recall: $e^{ikx} = \cos(kx) + i\sin(kx)$
$e^{-ikx} = \cos(kx) - i\sin(kx)$ –

so: $e^{ikx} - e^{-ikx} = 0 + i\sin(kx) - (-i\sin(kx)) = i\sin kx + i\sin kx$
$= 2i\sin(kx)$

c] so: $\psi(x) = A2i\sin(kx) = C\sin(kx)$ so $\boxed{\psi = C\sin(kx)}$
where $\boxed{C = 2iA}$

4] Boundary conditions:
a] $0 < x < L$ so: $\psi(0) < \psi(x) < \psi(L)$
but $\psi(0) = 0$ so: $0 < \psi(x) < \psi(L)$

But: $\boxed{\begin{matrix}\psi(L) = 0 \\ \text{and } C \neq 0 \\ \text{so } \sin(kL) = 0\,!\end{matrix}}$

and $\psi(L) = C\sin(kx) = C\sin(kL)$. Since $k = \frac{n\pi}{L}$

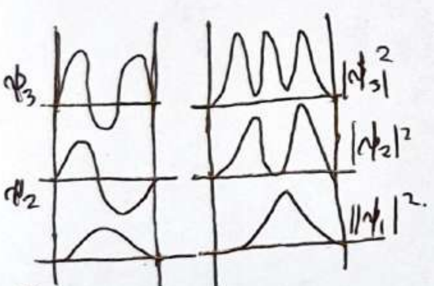
then $\boxed{kL = n\pi}$

1. $n \in \mathbb{Z}^+$ where $n > 0$.

So: $\boxed{k_n = \frac{n\pi}{L}}$

so: $\boxed{\psi(L) = C\sin(kL) = C\sin(n\pi)}$

So



5] Recall That: $E = \frac{\hbar^2}{2m}k^2$ so: $E_n = \frac{\hbar^2(n\pi)^2}{2m(L)^2}$ here $P = \hbar k$
$P^2 = \hbar^2 k^2$
and $T = \frac{P^2}{2m}$
$= \frac{\hbar^2 k^2}{2m}$

and: $\boxed{E_n = \frac{\hbar^2 n^2 \pi^2}{2mL^2}}$

6] NORMALIZATION: $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$

a] so: $\int_{-\infty}^{\infty} |\psi|^2 dx = C_n^2 \int_0^L \sin^2\left(\frac{n\pi}{L}x\right) dx$

Let $u = \sin\left(\frac{n\pi}{L}x\right)$ $x = \frac{n\pi}{L}x$
$du = \frac{n\pi}{L}\cos\left(x\right) dx$
$dx = \frac{L}{n\pi}\frac{du}{u}$

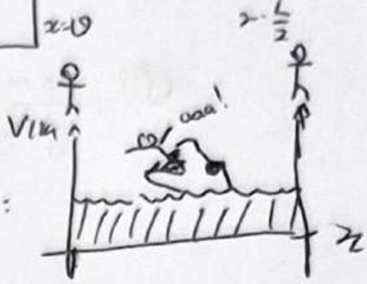
$C_n^2 \int_0^L \left[\frac{1 - \cos 2x\frac{n\pi}{L}}{2}\right] dx$

$= C_n^2 \left[\frac{x}{2} - \frac{\sin\left(\frac{2n\pi}{L}x\right)}{4n\pi/L}\right]_{x=0}^{L} = C_n^2 \frac{L}{2}$

$\frac{d\sin^2 x}{dx} = \frac{d}{dx}\sin x \sin x$
$\int \sin^2 x\, dx$ $= \frac{d\sin x}{dx}\sin x + \frac{d\sin x}{dx}$
$\int \sin x \sin x\, dx$ $= \cos x \sin x + \cos x \sin x$
$= 2\cos x \sin x$
$\sin x\cos x - \sin x \cos x$
$\sin x\cos x - \cos x\sin x$

iv) Since $C_n^2 \frac{L}{2} = 1$ then: $\boxed{C_n = \sqrt{\frac{2}{L}}}$

v) So: $\boxed{\psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) & 0 < x < L \\ & \text{Elsewhere} \end{cases}}$



[b] Example: The ILLEGAL wave function

1) $\psi(x,0) = \begin{cases} \sqrt{\frac{2}{L}} & 0 < x < \frac{L}{2} \\ 0 & \text{else} \end{cases}$ So:

$\cos\left(\frac{n\pi}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$

So: $\cos\left(\frac{n\pi}{2}\right) = \sum_{n=0}^{\infty} (-1)^k \frac{\left(\frac{n\pi}{2}\right)^k}{(2k)!}$

$= \sum_{n=0}^{\infty} (-1)^k \frac{n^k \pi^k}{(2k)!\, 2^k}$

2) $C_n = \int_0^{\infty} \psi_n^*(x)\, \Psi(x,0)\, dx$

$= \frac{2}{L} \int_0^{\frac{L}{2}} \sin\left(\frac{n\pi x}{L}\right) dx = \frac{-2}{n\pi} \cos\left(\frac{n\pi x}{L}\right)\Big|_0^{\frac{L}{2}}$

$= -\frac{2}{n\pi}\left[\cos\left(\frac{n\pi}{2}\right) - 1\right]$

3) So: $\boxed{C_n = \begin{cases} 2/n\pi & n \in (2k+1) \text{ odd} \\ 4/n\pi & n = 2, 6, 10 \\ 0 & n = 4, 8, 12 \end{cases}}$

4) The wave func:

Energy → A) $\Psi(x,t) = \sum_n C_n \psi_n(x) e^{-i\omega_n t} = \frac{2}{\pi}\sqrt{\frac{2}{L}} \sum_{n=1,3,5} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right) e^{i\left(\frac{n^2\pi^2\hbar}{2mL^2}\right)}$
$+ \frac{4}{\pi}\sqrt{\frac{2}{L}} \sum_{2,6,10} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right) e^{-i\frac{n^2\pi^2\hbar t}{2mL^2}}$

$P(E_n,t) = |C_n|^2$

$\begin{cases} \frac{4}{(n\pi)^2} & \text{odd} \\ \frac{16}{(n\pi)^2} & 2, 6, 10 \\ 0 & 4, 8, 12 \end{cases}$

Probability → B) $\langle \psi_n | \psi(t) \rangle = \sum_n C_n e^{-i\omega t} \langle \psi_n | \psi_n \rangle = \sum_n C_n e^{-i\omega_n t} \delta_{nm}$
$= C_n e^{-i\omega_n t}$

∴ So: $\left[\begin{array}{l} \text{ENERGY} = \Psi(x,t) = \frac{2}{\pi}\sqrt{\frac{2}{L}} \sum_{1,3,5} \frac{1}{n} \sin(u) e^{i\theta_n} + \frac{4}{\pi}\sqrt{\frac{2}{L}} \sum_{2,6,10} \frac{1}{n} \sin(u) e^{-i\theta_n} \\ \text{Probability} = \langle \psi_n | \psi(t) \rangle = C_n e^{-i\omega_n t} \end{array}\right.$

② TIME EVOLUTION

[a] PREFACE –

Problems 1.4

(1.1) Use separation of variables to split PDE into 2 ODEs.

Ans: 1) $\Psi = \psi(x)\,\phi(t)$

$$\nabla^2\Psi = \frac{\partial^2\psi}{\partial x^2}\phi(t) = \psi''\phi \qquad \frac{\partial\Psi}{\partial t} = \psi(x)\frac{d\phi}{dt} = \psi\dot{\phi}$$

So: $$\boxed{-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}}$$

2.)
$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}\phi + V\psi\phi = i\hbar\psi\dot{\phi}$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}\phi = i\hbar\psi\frac{d\phi}{dt} - V\psi\phi = i\hbar\psi\left[\frac{d\phi}{dt} - V\phi\right]$$

$$\Rightarrow -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}\frac{\phi}{\psi} = i\hbar\frac{dt}{dt} - V\phi \qquad \text{Let}$$

$$\rightarrow -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = i\hbar\frac{d\phi}{dt}\frac{1}{\phi} - V$$

$$-\frac{\hbar^2}{2m}\cancel{\nabla^2} i\hbar\frac{d\phi}{dt}\frac{1}{\phi} + V = 0$$

$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \frac{i\hbar\, d\phi}{dt}\frac{1}{\phi} + V = 0$$

3.) Let $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}\frac{1}{\psi} = -a$ and $i\hbar\frac{1}{\phi}\frac{d\phi}{dt} - V = a$ so $-a + a = 0$.

so: a.) $$\boxed{i\hbar\frac{d\phi}{dt}\frac{1}{\phi} - V = -a} \rightarrow i\hbar\frac{d\phi}{dt}\frac{1}{\phi} = a + V \qquad \boxed{E = a - V}$$

$$\rightarrow \frac{d\phi}{dt}\frac{1}{\phi} = \frac{-a+V}{i\hbar} \rightarrow \int\frac{d\phi}{\phi} = \int\frac{-a+V}{i\hbar}dt$$

$$\Rightarrow \phi(t) = e^{-\frac{Et}{i\hbar}} = e^{-\frac{iEt}{\hbar}}$$

$$(2m)(2m)^{-\frac{1}{2}} \quad (2m)^{1-\frac{1}{2}} \quad (2m)^{\frac{2}{2}-\frac{1}{2}} = 2m^{\frac{1}{2}} = \sqrt{2m}$$

$$\left((-1)\cdot\frac{\hbar^2}{2m}\right)^{\frac{1}{2}} \qquad (-1)^{\frac{1}{2}}\cdot\left(\frac{\hbar^2}{2m}\right)$$

b.) $$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}\frac{1}{\psi} = +a} \rightarrow \frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + a\psi = 0$$

Solves Quadratic Charact.: $$\frac{-b \pm\sqrt{b^2-4ac}}{2a} = \frac{0\pm\sqrt{0-\frac{\hbar^2}{2m}a}}{2\frac{-\hbar^2}{2m}} = \frac{\sqrt{-\frac{\hbar^2}{2m}a}}{2\frac{\hbar^2}{2m}}$$

$$= i\sqrt{\frac{\hbar^2}{2m}}\left(\frac{2m}{\hbar^2}\right)^1 = i\frac{\hbar}{\sqrt{2m}}\frac{2m}{\hbar^2} = i\frac{2m}{\sqrt{2m}}\frac{1}{\hbar} = \frac{i\sqrt{2m}}{\hbar}$$

$$T(t) = \phi(t)$$

$$i\hbar\psi\frac{dT}{dt} = -\frac{\hbar^2}{2m}T\frac{d^2\psi}{dx^2} + V\psi T$$

$$i\hbar\psi\frac{dT}{dt}\frac{1}{T} = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi$$

So: $$\psi(x) = e^{i\frac{\sqrt{2m}}{\hbar}x} = \cos\left(\sqrt{\frac{2m}{\hbar}}x\right) + i\sin\left(\sqrt{\frac{2m}{\hbar}}x\right) = \boxed{e^{ikx}} \checkmark$$

$$\frac{dT}{dt} = \frac{iE\,T(t)}{\hbar} \qquad T(t) = e^{\frac{iE\,T(t)}{\hbar}}$$

3.) So: $$\boxed{\Psi(x,t) = e^{i(kx-Et)}} \text{ Typical Solution!}$$

$$\frac{\partial\psi}{\partial t} = \frac{i\hbar}{i\hbar}E\psi(x) = E\psi(x)$$

4.) The PDE $\longrightarrow$ SODE is:

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x)}$$

(15) Verifying (HMP).

Ans: 1) $v_1 |A_1|^2 = v_1 \left| \frac{(k_1+k_2)}{(k_1-k_2)} B_1 \right|^2 = v_1 \frac{(k_1+k_2)^2}{(k_1-k_2)^2} B_1^2.$

$k_1 = \sqrt{\frac{2mE}{\hbar^2}}$

$k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$

$(xy)^{\frac{1}{2}} = x^{\frac{1}{2}} y^{\frac{1}{2}}$

$(16 \cdot 4)^{\frac{1}{2}} = 16^{\frac{1}{2}} 4^{\frac{1}{2}}$

$(64)^{\frac{1}{2}} = 4 \cdot 2$

$64 = 4^2 \cdot 2^2$

$64 = 16 \cdot 4$

$64 = 64$ ✓

It follows also that:

$(k_1 - k_2)^2$

$k_1^2 - 2k_1k_2 + k_2^2$

$(k_1+k_2)^2 - 2k_1k_2$

$\frac{2m}{\hbar^2}\left\{ (\sqrt{E} + \sqrt{E-V_0})^2 - \sqrt{E(E-V_0)} \right\}$

let = ...

and: $(k_1+k_2)^2 = k_1^2 + 2k_1k_2 + k_2^2$

$= \left(\sqrt{\frac{2mE}{\hbar^2}}\right)^2 + 2\sqrt{\frac{2mE}{\hbar^2}}\sqrt{\frac{2mE-V_0}{\hbar^2}} + \left(\sqrt{\frac{2m(E-V_0)}{\hbar^2}}\right)^2$

$= \frac{2mE}{\hbar^2} + \frac{2m(E-V_0)}{\hbar^2} + 2\left(\frac{2mE}{\hbar^2}\right)^{\frac{1}{2}}\left(\frac{2mE-V_0}{\hbar^2}\right)^{\frac{1}{2}}$

$= \frac{2m}{\hbar^2}\left(E + (E-V_0)\right) + 2\left(\frac{2mE}{\hbar^2} \cdot \frac{2m(E-V_0)}{\hbar^2}\right)^{\frac{1}{2}}$

$= \frac{2m(\sqrt{E}+\sqrt{E-V_0})^2}{\hbar^2} + 2\left(\frac{2m}{\hbar^2}\sqrt{E(E-V_0)}\right) = K + 2K'$

let: $k_1^2 + k_2^2 = K$

$k_1 k_2 = K'$

2.) $\cancel{2m(\ldots)} \to \frac{4m}{\hbar^2}\underbrace{\sqrt{E(E-V_0)}}_{K'}$ or $\underbrace{\frac{4mE - 2mV_0}{\hbar^2}}_{K}$

or: $\frac{k_1k_2}{k_1^2+k_2^2} = \frac{K'}{K} = \frac{2m/\hbar^2 \sqrt{E(E-V_0)}}{2m/\hbar^2 (\sqrt{E}+\sqrt{E-V_0})^2} = \boxed{\frac{\sqrt{E(E-V_0)}}{(\sqrt{E}+\sqrt{E-V_0})^2}}$

3. let: $T = 1 - R = 4\frac{\left(2m/\hbar^2 \sqrt{E(E-V_0)}\right)}{\left(2m/\hbar^2(\sqrt{E}+\sqrt{E-V_0})^2\right)} = \boxed{\frac{4\sqrt{(E-V_0)E}}{(\sqrt{E}+\sqrt{E-V_0})^2}}$ ✓

Problems N.

(13) Infinite square well potential:

$$\boxed{\Psi(x,0) = \begin{cases} \sqrt{\frac{2}{3L}} \cos\left(\frac{2\pi}{L}x\right) + i\sqrt{\frac{4}{3L}} \sin\left(\frac{3\pi}{L}x\right) & 0 < x < L \\ 0 & \text{else} \end{cases}}$$

(c) Probability $P(E_n,t)$?

Ans: $P(E_n,t) = |\langle \psi_n | \psi(t)\rangle|^2 = \sum_m C_m e^{-i\omega_m t} \langle \psi_n | \psi_m\rangle$

1st) $\langle \psi_n^* | \Psi(x,0)\rangle = \int_a^b \psi_n^*(x)\, \Psi(x,0)\, dx$

$$= \int_a^b \left\{\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)\right\} \cdot \left\{\sqrt{\frac{2}{3L}} \cos\left(\frac{2\pi}{L}x\right) + i\sqrt{\frac{4}{3L}} \sin\left(\frac{3\pi}{L}x\right)\right\} dx.$$

$$= \sqrt{\frac{2}{L}}\left(\sqrt{\frac{2}{3L}} + i\sqrt{\frac{4}{3L}}\right) \int_a^b \left[\sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{2\pi}{L}x\right) + i\sqrt{\frac{4}{3L}} \cos\left(\frac{3\pi}{L}x\right) \sin\left(\frac{n\pi L}{x}\right)\right] dx$$

$$= \int_0^L \frac{4}{3L^2} \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{2\pi}{L}x\right) + i\frac{8}{3L^2} \sin\left(\frac{3\pi}{L}x\right) \sin\left(\frac{n\pi L}{x}\right) \Big\} dx.$$

$\bar{R} = \frac{4}{3L^2}$

$\bar{C} = i\frac{8}{3L^2}$

Let $\alpha = \frac{n\pi x}{L}$, $\beta_1 = \frac{2\pi}{L}x$, $\beta_2 = \frac{3\pi}{L}x$.

Trig: $\sin\alpha \sin\beta_1 = \frac{1}{2}\left[\cos(\alpha-\beta_1) - \cos(\alpha+\beta_1)\right] = \frac{1}{2}\cos\left(\frac{n\pi x}{L} - \frac{2\pi x}{L}\right) - \cos\left(\frac{n\pi x}{L} + \dots\right)$

$\sin\alpha \sin\beta_2 = \frac{1}{2}\left[\cos(\alpha-\beta_2) - \cos(\alpha+\beta_2)\right] = \frac{1}{2}\cos\left(\frac{n\pi x}{L} - \frac{3\pi x}{L}\right) - \cos\left(\frac{n\pi x}{L} + \frac{3\pi x}{L}\right)$

So: $\int_0^L \bar{R}\left[\frac{1}{2}\left[\cos\left(\frac{n\pi x}{L} - \frac{2\pi x}{L}\right) - \cos\left(\frac{n\pi x}{L} + \frac{2\pi x}{L}\right)\right]\right] + \bar{C}\left[\frac{1}{2}\left[\cos\left(\frac{n\pi x}{L} - \frac{3\pi x}{L}\right) - \cos\left(\frac{n\pi x}{L} + \frac{3\pi x}{L}\right)\right]\right]$

$$= \frac{1}{2}\int_0^L \left[\bar{R}\left[\cos\left(\frac{\pi(n-2)x}{L}\right) - \cos\left(\frac{\pi(n+2)x}{L}\right)\right] + \bar{C}\left[\cos\left(\frac{\pi(n-3)x}{L}\right) - \cos\left(\frac{\pi(n+3)x}{L}\right)\right]\right] dx.$$

$\frac{d \sin(kx)}{dx}$

$\frac{dy}{du}\frac{du}{dx}$

$\frac{d}{du}(kx)\frac{d}{dx}(\sin(u))$

$= \cos(u)$.

$\boxed{C}$ Recall: $\int_0^L \cos\left(\frac{k\pi x}{L}\right) dx = \frac{1}{\frac{k\pi}{L}} \sin\left(\frac{k\pi x}{L}\right)\Big|_0^L = \frac{L}{k\pi} \sin\left(\frac{k\pi L}{L}\right) = \frac{L}{k\pi}\sin(k\pi$

So $\frac{1}{2}\left[\bar{R}\left\{\frac{L}{\pi(n-2)} \sin(\pi(n-2)) - \frac{L}{\pi(n+2)} \sin[\pi(n+2)]\right\} + \bar{C}\left\{\frac{L}{\pi(n-3)} \sin(\pi(n-3)) - \frac{L}{\pi(n-3)} \sin(\pi(n-3))\right\}\right.$

Conditions:

$\boxed{C_1 + C_2 + C_3 = 1}$

The form ex: $\frac{1}{2}\frac{L}{\pi(n-m)} \sin(\pi(n-m)) \Rightarrow \begin{cases} 0 & \text{if } n \neq m \\ \frac{L}{2} & \text{if } n = m \end{cases}$

Let $(3,2) = m$

2) So: $P(E_n,t) = \bar{R}\psi_1 + \bar{\bar{C}}_2\psi_2 = \bar{\bar{R}} + \bar{\bar{C}}$

$C_1 = 0$; $C_2 = \sqrt{\frac{2}{L}} \cdot \sqrt{\frac{2}{3L}}\, \frac{L}{2} = \sqrt{\frac{1}{3}}$ $\quad C_3 = \sqrt{\frac{2}{L}} \cdot \sqrt{\frac{4}{3L}}\, \frac{L}{2} = i\sqrt{\frac{2}{3}}$

So: $\left\{ \begin{array}{l} P(E_1) = 0 \\ P(E_2) = \sqrt{\frac{1}{3}} \\ P(E_3) = i\sqrt{\frac{2}{3}} \end{array} \right.$ now: $\boxed{P(E_1) + P(E_3) = \sqrt{\frac{1}{3}} + i\sqrt{\frac{2}{3}}}$

(b) $\langle E(t)\rangle = |c_2|^2 E_2 + |c_3| E_3$

$$= \frac{1}{3}\cdot\frac{2^2\pi^2\hbar^2}{2mL^2} + \frac{2}{3}\cdot\frac{3^2\pi^2\hbar^2}{2mL^2} = \frac{4}{3}\frac{\pi^2\hbar^2}{2mL^2} + \frac{18}{3}\frac{\pi^2\hbar^2}{2mL^2}$$

$$= \left(\frac{4+18}{3}\right)\frac{\pi^2\hbar^2}{2mL^2} = \frac{22}{3}\frac{\pi^2\hbar^2}{2mL^2} = \frac{11}{3}\frac{\pi^2\hbar^2}{mL^2}$$

so: $\boxed{\langle E\rangle = |c_2|^2 E_2 + |c_3| E_3 = \frac{11}{3}\frac{\pi^2\hbar^2}{mL^2}}$

(c) $\langle x(t)\rangle = \langle x(0)\rangle = \frac{1}{m}\int \langle \psi(t)|\hat{x}|\psi(t)\rangle\, dt.$

$= \frac{1}{m}\int x|\psi(x,0)|^2\, dx$? So: $\boxed{\langle x\rangle = \frac{1}{m}\int x|\psi|^2\, dx}$

we see that $\int |\psi|^2 dx \triangleq \int \phi_n \Phi(x,t=0)\, dx = \frac{L}{2}$.

so: $\langle x\rangle = \int_0^L x\cos\left(\frac{k\pi x}{L}\right) = \frac{L^2}{k^2\pi^2}\cos(k\pi) - \cos(0)$.

$$= \frac{L^2}{(k\pi)^2}\left((-1)^k - 1\right)$$

$$\Rightarrow \boxed{\int_0^L x\cos\left(\frac{k\pi x}{L}\right) = \begin{cases} 0 & \text{even} \\ \frac{-2L^2}{k^2\pi^2} & \text{odd} \end{cases}}$$

but intuitively $\langle x\rangle = c_1 x_1 + c_2 x_2 = \frac{1}{3}\frac{L}{2} + \frac{2}{3}\frac{L}{2} = \boxed{\frac{L}{2}}$

(d) $\langle p\rangle = \int_0^L \psi^*\left(\frac{\hbar}{i}\nabla\right)\psi\, dx$

or $\langle p\rangle = c_2 p_2 + c_3 p_3$ and see that $p = m\frac{dx}{dt}$ so $\langle p\rangle = m\langle \dot{x}\rangle$

But we know that $\langle \dot{x}\rangle = 0$. So: $\boxed{\langle p\rangle = 0}$

SOLVING THE SCHRÖDINGER EQS DIRECTLY (12.A)

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}m\omega^2x^2\psi = E\psi(x)}\quad (12.A.1)$$

① ... let $\chi = \beta x$ s.t. $\beta = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}}$. $\beta^2 = \frac{m\omega}{\hbar}$ (12.A.2)

1) $\frac{d^2\psi(\chi)}{dx^2} = \frac{d}{dx}\left(\frac{d\psi}{d\chi}\frac{d\chi}{dx}\right) = \frac{d}{dx}\left(\frac{d\psi}{d\chi}\beta\right) = \frac{d^2\psi}{d\chi^2}\frac{d\chi}{dx}\beta = \boxed{\frac{d^2\psi}{dx^2}}$

$\beta = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}} \rightarrow \beta^2 = \frac{m\omega}{\hbar}$

2) $T = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = -\frac{\hbar^2}{2m}\frac{m\omega}{\hbar}\chi^2\frac{d^2\psi}{d\chi^2} = -\frac{\hbar\omega}{2}\frac{d^2\psi}{d\chi^2} = -\frac{\hbar\omega}{2}\frac{d^2\psi}{d\chi^2}$

3) $V = \frac{1}{2}m\omega^2x^2\psi = \frac{1}{2}m\omega^2\frac{\chi^2}{\beta^2} = \frac{1}{2}m\omega^2\frac{\chi^2}{\beta^2}\psi(\chi)$.

