ADVANCE ENGINEERING MATHEMATICS

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DIFFERENTIAL OPERATORS & OSCILLATIONS — 2.6

$\boxed{Dy = y'}$

① OPERATORS:

1) Example: $D(x^2) = 2x$ $\quad Dy = y'$ $\quad D^3 = y'''$

$D(\sin x) = \cos x$ $\quad D^2 y = y''$

So: $\boxed{L = P(D) = D^2 + aD + b = y'' + ay + b\ldots}$

2) $L[y] = p(D)[y] = 0$

$L[y] = (D^2 + D - 6)y = y'' + y' - ay = 0.$ $\quad D[e^{\lambda x} = y] = \lambda y$

and $D[e^{\lambda x}] = \lambda e^{\lambda x}$ $\quad D^2[e^{\lambda x}] = \lambda^2 e^{\lambda x}$ $\quad$ Th $\quad D^2[e^{\lambda x} = y] = \lambda^2 y.$

so: $(\lambda^2 + a\lambda - b)e^{\lambda x} = p(\lambda)e^{\lambda x} = 0$

when $a = 1, b = 6 \rightarrow \boxed{(\lambda^2 + \lambda - 6)y = 0}$

we solve it: $\lambda = \dfrac{\pm b + \sqrt{b^2 + 4ac}}{2a} = \dfrac{\pm 1 + \sqrt{1 - 4(1)(6)}}{2a} = \dfrac{\pm 1 + \sqrt{1-24}}{-2} = \dfrac{1 \pm i\sqrt{24}}{2}.$

$= \dfrac{1 + i\sqrt{4}\sqrt{6}}{2} = \dfrac{1 + i2\sqrt{6}}{2} = \dfrac{1}{2} + \dfrac{i2\sqrt{6}}{2} = \dfrac{1}{2} + i\sqrt{6}.$

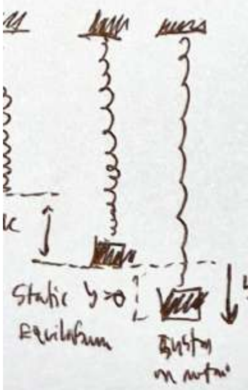
Sum: $\boxed{y(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}}$ or: $\boxed{y(x) = Ae^{\frac{1}{2}+i\sqrt{6}\cdot x} + Be^{-\frac{1}{2}+i\sqrt{6}\cdot x}}$

* More Accurately we: $(\lambda^2 + \lambda - 6) \Rightarrow (\lambda + 2)(\lambda - 3) \neq (\lambda + 3)(\lambda - 2)$

the result $\lambda_1 = -2, \lambda_2 = +3$ So: $\boxed{y(x) = Ae^{3x} + Be^{-2x}}$

3) $\boxed{xDy = xy'}$ so: $D(xy) = (xy)' = y + xy$ and $D(xy') = D(x)y' + D(y')x$

② FREE OSCILLATIONS



1) $F_1 = mg$ $\quad$ So $F_1 + F_2 = mg - ks$ $\quad$ and $F(t) = m\ddot{x}(t)$

$F_2 = -ks$ $\quad\quad F(t) \equiv ky$

hence: $\boxed{m\ddot{y}(t) + ky = 0}$

2) $(mD^2 + k)y = 0 \Rightarrow mD^2 + k = 0 \Rightarrow D^2 = \dfrac{\sqrt{k}}{\sqrt{m}} = D^2 = \omega^2$

we have the form $y(t) = e^{i\lambda t}$ Then: $D^2 e^{\lambda t} = -\lambda^2 e^{\lambda t}$ and $\lambda^2 = \omega^2$

here: $y(t) = Ae^{i\lambda t} + Be^{i\lambda t} \equiv Ae^{i\lambda t} = A\cos(\lambda t) + \sin(\lambda t)$

so: $\boxed{y(t) = A\cos\omega_0 t + B\sin\omega_0 t.}$

PROBLEMS 4.4

(11) $\boxed{xy'' + 2y' + 4xy = 0}$

$y' = rx^{r-1}$ $y'' = r(r-1)x^{r-2}$

Ans: let $x^r = y$. so

1.) so: $xy'' + 2y' + 4xy = x \cdot r(r-1)x^{r-2} + 2rx^{r-1} + 4x x^{r}$

$= r(r-1)x^{r-1} + 2rx^{r-1} + 4x^{r+1} = [r(r-1) + 2r]x^{r-1} + 4x^{r+1}$

$= (r^2 - r + 2r)x^{r-1} + 4x^{r+1} = (r^2 + r)x^{r+1} + 4x^{r+1} = 0$

$= r^2 x^{r-1} + rx^{r+1} + 4x^{r+1} = 0 \rightarrow r = \dfrac{-x^{r-1} \pm \sqrt{(x^{r-1})^2 - 16x^{r+1} \cdot x^{r-1}}}{2x^{r-1}}$

so $r = \dfrac{-x^{r-1} \pm \sqrt{x^{r-1+r-1} - 16x^{r+1+r-1}}}{2x^{r-1}} = \dfrac{-x^{r-1} \pm \sqrt{x^{2(r-1)} - 16x^{2r}}}{2x^{r-1}}$

$= \dfrac{-x^{r-1}}{2x^{r-1}} \pm \dfrac{(x^{2r-1} - 16x^{2r})^{\frac{1}{2}}}{2x^{r-1}} = -\dfrac{1}{2} \pm (x^{2r-1} + 16x^{2r})^{\frac{1}{2}}(x^{r-1})^{-1}$

$\sqrt{4 \cdot 4} = \sqrt{4} \cdot \sqrt{4}$

$\sqrt{16} = 2 \cdot 2$

$4 = 4$ ✓

$\sqrt{4+4} \neq \sqrt{4} + \sqrt{4}$

$\sqrt{8} \neq 2+2$

$\sqrt{4}\sqrt{2} \neq 4$

$2\sqrt{2} \neq 4$

$\sqrt{17} = \sqrt{(\sqrt{8}\sqrt{2})+1}$

$= \sqrt{\sqrt{8} \cdot \sqrt{2} + 1}$

$((x)^{\frac{1}{2}})^{\frac{1}{2}} = (x)^{\frac{1}{4}}$

$r_1 = \dfrac{-1 + \sqrt{-17}}{2} = \dfrac{-1 + i\sqrt{17}}{2}$

* dropping the xs: $= -\dfrac{1}{2} \pm \dfrac{\sqrt{1 - 16}}{2} = -\dfrac{1}{2} \pm \dfrac{\sqrt{-17}}{2} =$

$r_2 = -\dfrac{1 - \sqrt{17}}{2} = -\left(\dfrac{1 + i\sqrt{17}}{2}\right)$

2.) so: $e^{\left(\frac{i\sqrt{17}-1}{2}\right)} = e^{r_1}$, $e^{-\left(\frac{i\sqrt{17}+1}{2}\right)} = e^{-r_2}$ so: $e^{\frac{i\sqrt{17}}{2}x} \cdot e^{-\frac{1}{2}x} =$ $\left(\cos\left(\frac{\sqrt{17}}{2}x\right) + i\sin\left(\frac{\sqrt{17}}{2}x\right)\right)e^{-\frac{1}{2}x}$

$e^{-i\frac{\sqrt{17}}{2}x} \cdot e^{-\frac{1}{2}x} =$ $\left(\cos\left(\frac{\sqrt{17}}{2}x\right) - i\sin\left(\frac{\sqrt{17}}{2}x\right)\right)e^{-\frac{1}{2}x}$

not full solution

2.) so: using power series:

a/ $xy'' = \sum_{m=2}^{\infty} a_m m(m-1)x^{m-1} = \sum_{m=2}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1}$

b/ $2y' = 2 \cdot \sum_{n=1}^{\infty} a_m m x^{m-1} = 2\sum_{n=1}^{\infty} a_n(n+r)x^{n+r-1}$

c/ $4xy = 4\sum_{n=0}^{\infty} a_m x^{m+1} = 4\sum_{n=1}^{\infty} a_{n+r}(n+r)x^{n+r+1}$

So: a) + c) $= \sum_{m=0}^{\infty} a_{n+r}(n+r)(n+r+1)x^{n+r-1} + 2\sum_{m=1}^{\infty} a_{n+r}(n+r)x^{n+r} = 0$

$(n+r)(n+r+1) + 2(n+r)$

$m+r-1 = n+r-1$

$m = n+2$

so: $\sum_{m} a_m(n+r)(n+r+1)x^{n+r-1} + 4\sum a_n x^{n+r+1} = 0$

So: $\sum_{n} [a_m(n+r)(n+r+1) + 4a_{n-2}]x^{n+r-1}$

③ $\boxed{xy'' + (1-2x)y' + (x-1)y = 0}$

w/ $y = \sum_n a_n x^{n+r}$

1) $xy'' = x \sum_{n=0}^{\infty} a_n(n+r)(n+r-1) x^{n+r-2}$

$(1-2x)y' = (1-2x) \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$

$(x-1)y = (x-1) \sum_{n=0}^{\infty} a_n x^{n+r}$

2.) $xy'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1) x^{n+r-1}$

$(1-2x)y' = y' - 2xy' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1} - 2\sum_{n=0}^{\infty} a_n(n+r) x^{n+r}$

$(x-1)y = yx - y = \sum_{n=0}^{\infty} a_n x^{n+r+1} - \sum_{n=0}^{\infty} a_n x^{n+r}$

3) a) $\sum_{n=0}^{\infty} a_n(n+r)(n+r-1) x^{n+r-1} + \sum_{n=0}^{\infty} a_n(n+r) x^{n+r-1} = \sum a_n x^{n+r-1}[(n+r)(1-(n+r-1))]$

$xy'' + (1-2x)y' = \sum_{n=0} a_n x^{n+r-1}[(n+r)(n+r-1) - (n+r)] - 2\sum_{n=0}^{\infty} a_n x^{n+r}(n+r)$

b) $-2\sum_{n=0}^{\infty} a_n(n+r) x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = -\left(\sum_{n=0}^{\infty} a_n x^{n+r}[2(n+r) + 1]\right)$

So: altogether: $\sum_{n=0}^{\infty} a_n x^{n+r-1}[(n+r)(n+r-1)-(n+r)] - \left(\sum_{n=0}^{\infty} a_n x^{n+r}[2(n+r)+1]\right)$

$+ \sum_{n=0}^{\infty} a_n x^{n+r+1} = 0$

c) $\sum_n a_n x^{n+r-1} \mapsto \sum_m a_{m+2} x^{m+r+1}$ $\quad (n+r)(n+r-1)-(n+r) = (m+2+r)(m+r+1)(m+2+r)$

$\sum_n a_n x^{n+r} \mapsto \sum_n a_{m+1}$ x^{m+r+2}

$= m$

d.) $x^{n+r-1} : a_{n+1}(n+1+r)(n+r) + a_{n+1}(n+1+r)$

$x^{n+r} : -2a_n(n+r) - a_n$

$x^{n+r+1} : a_{n-1}$

Margin notes:

$r+1 = m+r-1$

$n+r-r+1-1$

$= n+0$

$r-1$

$n+r+1$

$n = 2$

$r-1$ ✓

$m+2$

$= m+r+2$

1^{st} k: BESSEL'S EQS (4.5)

① $$\boxed{x^2 y'' + xy' + (x^2 - \nu^2)y = 0}$$ w/ ν = Parameter

[I] DERIVATION : $$\boxed{y(x) = \sum_{m=0}^{\infty} a_m x^{m+r}}$$ w/ $x^m x^r$

1) $$\sum_{m=0}^{\infty}(m+r)(m+r-1)a_m x^{m+r} + \sum_{m=0}^{\infty}(m+r)a_m x^{m+r} + \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

w/ $s = m$

a.) $\sum_{m=1}^{\infty}(m+r)(m+r-1)a_m x^{m+r} = 0$

b.) $\sum_{m=1}^{\infty}(m+r)a_m x^{m+r} = 0$

c.) $\sum_{m=0}^{\infty} a_m x^{m+r+2} = 0$

d.) $\nu^2\sum_{m=0}^{\infty} a_m x^{m+r} = 0$

Observe a.), b.), c.) has x^{m+r} term

So: if $x^{s+r} \mapsto$ zero w/ $x^{s+r} = x^{m+r}$

Then the 3^{rd} sum doesn't contribute.

2.) $$\sum_{m=0}^{\infty}\left[(m+r)(m+r-1) + (m+r) + \nu^2\right]a_m x^{m+r} = \sum_{m=0}^{\infty}\left[(m+r)(1+(m+r-1)) + \nu^2\right]a_m x^{m+r} = 0.$$

a.) $r(r-1)a_0 + a_0 - \nu^2 a_0 = 0$

b.) $(r+1)a_1 + (r+1)a_1 - \nu^2 a_1 = 0$

c.) $(s+r)(s+r-1)a_s + (s+r)a_s + a_{s-2} - \nu^2 a_s = 0$ $\cancel{\forall m \geq 0}$

w/ $a_m \neq 0 \Rightarrow (r(r-1) + r - \nu^2)a_0 x^r = 0$ } $s = 0$

$a_0(r(r-1) + r - \nu^2) = 0$

$m = 1 \Rightarrow \left[(1+r)(r) + (r+1) + \nu^2\right]a_1$ } $s = 1$

$m = s \Rightarrow \left[(s+r)(s+r-1) + (s+r) + a_{s-2} - \nu^2 a_s\right.$ } $\forall s = 2, 3 \ldots$

3) Indicial Eqns: $$\boxed{(r-\nu)(r+\nu) = 0}$$

[i] Solutions: $r_1 = \nu$. ($r_1 = r = \nu$):

$r = \nu$ 1) $(s+r+\nu)(s+r-\nu)a_s + a_{s-2} = 0$ Say $s = 2m$

$(s+2\nu)(s)a_s + a_{s-2} = 0$ so: $s-2 = 2m-2$

$(2m+2\nu)2m + a_{2m-2} = 2^2(m+\nu)m + a_{2m-2} = 0$

ab: $4r(r-1)+2r=0$
$$r^2+\frac{1}{2}r=0$$

The sum of coef. in the last sum becomes

$$\boxed{4(s+r+1)(s+r)a_{s+1}+2(s+r+1)a_{s+1}+a_s=0}$$

w/ $m+r-1=r+s$ Thus $\begin{cases} m=s+1 \to 1^{st}\ \text{two} \\ m=s \to 2^{nd}\ \text{two}\end{cases}$

formula like
$\sum_s [(s+1)(s+2)a_{s+2} - 2(s+1)a_{s+1} + n(n+1)a_s]x^s$
quicker legendre?

so: $$\boxed{a_{s+1}=-\frac{a_s}{(2s+2r+2)(2s+2r+1)}}$$

[i] 1^{st} solution ($r_1=\frac{1}{2}$, $r=r_1$)..

$$\boxed{a_{s+1}=-\frac{a_s}{(2s+3)(2s+2)}} \Rightarrow$$

$a_1=\frac{a_0}{3!}$
$a_2=\frac{a_0}{5!}$
$a_3=-\frac{a_0}{7!}$

$a_0=1$
$m\in\mathbb{Z}^+$

$$a_m=\frac{(-1)^m}{(2m+1)!}$$

So the 1^{st} solution is: $$\boxed{y_1(x)=x^{\frac{1}{2}}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)!}x^m}$$

$$\to \sqrt{x}\left(1-\frac{1}{6}x+\frac{1}{120}x^2\pm\cdots\right)$$

[ii] 2^{nd} Solution ($r=r_2=0$)

$$\boxed{A_{s+1}=-\frac{A_s}{(2s+2)(2s+1)}}$$

$r=0$ means the r disappears.

we use formula of $a_{s+1}=\frac{a_s}{(2s+2r+1)(2s+2r)}$ and replace $a\to A$ (coeffs)

$$A_1=\frac{A_0}{2} \quad A_2=\frac{A_1}{12} \quad A_3=-\frac{A_2}{30}$$

$$\to A_1=\frac{A_0}{2!} \quad A_2=\frac{A_0}{4!} \quad A_3=-\frac{A_0}{6!}$$

$$\to \boxed{A_m=\frac{(-1)^m}{(2m)!}}$$

$$\boxed{y_2(x)=\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m)!}x^m}$$

$$\Rightarrow \quad a_{2m-2} = a_{2m}(m+\nu)2^2 m$$

$$\boxed{a_{2m} = a_{2m-2}\frac{1}{(m+\nu)2^2 m}} \quad \forall m = 1, \dots 2.$$

2.) if $m=0$:

$$a_{2(0)} = a_0 = a_{-2}\cancel{\frac{1}{\nu 2^2 0}}$$ can't divide by zero

$m=1$ $\quad a_2 = a_0 \frac{1}{(1+\nu)2^2} \Rightarrow a_0 = a_2(1+\nu)2^2$

and generally: $\boxed{a_0 = \frac{1}{\Gamma(\nu+1)2^\nu}}$ w/ $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$

3.) a) $\Gamma(\alpha+1) = \int_0^\infty e^{-t} t^{\alpha} dt = -e^{-t}t^\alpha \Big|_0^\infty + \alpha \int e^{-t} t^{\alpha-1} dt$

$$= 0 + \alpha \int e^{-t} t^{\alpha-1} dt = \alpha\Gamma(\alpha)$$

so: $\boxed{\Gamma(\alpha+1) = \alpha\Gamma(\alpha)}$

$$\Gamma(1) = \int e^{-t} dt = 1$$

so: $\boxed{\Gamma(\alpha+1) = k!}$ since: $\Gamma(1+1) = 1\Gamma(1) \Rightarrow \Gamma(2) = 2 = 2!$

$\Gamma(2+1) = 2\Gamma(2) \Rightarrow \Gamma(3) = 6 = 3!$

$\Gamma(0+1) = \Gamma(0) \Rightarrow \Gamma(1) = 1 = 1!$

4.) so: $a_2 = \frac{a_0}{2^{2+\nu}1!\,\Gamma(\nu+2)}$ $\quad a_4 = \frac{1}{2^{4+\nu}2!\,\Gamma(\nu+3)} \cdots$

$$\boxed{a_{2m} = \frac{(-1)^m}{2^{m+\nu} m!\,\Gamma(\nu+m+1)}}$$

(2.) so: $y(x) = a_m x^{m+r} = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{m+\nu} m!\,\Gamma(\nu+m+1)}$

so. $$\boxed{J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{m+n} m!\,\Gamma(\nu+m+1)}}$$

[i] such that: $$\boxed{J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!\,(n+m)!}}$$ w/ $J_n(x)$: $x^{n} \to x^{-\nu}$

[ii] Solution: $$\boxed{y(x) = c_1 J_\nu(x) + c_2 J_{-\nu}(x)}$$

APPLICATION OF BESSEL'S EQS

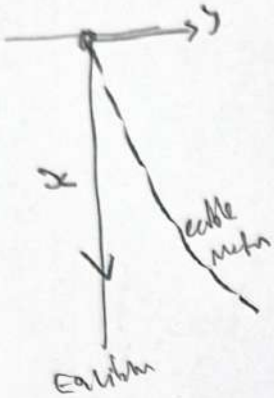
① VIBRATING CABLE:

[i] DERIVATION:

1.) $\boxed{W(x) = \rho g (L - x)}$ w/ $\rho = \frac{M}{V} = \frac{[M]}{[L]^3}$ $g = \frac{[L]}{[T]}$ $x = [L]$

$\frac{[M][L]}{[L]^3[T]^2}\frac{[L]}{} - \frac{[M][L][L]}{[L]^3[T]^2}$

$= \frac{[M]}{[L][T]^2}$



$F(x) = W \sin\alpha \sim W \tan\alpha$

$= W \frac{\partial u}{\partial x}$

so: $F(x) \sim W \frac{\partial u}{\partial x}$, $u(x,t) = u$

$\rho \Delta x = \frac{\partial^2 u}{\partial t^2}$

2.) $F(x + \Delta x) - F(x) \approx \Delta x \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$

$= \Delta x \frac{\partial^2 u}{\partial x^2}$ $\Delta x \rho g \left[(L - X) u_x\right]_x$

so: $F(x) = \rho \Delta x \frac{\partial^2 u}{\partial t^2} = \Delta x \frac{\partial}{\partial x}\left(W \frac{\partial u}{\partial x}\right)$

$\frac{(V_2 - V_1)(X_2 - X_1)}{(X_1 - X_2)^2} = 0$

3.) $F(x) : \Delta x \frac{\partial^2 u}{\partial t^2} = \left\{\frac{\partial W}{\partial x}\frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x} W\right\} \Delta x$

$= \Delta x \left\{ \rho g \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x}\{[L - \Delta x]\rho g\}\right\} = \rho g \Delta x \left(\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} z\right)$

$= \rho g \left(\frac{\partial u}{\partial x}\Delta x - \frac{\partial^2 u}{\partial x^2}\Delta x\right) \sim \rho g \left(\int \frac{\partial u}{\partial x} dx - \int \frac{\partial^2 u}{\partial x} dx\right)$

$\int u \, \Delta x$

$= \frac{U(x_1) - U(x_2)}{U(x_1) - V(x_2)}$

$= \rho g \left(u(x,t) - \frac{\partial u}{\partial x}\right) = \rho \, dx \frac{\partial^2 u}{\partial t^2}$

$\frac{\Delta x \, u - \Delta u z}{\partial u \Delta x} = \frac{\partial^2 u}{\partial t^2}$

$u - \frac{\Delta u}{\Delta x} z = u - \frac{\partial u}{\partial x} z = \frac{\partial^2 u}{\partial t^2} \rightarrow$

$= \frac{u \Delta x - \Delta u z}{\Delta x} = \boxed{\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}\frac{1}{z}}$

$= \frac{u(x_2 - x_1) - (u_2 - u_1) z}{(x_2 - x_1)}$

$\frac{U(x_1) - U(x_2) - U(x_1) + U(x_2)}{X(x_1) - X(x_2)}$

$= 0$

4.) Let $\boxed{u = y(x) \cos(\omega t + \delta)} \rightarrow \frac{\partial u}{\partial x} = y(x)$

$\frac{\partial^2 V}{\partial x} = y'' \rightarrow$

$-\omega^2 y \cos(\omega t + \delta) = \frac{\partial^2 u}{\partial t^2} = -\omega^2 u$

so:

$\frac{\partial U}{\partial x}(X_2 - X_1) - \frac{\partial^2 U}{\partial x^2}(X_2 - X_1) = 0$

$g[(L - x)y']' \cos(\omega t + \delta) = \frac{\partial^2 u}{\partial x^2}$

$-\omega^2 y = g[(L - x)y']' \frac{y}{y} \Rightarrow \frac{\omega^2}{g} y = -y' - (L - x)y''$

$\partial V = U_2 - U_1$

so:

$\frac{V_2 - V_1}{X_2 - X_1} - \frac{(U_1 - U_2)(X_2 - X_1)}{(X_2 - X_1)^2}$

$\frac{(V_2 - V_1)(X_2 - X_1) - (U_2 - U_1)(X_2 - X_1)}{(X_2 - X_1)^2}$

Let $\frac{\omega^2}{g^2} = \lambda^2$. So: $\lambda^2 y = -y' - (L - x)y'' \Rightarrow$ $\boxed{y''(L - x) - y' + \lambda^2 y = 0}$

[1] solve: $\boxed{z\frac{d^2y}{dz^2}+\frac{dy}{dz}+\lambda^2 y=0}$ w/ $\boxed{S=2\lambda z^{\frac{1}{2}}}$

$\left(\frac{2}{z}\frac{1}{z}\right)^{\frac{1}{2}}$

1) a) $\frac{dy}{dz}=\frac{dy}{ds}\lambda z^{-\frac{1}{2}}$

b) $\frac{d^2y}{dz^2}z=\frac{d^2y}{ds^2}\lambda^2 z^{-1}-\frac{1}{2}\frac{dy}{ds}\lambda z^{-\frac{3}{2}}$

$2\lambda=\frac{\partial}{\partial\lambda}\lambda^2$

$\frac{\partial R}{\partial\lambda}=2\lambda$

$R=\lambda^2$

$R=\frac{\omega^2}{g}$

$2\lambda=2\frac{\omega}{g}$

$\frac{\partial R}{\partial\lambda}=\frac{\partial R}{\partial\omega}$

2.) $\lambda^2\frac{d^2y}{ds^2}+\left(-\frac{1}{2}\lambda z^{-\frac{1}{2}}+\lambda z^{-\frac{1}{2}}\right)\frac{dy}{ds}+\lambda^2 y=0$

$\lambda=\frac{s}{2z^{\frac{1}{2}}}$

$z^{\frac{1}{2}}=\frac{s}{2\lambda}$

$\frac{d^2y}{ds^2}+\left(-\frac{1}{s}+\frac{2}{s}\right)\frac{dy}{ds}+y=0$

$-\frac{\lambda}{2z^{\frac{1}{2}}}+\frac{\lambda}{z^{\frac{1}{2}}}=\frac{\lambda}{z^{\frac{1}{2}}}-\frac{\lambda}{2z^{\frac{1}{2}}}=\frac{2\lambda-\lambda}{2z^{\frac{1}{2}}}=\frac{\lambda}{2z^{\frac{1}{2}}}=\lambda\left(2z^{-\frac{1}{2}}\right)$

$\frac{1}{s}=\frac{1}{2\lambda z^{\frac{1}{2}}}\Rightarrow\frac{\lambda^2}{s}=\frac{\lambda}{2z^{\frac{1}{2}}}\Rightarrow\frac{\lambda}{2z^{\frac{1}{2}}}=\frac{\lambda^2}{s}$

so: $\lambda^2\frac{d^2y}{ds^2}+\frac{\lambda^2}{s}\frac{dy}{ds}+\lambda^2 y=\boxed{\frac{d^2y}{ds^2}+\frac{dy}{ds}\frac{1}{s}+y}$

3.) observe that Bessel's eqs: $y(x)=C_1 J_\nu(x)+C_2 J_{-\nu}(x)$
and: $J_{-n}(x)=(-1)^n J_n(x)$ so $J_0=\sum_{m=1}^{\infty}\frac{(-1)^m x^{2m}}{2^{2m}m!(m)!}$

and: $S=2\lambda z^{\frac{1}{2}}=2\frac{\omega}{\sqrt{g}}\sqrt{L-x}$, Then

$$\boxed{y(x)=J_0\left(2\omega\sqrt{\frac{L-x}{g}}\right)}$$

(2) EXPERIMENT:

$L=228$ cm gave: 24.0; 24.5; 24.0, 24.0 vibes/m

hence: $2\omega\sqrt{\frac{L}{g}}=2.405$ (1st zero J_0)

so: $\boxed{f=\frac{\omega}{2\pi}}$ st: $f=\frac{\omega}{2\pi}=\frac{2.405}{2\cdot 2\pi\sqrt{\frac{L}{g}}}=\frac{2.405}{4\pi\sqrt{2.28/9.80}}=0.397$ sec

Which is $0.397\times 60=\boxed{2.38\ \frac{\text{cycles}}{\text{minute}}}$

PROBLEMS 4.5

(5) Show that:

$$J_0(x) = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \dots$$

Ans: 1) $J_{n=0}(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} n!(n+m)!}\Big|_{n=0}$

$$= x^0 \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+0} m!(0+m)!} = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m} m!(m)!}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}$$

2) $\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^m (m!)^2} = \frac{(-1)^0 x^{2(0)}}{2^0(0!)^2} + \frac{(-1)^1 x^{2(1)}}{2^2(1!)^2} + \frac{(-1)^2 x^{2(2)}}{2^4(2!)^2} + \dots$

$$= 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} + \dots$$

so: $\boxed{J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^m (m!)^2}}$

(2) Show that: $\left[x^\nu J_\nu(x)\right]' = x^\nu J_{\nu-1}(x)$

Ans: [i] $\left[x^\nu J_\nu(x)\right]' = \frac{\partial x^\nu}{\partial x} J_\nu(x) + x^\nu \frac{\partial J_\nu(x)}{\partial x}$ $\quad x^\nu x^{2m} = x^{2m+\nu}$

$$= \nu x^{\nu-1} J_\nu(x) + x^\nu \frac{\partial}{\partial x}\left\{ x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m!\, \Gamma(\nu+m+1)}\right\}$$

$x^{2\nu-1} + x^\nu$

$$= \nu x^{\nu-1} J_\nu(x) + x^{2\nu}(2m+2\nu) \sum_{m=0} \frac{(-1)^m x^{2m+\nu-1}}{2^{2m+\nu} m!\, \Gamma(\nu+m+1)}$$

$$= \nu x^{\nu-1} J_\nu(x) + x^{2\nu} \sum_{m=0}^{\infty} (2m+2\nu)\, x^{2m+2\nu-1} \phi_{m\nu}$$

$$= \nu x^{\nu} J_{\nu-1}(x) + \sum J_{\nu-1}(x)(2m+2\nu)_{-1}$$

$$= (\nu + 2m + \nu) J_{\nu-1}(x)\, x$$

$$= (2m+2\nu) J_{\nu-1} x^\nu \rightarrow \boxed{x^\nu J_\nu(x) = (2m+2\nu) J_{\nu-1} x^2}$$

[ii] $\frac{\partial}{\partial x} x^\nu J_\nu(x) = \sum_m \phi_{m\nu} \frac{d}{dx} x^{2m+2\nu} = \sum \phi_{m\nu}(2m+2\nu)\, x^{2m+2\nu-1}$

$= x^\nu$ for: $\phi_{m,\nu-1} = (2m+2\nu)\phi_{m\nu}$

and: $\boxed{\frac{\phi_{m\,\nu-1}}{\phi_{m\nu}} = 2\frac{\Gamma(m+\nu+1)}{\Gamma(m+\nu)}} \Rightarrow$ so $\boxed{(x^\nu J_\nu)' = x^\nu J_{\nu-1}(x)}$

(8.1) Show that: $\left[x^{-\nu} J_\nu(x)\right]' = -x^{-\nu} J_{\nu+1}(x)$.

