Response to Trevor

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1 Trevor's points

Trevor makes 3 points:

1. Because the average flux with constant diffusivity is $\kappa\bar{c}_z$, squeeze dispersion cannot exist;
2. Section 2 is wrong because diffusivity is calculated in height coordinates rather than isopycnal coordinates.
3. the GM90 scheme is not a parameterization of the bolus velocity; ie,

2 A simple example of squeeze dispersion

Consider a tracer squeezed and stretched by the two-dimensional incompressible flow

$$(u, w) = (-x, z)\, s_t(t)\,, \tag{1}$$

where $s(t) = w_z = -u_x$ is the amplitude of the linear strain field. A tracer advected by this flow while diffusing with constant diffusivity κ obeys

$$c_t - s_t\left(xc_x - zc_z\right) = \kappa\left(c_{xx} + c_{zz}\right)\,. \tag{2}$$

2.1 Solution for an infinite tracer stripe

If $c_x = 0$ initially, it remains so for all time, and (2) reduces to

$$c_t + s_t z c_z = \kappa c_{zz}\,. \tag{3}$$

This problem can be solved without too much trouble. With a change of coordinates to

$$Z = \mathrm{e}^{-s} z\,, \quad T = \int_0^t \mathrm{e}^{-2s}\,\mathrm{d}t' \qquad \text{such that} \qquad \partial_z = \mathrm{e}^{-s}\partial_Z\,, \quad \partial_t = \mathrm{e}^{-2s}\partial_T - s_t Z \partial_Z\,, \tag{4}$$

equation (3) transforms into

$$c_T = \kappa c_{ZZ}\,. \tag{5}$$

This equation can be solved for any initial condition. When $c(t=0) = \delta(z)$, we obtain the Gaussian-type solution,

$$c = \frac{1}{\sqrt{4\pi\kappa T}} \exp\left[-\frac{Z^2}{4\kappa T}\right]\,. \tag{6}$$

The second moment $\sigma = \int_{-\infty}^{\infty} z^2 c\,\mathrm{d}z$ is

$$\sigma = 2\kappa \mathrm{e}^{2s} \int_0^t \mathrm{e}^{-2s}\,\mathrm{d}t'\,, \tag{7}$$

in terms of which the solution becomes

$$c = \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{z^2}{2\sigma}\right]\,. \tag{8}$$

2.2 Analysis

We define the time-average of c as

$$\langle c \rangle = \frac{1}{\tau} \int_0^\tau c \, \mathrm{d}t \,, \tag{9}$$

where τ is the period of s_t. In terms of σ, the effective diffusivity defined via

$$\kappa_e = \frac{\sigma(\tau)}{2\tau} \tag{10}$$

for an initially δ-distributed c becomes

$$\kappa_e = \kappa \langle \mathrm{e}^{-2s} \rangle \,. \tag{11}$$

when $s(0) = 0$. An alternative definition of the effective diffusivity is $\kappa_e = \kappa + \kappa_{eddy}$ in terms of the 'eddy' diffusivity

$$\kappa_{eddy} = \frac{\langle wc \rangle}{\langle c_z \rangle} \tag{12}$$

Note that $c_z = -zc/\sigma$. With $w = s_t z$ the eddy diffusivity becomes

$$\kappa_{eddy} = -\frac{\langle s_t \sigma^{-1/2} \mathrm{e}^{-z^2/2\sigma} \rangle}{\langle \sigma^{-3/2} \mathrm{e}^{-z^2/2\sigma} \rangle} \,. \tag{13}$$

Glenn argues that the the time-averages can be evaluated at $z = 0$, so that

$$\kappa_{eddy} = \frac{\langle s_t \sigma^{-1/2} \rangle}{\langle \sigma^{-3/2} \rangle} \,. \tag{14}$$

We hope to find that $\kappa_{eddy} = \kappa_e - \kappa$. Our best shot at evaluating the averages in (14) is to choose an $s(t)$ for which we can evaluate $\sigma(t)$ analytically, I think.