$H = T+V \rightarrow \boxed{-\frac{\hbar\omega}{2}\frac{d^2\psi}{d\chi^2}+\frac{1}{2}m\omega^2\frac{\chi^2}{\beta^2}\psi(\chi) = E\psi(\chi).}$

4) $-\frac{\hbar\omega}{2}\frac{d^2\psi(\chi)}{d\chi^2},\ \frac{1}{2}m\omega^2\chi^2\frac{1}{\frac{m\omega}{\hbar}}\psi(\chi) \leftrightarrow \frac{1}{2}\cancel{m}\omega^{\cancel{2}}\chi^2\frac{\hbar}{\cancel{m\omega}} = \frac{1}{2}\chi^2\hbar\omega$

$$\boxed{-\frac{\hbar\omega}{2}\frac{d^2\psi(\chi)}{d\chi^2}+\frac{\hbar\omega}{2}\chi^2\psi(\chi) = E\psi(\chi)}\quad (12.A.3)$$

② let $\varepsilon = \frac{2E}{\hbar\omega}$. So,

1) $\frac{\hbar\omega}{2}\left[-D_\chi^2\psi(\chi)+\chi^2\psi(\chi)\right] = E\cancel{\psi(\chi)}$

$-D_\chi^2\psi(\chi)+\chi^2\psi(\chi) = \frac{2E}{\hbar\omega}\psi(\chi) = \varepsilon\psi(\chi)$.

$$\boxed{\frac{d^2\psi}{d\chi^2} = (\chi^2-\varepsilon)\psi(\chi).}\quad (12.A.5).\ (12.A.6).$$

2. $\boxed{\psi(\chi) = Ae^{-\frac{\chi^2}{2}}+Be^{-\frac{\chi^2}{2}}}$.

$\psi(\chi) \xrightarrow[\chi\to\infty]{} Ae^{-\frac{\chi^2}{2}}$ So $\psi(\chi) = h(\chi)e^{-\frac{\chi^2}{2}}$. $\left(\frac{dh(\chi)}{d\chi}\chi - h(\chi)\right)e^{-\chi^2}$

3) $\frac{d\psi(\chi)}{d\chi} = \frac{dh(\chi)}{d\chi}e^{-\frac{\chi^2}{2}}+h(\chi)(-\chi)e^{-\frac{\chi^2}{2}}$

$= \left(\frac{dh(\chi)}{d\chi}-\chi h(\chi)\right)e^{-\frac{\chi^2}{2}} \cdot x(\chi)e^{-\frac{\chi^2}{2}}$

$\frac{d^2\psi(\chi)}{d\chi^2} = \left(\frac{d^2h(\chi)}{d\chi^2}-\frac{d}{d\chi}(\chi h(\chi))\right)e^{-\frac{\chi^2}{2}} - \left(\frac{dh(\chi)}{d\chi}-h(\chi)\right)\chi e^{-\frac{\chi^2}{2}}.$

$= \left(\frac{d^2h(\chi)}{d\chi^2}-\frac{d\chi}{d\chi}h(\chi)-\frac{dh(\chi)}{d\chi}\chi\right)e^{-\frac{\chi^2}{2}} - \left(\frac{dh(\chi)}{d\chi}-h(\chi)\right)\chi e^{-\frac{\chi^2}{2}}$

$= \left(\frac{d^2h(\chi)}{d\chi^2}-h(\chi)+\chi h(\chi)-2\chi\frac{dh(\chi)}{d\chi}\right)e^{-\frac{\chi^2}{2}}$

chain rule

$\frac{d}{d\chi}\left(X(\chi)e^{-\frac{\chi^2}{2}}\right)$

$\frac{dX}{d\chi}e^{-\chi^2} - \chi X e$

LHS

$e^{\frac{U}{2}} = e^{-\frac{\chi^2}{2}}$

$\psi\ \frac{\partial U}{\partial \chi}\quad U = -\chi^2$

$\left(e^{\frac{U}{2}}\right)\frac{\partial}{\partial\chi}(-\chi^2)$

$e^{U}\ -2\chi$

$e^{U}\chi = e^{-\frac{\chi^2}{2}}\chi$

3.) so: $\frac{d^2\psi(x)}{dx^2} = (x^2-\varepsilon)\psi(x)$

$$\cancel{e^{-\frac{x^2}{2}}}\left(\frac{d^2h(x)}{dx^2} - h(x) - 2x\frac{dh(x)}{dx}\right) = (x^2-\varepsilon)\,h(x)\cancel{e^{-\frac{x^2}{2}}}$$

$$\frac{d^2h(x)}{dx^2} + x^2h(x) - h(x) - 2x\frac{dh(x)}{dx} = (x^2-\varepsilon)\,h(x)$$

$$\frac{d^2h(x)}{dx^2} - 2x\frac{dh(x)}{dx} + h(x)(x-1) = (x^2-\varepsilon)\,h(x)$$

then $\boxed{\frac{d^2h(x)}{dx^2} - 2x\frac{dh(x)}{dx} + (\varepsilon-1)\,h(x) = 0}$ (12.A.10).

(3) SERIES SOLUTION: $\boxed{h(x) = \sum_j a_j x^j}$ (12.A.11) then: $\partial_x h = \sum_j j x^{j-1} a_j$

$\partial_{xx} h = \sum_j j(j-1) x^{j-2} a_j$

1) So: $\frac{d^2h(x)}{dx^2} - 2x\frac{dh(x)}{dx} + (\varepsilon-1)\,h(x) = 0$ and $\partial_x h \longrightarrow \sum_j (j+1)x^j a_{j+1}$

$\partial_{xx} h \longrightarrow \sum_j (j+2)(j+1)x^j a_{j+2}$

$$= \sum_j j(j-1)a_j x^{j-2} - 2x\sum_j j a_j x^{j-1} + (\varepsilon-1)\sum_j a_j x^j = 0$$

$$= \sum_{j=0} (j+2)(j+1)\,a_{j+2}\cancel{x^j} + \sum_{j=0}\left[-2j+(\varepsilon-1)\right]a_j\cancel{x^j} = 0$$

so: $\sum_{j=0} (j+2)(j+1)a_{j+2} + (-2j+\varepsilon-1)a_j = 0 \rightarrow \therefore a_{j+2}[(j+2)(j+1)] - a_j[-2j+\varepsilon-1]$

2) and: $a_{j+2} = \frac{-a_j[-2j+\varepsilon-1]}{(j+2)(j+1)} = \frac{2j+1-\varepsilon}{(j+2)(j+1)}a_j$

then $\boxed{a_{j+2} = \frac{2j+1-\varepsilon}{(j+2)(j+1)}a_j}$ (12.A.16)

As long as this relationship is satisfied, the series $h(x)$ in (12.A.11), is a solution to the DE of (12.A.10).

3.) $\frac{a_{j+2}}{a_j} \rightarrow \frac{2}{j}$ as $j\to\infty$, so: $\frac{2}{j} = \frac{2j+1-\varepsilon}{(j+2)(j+1)}\Big|\lim_{j\to\infty}$.

and: $e^{x^2} = \sum_{n=0}\frac{1}{n!}(x^2)^n = \sum_{j=0,\ j=\text{even}} b_j x^j$ letting $2n=j$

and $\boxed{\frac{b_{j+2}}{b_j} = \frac{\left(\frac{j}{2}\right)!}{\left(\frac{j}{2}+1\right)!}}$ and $\boxed{\frac{b_j}{b_j}\xrightarrow[j\to\infty]{}\frac{2}{j}}$

(4) FINALE:

a) $h_n(x) = \sum_{j=0}^{n} a_j x^j$

b) $\psi_n(x) = h_n(x)\,e^{-\frac{x^2}{2}}$

c) and, $\boxed{\varepsilon_n = 2n+1}$ so: $\boxed{E_n = \hbar\omega\left(n+\frac{1}{2}\right)}$

① W.H.Ψ?

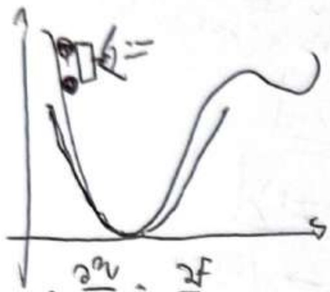
HARMONIC OSCILLATOR

[$\ell=1$] [$n=2$]

$$\boxed{V(x) = \sum_k \frac{1}{k!}\frac{d^k}{dx_0^k}\left(V(x_0)\right)[x-x_0]^k}$$

1) so $V(x) = V(x_0) + V'(x_0)(x-x_0) + \frac{1}{2}V''(x_0)\ldots$

V(x)



and $\frac{\partial V}{\partial x} = F(x)$ $\frac{\partial^2 V}{\partial x^2} = \frac{\partial F(x)}{\partial x}$ where $F(x) = \omega^2 x$

so $\frac{\partial F}{\partial x} = \omega^2$, $F(x) = \omega^2 x$ ...

2) $V(x) = \sum_k \frac{1}{k!} D^k(V(x_0))[x-x_0]^k = -\frac{1}{2}\omega^2 x^2 + \omega^2 x(x-x_0) + \omega^2(x-x_0)$

$= -\left(\frac{1}{2}\omega^2 x^2 + \omega^2 x^2 - \omega^2 x_0 + \omega^2 x + \omega^2 x_0\right)$ Let $x_0 = 0$ so.

$= -\frac{1}{2}\omega^2 x^2 + \omega^2 x^2 + \omega^2 x$

$= -\left(\frac{1}{2}\omega^2 x^2 + \frac{2}{2}\omega^2 x^2 + \omega^2 x\right) = -\left(\frac{3}{2}\omega^2 x^2 + x\omega^2\right)$

$= -\omega^2\left(\frac{3}{2}x^2 + x\right)$ so $V \sim \left(\frac{3}{2}\omega^2 x^2 - \omega^2 x\right)$

- $\frac{\partial^2 V}{\partial x^2} = \frac{\partial F}{\partial x}$
- $\frac{\partial V}{\partial x} = -F(x)$
- $V = -\int F(x)\,dx$

- $V'' = \omega^2$
- $V' = -\omega^2 x$
- $V = -\frac{1}{2}\omega^2 x^2$

3) $\boxed{V(x) = \frac{1}{2}k(x-x_0)^2}$ or $\boxed{V = \frac{1}{2}\omega^2 x^2}$

② CREATION, ANNIHILATION & NUMBER OPERATORS.

$$\boxed{\hat{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2}$$ Hamiltonian w/ Harmonic oscillator

here if $\hat{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 = \frac{1}{2}\left(\frac{p^2}{m} + m\omega^2 x^2\right) = \frac{1}{2}\left(\frac{p^2 + m^2\omega^2 x^2}{m}\right)$

$= \frac{1}{2m}\left(p^2 + m^2\omega^2 x^2\right) = \frac{1}{2}m\omega^2\left[x^2 + \frac{1}{(m\omega)^2}p^2\right]$

so $\boxed{\hat{H} = \frac{1}{2}m\omega^2\left[x^2 + \frac{1}{(m\omega)^2}p^2\right]}$

2) $\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y$?? $\hat{a} = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}}\left(x + \frac{i}{m\omega}\hat{p}\right)$

$\hat{a}^\dagger = \left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}}\left(x - \frac{i}{m\omega}\hat{p}\right)$

Find such that $\boxed{\hat{n} \equiv a^\dagger a}$

3) we proceed: $\hat{n} = \frac{m\omega}{2\hbar}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$

$= \frac{m\omega}{2\hbar}\left[x^2 + \frac{1}{(m\omega)^2}p^2 + \frac{i}{m\omega}(\hat{x}\hat{p} - \hat{p}\hat{x})\right] = \frac{m\omega}{2\hbar}\left[x^2 + \frac{p^2}{(m\omega)^2} + \frac{i[x,p]}{m\omega}\right]$

$= \frac{1}{\hbar\omega}\left[\frac{1}{2}m\omega^2\hat{x}^2 + \frac{p^2}{2m}\right] + \frac{i}{2\hbar}[\hat{x},\hat{p}] = \frac{1}{\hbar\omega}\left(\hat{H} - \frac{1}{2}\right)$

and $H = \hbar\omega\left(\hat{n} + \frac{1}{2}\right) = \hbar\omega\left(\hat{a}^\dagger\hat{a}\right) + \frac{1}{2}$

so $\boxed{\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)}$ and $[\hat{a}, \hat{a}^\dagger] = 1]$

3) $[\hat{n}, \hat{a}] = [\hat{a}^\dagger\hat{a}, \hat{a}]$

$= \hat{a}^\dagger[\hat{a},\hat{a}] + [\hat{a}^\dagger,\hat{a}]\hat{a} = 0 + [\hat{a}^\dagger\hat{a}]^*\hat{a} = -(1)\hat{a} = -\hat{a}$

so $\boxed{[\hat{n},\hat{a}] = -a}$

iv) $[\hat{n}, \hat{a}] = \hat{n}\hat{a} - \hat{a}\hat{n} = -\hat{a}$

$\boxed{\hat{n}\hat{a} = \hat{a}\hat{n} - \hat{a}}$

$\hat{H}|\phi\rangle = \hat{E}|\phi\rangle$
$\hat{H}|n\rangle = \hat{E}|n\rangle$ } so $\boxed{|n\rangle = |\phi\rangle}$

[1] EIGENVALUES & EIGENSTATES

1) $\boxed{\begin{aligned} \hat{H}|n\rangle &= \hbar\omega\left(n+\tfrac{1}{2}\right)|n\rangle \\ E_n|n\rangle &= \hbar\omega\left(n+\tfrac{1}{2}\right)|n\rangle \end{aligned}}$

w/ :
- $\hat{n}|n\rangle = n|n\rangle$
- $\hbar\omega\left(n+\tfrac{1}{2}\right) \geq 0$
- $n \geq \tfrac{1}{2}$..

$\to$

Let: $\boxed{\hat{a}|n\rangle = |\phi\rangle}$

w/ : $\hat{a}$ = annihilation operator
: ϕ = particle wave state

2) $\hat{n}|\phi\rangle = \hat{n}\hat{a}|n\rangle = (\hat{a}\hat{n} - \hat{a})|n\rangle = (\hat{a}n - \hat{a})|n\rangle = (n-1)\hat{a}|n\rangle$
$= (n-1)|\phi\rangle$

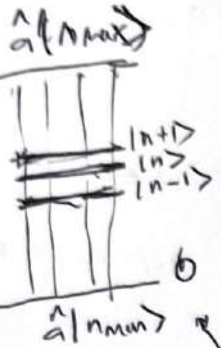
here: $\boxed{\hat{n}|\phi\rangle = (n-1)|\phi\rangle}$

3) $\begin{cases} \hat{a}|n\rangle = C_-|n-1\rangle \\ \hat{a}^\dagger|n\rangle = C_+|n+1\rangle \end{cases}$

then $\boxed{\hat{a}^{\gamma}|n\rangle = C_\pm|n\pm1\rangle}$

w/ $\gamma = \begin{cases} \dagger \to :(C_\pm \to C_-) \\ 1 \to :(C_\pm \to C_+) \end{cases}$

- The LADDER GRAPH



4) $\hat{n}|n_{min}\rangle = n_{min}|n_{min}\rangle$
$a^\dagger\hat{a}|n_{min}\rangle = n_{min}|n_{min}\rangle$
$a^\dagger[0] = n_{min}|n_{min}\rangle$

$\boxed{0 = n_{min}}$

$\to$ $\hat{a}|n_{min}\rangle = 0.$

So: $E_n = \hbar\omega\left(n+\tfrac{1}{2}\right) \in [0, T]$. So $\boxed{E_n \in \mathbb{R}^+}$

- MINIMUM ENERGY FOR THE UNCERTAINTY PRINCIPLE

$\langle H\rangle = \dfrac{(\Delta p)^2}{2m} + \dfrac{1}{2}m\omega^2(\Delta x)^2$

$\to (\Delta p)^2 \geq \dfrac{\hbar^2}{4(\Delta x)^2}$

$\langle H\rangle \gtrsim \dfrac{1}{2m}\dfrac{\hbar^2}{4(\Delta x)^2} + \dfrac{1}{2}m\omega^2(\Delta x^2)$

⚠ CAUTION: $\boxed{E_0 \neq 0}$ since $E_0 = \dfrac{\hbar\omega}{2}$ ~~is~~ the minimal reached!

So, $\Delta x \Delta p \geq \dfrac{\hbar}{2}$ $\to$ so $\Delta x\,\Delta p = 0$ is not true!
since $0 \geq \dfrac{\hbar}{2}$! $\leftrightarrow$ since $\Delta x\,\Delta p = 0$ is $E_0 = 0$ (PARTICLE AT REST).

$\boxed{\text{ii}}$

HARMONIC OSCILLATORS – wave funcs & Fock states

|12.3| |n.4|

3. WAVE FUNCTION

GROUND STATE:

• The Ground state of a particle in:

$$\boxed{\hat{a}|0\rangle = 0} \quad \text{why} \quad \hat{a}|n\rangle = a|n+1\rangle\sqrt{0} = a|0\rangle\sqrt{0} = 0.$$

• The ground state wave function has property:

$$\psi_0(x) \quad \boxed{\langle x|\hat{a}|0\rangle = 0} \text{ \& ground state wave}$$

>> w/ $\langle x|\hat{a}|0\rangle = \langle x|\frac{\beta}{\sqrt{2}}\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right)|0\rangle = \langle x|0\rangle\frac{\beta}{\sqrt{2}}\left|\hat{x}+\frac{i}{m\omega}\hat{p}\right).$

$$= \psi_0(x)\frac{\beta}{\sqrt{2}}\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right) = \frac{\beta}{\sqrt{2}}\left(\psi_0\hat{x}+\frac{i}{m\omega}\hat{p}\psi_0\right) = \frac{\beta}{\sqrt{2}}\left(\psi_0\hat{x}+\frac{i}{m\omega}(-i\hbar\nabla\psi_0)\right)$$

use EX-12.1

$$= \frac{\beta}{\sqrt{2}}\left(\langle x\rangle_0+\frac{\hbar}{m\omega}\nabla\psi_0\right) = \frac{\beta}{\sqrt{2}}\left(\langle x\rangle_0+\frac{\hbar}{m\omega}\left(\frac{\partial x}{\partial x}\frac{\partial\psi}{\partial x}+x\frac{\partial\psi}{\partial x}-\frac{\partial(x\psi)}{\partial x}\right)\right)$$

$$= \frac{\beta}{\sqrt{2}}\left(\langle x\rangle_0+\frac{\hbar}{2m}\frac{\partial\langle x\rangle}{\partial x}\right) = \frac{\beta}{\sqrt{2}}\left(0+\frac{\hbar}{2m}\frac{\partial 0}{\partial x}\right) = \frac{\beta}{\sqrt{2}}(0) = \boxed{0}$$

GAUSSIAN WAVE.

• The Differential eqs yields soln:

$$\boxed{\psi_0 = \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}} e^{-\frac{\beta^2x^2}{2}}} \Longleftrightarrow \frac{\partial\psi_0}{\partial x} = -\beta^2\psi_0(x)x$$

>> w/: $\frac{d\psi_0}{dx} = -\beta^2\psi_0 x$ $\quad$ * $\int\frac{d\psi}{\psi} = \int-\beta^2x\,dx = \ln\psi_0 - \ln A = -\frac{1}{2}\beta^2x^2$

* $\ln\psi_0 = \ln A - \frac{1}{2}\beta^2x^2$ $\quad$ * $\boxed{\psi_0(x) = Ae^{-\frac{1}{2}\beta^2x^2}}$

>> Normalization:

$$\int|\psi|^2dx = A^2\int e^{-\beta^2x^2}dx = A^2\sqrt{\frac{\pi}{\beta^2}} = 1$$

$$\star\ A^2 = \sqrt{\frac{\beta^2}{\pi}} \quad \star\ A^2 = \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{2}\cdot\frac{1}{2}} = \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}}$$

* ergo: $\psi_0(x) = \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}}\exp\left(-\frac{1}{2}\beta^2x^2\right)$

HERMITE SOLUTION:

• We previously used the fact that:

$$\boxed{\hat{a}^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle} \text{ \& "Harmonic" series}$$

$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$

$\frac{1}{\sqrt{n}}\hat{a}|n\rangle = |n-1\rangle$

• The same technique can be applied:

$$\boxed{\hat{a}^\dagger|n\rangle \equiv \psi_{n+1}(x)} \text{ \& "raised" wave}$$

$$\frac{\beta}{\sqrt{2(n+1)}}\langle x|\left(x-\frac{1}{m\omega}p\right)|n\rangle = \frac{\beta}{\sqrt{2n+1}}\left(x\psi_n-\frac{1}{\beta^2}\frac{\partial\psi_n}{\partial x}\right) \equiv \psi_{n+1}(x)$$

>> w/:

* $\hat{a}^\dagger|n\rangle = \frac{\beta}{\sqrt{2}}\left(x-\frac{i}{m\omega}\hat{p}\right).$

* $\frac{\beta}{\sqrt{2}}\left(x-\frac{1}{m\omega}\hat{p}\right)|n\rangle = \frac{\beta}{\sqrt{2}}\left(x|n\rangle-\frac{i}{m\omega}p|n\rangle\right)$

* $\langle x|\frac{\beta}{\sqrt{2}}\left(x|n\rangle-\frac{1}{m\omega}p|n\rangle\right) = \langle x|n+1\rangle\sqrt{n+1}$

* $\frac{\beta}{\sqrt{2}\sqrt{n+1}}\left(\langle x|x|n\rangle-\frac{1}{m\omega}\langle x|p|n\rangle\right) =$

• For example, $n=1$ is:

$$\boxed{\psi_1(x) = \frac{\beta}{\sqrt{2}}\left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}}2x\,e^{-\frac{\beta x^2}{2}}} \text{ \& 1st order polynom}$$

>> w/ $\frac{\beta}{\sqrt{2}}\left(x\psi_0-\frac{1}{\beta^2}\frac{\partial\psi}{\partial x}\right)$

$$= \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}}\frac{\beta}{\sqrt{2}}\left(xe^{-\frac{\beta^2x^2}{2}}-\frac{1}{\beta^2}\frac{\partial e^{-\beta x^2}}{\partial x}\right)$$

$$= \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}}\frac{\beta}{\sqrt{2}}\left(xe^{-\frac{\beta^2x^2}{2}}-\frac{1}{\beta^2}\left(\frac{-2x\beta^2}{2}\right)e^{-\frac{\beta^2x^2}{2}}\right)$$

$$= \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}}\frac{\beta}{\sqrt{2}}\,2x\,e^{-\beta x^2}$$

• Thus, born is the Hermite polynomial Generalized wave func:

$$\boxed{\psi_n(x) = \left(\frac{\beta^2}{\pi}\right)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(\beta x)\,e^{-\frac{\beta^2x^2}{2}}}$$

\& HARMONIC OSCILLATOR.