Ans. Recall: $\left[x^{\nu} J_\nu(x)\right]' = x^{\nu} J_{\nu-1}(x)$

(i)

and: $x^\nu J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \phi_{m,\nu} x^{2m+\nu}$

so: $x^{-\nu} J_\nu(x) = x^{-\nu} \sum_{m=0}^{\infty} \phi_{m,\nu} x^{2m+\nu} = \sum_{m=0}^{\infty} \phi_{m,\nu} x^{2m}$.

and: $\frac{\partial}{\partial x} \sum_{m=0}^{\infty} \phi_{m,\nu} x^{2m} = 2m \sum_{m=0}^{\infty} \phi_{m,\nu} x^{2m-1}$

add: $-x^{-\nu} \sum_{m=0}^{\infty} (2m) \phi_{m,\nu} x^{2m-1} = -2m \sum_{m=0}^{\infty} \phi_{m,\nu} x^{2m-1-\nu} = -2m \sum_{m}^{\infty} \phi x^{2m+(1+\nu)}$

$\equiv \boxed{-x^{-\nu} J_{\nu+1}}$ ✓

(ii) Quickie; Recall: $\phi_{m,\nu-1} = 2(m+\nu)\phi_{m,\nu}$

Then: $\phi_{m,\nu+1} = 2(m+\nu-\nu)\,\phi_{m,\nu}$

$\phi_{m,\nu+1} = 2m\,\phi_{m,\nu}$

so $\frac{\phi_{m,\nu+1}}{\phi_{m,\nu}} = 2\,\frac{\Gamma(m+\nu-1)}{\Gamma(m+\nu)} = 2m$ ✓

(9.) Show that: $\boxed{J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{x} J_\nu(x)}$

Ans: See that: $J_{\nu+1}(x) = -\frac{\partial_x\left(x^{-\nu} J_\nu(x)\right)}{x^{-\nu}}$

$J_{\nu-1}(x) = \frac{\partial_x\left(x^{\nu} J_\nu(x)\right)}{x^{\nu}}$

$\phi_{m,\nu}$

so: $J_{\nu-1} + J_{\nu+1} = \frac{\partial_x\left(x^{\nu} J_\nu\right)}{x^{\nu}} - \frac{\partial_x\left(x^{-\nu} J_{\nu+1}\right)}{x^{-\nu}}$

$x^{2\nu} x^{-\nu} = x^{\nu}$

$= \frac{\partial_x(x^\nu J_\nu)}{x^\nu} - \frac{x^{2\nu}\partial_x(x^{-\nu} J_\nu)}{x^\nu}$

$= \frac{\partial_x(x^\nu J_\nu) - x^{2\nu}\partial_x(x^{-\nu} J_\nu)}{x^\nu}$

$\frac{1}{1} - \frac{1}{2}$

$\frac{1\times 2}{1\times 2} - \frac{1}{2}$

$\frac{2-1}{2} = \frac{1}{2}\nu$

Fast:

$x^\nu J_{\nu-1} = \sum_m \phi_{m,\nu-1} x^{2m+2\nu-1}$

$\phi_{m,\nu-1} = 2(m+\nu)\,\phi_{m,\nu}$

$-x^\nu J_{\nu+1} = -\sum \phi_{m,\nu+1} x^{2m+1}$

$\phi_{m,\nu+1} = (2m+1)\phi_{m,\nu}$

$\frac{\phi_{m,\nu-1}}{\phi_{m,\nu+1}}$

$= \frac{\partial_x(x^\nu) J_\nu - \partial_x(x^{2\nu}) J_\nu + (\partial_x J_\nu)\, x^\nu + (\partial_x J_\nu)\, x^{-\nu}}{x^\nu}$

$= \frac{(x^\nu J_{\nu+1}) + x^{2\nu}\left(J_{\nu+1}\, x^{-2\nu}\right)}{x^\nu}$

$= \frac{x^\nu\left(J_{\nu-1} + J_{\nu+1}\right)}{x^\nu} =$

SHIFTING ON THE S-AXIS
SHIFTING ON THE t-AXIS W/ UNITSTEP

(6.3)

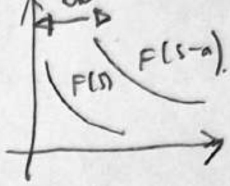
(1) TH:1 - 1st SHIFTING THEOREM:

- If $f(t)$ has transform $\forall s > \gamma$, $F(s)$,
  The $e^{at} f(t)$ has transform $F(s-a)$ w/ $s-a > \gamma$
- Thus if $\mathcal{L}\{e^{at} f(t)\} = F(s)$ then:

$$\boxed{\mathcal{L}\{e^{at} f(t)\} = F(s-a)}$$

- w/ Inverse:

$$\boxed{\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)}$$



[i] PROOF:

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\, dt = \int_0^\infty e^{-st}\left[e^{at} f(t)\right]$$

$$= \mathcal{L}\{e^{at} f(t)\}$$

[ii] EXAMPLE I

Small Body Mass $m=2$ attached at lower end of an elastic spring whose upper end is fixed, the spring Modulus being $\boxed{k=10}$. Let $\boxed{y(t)}$ be the static displacement from static equil. Determine free vibrations of the body w/ Initial velocity $\boxed{y'(0) = -4}$ Initial Position $\boxed{y(0)=2}$

ANS:

$$\ddot{y} + 2\dot{y} + 5y = 0 \quad \text{w/} \quad \begin{aligned} y(0) &= 2 \\ y'(0) &= -4 \end{aligned}$$

$$\mathcal{L}\{\ddot{y}\} + 2\mathcal{L}\{\dot{y}\} + 5\mathcal{L}\{y\} = s^2 Y - 2s + 4 + 2(sY - 2) + 5Y = 0$$

So:

$$Y(s) = \frac{2s}{(s+1)^2 + 2^2} = 2\frac{s+1}{(s+1)^2 + 2^2} - \frac{2}{(s+1)^2 + 2^2}$$

since $\mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} = \cos 2t$

$\mathcal{L}^{-1}\left\{\frac{2}{s^2+2^2}\right\} = \sin 2t$

$y(t) = \mathcal{L}^{-1}(Y) = e^{-t}(2\cos 2t - \sin 2t)$

so: $\boxed{y(t) = 2\cos 2t - \sin 2t}$

② Th:2 – 2^{nd} SHIFTING THEOREM ON THE t-AXIS

$\boxed{\forall \tilde{f}(t) = \begin{cases} 0 & t<a \\ f(t-a) & t>a \end{cases}}$ Is the transform iff $f(t)$ has transfcs

w/ Arbitrary $a \geqq 0$ has the transform $e^{-as}F(s)$

[i] PROVE:

$$e^{-as}F(s) = e^{-as}\int_0^\infty e^{-s\tau} f(\tau)\, d\tau$$

sub: $\tau + a = t$

$$= \int_0^\infty e^{-s(\tau+a)} f(\tau)\, d\tau$$

$$= \int_0^\infty e^{-st} f(t-a)$$

$$= \int_0^\infty e^{-st} f(t-a)\, u(t-a)\, dt = \mathcal{L}\{f(t-a)\, u(t-a)\}$$

$\boxed{e^{-as}F(s) = \mathcal{L}\{f(t-a)\, u(t-a)\}}$ w/ $u(t-a) = \begin{cases} 1 & t>a \\ 0 & t<a \end{cases}$

Heaviside function

$\boxed{\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}}$

w/ $\mathcal{L}\{u\} = \int 0\, e^{-st}\, dt + 1\int e^{-st} dt$

$= \frac{1}{s} e^{-st}\Big|_0^\infty = \boxed{\frac{e^{-as}}{s}}$

[ii] EXAMPLES

1) e^{-3s}/s^3 ANS: $\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2}$

$\boxed{\mathcal{L}^{-1}(e^{-3s}/s^3) = \frac{1}{2}(t-3)^2 u(t-3)}$

2.) $f(t) = \begin{cases} 1 \\ 0 \\ \sin t \end{cases}$ ANS: $f(t) = u(t) - u(t-\pi) + u(t-2\pi)\sin t$

since $\sin t = \sin(t-2\pi)$

so $\boxed{\mathcal{L}\{f\} = \frac{1}{s} - \frac{e^{-\pi s}}{s} + \frac{e^{-2\pi s}}{s^2+1}}$

PROBLEMS 8.3A

① $\boxed{\mathcal{L}\{0.4te^{2.5t}\}}$ $= 0.4\,\mathcal{L}\{te^{2.5t}\}$

$= 0.4\int_0^\infty te^{2.5t}e^{-st}\,dt = 0.4\int_0^\infty te^{(2.5-s)t}\,dt$

Ans

$= \boxed{0.4\frac{1}{(S-2.5)^2} = F(s-a)}$

Since by the 1st theorem: $F(s-a) : \mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$

② $\boxed{\mathcal{L}\{e^t \cos t\}}$ $= \int_0^\infty (e^t\cos t)e^{-st}\,dt$

Ans $= \int_0^\infty \cos t\, e^{t-st}\,dt = \int_0^\infty e^{t(1-s)}\cos t\,dt$

The table:

$f(t) = e^{at}\cos\omega t \Rightarrow \mathcal{L}\{f\} = \frac{s-a}{(s-a)^2+\omega^2}$

so: $\mathcal{L}\{e^t\cos t\} = \frac{S-1}{(S-1)^2+1^2}$ $\quad e^{at} = e^t \quad a=1$, $\cos\omega t = \cos t \quad \omega = 1$

$= \frac{S-1}{S^2-2S+1^2+1^2} = \boxed{\frac{S-1}{S^2+2S+2}}$

③ $\boxed{\mathcal{L}\{e^{-2t}\sin n\pi t\}}$

Ans: $F(S-a) \Rightarrow \mathcal{L}\{f = e^{+at}\sin\omega t\} = \frac{\omega}{(S-a)^2+\omega^2}$

Now: $\int e^{at}e^{-st}\,dt = \int e^{(a-s)t}\,dt = -\frac{1}{a-s} = \frac{1}{s-a}$

so: $\int e^{-at}e^{-st} = \int e^{-(at+st)} = \int e^{-(a+s)t} = -\frac{1}{a+s} = \frac{1}{s+a}$ ↓ still the same!

So: $\boxed{\mathcal{L}\{e^{-2t}\sin n\pi t\} = -\frac{n\pi}{(2+s)^2 + n^2\pi^2}}$

w/ $\omega = n\pi$, $a=2$

SYSTEMS of LINEAR EQUATIONS and

$\boxed{\text{GAUSS ELIMINATION}}$ — $\boxed{7.5}$

① INTRODUCTION

NON HOMOGENEOUS

• If we have a system of eqs as:

$$\left.\begin{aligned} \sum_{i,j} a_{1j} x_j &= b_1 \\ \sum_{i,j} a_{2j} x_j &= b_2 \\ \sum_{i,j} a_{3j} x_j &= b_3 \\ &\vdots \\ \sum_{i,j} a_{ij} x_{ij} &= b_i \end{aligned}\right\} = \vec{\Sigma} \begin{bmatrix} a_1 \\ \vdots \\ a_j \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_j \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_i \end{bmatrix} \equiv \boxed{\vec{\Sigma}\vec{a}\,\vec{x} = \vec{b}} \Rightarrow \boxed{S : \vec{\Sigma}\vec{a}\,\vec{x} = \vec{b}}$$

i, j

• Then if $\forall \vec{b} \neq 0$, then the system S is NON HOMOGENEOUS.

HOMOGENEOUS

• Assume the same $S : \vec{a}\,\vec{x} = \vec{b}$ w/ $\vec{a} = [a_1 \text{ --- } a_j]^T$, $\vec{x} = [x_1 \text{ --- } x_j]^T$, $\vec{b} = [b_1 \text{ --- } b_i]$, $\vec{\Sigma}$ = sum matrix.

• Then $\forall (\vec{b} = 0)$, S is Homogeneous

so: $\boxed{\begin{aligned} S\{b = 0\} &= \text{Homogeneous} \\ S\{b \neq 0\} &= \text{Non Homogeneous} \end{aligned}}$ $\Rightarrow \vec{A}\vec{x} = \vec{b} \rightarrow \tilde{\vec{A}} = \begin{bmatrix} a_{11} & & b_1 \\ & \ddots & \vdots \\ & & a_{mn} \; b_m \end{bmatrix}$ AUGMENTED MATRIX

GEOMETRIC INTERPRETATION - EXAMPLE 1

• If $m = n = 2$, then we have (x_1, x_2) cont.

$$\boxed{\begin{aligned} a_{11} \cdot x_1 + a_{12} \cdot x_2 &= b_1 \\ a_{21} \cdot x_1 + a_{22} x_2 &= b_2 \end{aligned}}$$

• Doing a little algebra we know that $\boxed{x_1 = \dfrac{b_1 - a_{12} x_2}{a_{11}}}$

$$a_{11} x_1 = b_1 - a_{12} x_2 \rightarrow$$

how: $a_{21} x_1 + a_{22} x_2 = a_{21}\left[\dfrac{(b_1 - a_{12} x_2)}{a_{11}}\right] + a_{22} x_2 = b_2$

$$\Rightarrow \frac{a_{21} b_1 - a_{21} a_{12} x_2 + a_{11} a_{22} x_2}{a_{11}} = b_2$$

• Here if $\forall b = 0$, then: $\quad x_2 a_{11} a_{22} - a_{21} a_{12} x_2 = 0$

$$-\frac{a_{21} a_{12} x_2 + a_{11} a_{22} x_2}{a_{11}} = 0 \rightarrow \quad x_2 a_{11} a_{22} = a_{21} a_{12} x_2 \Rightarrow \boxed{a_{11} a_{22} = a_{21} a_{12}}$$

• We can see that

$$a_{11} a_{22} - a_{21} a_{12} \equiv \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0$$

$$\text{or } |a_{1j} \times a_{2j}|_{j=2} = \det(a_{1j}, a_{2j})|_{j=2}$$

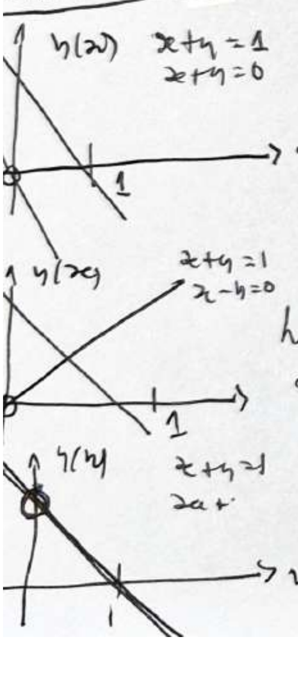
how:

$$\boxed{\begin{aligned} S_{HOM}\{j=2\} &: a_{11} a_{22} - a_{21} a_{12} = 0 \\ S_{NHOM}(j=2) &: \frac{a_{21} b_1 - a_{21} a_{12} x_2 + a_{22} a_{11} x_2}{a_{11}} = b_2 \end{aligned}}$$

Remember: if $\det(a_{1j}, a_{2j}) = 0$ then the 2 vectors are linearly dependent

• This means both lines go through the origin and lie on top of each other.

EQUATIONS



SERIES SOLUTIONS - POWER SERIES (4.1-4.2)

(1)

$$(3)\quad y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \dots$$

$$(4)\quad y' = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

$$(4)\quad y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} = 2a_2 + 3\cdot 2 a_3 x + \dots$$

Examples:

(i) $\boxed{y' - y = 0}$ → simply $\frac{dy}{dx} = y \Rightarrow \frac{dy}{y} = dx$ $\Rightarrow \boxed{y(x) = a_0 e^x}$

Sol: 1.) $y' - y = \sum_{m=1}^{\infty} m \cdot a_m x^{m-1} - \sum_{m=0}^{\infty} a_m x^m$

$$= \sum_{m=0,1}^{\infty} \left[m a_m x^{m-1} - a_m x^m \right]$$

$$= (a_1 - a_0) + (a_1 x + 2a_2 x) + (a_2 x^2 + 3a_3 x^2) + \dots$$

$$= (a_1 - a_0) + (a_1 + (-2a_2))x + (a_2 - 3a_3)x^2 + \dots$$

2.)
$$\begin{cases} a_1 - a_0 = 0 \\ a_1 + (-2a_2) = 0 \\ a_2 + (-3a_3) = 0 \end{cases} \Rightarrow \begin{cases} a_1 = a_0 \\ a_2 = \frac{1}{2}a_1 = \frac{a_0}{2!} \\ a_3 = \frac{1}{3}a_2 = \frac{a_0}{3!} \end{cases}$$ Same

So: $y = a_0 + a_0 x + \frac{a_0 x^2}{2!} + \frac{a_0 x^3}{3!}$

$$= a_0\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}\right) = a_0 e^x \quad \text{so } \boxed{y(x) = a_0 e^x}$$

(ii) $\boxed{y' = 2xy}$

$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$

$2a_0 x + 2a_1 x^2 + 2a_2 x^3$

↓

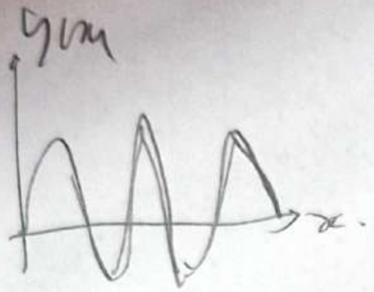
$a_2 = a_0$

1.) $y' - 2xy = \sum_{m=1} m \cdot a_m x^{m-1} - 2x \sum_m a_m x^m$

$$= (a_1 + 2a_2 x + 3a_3 x^2 + \dots) - (2a_0 x + 2a_1 x^2 + \dots$$

2.) $a_2 = a_0$, $a_4 = \frac{a_2}{2!} = \frac{a_0}{2!}$, $a_6 = \frac{a_4}{3} = \frac{a_0}{3!}$

$$y = a_0 + a_1 x - \frac{a_0}{2!}x^2 - \frac{a_1}{3!} + \frac{a_0}{4!}x^4$$



$$= a_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \dots\right) + a_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \dots\right)$$

$$= a_0\left(\cos(x)\right) + a_1\left(\sin(x)\right)$$

So: $\boxed{y(x) = a_0 \cos(x) + a_1 \sin(x)}$

(2) Theories

[i] BASIC

1.) $\sum_m^{\infty} a_m (x_m - x_0)^m \longrightarrow S_n(x) = \sum_{i=0}^{n} a_i (x - x_0)$

$$\boxed{R_n(x) = a_{n+1}(x - x_0)^{n+1} + \dots}$$

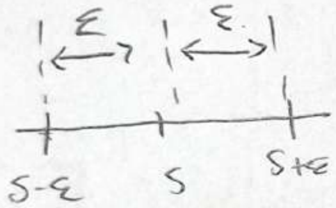
In (4), The Convergence at $x = x_1$ means we can make $R_n(x)$ as small as we please, by taking n large enough

$S_n(x)$ = Partial sum

$R_n(x)$ = Remainder

2.) Convergence: $\lim_{n \to \infty} S_n(x_1) = S(x_1) = \sum_{m=0}^{\infty} a_m (x_1 - x_0)^m$

Diverges : otherwise

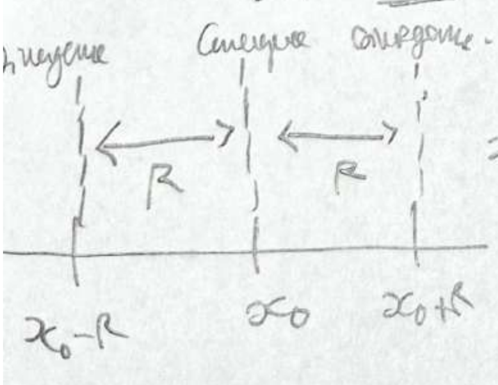


OR: $\boxed{|S_n - S| < \varepsilon \quad \forall n > N}$ (4)

3.) It follows: $\forall S_n (n > N)$; at $S-\varepsilon$ and $S+\varepsilon$, N will depend on:

$$\boxed{|S_n - S| = |R_n|}$$

[ii] RADIUS:



1.) The series Converges at $x = x_0$ because all its terms except the 1st (a_0) are zero.

2.) If There's further values of x which all series Converges, There's a range called the Convergence Interval:

$$\boxed{|x - x_0| < R}$$

3.) The Radius is basically by calculating either formula

$$\boxed{R = \frac{1}{\lim_{m \to \infty} \sqrt[m]{|a_m|}} \quad \text{or} \quad R = \frac{1}{\lim_{m \to \infty} \left|\frac{a_{m+1}}{a_m}\right|}}$$

PROBLEMS 4.1 (I)

(1) $\boxed{y' = 3y}$

Ans: w/ $\frac{dy}{dx} = 3y$ $\quad \frac{dy}{y} = 3\,dx$ $\quad \int_{y_0}^{y} \frac{dy}{y} = \int 3\,dx$

[i] $\ln y \Big|_{y_0}^{y} = 3x + C$

$\ln(y) - \ln(y_0) = 3x + C$

$\ln(y) = \ln(y_0) + 3x + C$

$\boxed{y(x) = y_0 e^{3x + C}}$

[ii] let: $y(x) = \sum_{n=0} a_n x^n$

so: $y'(x) = \sum_{n=1} a_n n x^{n-1} \Rightarrow \sum_{n=0} a_{n+1}(n+1) x^n$

1) so: $y' - 3y = \sum_{n=0} a_{n+1}(n+1) x^n - 3 \sum_{n=0} a_n x^n = 0$

$= \sum_{n=0} \left[a_{n+1}(n+1) - 3a_n \right] x^n = 0$

$= a_{n+1}(n+1) - 3a_n$

$\frac{2}{9} \cdot \frac{3}{2} =$

2.) $\boxed{a_n = \frac{a_{n+1}(n+1)}{3}}$ and $\boxed{a_{n+1} = \frac{3}{(n+1)} a_n}$ $\quad \boxed{a_1 = 3a_0}$

$a_0 = \frac{a_{0+1}(1)}{3} = \frac{a_1}{3}$

a) $a_1 = \frac{a_2(2)}{3} = \frac{2}{3} a_2$

$a_2 = \frac{a_3(3)}{3} = \frac{3}{3} a_3$

b) $a_3 = \frac{a_4(4)}{3} = \frac{4}{3} a_4$

$a_0 = \frac{a_1}{3} = \frac{2}{3}\frac{1}{3} a_2$

$= \frac{2}{9} a_2 = \frac{2}{9} a_3$

so: $\boxed{a_3 = \frac{9}{2} a_0}$

$\boxed{a_2 = \frac{3}{2} a_1}$

$a_3 = a_2 = \frac{3}{2} a_1$

$\frac{1}{2} a_1 = \frac{9}{2} a_0$

$3a_1 = 9a_0$

$3a_1 = 3^2 a_0$

$a_1 = 3a_0$ ✓

$\frac{4}{3} a_4 = \frac{9}{2} a_0$ so: $4a_4 = \frac{27}{2} a_0$

$\boxed{a_4 = \frac{27}{8} a_0}$

3.) so: $\boxed{a_n = \frac{3^n}{n!} a_0}$

and: $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{3^n}{n!} a_0 x^n = a_0 \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = a_0 e^{3x}$

so: $\boxed{y(x) = a_0 e^{3x}}$

(2) $\boxed{y' + 2y = 0}$

Ans. 1) $y'(x) + 2y(x) = \sum \left[(n+1)a_{n+1} + 2a_n \right] x^n = 0$

$= (n+1)a_{n+1} + 2a_n = 0$

so: $\boxed{a_{n+1} = \frac{2a_n}{n+1}}$

2.) $a_1 = 2a_0 \quad a_2 = \frac{2a_1}{2} \quad a_3 = \frac{2a_2}{3} \quad a_4 = \frac{2a_3}{4}$ ----

so: $a_0 = \frac{1}{2}a_1 \Rightarrow a_0 = \frac{1}{2}\frac{1}{2}a_2 = \frac{1}{2}a_2 = \frac{3}{2}a$

$a_1 = \frac{2}{2}a_2 \rightarrow a_1 = \frac{3}{2}a_3 = \frac{3}{2}a_3$

$a_{n+1} = \frac{a_n 2}{(n+1)}$ $\quad a_2 = \frac{3}{2}a_3 \rightarrow$

$$\boxed{a_n = \frac{(-2)^n}{n!}a_0}$$

so: $y' + 2y \rightarrow \boxed{y(x) = a_0 e^{-2x}}$

③ $\boxed{y' = ky}$ simple! , $y' - 3y$ or $y' - ky$

so: $y(x) = a_0 e^{3y}$ or $y(x) = a_0 e^{ky}$

so: $\boxed{y(x) = a_0 e^{ky}}$ $\quad y' = ka_0 e^{ky} = ky$

$\boxed{y' = ky}$ ✓

⑤ $\boxed{(1-x)y' = y}$

i) Direct method:

