Dear All, 11 December 2018

These are some notes following an enjoyable couple of hours that I spent with Greg Wagner talking about Squeeze diffusion last Friday.

The extra advection of GM is not the bolus velocity

In no particular order, I will start with the fact that the extra advection of GM is not the bolus velocity. When averaging the equations of motion it is best, as de Szoeke and Bennett did, to first do an instantaneous spatial average over a sphere of radius of the Thorpe scale. This will achieve static stability everywhere, and so one has a buoyancy coordinate. Next you performing averaging with respect to the buoyancy coordinate. The way to do this is to perform thickness-weighted averaging in this buoyancy coordinate. As emphasised by McDougall and McIntosh (2001) [the TRM-II paper], the temperature and salinity that one then obtains is the thickness-weighted temperature and salinity. Also, the horizontal velocity is the thickness-weighted horizontal velocity, $\hat{\mathbf{V}}$, of this buoyancy-coordinate averaging procedure. This velocity can be split up in two ways, one in Cartesian coordinates and one in density coordinates, so that (with errors that are cubic in perturbation quantities)

$$\hat{\mathbf{V}} = \tilde{\mathbf{V}} + \mathbf{V}^{\text{bolus}} = \overline{\mathbf{V}} + \mathbf{V}^{\text{quasi-Stokes}} \tag{1}$$

The Eulerian Mean velocity $\overline{\mathbf{V}}$ is non-divergent as is the quasi-Stokes velocity $\mathbf{V}^{\text{quasi-Stokes}}$ of the Temporal-Residual-Mean scheme. $\tilde{\mathbf{V}}$ is the horizontal velocity averaged ON an isopycnal surface, and $\mathbf{V}^{\text{bolus}}$ is the horizontal component of the bolus velocity. In the absence of small-sale diapycnal mixing, both of these velocities point along the isopycnal surface. That is, they are both adiabatic, as is $\hat{\mathbf{V}}$. When one constructs three-dimensional velocities from $\tilde{\mathbf{V}}$ and $\mathbf{V}^{\text{bolus}}$ one finds that both of these are divergent velocity vectors.

Note also the contrast between the Eulerian mean & quasi-Stokes velocities compared with the isopycnally-average velocity and the bolus velocity. The first pair of velocities are both diabatic while the second pair are both adiabatic.

Since the extra velocity of GM is added to the Eulerian mean velocity, we conclude that it is the quasi-Stokes velocity that is added, since it is a non-divergent velocity that is added, and it is a diabatic velocity that is added. All of this is discussed in the 2001 paper by McDougall and McIntosh in JPO. The main players (Peter Gent, Jim McWilliams etc) well understand this.

I know that it is very common to read that GM implements the bolus velocity but this is wrong. It's fake news. Being common doesn't make it right.

So this discussion establishes theoretically that the extra advection of GM is not the bolus velocity. But am I just having an academic discussion of no practical relevance? No. Consider an idealized ACC region where all the neutral density surfaces have the same slope at all depths in the region of the ACC. With parallel slopes of density surfaces, there is no northward gradient of the thickness between isopycnals, and hence we would expect a zero bolus velocity there (no down-gradient gradient of thickness). But what do we do with GM? We calculate the extra velocity of GM by multiplying κ times slope and κ has a vertical gradient, being smaller at depth than near the surface. In this way we have a non-zero $\mathbf{V}^{\text{quasi-Stokes}}$ even though zero values of $\mathbf{V}^{\text{bolus}}$ are indicated.

The need for the mean diapycnal flux ACROSS isopycnals (i.e. not thickness-weighted)

So we agree that the correct coordinate frame and averaging procedure for analysing the ocean is the thickness-weight averaging procedure in density coordinates. This means that the diapycnal fluxes that we want are the average diapycnal flux of a scalar quantity across density surfaces. This is a more fundamental need than a turbulent diapycnal diffusivity.

Fortunately these days the diapycnal diffusivities that we use are all based-on or ground-truthed via microscale observations of the dissipation rate of turbulent kinetic energy ε. Let us make a huge leap and assume that we have 100% certainly in the value of the mixing efficiency Γ so that we will say that the mixing observational community have given us estimates of $\Gamma\varepsilon$.