• AND

$$\boxed{E_n = \hbar\omega\left(n+\frac{1}{2}\right) = \frac{1}{2}m\omega^2x_n^2}$$

③ FOCK STATES & PHOTONS.

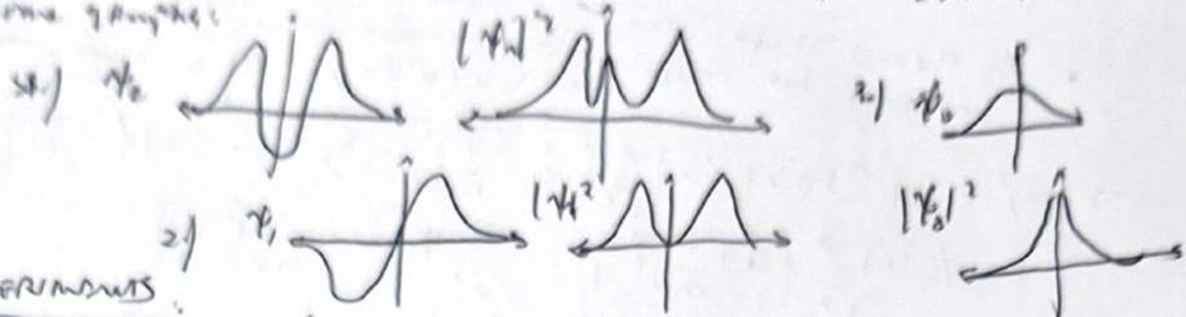
TIME DEPENDENT WAVE:

$$\boxed{\Psi(x,t) = \psi_n(x)\, e^{-\frac{iE_n t}{\hbar}}}$$

- It's measurable:

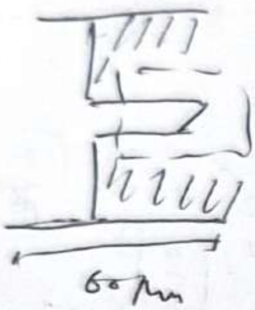
$$|\Psi|^2 = \psi^* e^{-\frac{iE_n t}{\hbar}} \psi\, e^{\frac{iE_n t}{\hbar}} = \psi\psi^* = |\psi_n(x)|^2 \checkmark$$

- some graphs:



EXPERIMENTS:

- The Harmonic oscillator doesn't oscillate, since the $\boxed{\text{HAMILTONIAN-TIME-INDEPENDENT}}$
- The Hamiltonian is in a superposition of states corresp. to $\boxed{\text{2-ADJACENT-ENERGY-LEVELS}}$
- The separation of the energy levels is long long. $\boxed{\Delta E = \hbar\omega}$ as n increases, or $\boxed{\frac{dE}{dn} = \hbar\omega}$
- But if the classical limit applies at large energies $\boxed{\lim_{n\to\infty} \Delta P_n = \text{CLA}}$, No matter how large $n \to \infty$.
- The $\boxed{\text{FOCK STATES}}$ are $\boxed{\text{NON-CLASSICAL}}$ since $\boxed{E_n = n\hbar\omega}$ the particle grows but with const or large



:

$\omega \sim 6.175$ GHz

$$\boxed{\omega \propto \frac{1}{T}}$$

12.5

(5) COHERENT STATES

SERIES ANALYSIS: • The Annihilation operator α:

$$\boxed{\hat{a}|\alpha\rangle = \alpha|\alpha\rangle}$$

• The series & coupling up than Annihilation operator n: $\boxed{C_n = \frac{\alpha^n}{\sqrt{n}} C_0}$ $\boxed{|\alpha\rangle = C_0 \sum_{n=0} \frac{\alpha^n}{\sqrt{n!}} |n\rangle}$

COEFFICIENT w/ "DRIFT" — ANNIHILATOR SERIES

>> The coefficient is derived:

* $|\alpha\rangle\alpha = \hat{a}|\alpha\rangle$
* $\alpha \sum_n C_n \alpha^n = \hat{a} \sum_n C_n \alpha^n$
* $\alpha \sum_n C_n \alpha^n = \hat{a} \sum_{n-1} C_n \alpha^{n-1} = \hat{a} \sum_{n=1} C_{n-1}\alpha^n$
* $\alpha C_n = \hat{a} C_{n-1} \rightsquigarrow \frac{\alpha}{\hat{a}} = \frac{C_{n-1}}{C_n} \rightsquigarrow \frac{C_n}{C_{n-1}} = \frac{\hat{a}}{\alpha}$

$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$

$$\boxed{|\alpha\rangle\hat{a} = \sum_{n=1} C_n \sqrt{n} |n-1\rangle}$$

* $\hat{a}|\alpha\rangle = \sum C_{n-1} \alpha |n-1\rangle$
* $\sqrt{n} C_n = C_{n-1}\alpha$ w/ $\langle\alpha|n-1\rangle$ $= \langle\alpha|\frac{1}{\sqrt{n}}\hat{a}|n\rangle$ $= \alpha\frac{1}{\sqrt{n}}\hat{a}|n\rangle$
* $\boxed{C_n = \frac{\alpha}{\sqrt{n}} C_{n-1}}$

Similar to: $\sigma_T = \frac{\sigma_0}{\sqrt{T}}$ ...

$\frac{\sigma_T\sqrt{T} - \sigma_T}{\sqrt{T}} = \varepsilon_T$

Random walk w/ drift

* Thus:

$$|\alpha\rangle = C_0 \sum_{n=0}^{\hat{}} \frac{\alpha^n}{\sqrt{n}} |n\rangle$$

• One of the fundamentals: $\boxed{\langle\alpha|\alpha\rangle = C_0^2 e^{|\alpha|^2} = 1}$ & orthogonality condition

>> wh/:

$$\langle\alpha|\alpha\rangle = \left(C_0 \sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle\right)\left(C_0 \sum_{n'} \frac{\alpha^{*n'}}{\sqrt{n'!}}|n'\rangle\right) = C_0^2 \sum_{nn'} \frac{\alpha^n \alpha^{*n'}}{\sqrt{n!}\sqrt{n'!}}\langle n|n'\rangle$$

$$= C_0^2 \sum_n \frac{\alpha^n \alpha^{*n}}{n!}(1) = C_0^2 \sum_n \frac{(\alpha\alpha^*)^n}{n!} = C_0^2 e^{\alpha^2}$$

• The normalization factor requires: $\boxed{|\alpha\rangle = e^{-\frac{\alpha^2}{2}} \sum_{n=0} \frac{\alpha^n}{\sqrt{n}} |n\rangle}$ NORMALIZED ANNIHILATOR

>> wh/ $\langle\alpha|\alpha\rangle = C_0^2 e^{\alpha^2} = 1$

* $C_0^2 = e^{-\alpha^2}$ * $C_0^{2\frac{1}{2}} = e^{-\alpha^2\frac{1}{2}}$ * $\boxed{C_0 = e^{-\frac{\alpha^2}{2}}}$

and $|\alpha=0\rangle = |n=0\rangle$

Since $|\alpha=0\rangle = |0\rangle = C_0 \sum \frac{0^n}{\sqrt{n!}}|n\rangle$

$0 = 0|n\rangle = \boxed{0}$

STATISTICS & EXPECTATIONS:

• Dimensionless operator of space: $\boxed{\hat{x} = \frac{\beta}{\sqrt{2}} \xi = \frac{1}{2}(\hat{a} + \hat{a}^\dagger)}$ (real)

• Dimensionless operator of momentum: $\boxed{\hat{P} = \frac{1}{\hbar\beta\sqrt{2}} \hat{p} = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger)}$ (imag)

• The creation & Annihilation operators is linear: $\boxed{\begin{cases} \hat{a} = (\hat{x} + i\hat{P}) \\ \hat{a}^\dagger = (\hat{x} - i\hat{P}) \end{cases}}$ Dimensionless Annihilator

$\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$
$\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + 1$
$\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a} = 2\hat{a}^\dagger\hat{a} + 1$

>> MEAN: $\langle x\rangle = \langle\alpha|\hat{x}|\alpha\rangle = \frac{1}{2}\langle\alpha|(a + a^\dagger)|\alpha\rangle = \frac{1}{2}(\alpha + \alpha^*)\langle\alpha|\alpha\rangle = \mathrm{Re}(\alpha)$

>> STD. DEV: $\langle x^2\rangle = \langle\alpha|\hat{x}^2|\alpha\rangle = \frac{1}{4}\langle\alpha|(\hat{a} + \hat{a}^\dagger)(a + \hat{a}^\dagger)|\alpha\rangle = \frac{1}{4}\langle\alpha|\hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}|\alpha\rangle$

$= \frac{1}{4}($

EXERCISE 12

(1) PROOF $\boxed{[a^\dagger \hat{a}] = 1}$

Ans: $[\hat{a}^\dagger \hat{a}] = \hat{a}^\dagger \hat{a} - \hat{a}^\dagger \hat{a} = \hat{n} - \hat{n}^\dagger = \left(\frac{1}{\hbar\omega}\hat{H} + \frac{1}{2}\right) - \left(\frac{1}{\hbar\omega}\hat{H} - \frac{1}{2}\right)$

$= \frac{1}{\hbar\omega}\hat{H} + \frac{1}{2} - \frac{1}{\hbar\omega}\hat{H} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = \boxed{1}$

(2) PROOF $\boxed{[\hat{n}, \hat{a}^\dagger] = \hat{a}^\dagger}$

Ans: $[\hat{n}, \hat{a}^\dagger] = \hat{n}\hat{a}^\dagger - \hat{a}^\dagger\hat{n} = \hat{a}\hat{a}^\dagger a^\dagger - \hat{a}^\dagger \hat{a}\hat{a}^\dagger = a^\dagger(\hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a})$

$= \hat{a}^\dagger[\hat{a}, a^\dagger] = \hat{a}^\dagger(1) = \hat{a}^\dagger$

(3) PROOF $\boxed{\hat{a}^\dagger |n\rangle = \sqrt{n+1}\,|n+1\rangle}$

Ans:
- Basically if $n = |C_-|^2$. Then:
- So $a^\dagger |n\rangle = C_+ |n+1\rangle$
  $= \sqrt{n+1}\,|n+1\rangle$ QED

$n = |C_-|^2$

$n+1 = C_-^2 + 1$

$(n+1)^{\frac{1}{2}} = C_-^{2\frac{1}{2}} + 1^{\frac{1}{2}}$

$\sqrt{n+1} = C_- + 1$

$\sqrt{n+1} = C_+$

$C_- + 1 = C_+$

$C_- - C_+ = 1$

$\sqrt{n} - \sqrt{n+1} = 1$

$\sqrt{n}^2 - \sqrt{n+1}^2 = 1^2$

$n - n + 1 = 1$ ✓

$\boxed{1 = 1}$

$\boxed{QED}$

SCHRÖDINGER 3D - SEPARATION I (17.1) (17.2)

1. The 3D Schrödinger eqs

[1] BASICS: $\hat{H} = \frac{\vec{P}^2}{2m} + V(\vec{r}) = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + V(\vec{r})$ $H|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle$

1) Car: $\boxed{\hat{H}\psi = \left(\frac{P^2(x,y,z)}{2m} + V(x,y,z)\right)\psi = i\hbar \frac{\partial \psi(\vec{r},t)}{\partial t}}$

2) $\langle \vec{r}|H|\psi(t)\rangle = \langle r| i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle$

$= \langle \vec{r}| \frac{P^2(x,y,z)}{2m} + V|\psi(t)\rangle = \langle r| i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle$

$= \frac{(-i\hbar)^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) + V(\vec{r}) \Big| \psi(\vec{r},t) = i\hbar\frac{\partial\psi(\vec{r},t)}{\partial t}$.

3) we: $\boxed{-\frac{\hbar^2}{2m}\nabla^2\Phi(\vec{r},t) + V\Phi(\vec{r},t) = i\hbar\frac{\partial}{\partial t}\Phi(\vec{r},t)}$

4) solutions & normalization:

a) $\Phi(\vec{r},t) = \sum_n C_n\psi_n(\vec{r})\cdot e^{-i\omega_n t}$ b) $\iiint_{r\varphi\theta} |\psi(\vec{r})|^2 = 1$.

[2] PDE Solutions:

$\psi(\vec{r}) = X(x)\,Y(y)\,Z(z) = XYZ$

we: $-\frac{\hbar^2}{2m}\left[YZ\frac{d^2X}{dx^2} + XZ\frac{d^2Y}{dy^2} + XY\frac{d^2Z}{dz^2}\right] = E\,XYZ$

$-\frac{\hbar^2}{2m}\left[\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2}\right] = E \rightarrow$ $-\frac{\hbar^2}{2m}\frac{1}{X}\frac{d^2X}{dx^2} = E_x$, $-\frac{\hbar^2}{2m}\frac{1}{Y}\frac{d^2Y}{dy^2} = E_y$, $-\frac{\hbar^2}{2m}\frac{1}{Z}\frac{d^2Z}{dz^2} = E_z$.

$-\frac{\hbar^2}{2m}\left[E_x + E_y + E_z\right] = E$. $\leftarrow$

so: $\boxed{E_{n_x} = \frac{n_x^2\pi^2\hbar^2}{2mL_x^2}}$ $\rightarrow$ $\boxed{X_{n_x}(x) = \begin{cases}\sqrt{\frac{2}{L_x}}\sin\left(\frac{n_x\pi}{L}x\right) & 0 < x < L \\ 0\end{cases}}$

- Boundary conditions
- Forces free

Since $E(n_x n_y n_z) = \frac{\pi^2\hbar^2}{2m}\left(\frac{n_x^2}{L_x^2} + \frac{n_y^2}{L_y^2} + \frac{n_z^2}{L_z^2}\right)$

h.w: $\boxed{\psi_{n_x n_y n_z}(\vec{r}) = \sqrt{\frac{8}{L_x L_y L_z}}\sin\left(\frac{n_x\pi}{L_x}x\right)\sin\left(\frac{n_y\pi}{L_y}y\right)\sin\left(\frac{n_z\pi}{L_z}z\right)}$

SPHERICAL COORDINATES

2. CENTRAL POTENTIALS - PART I

1) Assumptions:

a) $r = \sqrt{x^2+y^2+z^2}$ $\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$ b) $\boxed{\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta\end{aligned}}$

$\phi = \tan^{-1}\left(\frac{y}{x}\right)$ $\theta = \cos^{-1}\left(\frac{z}{r}\right)$

2) $d^3r = r^2\sin\theta\cdot dr\cdot d\theta\, d\phi$ so: $\int d^3r = \int r^2\sin\theta\,|\psi(\vec{r})|^2\,dr\,d\theta\,d\phi = 1$

3) $\boxed{\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}}$

Make in the APPENDIX!

w/: $\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} \rightarrow$ $\nabla\varphi = \frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y} + \frac{\partial\varphi}{\partial z} = \frac{\partial}{\partial r}$

as: $\frac{\partial\varphi}{\partial x_i} = \frac{\partial\varphi}{\partial r}\frac{\partial r}{\partial x_i} + \frac{\partial\varphi}{\partial\theta}\frac{\partial\theta}{\partial x_i} + \frac{\partial\varphi}{\partial\phi}\frac{\partial\phi}{\partial x_i}$

II) SEPARATION OF VARIABLES

1) $-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \Rightarrow -\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\right] + V\psi = E\psi$

Let $\boxed{\psi(r,\theta,\phi) = R(r)\,Y(\theta,\phi)}$ or $RY = \psi$

so: $-\frac{\hbar^2}{2m}\left[\frac{Y(\theta,\phi)}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{R(r)}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{R}{r^2\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] + VRY = ERY$

2) $-\frac{2mr^2}{\hbar^2 RY}\left(-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi\right) = \frac{-2mr^2}{\hbar^2 \cancel{RY}}E\cancel{RY} \rightarrow +\frac{\cancel{2m}r^2\cancel{\hbar^2}}{\cancel{\hbar^2}RY\cancel{2m}}\nabla^2\psi + \frac{V\cancel{RY}}{\cancel{RY}}\frac{2mr^2}{\hbar^2} = -E\frac{2mr^2}{\hbar^2}$

so: $\boxed{\frac{r^2}{RY}\nabla^2\psi - \frac{2m}{\hbar^2}r^2\left[V(r) - E\right] = 0}$

3) so: $\frac{r^2}{RY}\left[\frac{Y}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{R}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{R}{r^2\sin\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] = \frac{\cancel{r^2}\cancel{Y}}{\cancel{r^2}R\cancel{Y}}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{\cancel{R}\cancel{r^2}}{\cancel{r^2}\cancel{R}Y}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\ldots\right) + \frac{\partial^2 Y}{\partial\phi^2}\frac{\cancel{R}\,\cancel{r^2}}{\cancel{r^2}\sin^2\theta\cancel{RY}}$

$= \frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{Y}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{Y\sin^2\theta}\frac{\partial Y}{\partial\phi}$

so: $\boxed{\frac{1}{R}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)\right] + \frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial Y}{\partial\phi}\right] - \frac{2m}{\hbar^2}r^2\left[V - E\right] = 0}$

4) Let: $\begin{cases} \frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{2m}{\hbar^2}r^2(V-E) = l(l+1) \rightarrow \text{Radial part} \\ \frac{1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial Y}{\partial\phi}\right] = -l(l+1) \rightarrow \text{Angular part} \end{cases}$

so: $\boxed{l(l+1) - l(l+1) = 0}$

[ii] ANGULAR EQS:

1) $\sin\theta\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial Y}{\partial\theta}\right] + l(l+1)\sin^2\theta\, Y + \frac{\partial^2 Y}{\partial\phi^2} = 0$ and $\boxed{Y(\theta,\phi) = \Theta(\theta)\,\Phi(\phi)}$

so: $\boxed{\frac{\sin\theta}{\Theta(\theta)}\frac{d}{d\theta}\sin\theta\frac{d\Theta}{d\theta} + l(l+1)\sin^2\theta\,\frac{1}{\Phi(\phi)}\frac{d^2\Phi}{d\phi^2} = 0}$

2) and: $\left(\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m_l^2\Phi\right) \wedge \left(\left(\frac{\sin\theta}{\Theta(\theta)}\frac{d}{d\theta}\sin\theta\frac{d\Theta}{d\theta}\right) + l(l+1)\sin^2\theta = m_l^2\right)$

[iii] AZIMUTHAL EQS:

1) $\boxed{\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\phi^2} + m_l^2\Phi = 0} \rightarrow \frac{-b \pm\sqrt{b^2-4ac}}{2a} = \frac{0 \pm\sqrt{0-4(1)(m_l^2)}}{2} = \frac{\pm\sqrt{-4m_l^2}}{2}$

TYPICAL CHARACTERISTIC SOLN

$= \pm i\frac{2}{2}m_l = \boxed{\pm i m_l}$ $\quad \sqrt{-4m_l^2} = \sqrt{-1}\sqrt{4m_l^2} = i\sqrt{4m_l^2} = i2m_l$

so: $\boxed{\Phi(\phi) = e^{im_l\phi}}$

and $\boxed{\Phi(\phi) = \Phi_{m_l}(\phi + 2\pi)}$ wh/ $2\pi = 360° = 0$

so: $e^{im_l\phi} = e^{im_l(\phi+2\pi)} = e^{im_l\phi}e^{im_l 2\pi}$ and $e^{im_l 2\pi} = 1$.

2) is the orthonormality:

$$\boxed{\int_0^{2\pi}\Phi^*_{m_l}(\phi)\,\Phi_{m_l'}(\phi) = 2\pi\,\delta_{m_l m_l'}}$$

SCHRÖDINGER 3D - SEPARATION II

② CENTRAL POTENTIALS - PART II

[i] POLAR:

1) $$\sin\theta \frac{\partial}{\partial\theta}\left[\sin\theta \frac{\partial\Theta}{\partial\theta}\right] + \left[l(l+1)\sin^2\theta - m_l^2\right]\Theta = 0$$

$$\boxed{\Theta_l^{m_l}(\theta) = P_l^{m_l}\cos(\theta)}$$ in which $\forall\, 0 \le \theta \le \pi,\ l \in \mathbb{Z}^+$

$P_l^{m_l}(x) = (1-x)^{\frac{m}{2}}(D_x)^{m_l} P_l(x) \rightarrow$ Rodrigues

$P_l(x) = \frac{1}{2^l l!}(D_x)^l (x^2-1)^l. \rightarrow$ Legendre.