1.) $(1-x)\frac{dy}{dx} = y \Rightarrow \frac{dy}{dx} = \frac{y}{(1-x)}$ so: $\int \frac{dy}{y} = \int \frac{1}{(1-x)}$

2.) let $u = 1-x$ then $\frac{du}{dx} = \frac{d}{dx}(1) - \frac{d}{dx}(x) = 0 - 1 = -1$

so: $\frac{du}{dx} = -1 \rightarrow du = -dx \Rightarrow \boxed{dx = -du}$

3.) Then: $\ln y = \int \frac{1}{(1-x)} = -\int \frac{1}{u}du = -\ln u = -\ln(1-x)$

so: $\boxed{\ln y = -\ln(1-x)}$

4.) Take exponentials: $e^{\ln(y)} = e^{-\ln(u)} \Rightarrow \boxed{y = \frac{1}{1-x}}$

$\boxed{y(x) = \frac{a_0}{1-x}}$

ii) SERIES:

1.) $y(x) = \sum_{n=0}^{\infty} a_n x^n$

$y'(x) = \sum_{n=1}^{\infty} a_n n x^{n-1}$

2.) So: $(1-x)\frac{dy}{dx} - y = \sum_{n=1}^{\infty} na_n x^{n-1}(1-x) - \sum_{n=0}^{\infty} a_n x^n = 0$

$= \boxed{\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=1}^{\infty} na_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.}$

3.) shift $n-1 \rightarrow n$ so $a_n \rightarrow a_{n+1}$ s.t.:

$\stackrel{\Delta}{=} \sum_{n=0}^{\infty}(n+1)a_{n+1}\cancel{x^n} - \sum_{n=1}^{\infty} na_n \cancel{x^n} - \sum_{n=0}^{\infty} a_n \cancel{x^n} = 0$

$= (n+1)a_{n+1} - na_n - a_n = \cancel{(n+1)}a_{n+1} - a_n\cancel{(n+1)} = 0$

$\stackrel{\Delta}{=} \boxed{a_{n+1} = a_n}$

$a_{n+1} = a_n = \boxed{a_0}$

$a_0 = a_1$
$a_1 = a_2$
$a_3 = a_4$
$a_5 = a_6$

$a_0 = a_1 = a_5 = \cdots a_{n+1}$

$\boxed{a_0}$

4.) So: $y(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_0 x^n$

$= \frac{a_0}{1-x}$

PROBLEMS 4.1 (1)

(8) $\boxed{(1-x^2)y' = y}$

ans: $(1-x^2)y' = y' - x^2y' = y$

so $\boxed{y' - x^2y' - y = 0}$

(I) Direct:

1.) $\boxed{\frac{dy}{dx} = \frac{y}{(1-x^2)}}$

So: $\int \frac{dy}{y} = \int \frac{1}{(1-x^2)} = \int \frac{1}{2(1-x)}dx + \int \frac{1}{2(1+x)}dx$

let $u = 1-x$, $v = 1+x$

So: $\boxed{\ln(y) = \int \frac{1}{2(1-x)}dx + \int \frac{1}{2(1+x)}dx}$

Then $du = -dx$, $dv = dx$

Then $dx = -du$, $dx = dv$ $\longrightarrow$ 2.) $\int \frac{1}{2(1-x)}dx + \int \frac{1}{2(1+x)}dx = \frac{1}{2}\left[-\int \frac{1}{u}du + \int \frac{1}{v}dv\right]$

$= \frac{1}{2}\left[\int \frac{1}{v}dv - \int \frac{1}{u}du\right] = \frac{1}{2}\left[\ln v - \ln u\right] = \frac{1}{2}\ln\left(\frac{v}{u}\right).$

$= \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|$ so $\boxed{\ln(y) = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|}$

3.) Then: $e^{\ln y} = e^{\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|}$

$\therefore \boxed{y(x) = C\sqrt{\frac{1+x}{1-x}}}$ w/ $C = $ Const.

(II) Series:

1.) $(1-x^2)\sum_{n=0}^{\infty} a_n n x^{n-1} = \sum_{n=0}^{\infty} a_n x^n$

$= \sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n-1+2} - \sum_{n=0}^{\infty} a_n x^n = 0$

$= \sum_{n=0}^{\infty} a_n n x^{n-1} - \sum_{n=0}^{\infty} a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$

$= \sum_{n=0}^{\infty} a_{n+1}(n+1) x^{n} - \sum_{n=0}^{\infty} a_{n-1}(n-1) x^n - \sum_{n=0}^{\infty} a_n x^n = 0$

$= \boxed{(n+1)a_{n+1} - a_{n-1}(n-1) - a_n = 0}$

$\frac{1}{2}a_3 = 2a_4 - a_2$

2.) $\boxed{a_{n+1} = \frac{a_{n-1}(n-1) + a_n}{(n+1)}}$ or $a_n = (n+1)a_{n+1} - a_{n-1}(n-1)$

$\frac{1}{2}a_3 = 2a_4 - 3a_3 - a_0$

$\frac{1}{2}a_3 + 3a_3 = 2a_4 - a_0$

$\frac{7}{2}a_3 = 2a_4 - a_0 \ldots$

So: $a_0 = a_1$ $a_1 = a_0$. $a_2 = 3a_3 - a_1 = 3a_3 - a_0$

$a_3 = 4a_4 - 2a_2 - a_1 - a_2 = \frac{a_2}{2}$ $a_3 = \frac{a_0}{2}$

so: $\boxed{y(x) = C\sqrt{\frac{1+x}{1-x}} = \sum a_n x^n}$

PROBLEMS 4.2

Radius of Convergence : Find It!

(17) $\boxed{\sum_{m=0}^{\infty} \frac{x^m}{3^m}}$: $\lim_{m\to\infty} \left|\frac{a_{m+1}}{a_m}\right| = \frac{\frac{1}{3^{m+1}}}{\frac{1}{3^m}} = \frac{3^m}{3^{m+1}} = 3^{m-m+1} = \boxed{\frac{1}{3}}$

$\sum_{m=0}^{\infty} \frac{x^m}{3^m} = \frac{x^0}{3^0} + \frac{x^1}{3} + \frac{x^2}{9} + \frac{x^3}{27} + \ldots$

hence $R = \frac{1}{\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|} = \frac{1}{\frac{1}{3}} = \boxed{3}$ so

so : $\boxed{R = 3 \text{ hence Converges for } |t| < 3}$

(19) $\boxed{\sum_{m=0}^{\infty} (-1)^m x^{2m}}$: $\lim_{m\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{m+1}}{(-1)^m}\right| = \left|(-1)^{m-m+1}\right| = \left|(-1)^1\right|$

$= |-1|$

$(-1)^0 x^0 + x^1 + x^2 - x^3$

so $R = \frac{1}{\lim\left|\frac{a_{n+1}}{a_n}\right|} = \frac{1}{1} = \boxed{1}$

so : $\boxed{R = 1 \text{ hence converges for } |t| < 1}$

(21) $\boxed{\sum_{m=2}^{\infty} \frac{m(m-1)}{3^m} x^m}$ $\rightarrow$ The only Thing Matters is 3^m. Since Its : $\frac{3^{m+1}}{3^m} = 3^{m-m+1} = \boxed{3}$ ✓

Then Its also $\boxed{R = 3}$ as (17)

(23) $\boxed{\sum_{m=0}^{\infty} \frac{(-1)^m}{k^m} x^{2m}}$: $\lim_{m\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{m\to\infty}\left|\frac{(-1)^{m+1}/k^{m+1}}{(-1)^m/k^m}\right|$

$= \left|\frac{(-1)^{m+1}}{k^{m+1}} \cdot \frac{(k)^m}{(-1)^m}\right| = \left|(-1)^{m+1-m} \cdot k^{m-m-1}\right| = \left|(-1) \cdot k^{-1}\right| = \frac{1}{|k|}$

so $R = \frac{1}{\lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right|} = \frac{1}{\frac{1}{|k|}} = |k|$.

For Absolute Convergence $\Rightarrow$ $\left|\frac{x^{2m}}{k^m}\right| < 1 \rightarrow |x^2| < |k|$ $\boxed{|x| < \sqrt{|k|}}$

or Root test

$\frac{1}{R} = \lim_{m\to\infty} \sup \left|\frac{1}{k^m}\right|^{\frac{1}{2m}}$

$= \lim_{k\to\infty} \sup \left|\frac{1}{k}\right|^{\frac{1}{2}}$

$= \boxed{\frac{1}{\sqrt{|k|}}}$

agrees

so $\boxed{R = \sqrt{|k|}}$

LEGENDRE POLYNOMIALS

43

1) $$\boxed{(1-x^2)y'' - 2xy' + n(n+1)y = 0}$$

Legendre's differential eqs.

i) Solve using: $\boxed{y(x) = \sum_{m=0}^{\infty} a_m x^m}$

1.

$k = n(n+1)$

So: $(1-x^2)y'' - 2xy' + n(n+1)y$

$$= (1-x^2)\left(\sum_{m=2}^{\infty} m(m-1)x^{m-2}\right) - 2x\left(\sum_{m=1}^{\infty} m a_m x^{m-1}\right) + k\left(\sum_{m=0}^{\infty} a_m x^m\right) = 0$$

$$= (1-x^2)\left[2a_2 + 3\cdot 2 a_3 x + 4\cdot 3 a_4 x^2\right]$$
$$- 2x\left[a_1 + 2a_2 x + 3a_3 x^2\right]$$
$$k\left[a_0 + a_1 x + a_2 x^2 + \dots\right] +$$

So: $2a_2(1-x^2) - 2x a_1 - n(n+1)a_0 = 0$

$6a_2(1-x^2) - 4xa_2 - n(n+1)a_1 x = 0$

- We can manipulate the 1st term from the expression so that:

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m$$

so that: $$\begin{cases} 2a_2 + n(n+1)a_0 = 0 \\ 6a_3 + [-2+(n(n+1))]a_1 = 0 \end{cases}$$

2) Continuing on; when $s = 2, 3$

$$(s+2)(s+1)a_{s+2} + [-s(s-1) - 2s + n(n+1)]a_s = 0$$

$$a_{s+2} + \frac{[-s(s-1) - 2s + n(n+1)]}{(s+2)(s+1)}a_s = 0$$

$$\boxed{a_{s+2} = \frac{(n-s)(n+s+1)}{(s+2)(s+1)}a_s} \quad \forall s = 0, 1, \dots$$

Recurrence relation

$a_2 = -\frac{n(n+1)}{2!}a_0$

$a_4 = -\frac{(n-2)(n+3)}{4\cdot 3}a_2$

$a_3 = \frac{(n-1)(n+2)}{3!}a_1$

$a_5 = \frac{(n-3)n+4}{5!}a_3$

3) So: $\boxed{y(x) = a_0 y_1(x) + a_1 y_2(x)}$

The solution that we plug in these stuff and: $y_1(x) = 1 - \frac{n(n+1)}{2!} + \frac{(n-2)n(n+1)(n+3)}{4!} + \cdots$

$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 + \cdots$

so:
$$\boxed{\begin{aligned} y_1(x) &= \sum_{k=0}^{n}\left[(-1)^{2k} x^{2k} a_{2k}\right] \to n = \cancel{\text{odd}}\ \text{even} \\ y_2(x) &= \sum_{k=0}^{n}\left[(-1)^{2k+1} x^{2k+1} a_{2k+1}\right] \to n = \text{odd.} \end{aligned}}$$

(2)

$$a_s = -\frac{(s+2)(s+1)}{(n-s)(n+s+1)} a_{s+2} \qquad \forall s \leq n-2$$

$$a_n = \frac{(2n)!}{2^n (n!)^2} = \frac{1\cdot 3\cdot 5\cdot \ldots (2n-1)}{n!}$$

$$a_{n-2} = \frac{n(n-1)}{2(2n-1)} a_n \to \frac{n(n-1)}{2(2n-1)} \frac{2n!}{2^n (n!)^2}$$

$$= -\frac{n(n-1)\,2n(2n-1)(2n-2)!}{2(2n-1)\,2^n n(n-1)!\,n(n-1)(n-2)!}$$

$$= \boxed{-\frac{(2n-2)!}{2^n(n-1)!(n-2)!}}$$

$a_{n-2} = -\frac{2n-2!}{2^n(n-1)!(n-2)!}$

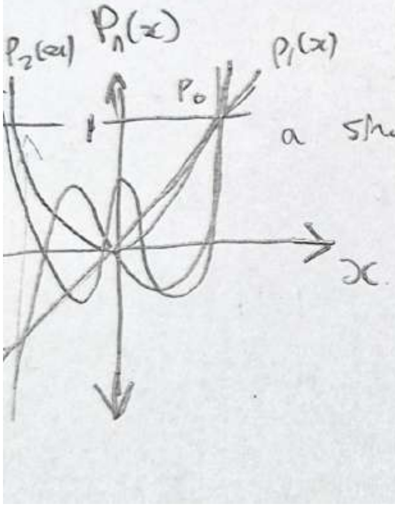
$a_{n-4} = -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2}$

$\vdots$

$a_{n-2m} = (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!}$

so:
$$\boxed{P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m}}$$

$M: \begin{cases} \frac{n}{2} \\ \frac{(n-1)}{2}. \end{cases}$



a short table:

even	odd
$P_0(x) = 1$	$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2-1)$	$P_3(x) = \frac{1}{2}(5x^3-3x)$
$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$	$P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$

PROBLEMS 4.3

quickie, Routines

Quickie

(4.) Show that we can obtain the series more quickly from the DE. by subbing $m-2=s$, $m=s$ in others.

Ans: See that:

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x\sum_{m=1}^{\infty} m a_m x^{m-1} + k\sum_{m=0}^{\infty} a_m x^m = 0$$

w/
$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1}$$
$$y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

[i] and: $m-2=s \rightarrow m-1=s+1$

$m = s+2$

so: $\boxed{y'' \rightarrow \sum_{s=0}^{\infty}(s+1)(s+2)a_{s+2}x^s}$

[ii] and for y': $m-1=s$, so $m=s+1$

$\boxed{y' \rightarrow \sum_{s=0}^{\infty}(s+1)s\, a_s x^{s}}$

[iii] The 1st term is:

$$(1-x^2)y'' = y'' - x^2y''$$
$$= \sum_{s=0}^{\infty}(s+1)(s+2)a_{s+2}\left[x^s - x^{s+2}\right]$$
$$= \sum_{s=0}^{\infty}(s+1)(s+2)a_{s+2}\left[x^s - x^{m-2+2}\right]$$
$$= \sum_{s=0}^{\infty}(s+1)(s+2)a_{s+2}\left[x^s - x^m\right]$$

so: $(1-x^2)y'' = \sum_{s=0}^{\infty}(s+1)(s+2)a_{s+2}\left[x^s - x^m\right]$

$$\boxed{=\sum_{s=0}^{\infty}(s+1)(s+2)a_{s+2}x^s + \sum_{m=0}^{\infty} m(m-1)x^m}$$

[iv] The 2nd term:

$$-2xy' = -2x\left\{\sum_{s=0}^{\infty} s(s+1)a_s x^s\right\}$$
$$= -2\sum_{s=0}^{\infty} s(s+1)a_s x^{s+1} = -2\sum_{s=0}^{\infty} s(s+1)a_s x^{m-1+1}$$
$$= -2\sum_{s=0}^{\infty} s(s+1)a_s x^{m}$$

so: $-2xy'$ is:

$$\boxed{\sum_{s=0}^{\infty} 2(s+1)a_{s+1}x^s}$$

[11] The 3rd term stays so:

$0 = \sum_{s=0}^{\infty}(s+2)(s+1)a_{s+2}x^{s} - \sum_{m=2}^{\infty} a_m x^m - \sum_{s=0}^{\infty} 2(s+1)a_{s+1}x^{s} + \sum_{m=0}^{\infty} n(n+1)a_m x^m$ It represents the same term

$$0 = \underbrace{\sum_{s=0}^{\infty}(s+2)(s+1)a_{s+2}x^{s}}_{1^{st}\text{ term}} \underbrace{- \sum_{m=2}^{\infty} m(m-1)a_m x^m}_{\text{Drop this one}} \underbrace{- \sum_{s=0}^{\infty} 2(s+1)a_{s+1}x^{s}}_{2^{nd}\text{ term}} \underbrace{+ \sum_{m=0}^{\infty} n(n+1)a_m x^m}_{3^{rd}\text{ one}}$$

$$\rightarrow \boxed{0 = \sum_{s=0}^{\infty}\left[(s+2)(s+1)a_{s+2} - 2(s+1)a_{s+1} + n(n+1)\right]x^{s}}$$

$m(m-1) = m^2 - m \quad (n^2-m)(n^2-n)$

$n(n+1) = n^2 + n \quad m^2n^2 - 2m^2n + mn$

(12) Rodrigues Formula

Proof: $\boxed{P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left[(x^2-1)^n\right]}$ using The Legendre Polynomials, Binomial theorem on $(x^2-1)^n$ and differentiating n-times.

Ans:

$(x+y)^n = \binom{n}{k} x^{n-k} y^k \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!}$

$(x^2-1)^n = \binom{n}{k} x^{2(n-k)} y^{-k}$

$\frac{d}{dx}(x^2-1)^n \xrightarrow{m=0} u = x^2-1 \rightarrow \frac{d}{du}u^n = nu^{n-1} = n(x^2-1)^{n-1}$

$\frac{d^j}{dx^j}(x^2-1)^{n-j} = (n-j)(x^2-1)^{n-j} = \frac{d^j}{dx^j}\binom{n-j}{k} x^{2(n-k-j)} y^{k}$

$= \binom{n}{k}\frac{(2((n-k)))!}{(2(n-k)-n)!} x^{2(n-k-j)} y^{k}$

$= \sum_{k=0}^{n}\binom{n}{k}\frac{(2((n-k)))!}{(2(n-k)-n)!k!} x^{2(n-k)-n}(-1)^k \quad y=-1$

So: $\sum_{k=0}^{n}\binom{n}{k}\frac{[2(n-k)]!}{[2(n-k)-n]!} x^{2(n-k)-n}(-1)^k \cdot \frac{1}{2^n n!} = P_n(x)$

and $\boxed{P_n(x) = \frac{1}{2^n n!}\left[(x^2-1)^n\right]\frac{\partial^n}{\partial x^n}}$

PROBLEMS 4.3

RODRIGUEZ FORMULA Integral ~~~

(13) RODRIGUEZ FORMULA SUM

Using the Rodriguez formula and Integrate n-times. Show that

$$\boxed{\int_{-1} P_n^2(x) = \frac{2}{2n+1}}$$

Ans:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)\right]$$

$$P_n^2(x) = \left(\frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]\right)\left(\frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right]\right)$$

$$= \frac{1}{2^n n!}\frac{1}{2^n n!} \cdot \frac{d^n}{dx^n}\left((x^2-1)^n\right) \cdot \frac{d^n}{dx^n}\left((x^2-1)^n\right)$$

$$= \frac{1}{2^{2n} n!^2} \cdot \left(\frac{d^n}{dx^n}(x^2-1)^n\right)^2$$

so: $\displaystyle\int_{-1}^{1} P_n^2(x)\, dx = \frac{1}{2^n n!^2}\int_{-1}^{1}\left(\frac{d^n}{dx^n}\left((x^2-1)^n\right)\right)^2 dx \longrightarrow$ wtf?

This is to complicated! Let's take a couple of results of the legendre:

$$P_0(x) = 1 \rightarrow P_0^2(x) = 1 \rightarrow \int_{-1}^{1} P_0^2(x) = x\Big|_{-1}^{1} = \boxed{2} \checkmark$$

$$P_1(x) = x \rightarrow P_1^2(x) = x^2 \rightarrow \int_{-1}^{1} P_1^2(x) = \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{1}{3}(1+1) = \boxed{\frac{2}{3}} \checkmark$$

$$P_2(x) = \frac{1}{2}(3x^2-1) \rightarrow P_2^2(x) = \frac{1}{4}(9x^4-6x^2+1)$$

$$\int_{-1}^{1} P_2^2(x) = \frac{1}{4}\int_{-1}^{1} 9x^4-6x^2+1 = \frac{1}{4}\left[\frac{9}{5}x^5 - \frac{6}{3}x^3 + x\right]_{-1}^{1}$$

$$= \frac{1}{4}\left(\frac{9}{5}(1+1) - 2(1+1) + (1+1)\right) = \frac{1}{4}\left(\frac{18}{5} - 4 + 2\right) = \frac{1}{4}\left(\frac{18-20+10}{5}\right)$$

$$= \frac{1}{4}\left(\frac{8}{5}\right) = \left(\boxed{\frac{8}{5}} \times \frac{1}{4}\right) = \frac{8}{20} = \boxed{\frac{2}{5}} \checkmark$$

so $\left\{\int P_0^2(x), \int P_1^2(x), \int P_2^2(x)\right\} = \left(2, \frac{2}{3}, \frac{2}{5}\right)$

Obtain That: $\left\{ 2 = \frac{2}{2(0)+1},\ \frac{2}{3} = \frac{2}{2(1)+1},\ \frac{2}{5} = \frac{2}{2(2)+1} = \frac{2}{5} \right\}$

so basically:

$$\boxed{\int_{-1}^{1} P_n(x)\, dx = \frac{2}{2n+1}}$$

(14) GENERATING FUNCTION

Show that: $\boxed{\frac{1}{\sqrt{1-2xu+u^2}} = \sum_{n=0}^{\infty} P_n(x)\, u^n}$ QED?

Ans: start w/ binomial expansion of $\frac{1}{\sqrt{1-v}}$ w/ $v = 2x - u^2$

so: $\frac{1}{\sqrt{1-v}} = (1-v)^{-\frac{1}{2}} \Leftrightarrow \sum_{k=1}^{\infty} \binom{n}{k} x^{n-k} y^k$

so: $(1-v)^{n=\frac{1}{2}} = \sum_{k=1} \binom{-\frac{1}{2}}{k} 1^{n-k} v^k = \sum_{k=1} \frac{(-\frac{1}{2})!}{k!(-\frac{1}{2}-k)!} 1^{n-k} v^k.$

$\sim 1 + \frac{1}{2}v \pm \frac{-\frac{1}{2}\cdot-\frac{3}{2}}{2!} v^2 \pm \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!} v^3 \pm \ldots$

$= 1 \pm \frac{1}{2}(2x-u^2) \pm \frac{3}{8}(2x-u^2)^2 \pm \frac{5}{16}(2x-u^2)^3$

$\equiv \sum_{n=0}^{\infty} \frac{(\lambda+n-1)!}{n!(\lambda-1)!} v^n$ wh/ $\lambda = \frac{1}{2}$

$\lambda + n - 1 = \frac{1}{2} + n - 1 = \frac{1}{2} + \frac{2n}{2} - \frac{2}{2} = \frac{1+2n-2}{2} = \frac{2n-1}{2}$

$\lambda - 1 = \frac{1}{2} - 1 = \frac{1-2}{2} = -\frac{1}{2}$

$(1-v)^{\lambda} = \sum_{n=0}^{\infty} \frac{(2n-1)!}{n!\cdot 2^n} v^n = \sum_{n=0} C_n v^n$ w/ $C_n = \frac{(2n-1)!}{n!\,2^n}$

and subbing $v = 2xu - u^2$ we get:

$$\boxed{(1-v)^{\lambda} = \sum_{n=0}^{\infty} C_n (2xu - u^2)^n}$$

Now the v term is expandable via the binomial theorem:

$((2xu - u^2)^n = \sum_{n=1}^{\infty} \binom{n}{k} (2xu)^{n-k} (u)^{2k} = \sum_{n=1} \binom{n}{k} (2x)^{n-k} (-1)^k u^{n+k}.$

so: $(1-v)^{\lambda} = \sum_{n=0}^{\infty} C_n \left\{ \sum_{n=1}^{\infty} \binom{n}{k} (2x)^{n-k} (-1)^k u^{n+k} \right\}$ Let $k = m-n$ so:

$= \sum_{m=0}^{\infty} C_n \left\{ \sum_{n=0}^{\infty} \binom{n}{m-n} 2x^{n-m-n} (-1)^{m-n} u^{n+m-n} \right\}$

$= \sum_{n=0}^{\infty} C_n \binom{n}{m-n} 2x^{-m} (-1)^{m-n} u^m = \sum_{m=0}^{\infty} u^m \cdot \sum_{n=0}^{\infty} C_n \binom{n}{m-n} 2x^{-m} (-1)^{m-n}.$

$= \sum_{n=0}^{\infty} u^m \cdot \sum_{m=0}^{\infty} \frac{(2n-1)!}{n!\,2^n} \cdot \frac{n!}{(m-n)!(m-2n)!} 2x^{-m} (-1)^{m-n}$

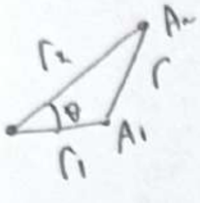
$= \boxed{\sum_{n=0}^{\infty} P_n(x)\, u^n}$

PROBLEMS 4.3

(15) POTENTIAL theory & Right angle triangle

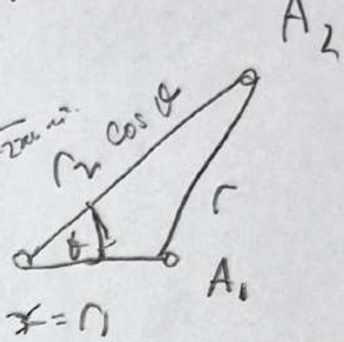
Let A_1, A_2 be 2 points in space s.t $r_2 > 0$ (figure below).
Show that

$$\boxed{\frac{1}{r} = \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\theta}} = \frac{1}{r_2}\sum_{m=0}^{\infty} P_m(\cos\theta)\left(\frac{r_1}{r_2}\right)^m.}$$



using: $\dfrac{1}{\sqrt{1-2xu+u^2}} = \sum_{n=0}^{\infty} P_n(x)\,u^n.$

ANS: we proofed $\dfrac{1}{\sqrt{1-2xu+u^2}} = \sum_{n=0}^{\infty} P_n(x)\,u^n$



by using $v = 2xu - u^2$ so $\dfrac{1}{\sqrt{1-v}} = \dfrac{1}{\sqrt{1-2xu+u^2}}$

$x = r_1$

Let: $2xu = 2(r_1)(r_2\cos\theta)$

$u^2 = r^2 = r_1^2 + r_2^2$

so: $v = 2r_1r_2\cos\theta - (r_1^2 + r_2^2).$

so: $\dfrac{1}{\sqrt{1-v}} = \dfrac{1}{\sqrt{1-2r_1r_2\cos\theta - r_1^2 + r_2^2}} = \left(\sqrt{1-2r_1r_2\cos\theta - r_1^2 + r_2^2}\right)^{-1}$

and: $$\boxed{\sum_{n=0}^{\infty} C_n v^n = \sum_{n=0}^{\infty} C_n\left(-2r_1r_2\cos\theta + r_1^2 + r_2^2\right)^n}$$

so: $\sum_{n=0}^{\infty} C_n \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{m}(-1)^m\left(2r_1r_2\cos\theta\right)^m\left(r_1^2 + r_2^2\right)^{-m}.$

$\frac{1}{r_2}\sum^{\infty}\left(\frac{r_1}{r_2}\right)^m\sum C_n\binom{n-m}{m}\left(\frac{r_1}{r_2}\cos\theta\right)^m$

$= \sum_{m=0}^{\infty}\left(r_1^2 + r_2^2\right)\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{m} C_n(-1)^n\left(2r_1r_2\cos\theta\right)^n.$

Now: $\left[r_1^2 + r_2^2 + 2r_1r_2\cos\theta\right]^{-\frac{1}{2}} = \left[r_2^2\left(1 + \left(\frac{r_1}{r_2}\right)^2 - 2\frac{r_1}{r_2}\cos\theta\right)\right]^{\frac{1}{2}}$

$= \left[r_2\sqrt{1 + \frac{r_1}{r_2} - \frac{2r_1}{r_2}\cos\theta}\right]^{-1}$ and $2xu$ $2\left(\left(\frac{r_1}{r_2}\right)^2 + 2\frac{r_1}{r_2}\cos\theta\right)\cdot\left(r_1^2 + r_2^2\right)$.

so: $$\boxed{\frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\theta}} = \frac{1}{r_2}\sum_{m=0}^{\infty} P_m(\cos\theta)\left(\frac{r_1}{r_2}\right)^m}$$

INDICIAL EQS (4.5)

(1)

$$\boxed{y'' + p(x)\, y' + q(x)\, y = 0}$$

Power series method in Singular / General form

- singular point: Point x for wh/ p & q isn't analytic
- Regular Points = opposite:

$$\Rightarrow \boxed{y'' + \frac{1}{x} y' + \left(1 - \frac{v^2}{x^2}\right) y = 0}$$

Bessel eqs

(2) Frobenius method : $\boxed{Th-1}$!