2). $P_l^{m_l}(x) = (1-x)^{\frac{m}{2}}(D_x)^{m_l} P_l(x) = (1-x)^{\frac{m}{2}}(D_x)^{m_l}\left[\frac{1}{2^l l!}(D_x)^l(x^2-1)^l\right]$

$= \frac{(1-x^2)^{\frac{m}{2}}}{2^l l!}(D_x)^{m_l}(D_x)^l(x^2-1)^l = \frac{1}{2^l l!}(1-x^2)^{\frac{m_l}{2}}(D_x)^{l+m_l}(x^2-1)^l$

$\therefore$ $$\boxed{P_l^m(x) = \frac{1}{2^l l!}(1-x^2)^{\frac{m_l}{2}}(D_x)^{l+m_l}(x^2-1)^l}$$

3) It follows for negative value m_l as long as $(l+m_l) \ge 0$ (well posed!)

in which: $$\boxed{P_l^{-m_l}(x) = (-1)^{m_l}\frac{(l-m_l)!}{(l+m_l)!}P_l^{m_l}(x)}$$

4) It's also orthogonal!

$$\int_{-1}^{1} P_l^m(x) P_{l'}^m(x) = \int_0^{\pi} P_l^{m_l}[\cos\theta]\sin\theta\, P_{l'}^{m_l}[\cos\theta] = \frac{2}{2l+1}\frac{(l+m_l)!}{(l-m_l)!}\delta_{ll'}$$

$\therefore$ $$\boxed{\langle P_l^m(x) | P_{l'}^m(x)\rangle = \frac{2}{2l+1}\frac{(l+m_l)!}{(l-m_l)!}\delta_{ll'}}$$

[ii] SPHERICAL HARMONICS:

$$\boxed{Y_l^{m_l}(\theta,\phi) = (-1)^{m_l}\left[\frac{(2l+1)(l-m_l)!}{4\pi(l+m_l)!}\right]^{\frac{1}{2}} P_l^{m}[\cos\theta]\, e^{i m_l \phi}}$$

and so: $\psi(r,\theta,\phi) = R(r)Y(\theta,\phi) \equiv R(r)Y_l^{m_l}(\theta,\phi)$

$= R(r)(-1)^{m_l}\left[\frac{(2l+1)(l-m_l)!}{4\pi(l+m_l)!}\right]^{\frac{1}{2}} P_l^{M}[\cos\theta]\, e^{i m_l\phi}$

$\therefore$ $$\boxed{\psi(r,\theta,\phi) = R(r)(-1)^{m_l}\left[\frac{(2l+1)}{4\pi}\frac{(l-m_l)!}{(l+m_l)!}\right]^{\frac{1}{2}} P_l^{M}[\cos\theta]\, e^{i m_l\phi}}$$

2) $Y_l^{-m_l} = (-1)^m Y_l^{m_l *}(\theta,\phi)$.

$\therefore$ $\int_0^{2\pi}\int_0^{\pi} Y_l^{m_l *}(\theta,\phi)\sin\theta\, Y_l^{m_l}(\theta,\phi) = \delta_{ll}\delta_{m_l l}.$ $\therefore$ $\boxed{\langle Y_l^{m_l} | Y_l^{m_l}\rangle = \delta_{ll}\delta_{m_l}}$

3) Common ones:

$Y_0^0 = \frac{1}{\sqrt{4\pi}}$ $\quad Y_l^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$ $\quad Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\theta}$ $\quad Y_2^0 = \sqrt{\frac{5}{16\pi}}(3\cos^2\theta - 1)$

PROBLEMS 13

(1) Solve the harmonic oscillator of a 3D Schrödinger!

ANS: Let $r^2 = x^2 + y^2 + z^2$ and $\psi(r) = R(r)$

Then $\boxed{-\frac{\hbar^2}{2m}\nabla^2 R - \frac{1}{2}m\omega^2 r^2 = E}$ or $\boxed{\frac{\hbar^2}{2m}\nabla^2 R + \frac{1}{2}m\omega^2 r^2 - E = 0}$

1) Recall: $\nabla^2 R \equiv \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{d^2R}{dr^2}\frac{r^2}{r^2} + \frac{1}{r^2}\frac{dr^2}{dr}\frac{dR}{dr} = \frac{d^2R}{dr^2} + \frac{2r}{r^2}\frac{dR}{dr}$

$$= \frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}$$

2) So: $\frac{\hbar^2}{2m}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right) + \frac{1}{2}m\omega^2 r^2 R - ER = 0$

$$\frac{2m}{\hbar^2}\frac{\hbar^2}{2m}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right) + \frac{1}{\cancel{2}}\frac{m\omega^2 r^2 \cancel{2}m}{\hbar^2}R - \frac{2mE}{\hbar^2}R = 0$$

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \frac{m^2\omega^2 r^2}{\hbar^2}R - \frac{2mE}{\hbar^2}R = 0$$

Let $\frac{m\omega}{\hbar} = \alpha$ or $\frac{m^2\omega^2}{\hbar^2} = \alpha^2$

$\frac{2mE}{\hbar^2} = \varepsilon$

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \alpha^2 r^2 R - \varepsilon R = 0$$

$$\boxed{\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left[\alpha r^2 - \varepsilon\right]R = 0}$$

GAUSSIAN APPROXIMATION

as $r \to \infty$,

$\frac{d^2R}{dr^2} + [\alpha r^2 - \varepsilon]R \approx 0$

GAUSSIAN EQ.

3.) Let $R(r) \approx e^{-\frac{\alpha r^2}{2}}$ then $R(r) = e^{-\frac{\alpha r^2}{2}}u(r)$

so $\frac{dR}{dr} = -\alpha r u(r) e^{-\frac{\alpha r^2}{2}} + \frac{du}{dr}e^{-\frac{\alpha r^2}{2}} = \left(-\alpha r u(r) + \frac{du}{dr}\right)e^{-\frac{\alpha r^2}{2}}$

$$\frac{d^2R}{dr^2} = \left(-\alpha u(r) + \alpha r\frac{du}{dr} + \frac{d^2u}{dr^2}\right)e^{-\frac{\alpha r^2}{2}} + -\alpha r\left(-\alpha r u + \frac{du}{dr}\right)e^{-\frac{\alpha r^2}{2}}$$

$$= \left(-\alpha u + \alpha r\frac{du}{dr} - \alpha^2 r^2 u + \frac{du}{dr}\alpha r + \frac{d^2u}{dr^2}\right)e^{-\frac{\alpha r^2}{2}}$$

$$= \left(\frac{d^2u}{dr^2} + 2\alpha r\frac{du}{dr} + (\alpha^2 r^2 - \alpha)u\right)e^{-\frac{\alpha r^2}{2}}$$

So: $\boxed{\frac{d^2u}{dr^2} + \left(\frac{2}{r} - 2\alpha r\right)\frac{du}{dr} + \left(\varepsilon - 3\alpha - \alpha^2 r^2\right)u = 0}$

4) Finally: Let $\sum_{n=0}^{?} a_n r^n = u(r)$

Then: $\sum_n^{?}(n-1)n\, a_n r^{n-2} + \left(\frac{2}{r} - 2\alpha r\right)\sum_n^{?} n a_n r^{n-1} + \left(\varepsilon - 3\alpha - \alpha^2 r^2\right)\sum_n^{?} a_n r^n = 0$

$$\sum_n^{?}(n-1)(n+1)a_n r^n + \left(\frac{2}{r} - 2\alpha r\right)\sum_n^{?}(n+1)a_{n+1}r^n + \left(\varepsilon - 3\alpha - \alpha^2 r^2\right)\sum_n^{?} a_n r^n$$

$$a_{n+2}(n-1)(n+1) + \left(\frac{2}{r} - 2\alpha r\right)(n+1)a_{n+1} + \left(\varepsilon - 3\alpha - \alpha^2 r^2\right)a_n$$

... (skipping algebra)

$$\boxed{a_{n+2} = \frac{2\alpha(n+1) - \varepsilon + 3\alpha}{(n+2)(n+3)}a_n}$$

v.) The series must terminate at $n_x = N$:

$n_x + 2 = 0 \Rightarrow \varepsilon = 2\alpha(N+1) + 2\alpha \Rightarrow \boxed{\varepsilon = 2\alpha(N+1)}$

where $\varepsilon = \frac{2mE}{\hbar^2}$, $\alpha = \frac{m\omega}{\hbar}$

so, $\frac{2mE}{\hbar^2} = 2 \cdot \frac{m\omega}{\hbar}(N+2) \Rightarrow \boxed{E = \hbar\omega(N+2)}$

so, $\boxed{E_n = \hbar\omega\left(n + \frac{1}{2}\right)}$

and since ~~$n = n_x + n_y + n_z$ then~~ :

$\boxed{E_n = \hbar\omega\left(n_x + n_y + n_z + \frac{3}{2}\right)}$

$\tan\theta = \frac{y}{x}$

$\theta = \tan^{-1} y/x$

(3) ~~Differentiate~~ 1st Derivative of the cylindrical coordinate (r, θ, ϕ).

Ans : Recall that :

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$

$\varphi(x(r,\theta,\phi)) = r\sin\theta\cos\phi$ =.

$\boxed{\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}}$

so, $\nabla_\theta \varphi(x) = \frac{\partial\varphi}{\partial r} + \frac{\partial\varphi}{\partial\theta} + \frac{\partial\varphi}{\partial\phi} = \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\theta} + \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial\phi}$

$= \sin\theta\cos\phi \left.\frac{\partial\varphi}{\partial x}\right|_r + r\cos\theta\cos\phi \left.\frac{\partial\varphi}{\partial x}\right|_\theta - r\sin\theta\sin\phi \left.\frac{\partial\varphi}{\partial x}\right|_\phi$

since $r =$

$\theta = \cos^{-1}\frac{z}{r}$

$r\cos\theta = \frac{z}{r}$

$\nabla_x = \sin\theta\cos\phi\frac{\partial}{\partial r} + r\cos\theta\cos\phi\frac{\partial}{\partial\theta} - r\sin\theta\sin\phi\frac{\partial}{\partial\phi}$

$r\cos\theta = z$

$r\cos\theta = r\cos\theta$

(i) Non-Degenerate Theory

Formulation from Hamiltonian

- Let $\hat{H}$ be the total Hamiltonian. Then

$$(14.1)\quad \boxed{\hat{H} = H_0 + \hat{H}_p} \quad w/\ \hat{H}_p = \text{perturbed state},\ H_0 = \text{initial state}$$

- The exact energies is:

$$(14.2)\quad \boxed{\hat{H}_0 |\psi_n^{(1)}\rangle = E_n^{(1)} |\psi_n^{(0)}\rangle} \quad w/\ E^{(0)} = 0^{th} \text{ order unperturbed energies. } E_n^{(1)} = \text{unique eigenstate. } |\psi_n\rangle = 2^{nd} \text{ piece of } \hat{H}_p.$$

- We modify it w/:

$$(14.3)\quad \boxed{H = \hat{H}_0 + \lambda \hat{H}_p} \text{ and } \lambda \in [0,1]$$

- To relate the eigenstates, we sum over power series in terms of λ^k so –

$$(14.4)\quad \hat{H}|\psi_n\rangle = E_n|\psi_n\rangle$$

and:
$$E_n = E_n^{(0)} + \lambda E_n^{(1)} + \lambda^2 E_n^{(2)} + \cdots \doteq \sum_k \lambda^k E_n^{(k)}.$$
$$|\psi_n\rangle = |\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle + \lambda^2|\psi_n^{(2)}\rangle + \cdots \doteq \sum_k \lambda^k |\psi_n^{(k)}\rangle.$$

$\therefore$ Thus:
$$\boxed{\begin{aligned} E_n &\sim \textstyle\sum_k \lambda^k E_n^{(k)} \quad (14.5)\\ |\psi_n\rangle &\sim \textstyle\sum_k \lambda^k |\psi_n^{(k)}\rangle \quad (14.6) \end{aligned}}$$

N^{th} order

- Continuing in substituting (14.5):

1.) $\hat{H}|\psi_n\rangle = \hat{H}\sum_k \lambda^k |\psi_n^{(k)}\rangle = (H_0 + \lambda H_p)\sum_k \lambda^k |\psi_n^{(k)}\rangle.$

$E_n|\psi_n\rangle = \left(\sum_k \lambda^k E_n^{(k)}\right)\left(\sum_k \lambda^k |\psi_n^{(k)}\rangle\right)$

2.) So;
$$(\hat{H}_0 + \lambda \hat{H}_p)(|\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle + \cdots) = (E_n^{(0)} + \lambda E_n^{(1)} + \cdots)(|\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle)$$
$$\Rightarrow \hat{H}_0(|\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle) + \hat{H}_p(\lambda|\psi_n^{(0)}\rangle + \lambda^2|\psi_n^{(1)}\rangle + \cdots) = (E_n^{(0)} + \lambda E_n^{(1)} + \cdots)(|\psi_n^{(0)}\rangle + \lambda|\psi_n^{(1)}\rangle$$
$$\Rightarrow \hat{H}_0|\psi_n^{(0)}\rangle + \lambda(\hat{H}_0|\psi_n^{(1)}\rangle + \hat{H}_p|\psi_n^{(0)}\rangle) + \lambda^2(\hat{H}_0|\psi_n^{(2)}\rangle + \hat{H}_p|\psi_n^{(1)}\rangle)$$
$$= E_n^{(0)}|\psi_n^{(0)}\rangle + \lambda(E_n^{(0)}|\psi_n^{(1)}\rangle + E_n^{(1)}|\psi_n^{(0)}\rangle) + \lambda^2(E_n^{(0)}|\psi_n^{(2)}\rangle + E_n^{(1)}|\psi_n^{(1)}\rangle + E_n^{(2)}|\psi_n^{(0)}\rangle)$$

or $\Rightarrow$
$$\sum_k \lambda^k H_0 + \sum_k \lambda^{k+1} H_p = \sum_k \lambda^{2k} \lambda |\psi_n^{(k)}\rangle \quad w/\ \Lambda(k) = \sum_k \lambda^k.$$
$$\equiv \Lambda(k) H_0 + \Lambda(k+1) H_p = \Lambda(2k) E_n|\psi_n\rangle.$$

So:
$$\boxed{\Lambda_k H_0 + \Lambda_{k+1} H_p = \Lambda_{2k} E_n|\psi_n\rangle}$$

- 1^{st} order term yields:

$$\boxed{\hat{H}_0|\psi_n^{(1)}\rangle + \hat{H}_p|\psi_n^{(0)}\rangle = E_n^{(0)}|\psi_n^{(1)}\rangle + E_n^{(1)}|\psi_n^{(0)}\rangle}. \quad (14.9)$$

- 2^{nd} order term yields:

$$\boxed{\hat{H}_0|\psi_n^{(2)}\rangle + \hat{H}_p|\psi_n^{(1)}\rangle = E_n^{(0)}|\psi_n^{(2)}\rangle + E_n^{(1)}|\psi_n^{(1)}\rangle + E_n^{(2)}|\psi_n^{(0)}\rangle}. \quad (14.10)$$

[1] 1^{st} Order Corrections

- Suppose we project $\langle\psi_n^{(0)}|$ into (14.9) s.t.

$$\langle\psi_n^{(0)}|H_0|\psi_n^{(1)}\rangle + \langle\psi_n^{(0)}|\hat{H}_p|\psi_n^{(1)}\rangle = \langle\psi_n^{(0)}|E_n^{(0)}|\psi_n^{(1)}\rangle + \langle\psi_n^{(0)}|E_n^{(1)}|\psi_n^{(0)}\rangle$$

$\Rightarrow E_n^{(0)}\langle \psi_n^{(0)}|\psi_n^{(1)}\rangle + \langle \psi_n^{(0)}|\hat H_p|\psi_n^{(0)}\rangle = E_n^{(0)}\langle\psi_n^{(0)}|\psi_n^{(1)}\rangle + E_n^{(1)}$

$\Rightarrow E_n^{(0)}\langle \psi_n^{(0)}|\psi_n^{(1)}\rangle - E_n^{(0)}\langle\psi_n|\psi_n^{(1)}\rangle + \langle\psi_n^{(0)}|\hat H_p|\psi_n^{(0)}\rangle = E_n^{(1)}$

$\Rightarrow 0 + \langle\psi_n^{(0)}|\hat H_p|\psi_n^{(0)}\rangle = E_n^{(1)}$

Thus: $\boxed{E_n^{(1)} = \langle\psi_n^{(0)}|\hat H_p|\psi_n^{(0)}\rangle}$

■ THE CORRECTION

- Let $E_n = E_n^{(0)} + E_n^{(1)}$ be the total Energy s.t. we will employ this with the fact that: $\boxed{|\psi_n^{(1)}\rangle = \sum_m' C_{mn}|\psi_{mn}\rangle}$ (14.13)

- Our goal is to find Eigenstates $|\psi_n\rangle$ of Full Hamiltonian $\hat H$ ew.
  1.) States are given ($\lambda = 1$) and contains $|\psi_n^{(0)}\rangle$. So No Inclusion of $|\psi_n^{(0)}\rangle$.
  2.) By assuming that $\langle\psi_n^{(0)}|\psi_n^{(1)}\rangle = 0$, we can find the expression

* Thus it follows:

1.) $|\psi_n^{(1)}\rangle = \sum_m' C_{mn}^{(1)}|\psi_m^{(0)}\rangle$ 2.) $C_{mn}^{(1)} = \langle\psi_m^{(0)}|\psi_n^{(1)}\rangle$.

- Applying the Bra of $\psi_m^{(0)}$ we can derive:

$\langle\psi_m^{(0)}|\hat H_0|\psi_n^{(1)}\rangle + \langle\psi_m^{(0)}|\hat H_p|\psi_n^{(0)}\rangle = \langle\psi_m^{(0)}|E_n^{(0)}|\psi_n^{(1)}\rangle + \langle\psi_m^{(0)}|E_n^{(1)}|\psi_n^{(0)}\rangle$

$\Rightarrow E_m^{(0)}\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle + \langle\psi_m^{(0)}|\hat H_p|\psi_n^{(0)}\rangle = E_n^{(0)}\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle$

$\Rightarrow E_m^{(0)} - E_n^{(0)} + \dfrac{\langle\psi_m^{(0)}|H_p|\psi_n^{(0)}\rangle}{\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle} = 0$

$\Rightarrow E_n^{(0)} - E_m^{(0)} = \dfrac{\langle\psi_m^{(0)}|H_p|\psi_n^{(0)}\rangle}{\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle} \Rightarrow \boxed{C_{mn} = \dfrac{\langle\psi_m^{(0)}|H_p|\psi_n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}}}$ (14.18)

[ii] 2nd ORDER CORRECTION

■ Project: $\langle\psi_n^{(0)}|$ to (14.10) s.t..

- Same as before, we project

$\Rightarrow E_n^{(0)}\langle\psi_n^{(0)}|\psi_n^{(2)}\rangle + \langle\psi_n^{(0)}|\hat H_p|\psi_n^{(1)}\rangle = E_n^{(0)}\langle\psi_n^{(0)}|\psi_n^{(2)}\rangle + E_n\underbrace{\langle\psi_n|\psi_n^{(1)}\rangle}_{0} + E_n^{(2)}$

$\Rightarrow E_n^{(0)}\cancel{\langle\psi_n^{(0)}|\psi_n^{(2)}\rangle} + \langle\psi_n^{(0)}|\hat H_p|\psi_n^{(1)}\rangle = \cancel{E_n^{(0)}\langle\psi_n^{(0)}|\psi_n^{(2)}\rangle} + E_n^{(2)}$

$\boxed{E_n^{(2)} = \langle\psi_n^{(0)}|\hat H_p|\psi_n^{(1)}\rangle}$ (14.27)

- Using (14.18) s.t. $(E_n^{(0)} - E_m^{(0)})C_{mn}$ is substitution on RHS of (14.27).

$E_n^{(2)} = \langle\psi_n^{(0)}|\hat H_p|\psi_n^{(1)}\rangle = (E_n^{(0)} - E_m^{(0)})C_{mn} = (E_n^{(0)} - E_m^{(0)})\langle\psi_m^{(0)}|\psi_n^{(1)}\rangle$

$= \sum_{m\neq n}' \dfrac{|\langle\psi_m^{(0)}|\hat H_p|\psi_n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$

∴ So:

$E_n^{(1)} = \dfrac{\langle\psi_n^{(0)}|\hat H_p|\psi_n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}}$ # FIRST-ORDER.

$E_m^{(1)} = \dfrac{|\langle\psi_m^{(0)}|\hat H_p|\psi_n^{(0)}\rangle|^2}{E_n^{(0)} - E_m^{(0)}}$ # 2nd ORDER

TIME INDEPENDENT PERTURBATION
– DEGENERATE THEORY & RELATIVISTIC CORRECTION

(12.) **DEGENERATE THEORY:**

- Philosophy: $\begin{cases} \text{Degenerate} = \text{split in energy levels} \\ \text{Non degenerate} = \text{no split in energy} \end{cases}$

- MATH: • We assume that: $E_n^{(0)}$ is n-fold degenerate and have $|\psi_{nij}^{(0)}\rangle$ corresp. eigenvectors w/ $j \in \mathbb{Z}^+$. The particular linear combination:

$$\boxed{\hat{H}_0 \sum_{i=1}^{n} b_{nij} |\psi_{nij}^{(0)}\rangle = E_n^{(0)} \sum_{i=1}^{n} b_{nij} |\psi_{nij}^{(0)}\rangle} \quad (14.3m)$$

• observe that: $|\psi_n^{(0)}\rangle \mapsto \sum^{n} b_{nij} |\psi_{nij}^{(0)}\rangle$ · (14.2s)

Kinda like: $f(x) \to \epsilon(x)$ where $\epsilon(x) = f_r(x) \approx$ or $:= b_{nij}|\psi_{nij}^{(0)}\rangle$

• we substitute this to eqs (14.1) in

for terms: $\hat{H}_0|\psi_n^{(1)}\rangle + \hat{H}_p|\psi_n^{(0)}\rangle = E_n^{(0)}|\psi_n^{(1)}\rangle + E_n^{(1)}|\psi_n^{(0)}\rangle$

$$\Rightarrow \langle\psi_{nii}^{0}|H_0|\psi_n^{1}\rangle + \langle\psi_{nii}^{0}| H_p \sum_{j=1}^{n} b_{nij} |\psi_{nij}\rangle$$
$$= \langle\psi_{nii}^{0}| E_n^{0} |\psi_n^{(1)}\rangle + \langle\psi_{nii}^{(0)}| E_n^{(1)} \sum^{n} b_{nij} |\psi_{nij}^{(0)}\rangle$$

• The matrix elements of $\hat{H}_p$ in the n-dimensional subspace of the degenerate states to be: $\boxed{H_{pij} = \langle\psi_{nii}^{0}|\hat{H}_p|\psi_{nij}^{0}\rangle}$ and $\boxed{\sum_{i=1}^{n} b_{nij} \langle\psi_{nii}^{0}|\hat{H}_p|\psi_{nij}^{0}\rangle = E_n^{1} b_{ni}}$

• using this we can write:

$$\boxed{\sum_{i=1}^{n} H_{pij} b_{nij} = E_n^{(1)} b_{nni}}$$

- SIMPLIFIED:

• Basically: $H_0 + H_p = E_0 + E_1 \Rightarrow H_0|\psi\rangle + H_p|\psi\rangle = E_0|\psi\rangle + E_1|\psi\rangle$

$\Rightarrow H_0 B_0|\psi_0\rangle + H_p \underbrace{b_k|\psi_k\rangle}_{k>0} = E_0 b_0|\psi_0\rangle + E_1 b_k|\psi_k\rangle_{k>0}$

• where: $\hat{H}_p\langle\psi_i|\psi_j\rangle = H_{pij}$ then: $H_0 + H_{pij} b_k = E_0 + E_1 b_k|\psi_k\rangle$

• or just: if $H_0, E_0 = 0 \Rightarrow \boxed{H_{pij} b_k = E_1 b_k |\psi_k\rangle}$

$\Rightarrow H_0 = E_n$

$\Rightarrow H_0 + H_p = E_n$

$\Rightarrow H_0 + \sum b H_p = \sum b E_n$; $\langle\psi_i|b|\psi_j\rangle H_p = \langle\psi_i|b|\psi_j\rangle E$

$\Leftrightarrow \langle\psi_i|H_0|\psi_j\rangle + \langle\psi_i|\hat{H}_p b|\psi_j\rangle = \langle\psi_i|bE|\psi_j\rangle$

$\Rightarrow \sum b_n \langle\psi_i|H_p|\psi_j\rangle = E_n^{(1)} b_{nij}$

$= \sum^{n} H_{pij} b_{ij}^{0} = E_n b_{ij}$

$\Rightarrow \hat{H}_p b_n = E_n b_n$

(13.) **RELATIVISTIC CORRECTION – HYDROGEN**

- Fine structure of Hydrogen:

• The Einstein energy: $\underline{(K + mc^2)^2 = m^2c^4 - p^2c^2 + mc^2} = mc^2\left(1 + \frac{p^2c^2}{m^2c^2}\right)$ and

then: $\boxed{K = mc^2\left(1 + \frac{p^2c^2}{mc^2}\right) - mc^2}$

• This is basically a binomial expansion of: $\text{Binom}\left(\frac{p^2}{2m}, \frac{1}{c^2}\right) \approx H_R^K = \frac{p^2}{2m} + \frac{p^4}{8m^3c^2} + \cdots$

• The kinetic energy is:

$$\boxed{\hat{H}_R = -\frac{\hat{p}^4}{8m^3c^2} = -\frac{1}{2mc^2}\left(\frac{\hat{p}^2}{2m}\right)^2}$$

• Some caveats:

>> Hydrogen mass is literally: $\boxed{m = m_e}$

since 1

>> so we'll use a $\boxed{\text{Non degenerate}}$ theory w/ $|n, l, m_e\rangle$ and fix n.

>> computing hydrogen requires computing $\langle p^2 nlm|\cdot|nlm\rangle$.

>> Avoiding degeneracy matrices requires we use $[L_z, L^2]$ as roots

s.t. $[p^2, \hat{L}] = [p_x^2, [p_x x]] = [p_x^2, i\hbar]$ $\quad p_x = i\hbar\nabla$

$= [-p_{xx}\hbar, i\hbar]\phi = -\nabla^2\phi\,\hbar^2 i + \nabla^2\phi\,\hbar^2 i$ $\quad p_x^2 = -\hbar\nabla^2$

$= \boxed{0}\ V$

• Defining: >> $H_0 = \frac{p^2}{2m} + V(r) \Rightarrow \frac{p^2}{2m} = V(r) - H_0$ >> $H_R = -\frac{1}{2mc^2}(H_0 - V)(H - V)$

>> on: $\boxed{\frac{p^2}{2m} = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} - H_0}$ ★ do: $H_R = -\frac{1}{2mc^2}(H_0 - V)^2$

$= -\frac{1}{2mc^2}(H_0^2 - 2H_0V + V^2)$

• The perturbed state:

$$★\ H_R = -\frac{1}{2mc^2}\left(H_0^2 - 2H_0V + V^2\right) = -\frac{1}{2mc^2}\left[\left(\frac{p^4}{4m} + \frac{p^2}{2m}\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} + \frac{e^4}{16\pi^2\varepsilon_0^2 r^2}\right) + \left(2\frac{p^2}{2m}\frac{e^2}{4\pi\varepsilon_0}\frac{1}{r} + \frac{e^4}{16\pi^2\varepsilon_0^2}\frac{1}{r^2}\right) + \left(\frac{e^4}{16\pi^2\varepsilon_0^2}\frac{1}{r^2}\right)\right]$$

• This energy is expected as:

$$>>\ E_R^{(1)} = \langle n, l, m_e| H_R |n l m_e\rangle$$

$$= \left\langle n l m_e\left| -\frac{1}{2mc^2}\left(H_0^2 - 2H_0V + V^2\right)\right| n l m_e\right\rangle = -\frac{1}{2mc^2}\left[\langle nlm|H_0^2|nlm\rangle - \langle nlm_e|2H_0V|nlm_e\rangle + \langle nlm_e|V^2|nlm_e\rangle\right]$$

$$= -\frac{1}{2mc^2}\left[E^{0\,2} - 2E^0\frac{e^2}{4\pi\varepsilon_0}\langle nlm_e|\frac{1}{r}|nlm_e\rangle + \left(\frac{e^2}{4\pi\varepsilon_0}\right)^2\langle nlm_e|\frac{1}{r^2}|nlm_e\rangle\right]$$

$$★\ \boxed{\left\langle\frac{1}{r}\right\rangle = \frac{1}{n^2 a_0}} \quad ★\ \boxed{\left\langle\frac{1}{r^2}\right\rangle = \frac{1}{n^3 a_0^2(l + \frac{1}{2})}}$$

$$>>\ E_R^{(1)} = -\frac{1}{2mc^2}\left(-2E^0\frac{e^2}{4\pi\varepsilon_0}\left(\frac{1}{n^2 a_0}\right) + (E^0)^2 + \frac{e^4}{16\pi^2\varepsilon_0}\left(\frac{1}{n^3 a_0^2(l+\frac{1}{2})}\right)\right)$$

$$= \frac{(E_n^{(0)})^2}{2mc^2}\left[\frac{4n}{(l+\frac{1}{2})} - 3\right]$$

• The fine structure and unperturbed energy:

$$>>\ \alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{1}{137}$$

• Our final equation:

$$\boxed{E_R^{(1)} = -\frac{mc^2\alpha^4}{8n^4}\left(\frac{4n}{(l+\frac{1}{2})} - 3\right)}$$

$$\Rightarrow E_n^0 = \frac{m_e c^2\alpha^4}{8n^4}\left[\frac{4n}{(l+\frac{1}{2})} - 3\right]$$

1st PERTURBATION of EINSTEIN'S ENERGY

TIME INDEPENDENT PERTURBATION EXAMPLES

(14.Ex)

(1) 1^{st} ORDER EXAMPLE - 14.1 :

- SCENARIO:

Suppose an electric field $\vec{E} = E\hat{x}$ is applied to a particle w/ charge q in an infinite potential well:

$$\boxed{V(x) = \begin{cases} -qEx & 0 < x < L \\ \infty & \text{elsewhere} \end{cases}}$$

(?) Determine the Allowed Energies!

- ANSWER:
  - The unperturbed states is basically:

$$\boxed{E_n^{(0)} = \frac{n^2 \pi^2 \hbar^2}{2mL^2}} \ \forall n \in \mathbb{Z}^+ \qquad (14.21)$$

  - The wave function on the other hand is:

$$\boxed{\psi_n^{(0)}(x) = \sqrt{\frac{2}{L}} \cdot \sin\left(\frac{n\pi}{L}x\right)} \text{ and (0) elsewhere } (14.22)$$

$\forall \psi_n \in 0 < x < L$

  - The perturbed Hamiltonian is:

$$\boxed{H_p \equiv V(x) = -qE\hat{x}} \ (14.23)$$

$\therefore$ The 1^{st} order correction is:

$$E_n^{(1)} = \langle \psi_n^{(0)} | \hat{H}_p | \psi_n^{(0)} \rangle = \frac{2}{L}\int_0^L \sin^2\left(\frac{n\pi}{L}x\right) \cdot -qEx \, dx.$$

$$= \frac{2}{L}\int_0^L \left(\frac{1-\cos\left(\frac{2n\pi x}{L}\right)}{2}\right)(-qEx)\, dx = \frac{2}{L}\int_0^L -\frac{1}{2}qEx + \frac{1}{2}\cos\left(\frac{2n\pi}{L}\right) qEx \ldots$$

$$= \frac{-qEL}{2} \quad \text{so} \quad \boxed{E_n = -\frac{qEL}{2}}$$

(2) 2^{nd} ORDER EXAMPLE :

- SCENARIO :

A uniform electric field of $\tilde{E} = \tilde{E}\hat{x}$ is applied to a particle w/ charge of q is situated in a harmonic oscillator potential. Determine the energies of the particle.

$n = |c-1|^2$

- ANSWER: We apply: $\boxed{V(x) = \frac{1}{2}m\omega^2 x^2 - qEx}$ $\Leftarrow$ NET POTENTIAL OF HARMONIC OSCILLATOR and CHARGE FORCE
  - The perturbation shifts: $E_n^0(x) \rightarrow E_n^{0'}(x)$; $-\frac{q^2E^2}{2m\omega^2} \rightarrow -\frac{qE^2}{m\omega^3}$.
  - The Perturbation's solution for:

$$\boxed{\hat{H}_p(x) = -qE\hat{x}} \text{ w/ the eigenstates: } \boxed{E_n^0 = \hbar\omega\left(n + \frac{1}{2}\right)}$$

  - The matrix element:

$$\langle \psi_j^* | \hat{H}_p | \psi_n^0 \rangle = -qE\langle j|\hat{x}|n\rangle = -qE\sqrt{\frac{\hbar}{2m\omega}} \cdot \langle j|\hat{a} + \hat{a}^\dagger|n\rangle$$

$$= -qE\sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\langle j|n-1\rangle + \sqrt{n+1}\langle j|n+1\rangle\right)$$

$$= -qE\sqrt{\frac{\hbar}{2m\omega}}\left[\sqrt{n}\cdot\delta_{j,n-1} + \sqrt{n+1}\cdot\delta_{j,n+1}\right]$$

$\langle j|\hat{a} + a^\dagger|n\rangle$

$\langle j|n\rangle + a^\dagger\langle j|n\rangle$

$(\hat{a}|n\rangle) + (a^\dagger|n\rangle)\langle j|$

$(c_-|n\rangle) + (c_+|n+1\rangle)\langle j|$

$\sqrt{n}\langle j|n-1\rangle + \sqrt{n+1}\langle j|n-1\rangle$

- The 1st order correction:

$$\boxed{E_n^{0} = \langle \psi_n^{0} | \hat{H}_p | \psi_n^{0} \rangle = 0}$$

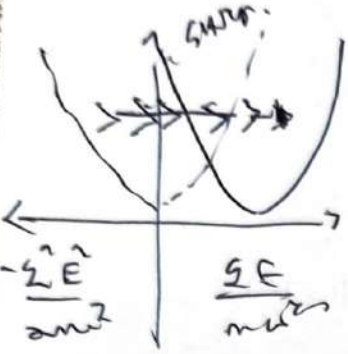
- So up to 1st order of correction:

$$\begin{cases} E_n^{(0)} = 0 \\ E_n^{(1)} = 0 \\ E_n^{(2)} = ? \end{cases}$$

- This means that:

$$E_n^{(2)} = \sum_{j \neq n} \frac{|\langle \psi_j^{(0)} | \hat{H}_p | \psi_n^{(0)} \rangle|^2}{E_n^0 - E_j^0} = \sum_{j \neq n} \frac{1}{E_n^0 - E_j^0} \left| qE\sqrt{\frac{\hbar}{2m\omega}} \langle j | n-1 \rangle \sqrt{n} + \langle j | n+1 \rangle \sqrt{n+1} \right|$$

$$= \frac{q^2E^2\hbar}{2m\omega} \sum_{j \neq n} \frac{\left(\langle j | n-1 \rangle \sqrt{n} + \langle j | n+1 \rangle \sqrt{n+1} \right)^2}{E_n^0 - E_j^0} = \frac{q^2E\hbar}{2m\omega} \sum_{j \neq n} \left[\frac{(\sqrt{n}\,\delta_{j,n-1} + \sqrt{n+1}\,\delta_{j,n+1})}{E_n^0 - E_j^0} \right]$$



$$= \frac{q^2E\hbar}{2m\omega} \sum_{j \neq n} \frac{1}{E_n^0 - E_j^0} \sqrt{n}\,\delta_{j,n-1}\,\delta_{j,n-1}\sqrt{n} + \ldots + \sqrt{n+1}\,\delta_{j,n+1}\,\delta_{j,n+1}\sqrt{n+1}$$

*we know this

$$= \frac{q^2E\hbar^2}{2m\omega} \sum_{j \neq n} \frac{1}{E_n^0 - E_j^0} \left((n) + (n+1) \right) = \frac{q^2E\hbar^3}{2m\omega} \frac{n}{E_n^0 - E_{n-1}^0} + \frac{n-1}{E_n^0 - E_{n+1}^0}$$

$$= \frac{q^2E^2\hbar}{2m\omega} \left[\frac{n}{\hbar\omega} - \frac{n+1}{\hbar\omega} \right] = -\frac{qE^2}{2m\omega^2}$$

$$\boxed{E_n^{(2)} = -\frac{qE^2}{2m\omega^2}}$$

2nd order correction is the minimum shift!

PROBLEM 6

(13.) Find the state $|-n\rangle$ which corresps ~~to the~~

<u>Ans</u>: If: $\boxed{|+n\rangle = \cos\left(\frac{\theta}{2}\right)|+z\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|-z\rangle}$

Then: an analogy like $ds = dx + dy :=$ $dx = \sin\frac{\theta}{2} + \cos\left(\frac{\theta}{2}\right)$
$dy = -\cos\frac{\theta}{2} + \sin\left(\frac{\theta}{2}\right)$

It should be:

$$\boxed{|-n\rangle = -\sin\left(\frac{\theta}{2}\right)|+z\rangle + e^{i\phi}\cos\left(\frac{\theta}{2}\right)|-z\rangle}$$

(14.) Proof that $|-n\rangle$ is orthogonal to $+n\rangle$

<u>Ans</u>: 1.) $\langle -n| = |-n\rangle^{\dagger} = -\sin\left(\frac{\theta}{2}\right)|+z\rangle^{\dagger} + (e^{i\phi})^{\dagger}\cos\left(\frac{\theta}{2}\right)|-z\rangle^{\dagger}$

$= -\sin\left(\frac{\theta}{2}\right)\langle +z| + e^{-i\phi}\cos\left(\frac{\theta}{2}\right)\langle -z|.$

So: 2.) $\langle -n|+n\rangle = \left[-\sin\left(\frac{\theta}{2}\right)\langle +z| + e^{-i\phi}\cos\left(\frac{\theta}{2}\right)\langle -z|\right]\left[\cos\left(\frac{\theta}{2}\right)|+z\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|-z\rangle\right]$

$= -\sin\left(\frac{\theta}{2}\right)\underbrace{\langle +z|+z\rangle}_{1}\cos\left(\frac{\theta}{2}\right) + \underbrace{e^{-i\phi}e^{i\phi}}_{e^{-i\phi+i\phi}=e^{0}=1}\cos\left(\frac{\theta}{2}\right)\underbrace{\langle -z|-z\rangle}_{1}\sin\left(\frac{\theta}{2}\right)$

$= -\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) = \cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$

$= \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = \boxed{0}$ ✓ <u>QED</u>

3.)

ANGULAR MOMENTUM & COMMUTATION BASICS (7.1) (7.2) (7.3)

① COMMUTING OBSERVABLES:

$$\boxed{[\hat{A},\hat{B}]=0}$$

This means: $\hat{A}\hat{B}-\hat{B}\hat{A} = \hat{A}^2 - A^2 = 0 = \hat{B}^2-\hat{B}^2 = \hat{A}\hat{B}-\hat{B}\hat{A}$ $\forall \begin{vmatrix}\hat{A}_x & \hat{A}_y \\ \hat{B}_x & \hat{B}_y\end{vmatrix}$

1) For Commutation $[\hat{A},\hat{B}]=0$, There exists a Complete set of States $|\lambda\rangle$ That are Simultaneously eigenstates of Both $\hat{A}$, $\hat{B}$ st:

$$\boxed{\begin{aligned}\hat{A}|\lambda\rangle &= a_\lambda|\lambda\rangle\\ \hat{B}|\lambda\rangle &= b_\lambda|\lambda\rangle\end{aligned}}\ \forall [\hat{A},\hat{B}]=0\ \exists|\lambda\rangle \text{ s.t. } \left\{\begin{aligned}\hat{A}|\lambda\rangle &= a_\lambda|\lambda\rangle\\ \hat{B}|\lambda\rangle &= b_\lambda|\lambda\rangle\end{aligned}\right]$$ • COMPLETENESS OF EIGENSTATES

2) Let $|\lambda\rangle = |a,b\rangle$. Thus:

$\hat{A}|\lambda\rangle = \hat{A}|a,b\rangle \rightarrow \langle\lambda|\hat{A}|\lambda\rangle = \langle a,b|\hat{A}|a,b\rangle$

$\hat{A}=\hat{A}$ and $a=\hat{A}$ so: $A|\lambda\rangle = a|a,b\rangle$

similarly $\hat{B}|\lambda\rangle = \hat{B}|a,b\rangle \rightarrow \langle\lambda|\hat{B}|\lambda\rangle = \langle a,b|\hat{B}|a,b\rangle$

$\langle a,b|\hat{B}\hat{B}^\dagger|a,b\rangle = [\ ][\ ]\ \hat{B}\hat{B}^\dagger = 1.$

thus $\boxed{\begin{aligned}\hat{A}|a,b\rangle &= a|a,b\rangle\\ \hat{B}|a,b\rangle &= b|a,b\rangle\end{aligned}}$

INVERSE:

$\forall [\hat{A},\hat{B}]\neq 0,\ \nexists|\lambda\rangle$ s.t. $\left\{\begin{aligned}\hat{A}|\lambda\rangle &\neq a_\lambda|\lambda\rangle\\ \hat{B}|\lambda\rangle &\neq b_\lambda|\lambda\rangle\end{aligned}\right]$ • INCOMPLETENESS OF EIGENVALUES

② ANGULAR MOMENTUM OPERATORS

[1] TOTAL:

1) a) $\vec{S} = S_x\vec{u}_x + \hat{S}_y\vec{u}_y + \hat{S}_z\vec{u}_z$ → SPIN VECTOR

b) $\boxed{\vec{J} = \vec{S}+\vec{L}}$ → TOTAL ANGULAR MOMENTUM.

c) $\vec{J} = \hat{J}_x\vec{u}_x + \hat{J}_y\vec{u}_y + \hat{J}_z\vec{u}_z$ →

2) $[\hat{J}_x,\hat{J}_y] = i\hbar J_z$

$[\hat{J}_y,J_z] = i\hbar J_x$ • here: $\boxed{[J_i,J_j] = i\hbar J_k}$ w/ $i\neq j\neq k$.

$[\hat{J}_z,J_x] = i\hbar J_y$ • If $\vec{L} = \int(r\times\omega)\,d\vec{r} = 0$

Since $i=j$ Then $[J_i,J_i]=0$.

Then $\vec{J} = S+\vec{L} = S+0 = S$ so $\boxed{J=S}$ iff $L=0$.

3) $\hat{J}_z|+z\rangle = \frac{\hbar}{2}|+z\rangle$

$\hat{J}_z|-z\rangle = -\frac{\hbar}{2}|-z\rangle$

[ii] The J^2 OPERATOR:

1) $\boxed{\hat{J}^2 = J\cdot J = \hat{J}_x^2+\hat{J}_y^2+\hat{J}_z^2}$ w/ $\begin{bmatrix}J_x\\J_y\\J_z\end{bmatrix}\cdot\begin{bmatrix}J_x\\J_y\\J_z\end{bmatrix} = \begin{bmatrix}J_x\begin{bmatrix}J_x\\J_y\\J_z\end{bmatrix}\\ J_y\begin{bmatrix}J_x\\J_y\\J_z\end{bmatrix}\\ J_z\begin{bmatrix}J_x\\J_y\\J_z\end{bmatrix}\end{bmatrix}$

$= \begin{bmatrix}J_x^2+J_xJ_y+J_xJ_z\\ J_xJ_y+J_y^2+J_yJ_z\\ J_xJ_z+J_zJ_y+J_z^2\end{bmatrix} = \begin{bmatrix}J_x^2+0+0\\0+J_y^2+0\\0+0+J_z^2\end{bmatrix}$

$= \begin{bmatrix}J_x^2\\J_y^2\\J_z^2\end{bmatrix} \equiv J_x^2+J_y^2+J_z^2$

2) $[\hat{A}\hat{B},\hat{C}] = \hat{A}\hat{B}(\hat{C}) - \hat{C}(\hat{A}\hat{B})$

$= \hat{A}(\hat{B}\hat{C}) - (\hat{A}\hat{C})\hat{B}$

$= \hat{A}(\hat{B}\hat{C}-\hat{C}\hat{B}) - \hat{B}(\hat{A}\hat{C}-\hat{C}\hat{A})$

$= \hat{A}[\hat{B},\hat{C}] + \hat{B}[\hat{A},\hat{C}]$

thus $\boxed{[\hat{A}\hat{B},\hat{C}] = \hat{A}[\hat{B},\hat{C}]+\hat{B}[\hat{A},\hat{C}]}$

3) $\hat{J}^2, J_z = [(J_x^2+J_y^2+J_z^2), J_z] = [J_x^2,\hat{J}_z] + [J_y^2,J_z] + \cancel{[J_z^2,J_z]}=0.$

$= \hat{J}_x[\hat{J}_x,\hat{J}_z] + [\hat{J}_x,\hat{J}_z]\hat{J}_x + \hat{J}_y[\hat{J}_y,\hat{J}_z] + [\hat{J}_y,\hat{J}_z]\hat{J}_y \ldots$ $=0$

so: $\boxed{[\hat{J}^2,J_z] = [J_x^2,\hat{J}_z] + [J_y^2,J_z] = 0}$

$$\hat{J}^2 = J_x^2 + J_y^2 + J_z^2 = \frac{\hbar^2}{4}\left\{\sigma_x^2 + \sigma_y^2 + \sigma_z^2\right\} = \frac{\hbar^2}{4}\left\{\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix} + \begin{bmatrix}0&-i\\i&0\end{bmatrix}\begin{bmatrix}0&-i\\i&0\end{bmatrix} + \ldots\right\}$$

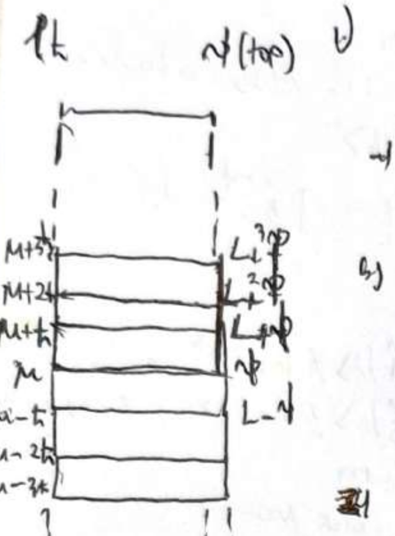
$$= \frac{\hbar^2}{4}\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix} + \begin{bmatrix}1&0\\0&1\end{bmatrix} + \begin{bmatrix}1&0\\0&1\end{bmatrix}\right\} = \frac{3}{4}\hbar^2\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

hence $\boxed{\hat{J}^2 = J_x^2 + J_y^2 + J_z^2 = \frac{3}{4}\hbar^2\begin{bmatrix}1&0\\0&1\end{bmatrix}}$

with $\sigma_x^2 = \begin{bmatrix}0&1\\1&0\end{bmatrix}^2 \quad \sigma_y^2 = \begin{bmatrix}0&-i\\i&0\end{bmatrix}^2 \quad \sigma_z^2 = \begin{bmatrix}1&0\\0&-1\end{bmatrix}^2$.