Any differential eqs of the form:

$$\boxed{y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0}$$

wh/ $b(x)$, $c(x)$ are analytic at $x=0$, at least one solution can be found as the form:

$$\boxed{y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r \left(a_0 + a_1 x + a_2 x^2 + \dots\right)}$$

wh/ exponent $r \in (\mathbb{R}$ or $\mathbb{C})$, r is chosen st $a_0 \neq 0$.

(3) To solve the DE. we have:

$$\boxed{x^2 y'' + x b(x) y' + c(x) y = 0}$$

So:

$$b(x) = b_0 + b_1 x + b_2 x^2 + \dots$$

$$c(x) = c_0 + c_1 x + c_2 x^2 + \dots$$

differentiate $y(x)$:

$$y'(x) = \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} = x^{r-1}\left[r a_0 + (r+1) a_1 x + \dots\right.$$

$$y''(x) = \sum_{m=0}^{\infty} (m+r)(m+r-1) x^{m+r-2} = x^{r-2}\left[r(r-1) a_0 + (r+1) r a_1 x\right]$$

So:

$$x^2\left[\sum_{m=0}^{\infty} (m+r)(m+r-1) x^{m+r-2}\right] + x\left[\sum_{m=0} b_m x^m\right]\left[\sum_{m=0} (m+r) a_m x^{m+r-1}\right]$$

$$+ \left[\sum_{m=0}^{\infty} c_m x^m\right]\left[\sum_{m=0}^{\infty} x^r x^m a_m\right]$$

$$= x^r[r(r-1)a_0 + \cdots] + (b_0 + b_1 x + \cdots) x^r (r a_0 + \cdots) + (c_0 + c_1 x + \cdots) x^r (a_0 + a_1 x + \cdots) = 0$$

use the lowest terms?

$$= (r(r-1) + b_0 r + c_0) a_0 = \quad \text{w/ } a_0 > 0$$

$$= r(r-1) + b_0 r + c_0 = \boxed{r^2 + (b_0 - 1) r + c_0 = 0}$$

1st Solution Problems

(3) 2^{nd} solution $\boxed{\text{Thm}}$

Distinct roots differing by an Integer.

[i] Case I - Distinct roots differing by an Integer

$$\boxed{\begin{aligned} y_1(x) &= x^{r_1}[a_0 + a_1 x + a_2 x^2 + \cdots] \\ y_2(x) &= x^{r_2}[A_0 + A_1 x + A_2 x^2 + \cdots] \end{aligned}}$$

$r = r_1$ so $r_1 \neq r_2$

$r = r_2$

w/ coefficients successively from before w/ $r = r_1$, $r = r_2$

[ii] Case II - $(r_1 = r_2 = r)$ or Double Root

$$\boxed{\begin{aligned} y_1(x) &= x^r (a_0 + a_1 x + a_2 x^2 + \cdots) \\ y_2(x) &= y_1(x) \ln x + x^r (A_1 x + A_2 x^2 + \cdots) \end{aligned}}$$

$r = \frac{1}{2}(1 - b_0)$

$x > 0$

[iii] CASE III - Roots differing by Integer:

$$\boxed{\begin{aligned} y_1(x) &= x^{r_1}(a_0 + a_1 x + a_2 x^2 + \cdots) \\ y_2(x) &= k y_1(x) \ln x + x^{r_2}(A_0 + A_1 x + A_2 x^2 + \cdots) \end{aligned}}$$

with $r_1 - r_2 > 0$

Typical Applications of Indicial eqs

① Euler-Cauchy ODEs :- Ex 1

$b_0, c_0 =$ Cons.

$$\boxed{x^2 y'' + b_0 x y' + c_0 y = 0}$$

↳ this satisfies the assumptions of the Frobenius method

$\Rightarrow$ Let $y = x^r$, Then:

$$y' = r x^{r-1} \quad y'' = r(r-1) x^{r-2}$$

$$x^2 \{ r(r-1) x^{r-2} \} + b_0 \{ r x^{r-1} \} x + c_0 x^r = 0$$

$$r(r-1) x^r + b_0 r x^r + c_0 x^r = 0$$

$$(r(r-1) + b_0 r + c_0) x^r = 0$$

$$r(r-1) + b_0 r + c_0 = 0$$

$$r^2 - r + b_0 r + c_0 = \boxed{r^2 - r(1+b_0) + c_0 = 0}$$

Sol: (i) $y_1 = x^{r_1}$, $y_2 = x^{r_2}$ iff $r_1 \neq r_2$

(ii) $y_1 = x^r$, $y_2 = x^r \ln x$ iff $r_1 = r_2 = r$.

② Eqs leads to case 1

$$\boxed{y'' + \frac{1}{2x} y' + \frac{1}{4x} y = 0}$$

↳ satisfies Th1, Th2

$\Rightarrow 4x y'' + 2y' + y = 0$

$$\Rightarrow 4 \sum_{m=0}^{\infty} (m+r)(m+r-1) a_m x^{m+r-1} + 2 \sum_{m=0}^{\infty} (m+r) a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0$$

$r_1 = \frac{1}{2}$

$r_2 = 0$

$$\Rightarrow 4r(r-1) x^{r-1} + 4(r+1) x^r + 4(r+2)(r+1) x^{r+1} = 0$$

$$2r x^{r-1} + 2(r+1) x^r + 2(r+2) x^{r+1} = 0$$

$$x^r + a_1 x^{r+1}$$

(16) $u(x,b) = \sum_{n=1}^{\infty} \left(A_n^* \sinh \frac{n\pi b}{a}\right) \sin \frac{n\pi x}{a}$.

* $b_n = A_n^* \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} \, dx$.

$$\boxed{\begin{aligned} u(x,y) &= \sum_{n=1}^{\infty} A_n^* \sin \frac{n\pi x}{a} \sinh \frac{n\pi y}{a} \\ \text{w/ } A_n^* &= \frac{2}{a \sinh\left(\frac{n\pi b}{a}\right)} \int_0^a f(x) \frac{n\pi x}{a} \, dx \end{aligned}}$$

* Recall: $b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$.

HERMITIAN, SKEW-HERMITIAN, & UNITARY MATRICES $\boxed{7.?3}$

1) DEFINITIONS & BASICS

- COMPLEX SQUARE MATRICES
  - We consider that:
$$\bar{A} = [\bar{a}_{jk}] \text{ and } \bar{A}^T = [\bar{a}_{kj}]$$
  - Then for example below, the complex conjugate of the
$$\boxed{\vec{A} = \begin{bmatrix} 3+4i & -5i \\ 7 & 6-2i \end{bmatrix}} \rightarrow \boxed{\bar{A}^T = \begin{bmatrix} 3-4i & -7 \\ 5i & 6+2i \end{bmatrix}}$$
  - As we can see the below:
$$\bar{A}^T = [\bar{a}_{kj}]^T = [a_{jk}]^T = \begin{bmatrix} 3+4i & -5i \\ 7 & 6-2i \end{bmatrix}^T = \left(\begin{bmatrix} 3+4i \\ -7 \end{bmatrix}[1\ 0] + \begin{bmatrix} -5i \\ 6-2i \end{bmatrix}[0\ 1]\right)$$
$$= \begin{bmatrix} 3+4i \\ 7i \end{bmatrix}^T [1\ 0]^T + \begin{bmatrix} -5i \\ 6-2i \end{bmatrix}[0\ 1]^T$$
$$= [3+4i\ \ 7]\begin{bmatrix} 1 \\ 0 \end{bmatrix} + [-5i\ \ 6-2i]\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3+4i & 7 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -5i & 6-2i \end{bmatrix} = \begin{bmatrix} 3+4i & 7 \\ -5i & 6-2i \end{bmatrix} = \begin{bmatrix} 3-4i & -7 \\ 5i & 6+2i \end{bmatrix}$$
  Then $\bar{A}^T = \left(\begin{bmatrix} 3+4i \\ 7 \end{bmatrix}[1\ 0] + \begin{bmatrix} -5i \\ 6-2i \end{bmatrix}[0\ 1]\right)^T \Leftrightarrow \vec{A}$ with $\overline{(\cdot)}$ = the complex conjugate

- THE HERMITIAN: $\forall \vec{A},\ \vec{A} = [a_{jk}]$ is:
  - We see that:

$$\boxed{\begin{array}{ll} 1)\ \text{HERMITIAN}: & \bar{A}^T = \vec{A} \\ 2)\ \text{SKEW-HERMITIAN}: & \bar{A}^T = -\vec{A} \\ 3)\ \text{UNITARY}: & \bar{A}^T = A^{-1} \end{array}}$$

  or 1) $A^H = \vec{A} \rightarrow A^H\vec{A}^{-1} = 1$; 2) $A^H = -\vec{A} \rightarrow A^H\vec{A} = 1$; 3) $\bar{A}^T = A^{-1} \rightarrow \bar{A}^T A = 1$.

  - Example:
$$\underbrace{\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}}_{1)\ \text{HERMITIAN}} ; \underbrace{\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}}_{2)\ \text{SKEW HERMITIAN}} , \underbrace{\begin{bmatrix} i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & i/2 \end{bmatrix}}_{3)\ \text{UNITARY}}$$
  - To verify:

1) $\vec{A} = \begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix} \rightarrow \bar{A}^T = \begin{bmatrix} 7 & 1-3i \\ 1+3i & 4 \end{bmatrix} = \boxed{A}$

2) $\vec{A} = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix} \rightarrow \bar{A}^T = \begin{bmatrix} -i & -2+i \\ 2+i & 3i \end{bmatrix} = (-1)\begin{bmatrix} i & -2+i \\ -2+i & 3i \end{bmatrix} = \boxed{-A.}$

3) $\vec{A} = \begin{bmatrix} i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & i/2 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & i/2 \end{bmatrix} \Rightarrow \bar{A}^T \Rightarrow \bar{A}^T = A = A^{-1}$ hence $\boxed{\bar{A}^T = A^{-1}}$

2) UNITARY & ORTHOGONAL MATRICES

- ORTHOGONALITY:
  - The previous definition satisfies:
$$\boxed{\bar{A}^T = A^{-1}} \quad (1)$$
  - A real unitary matrix is an orthogonal matrix then (1) becomes
$$\boxed{A^T = A^{-1}}$$

- We can say a system of orthogonal matrices can be described as:

$$\boxed{\bar{x}_j^T x_k = \delta_{jk} = \begin{cases} 0 & \text{if } j \neq k \\ 1 & \text{if } j = k \end{cases}}$$

I. THEOREM 1 - UNITARY MATRIX.

- STATEMENT:

A square Matrix $\vec{A}$ is unitary iff its column vectors (and also its Row vectors) form a unitary system

- PROOF:

Let $\vec{A}$ be a Unitary MATRIX. Then:

$$\vec{\bar{A}}^{-1}\vec{A} = \bar{A}^T A = \begin{bmatrix} \bar{a}_1^T \\ \vdots \\ \bar{a}_n^T \end{bmatrix} \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} = \begin{bmatrix} \bar{a}_1^T a_1 & \cdots & \bar{a}_n^T a_1 \\ \vdots & & \vdots \\ \bar{a}_n^T a_n & \cdots & \bar{a}_n^T a_n \end{bmatrix}$$

- Since $\bar{a}\,\bar{a}^T = 1 = I$. Then we can say:

$$\bar{A}^T = A = \begin{bmatrix} \bar{a}_1^T a_1 & \cdots & \bar{a}_1^T a_n \\ \vdots & & \vdots \\ \bar{a}_n^T a_1 & \cdots & \bar{a}_n^T a_n \end{bmatrix} = \vec{I} = \begin{cases} 0 \\ 1 \end{cases} = \delta_{jk} = \vec{a}\,\vec{a}^T$$

Hence $\boxed{\bar{\vec{x}}_j^T \vec{x}_k = \delta_{jk} = \vec{\bar{A}}^T \vec{A}}$ QED

II. THEOREM 2 - DETERMINANTS.

- STATEMENT:

The Determinant of a unitary matrix $\vec{A}$ has absolute value 1

- PROOF:

- Obviously that: $\det(\bar{A}\bar{A}^T) = \det(AA^{-1}) = \det(I) = 1.$
- This means that: $\det(A) \cdot \det(\bar{A}^T) = 1.$
- Observe that: $\det(A) = \overline{(\det A)}$ since $\begin{cases} \bar{a}+\bar{b} = \overline{(a+b)} \\ \bar{a}\bar{b} = \overline{(ab)}. \end{cases}$
- So the Result is:

$$\det(A)\,\overline{\det(A)} = 1 \quad \text{and} \quad |\det(A)|^2 = \det(A) = 1$$

hence $\boxed{|\det(A)|^2 = 1 \text{ iff unitary}}$

III. THEOREM 3 - ORTHOGONAL MATRIX: $\boxed{\vec{a}_j^T \vec{a}_k = \delta_{jk} = \begin{cases} 0 \\ 1 \end{cases}}$ if $\begin{cases} j \neq k = 0 \\ j = k = 1. \end{cases}$

3. ORTHOGONAL TRANSFORMATION.

- Basically an orthogonal Transformation does $\boxed{(\vec{x}, \vec{y}) = \vec{x}^T \vec{y}}$ Thus $\boxed{\|\vec{x}\| = \sqrt{\vec{x}^T \vec{x}}}$
- The Proof is quite simple. Let $\vec{A}$ be orthogonal, $\vec{u} = A\vec{x}$ and $\vec{v} = A\vec{y}$. Then

$$(\vec{u}, \vec{v}) = \vec{u}^T \vec{v} = (A\vec{x})^T A\vec{y} = \vec{x}^T A^T A y = \vec{x}^T A^{-1} A\vec{y} = \vec{x}^T I \vec{y} = \vec{x}^T \vec{y} = (\vec{x}, \vec{y})$$

Thus $\boxed{(\vec{u}, \vec{v}) = (A\vec{x}, A\vec{y}) = \vec{x}^T \vec{y}}$

Problems 7.13.

Orthogonal matrices

(1) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

here $A^T = A \rightarrow \boxed{A^T A^{-1} = 1}$

(2) $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \Rightarrow A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$ and $A^T = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ $A^T = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$.

$\Rightarrow A$

(3) $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Rightarrow A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

and $e^{i\theta} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \cos\theta \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$a + ib = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with $1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

so $\boxed{\cos\theta + i\sin\theta}$ and $\det(1) = 1$ $\det(i) = -1$

Reinforcement Problems

(12) Find A such that $y = Ax$ is a counterclockwise rotation thru $30°$ in the plane:

ans: • well we can say: $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ w/ $\theta = \frac{\pi}{6}$?

• Now the equation is: $y = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} x$

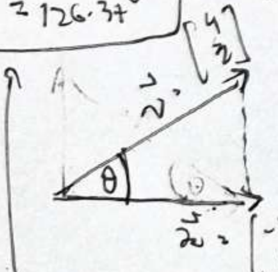
• Then: $\begin{bmatrix} \cos(\frac{\pi}{6}) & -\sin(\frac{\pi}{6}) \\ \sin(\frac{\pi}{6}) & \cos(\frac{\pi}{6}) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$.

So $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ and $\boxed{y = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} x}$

(13) Let $v^T = [4, 2]$, $x^T = [-2, 1]$, $w = Av$, $y = Ax$ w/ A in Problem 3. Find $|v|, |x|, |w|, |y|$ and angles between (v, x) and (w, y).

$\boxed{\theta = \cos^{-1}(-0.6) = 126.87°}$

Ans. $v = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ $x = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$



so $|v| = \sqrt{4^2 + 2^2} = \sqrt{16+4} = \sqrt{20} = \sqrt{4}\sqrt{5} = 2\sqrt{5}$

$|x| = \sqrt{(-2)^2 + 1^2} = \sqrt{4+1} = \sqrt{5}$

$W = Av = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} v = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2\sqrt{3} - 1 \\ 2 + \sqrt{3} \end{bmatrix}$

$y = Ax = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} x = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sqrt{3} - \frac{1}{2} \\ -1 + \frac{\sqrt{3}}{2} \end{bmatrix}$

$\cos\theta = \frac{v \cdot x}{|v||x|} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \frac{1}{2\sqrt{5}\sqrt{5}}$

$= -6 \frac{1}{10} = -0.6$

THE FOURIER SERIES (10.1) (10.2)

① EULER FORMULAS:

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(nx) + b_n \sin(nx)\right]}$$

$\frac{\partial \cos x}{\partial x} = -\sin x$

$\sim \frac{\partial -\cos x}{\partial x} = \sin x$

$-\cos x = \int \sin x\, dx$

② Deriving the coefficients? Ans:

[I] $\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} a_0\,dx + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos nx\,dx + b_n\int_{-\pi}^{\pi}\sin(nx)\,dx\right]$

$= a_0[\pi+\pi] + \sum_{n=1}^{\infty}\left[a_n \sin(n(\pi+\pi)) + b_n(\cos n(\pi+\pi))\right)$

$= a_0[2\pi] + \sum^{\infty} a_n \sin(n2\pi) - b_n\cos(n2\pi)$

$= a_0 2\pi + \sum_{n=0}^{\infty} a_n(0) - b_n(1) = a_0 2\pi$

So: $\displaystyle\int_{-\pi}^{\pi} f(x) = 2\pi a_0 \Rightarrow \therefore \boxed{a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx}$

[II] Deriving a_m:

$\displaystyle\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n \sin nx)\right]\cos mx$

observe: $\displaystyle\int_{-\pi}^{\pi}\cos(nx)\cdot\cos(mx) = \frac{1}{2}\int_{-\pi}^{\pi}\cos[(n+m)x]\,dx + \frac{1}{2}\int_{-\pi}^{\pi}\cos[(n-m)x]\,dx.$

$\displaystyle\int_{-\pi}^{\pi}\sin(nx)\cdot\cos(mx) = \frac{1}{2}\int_{-\pi}^{\pi}\sin[(n+m)x]\,dx + \frac{1}{2}\int_{-\pi}^{\pi}\sin[(n-m)x]\,dx$ +

$\displaystyle\int_{-\pi}^{\pi}\cos(nx)\cdot\cos(mx) + \sin(nx\cdot\cos(mx))\,dx = 0.$

So: $\displaystyle\int_{-\pi}^{\pi} f(x)\cos(mx)\,dx = a_0 \sin\left[m[2\pi]\right] = a_0\pi$

$$\boxed{a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)}$$

[III] Deriving b_m:

$\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx = b_m\pi \Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin dx = b_m$

So: $\boxed{b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx.}$

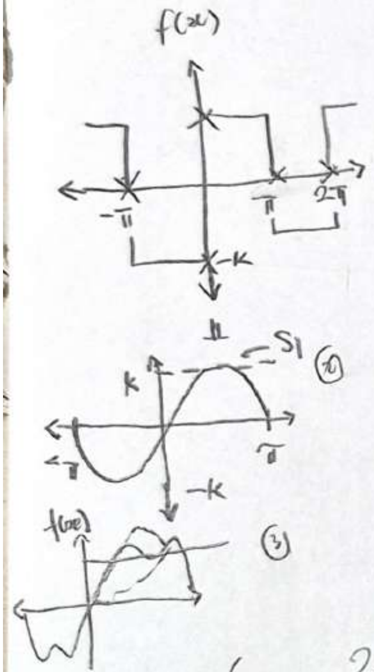
[V] Euler formulas (Summary)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \quad (1) \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad (2) \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad (3)$$

$$\boxed{f(x) = \underbrace{a_0}_{(1)} + \sum_{n=1}^{\infty} \underbrace{a_n\cos nx}_{(2)} + \underbrace{b_n \sin nx}_{(3)}}$$

(2) Example: [1] $f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi \end{cases}$

Ans:



$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,dx + \int_{0}^{\pi} k\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-k\,\frac{\sin(nx)}{n}\Big|_{-\pi}^{0} + k\,\frac{\sin(nx)}{n}\Big|_{0}^{\pi}\right] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[+k\,\frac{\cos nx}{n}\Big|_{-\pi}^{0} - k\,\frac{\cos nx}{n}\Big|_{0}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[k\left\{\frac{\cos n(0)}{n} - \frac{\cos n(-\pi)}{n}\right\} - k\left\{\frac{\cos n(\pi)}{n} + \frac{\cos(0)}{n}\right\}\right]$$

$$= \frac{k}{\pi n}\left[\underbrace{\cos n(0) + \cos n(0)}_{1+1=2} - \underbrace{\cos n(-\pi) - \cos n(\pi)}_{2\cos(n\pi) \text{ since } \cos(x)=\cos(-x)}\right]$$

$$= \frac{2k}{\pi n}\left[1 - \cos(n\pi)\right]$$

So: $\frac{2k}{\pi n}\left[1-\cos(n\pi)\right] \Leftrightarrow \begin{matrix}\cos(2j+1) = -1 \\ \cos(2j) = 1\end{matrix}$

$n \in \{2j+1,\ 2j\}$

So: $\boxed{\cos n\pi \begin{cases} 1 & \text{even}(n) \\ -1 & \text{odd}(n)\end{cases}}$

and: $(b_1, b_2, b_3, b_4) = \left(\frac{4k}{\pi}, 0, \frac{4k}{3\pi}, 0, \frac{4k}{5\pi}\right)$

only Exists for odd functions.

So: $\boxed{f(x) = \sum_{n=1}^{\infty} \frac{4k}{n\pi}\sin(nx)}$

Observe That $f(x)$ discontinues at $x=0,\ x=\pi$.

$$f\left(\frac{\pi}{2}\right) = k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} \pm \cdots\right) = \boxed{\frac{\pi}{4}} \rightarrow \frac{4k}{\pi}\frac{\pi}{4} = \frac{4k}{4} = \boxed{k}$$

PROBLEMS 10.1

(22) $\boxed{\int_0^{\frac{\pi}{2}} \cos(nx)\,dx}$

Ans 1.) $\int \cos(nx)\,dx = \frac{1}{n}\sin nx + C$

So: $\int_0^{\frac{\pi}{2}} \cos(nx)\,dx = \frac{1}{n}\sin nx\Big|_0^{\frac{\pi}{2}} = \frac{1}{n}\left\{\sin\left(\frac{n\pi}{2}\right) - \sin(0n)\right\}$

$= \frac{1}{n}\sin\left(\frac{n\pi}{2}\right)$

So: $\boxed{I = \frac{1}{n}\sin\left(\frac{n\pi}{2}\right)}$

2.) Observe: $\frac{1}{1}\sin\left(\frac{\pi}{2}\right) = \frac{1}{1}$ $\quad \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) \triangleq -\frac{1}{3}$

$0\sin(\pi) = 0$ $\quad \frac{1}{4}\sin(2\pi) \triangleq 0$

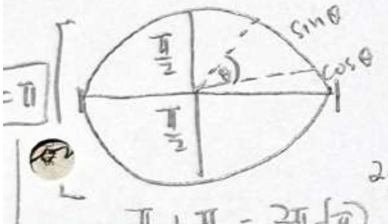
So: pattern: $\boxed{\frac{(-1)^{\frac{n-1}{2}}}{n}}$

(23) $\boxed{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos(nx)\,dx}$

Ans: 1.) $\int x\cos(nx) = \frac{1}{n}x\sin(nx) - \int \frac{1}{n}\sin(nx)\,dx$

$= \frac{1}{n}x\sin(nx) + \frac{1}{n^2}\cos(nx)\,dx.$

So: $\boxed{I = \frac{1}{n}x\sin(nx) + \frac{1}{n^2}\cos(nx)\,dx}$



$\frac{\pi}{2} + \frac{\pi}{2} = \frac{2\pi}{2} = \boxed{\pi}$

$\Delta r = 0$ since 360°!

2.) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos(nx) = \frac{1}{n}\frac{\pi}{2}\left\{\sin\left(\frac{\pi}{2}n\right) - \sin\left(-\frac{\pi}{2}n\right)\right\} + \frac{1}{n^2}\left\{\cos\left(\frac{\pi}{2}n\right) - \cos\left(-\frac{\pi}{2}n\right)\right\}$

$= \frac{1}{n}\frac{\pi}{2}\left\{\sin\left(\frac{n\pi}{2}\right) + \cos\left(\frac{n\pi}{2}\right)\right\} + \frac{1}{n^2}\left\{\cos\left(\frac{\pi}{2}n\right) + \sin\left(\frac{\pi}{2}\right)\right\}$

$= \left(\frac{\pi/2\,\sin(\frac{n\pi}{2})}{n} + \frac{\cos(n\pi/2)}{n^2}\right) - \left(\frac{\pi/2\,\sin(\frac{n\pi}{2})}{n} + \frac{\cos(n\pi/2)}{n^2}\right)$

$= \boxed{0}$ ✓

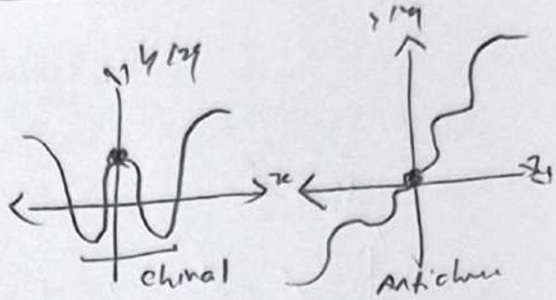
So: $\boxed{I[\frac{\pi}{2}, -\frac{\pi}{2}] = 0}$

(24) $\boxed{\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin(nx)\,dx}$

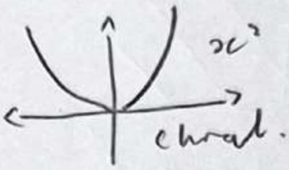
EVEN & ODD FUNCTIONS

① Even & odd functions

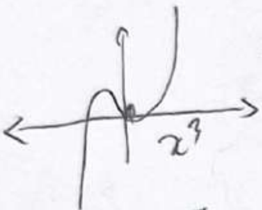
[i] DEF
1.) Even : $g(-x) = g(x)$
2.) Odd : $h(-x) = -h(x)$



[ii] Example :
1.) Even: $g(x) = x^2$
$g(-x) = (-x)^2 = x^2$



2.) Odd : $h(x) = x^3$
$h(-x) = (-x)^3 = (-x)(-x)(-x) = x^2(-x) = -x^3 = -h(x)$



[iii] Consequence :

$$\boxed{\begin{array}{ll} 1.)\ \text{Even} : & \int_{-L}^{L} g(x)\,dx = 2\int_0^L g(x)\,dx \\ 2.)\ \text{odd} : & \int_{-L}^{L} h(x)\,dx = 0 \end{array}}$$

② THEOREMS

[i] THEOREM 1 - ODDNESS / EVENNESS

1.) $\boxed{g(x) = a_0 + \sum_{i=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)}$ if EVEN.

w/ $a_0 = \frac{1}{L}\int_0^L f(x)\,dx$

$a_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi}{L}x$.