3) Eigenvalues and Eigenstates

LADDER



- Each neighbours are separated via one unit ħ.
- Eventually we're gonna reach a state for which the z-component exceeds the total.
- hence $\boxed{\exists\ \varphi(top),\ L_+\varphi(top) = 0}$

[i] BASICS:

1) $\boxed{\begin{aligned}\hat{J}^2|j,m_j\rangle &= j(j+1)\hbar|j,m_j\rangle \\ \hat{J}_z|j,m_j\rangle &= m_j\hbar|j,m_j\rangle\end{aligned}}$

Allowed values of j is $j = \frac{n}{2}$, $n \in \mathbb{Z}$ so: $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, 3, \ldots$

→ Allowed values on m_j are $m_j \in [-j, j] = -j, -j+1, -j+2, \ldots\ldots j-2, j-1, j$.
$= 0-j, 1-j, 2-j, \ldots\ldots j-2, j-1, j-0$.

By matrix formatting:

$$\hat{J}^2 = \hbar^2 j(j+1) I = \hbar^2 2\cdot I = \hbar^2\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}$$

$$J_z = \hbar\begin{bmatrix}1&0&0\\0&0&0\\0&0&-1\end{bmatrix}$$

hence $\boxed{J_z = \hbar\begin{bmatrix}1&&\\&0&\\&&-1\end{bmatrix} \quad J^2 = \hbar^2\begin{bmatrix}2&&\\&2&\\&&2\end{bmatrix}}$

2) Derivation:

- $[L^2, L_z] = 0 \rightarrow \begin{cases} L^2\varphi = \lambda\varphi \\ L_z\varphi = \mu\varphi \end{cases}$ with φ is a test function, $(\lambda, \mu) :=$ Eigenstates (L^2, L).

and $\boxed{L_\pm = L_x \pm iL_y}$

- $[L_z, L_\pm] = [L_z, L_x] \pm i[L_z, L_y] = i\hbar L_y \pm i(-i\hbar L_x) = \pm\hbar(L_x \pm iL_y)$.

$$L_z(L_\pm\varphi) = (L_zL_\pm - L_\pm L_z)\varphi + L_\pm L_z\varphi = \pm\hbar L_\pm\varphi + L_\pm(\mu\varphi)$$
$$= (\mu \pm \hbar)L_\pm\varphi$$

- $\begin{cases} L_z\varphi(top) = l\hbar\varphi(top) \\ L^2\varphi(top) = \lambda\varphi(top) \end{cases}$ → $L_\pm L_\mp = (L_x \pm iL_y)(L_x \mp iL_y) = L_x^2 + L_y^2 \ldots$
$= L_x^2 + L_y^2 \mp i(L_xL_y - L_yL_x)$.
$= L^2 - L_z^2 \mp i(i\hbar L_z)$.

→ $\boxed{L^2 = L_\pm L_\mp + L_z^2 \mp \hbar L_z}$

hence $L^2\varphi(top) = (L_-L_+ + L_z^2 + \hbar L_z)\varphi(top) = (0 + \hbar^2 l^2 + \hbar^2 l)\varphi(top)$
$= \hbar^2 l(l+1)\varphi(top)$

[ii] RAISING and LOWERING OPERATORS:

1) $\hat{J}_\pm = J_x \pm iJ_y \rightarrow \begin{cases}\hat{J}_+|j,m_j\rangle = \hbar[j(j+1) - m_j(m_j+1)]^{\frac{1}{2}}|j,m_j+1\rangle \\ \hat{J}_-|j,m_j\rangle = \hbar[j(j+1) - m_j(m_j-1)]^{\frac{1}{2}}|j,m_j-1\rangle\end{cases}$

hence $\boxed{J_\pm|j,m_j\rangle = \hbar\sqrt{j(j+1) - m_j(m_j\pm1)}\times|j,m_j\pm1\rangle}$

2) If $m_j = j$ then $J_+|j,j\rangle = \hbar|j(j+1) - j(j+1)|^{\frac{1}{2}}|j,j\rangle = \hbar\,0\,0 = \boxed{0}$ → $\boxed{J_+|j,j\rangle = 0}$

[iii] SPINS:

1) $S^2|+z\rangle = \frac{3}{4}\hbar|+z\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2|+z\rangle$
$S^2|-z\rangle = \frac{3}{4}\hbar|-z\rangle = \frac{1}{2}\left(\frac{1}{2}+1\right)\hbar^2|-z\rangle$ and $\boxed{\begin{aligned}\hat{S}_z &= \frac{1}{2}\hbar|+z\rangle \\ \hat{S}_z &= -\frac{1}{2}\hbar|-z\rangle\end{aligned}}$

2) $|+z\rangle = \left|\frac{1}{2},\frac{1}{2}\right\rangle,\ |-z\rangle = \left|\frac{1}{2},-\frac{1}{2}\right\rangle$ or $\begin{array}{cc} z+ & z- \\ |++\rangle, & |+-\rangle\end{array}$

Problem 2

(a) Find the column vector representing

$\boxed{|\psi\rangle = \cos(\theta)|H\rangle + e^{i\phi}\sin(\theta)|V\rangle}$ Let $|\psi\rangle = \}$

Ans:

1) $\langle +45|H\rangle = \frac{1}{\sqrt{2}}[1\ 1]\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}$

$\langle +45|V\rangle = \frac{1}{\sqrt{2}}[1\ 1]\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}$

So $\langle +45|\cos\theta|H\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}\cos\theta = \frac{1}{\sqrt{2}}\begin{bmatrix}\cos\theta & 0\\\cos\theta & 0\end{bmatrix}$

$\langle +45|e^{i\phi}\sin\theta|V\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}e^{i\phi}\sin\theta = \frac{1}{\sqrt{2}}\begin{bmatrix}0 & 0\\e^{i\phi}\sin\theta & e^{i\phi}\sin\theta\end{bmatrix}$

2) So $\langle +45|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}^T\begin{bmatrix}\cos\theta\\e^{i\phi}\sin\theta\end{bmatrix} = \frac{1}{\sqrt{2}}(\cos\theta + e^{i\phi}\sin\theta)$

Similarly: $\langle -45|\psi\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}^T\begin{bmatrix}\cos\theta\\e^{i\phi}\sin\theta\end{bmatrix} = \frac{1}{\sqrt{2}}(\cos\theta - e^{i\phi}\sin\theta)$

3) So: $\begin{bmatrix}\langle 45|\psi\rangle\\\langle -45|\psi\rangle\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}(\cos\theta + e^{i\phi}\sin\theta)\\\frac{1}{\sqrt{2}}(\cos\theta - e^{i\phi}\sin\theta)\end{bmatrix}$

So: $\boxed{\langle \pm 45|\psi\rangle = \frac{1}{\sqrt{2}}\cos\theta \pm e^{i\phi}\sin\theta}$

(b) Work out the matrix representation of $\hat{P}_H = |H\rangle\langle H|$, $\hat{P}_V = |V\rangle\langle V|$ using $|L\rangle\ |R\rangle$ states as basis;

(a) Check that the relationships $\hat{P}_H^2 = \hat{P}_H$, $\hat{P}_V^2 = \hat{P}_V$ and $\hat{P}_H\hat{P}_V = \hat{P}_V\hat{P}_H = 0$!

Ans: (i) Answer 1:

If, $|R\rangle = \frac{1}{\sqrt{2}}(|H\rangle - i|V\rangle)$; $|L\rangle = \frac{1}{\sqrt{2}}(|H\rangle + i|V\rangle)$.

1) Then: $|H\rangle = \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle)$; $\frac{i}{\sqrt{2}}(|R\rangle - |L\rangle)$.

So $|H\rangle = \frac{1}{\sqrt{2}}(|R\rangle + |L\rangle) \rightarrow \hat{P}_H: \left(\frac{1}{\sqrt{2}}(|R\rangle + |L\rangle)\right)\left(\frac{1}{\sqrt{2}}(\langle R| + \langle L|)\right)$

$= \frac{1}{2}(\langle R|R\rangle + \langle L|R\rangle + \langle R|L\rangle + \langle L|L\rangle)$

$(1 + 0 + 0 + 1) = \frac{1}{2}(2) = \boxed{1}$ ✓

2) Similarly: $|V\rangle = \frac{1}{2}(\langle R|R\rangle - |R\rangle\langle L| - |L\rangle\langle R| + \langle L|L\rangle)$

$= \frac{1}{2}(1 - 0 - 0 + 1) = \frac{1}{2}(1) = \frac{2}{2} = \boxed{1}$ ✓

3) Now $\frac{1}{2}(\langle R|R\rangle + \langle L|R\rangle\langle R|L\rangle + \langle L|L\rangle)(\langle R|R\rangle - \langle L|R\rangle - \langle R|L\rangle + \langle L|L\rangle)$

$= \frac{1}{2}(1)(1)$

(ii) Answer 2

1) $\hat{P}_V = |V\rangle\langle V| = \frac{1}{2}\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}$ $\quad \hat{P}_H = \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = |H\rangle\langle H|$

w/ $|V\rangle\langle V| = \begin{bmatrix}1\\-1\end{bmatrix}[1\ -1]\frac{1}{2}$ $\quad |H\rangle\langle H| = \begin{bmatrix}1\\1\end{bmatrix}[1\ 1]\frac{1}{2}$

here $\boxed{\begin{matrix}\langle R|\hat{P}_H = \frac{1}{2}[1\ \ 1]\\ \langle L|\hat{P}_H = \frac{1}{2}[1\ \ 1]\end{matrix}}$ $\boxed{\begin{matrix}\langle R|\hat{P}_V\ \frac{1}{2}[1\ -1]\\ \langle L|\hat{P}_V\ \frac{1}{2}[1\ -1]\end{matrix}}$

3) to verify: $\hat{P}_H^2 = \frac{1}{2^2}\begin{bmatrix}1&1\\1&1\end{bmatrix}^2 = \frac{1}{4}\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1&1\\1&1\end{bmatrix}$

$= \begin{bmatrix}1\begin{bmatrix}1&1\\1&1\end{bmatrix} & 1\begin{bmatrix}1&1\\1&1\end{bmatrix}\\ 1\begin{bmatrix}1&1\\1&1\end{bmatrix} & 1\begin{bmatrix}1&1\\1&1\end{bmatrix}\end{bmatrix} = \begin{bmatrix}2&2\\2&2\end{bmatrix}$

so $\boxed{\hat{P}_H^2 = P_H}$

as $\frac{1}{4}\begin{bmatrix}2&2\\2&2\end{bmatrix} = \begin{bmatrix}\frac{2}{4}&\frac{2}{4}\\\frac{2}{4}&\frac{2}{4}\end{bmatrix} = \begin{bmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1&1\\1&1\end{bmatrix} =$

b) Similarly: $\hat{P}_V^2 = \frac{1}{2^2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix}1\cdot1+(-1)(-1) & 1(-1)+(-1)(1)\\(-1)(1)+(1)(-1) & (-1)(-1)+(1)\end{bmatrix} = \frac{1}{4}\begin{bmatrix}2&-2\\-2&2\end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix} = P_V$ so $\boxed{\hat{P}_V^2 = P_V}$

4) Last but not least:

$P_HP_V = \begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1&-1\\-1&1\end{bmatrix} = \begin{bmatrix}1\begin{bmatrix}1&-1\\-1&1\end{bmatrix} & 1\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\\ 1\begin{bmatrix}1&-1\\-1&1\end{bmatrix} & 1\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\end{bmatrix}$

for $\boxed{\begin{bmatrix}P_H&0\\0&P_V\end{bmatrix}^2 = \begin{bmatrix}P_HP_H & P_HP_V\\P_HP_V & P_HP_H\end{bmatrix}}$

$= \begin{bmatrix}1\cdot1+1\cdot(-1) & 1\cdot(-1)+1\cdot1\\1\cdot1+1(-1) & 1\cdot(-1)+1\cdot1\end{bmatrix} = \begin{bmatrix}1-1&1-1\\1-1&1-1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}0&0\\0&0\end{bmatrix}$

$= \boxed{0}$ ✓ so $P_HP_V = 0$

Problems 4

① Show that: $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B}$

Ans: 1) $[\hat{A}\hat{B}, \hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{A}\hat{C} = \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} + \hat{B}\hat{A}\hat{C} - \hat{B}\hat{C}\hat{A}$

$= \hat{A}(\hat{B}\hat{C} - \hat{C}\hat{B}) + \hat{B}(\hat{A}\hat{C} - \hat{C}\hat{A})$

$= \hat{A}[\hat{B}, \hat{C}] + \hat{B}[\hat{A}, \hat{C}]$

2) Reverse:

$\hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B} = \hat{A}(\hat{B}\hat{C} - \hat{C}\hat{B}) + \hat{B}(\hat{A}\hat{C} - \hat{C}\hat{A})$

$= (\hat{A}\hat{B} - \hat{B}\hat{A})\hat{C} + (\hat{B}\hat{A} - \hat{A}\hat{B})\hat{C} = [A, B]\hat{C} + \hat{C}[\hat{B}, \hat{A}]$

$= ([\hat{A}, \hat{B}] + [\hat{B}, \hat{A}])\hat{C} = [\hat{A}\hat{B}, \hat{C}] \neq 0$ unless $\hat{C} = 0$. ✓

② Show that $\hat{J}^2$ is hermitian

Ans: we need show $\boxed{(\hat{J}^2)^\dagger = \hat{J}^2}$

Ways 1) $(\hat{J}^2)^\dagger = (\hat{J}\cdot\hat{J})^\dagger = \hat{J}^\dagger \hat{J}^\dagger = \hat{J}\cdot\hat{J} = \hat{J}^2$

2) $(\hat{J}^2)^\dagger = (\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2)^\dagger = \hat{J}_x^{2\dagger} + \hat{J}_y^{2\dagger} + \hat{J}_z^{2\dagger} = \hat{J}^2.$

So, Indeed $\boxed{\hat{J}^2 \text{ is Hermitian!}}$

3) Also: $\langle\phi|\hat{A}|\psi\rangle = (\langle\psi|A|\phi\rangle)^* \longleftrightarrow \boxed{A^\dagger = A}$

so J^2 is Hermitian because it has a complete set of Eigenstates of real values.

③ Compute $[\hat{J}^2, \hat{J}_x]$ and $[\hat{J}^2, J_y]$

Ans: 1. $[\hat{J}^2, \hat{J}_x] = [(\hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2), \hat{J}_x]$

$= \hat{J}_x^2\hat{J}_x + \hat{J}_y^2\hat{J}_x + \hat{J}_z^2\hat{J}_x = J_x \cdot J_x \cdot J_x + 0 + 0 = 0$

and $[\cancel{J_x^2}, J_x] + [J_y^2, \hat{J}_x] + [J_z^2, \hat{J}_x]$

$= \begin{cases} \hat{J}_y[J_y, J_x] + J_z[J_z J_x] = \{-i\hbar J_z J_y + i\hbar J_y J_z\} \\ J_y[J_x, J_y] + J_z[J_z J_x] = \{-i\hbar J_z J_y + i\hbar J_y J_z\} \end{cases}$ +

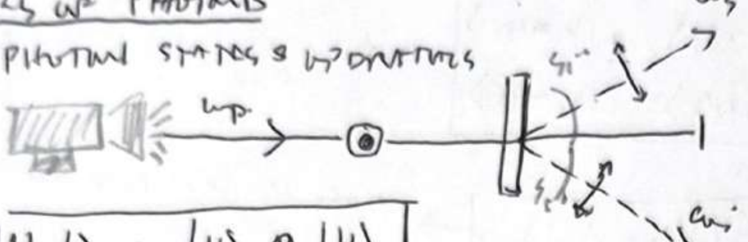
$= 0$ ✓

So: $\boxed{\begin{array}{l}[\hat{J}^2, \hat{J}_x] = 0 \\ [\hat{J}^2, \hat{J}_y] = 0\end{array}}$

Two PARTICLE Systems
ENTANGLEMENT - PAIRS OF PHOTONS (8.1)

1) PAIRS OF PHOTONS

1) 2 PHOTON STATES & NOTATIONS



SPONTANEOUS PARAMETRIC DOWNCONVERSION
- One Pump photon at angular Frequency ($\omega_p \xrightarrow{\Delta} (\omega_s, \omega_i)$) Inside a crystal.
- Polarization of the signal and idler photons are orthogonal of the pump.

$$\boxed{|H,H\rangle = |H\rangle_s \otimes |H\rangle_i}$$ HILBERT SPACE METHOD OF POLARIZATION

w/ $\otimes$ = Direct vector product That combines in the 2 Particle System in an Enlarged Hilbert space.

2.) $|H,H\rangle = |H\rangle_s |H\rangle_i = |HH\rangle$

and $|HH\rangle = (C_H^s |H\rangle_s + C_V^s |V\rangle_s)(C_H^i |H\rangle_i + C_V^i |V\rangle_i)$

EIGENVALUE

$|\Psi\rangle = \sum_n C_n |\psi_0(x)\rangle = \sum_n e^{iEt} |\psi_0(x)\rangle$

$= C_H^s C_H^i |H\rangle_s |H_i\rangle + 2 C_H^s C_V^i |H\rangle_s |V\rangle_s + C_V^s C_V^i |V\rangle_s |V\rangle_i$

$= C_H^s C_H^i |H,H\rangle + C_V^s C_S^i |V,V\rangle + C_H^s C_V^i |H\rangle_s |V\rangle_i + C_V^s C_H^i |V\rangle_s |H\rangle_i$

$= C_H^s C_H^i |H,H\rangle + C_V^s C_V^i (V,V\rangle + C_H^s C_V^i |H,V\rangle + C_V^s C_H^i |V,H\rangle$

and Since: $C_H^s C_H^i \equiv P_H(s,i)$. $C_V^s C_V^i \equiv P_V(s,i)$ $\begin{cases} C_H^s C_V^i = P_H^s(s) P_V^i(s) \\ C_V^s C_H^i = P_V(s) P_H(i). \end{cases}$

Thus: $$\boxed{|HH\rangle = P_H(s,i)|H,H\rangle + P_V(s,i)|V,V\rangle + P_H^s(s) P_V(s)|H,V\rangle + P_V^s(s) P_H(i)|V,H\rangle}$$

3.) When The Beam is $|+45,H\rangle$, Then:

$|+45,H\rangle = |+45\rangle_s \otimes |H\rangle_i = \frac{1}{\sqrt{2}}(|H\rangle_s + |V\rangle_s) \otimes |H\rangle_i = \frac{1}{\sqrt{2}}(|H\rangle_s \otimes |H\rangle_i + |V\rangle_s \otimes |H\rangle_i)$

$= \frac{1}{\sqrt{2}}(|H,H\rangle + |V,H\rangle)$

so: $$\boxed{|+45,H\rangle = \frac{1}{\sqrt{2}}[|H,H\rangle + |V,H\rangle]}$$

4.) Suppose we Modify The polarization state of The Idler Beam.

$|+45,R\rangle = |+45\rangle_s \otimes |R\rangle_i = \frac{1}{\sqrt{2}}[|H\rangle_s + |V\rangle_s] \otimes \frac{1}{\sqrt{2}}[|H\rangle_i - i|V\rangle_i]$

$= \frac{1}{2}[|H\rangle_s |H\rangle_i - i|H\rangle_s |V\rangle_i + |V\rangle_s |H\rangle_i - i|V\rangle_s |V\rangle_i]$

$= \frac{1}{2}[|H,H\rangle - i|H,V\rangle + |V,H\rangle - i|V,V\rangle]$

5.) $\langle V,+45 | R,H\rangle = ({}_s\langle V|\,{}_i\langle +45|)(|R\rangle_s |H_i\rangle) = {}_s\langle V|R\rangle_s\,{}_i\langle +45|H_i\rangle$

$= \left(\frac{-i}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{-i}{2}$ w/ $\langle V| = \begin{bmatrix} -i \\ 0 \end{bmatrix}\frac{1}{\sqrt{2}}$ $|H\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}^T \frac{1}{\sqrt{2}}$

6.) Suppose That $(\hat{\wp}_{HV}^s, \hat{\wp}_{HV}^i)$ Be a photon state That acts on The (signal, idler) Resp. Then: $\boxed{[\hat{\wp}_{HV}^s, \hat{\wp}_{HV}^i] = 0}$. For Example: ? Calculate The action operators such as $\hat{\wp}_{HV}^s$, $\hat{\wp}_{HV}^i$, and $\hat{\wp}_{HV}^{si}$ on $|V,+45\rangle$

Ans: a) $\hat{\wp}_{HV}^s |V,+45\rangle = (\hat{\wp}_{HV}^s |V\rangle_s) \otimes |+45\rangle_i = (-1)|V\rangle_s \otimes |+45\rangle_i = -|V,+45\rangle$.

b) $\hat{\wp}_{HV}^i (V,+45\rangle = |V\rangle_s \otimes (\hat{\wp}_{HV}^i |+45\rangle_i) = |V\rangle_s \otimes [\hat{\wp}_{HV}^i \frac{1}{\sqrt{2}}[|H\rangle_i + |V\rangle_i]]$.

$= |V\rangle_s \otimes [\frac{1}{\sqrt{2}}[|H\rangle_i - |V\rangle_i]] = |V\rangle_s \otimes |-45\rangle_i = |V,-45\rangle$.

c) $\hat{\mathcal{P}}^{si}_{HV} |V, +45\rangle = \hat{\mathcal{P}}^{s}_{HV} \hat{\mathcal{P}}^{i}_{HV} |V, +45\rangle = (\hat{\mathcal{P}}^{s}_{HV}|V\rangle_s) \otimes (\hat{\mathcal{P}}^{i}_{HV}|+45\rangle_i)$

$= (-1)|V\rangle_s \otimes |-45\rangle_i = -|V, -45\rangle.$

So: $\boxed{\begin{aligned} \hat{\mathcal{P}}^{s}_{HV}|V, +45\rangle &= -|V, +45\rangle \\ \hat{\mathcal{P}}^{i}_{HV}|V, +45\rangle &= +|V, +45\rangle \\ \hat{\mathcal{P}}^{si}_{HV}|V, +45\rangle &= -|V, -45\rangle \end{aligned}}$

$(\hat{\mathcal{P}}^s_{HV} \; \hat{\mathcal{P}}^i_{HV}) = (-1, +1).$

* wt: $\hat{\mathcal{P}}^i_{HV}(|H\rangle_i + |V\rangle_i) = (|H\rangle_i + V_i\rangle)$
$= |H\rangle_i + V_i\rangle$

iii) PROBABILITIES

1) w/ $\boxed{\hat{O}|\lambda_n\rangle = \lambda_n|\lambda_n\rangle} \rightarrow$ make measurements on observable $\hat{O}$.

$P(\lambda_n | |\psi\rangle) = |\langle\lambda_n|\psi_n\rangle|^2 = \langle\psi|\lambda_n\rangle\langle\lambda_n|\psi\rangle = \langle\psi|\psi\rangle\langle\lambda_n|\lambda_n\rangle.$

$= \langle\psi|\hat{P}_{\lambda_n}|\psi\rangle = \langle\hat{P}_{\lambda_n}\rangle$

So, $\boxed{P(\lambda_n||\psi\rangle) = \langle\hat{P}_{\lambda_n}\rangle}$

the $\langle x\rangle = \sum_i^N p(x) \delta x_i$
$\sim \alpha x - (1-\alpha) x_i$

and $\langle\hat{P}_{\lambda_n}\rangle \equiv \alpha\lambda_1 + (1-\alpha)\lambda_2 \ldots$
$= \alpha\lambda_1 + \lambda_2 - \alpha\lambda_2 = \alpha(\lambda_1 - \lambda_2) + \lambda_2.$

* EXAMPLE PROBLEMS

then: $P(\lambda_n) = P(\lambda_n||\psi\rangle)$

2) Calculate the probability that the signal photon will be measured to have vertical polarization and the idler photon will be measured to have horizontal polarization in a state $|R, +45\rangle$

Ans: a) If $\hat{P}(V_s, H_i)$ is a proj-operator. then:
$\hat{P}_{V_s, H_i}(\cdot) = |V, H\rangle\langle V, H| \rightarrow P(V_s, H_i) = \langle P_{V_s}; H_i\rangle = \langle V, H|\langle R, +45| V, H\rangle$

b) so: $\langle P_{V_s}, H_i\rangle = \langle R, +45|V, H\rangle\langle V, H|R, +45\rangle = |\langle R, +45|V, H\rangle|^2$

$= |{}_s\langle R|V\rangle_s \; {}_i\langle +45|H\rangle_i|^2 = \left|\frac{i}{\sqrt{2}}\frac{1}{\sqrt{2}}\right|^2 = \left|\frac{i^2}{(\sqrt{2})^2}\frac{1^2}{(\sqrt{2})^2}\right| = \left|\frac{-1}{2}\frac{1}{2}\right| = \left|-\frac{1}{4}\right|$

$= \frac{1}{4}$

So: $\boxed{\langle P_{V_s}, H_i\rangle = \frac{1}{4}}$

w/ $\langle R|$

3) Calculate the probability that the idler photon will be measured to have horizontal polarization on a system prepared in state $|R, +45\rangle$

Ans: $P(H_i) = \langle\hat{P}_{H_i}\rangle = \langle R, +45||H\rangle_i\langle H||R, +45\rangle = {}_s\langle R|R\rangle_s \; {}_i\langle +45|H\rangle_i \; {}_i\langle H|+45\rangle_i$

$= (1)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$ so: $\boxed{P(H_i) = \frac{1}{2}}$

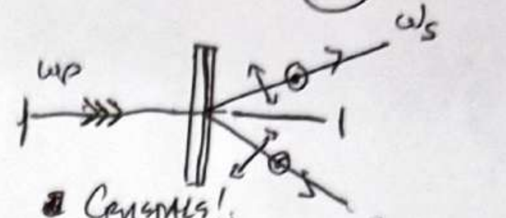
4) Calculate the probability that the signal photon will be measured to have vertical polarization, given that the idler photon is measured to be horizontally polarized, for a system prepared in $|R, +45\rangle$.

Ans: $P(V_s|H_i) = \frac{P(V_s, H_i)}{P(H_i)} = \frac{1/4}{1/2} = \frac{1}{2}$ s.t. $\langle\hat{P}_{V_s}, H_i\rangle \equiv P(V_s, H_i) = \frac{1}{4}$
$\langle\hat{P}_{H_i}\rangle = P(H_i) = \frac{1}{2}.$

PARTICLE SYSTEMS ENTANGLEMENT

(2) ENTANGLED STATES :

i) $$\boxed{|\phi^+\rangle = \frac{1}{\sqrt{2}}\left[|H,H\rangle + |V,V\rangle\right]}$$ (8.19)



ii) $$\boxed{|\psi_A,\psi_B\rangle = |\psi_A\rangle_A \otimes |\psi_B\rangle_B}$$ (8.20)

w/ ψ_A = State particle A, ψ_B = State particle B

■ CRYSTALS!:
- A source that uses 2 crystals whose axes is rotated 90° w.r.t. each other.
- Pump is polarized at 45°

3) For 2 photons prepared in state $|\phi^+\rangle$ of Eqs (8.19), determine the probabilities of obtaining (3 a.) The signal photon is polarized Horizontally
b) The signal photon is measured Horizontally polarized given idler pho

Ans: a) $P(H_s)$ = Probability of signal photon polarized.

so: $$P(H_s) = \frac{1}{\sqrt{2}}\left[\langle H,H| + \langle V,V|\right]\left[|H\rangle_s {}_s\langle H|\right]\frac{1}{\sqrt{2}}\left[|H,H\rangle + |V,V\rangle\right]$$

$$= \frac{1}{\sqrt{2}}\left[\langle H,H|H\rangle_s {}_s\langle H| + \langle V,V|H\rangle_s {}_s\langle H|\right]\frac{1}{\sqrt{2}}\left[|H,H\rangle + |V,V\rangle\right]$$

$$= \frac{1}{2}\left[\langle H,H|H,H\rangle|H\rangle_s {}_s\langle H| + \langle H,H|H,H\rangle|V,V\rangle + \langle V,V|V,V\rangle|H\rangle_s {}_s\langle H| + \langle V,V|V,V\rangle|H,H\rangle\right]$$

$$= \frac{1}{2}\left[{}_s\langle H|H\rangle_s\, {}_i\langle H| + {}_s\langle V|H\rangle_s\, {}_i\langle V|\right]\left[{}_s\langle H|H\rangle_s|H\rangle_i + {}_s\langle H|V\rangle_s|V\rangle_i\right]$$

$$= \frac{1}{2}\,{}_i\langle H|H\rangle_i = \boxed{\frac{1}{2}}$$ so: $$\boxed{P(H_s) = \frac{1}{2}}$$

b) : we proceed to use $P(H_s|H_i) = \frac{P(H_s,H_i)}{P(H_i)}$ such that $P(H_i) = \frac{1}{2}$ so:

$$P(H_s,H_i) = \langle \hat{P}_{H_s,H_i}\rangle = \frac{1}{\sqrt{2}}\left[\langle H,H| + \langle V,V|\right]\left[|H,H\rangle\langle H,H|\right]\frac{1}{\sqrt{2}}\left[|H,H\rangle + |V,V\rangle\right]$$

$$= \frac{1}{2}\left[\langle H,H|H,H\rangle\langle H,H| + \langle V,V|H,H\rangle\langle H,H|\right]\left[|H,H\rangle + |V,V\rangle\right]$$

$$= \frac{1}{2}\left[\langle H,H|H,H\rangle\langle H,H|H,H\rangle + \langle H,H|H,H\rangle\langle H,H|V,V\rangle + \langle V,V|H,H\rangle\langle H,H|H,H\rangle + \langle V,V|H,H\rangle\langle H,H|V,V\rangle\right] = \frac{1}{2}\left[\langle H,H\rangle + \langle V,V|H,H\rangle\right]\left[\langle H,H|H,H\rangle + \langle H,H|\right.$$

$$= \frac{1}{2}\left[1+0\right]\left[1+0\right] = \frac{1}{2}\left[1\right]\left[1\right] = \frac{1}{2}\left[1\right] = \boxed{\frac{1}{2}}$$

so: $$P(H_s|H_i) = \frac{P(H_s,H_i)}{P(H_i)} = \frac{1/2}{1/2} = \frac{1}{2}\cdot\frac{2}{1} = \frac{2}{2} = \boxed{1}$$ so: $$\boxed{P(H_s|H_i) = 1}$$

c). Interpretation:
- The signal photon is measured to have Horizontal Polarization at half
- The results of the Idler are the same : It

Problems 8

(1) Calculate the expected value of $\mathcal{P}^s_{HV}$, $\mathcal{P}^i_{HV}$, and $\mathcal{P}^{si}_{HV}$ for photons prepared in state $|V,L\rangle$

Ans. $|V,L\rangle = \frac{1}{\sqrt{2}}\left[|V\rangle_s \otimes |H\rangle_i + i|V\rangle_s \otimes |V\rangle_i\right]$

$\langle V,L| = \frac{1}{\sqrt{2}}\left[\langle V|_s \otimes \langle H|_i - i\langle V|_s \otimes \langle V|_i\right]$

so $\langle V,L|V,L\rangle = \left(\frac{1}{\sqrt{2}}\right)^2\left[\langle V|V\rangle\langle H|H\rangle + i\langle V|V\rangle\langle H|V\rangle - i\langle V|V\rangle\langle V|H\rangle + (-i)(i)\langle V|V\rangle\langle V|V\rangle\right]$

$= \frac{1}{2}\left[1+0-0+1\right] = \frac{1}{2}2 = \boxed{1}$ ✓

and also: $\langle \mathcal{P}^s_{HV}\rangle = \langle V,L|\mathcal{P}^s_{HV}|V,L\rangle = \langle V,L|V,L\rangle = 1$

Similarly $\langle \mathcal{P}^i_{HV}\rangle = \langle V,L|\mathcal{P}^i_{HV}|V,L\rangle = \langle V,L|V,L\rangle = 1$

$\langle \mathcal{P}^s_{iHV}\rangle = \langle \mathcal{P}^s_{HV}\rangle\langle \mathcal{P}^i_{HV}\rangle = 1\times 1 = 1$

so $\boxed{\langle \mathcal{P}^s\rangle = \langle \mathcal{P}^i_{HV}\rangle = \langle \mathcal{P}^{si}_{HV}\rangle = 1}$ → The operator is basically doing the:

- Projects only relevant settings
- Projected component lies entirely on the basis you're projecting on.
- If the state is orthogonal then it's 0! $\langle V|L\rangle = 0$!

(2) Calculate the expectation values of $\mathcal{P}^s_{HV}$, $\mathcal{P}^i_{HV}$, $\mathcal{P}^{si}_{HV}$ for photons prepared in state

$$\boxed{|\psi\rangle = \left(\frac{1}{3}\right)^{\frac{1}{2}}|H,+45\rangle + \left(\frac{2}{3}\right)^{\frac{1}{2}}|H,-45\rangle}$$

Ans. $\langle \mathcal{P}^s_{HV}\rangle = \langle\psi|\mathcal{P}^s_{HV}|\psi\rangle = \mathcal{P}^s_{HV}\langle\psi|\psi\rangle$ ⑦ $\langle\psi|$?..

since $|\psi\rangle = \left(\frac{1}{3}\right)^{\frac{1}{2}}|H,+45\rangle + \left(\frac{2}{3}\right)^{\frac{1}{2}}|H,-45\rangle = C_1|H,+45\rangle + C_2|H-45\rangle$

$= C_1|H\rangle\otimes|+45\rangle + C_2|H\rangle\otimes|-45\rangle$

now $\langle\psi| = |\psi\rangle^\dagger = \left(\frac{1}{3}\right)^{\frac{1}{2}}|H,+45\rangle^* - \left(\frac{2}{3}\right)^{\frac{1}{2}}|H,-45\rangle$

$= \left(\frac{1}{3}\right)^{\frac{1}{2}}\langle H,+45| - \left(\frac{2}{3}\right)^{\frac{1}{2}}\langle H,+45| = C_1\langle H,+45| - C_2\langle H-45|$

$= C_1\langle H|\otimes\langle +45| - C_2\langle H|\otimes\langle -45|$

now $\langle\psi|\psi\rangle = \left[C_1|H,+45\rangle + C_2|H,-45\rangle\right] =$

① TIME EVOLUTION OPERATOR

BASICS:

- The Schrodinger Eqn evolves in time s.t.:

$$|\psi\rangle \longrightarrow |\psi(t)\rangle$$

- If some operator changes one state then evolved the $|\psi(t_0)\rangle$ then:

$$\boxed{|\psi(t)\rangle = \hat{U}(t,t_0)\,|\psi(t_0)\rangle}$$

- and if $t_0 = 0$

$$|\psi(t)\rangle\Big|_{t_0=0} = \hat{U}(t,t_0)\Big|_{t_0=0} |\psi(t_0=0)\rangle = \hat{U}(t,0)\,|\psi(0)\rangle = \hat{U}(t)\,|\psi(0)\rangle$$

Since: $\boxed{|\psi(t)\rangle = \hat{U}(t)\cdot|\psi(0)\rangle}$

- Recall That:

$$\frac{\partial}{\partial t}|\psi(t)\rangle = \frac{\partial\psi(t)}{\partial t} \to i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \mathcal{H}|\psi(t)\rangle \text{ and } \mathcal{H} = -\frac{\hbar^2}{2m}\nabla^2 + V(\vec{x}).$$

and That: $\langle x|\psi(t)\rangle = \psi(x,t)$ and $\mathcal{H}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec{x})$.

- Then:

$$\langle x|\psi(t)\rangle = \hat{U}\langle x|\psi(0)\rangle \to \psi(x,t) = \hat{U}\,\psi(x,0) = \hat{U}(t)\,\psi(x).$$

So: $\boxed{\psi(x,t) = \hat{U}(t)\,\hat{\psi}(x)}$ $\Rightarrow$ The Rationale for T.I.S.E

- Follows w/-out saying:

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|U(t)^\dagger U(t)|\psi(0)\rangle = \langle\psi(0)|\psi(0)\rangle U(t)^\dagger U(t) = 1\cdot 1 = \boxed{1}.$$

THE PHILOSOPHY / RELEVANCE

- If we let a system evolve for infinitesimal time dt and letting $\hat{U}(t)$ to be the "TIME operator", Then:

$$\boxed{\hat{U}(dt) = \hat{1} - i\,\hat{G}_t\,dt}$$ w/ $\hat{G}_t$ = Generator for Infinitesimal Time Evolution

- This is also unitary!

$$\hat{U}^\dagger(dt)\,\hat{U}(dt) = (\hat{1} + i\hat{G}_t^\dagger dt)(\hat{1} - i\hat{G}_t dt) = \hat{1} + (\hat{G}_t + i\hat{G}_t^\dagger)dt = \hat{1} - i(\hat{G}_t^\dagger - \hat{G}_t)dt.$$

$$\hat{=}\ 1 + (0)dt = \boxed{1}$$ so That $\hat{G}_t^\dagger = \hat{G}_t$

②. THE SCHRODINGER EQS

[i] THE HAMILTONIAN:

PLANK'S ENERGY:

- $E = hf = \hbar\omega$ s.t. $\to$ $E = \frac{f}{\omega} = \frac{\hbar}{h}$ and $\frac{f}{\omega} \propto \frac{1}{\sqrt{\hbar\omega\varepsilon_0}}$

So $\boxed{E = \hbar\omega}$ w/ $\omega = \sqrt{\frac{k}{m}}$.

T.D.S.E

- We can write:

$$\hat{U}(dt) = 1 - \frac{i}{\hbar}\hat{H}\,dt \to \boxed{\frac{\partial}{\partial t}\hat{U} = -\frac{i}{\hbar}\hat{H}\,\hat{U}(t)}$$

w/ $\hat{U}(dt) - 1 = -\frac{i}{\hbar}\hat{H}$ and $\hat{U}(dt) - U(t) = -\frac{i}{\hbar}\hat{U}(t)$ so $\hat{U}(dt) - U(t)$
$= \hat{U}(\Delta t) - U(t)$
$= \frac{dU}{dt}$ ✓

- Continuing:

$$|\psi_0\rangle\left(\frac{\partial\hat{U}}{\partial t} = -\frac{i}{\hbar}\hat{H}\,\hat{U}(t)\right) \to -\frac{i}{\hbar}H\hat{U}(t)|\psi_0\rangle \to \frac{\partial}{\partial t}|\psi(t)\rangle = -\frac{i}{\hbar}H|\psi(t)\rangle$$

so: $$\boxed{\frac{\partial}{\partial t}|\psi(t)\rangle = -\frac{i}{\hbar}H|\psi(t)\rangle}$$

SOLUTIONS TO THE SCHRÖDINGER EQS

$$\boxed{\hat{U}(t) = e^{-iHt/\hbar}}$$

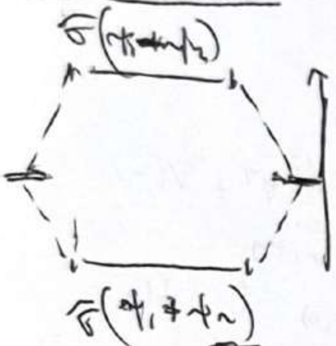
w/ $\frac{\partial \hat{U}}{\partial t} = -\frac{iH}{\hbar} e^{-iHt} = -\frac{iH}{\hbar}\hat{U}(t) = -\frac{i}{\hbar}\hat{H}\hat{U}(t)$ so $\boxed{\frac{\partial \hat{U}}{\partial t} = -\frac{i}{\hbar}\hat{H}\hat{U}}$ ✓

ENERGIES & DISCRETE ENERGIES

- have $\frac{\partial \hat{U}}{\partial t} = \frac{\partial}{\partial t}|\psi(t)\rangle$. This means that:

$$\boxed{|\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|\psi(0)\rangle} \quad (9.13)$$

CHEMICAL BONDING



$P_1(\psi_1) \div C_1 = \sqrt{\frac{1}{3}}$

$P_2(\psi_2) \div C_2 = \sqrt{\frac{2}{3}}$

$|E_n\rangle = E_n$

- The Hamiltonian is an Energy operator so:

$\hat{H}|E_n\rangle = E_n|E_n\rangle = E_n|E_n\rangle$ (9.14) so: $\boxed{\hat{H}|E_n\rangle = E_n|E_n\rangle}$

$\langle x|E_n\rangle = \sum_n C_n\psi_n = C_1\psi_1 + C_2\psi_2 = \frac{1}{3}\psi_1 + (1-\frac{1}{3})\psi_2 \therefore \langle E_n\rangle$

- If the initial energy eigenstate $|\psi(0)\rangle = |E_n\rangle$ then

$|\psi(t)\rangle = e^{-\frac{i\hat{H}t}{\hbar}}|E_n\rangle = \exp\left(-\frac{iE_nt}{\hbar}\right)|E_n\rangle \equiv \exp\left(-\frac{i\omega_nt}{\hbar}\right)|E_n\rangle$

so: $\boxed{|\psi(t)\rangle = \exp\left(-\frac{i\omega_nt}{\hbar}\right)|E_n\rangle}$ (9.15)

- Consequently: if we take $\langle x|$ to EQS (9.15), then

$\langle x|\psi(t)\rangle = \exp\left(-\frac{i\omega_nt}{\hbar}\right)\langle x|E_n\rangle$ let $a_n = \frac{i\omega_n}{\hbar}$

$\Rightarrow$ $\psi(x,t) = \sum_n \exp(-a_nt)\psi_n = e^{-a_nt}[C_1\psi_1 + C_2\psi_2]$.

$\Rightarrow \therefore \boxed{\psi(x,t) \equiv e^{-a_nt}[C_1\psi_1 + C_2\psi_2]} \longrightarrow |\psi(t)\rangle = e^{i\omega_nt}|E_n\rangle$

doesn't change in time! The overall phase changes but the state is $|E_n\rangle$

EIGENSTATE EXAMPLE

- we let: $|\psi(0)\rangle = C_1|E_1\rangle + C_2|E_2\rangle$. Then in later times:

$|\psi(t)\rangle = \exp\left(-\frac{iHt}{\hbar}\right)|\psi(0)\rangle = \exp(-a_nt)[C_1|E_1\rangle + C_2|E_2\rangle]$

$\rightarrow C_1|E_1\rangle e^{-a_1t} + C_2|E_2\rangle e^{-a_2t} = e^{-i\omega_1t}[C_1|E_1\rangle + C_2e^{-i(\omega_2-\omega_1)t}|E_2\rangle]$

so: $\boxed{|\psi(t)\rangle = e^{-i\omega_1t}[C_1|E_1\rangle + C_2e^{-i(\omega_2-\omega_1)t}|E_2\rangle]}$

(3) EXPECTATION VALUES

1.) $\langle H\rangle(t) = \langle\psi(t)|\hat{H}|\psi(t)\rangle = \langle\psi(0)|e^{-i\frac{\hat{H}t}{\hbar}}\cdot\hat{H}\cdot e^{+i\frac{\hat{H}t}{\hbar}}|\psi(0)\rangle$

$= \langle\psi(0)|\hat{H}|\psi(0)\rangle = \langle H\rangle(t=0)$ so: $\boxed{\langle H\rangle(t) = \langle H\rangle(t=0)}$

2.) How about other observables? well:

$\frac{d}{dt}\langle A\rangle = \frac{d}{dt}\langle\psi(t)|\hat{A}|\psi(t)\rangle = \underbrace{\left[\frac{d}{dt}\langle\psi(t)|\right]\hat{A}|\psi(t)\rangle + \langle\psi(t)|\hat{A}\left[\frac{d}{dt}|\psi(t)\rangle\right]}_{\text{chain rule!}}$

$= \frac{i}{\hbar}\langle\psi(t)|\hat{H}\hat{A}|\psi(t)\rangle - \frac{i}{\hbar}\langle\psi(t)|\hat{A}\hat{H}|\psi(t)\rangle$

$= \frac{i}{\hbar}\langle\psi(t)|[\hat{H},\hat{A}]|\psi(t)\rangle$

so: $\boxed{\frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}\langle\psi(t)|[\hat{H},\hat{A}]|\psi(t)\rangle}$

(4) SPIN $\frac{1}{2}$ IN A MAGNETIC FIELD

BASICS

- Energy of a magnetic dipole w/ $\vec{\mu}$ moment in $\vec{B}$ magnetic field is:
$$\boxed{H = -\vec{\mu}\cdot\vec{B}}$$
- Assume it points at z-direction w/ $(\vec{B} = B_0\hat{k}_z)$ and $(\vec{\mu} = \gamma\vec{S})$. Then
$$\boxed{H = -\gamma S_z \vec{B}}$$
so: $\boxed{\hat{H} = -S_z\Omega}$ w/ s^{-1} units — The LARMOR FREQUENCY
- So the Hamiltonian is:
$$\hat{H} = -\gamma\hat{S}_z\vec{B} = -\gamma B\hat{S}_z = -\hat{S}_z\Omega$$