$g(x) = f^{even}(x)$
$h(x) = f^{odd}(x)$

2.) $\boxed{h(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)}$

w/ $b_n = \frac{2}{L}\int_0^L f(x) \sin nx$. if ODD

[ii] THEOREM 2 - MULTIPLICITY OF FOURIER SERIES.
Let $\boxed{f = f_1 + f_2}$ then the multiplicity of the Fourier series is defined by. cf w/ its C (Fourier Coefficients) & the Fourier series

③ EXAMPLES:
1) $f^*(x) = k + \frac{4k}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right)$

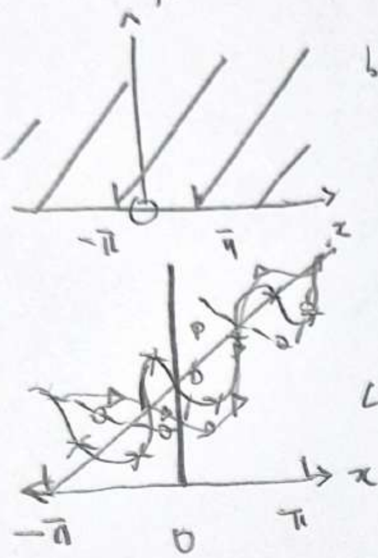
2.) $\boxed{f(x) = x + \pi}$ wh/ $\pi < x < \pi$ and $f(x+2\pi) = f(x)$

a) $f(-x) = -x+\pi = \pi - x = -(-\pi + x)$.

$= -(x+\pi) = -f(x)$

w: $\boxed{f(x) \text{ is odd}}$ so $a_n = 0$ w/ $\boxed{f = x+\pi = f_1 + f_2}$

The Function $f(x)$



S1 = Δ
S2 = O
S3 = x

The sum

b.) $b_n = \frac{2}{\pi}\int_0^L f(x)\Big|_{f_1} \sin(nx)$

$= \frac{2}{\pi}\int_0^{L=\pi} x \sin nx = \left[-\frac{1}{n}x\cos nx\Big|_0^{\pi} + \frac{1}{n}\int \cos nx\,dx\right]\frac{2}{\pi}$

$= \frac{2}{n}\cos n\pi$

c.) $b_1 = 2\cos\pi$ $\quad b_3 = \frac{2}{3}\cos 3\pi$ $\quad b_5 = \frac{2}{5}\cos 5\pi$

$b_2 = \cos 2\pi$ $\quad b_4 = \frac{1}{2}\cos 4\pi$ $\quad b_6 = \frac{1}{3}\cos 3\pi$

$b_1 = 2,\ b_2 = -\frac{2}{2},\ b_3 = \frac{2}{3},\ b_4 = -\frac{2}{4}$.

d.) w: $\boxed{f(x) = \pi + 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x\right)}$

Functions of Any Period $P = 2L$ (11.3)

(1) Preface:

[i] Fourier Series:

$$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]}$$

[ii] Fourier Coeff:

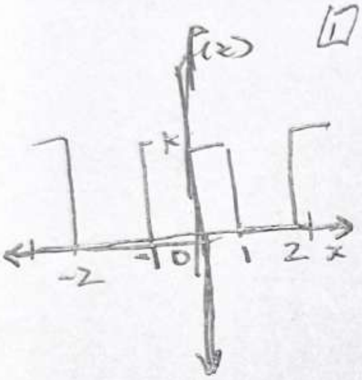
1) $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$

2) $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\,C\left(\frac{n\pi x}{L}\right)$ $C = \cos$

3) $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\,S\left(\frac{n\pi x}{L}\right)$ $S = \sin$

(2) Applications of Fourier Series

[i] Periodic Square Wave:



$$\boxed{f(x) = \begin{cases} 0 & -2 < x < -1 \\ k & -1 < x < 1 \\ 0 & 1 < x < 2 \end{cases}}$$

$P = 2L = 4$
$L = 2$.

1) $a_0 = \frac{1}{4}\int_{-2}^{2} f(x) = \frac{1}{4}\int_{-1}^{1} k\,dx = \frac{1}{4}\left[kx\right]_{-1}^{1} = \frac{1}{4}k\left[1+1\right] = \frac{1}{\cancel{4}}\cancel{2k}$
$= \frac{1}{2}k$

Thus $\boxed{a_0 = \frac{1}{2}k}$

2) $a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2} = \frac{1}{2}\int_{-1}^{1} k\cos\frac{n\pi x}{2}\,dx.$
$= \frac{2k}{n\pi}\sin\frac{n\pi}{2}$

Thus $\boxed{\begin{cases} a_n = 0 & \text{if even.} \\ a_n = \frac{2k}{n\pi} & \text{if } 1, 5, 9, \ldots \\ a_n = -\frac{2k}{n\pi} & \text{if } 3, 7, 11, \ldots \end{cases}}$

~~[ii] Half Wave Rectifier~~

3) $b_n = 0 \ \forall n = 1, 2, \ldots$

So: $f(x) = \frac{k}{2} + \frac{2k}{\pi}\left(\cos\frac{\pi}{2}x - \frac{1}{3}\cos\frac{3\pi}{2}x + \frac{1}{5}\cos\frac{5\pi}{2}x - \right)$

So: $\boxed{f(x) = \frac{k}{2} + \frac{2k}{\pi}\sum_{k=1}^{\infty}(-1)\frac{\cos\left(\frac{(2k+1)\pi}{2}\right)}{(2k+1)}}$

[ii] Half-Wave Rectifier

$$\boxed{u(t) = \begin{cases} 0 & \text{if } -L < t < 0 \\ E\sin\omega t & \text{if } 0 < t < L \end{cases}}$$

$P = 2L = \frac{2\pi}{\omega}$ ∴
so $L = \frac{\pi}{\omega}$

1) $a_0 = \frac{\omega}{2\pi}\int_{0}^{\frac{\pi}{\omega}} E\sin\omega t\,dt = \frac{E}{\pi}$. so $a_0 = \frac{E}{\pi}$

2.) $a_n = \frac{\omega}{\pi} \int_0^{\frac{\pi}{\omega}} E \sin(\omega t) \cos(n\omega t)\, dt$

Recall: if $x = \omega t$

$\sin(m\omega t) \cos(n \cdot \omega t)$

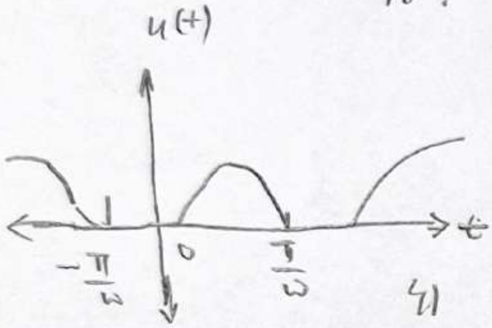
$= \frac{1}{2}\left[\sin((m+n)x) + \sin((m-n)x)\right]$

$= \frac{\omega E}{2\pi} \int_0^{\frac{\pi}{\omega}} \left[\sin((1+n)\omega t) + \sin((1-n)\omega t)\right] dt$

$= \frac{E}{2\pi} \left(\frac{-\cos((1+n)\pi) + 1}{1+n} + \underbrace{\frac{-\cos((1-n)\pi) + 1}{1-n}}_{0 \text{ iff even.}} \right)\Bigg|_0^{\frac{\pi}{\omega}}$

3.) $a_n = \frac{E}{2\pi} \left(\frac{2}{1+n} + \frac{2}{1-n} \right) = \frac{2}{(n-1)(n+1)\pi} \Big\}$ odd.

so: $a_n = \left(\frac{E}{2\pi}\right) \frac{-\cos((1+n)\pi) + 1}{1+n} - \frac{\cos(1-n)\pi + 1}{1-n} = \begin{cases} 0 & \text{odd} \\ \frac{2}{(n-1)(n+1)\pi} & \text{even} \end{cases}$



so: $\boxed{a_n = \begin{cases} 0 & \text{odd} \\ \frac{2}{(n-1)(n+1)} & \text{even} \end{cases}}$

4.) $u(t) = \frac{E}{\pi} + \frac{E}{2} \sin \omega t - \frac{2E}{\pi} \left[\frac{1}{1 \cdot 3} \cos(2\omega t) + \frac{1}{3 \cdot 5} \cos(4\omega t) \right]$

now $1 \cdot 3,\ 3 \cdot 5 = (2-1)(2+1) = 1 \cdot 3.$

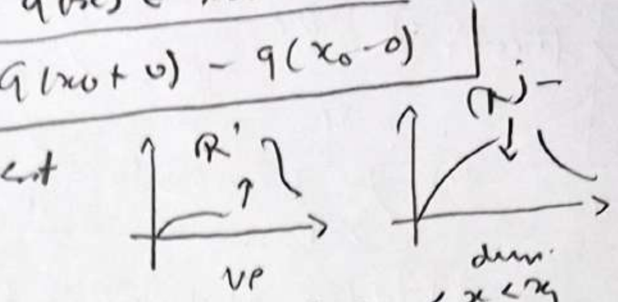
$2\omega t, 4\omega t = 2n \cdot \omega.$

5.) $\boxed{u(t) = \frac{E}{\pi} + \frac{E}{2} \sin(\omega t) - \frac{2E}{\pi} \sum_{n=1}^{\infty} \cos\left(\frac{2n}{(n+1)(n-1)}\right)}$

Function coefficients
No Inequality ? (10.6)

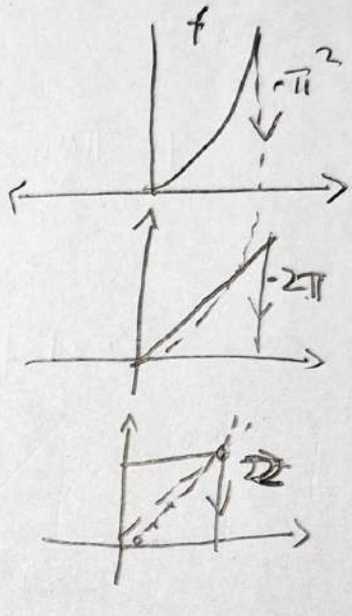
(1) Jump = $\forall\ g(x) \in x_0$

$$\boxed{j = g(x_0+0) - g(x_0-0)}$$



(1) Polynomials:
$$f(x) = \begin{cases} P_1(x) & \text{If } x_0 < x < x_1 \\ P_2 & \text{If } x_1 < x < x_2 \\ \vdots & \\ P_n(x) & \text{If } x_{n-1} < x < x_n \end{cases}$$

(2) f's Jump:
j_s = Jump of f at x_s
j'_s = Jump of f' at x_s
j''_s = Jump of f'' at x_s.



(2) Examples:

1) $$\boxed{f(x) = \begin{cases} x^2 & \pi < x < 0 \\ 0 & 0 < x < \pi \end{cases}}$$

Sol: $f_1(x) = x^2$, $f'(x) = 2x$, $f''(x) = 2$ | $f(\pi) = \pi^2$, $f(\pi) = 2\pi$, $f''(\pi) = 2$

2) $$\boxed{f(x) = \begin{cases} -k & \text{If } -\pi < x < 0 \\ k & \text{If } 0 < x < \pi \end{cases}}$$

Jump at $x_1 = 0$: f $j_1 = 2k$ | $x_2 = \pi$: $j_2 = -2k$

here $f(x)$ is odd.
Sol:

$$b_n = \frac{1}{n\pi}\left[j_1 \cos nx_1 + j_2 \cos nx_2\right] = \frac{1}{n\pi}\left[2k\cos(0) - 2k\cos n\pi\right]$$

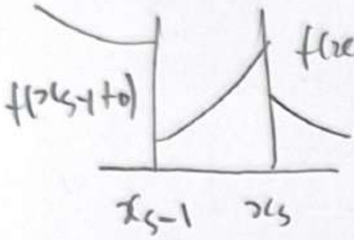
$$= \frac{2k}{n\pi}(1-\cos n\pi) = \begin{cases} 4k/n\pi & \text{odd } n \\ 0 & \text{even } n \end{cases}$$

Sol: $$\boxed{b_n = \frac{2k}{n\pi}(1-\cos n\pi) = \begin{cases} 4k/n\pi \\ 0 \end{cases}}$$

(8) EULER FORMULA:

[i] 1) $\pi a_n = \int_{-\pi}^{\pi} f \cos nx \, dx$ (4)

2) $\pi a_n = \int_{x_0}^{x_1} + \int_{x_1}^{x_2} + \dots \int_{x_{m-1}}^{x_m} = \sum_{s=1}^{m} \int_{x_{s-1}}^{x_s} f \cos nx \, dx$ (5)



3) $\int_{x_{s-1}}^{x_s} f \cos nx \, dx = \frac{f}{n} \sin nx \Big|_{x_{s-1}}^{x_s} - \frac{1}{n} \int_{x_{s-1}}^{x_s} f' \sin nx \, dx$ (6)

4) $\boxed{\frac{1}{n}\left[f(x_s-0) \sin nx_s - f(x_{s-1}+0) \sin nx_{s-1}\right]}$

(v) Preliminary $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f \cos nx$

$\pi a_n = \sum \int_{n}^{x_s} f \cos nx = \sum_n \frac{f}{n} \sin nx \Big|_{x_{s-1}}^{x_s} - \frac{1}{n}\int_{x_{s-1}}^{x_s} f' \dots$

$\therefore \frac{1}{n}\left[f(x_s) \sin nx_s - f(x_{s-1}) \sin nx_{s-1}\right]$

[ii] FINAL FORMULA.

1) $\pi a_n = \frac{1}{n}\left[f(x_1-0)S_1 - f(x_0+0)S_0 + f(x_2-0)S_2 - f(x_1+0)S_1 + \dots - f(x_{m-1}+0)S_{m-1}\right]$

$- \frac{1}{n}\sum_{s=1}^{m}\int_{x_{s-1}}^{x_s} f' \sin nx \, dx$ (7)

$\Rightarrow$ 2) $f(x_0+0)S_0 + [f(x_1-0) - f(x_1+0)]S_1 + [f(x_2-0) - f(x_2+0)]S_2 + \dots$

OR: $f(x_0+0)S_0$
$(f(x_1-0) - f(x_1+0))S_1$
$(f(x_2-0) - f(x_2+0))S_2$

3.) $\boxed{\pi a_n = -\frac{1}{n}\sum_{s=1}^{m} J_s \sin nx_s - \frac{1}{n}\sum_{s=1}^{m}\int_{x_{s-1}}^{x_s} f' \sin nx \, dx.}$ (9)

4.) OR: $\sum_{s=1}^{m}\int_{x_{s-1}}^{x_s} f' \sin nx \, dx = \frac{1}{n}\sum_{s=1}^{m} J'_s \cos nx_s + \frac{1}{n}\sum_{s=1}^{m}\int_{x_{s-1}}^{x_s} f'' \cos nx.$

(1) FOURIER COSINE & SINE TRANSFORM

[i] COSINE:

Fourier Integrals

1) $\begin{cases} f(x) = \int_0^\infty A(w) \cos wx \, dw \\ A(w) = \frac{2}{\pi}\int_0^\infty f(v) \cos wv \, dv \end{cases}$ The sine Integral

2) $f(x) = \int_0^\infty A(w) \cos wx \, dw$; $f(x) = \int_0^\infty B(w) \sin wx \, dx$ — prev. chapter statement

3) $f_L = \frac{1}{2L}\int_{-L}^{L} f_L(v)\, dv + \frac{1}{L}\sum_1^\infty \left[\cos w_n x \int_{-L}^{L} f_L \cos w_n v \, dv + \sin w_n x \int_{-L}^{L} f_L \sin w_n v \, dv\right]$

$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \sin nx + b_n \cos nx$

hence: $\boxed{f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos wx \, dw}$ & $\boxed{\begin{aligned} f(x) &= \int_0^\infty A(w) \cos wx \\ A(w) &= \frac{2}{\pi}\int_0^\infty f(v) \cos wv \, dv \end{aligned}}$

Cosine "Normalizing"

1) $A(w) = \sqrt{\frac{2}{\pi}}\, \hat{f}_c(w)$

2) $f_c(w) = \int_0^\infty f(x) \cos wx \, dx$

$\rightarrow \hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx \, dx$

[ii] SINE:

Sine Integrals

1) $f(x) = \int_0^\infty B(w) \sin wx \, dw$

2) $B(w) = \frac{2}{\pi}\int_0^\infty f(v) \sin wv \, dv$

Sine Transform

1) $\hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x) \sin wx \, dx.$

2) $f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(w) \sin wx \, dw.$

$\therefore$ hence: $\boxed{\begin{aligned} f(x) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos wx \, dw \\ f(x) &= \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin wx \, dw. \end{aligned}}$

The Fourier Transforms are similar ... or $\boxed{\begin{aligned} \mathcal{F}_c(f) &= \hat{f}_c \\ \mathcal{F}_s(f) &= \hat{f}_s \end{aligned}}$

[iii] EXAMPLES

1. Fourier Cosine - Sine Transforms: - Ex

Q/ Find the Fourier Transform of Sine and Cosine of $\boxed{f(x) = \begin{cases} k & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}}$

Ans: 1) $\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\, k \int_0^a \cos wx \, dx = \sqrt{\frac{2}{\pi}}\, k \left[\frac{\sin aw}{w}\right]$

$\hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\, k \int_0^a \sin wx \, dx = \sqrt{\frac{2}{\pi}}\, k \left[\frac{1 - \cos aw}{w}\right]$

2. Fourier Cosine Transform of the Exponential fncs -

Q/ Find the Fourier Cosine Transform of $\boxed{\mathcal{F}_c(e^{-x})}$!

Ans: $\mathcal{F}_c(e^{-x}) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \cos wx \, dx = \sqrt{\frac{2}{\pi}}\, \frac{e^{-x}}{1+w^2}\left[-\cos wx - w \sin wx\right]\Big|_0^\infty = \frac{\sqrt{\frac{2}{\pi}}}{1+w^2}$

hence $\boxed{\mathcal{F}_c(e^{-x}) = \sqrt{\frac{2}{\pi}}\, \frac{1}{1+w^2}}$

② LINEARITY

$$\boxed{\begin{aligned} \tilde{\mathcal{F}}_c(af+bg) &= a\,\tilde{\mathcal{F}}_c(f(x)) + b\,\tilde{\mathcal{F}}(g(x)) \\ \tilde{\mathcal{F}}_s(af+bg) &= a\,\tilde{\mathcal{F}}_c(f(x)) + b\,\tilde{\mathcal{F}}(g(x)) \end{aligned}}$$

wh/ $\tilde{\mathcal{F}}_c(af+bg) = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty}\left[af(x)+bg(x)\right]\cos wx\,dx$

$= \sqrt{\frac{2}{\pi}}\left[\int_{-\infty}^{\infty} af(x)\cos wx\,dx + \int_{-\infty}^{\infty} bg(x)\cos wx\right]dx$

$= a\,\tilde{\mathcal{F}}_c(f) + b\,\tilde{\mathcal{F}}_c(g)$ # Same goes f/ sine

ii) THM:1 - DERIVATIVES

$$\boxed{\begin{aligned} \tilde{\mathcal{F}}_c\{f'(x)\} &= w\,\tilde{\mathcal{F}}_s\{f(x)\} - \sqrt{\tfrac{2}{\pi}}\,f(0) \\ \tilde{\mathcal{F}}_s\{f'(x)\} &= -w\,\tilde{\mathcal{F}}_c\{f(x)\} \end{aligned}}$$

$\forall f(x)$ is continuous and integrable, if $f'(x) = \frac{df}{dx}$ is a piecewise continuous on each finite interval and let $f(x) \to 0$ as $f(x) \to \infty$.

Then the Fourier Differential is defined

PROOF

1) $\tilde{\mathcal{F}}_c\{f'\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\cos wx\,dx$

$= \sqrt{\frac{2}{\pi}}\left[f(x)\cos wx\Big|_0^\infty + w\int_0^\infty f(x)\sin wx\,dx\right]$

$= -\sqrt{\frac{2}{\pi}}f(0) + w\,\tilde{\mathcal{F}}_s\{f(x)\}$ ∴ wh/ $\cos(0)=1$

$u = \cos wx \to du = -w\sin wx$

$dv = f'\,dx \to v = f$

2) $\tilde{\mathcal{F}}_s\{f'\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f'(x)\sin wx\,dx = \sqrt{\frac{2}{\pi}}\left[f(x)\sin wx\Big|_0^\infty - w\int_0^\infty f(x)\cos wx\,dx\right]$

$= 0 - w\,\tilde{\mathcal{F}}_c\{f(x)\}$ wh/ $\sin(0) = 0$.

hence ∴ $\boxed{\begin{aligned} \tilde{\mathcal{F}}_c &= w\,\tilde{\mathcal{F}}_s\{f\} - \sqrt{\tfrac{2}{\pi}}f(0) \\ \tilde{\mathcal{F}}_s &= -w\,\tilde{\mathcal{F}}_c\{f(x)\} \end{aligned}}$

3.) $\tilde{\mathcal{F}}_c\{f''\} = -w^2\tilde{\mathcal{F}}_s(f) + \sqrt{\frac{2}{\pi}}\,w f(0)$

h $\boxed{\begin{aligned} \tilde{\mathcal{F}}_c\{f''\} &= -w^2\tilde{\mathcal{F}}_c\{f\} - \sqrt{\tfrac{2}{\pi}}\,f'(0) \\ \tilde{\mathcal{F}}_s\{f''\} &= -w^2\tilde{\mathcal{F}}_s\{f\} + \sqrt{\tfrac{2}{\pi}}\,w f(0) \end{aligned}}$

iii) EXAMPLES:

GAUSSIAN FUNC = EX 3

a/ Find the Fourier cosine transform of $\boxed{f(x) = e^{-ax}}$.

Ans: 1) $\left(e^{-ax}\right)'' = a^2 e^{-x} = f''(x)$.

2) $a^2\tilde{\mathcal{F}}_c(f'') = -w^2\tilde{\mathcal{F}}_c(f) - \sqrt{\frac{2}{\pi}}f'(0) = -w^2\tilde{\mathcal{F}}_c(f) + a\sqrt{\frac{2}{\pi}}$ wh $f(0) = ae^{-a0} = \boxed{a}$.

he ∴ $\boxed{a\sqrt{\frac{2}{\pi}} - w^2\tilde{\mathcal{F}}_c(f) = a^2\tilde{\mathcal{F}}_c(f'')}$ $\longrightarrow$ $\boxed{\tilde{\mathcal{F}}_c\{a^2+w^2\} = a\sqrt{\frac{2}{\pi}}}$

3) $\tilde{\mathcal{F}}_c\{f\} = \tilde{\mathcal{F}}_c(e^{-ax}) = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+w^2}\right)$

hw. $a\sqrt{\frac{2}{\pi}} - w^2\tilde{\mathcal{F}}_c(f) = a\sqrt{\frac{2}{\pi}} - w^2\frac{a}{a^2+w^2} = \tilde{\mathcal{F}}(f'')$

So: $\boxed{\tilde{\mathcal{F}}_c(f) = \sqrt{\frac{2}{\pi}}\,\frac{a}{(a^2+w^2)}}$ ✓

③ DES

Let $\frac{dy(x)}{dx} + y(x) = 0$. Then $-k\tilde{\mathcal{F}}_s + \tilde{\mathcal{F}}_c = 0$

$k\tilde{\mathcal{F}}_s = \tilde{\mathcal{F}}_c \to \boxed{k = \frac{\tilde{\mathcal{F}}_s}{\tilde{\mathcal{F}}_c}}$

hw. $k = \int_0^\infty \frac{f(x)\sin kx}{f(x)\cos kx}dx = \int_0^\infty \frac{\sin kx}{\cos kx}dx = \int_0^\infty \tan(kx)\,dx$

$= -\frac{1}{k}\ln|\cos kx|\Big|_0^\infty = -\frac{1}{k}(\infty) + \ln(0) =$ Doesn't DIVERGE!

PROBLEMS 10.10

(1) Does $(\mathcal{F}_c, \mathcal{F}_s)\{f\}$ with $f = e^{x}$ exist?

Ans: Recall: $\mathcal{F}_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+w^2}$

$\mathcal{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\frac{w}{a^2+w^2}$.

Now let $a=-1$, so: $f_N(x) = e^{-ax} = e^{-(-1)x} = e^{x}$.

so: $\mathcal{F}_c\{f_N\} = \sqrt{\frac{2}{\pi}}\frac{-1}{(-1)^2+w^2} = -\sqrt{\frac{2}{\pi}}\frac{1}{w^2+1}$ so $\boxed{\mathcal{F}_c = -\mathcal{F}_s}$ X

$\mathcal{F}_s\{f_N\} = \sqrt{\frac{2}{\pi}}\frac{w}{(-1)^2+w^2} = \sqrt{\frac{2}{\pi}}\frac{1}{1^2+w^2}$

and To test This:

$\hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} e^{x}\cos wx = \sqrt{\frac{2}{\pi}}\left[-e^{x}\cos wx + w\int^{\infty} e^{x}\sin wx\, dx\right]$.

Let $dv = \int e^{x}dx \to e^{x} = v$ as we iterate This. $x \to \infty$ doesn't occur

$u = \cos w \to -w\sin wx\, dx = du$

(2) Show That $f(x)$ has no Fourier cosine/sine Transform.