INTRODUCING THE TIME OPERATOR

$\{S_z = \frac{\hbar}{2}\}$

- At time zero, say we have a state $|+z\rangle$. Then:
$$|\psi(t)\rangle = e^{-\frac{iHt}{\hbar}}|+z\rangle = \exp\left(-\frac{i\Omega\hat{S}_z t}{\hbar}\right)|+z\rangle = \exp\left(-i\Omega\left(\frac{\hbar}{2}\right)t\frac{1}{\hbar}\right)|+z\rangle$$
$$= \exp\left(-i\frac{\Omega}{2}\right)|+z\rangle \quad \text{so: } \boxed{|\psi(t)\rangle = e^{-i\frac{\Omega}{2}}|+z\rangle}$$
- But this will be different if we start on $|+x\rangle$ s.t.:
$$|\psi(t)\rangle = \exp\left(-\frac{iH}{\hbar}t\right)|+x\rangle = \exp\left(-\frac{iHt}{\hbar}\right)\frac{1}{\sqrt{2}}\left[|+z\rangle + |-z\rangle\right]$$
$$= \frac{1}{\sqrt{2}}\left[e^{+\frac{i\Omega S_z t}{\hbar}}|+z\rangle + e^{\frac{i\Omega S_z t}{\hbar}}|-z\rangle\right]$$
$$= \frac{1}{\sqrt{2}}\left[e^{i\Omega\left(\frac{\hbar}{2}\right)\frac{t}{\hbar}}|+z\rangle + e^{-i\Omega\frac{\hbar}{2}t}|z\rangle\right]$$
$$= e^{\frac{i\Omega t}{2}}\frac{1}{\sqrt{2}}\left[|+z\rangle + e^{-i\Omega t}|-z\rangle\right]$$
so: $\boxed{|\psi(t)\rangle = e^{\frac{i\Omega t}{2}}\frac{1}{\sqrt{2}}\left[|+z\rangle + e^{-i\Omega t}|-z\rangle\right]}$
- At $\left(t = \frac{\pi}{2\Omega}\right)$, the state is (Ex):
$$\left|\psi\left(\frac{\pi}{2\Omega}\right)\right\rangle = e^{i\frac{\pi}{4}}\frac{1}{\sqrt{2}}\left[|+z\rangle + e^{-i\frac{\pi}{2}}|-z\rangle\right]$$
$$= \frac{1}{\sqrt{2}}\left(e^{i\frac{\pi}{4}}|+z\rangle + e^{-i\frac{\pi}{4}}e^{-i\frac{\pi}{2}}|-z\rangle\right) \rightarrow$$
$$= \frac{1}{\sqrt{2}}\left(e^{i\frac{\pi}{4}}|+z\rangle + e^{-i\frac{3\pi}{4}}|-z\rangle\right)$$
$$= e^{i\frac{\pi}{4}}|-y\rangle \quad \text{w/ } \boxed{\frac{\pi}{2} := y}$$
$$\exp\left(-\left(i\frac{\pi}{4} + i\frac{\pi}{2}\right)\right) = \exp\left(-\left(i\left(\frac{\pi}{4} + \frac{\pi}{2}\right)\right)\right) = \exp\left(-\left(i\left(\frac{\pi + 2\pi}{2}\right)\right)\right) = \exp\left(-\left(i\left(\frac{3\pi}{2}\right)\right)\right) = \exp\left(-i(4.71239)\right) = e^{-i4.71239}$$
- Whilst at $\left(t = \frac{\pi}{\Omega}\right)$ is:
$$\left|\psi\left(t = \frac{\pi}{\Omega}\right)\right\rangle = e^{i\frac{\pi}{2}}\frac{1}{\sqrt{2}}\left[|+z\rangle + e^{i\pi}|-z\rangle\right] \quad \boxed{\pi := -x}$$
$$= e^{i\frac{\pi}{2}}|x\rangle \qquad \text{since } \frac{\pi}{2} + \frac{\pi}{2} = \pi = x.$$
so: $\boxed{\left|\psi\left(\frac{\pi}{2\Omega}\right)\right\rangle = e^{i\frac{\pi}{4}}|-y\rangle, \quad \left|\psi\left(\frac{\pi}{\Omega}\right)\right\rangle = e^{i\frac{\pi}{2}}|-x\rangle}$

EXPECTED VALUE OF SPIN

- How do we confirm that the state is changing in time? Well one way to look at it is to take the expected value of $\vec{S}$ s.t.:
$$\boxed{\langle S\rangle(t) = \langle S_x\rangle(t)\,\hat{u}_x + \langle S_y\rangle(t)\,\hat{u}_y + \langle S_z\rangle(t)\,\hat{u}_z}$$
- The expectation value is thus for (x):
$$\langle S_x\rangle(t) = \langle\psi(t)|\hat{S}_x|\psi(t)\rangle = \left(e^{-i\frac{\Omega t}{2}}\frac{1}{\sqrt{2}}\left[\langle +z| + e^{i\Omega t}\langle -z|\right]\right)\hat{S}_x\left(e^{\frac{i\Omega t}{2}}\frac{1}{\sqrt{2}}\left[|z\rangle + e^{-i\Omega t}|-z\rangle\right]\right)$$

$$= \frac{\hbar}{4}\left(e^{-i\Omega t} + e^{i\Omega t}\right) = \frac{1}{2}\left[\langle +z| + e^{i\Omega t}\langle -z|\right]\left[\frac{\hbar}{2}|z\rangle + e^{-i\Omega t}\frac{\hbar}{2}|+z\rangle\right]$$

$$= \frac{1}{2}\left(\frac{\hbar}{2}\langle +z|-z\rangle + e^{n(t)}\langle +z|+z\rangle\frac{\hbar}{2} + \langle -z|-z\rangle\frac{\hbar}{2} + e^{-n(t)}\frac{\hbar}{2}\right)$$

$$= \boxed{\frac{\hbar}{2}\cos(\Omega t)}$$

$$= \frac{1}{2}\left(0 + e^{nt}\frac{\hbar}{2} + e^{-nt}\frac{\hbar}{2} + 0\right) = \frac{\hbar}{4}\left(e^{nt} + e^{-nt}\right) \checkmark$$

- For y it must be $\langle S_y\rangle_t = -\frac{\hbar}{2}\sin(\Omega t)$.

with $n(t) = n_t = \Omega t$.

$$\langle S_y\rangle(t) = \langle \psi(t)|\hat{S}_y|\psi(t)\rangle$$

$$= \left[e^{-\frac{i\Omega t}{2}}\frac{1}{\sqrt{2}}\left[\langle +z| + e^{-i\Omega t}\langle -z|\right]\right]\hat{S}_y\left[e^{\frac{i\Omega t}{2}}\frac{1}{\sqrt{2}}\left[|+z\rangle + e^{-i\Omega t}|-z\rangle\right]\right]$$

$$= \frac{1}{2}\left[\left[\langle +z| + e^{i\Omega t}\langle -z|\right]\left[i\frac{\hbar}{2}|-z\rangle - ie^{-i\Omega t}\frac{\hbar}{2}|+z\rangle\right]\right]$$

$$= \frac{1}{2}\frac{\hbar}{2}\left[-ie^{-i\Omega t} + ie^{i\Omega t}\right]$$

$$= \frac{\hbar}{4}\left[2ie^{-i\Omega t} + ie^{\Omega t}\right] = \boxed{-\frac{\hbar}{2}\sin(\Omega t)}$$

hw

$$\boxed{\begin{aligned}\langle S_x\rangle(t) &= \frac{\hbar}{2}(\cos(\Omega t))\\ \langle S_y\rangle(t) &= -\frac{\hbar}{2}(\sin(\Omega t))\end{aligned}}$$

WAVE EQS MECHANICS

11.1 11.2 11.3

① $\boxed{\hat{H}|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle}$ Schrödinger EQS. - Compact form

② whatever we do w...

[i] Assumptions:

1) $\hat{H} = \frac{p^2}{2m} + V(x)$; $p^2 = (i\hbar)^2 \frac{\partial^2}{\partial x} = -\hbar^2 \nabla^2$

$p = i\hbar\nabla$

2) $\psi(x,t) = \langle x|\psi(t)\rangle$

So: $\hat{H}\psi = \left[-\frac{\hbar \nabla^2}{2m} + V(x)\right]\psi(x,t) = \boxed{-\frac{\hbar}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}}$

[ii] TIME INDEPENDENT SOLUTIONS .. Schrödinger EQS - full form

1) $\Psi(x,t) = \psi(x)\,\phi(t)$

$|\psi(t)\rangle = e^{i\frac{Ht}{\hbar}}|\psi(0)\rangle$

Soln/ $\boxed{\Psi = \sum_n c_n\psi_n(x)\,\phi_n(t)}$

$c_n = \int_0^\infty \psi_n^*(x,0)\,\psi(x,0)\,dx = \int_0^\infty |\psi|^2\,dx = 1$

2) $|\psi(0)\rangle = \sum_n c_n|\psi_n\rangle$

$\psi(x,0) = \langle x|\psi(0)\rangle = \sum_n c_n\langle x|\psi_n\rangle = \sum_n c_n\psi_n(x)$

$\boxed{E = \hbar\omega_n}$

So: $\boxed{|\psi(0)\rangle = \sum_n c_n\langle x|\psi_n\rangle = \sum_n c_n\psi_n(x)}$

3) $c_n = \langle\psi_n|\psi_0\rangle = \int_0^\infty \langle\psi_n|x\rangle\langle x|\psi_0\rangle\,dx = \int_0^\infty \psi_n^*\psi(x,0)\,dx$.

4) $|\psi(t)\rangle = \sum_n c_n e^{-i\frac{Ht}{\hbar}}|\psi_n\rangle = \sum_n c_n e^{\frac{iE_nt}{\hbar}}|\psi_n\rangle = \sum_n e^{i\omega_n t}|\psi_n\rangle$

So: $\boxed{|\psi(t)\rangle = \sum_n e^{i\omega_n t}|\psi_n\rangle}$

5) $\Psi(x,t) = \langle x|\psi(t)\rangle \Rightarrow \boxed{\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\Psi + V\Psi = i\hbar\frac{\partial\Psi}{\partial t}}$ $\boxed{11.8}$

[iii] BOUNDARY CONDITIONS

• If uncertainty is finite [V(x) finite] ⟹ ∴ The Integral is finite

$$\boxed{\int_{x_0-\eta}^{x_0+\eta}\frac{\partial^2\psi}{\partial x^2} = \frac{d\psi_n}{dx}(x_0+\eta) - \frac{d\psi_n}{dx}(x_0-\eta) = \frac{2m}{\hbar}\int_{x_0-\eta}^{x_0+\eta}[V - E_n]\psi_n\,dx}$$

are bounded functions

⑤ FREE PARTICLE

$k = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}$

[i] $\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V_0\psi = E\psi \Rightarrow \frac{d^2\psi}{dx^2} = \frac{2m(E-V_0)\psi(x)}{\hbar}$

So: $\boxed{\psi(x) = Ae^{ikx} + Be^{ikx}}$

so $\boxed{\Psi(x,t) = Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}}$ ⟶ Full solution.

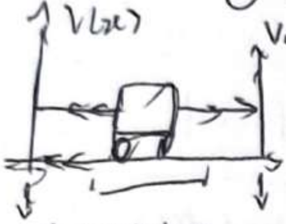
[B] IMPLICATIONS:

1) $\psi(A,B)$: A is travelling positive direction $\xrightarrow{+}$
B is travelling negative direction $\xleftarrow{-}$

2.) The Eigenstates are the same as $\boxed{k = \frac{p}{\hbar}}$ where h is a De-Broglie Relation: $\boxed{A = \frac{1}{\sqrt{2\pi\hbar}}}$

$V = \begin{cases} 0 & \text{If not push} \\ \infty & \text{If pushed} \end{cases}$

(3) POTENTIAL STEP



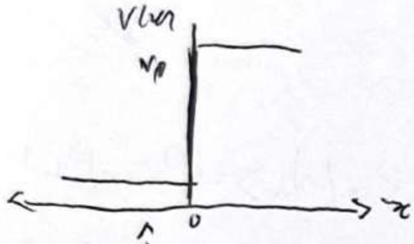
A wagon in an Infinite well. Energy is Infinite thus It always Bounces but 0 if its not pushed

[i] PREFACE / Assumptions

1) $V(x) = \begin{cases} 0 \\ V_0 \end{cases}$

2.) $\psi(x) = A_1 e^{ik_1 x} + B_1 e^{-ik_1 x}$
w/ $\boxed{k_1 = \sqrt{\frac{2ME}{\hbar^2}}}$

3.) $\psi(x) = A_2 e^{ik_2 x} + B_2 e^{-ik_2 x}$
w/ $\boxed{k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}}$ $\quad \boxed{E \to E - V_0}$



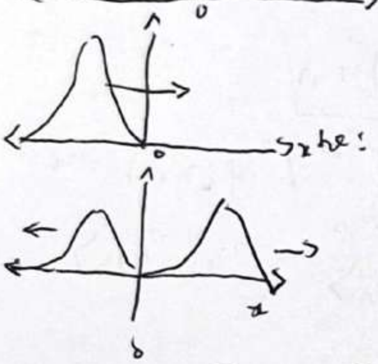
[ii] PROBABILITY FLUX

1.) $\frac{\partial |\psi(x,t)|^2}{\partial t} = \frac{\partial \psi^* \psi}{\partial t} = \psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t}$

2.) so: $\boxed{\frac{\partial |\psi|^2}{\partial t} = \psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial t}}$

3.) Recall: $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} \longrightarrow \boxed{\frac{\partial \psi}{\partial t} = -\frac{\hbar}{i2m} \frac{\partial^2 \psi}{\partial x^2}}$

$i\hbar \frac{\partial \psi^*}{\partial t} = \frac{-\hbar^2}{2m} \frac{\partial^2 \psi^*}{\partial x^2} \longrightarrow \boxed{\frac{\partial \psi^*}{\partial t} = -\frac{\hbar}{i2m} \frac{\partial^2 \psi^*}{\partial x^2}}$



so we: $\psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi}{\partial t} = \frac{\hbar}{i2m} \left[\psi^* \frac{\partial^2 \psi}{\partial x^2} - \psi \frac{\partial^2 \psi^*}{\partial x^2} \right].$

2.) Let $\boxed{J_x(x,t) = \frac{\hbar}{i2m} \left(\psi^* \nabla \psi - \psi \nabla \psi^* \right).}$

Thus $\frac{\partial J}{\partial x} = \frac{\hbar}{i2m} \left[\cancel{\left(\frac{\partial \psi^*}{\partial x}\right)\left(\frac{\partial \psi}{\partial x}\right)} + \psi^* \frac{\partial^2}{\partial x^2} \psi - \cancel{\left(\frac{\partial \psi}{\partial x} \frac{\partial \psi^*}{\partial x}\right)} - \psi \frac{\partial^2 \psi}{\partial x^2} \psi^* \right]$
$= \frac{\hbar}{i2m} \left[\psi^* \nabla \psi - \psi \nabla \psi^* \right]$

so: $\boxed{\frac{\partial |\psi(\vec{r},t)|^2}{\partial t} = \frac{\partial J}{\partial x} = \nabla J(x)}$

Divergence Theorem to Replace volume Interval Ω closed surface S that Bounds S.

3.) w/ : $\iiint_\Omega \frac{\partial |\psi|^2}{\partial t} d^3\vec{r} = -\iiint \nabla \cdot \vec{J} \, d^3\vec{r}.$

$\frac{\partial}{\partial t} \iiint_\Omega |\psi(\vec{r},t)|^2 d^3\vec{r} = -\oint J(\vec{r},t) \, d\vec{a}.$

here $\iiint \frac{\partial |\psi|^2}{\partial t} d^3\vec{r} = -\iiint \nabla \cdot \vec{J} \, d^3\vec{r}.$

MOSFET-I, Extensions, Tunneling (11.3) (11.4)

① Cases of the flux eqs.

recall: $\boxed{\nabla J(x,t) = \frac{\partial |\phi(x,t)|^2}{\partial t}}$

1) $E > V_0$.

recall that: $\begin{cases} k_1 = \sqrt{\frac{2mE}{\hbar^2}} \\ k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}} \end{cases}$

2) $k_2 = \sqrt{\frac{2m(E-V_0)}{\hbar^2}}\Big|_{E > V_0}$ s.t $E - V_0 > 0$.

here $\boxed{k_2 \in \mathbb{R}^+ \ \forall t \ E - V_0 \ \forall E > V_0.}$

3.) $A_1 + B_1 = A_2$

$\boxed{ik_1A_1 - ik_1B_1 = ik_2A_2.}$ → Applied the boundary condition of. $\frac{2m}{\hbar^2}\int_{x_0-\eta}^{x_0+\eta}[V(x) - E_n]\phi \, dx$

✓ PROBLEM 11.2

confirm EQS 11.44 and rederive

4.) $ik_1A_1 - ik_2A_2 = ik_1B_1$

$k_1A_1 - k_2A_2 = k_1B_1$

$k_1B_1 = k_1A_1 - k_2(A_1+B_1) = k_1A_1 - k_2A_1 - k_2B_1.$

$k_1B_1 + k_2B_1 = k_1A_1 - k_2A_1$

$B_1(k_1+k_2) = A_1(k_1-k_2)$

$\boxed{B_1 = \frac{(k_1-k_2)}{(k_1+k_2)}A_1}$ (11.44a).

recall that:

$\phi_1 = A_1e^{ik_1x} + B_1e^{-ik_1x}$

$\phi_2 = A_2e^{ik_2x} + B_2e^{-ik_2x}$

w/ $k_1B_1 - k_1A_1 = -k_2A_2$ $\quad B_1 = A_2 - A_1$

$k_1(A_2 - A_1) - k_1A_1 = -k_2A_2$

$k_1A_2 - k_1A_1 - k_1A_1 = -k_2A_2$

$+k_1A_1 - k_1A_1 = +k_2A_2 - k_1A_2$

$2k_1A_1 = A_2(k_2+k_1) \Rightarrow \boxed{A_2 = \frac{2k_1}{k_2+k_1}A_1}$ (11.44B)

$B_1 = \frac{k_1-k_2}{k_1+k_2}A_1$

$= \frac{k_1-k_2}{k_1+k_2}\frac{k_2+k_1}{2k_1}A_2$

$= \frac{k_1-k_2}{2k_1}$

6) Summary:

- $A_1 = \frac{k_2+k_1}{2k_1}A_2$
- $A_2 = \frac{2k_1}{k_2+k_1}A_1$
- $B_1 = \frac{(k_1-k_2)}{k_1+k_2}A_1$
- $B_1 = \frac{k_1-k_2}{2k_1}A_2.$

Recall $e^{i(k_1x-\omega t)}$

$F(x,t) = \ldots$

and $F \in (\mathbb{R}, \mathbb{C})$

but in this case $F \in \mathbb{R}$.

7.) $J_x^{A_1}(x,t) = \frac{\hbar}{m}\,\mathrm{Im}\left(A_1^* e^{-i(k_1x-\omega_1 t)} \cdot \frac{\partial}{\partial x}A_1e^{i(k_1x-\omega_1 t)}\right).$

$= \frac{\hbar}{m}\mathrm{Im}\left(A_1^* e^{i(k_1x-\omega_1 t)}\, ik_1A_1\, e^{i(k_1x-\omega t)}\right)$

w/ (11.46) $\frac{\hbar k_1}{m} = \frac{p_1}{m} = v$ or $\boxed{p = mv}$

$= \frac{\hbar}{m}\mathrm{Im}\left(A_1^* \cancel{F} ik_1A_1\cancel{F}\right) = \frac{\hbar}{m}k_1A_1^*A_1 = \frac{\hbar k_1}{m}|A_1|^2.$

$= v_1|A_1|^2.$ do: $\boxed{J^{A_1} = v|A_1|^2}$ (11.45)

4.) $R = \frac{J_x^{B_1}(x,t)}{J_x^{A_1}(x,t)}$ = probability that the particle will be found to reflect

R = Reflect, T = Transmit

w: $R = \left| \frac{J_x^{B_1}}{J_x^{A_1}} \right| = \frac{v_1 |B_1|^2}{v_1 |A_1|^2} = \frac{|B_1|^2}{|A_1|^2} = \frac{(k_1 - k)^2}{(k_1 + k_2)^2} = \frac{(\sqrt{E} - \sqrt{E - V_0})^2}{(\sqrt{E} + \sqrt{E - V_0})^2}$

5) $\boxed{T = 1 - R} = 1 - \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} = \frac{(k_1 + k_2)^2 - (k_1 - k_2)^2}{(k_1 + k_2)^2}$

$= \frac{[k_1^2 + 2k_1k_2 + k_2^2] - [k_1^2 - 2k_1k_2 + k_2^2]}{(k_1 + k_2)^2}$

$= k_1^2 + 2k_1k_2 + k_2^2 - k_1^2 + 2k_1k_2 - k_2^2 / (k_1 + k_2)^2$

$= (k_1^2 + k_2^2) - (k_1 + k_2^2) + 2k_1k_2 + 2k_1k_2 / (k_1 + k_2)^2$

$= 4k_1k_2 / (k_1 + k_2)^2 = \frac{4\sqrt{E(E - V_0)}}{(E + \sqrt{E - V_0})^2}$

w: $T = \frac{4\sqrt{E(E - V_0)}}{(E + \sqrt{E - V_0})^2}$, $R = \frac{(E - \sqrt{E - V_0})^2}{(\sqrt{E} + \sqrt{E - V_0})^2}$

R = 1, T = 0 if $E \approx V_0$

R = 0, T = 1 if $E > V_0$

- It transmits if there's "no V_0" since its wholly energym
- It reflects if the energy apparently matches the potential

[ii] $E < V_0$.

1) Thus $k_2 = \sqrt{\frac{2m(E - V_0)}{\hbar}}$ | $E < V_0$

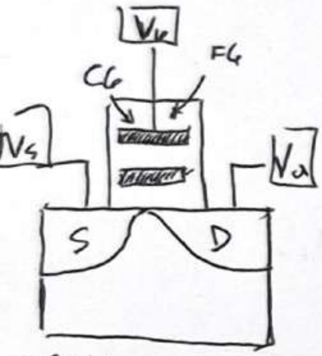
$= \sqrt{\frac{-2m(E - V_0)}{\hbar}} = i\sqrt{\frac{2m(E - V_0)}{\hbar}} = i\alpha$ w/ $\alpha \in \mathbb{R}^+$.

2) w: $\boxed{\phi(x) = Ae^{\alpha x} + Be^{\alpha x}}$

3) Thus: $B_1 = \frac{(ik_1 + \alpha)}{(ik_1 - \alpha)} A_1$ $A_2 = \frac{2ik_1}{(ik_1 - \alpha)} A$

4) $R = \left| \frac{J_x^{B_1}(x<0)}{J_x^{A_1}(x<0)} \right| = \frac{k|B_1|^2}{k_1|A_1|^2} = \left| \frac{(ik + \alpha)}{(ik - \alpha)} \right|^2 = 1.$

so clearly $T = 0$



S = Source
D = Drain
CG = Control Gate
V = Volts.

↓

MOSFET

Metal-oxide Field Effect Transistor

(2) QUANTUM TUNNELLING

So far: $\phi(x) = \begin{cases} A_1 e^{ikx} + B_1 e^{-ikx} & x < 0 \\ A_2 e^{-\alpha x} + B_2 e^{\alpha x} & 0 < x < L \\ A_3 e^{ikx} & x > L \end{cases}$

w/ $k = \sqrt{\frac{2mE}{\hbar^2}}$, $\alpha = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}}$

[i] Implications

1) TUNNEL = Doesn't have enough energy to go through barrier
2) FULL REFLECTION = $L \gg \alpha^{-2}$ so $P(R) = 1$
3) IMAGINARY = $i\alpha$ means that the particle doesn't "propogate" through the barrier in a traditional sense.
4) SUPERLUMINAL