Ans: $\mathcal{F}_c\{1\} = \sqrt{\frac{2}{\pi}}\int_0^{x}\cos wx\, dx = \sqrt{\frac{2}{\pi}}\frac{\sin wx}{w}\Big|_0^{x} = \sqrt{\frac{2}{\pi}}\left(\frac{\sin x}{w} - \frac{\sin(0w)}{w}\right)$

$= \sqrt{\frac{2}{\pi}}\left(\frac{\sin x}{w} - 0\right) = \sqrt{\frac{2}{\pi}}\frac{\sin wx}{w}$

$\mathcal{F}_s\{1\} = \sqrt{\frac{2}{\pi}}\int_0^{x}\sin wx\, dx = -\sqrt{\frac{2}{\pi}}\frac{\cos wx}{w}\Big|_0^{x} = \sqrt{\frac{2}{\pi}}\frac{-\cos wx + 1}{w} = \sqrt{\frac{2}{\pi}}\frac{1-\cos wx}{w}$.

so: But if $\forall x=1$ and let $w=1$ then

$\sqrt{\frac{2}{\pi}}\frac{\sin wx}{w} = \sqrt{\frac{2}{\pi}}\frac{\sin(1)}{w} = \sqrt{\frac{2}{\pi}}\frac{0.0175}{(w=1)} \to \sqrt{\frac{2}{\pi}}0.0175$

$\sqrt{\frac{2}{\pi}}\frac{1-\cos wx}{w} = \sqrt{\frac{2}{\pi}}\frac{1-\cos w(1)}{w} = \sqrt{\frac{2}{\pi}}\frac{\text{unknown } 2}{(w=1)} = \sqrt{\frac{2}{\pi}}\text{unknown}$

so $\boxed{\begin{matrix}\mathcal{F}_c\{1\}\sim 0\\ \mathcal{F}_s\{1\}\sim 0\end{matrix}} \longleftrightarrow$ since if $w\to\infty$, $(\mathcal{F}_c, \mathcal{F}_s) = 0$ and suppose $w=2$

$\frac{1-\cos(2)}{2}$ are 0.0003.

$\frac{\sin(2)}{2}$ at 0.0174 so $w\to\infty$, $\boxed{\mathcal{F}_s, \mathcal{F}_c = 0}$

(3) Find $\mathcal{F}_s(e^{-ax})$ by Integration

Ans:

$\mathcal{F}_s(e^{-ax}) = \sqrt{\frac{2}{\pi}}\int^{+\infty}\sin(wx)e^{-ax}dx$

$= \sqrt{\frac{2}{\pi}}\left[-\frac{1}{a}e^{-ax}\sin(wx)\Big|_0^{\infty} + \frac{1}{aw}\int_0^{\infty} e^{-ax}\cos(wx)\,dx\right]$

$= \sqrt{\frac{2}{\pi}}\left[-\frac{1}{a}e^{-ax}\sin(wx)\Big|_0^{\infty} + \frac{1}{a^2}e^{-ax}\cos wx\Big|_0^{\infty} - \frac{1}{a^2w^2}\int^{\infty} e^{-ax}\sin wx\right]$ -----

$dv = e^{-ax}dx$

$u = \sin wx$

$v = -\frac{1}{a}e^{-ax}dx$

$du = \frac{1}{w}\cos wx\, dx$.

$\int u\,dv = uv - \int v\,du$

To complicated! so we can assume that $a\leftarrow t$ so we can use The Laplacian transform so:

$\mathcal{L}\{\sin wx \mid f(t) = f(a)\}$

$\mathcal{F}_s(e^{-ax}) := \sqrt{\frac{2}{\pi}}\int^{\infty}\sin(wx)e^{-ax} :=$

$= \sqrt{\frac{2}{\pi}}\frac{w^2}{a^2+w^2}$. $\forall a>0$ so $\boxed{\mathcal{F}_s(e^{-ax}) = \mathcal{L}\{\sin(wx)\} = \sqrt{\frac{2}{\pi}}\frac{w^2}{a^2+w^2}}$

and further $\frac{w^2}{a^2+w^2} \sim \frac{1}{a}\frac{z}{1+z^2}$ w/ $\frac{w}{a} = z$ so: $\frac{z}{1+z^2} = z - z^3 + z^5 - z^7 \dots \to \boxed{\sum(-1)^n\frac{w^{2n+1}}{a^{2n+1}}}$

To Present:

$I(k) := \int_0^\infty e^{-ax}\sin(kx)\,dx = -\frac{1}{a}e^{-ax}\sin wx + \frac{w}{a}\int_0^\infty e^{-ax}\cos(wx)\,dx$ let $\boxed{k=w}$

$= \frac{k}{a}\left[-\frac{1}{a}\cos kx\, e^{-ax}\Big|_0^\infty - \frac{k}{a}\int_0^\infty e^{-ax}\sin(kx)\,dx\right] = \ldots$

$\frac{1}{a}\cos wx\, e^{-ax}\Big|_0^\infty$

$= \frac{1}{a}\cos(w(0))e^{-a0}$

$= \frac{1}{a}(1)(e^0)$

$= \frac{1}{a}\,1\cdot 1 = \frac{1}{a}$

Thus $I(k) = \frac{k}{a^2} - \frac{k^2}{a^2}I(k) \Rightarrow I(k)\left(1+\frac{k^2}{a^2}\right) = \frac{k}{a^2} \rightarrow I(k) = \frac{k}{a^2+k^2}$

So: $\boxed{I(k) = \frac{k}{a^2+k^2}}$ So: $\boxed{\sqrt{\frac{2}{\pi}}\,I(k) = \frac{k}{a^2+k^2}}$ ✓

(15) Using $\Gamma(\frac{1}{2}) = \sqrt{\pi}\ldots$, obtain $\tilde{F}_c(x^{a-1}) = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(a)}{w^a}\sin\frac{a\pi}{2}$.

Ans: $\tilde{F}_c(x^{a-1}) = \sqrt{\frac{2}{\pi}}\int x^{a-1}\cos wx = \sqrt{\frac{2}{\pi}}\int x^a\frac{\cos wx}{x}\,dx$.

CM III SEPERATION OF VARIABLES 11.3

Q1.

$$\boxed{\frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}} \longleftrightarrow \text{Wave EQS.}$$

w/ • Boundary Conditions $u(0,t) = 0 \quad u(L,t) = 0 \quad \forall t$

• Initial Condition $u(x,0) = F(x) \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(x)$

(i) $u(x,t) = F(x)\,G(t)$

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2F}{dx^2}G(t) \qquad \frac{\partial^2 u}{\partial t^2} = \frac{d^2G}{dt^2}F(x)$$

So:

$$\frac{1}{v^2}F''G = \ddot{G}F$$

$$\frac{1}{v^2}\frac{F''}{F} = \frac{\ddot{G}}{G}$$

(ii)

$$\frac{F''}{F} = \frac{d^2F}{dx^2}\frac{1}{F} = k_1 \rightarrow \frac{d^2F}{dx^2} - k_1F = 0$$

$$\frac{\ddot{G}}{G} = \frac{d^2G}{dt^2}\frac{1}{G} = k_2 \rightarrow \frac{d^2G}{dt^2} - v^2k_2G = 0$$

$$p = -k_2^2$$

(iii)

$$F(x) = A\cos px + B\sin px \quad (\mu x,\ -\mu x)$$

$$G_n(t) = B_n\cos\lambda_n t + B_n^*\sin\lambda_n t \qquad \lambda = \frac{v n\pi}{L}$$

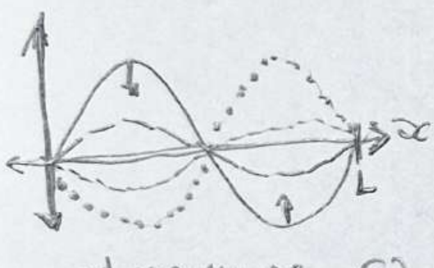
$$u: F(x=0) = A\cos(p0) + B\sin p(0) = A(1) + B(0) = \boxed{A} = 0$$

$$F(x=L) = A\cos(pL) + B\sin(pL) = 0(\cos(pL)) + B\sin(pL)$$

$$= B\sin(pL) = 0$$

∴ B can't be 0, since F = 0. So $\sin(pL) = 0$ and

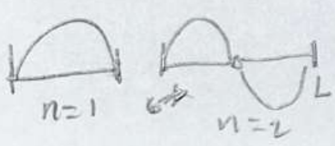
$$\boxed{pL = n\pi \quad \text{or} \quad p = \frac{n\pi}{L}}$$



2nd NORMAL MODE

(iv)

$$\boxed{u(x,t) = \sum_{i=0}^{n} u_n(x,t) = \sum_{i=1}^{n}\left[B_n\cos\lambda_n t + B_n^*\sin\lambda_n t\right]\sin\frac{n\pi}{L}x}$$



(2) (i) $u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{L}x = f(x)$

$$\rightarrow \boxed{B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi}{L}\, dx}$$

* Fourier sine series

(ii) $$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \frac{\partial}{\partial t}\left[\sum_{i=1}^{n} \left[B_n \cos \lambda_n t + B_n^* \sin \lambda_n t\right] \sin \frac{n\pi}{L}x\right]\Bigg|_{t=0}$$

$$= \sum_{n=1}^{\infty} B_n^* \lambda_n \sin \frac{n\pi x}{L} = g(x)$$

so: $$B_n^* \lambda_n = \frac{2}{L}\int_0^L g(x) \sin \frac{n\pi x}{L}$$

$$\boxed{B_n^* = \frac{2}{cn\pi}\int_0^L g(x) \sin \frac{n\pi x}{L}\, dx}$$

(iii) $$u(x,t) = \sum_{n=1}^{\infty} B_n \cos(\lambda_n t) \cdot \sin\left(\frac{n\pi x}{L}\right) \quad w/\ \lambda = \frac{cn\pi}{L}$$

* Use the trig trick:

$$\cos \frac{cn\pi}{L}t \sin \frac{n\pi}{L}x = \frac{1}{2}\sin\left\{\frac{n\pi}{L}(x-ct)\right\} + \sin\left\{\frac{n\pi}{L}(x+ct)\right\}$$

So: $$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x-ct)\right\} + \frac{1}{2}\sum_{n=1}^{\infty} B_n \sin\left\{\frac{n\pi}{L}(x+ct)\right\}$$

Finally $$\boxed{u(x,t) \triangleq \frac{1}{2} f^*(x-ct) + \frac{1}{2} f^*(x+ct)}$$

CH 11 : D'ALEMBERT'S SOLUTION OF THE WAVE EQS

11.4

$$\boxed{\frac{\partial^2 u}{\partial t^2} = v^2 \frac{\partial^2 u}{\partial x^2}}$$ The Wave eqs (again!)

wh/ $v = c$.
$c^2 = \frac{T}{\rho}$

(i)
$$v = x + ct$$
$$z = x - ct$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = (1)\frac{\partial u}{\partial v} + \frac{\partial u}{\partial z}(1)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial v} + \frac{\partial u}{\partial z}\right) = \frac{\partial(u_v + u_z)}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial(u_v + u_z)}{\partial z}\frac{\partial z}{\partial x}$$

$$= \frac{\partial^2 u}{\partial v^2} + 2\frac{\partial u}{\partial v \partial z} + \frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial v^2} + 2\frac{\partial u}{\partial v \partial z} + \frac{\partial^2 u}{\partial z^2}\right)$$

so: (ii)
$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = \frac{c^2}{c^2}\left(\frac{\partial^2 u}{\partial v^2} + 2\frac{\partial u}{\partial v \partial z} + \frac{\partial^2 u}{\partial z^2}\right) - c^2\left(\frac{\partial^2 u}{\partial v^2} + 2\frac{\partial u}{\partial v \partial z} + \frac{\partial^2 u}{\partial z^2}\right)$$
$$= 0$$

so: $$\boxed{\frac{\partial^2 u}{\partial v \partial z} = 0}$$

(iii)
$$\frac{\partial u}{\partial v} = h(v)$$
$$\partial u = h(v)\,\partial v$$
$$u(v) = \int h(v)\,dz + \psi(z) = \phi(v) + \psi(z)$$

so: $$\boxed{u(x,t) = \phi(x + ct) + \psi(x - ct)}$$ ⇒ D'alembert's Solution

(iv) $\frac{\partial u}{\partial t} = c\phi'(x+ct) - c\psi'(x-ct)$.

$u(x,0) = \phi(x) + \psi(x) = f(x)$

$\frac{\partial u}{\partial t}\Big|_{t=0} = c\phi'(x) - c\psi'(x) = 0$ so: $\phi'(x) = \psi'(x)$

$\phi(x) = \psi(x) + k$

so: $u(x,0) = \{\psi(x) + k\} + \psi(x) = 2\psi(x) + k = f(x)$

$\frac{1}{2}f(x) = \psi(x) + \frac{1}{2}k$

$\frac{1}{2}f(x) - \frac{1}{2}k = \psi(x)$

so: $\boxed{u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]}$ ✓

(v) The initial velocity isn't identically zero, instead of before, we get

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

w/ Initial velocity $= g(x)$

PROBLEMS 11.4

EXAMPLES OF D'Alembert's.

USING THE INDICATED Transformations, solve the following:

7. $\boxed{\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 u}{\partial x^2}}$ w/ $v = y$, $z = x+y$)

ANS $u(x,y) = \phi(v) + \psi(z) = \phi(y) + \psi(x+y)$

(i) $\dfrac{\partial u}{\partial x} = \dfrac{\partial \phi}{\partial v}\dfrac{\partial v}{\partial x} + \dfrac{\partial \psi}{\partial z}\dfrac{\partial z}{\partial x} = \phi'(0) + \psi'(1) = 0 + \psi'$

$= \psi'$

$\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\dfrac{\partial \psi}{\partial x} = \dfrac{\partial^2 \psi}{\partial x^2} = \boxed{\psi''}$

(ii) $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial (\partial u/\partial x)}{\partial y} = \dfrac{\partial}{\partial y}\psi'$

$= \dfrac{\partial}{\partial y}\psi'(x+y) + \psi'\dfrac{\partial \psi'}{\partial v}\dfrac{\partial v}{\partial y} = \psi''(1)$

$= \boxed{\psi''}$ or $\dfrac{\partial^2 \psi}{\partial x^2}$

(iii) $\dfrac{\partial^2 u}{\partial y \partial x} - \dfrac{\partial^2 u}{\partial x^2} = \psi'' - \psi'' = 0$

$\therefore$ This means the general Solution is $\boxed{u = \phi(y) + \psi(x+y)}$

9. $\boxed{\dfrac{\partial^2 u}{\partial x^2} - 2\dfrac{\partial^2 u}{\partial y \partial x} + \dfrac{\partial^2 u}{\partial y^2} = 0}$ w/ $(v = y,\ z = x+y)$

ANS. $u(x,y) = \phi(v) + \psi(z) = \phi(y) + \psi(xy)$

(i) $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial}{\partial x}\dfrac{\partial}{\partial x}\left(\phi(v) + \psi(z)\right) = \dfrac{\partial}{\partial x}\phi'(v)0 + \psi'(z)y$

$= \dfrac{\partial}{\partial x}\psi' y = \boxed{\psi''}$

(iii) $\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = \frac{\partial}{\partial y}\psi'(z) = \psi''(z)$ w/ $\frac{\partial \psi'}{\partial z}\frac{\partial z}{\partial y} = \psi'' y + 0$

(iv) $\frac{\partial u^2}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\phi'(v) + \psi'(z)x \right)$

$= \phi''(v) + \psi''(z)x$

(iv) $u_{xx} - 2u_{xy} + u_{yy} = \psi'' - 2\psi''(+) \phi'' + \psi'' x = 0$

so: $\boxed{\psi'' - 2\psi'' + \phi'' + \psi'' = 0}$

OR $-\psi'' + \phi'' + \psi'' = \boxed{\phi'' = 0}$

(v) $\phi''(v) = 0 \to$

$\int \phi''(v)\, dv = \phi'(v) = A$

$\int \phi'(v)\, dv = \int A\, dv \to \boxed{\phi(v) = Av + B}$

(vi) so putting them together:

$u(x,y) = \phi(v) + \psi(z) = Av + B + \psi(x+y)$

$= Ay + B + \psi(x+y)$

so: $\boxed{u(x,y) = Ay + B + \psi(x+y)}$

* check:

$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial \psi(x+y)}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \psi}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial \psi'}{\partial x} = \frac{\partial^2 \psi'}{\partial x^2} = \psi''$

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 \psi'}{\partial y^2} = \psi''$

$\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = \frac{\partial}{\partial y}\psi' = \psi''$

So: $u_{xx} - 2u_{xy} + u_{yy} = \psi'' - 2\psi'' + \psi'' = -\psi'' + \psi'' = \boxed{0}$

QED.

PROBLEMS 11.4B

NORMAL FORMS & ANY EQS.

Types and Forms of Linear partial DEs.

$$\boxed{Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)}$$

$\forall F(x, y, u, u_x, u_y)$, $F(x)$ has same properties

(a) $AC - B^2 > 0$, → Elliptic
(b) $AC - B^2 = 0$, → Parabolic
(c) $AC - B^2 < 0$, → Hyperbolic

(B) Show that using the properties of $F(\cdot)$

(a) $\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{is Elliptic}}$ w/ $\nabla^2 \phi(\vec{r}) = 0$.

Ans: $a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial y^2} \Longleftrightarrow a = 1$, $b = 1$

So: $AC - B^2 = (1)(1) - 0 = 1$

since $AC - B^2 = 1 > 0$, it's $\boxed{\text{elliptic}}$

(b) $\boxed{\frac{\partial u}{\partial t} = C^2 \frac{\partial^2 u}{\partial x^2} \quad \text{Is PARABOLIC}}$ w/ $A := u_{xx}$, $B := u_{xy}$, $C := u_{yy}$.

$A = 1$, $B = 0$, $C = 0$

So: $AC - B^2 = (1)(0) - (0) = \boxed{0}$

since $AC - B^2 = 0$, then it's $\boxed{\text{PARABOLIC}}$

(c) $\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{IS HYPERBOLIC}}$

$A = -c^2$, $B = 0$, $C = 1$ w/ $A := u_{xx}$, $B := u_{yx}$, $C := u_{tt}$

So: $AC - B^2 = -c^2(1) - 0 = -c^2 = c^2$

Since $AC - B^2 = c^2 > 0$, Then it's $\boxed{\text{HYPERBOLIC}}$

(d) $\boxed{y u_{xx} + u_{yy} = 0 \text{ is mixed type}}$

Ans: $A := u_{xx} \to A = y$

$B := u_{xy} \to B = 0$

$C := u_{yy} \to C = 1$

so: $B^2 - AC = 0^2 - y = -y = y$ or $\pm y$

If $y > 0 \to$ Eliptic
$y < 0 \to$ Parabolic
$y = 0 \to$ Hyperbolic

} Eliptic upper half
Lower half hyperbolic

(1b) Airy EQS:

Show that by seperation of variables of:

$\boxed{y u_{xx} + y u_{yy} = 0} \to$ TRICOMI EQS

we get the Airy eqs:

$\boxed{G'' - yG = 0}$

Ans: w/ $u(x,y) = X(x)Y(y)$

(i) $y \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$y \frac{X''}{X} + \frac{Y''}{Y}\frac{1}{y} = 0$ so: $k - k = 0$

(ii) $\frac{X''}{X} = \lambda \quad -\frac{Y''}{Y}\frac{1}{y} = -\lambda \to \frac{Y''(y)}{Y(y)} = -\frac{\lambda}{y}$

So: $\boxed{X'' - \lambda X = 0 \quad Y'' - Y\frac{\lambda}{y} = 0}$

(iii) $X(x) = C_1 e^{x} + C_2 e^{-x}$ w/ $\boxed{G(y) = \frac{\lambda}{y}}$

$Y'' - Y\frac{\lambda}{y} \longleftrightarrow G'' - yG = 0$

So: $u(x,y) = \left[C_1 e^{x} + C_2 e^{-x}\right] G(y)$

so: $\partial_x^2 u = \partial_x\, C_1 e^{x} + C_2 e^{-x} = \partial_x C_1 e^{x} + C_2 e^{-x} = u(x,y)$

$\partial_y^2 u = G''(y)$

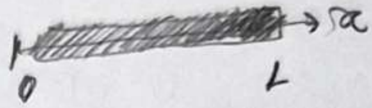
So: $y u_{xx} + u_{yy} = \boxed{yu + G''(y)}$ " Which is proportional to $\frac{d^2y}{dx^2} - xy = 0$

HEAT FLOW-A

(18)

1. * Heat eqs for Homogeneous Material $\boxed{\dfrac{\partial u}{\partial t} = c^2 \nabla^2 u}$ $\quad c^2 = \dfrac{K}{\sigma \rho}$.

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$



$$\boxed{\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}}$$ one Dimensional heat eqs.

w/
- Boundary Conditions : $u(0,t),\ u(L,t) = 0 \ \forall t$
- Initial Conditions : $u(x,0) = f(x)$

2. $\boxed{u(x,t) = F(x)\, G(t)}$

(i) $\dfrac{\partial^2 u(x,t)}{\partial x} = \dfrac{d^2 F}{dx^2} G(t) \qquad \dfrac{\partial u}{\partial t} = \dfrac{dG}{dt} F(x)$

or: $\boxed{\dfrac{\dot{G}}{G} = c^2 \dfrac{F''}{F}}$ → seperated by variables

(ii) and : $\dfrac{\dot{G}}{G}\dfrac{1}{c^2} = -p^2 \ ; \ \dfrac{F''}{F} = -p^2$.

$$\boxed{\begin{aligned} \dot{G} + Gp^2 &= 0 \\ F'' + Fp^2 &= 0 \end{aligned}}$$ → solving these differential eqs is next!

(iii) $\dfrac{dG}{dt}\dfrac{1}{G} = -p^2 \Rightarrow \ln G = -p^2 t$ $\quad \boxed{G(t) = e^{-p^2 t}} \triangleq e^{-\lambda_n t}$ w/ $\lambda = \dfrac{cn\pi}{L} = p$

$\dfrac{d^2 F}{dx^2} + F(x)\, p^2 = 0 \rightarrow \boxed{F(x) = A\cos px + B\sin px}$

(iv) $u(0,t) \triangleq F(0) = A\cos(0 \cdot p) + B\sin 0x = A = 0$.
$u(L,t) \triangleq F(L) = (0)\cos(Lp) + B\sin(Lp) = B\sin(Lp)$.

$$\boxed{F_n(x) = \sin \frac{n\pi x}{L}}$$

(v) $U_n(x,t) = B_n \sin \frac{n\pi x}{L} e^{-\lambda_n^2 t}$

$$\boxed{u(x,t) = \sum_{n=0}^{N} \left(B_n \sin \frac{n\pi x}{L} e^{-\lambda_n^2 t} \right)} \quad \lambda_n = \frac{cn\pi}{L}$$

(vi) IC : $u(x,0) = f(x)$

$$u(x,0) = \sum_{n=0}^{N} B_n \sin \frac{n\pi x}{L} e^{-\lambda_n^2 (0)}$$

$$= \sum_{n=0}^{N} B_n \sin \frac{n\pi x}{L} = f(x)$$

so : $\boxed{B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}}$ by the Fourier transform.

[3] (i) SINOSIDUAL INITIAL TEMP :

[A] (?) $u(x,t)$ is in an Insulted copper 80 cm long w/ initial temp $\sin\left(\frac{\pi x}{80}\right) \cdot 100\ ^\circ C$ w/ $0^\circ C$ ending.

[B] (?) How long will it take for the maximum temperature in the bar drops to $50^\circ C$?

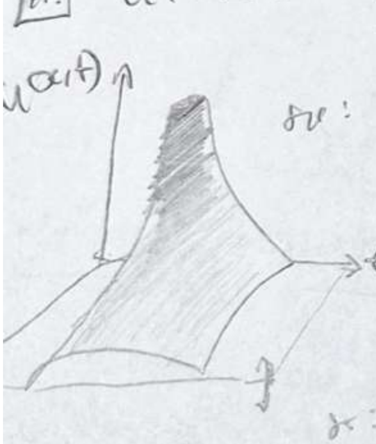
$\rho = 8.92 \frac{gm}{cm^3}$

heat (spec) $\sigma = 0.092 \frac{cal}{gm^\circ C}$

conductivity $k = 0.95 \frac{cal}{cm} \sec^{-1}$

ANS :

[A.] $u(x,0) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{80} = f(x) = \sin \frac{\pi x}{80} 100^\circ C$



so : $u(x,t) = 100 \sin \frac{\pi x}{80} e^{-\lambda_1 t}$ w/ $\lambda_1 = \frac{c^2 \pi^2}{L^2}$

$$c^2 = \frac{k}{\sigma\rho} = \frac{0.95}{8.92 \cdot 0.092} = \boxed{1.158 \frac{cm^2}{sec}}$$

$$e^{\frac{1.158 \cdot \pi^2}{80^2} t} = e^{\frac{1.152 \cdot 9870}{6400} t} = e^{0.01785 \sec^{-1}}$$

so : $\boxed{u(x,t) = 100 \sin \frac{\pi x}{80} e^{-0.017856t}}$ → Final Solution

[B] $100 e^{-0.0017856} = 50 \Rightarrow e^{-0.001785} = 0.5 \Rightarrow t = -\frac{\ln(0.5)}{0.001785} = 388$ seconds $= 6.5$ mins

HEAT FROM-B

MORE examples.

- SPEED OF DECAY
- TRIANGULARS.

1 SPEED OF DECAY

Solve problem 1 which was : $u(x,0) = 100 \sin \frac{\pi x}{80}$

By which $\boxed{f(x) = 100 \sin\left(\frac{3\pi x}{80}\right) \,^\circ C}$

ANS : **a.** $f(x) = 100 \sin\left(\frac{n\pi x}{L}\right) = 100 \sin\left(\frac{3\pi x}{80}\right)$

since $n=3$. $\lambda_3^2 = 3^2 \lambda_1^2 = 9 \cdot 0.001785 = 0.01607 \text{ seconds}^{-1}$

so: $\boxed{u(x,t) = 100 \sin\left(\frac{3\pi x}{80}\right) e^{-0.01607t}}$

b. $100 e^{-0.01607t} = 50 \rightarrow e^{-0.01607 \cdot t} = 0.5$

$t = -\frac{\ln 0.5}{0.01607} = \boxed{43 \text{ seconds}}$

before $t = 6.5$ minutes

so $\boxed{t_2 < t_1 \rightarrow t_2 \text{ is faster}}$

? Why is it faster? well as $n \rightarrow \infty$,

$\lim_{n\to\infty} u(x,t) = 100 \sin\left(\frac{\infty \pi x}{80}\right) e^{-\lambda_\infty t} = \boxed{0}$

2 "TRIANGULAR" INITIAL TEMPERATURE

1 Find the temperature in a Laterally Insulated bar of length L whose ends are kept at temperature 0.

$$\boxed{f(x) = \begin{cases} x & \rightarrow \text{if } 0 < x < \frac{L}{2} \\ L - x & \rightarrow \text{if } \frac{L}{2} < x < L \end{cases}}$$

ANS : 1 $B_n = \frac{2}{L}\left(\int_0^{\frac{L}{2}} x \sin\frac{n\pi x}{L}\, dx + \int_{\frac{L}{2}}^{L} x \sin\frac{n\pi x}{L}(L-x)\, dx\right)$

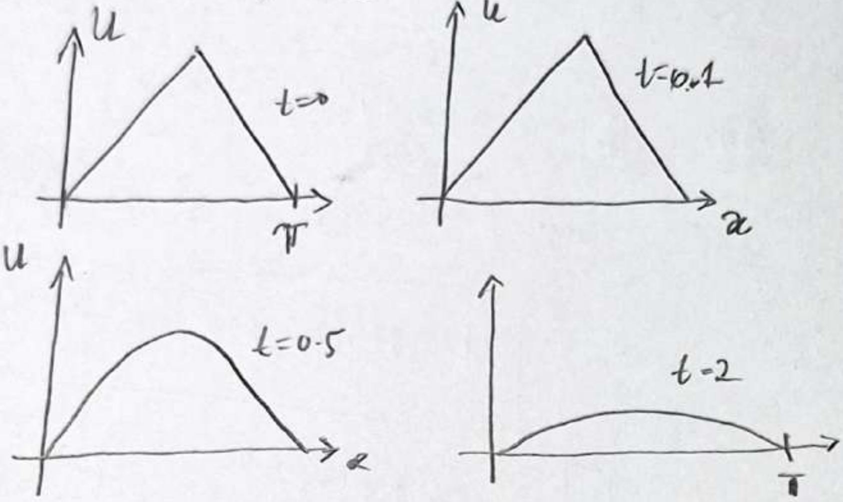
$$B_n^+ = \frac{4L}{n^2\pi^2} \quad \text{w/ } B_n = 0 \text{ iff } n \text{ is even}$$

$$B_n^- = \frac{-4L}{n^2\pi^2}$$

$$\searrow \quad B_n^+ \triangleq \{1, 5, 9, \dots\}$$

$$B_n^- \triangleq \{3, 7, 11, \dots\}$$

GRAPH:



(iv) The full solution..

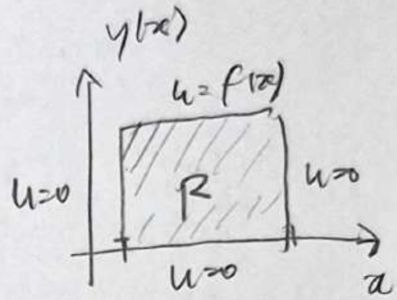
$$u(x,t) = \frac{4L}{\pi^3}\Bigg[\underbrace{\sin\frac{\pi x}{L}\, e^{-\left(\frac{c\pi}{L}\right)^2 t} - \frac{1}{9}\sin\frac{3\pi x}{L}\, e^{-\left(\frac{3c\pi}{L}\right)^2 t}}_{1^{st}\ 2\ \text{terms}} + \dots\Bigg]$$

STEADY State 2D HEAT (11.5)

(1) $$\frac{\partial u}{\partial t} = c^2 \nabla^2 u = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

This means also:

$$\boxed{\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$



- Dirichlet : $u \in C$
- Neuman : $u_n = \frac{\partial u}{\partial n} \in C$
- Mixed : $u \in (u_n \cap C)$

(2) $$u(x,y) = F(x)G(y)$$

$$\frac{1}{F}\frac{d^2F}{dx^2} = -\frac{1}{G}\frac{d^2G}{dy^2} = -k$$

$$\frac{d^2F}{dx^2} + kF = 0 \quad ; \quad \frac{d^2G}{dy^2} - \left(\frac{n\pi}{a}\right)^2 G = 0$$

$$\boxed{F(x) = F_n(x) = \sin\frac{n\pi}{a}x} \; ; \; \boxed{G(y) = A_n e^{\frac{n\pi y}{a}} + B_n e^{-\frac{n\pi y}{a}}}$$

(3)
- Boundary Conditions : $u = 0$, $\forall$ R-side
- Initial Conditions :

so: $$G(0) = A_n e^{0} + B_n e^{0} = A_n + B_n = 0$$

$$\boxed{A_n = -B_n \text{ or } B_n = -A_n}$$

so: $$G(y) = A_n e^{ky} + B_n e^{-ky} = A_n e^{ky} - A_n e^{-ky} = A_n(e^{ky} - e^{-ky})$$

$$= 2A_n \sinh\frac{n\pi y}{a}.$$

So: $$\boxed{u(x,y) = 2\sin\frac{n\pi}{a}x \,\sinh\frac{n\pi y}{a}}$$

CIRCULAR MEMBRANE - BESSEL'S EQS (V.10)

1. our old buddy: $\boxed{\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}}$ → Laplace wave in Cartesian

is NOW: $$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)}$$ → Laplace wave in Polar.

The solution is: $u(R,t) = 0$

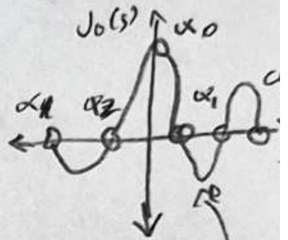
[1] Solve: via ODE separation:

1.) $\boxed{u(r,t) = W(r)\,G(t)}$ → $u_{rr} = W''(r)\,G(t)$, $u_{tt} = \ddot{G}(t)\,W(r)$

So: $$\frac{\ddot{G}}{c^2 G} = \frac{1}{W}\left(W'' + \frac{1}{r}W'\right) = -k^2$$

Briefly: $\boxed{\ddot{G} + \lambda^2 G = 0}$ and $\boxed{W'' + \frac{1}{r}W' + k^2 W = 0}$

2.) Consider $s = kr$. Now: $\frac{1}{r} = \frac{k}{s}$ so via chain rule: $\left(\frac{1}{R} = \frac{r}{s}\right.$.



$$W' = \frac{dW}{dr} = \frac{dW}{ds}\frac{ds}{dr} = \frac{dW}{ds}k \Rightarrow W'' = \frac{d}{dr}\frac{dW}{dr} = \frac{d^2W}{ds^2}k^2$$

so: $$W'' + \frac{1}{r}W' + k^2 W = \frac{d^2W}{ds^2}k^2 + \frac{k}{r}\frac{dW}{ds} + k^2 W = 0$$

$$= \left(\frac{d^2W}{ds^2} + \frac{r}{k}\frac{dW}{ds} + W\right)k^2 = \left(\frac{d^2W}{ds^2} + \frac{1}{s}\frac{dW}{ds} + W\right) = 0$$

and: $$\frac{d^2W}{ds^2} + \frac{1}{s}\frac{dW}{ds} + W \Rightarrow \boxed{W = C_1 J_1(s) + C_2 J_2(s)}$$

w/ J_i = Bessel eqs of the i^{th} kind.

$W(r) = J_0(s) = J_0(kr)$

b.) $W(R) = J_0(kR) = 0 \iff u(R,t) = W(R)\,G(t) = 0.$

Suppose $S = \{\alpha_1\ \alpha_2\ \alpha_3 \ldots\ldots\}$. The numerical values is $\{2.4048, 5.5201, 8.6537 \ldots\}$

The zeros are irregularly spaced, s.t.: $kR = \alpha_m$, $k = k_m = \frac{\alpha_m}{R}$.

type: $$\boxed{W_m(r) = J_0(k_m r) = J_0\left(\frac{\alpha_m}{R}r\right)}$$

c.) Since $G_m(t) = a_m \cos\lambda_m t + b_m \sin\lambda_m t$, The solution is. (pre-full).

$$\boxed{u_m(r,t) = \sum_{m=1}^{\infty}\left[a_m \cos(\lambda_m t) + b_m \sin(\lambda_m t)\right]J_0\left(\frac{\alpha_m}{R}r\right)}$$ w/

$\boxed{\text{ii}}$ Full solution?:

set $t=0$ s.t.

$$\boxed{u(r,0) = \sum_{m=1}^{\infty} a_m J_0\left(\frac{\alpha_m r}{R}\right) = f(r)}$$

The Fourier : kefor $\frac{2}{L}\int_0^{\frac{L}{2}} x \sin\frac{n\pi x}{L} + (L-x)\int_{\frac{L}{2}}^{L} \sin\left(\frac{n\pi x}{L}\right)dx \overset{\Delta}{\Rightarrow} b_m$

from problem $\boxed{11.5}$

Now first look at it as: $\sin(\cdot) \to J_0\left(\frac{\alpha_m r}{R}\right)$

$L \to R^2 J_1^2(\alpha_m)$, $f(x) \to f(r)$, $x \to r$,

So:

$$\boxed{a_m = \frac{2}{R^2 J_1^2(\alpha_m)}\int_0^R r f(r) J_0\left(\frac{\alpha_m}{R} r\right) dr}$$

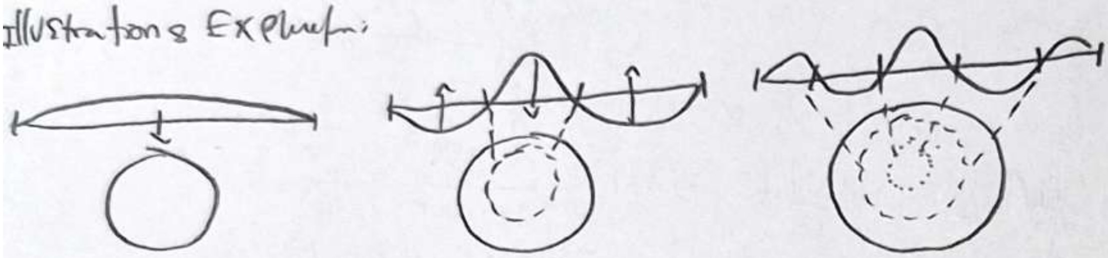
w/ δ_r = differentiability

$\exists\, \delta f(r)$, s.t

$\delta_r f(r) := 0 \leq r \leq R$

is sufficient for the Fourier series

Illustrations Explanation:



$m=1$ $m=2$ $m=3$

Example - vibrations of Circular membrane ...

Find the vibrations of a circular drumhead radius of 1ft & density 2 slugs/ft² If the tension is 8 lb/ft, w/ initial velocity and The initial displacement

$$\boxed{f(r) = 1 - r^2 \text{ (ft)}}$$

Soln: $c^2 = \frac{T}{\rho}^2 = \frac{8 \text{ lb}/\cancel{ft}}{2 \text{ slugs}/\cancel{ft}^2} = 4 \cdot \frac{lb^2}{slugs} \frac{1}{ft} = 4 \frac{ft^2}{sec^2}$ w/ $b_m = 0$

So: $a_m = \frac{2}{J_1^2(\alpha_m)}\int_0^1 r(1-r)^2 J_0(\alpha_m r)\, dr = \frac{4 J_2(\alpha_m)}{\alpha_m^2 J_1^2(\alpha_m)} = \frac{4}{\alpha_m^2 J_1(\alpha_m)}$

So: $\boxed{b_m = \frac{4}{\alpha_m^2 J_1(\alpha_m)}}$

Problem 11.10

(18) Show that the substitution of $u = F(r,\theta)G(t)$ into $u_{tt} = c^2 \nabla_{r,\theta} u$ leads to:

$$\boxed{\begin{array}{l} \ddot{G} + \lambda^2 G = 0 \quad w/ \ \lambda = ck \text{ and} \\ F_{rr} + \frac{1}{r} F_r + \frac{1}{r^2} F_{\theta\theta} + k^2 F = 0 \end{array}}$$

Ans: observe that: $u_{rr} = \frac{\partial^2 F(r)}{\partial r^2} G$ $\quad u_{tt} = F(r,\theta) \frac{d^2 G}{dt^2}$

$$u_{\theta\theta} = \frac{\partial^2 F}{\partial \theta^2} G$$

so: $c^2\left(u_{rr} + \frac{1}{r} u_r + \frac{1}{r} u_{\theta\theta}\right) \triangleq \left(G \frac{\partial^2 F}{\partial r^2} + G \frac{1}{r} \frac{\partial F}{\partial r} + \frac{1}{r^2} G \frac{1}{r^2} \frac{\partial F}{\partial \theta^2}\right) c$

and $u_{tt} = \frac{d^2 G}{dt^2} F$

obvious

so: $\frac{1}{c^2} \frac{d^2 G}{dt^2} F = G \frac{\partial^2 F}{\partial r^2} + G \frac{1}{r} \frac{\partial F}{\partial r} + \frac{1}{r^2} G \frac{\partial^2 F}{\partial \theta^2} = -k^2$

ergo: $\left(\frac{\partial^2 F}{\partial r^2} + \frac{1}{r} \frac{\partial F}{\partial r} + \frac{1}{r^2} \frac{\partial^2 F}{\partial \theta^2}\right) \frac{1}{F} = k^2$ and $\boxed{\ddot{G} + \lambda^2 G = 0}$

thus: $\boxed{F_{rr} + \frac{1}{r} F_r + \frac{1}{r^2} F_{\theta\theta} + F k^2 = 0}$ (18)

(19) Show that $F(r,\theta) = W(r) Q(\theta)$ in (18) leads to:

$\boxed{Q'' + n^2 Q = 0}$ (19) and $\boxed{r^2 W'' + r W' + (k^2 r^2 - n^2) W = 0}$ (20).

Ans: This means: $F_{rr} = W''(r) Q(\theta)$, $F_{\theta\theta} = Q''(\theta) W(r)$

so: $W'' Q + \frac{1}{r} W' Q + \frac{1}{r^2} Q'' W + W Q k^2 = -n^2 = 0$

$$W'' Q + \frac{1}{r} W' Q + k^2 W Q = -\frac{1}{r^2} Q'' W = -n^2 \text{ (19a)}$$

and: $W'' Q + \frac{1}{r} W' Q + k^2 W Q$

so: $\begin{cases} W'' + \frac{1}{r} W' + k^2 W + n^2 W = 0 \ \text{(19a)} \\ \frac{1}{r^2} Q'' + n^2 Q = 0 \ \text{(19b)} \end{cases}$

If multiply both eqs by the r^2: Then (19a) is:

$$\boxed{\begin{array}{c} r^2 W'' + r W' + k^2 r^2 W - n^2 W = 0 \\ Q'' + n^2 Q = 0 \end{array}}$$

■

LAPLACE'S TRANSFORM BASICS (5.1)

$$\boxed{F(s) = \int_0^{\infty} e^{-st} f(t) = at}$$

① $\boxed{f(t) = 1}$ $t \geq 0$. Find $F(s)$:

Ans: $\mathcal{L}\{f\} = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty}$

$= \frac{1}{s} e^{-s\infty} - e^{-0} = \frac{1}{s} e^{-0} = \boxed{\frac{1}{s}}$

② $\boxed{f(t) = e^{at}}$ when $t \geq 0$ w/ a = cons

Ans $\mathcal{L}(t) = \mathcal{L}\{e^{at}\} = \int_0^{\infty} e^{at} e^{-st} dt$

$= \int_0^{\infty} e^{(a-s)t} dt = \frac{1}{a-s} e^{(a-s)t} \Big|_0^{\infty} = -\frac{1}{a-s}$

$= \boxed{\frac{1}{s-a}}$ w/ $s - a > 0$

③ $\boxed{\text{TH : 1} \quad \mathcal{L}\{a f(t) + b g(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}}$

↳ Linearity

[i] Proof: $\mathcal{L}\{af(t) + bg(t)\} = \int_0^{\infty} e^{-st} [a f(t) + b g(t)] dt$

$= \int_0^{\infty} e^{-st} a f(t) dt + \int_0^{\infty} e^{-st} b f(t) dt$

$= a \int_0^{\infty} f(t) e^{-st} dt + b \int_0^{\infty} e^{-st} g(t)$

$\boxed{= a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}}$

As we have proven.

[ii] EXAMPLE:

$$\boxed{f(t) = \cosh at = \frac{e^{at}+e^{-at}}{2}}$$

ANS: $\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\frac{e^{at}+e^{-at}}{2}\right\} = \frac{1}{2}\int_0^{\infty}(e^{at}+e^{-at})e^{-st}$

$$= \frac{1}{2}\left[\int_0^{\infty} e^{(a-s)t}\,dt + \int_0^{\infty} e^{-a-s}\,dt\right]$$

$$= \frac{1}{2}\mathcal{L}\{e^{at}\} + \frac{1}{2}\mathcal{L}\{e^{-at}\} = \boxed{\frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right)}$$

(b) When $s > a\ (\geqq 0)$:

$$\boxed{\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2}}$$

w/ $\frac{s+a}{(s-a)s+a} + \frac{s-a}{(s+a)s-a} = \frac{s+a+s-a}{(s+a)(s-a)} = \frac{\cancel{2}s}{s^2-sa+sa-a^2}\frac{1}{\cancel{2}}$

$$= \boxed{\frac{s}{s^2-a^2}}$$

(4) Inverse Laplace

$$F(s) = \frac{1}{(s-a)(s-b)} \quad \text{find } \mathcal{L}^{-1}\{F\}$$

ANS: $\mathcal{L}^{-1}(F) = \mathcal{L}^{-1}\left\{\frac{1}{a-b}\left(\frac{1}{s-a}-\frac{1}{s-b}\right)\right\}$

$$= \frac{1}{a-b}\left[\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) - \mathcal{L}^{-1}\left(\frac{1}{s-b}\right)\right]$$

$$= \frac{1}{a-b}(e^{at}-e^{bt})$$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\mathcal{L}\{t^{-\frac{1}{2}}\} = \int_0^{\infty} e^{-st}t^{-\frac{1}{2}} = \frac{1}{\sqrt{s}}\int_0^{\infty} e^{-x}x^{-\frac{1}{2}}\,dx = \frac{1}{\sqrt{s}}\Gamma\left(\frac{1}{2}\right)$$

$$= \boxed{\sqrt{\frac{\pi}{s}}} \Longrightarrow \text{Bonus 'Ex'}$$

LAPLACIAN TRANSFORMS OF PDES

(1.) $$\frac{\partial W}{\partial x} + x\frac{\partial W}{\partial t} = 0 \qquad W(x,0)=0 \quad W(0,t)=t$$

Ans: [i] $$\mathcal{L}\left\{\frac{\partial W}{\partial x}\right\} + x\left[s\mathcal{L}\{W\} - W(x,0)\right] = 0$$ Take Laplace in resp. t

$$\mathcal{L}\left\{\frac{\partial W}{\partial x}\right\} = \int_0^\infty e^{-st}\frac{\partial W}{\partial x}\,dt = \frac{\partial}{\partial x}\int_0^\infty e^{-st}W(x,t)\,dt$$
$$= \frac{\partial}{\partial x}\mathcal{L}\{W(x,t)\}$$

suppose $W(x,s) = \mathcal{L}\{W(x,t)\}$. So:

$$\boxed{\frac{\partial W}{\partial x} + sWx = 0}$$

[ii] $$\frac{dW}{dx} + sWx = 0 \rightarrow \int\frac{dW}{W} = \int sx\,dx$$

$$\boxed{W(x,s) = e^{-\frac{sx^2}{2}}}$$

Since $\mathcal{L}\{t\} = \frac{1}{s^2}$. The condition $W(0,t)=t$ yields

$W(0,s) = \frac{1}{s^2}$. So:

$$W(0,s) = C(s) = \frac{1}{s^2} \Rightarrow \boxed{W(x,s) = \frac{1}{s^2}e^{-\frac{sx^2}{2}}}$$

[iii] $$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t \qquad a = \frac{x^2}{2} \text{ So:}$$

$$\boxed{W(x,t) = \left(t - \frac{x^2}{2}\right)u\left(t - \frac{1}{2}x^2\right) = \begin{cases} 0 & t < \frac{x^2}{2} \\ t - \frac{1}{2}x^2 & t > \frac{x^2}{2}\end{cases}}$$

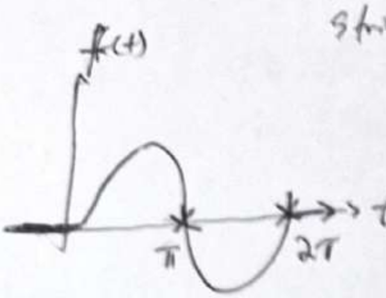
② SEMI - INFINITE STRING:

Find the Displacement of $W(x,t)$ of an elastic string subject to $[x=0, \infty]$

(i) string is Initially at Rest on x-axis,

(ii) $\forall t>0,\ W(0,t) = f(t) = \begin{cases} \sin(t) & \text{if } 0 \le t \le 2\pi \\ 0 & \text{otherwise} \end{cases}$

(iii) $\lim_{x \to \infty} W(x,t) = 0$ for $t \geq 0$



Ans:

$$\boxed{\frac{\partial^2 W}{\partial t^2} = c^2 \frac{\partial^2 W}{\partial x^2}}$$

w/ $W(0,t) = f(t) \rightarrow W(x,0) = 0$

$\lim_{x\to\infty} W(x,t) = 0 \rightarrow \left.\frac{\partial W}{\partial t}\right|_{t=0} = 0$

[i] $$\frac{\partial^2 W}{\partial t^2} - c^2 \frac{\partial^2 W}{\partial x^2} \triangleq \mathcal{L}\left\{\frac{\partial^2 W}{\partial t^2}\right\} - c^2 \mathcal{L}\left\{\frac{\partial^2 W}{\partial x^2}\right\} = 0$$

$$\mathcal{L}\left\{\frac{\partial^2 W}{\partial t^2}\right\} = s^2 \mathcal{L}\{W\} - sW(x,0) - \left.\frac{\partial W}{\partial t}\right|_{t=0} = c^2 \mathcal{L}\left\{\frac{\partial^2 W}{\partial x^2}\right\}$$

$$\mathcal{L}\left\{\frac{\partial^2 W}{\partial x^2}\right\} = \frac{\partial^2}{\partial x^2} \int_0^\infty e^{-st} W(x,t)\, dt = \frac{\partial^2}{\partial x^2} \mathcal{L}\{W(x,t)\}$$

[ii] $W(x,s) = \mathcal{L}\{W(x,t)\}$ # From Defn

$$s^2 W = c^2 \frac{\partial^2 W}{\partial x^2} \longrightarrow \boxed{\frac{\partial^2 W}{\partial x^2} - \frac{c^2}{s^2} W = 0}$$

or: $$\frac{d^2 W}{dx^2} - \frac{c^2}{s^2} W = 0$$

$$\boxed{W(x,s) = A(s)\, e^{\frac{sx}{c}} + B(s)\, e^{-\frac{sx}{c}}}$$

[iii] $$W(0,s) = \mathcal{L}\{W(0,t)\} = \mathcal{L}\{f(t)\} = F(s)$$

$$\lim_{x\to\infty} W(x,s) = \lim_{x\to\infty} \int_0^\infty e^{-st} W(x,t)\, dt = \int_0^\infty e^{-st} \lim_{x\to\infty} W(x,t)\, dt = \boxed{0}$$

[iv] $$\boxed{W(x,t) = f\left(t - \frac{x}{c}\right) u\left(t - \frac{x}{c}\right)}$$

w/ $\begin{cases} W(0,s) = B(s) = F(s) \\ W(x,s) = F(s)\, e^{-\frac{sx}{c}} \end{cases}$

CH 14.1: SEQUENCES & SERIES

/14.1

(1) $Z = \{z_1 \cdots z_n\} = \{z_n\}_{i=1}^{n}$

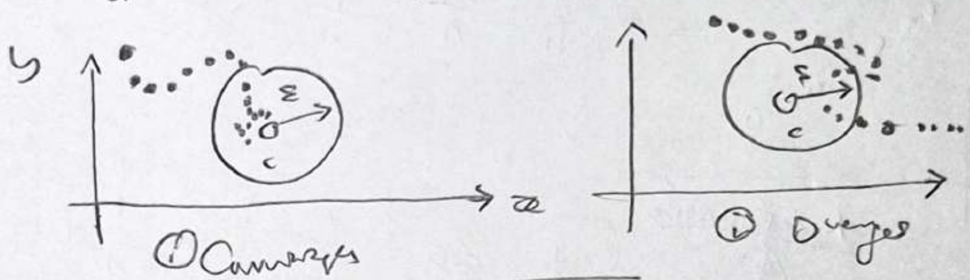
is a sequence

$$\boxed{|z_n - c| < \varepsilon} \Rightarrow \text{convergence}$$

$\lim_{n\to\infty} z_n = c$ or simply $z_n \to c$.

i.e. $\boxed{\forall\, \varepsilon > 0, \text{ we can find } N \text{ such that } |z_n - c| < \varepsilon.}$

or else it's Divergent



(2) $\boxed{Z = \left\{ z_n = 1 + \frac{2}{n} \;\middle|\; n \in \mathbb{Z} \right\}}$

So: $z_n : \frac{1}{4}, \frac{2}{4}, \frac{1}{5}, \frac{4}{5}, \frac{1}{c}, \frac{5}{f}, \ldots\ldots$

(1) Is it convergent?

ANS: Yes! $\lim_{n\to\infty} z_n = c = 1$

$z - c = 1 + \frac{2}{n} - 1$

$\frac{2}{n} < \varepsilon \rightarrow$

$n > \frac{\varepsilon}{2} \rightarrow$

(3) $Z = \{ Z_n = x_n + iy_n \}$

$\boxed{Z = \{ Z_n : 2 - \frac{1}{n} + i(1 + \frac{2}{n}) \}}$ w/ $c = 2+i$

So: Z_n : $1+3i$, $\frac{3}{2}+2i$, $\frac{5}{3}+\frac{5}{3}i$ ...

$|Z_n - c| < \varepsilon$ means $|2 - \frac{1}{n} + i(1+\frac{2}{n}) - c| < \varepsilon$

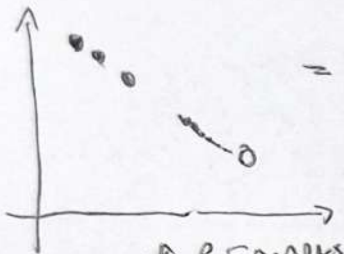
So: $|2 - \frac{1}{n} + i(1+\frac{2}{n}) - c| = |2 - \frac{1}{n} + i + \frac{2}{n} - c|$

$= |2 - \frac{1}{n} + i\frac{n+2}{n} - c| = |\frac{2n-1}{n} + \frac{i(n+2)}{n} - c| < \varepsilon$

$= |\frac{2n-1}{n} + \frac{i(n+2)}{n} - (2+i)| < \varepsilon$

$= -\frac{1}{n} + \frac{2i}{n} = \frac{\sqrt{5}}{n} < \varepsilon$

So: $\boxed{\frac{\sqrt{5}}{n} < \varepsilon \quad \frac{\sqrt{5}}{\varepsilon} < n}$



* REMARKS

(i) $\lim_{n \to \infty} Z_n = \lim_{n \to \infty} |2 - \frac{1}{n} + i\frac{n-2}{n}|$

$= |2 - \frac{1}{\infty} + i\frac{n-2}{\infty}|$

$= \boxed{2+i}$

or $\lim_{n\to\infty} Re(Z_n) = \lim_{n\to\infty} (2 - \frac{1}{n}) = 2$

$\lim_{n\to\infty} Im(Z_n) = \lim_{n\to\infty} (\frac{n+2}{n}) = 1$ $\Rightarrow \boxed{2+i}$

(ii) $\lim_{n\to\infty} 1 + \frac{2}{n} = 1 + \frac{2}{\infty} = \boxed{1}$

PROBLEMS 14.1

Plot the Terms for the following and find the 1st few sequences :

(1) $$\frac{i^n}{n^3} = \left\{ \frac{\left((-1)^{\frac{1}{2}}\right)^1}{1^3}, \frac{\left((-1)^{\frac{1}{2}\cdot 2}\right)}{2^3}, \frac{\left((-1)^{\frac{1}{2}\cdot 3}\right)}{3^3} \cdots \right\}$$

$$= \left\{ i, -\frac{1}{8}, \frac{i^3}{27}, \cdots \right\}$$

(2) $$\frac{in}{(n+1)} = \left\{ \frac{i}{2}, \frac{2i}{3}, \frac{3i}{4}, \frac{4i}{5}, \cdots \right\}$$

(3) $$\frac{i^n n^2}{(n+i)} = \left\{ \frac{i}{1+i}, \frac{(-1)4}{2+i}, \frac{i^3 9}{3+i}, \frac{-16}{4+i} \right\}$$

(5) $$e^{in\pi 4} = \left\{ e^{i\pi 4}, e^{i8\pi}, e^{i12\pi}, e^{i20\pi}, \cdots \right\}$$

$$\text{w/ } e^{in\pi 4} = e^{ix} = \cos x + i\sin(x) = \underline{\cos(n\pi 4) + i\sin(n\pi 4)}$$

$$\text{So: } \left\{ \frac{1+i}{\sqrt{2}} \right\}$$

Are the Following Sequences $z_1 \ldots z_n$ Convergent?

(4) $$z_n = \frac{i^n}{n} = \left\{ \frac{i}{1}, \frac{-1}{2}, \frac{i}{3}, -\frac{1}{4}, \cdots \right\}$$

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \frac{i^\infty}{\infty} = \underline{\text{Vanishes}}$$

$$\text{So: } \left| \frac{i^n}{n} - 0 \right| < \varepsilon \quad 0 < \varepsilon \qquad \frac{1}{n} + i^n \qquad \lim_{n\to\infty} \frac{1}{n} = 0 \qquad \lim_{n\to\infty} i^n = \infty$$

(9) $$z_n = \frac{n^2}{n+i} \qquad \lim_{n\to\infty} S_n = S$$

$$S = \sum_k z_n = \sum_n \frac{n^2}{n+i} = \frac{1}{1+i} + \frac{4}{2+i} + \frac{9}{3+i} + \cdots$$

$$S_{n-1} = \sum_n z_{n-1} = \sum_n \frac{(n-1)^2}{n-1+i} = 0 + \frac{1}{1+i} + \frac{4}{1+i} + \cdots$$

and as $n \to \infty$: $\frac{\infty^2}{\infty + i}$ = Diverges? So Not Conv

(11) $S = \sum_n e^{\frac{in\pi}{4}} = \sum_n \left[\cos\left(\frac{n\pi}{4}\right) + i\sin\left(\frac{n\pi}{4}\right)\right]$

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_n \left[\cos\left(\frac{n\pi}{4}\right) + i\sin\left(\frac{n\pi}{4}\right)\right]$

$= \sum_n \cos\left(\infty\frac{\pi}{4}\right) + i\sin\left(\infty\frac{\pi}{4}\right)$

$= 1 + \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} + i - \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} - 1 \ldots$

$\left(1 + \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} + i + \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) + 1$

and $\left|1 + \frac{\sqrt{2}}{2} + \ldots\right| = \left(1 + \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$ keeps on cycling then it means it ~~converges~~.

(8i) $\boxed{Z_n = i^n}$

$\sum_n Z_n = \sum_n i^n = (i - 1 - i + 1) + (i - 1 - i + \ldots)$

$= k(i - 1 - i + 1)$ as $k \to \infty$

$= \boxed{i - 1 - i + 1} \to$ diverges

(13) $\boxed{Z_n = \frac{(1+2i)^n}{n!}}$

(i) $\sum_n Z_n = \sum_n \frac{(1+2i)^n}{n!} = \frac{1+2i}{1!} + \frac{(1+2i)^2}{2!} + \frac{(1+2i)^3}{3!} + \ldots$

$= x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = e^x = e^{(1+2i)}$

so $\boxed{(1+2i) \to x}$ which shows it converges

(ii) $\left|\frac{Z_{n+1}}{Z_n}\right| = \left|\frac{(1+2i)^{n+1}/(n+1)!}{(1+2i)^n/n!}\right|$

$= \frac{(1+2i)^{n+1}}{(1+2i)^n} \cdot \frac{n!}{(n+1)!} = (1+2i)^n \cdot \frac{1}{n+1}$

$|1+2i| = \sqrt{1^2+2^2} = \sqrt{5}$ so $\frac{\sqrt{5}}{n+1}$ and $\boxed{n\to\infty : \frac{\sqrt{5}}{n+1} = 0}$

W4 Ma: Theorems in Sequences & Series

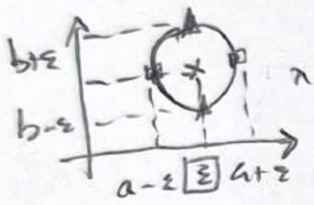
(1) Complex Convergence (2) Cauchy's convergence principle (3) Divergence (4) Comparison (5) General

(1) **TH 1:** $z_n = x_n + iy_n \to c = a + ib$ iff
$Re(x_1 \dots x_n) \to a$, $Im(y_1 \dots y_n) \to b$

(i) Proof: if $|z_n - c| < \varepsilon$ then $(z_n = x_n + iy_n) \in \varepsilon \sim c$

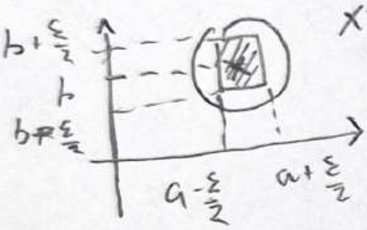
wh/ $c = a + ib$

So: $|x_n - a| < \varepsilon$, $|y_n - b| < \varepsilon$.



$\lim_{n \to \infty} |x_n - a| = |a - a| = 0 < \varepsilon$ given its true

$\lim_{n \to \infty} |y_n - b| = |b - b| = 0 < \varepsilon$ since $\forall \varepsilon > 0$



and: $|x_n - a| < \frac{\varepsilon}{2}$ $|y_n - b| < \frac{\varepsilon}{2}$

given n is large ■

(2) **TH 2:** $\sum_n z_n$ is convergent iff $\forall \varepsilon > 0$, $\exists N$ s.t.
$\sum_{m=1}^{\infty} |z_m| = \sum |z_1| + |z_2| \dots < \varepsilon$

Cauchy's Convergence Theorem

otherwise

Example: $(-1)^{2n}$ $\sum_n (-1)^{2n} \frac{1}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

So: $z_n = (-1)^{2n} \frac{1}{n}$

$|z_n| = \left|(-1)^{2n} \frac{1}{n}\right| = \frac{1}{n}$

(3) $\boxed{\text{TH 3: if } \sum_n z_n \text{ Converges then } \boxed{\lim_{n\to\infty} z_n = 0} \text{ if not then it } \underline{\text{DIVERGES}}}$ Divergence Theorem

(1) PROOF: $z_1 + \dots + z_n$ Converges since $z_n = S_n - S_{n-1}$ over sum S

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} (S_n - S_{n-1}) = S_\infty - S_{\infty-1} = S - S = \boxed{0}$$

(2) CAUTION ⚠ : $z_n \to 0$ Necessary but not sufficient

for ex: $\sum_n \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

(4) TH 4 : $z_1 + z_2 + \dots + z_n$ w/ convergent $b_1 + \dots + b_n$ w/ non-negative terms : $|z_n| \leq b_n$ for $n = 1, \dots, n$)
Then the series Converges, even Absolutely

Proof: By Cauchy : $b_1 + b_2 + b_3 + \dots$ Converges $\forall \varepsilon > 0$

so: $b_{n+1} + \dots b_{n+p} < \varepsilon, \forall n > N, p = 1, 2 \dots$

$|z_1| \leq b_1 ; |z_2| \leq b_2$

$|z_{n+1}| + \dots |z_{n+p}| \leq b_{n+1} + \dots + b_{n+p} < \varepsilon$

$|z_1| + |z_2| + |z_3| + \dots$ Converges since from Cauchy's principle $b_1 + b_2 + b_3 + \dots$ also Converges since it's proportional to $|z_n|$

ergo $\boxed{|z_n| \leq b_n \text{ Series Converges}}$

(5) TH 5 (Geometric Series) : $1 + \dots$

$$\sum_{k=1}^{\infty} q^k = \frac{q^{k+1}}{1-q}$$

since $S_n = 1 + \dots q^k$

$qS_n = q + \dots q^{k+1}$

$S_n - qS_n = (1-q) S_n = 1 - q^{k+1}$

$$\boxed{S_n = \frac{1-q^{k+1}}{1-q}}$$

ON: M2 RATIO TEST TEST

(1.) TH 5 : $\left|\frac{Z_{n+1}}{Z_n}\right| \leqq q < 1$ Converges $\forall n > N$

$\left|\frac{Z_{n+1}}{Z_n}\right| \geqq 1$ Diverges iff $\forall n > N$

Proof:

$|Z_{n+1}| \geqq |Z_n|$ iff $\left|\frac{Z_{n+1}}{Z_n}\right| \geqq 1$ holds

$|Z_{n+1}| \geqq q|Z_n|$ iff $\left|\frac{Z_{n+1}}{Z_n}\right| \geqq q$ holds

then: $\sum_n |Z_{n+1}| \geqq \sum_n q|Z_n|$

$$\sum_p |Z_{n+p}| \leqq \sum_p q^{p-1}|Z_{n+1}|$$

$$|Z_{n+1}| + |Z_{n+2}| + \ldots \leqq |Z_{n+1}|(1+q+q^2+\ldots)$$

$$\leqq |Z_{n+1}| \frac{1-q^{n+1}}{1-q}$$

$$\boxed{\frac{|Z_{n+1}|+|Z_{n+2}|+\ldots}{|Z_{n+1}|} \leqq \frac{1}{1-q}}$$

$Z_p = \frac{\sum Z_p}{Z_{n+1}}$

Comparison test : $Z_p \leqq b_p$ is convergent

Geometric test : $\frac{1}{1-q}$ is convergent ∎

(2.) $\boxed{TH\ 6 : \lim_{n\to\infty}\left|\frac{Z_{n+1}}{Z_n}\right| = L}$ ⇒ Ratio test

- If $L<1$: Converges Absolutely → (a.)
- If $L>1$: Diverges Absolutely → (b.)
- If $L=1$: Test fails! → (c.)

① Proof:

(a) $k_n = \left|\frac{z_{n+1}}{z_n}\right|$ and let $(L = 1-b) < 1$.

$$\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = 1-b \quad \text{or} \quad \lim_{n\to\infty} k_n = L$$

Say $k_n \leqq q = 1 - \frac{1}{2}b < 1 \quad \forall n > N$

Then $\frac{|z_{n+1}|}{|z_n|} \leqq q < 1$ follows by theorem 5

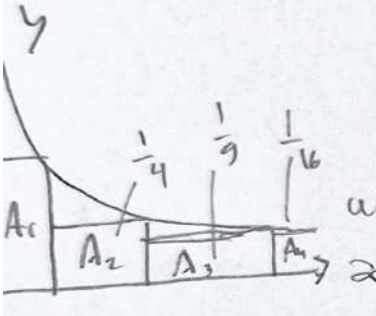
(b) $L = 1+c$ and $k_n \geqq \left(1 + \frac{1}{2}c\right) > 1 \quad \forall n \geq N^*$

Then $\frac{|z_{n+1}|}{|z_n|} > 1$ diverges via theorem 5

(c) $L = 1$, Then $\lim_{n\to\infty} k_n = 1$ or $\lim_{n\to\infty}\left|\frac{z_{n+1}}{z_n}\right| = 1$

and $\frac{\sum z_{n+1}}{\sum z_n} = \sum_n \frac{z_{n+1}}{z_n} = \frac{n^2}{(n+1)^2}$ converges

but also $\frac{z_{n+1}}{z_n} = \frac{n}{n+1}$; $L = 1$ Diverges



and $S_n = 1 + \frac{1}{4} + \frac{1}{9} + \dots \frac{1}{n^2} = \sum_n \frac{1}{n^2} \triangleq \sum_x \frac{1}{x^2}$

then $\lim_{x\to\infty} \sum_x \frac{1}{x^2} = \int_1^n \frac{1}{x}\,dx = \boxed{2 - \frac{1}{n}}$

(ii) Example:

$$\sum_{n=0}^{\infty} \frac{(100+75i)^n}{n!}$$

$$\frac{|z_{n+1}|}{z_n} = \frac{(100+75i)^{n+1}}{(n+1)!} \cdot \left(\frac{(100+75i)^n}{n}\right)^{-1}$$

$\sqrt{100^2 + 75^2} = \sqrt{10000 + 5625} = \boxed{125}$

$$= \frac{100+75i}{n+1} = \frac{125}{n+1} \to \lim_{n\to\infty} \frac{125}{\infty+1} = \boxed{0}$$

$125 =$

$\boxed{L = 0}$

Root Test

$$\boxed{\begin{array}{l} \text{Th 7:} \quad \sqrt[n]{z_n} \leq q < 1 \Rightarrow \text{Converges} \\ \qquad\quad \sqrt[n]{z_n} \geq 1 \Rightarrow \text{Diverges.} \end{array}}$$

(1) Proof: if $\lim_{n\to\infty} \sqrt[n]{z_n} = L$ holds w/ $L < 1$.

Then $\sqrt[n]{z_n} \leq q < 1$ *converges by geometric series

$\therefore |z_n| \leq q^n < 1 \quad \forall n > N \Rightarrow \boxed{\text{converges}}$ ∎

*similarly: if $\lim_{n\to\infty} \sqrt[n]{z_n} = L$ holds w/ $L > 1$

Then $\sqrt[n]{z_n} \leq 1$

$|z_n| \leq 1$ holds $\forall n > N \Rightarrow \boxed{\text{Diverges}}$ ∎

(2) $\boxed{\text{Th 8} : \sqrt[n]{|z_n|} = 1}$

$L<1 \to$ Converges
$L>1 \to$ Diverges
$L=1 \to$ fails

Proof: follows from Th 7, Th 6, Th 5. ∎

examples: $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}+3}(4-i)^n = \frac{1}{4} - \frac{1}{7}(4-i) + \frac{1}{19}(4-i)^2$

now: $\displaystyle\sqrt[n]{\frac{(-1)^n}{2^{2n}+3}(4-i)^n} = \frac{|4-i|}{\sqrt[n]{4^n+3}} = \frac{\sqrt{17}}{\sqrt[n]{4^n+3}}$

so $\displaystyle \frac{\sqrt{17}}{\sqrt[n]{4^n+3}} = \frac{\sqrt{17}}{(4^n+3)^{1/n}} \simeq \frac{\sqrt{17}}{(4^n)^{\frac{1}{n}}} = \boxed{\frac{\sqrt{17}}{4}}$

and $\frac{\sqrt{17}}{4} \simeq \boxed{1.0308 > 1}$ so it Diverges

PROBLEMS 14.2

Are the following sequences $z_1 \ldots z_n$ bounded? convergent? Find their limit points

(1.) $\boxed{z_n = \left\{(-1)^n + \frac{i}{n}\right\}} \triangleq \left\{\left(\frac{i}{n}+1\right), \left(\frac{i}{2}+1\right), \left(\frac{i}{3}-1\right)\right\}$

$(-1)^n$ Dosent converge $\qquad \lim_{n\to\infty} (-1)^\infty = \boxed{\pm 1}$

$\frac{i}{n} = i\frac{1}{n}$ = converges to 0

since $z_n \triangleq Re(-1^n) + im\left(\frac{1}{n}\right)$ and not Both of them converge, z_n $\boxed{\text{dosent converge.}}$

(2.) $\boxed{z_n = e^{\frac{in\pi}{2}}} = \cos\left(\frac{n\pi}{2}\right) + i\sin\left(\frac{n\pi}{2}\right)$

$\triangleq \{1, i, -1, -i, 1\ i, -1, -i, 1\}$

$\triangleq \{(1, i, -1, -i) ; \{1, i, -1, -i, 1\}\} \rightarrow$ keeps oscillating

so : $\boxed{\text{It dosen't converges}}$

(3.) $\boxed{z_n = i^n \cos n\pi.} \Rightarrow$ $i^{n\to\infty} \{i, -1, i, -1, i, -i\}$ Diverges

$\cos(n\pi)_{n\to\infty} = \{1, -1, \ldots\}$ Diverges

so : $z_n = i^n \cos n\pi$ Diverges

w/ $\lim_{n\to\infty} i^n = \pm i$

$\lim_{n\to\infty} \cos(n\pi) = \pm 1$

(4.) $z_n = (-1)^n + 2i = Re(-1^n) + im(2)$

$-1^{n\to\infty} = \{-1, 1, -1, \ldots\}$ Diverges

2 = converges

so $\lim_{n\to\infty} (-1)^n + 2i = \boxed{\pm(1 + 2i)}$

Problems 11.5 - A

Find the tempature $u(x,t)$ in a bar of silver w/ length 10 cm
cons. cross section area 1 cm², density 10.6 gm/cm³,
thermal conductivity 1.04 cal/cm sec °C, specific heat 0.056 cal/gm°C
that is perfectly insulated laterally, whose ends are kept at 0°C

(4) $\boxed{f(x) = \sin 0.2\pi x}$ w/ $\longrightarrow$

$\rho = 10$ gm/cm³
$k = 1.04$ cal/cm sec °C
$\sigma = 0.056$

Ans:

$$f(x) = \sin 0.2\pi x = \sin\left(\frac{2\pi x}{10}\right) °C$$

$$g(t) = e^{\lambda_n t} , = e^{\lambda_2 t} =$$

$$c^2 = \frac{k}{\sigma\rho} = \frac{1.04}{(0.056)(10)} = 1.752$$

$$\lambda_n = \frac{c n\pi}{L} := \frac{2c\pi^2}{L^2} = \frac{2\,1.752\pi^2}{100} =$$

So: $$u(x,t) = \sin\frac{2\pi x}{10}\, e^{-\frac{3.5040\pi^2}{100}}$$

(5) $\boxed{f(x) = \sin 0.1\pi x}$

Ans: $$f(x) = \sin 0.1\pi x = \sin\frac{1}{10}\pi x \rightarrow n = 1$$

So: $$c^2 = \frac{k}{\sigma\rho} = 1.8571$$

$$\lambda_{n=1} = \frac{c\pi^2}{L^2} = \frac{1.752\pi^2}{100.}$$

So: $$\boxed{u(x,t) = \sin\frac{\pi x}{10}\, e^{-\frac{1.752\pi^2}{100}t}}$$

(8.) $f(x) = \begin{cases} x & \text{if } 0 < x < 5 \\ 10 - x & \text{if } 5 < x < 10 \end{cases}$

Ans: We have $c^2 = 1.752$ so for $f(x)$..

[i] $f(x)\big|_{0<x<5} = x$ $\rightarrow$ $b_n = \frac{2}{10}\int_0^5 x \sin\frac{n\pi x}{10}\,dx + \int_5^{10} (10-x)\frac{n\pi x}{10}\,dx$

$f(x)\big|_{5<x<10} = 10 - x$

[ii] $\int_0^5 x \sin\left(\frac{n\pi x}{10}\right)dx = \frac{\sin(ax)}{a^2}\Big| - \frac{x\cos(ax)}{a}\Big|_0^5$

$\int_5^{10} 10 - x \sin\frac{n\pi x}{10}\,dx = \int_5^{10} 10\frac{\sin n\pi x}{10}\,dx + \int_5^{10} x\frac{\sin n\pi x}{10}\,dx$

$= 10\frac{\cos ax}{a}\Big|_5^{10} + \frac{\sin(ax)}{a^2}\Big|_5^{10} - \frac{x\cos(ax)}{a}\Big|_0^5$

algebra on other paper lol!

$= (10 - x)\frac{\cos ax}{a}\Big|_5^{10} + \frac{\sin(ax)}{a^2}\Big|_0^5$

$= \frac{40}{(n\pi)^2}\sin\left(\frac{n\pi}{10}\right)$ so: $b_n = \begin{cases} \frac{40}{n^2\pi^2}\sin & n \in \text{odd} \\ 0 & n \in \text{even} \end{cases}$

[iii] $u(x,t) = \frac{40}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)e^{-\frac{1.752(\pi n)^2}{100}}$

Ans key! w/ $= \frac{40}{\pi^2}\left(\sin\frac{\pi x}{10}e^{-c^2\pi^2 t} \ldots \frac{1}{9}\sin\frac{3\pi x}{10} \ldots\right)$

So: $\boxed{u(x,t) = \sum_{\substack{n=1 \\ n=\text{odd}}}^{\infty} \frac{40}{n^2\pi^2} e^{-\frac{1.752(n\pi)^2}{100}t}\sin\left(\frac{n\pi x}{10}\right)}$

LAPLACIAN IN POLAR (11.9)

(1) Let : $\begin{cases} y = r\sin\theta \\ x = r\cos\theta \end{cases}$. Then $\boxed{\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0}$

proof:

$\boxed{1}$ We see that. $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. so

1.) $u(y,x) = u(r\sin\theta, r\cos\theta) = u(x(r,\theta), y(r,\theta))$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} x(r,\theta) \sim \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$

so. $\boxed{u_x = u_r r_x + u_\theta \theta_x}$

Ergo: $u_{xx} = u_x(u_r r_x + u_\theta \theta_x) = (u_r)_x r_x + u_r r_{xx} + (u_\theta)_x \theta_x + u_\theta \theta_{xx}$

2.) $(u_r)_x = u_{rr} r_x + u_{r\theta}\theta_x$ and $r = \sqrt{x^2+y^2}$

$(u_\theta)_x = u_{\theta r} r_x + u_{\theta\theta}\theta_x$ $= \sin\theta + \cos\theta$

$\boxed{\theta = \arctan\frac{y}{x}}$

so. $\frac{dr}{dx} = \frac{d}{dx}\frac{1}{\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}} = \boxed{\frac{x}{r}}$

$\frac{d\theta}{dx} = \frac{d}{dx}\arctan\frac{y}{x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} - \frac{y}{x^2} = \boxed{-\frac{y}{r^2}}$

3.) again: $\frac{d^2 r}{dx^2} = \frac{r - x r_x}{r^2} = \frac{1}{r} - \frac{x^2}{r^3} = \boxed{\frac{y^2}{r^3}} =$ w/ $\frac{r^2 - x^2}{r^3} = \boxed{\frac{y^2}{r^3}}$ for $r^2 = x^2+y^2$, $r^2 - x^2 = y^2$

$\frac{d^2\theta}{dx^2} = -y\left(+\frac{2}{r^3}\right) r_x = \boxed{\frac{2xy}{r^4}}$ w/ $\frac{d\theta}{dr}\frac{dr}{dx} \to \frac{d}{dr} - \frac{y}{r^2} = \frac{d}{dr} - y r^{-2} = -(-2r^{-3}y) = 2\frac{y}{3}$

4). Back to u_{xx} w/ $u_{\theta r} = u_{r\theta}$ =

$u_{xx} = \frac{x^2}{r^2}u_{rr} - 2\frac{xy}{r^3}u_{r\theta} + \frac{y^2}{r^4}u_{\theta\theta} + \frac{y^2}{r^3}u_r + 2\frac{xy}{r^4}u_\theta$

$u_{yy} = \frac{y^2}{r^2}u_{rr} + \frac{2xy}{r^3}u_{r\theta} + \frac{x^2}{r^4}u_{\theta\theta} + \frac{x^2}{r^3}u_r - 2\frac{xy}{r^4}u_\theta$ +

$u_{xx} + u_{yy} = \frac{x^2+y^2}{r^2}u_{rr} + \frac{y^2+x^2}{r^4}u_{\theta\theta} + \frac{y^2+x^2}{r^3}u_r = u_{rr} + \frac{1}{r}u_{\theta\theta} + \frac{1}{r}u_{rr}$

Ergo: $\boxed{\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}}$

PROBLEMS 11.?

① Perform the Calculations but for u_{yy}?

Ans: $$u_{yy} = (u_r)_y r_y + u_r r_{yy} + (u_\theta)_y \theta_y + u_\theta \theta_{yy}.$$

$$r = \sqrt{x^2+y^2} \qquad \theta : \arctan\frac{y}{x}$$

1) So: $\frac{dr}{dy} = \frac{y}{\sqrt{x^2+y^2}} = \boxed{\frac{y}{r}}$ and $(u_r)_y = \frac{d^2u}{dr^2}\frac{dr}{dy} = u_{rr}\frac{y^2}{r^2}$

$$\frac{d\theta}{dy} = \frac{d}{dy}\arctan\frac{y}{x} = \frac{f'(y)}{1+[f(y)]^2} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot F'(y) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x}$$

and: $\left(1+\left(\frac{y}{x}\right)^2\right)x = x + \frac{y^2}{x^2}x = \frac{x^2+y^2}{x} = \boxed{\frac{r}{x}}$

So: $\frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \frac{1}{\frac{r}{x}} = \boxed{\frac{x}{r}}$ and: $(u_\theta)_y = \frac{\partial^2 u}{\partial\theta^2}\frac{\partial y}{\partial r^2} = u_{\theta\theta}\cdot\frac{y^2}{r^4}$

2.) $$\frac{d^2r}{dy^2} = \frac{d}{dy}\frac{y}{r} = \frac{dy}{dy}r^{-1} + \frac{dr^{-1}}{dy}y = \frac{r - y r_y}{r^2} = \frac{1}{r} - \frac{y^2}{r^3} = \boxed{\frac{x^2}{r^3}}$$

$$\frac{d^2\theta}{dx^2} = \frac{d\theta}{dr}\frac{dr}{dy} = \begin{cases} \frac{d}{dr}\left(\frac{x}{r}\right) = -\frac{2x}{r^3} \Rightarrow -\frac{2x}{r^3}\cdot\frac{y}{r} = \boxed{\frac{-2xy}{r^4}} \\ \frac{dr}{dy} = \frac{d}{dy}\sqrt{x^2+y^2} = \frac{y}{r} \end{cases}$$

Ergo: $u_{yy} = (u_r)_y r_y + u_r r_{yy} + (u_\theta)_y \theta_y + u_\theta \theta_{yy}$

$$\therefore \boxed{u_{yy} = u_{rr}\frac{y^2}{r^2} + 2\frac{xy}{r^3}u_{r\theta} + \frac{x^2}{4}u_{\theta\theta} + \frac{x^2}{r^3}u_r - 2\frac{xy}{r^4}u_\theta}$$

② Show That It can be written as: $\nabla^2 u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2}$ → by chain rule

Ans: $$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} \Longleftrightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) = \frac{1}{r}\left(\frac{\partial r}{\partial r}\frac{\partial v}{\partial r} + r\frac{\partial^2 v}{\partial r^2}\right)$$

$$= \frac{1}{r}\frac{\partial v}{\partial r} + \frac{r}{r}\frac{\partial^2 v}{\partial r^2} = \boxed{\frac{1}{r}\frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial r^2}}$$ ergo: $\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 v}{\partial r^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)$

Thus: $$\boxed{\nabla^2 v = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right) + \underbrace{\frac{1}{r^2}\frac{\partial v}{\partial\theta^2}}_{}}$$

This term doesn't change