So what we need in our thickness-weighted isopycnally averaged framework is isopycnal averages of the small-scale diapycnal turbulent fluxes of quantities. For buoyancy we need $\widetilde{\Gamma\varepsilon}$, the over-tilde meaning an average ALONG and isopycnal (not thickness-weighted). For some other variable, such as Conservative temperature, Θ, its isopycnal average of the small-scale diapycnal turbulent of Θ is

$$\widetilde{\Gamma\varepsilon}\frac{d\Theta}{db} \qquad (2)$$

where $d\Theta/db$ is found at constant latitude and longitude by going up and down the cast. In this way, it is independent of heave and thus independent of Squeeze (just like the slope $d\Theta/dS_{\mathrm{A}}$ of the $S_{\mathrm{A}}-\Theta$ curve is independent of heave and Squeeze on a water column). Because there are no complications in thinking about the fluxes of these other scalar fluxes (such as Θ), we can concentrate our discussion solely on buoyancy.

So it seems that the right way of making use of a cruise-worth of microstructure data is to average it all on buoyancy surfaces so that we end up with estimates of isopycnally-averaged buoyancy fluxes $\widetilde{\Gamma\varepsilon}$ on a series of density surfaces (note that this averaging is don ON an isopycnal, not thickness-weighted between isopycnals). Now how do we use this in

(1) an inverse model or budget study, (2) a layered-coordinate model or (3) a z-coordinate model. I now think that the discussion for all three uses of $\widetilde{\Gamma\varepsilon}$ is the same. That is, following on from McDougall and McIntosh (2001), once we use GM (which is actually TRM-II) in a z-coordinate model then we must interpret that model's Conservative Temperature as the thickness-weighted version, $\hat{\Theta}$, and, importantly for this discussion, the density of the model is not the Eulerian average density averaged over the unresolved temporal perturbations, $\bar{\gamma}$, but rather is the density value, $\tilde{\gamma}(z)$, whose surface has an average height of z when averaging the height of this density surface over the unresolved temporal scales. This is discussed at length in the first half of the McDougall and McIntosh (2001) paper. *This means that the average thickness between a pair of density surfaces is already known to the height-coordinate model.* It is fact just simply the vertical grid interval Δz. I realize that this is just the dead opposite of my first main objection to you guys in my review of the draft paper, for which I apologise. Some things take a while to sink in, even when this was one of the main point of this work of mine from twenty years ago!

To repeat, one would normally think that the vertical density gradient that one would get from the output of a z-coordinate model would be $\bar{\gamma}_z = d\bar{\gamma}/dz$, but in fact, due to the magic of GM and TRM theory, what one actually gets is

$$\tilde{\gamma}_z = d\tilde{\gamma}/dz = \lim_{\Delta\tilde{\gamma}\to 0} \frac{\Delta\tilde{\gamma}}{\Delta\overline{z}\left(\tilde{\gamma}\right)}. \tag{3}$$

Now, a caveat here would be that the heaving and Squeezing that GM=TRM accounts for ore could argue is the heaving and Squeezing of the mesoscale, and maybe not the heaving and Squeezing of the internal wave scales.

So the primary microscale measurement, to which all of our modern turbulent diapycnal diffusivities are tethered is ε. Now I am going to hope that the observationalists have given us a cruise-worth of data averaging along isopycnals as $\widetilde{\Gamma\varepsilon}$. In practice to date they probably average at fixed pressure (depth) so they probably more often provide us with $\overline{\Gamma\varepsilon}$. Sine I have argued above that in all of (1) an inverse model or budget study, (2) a layered-coordinate model or (3) a z-coordinate model, what we actually have access to in terms of a vertical buoyancy gradient is $d\tilde{\gamma}/dz$ as given in Eqn. (3), then the microstructure community should be doing the same "average thickness" type of N^2 calculation before evaluating a diapycnal diffusivity for export to modellers. That is, the appropriate calculation of the square of the buoyancy frequency in their data is

$$\tilde{b}_z \;=\; d\tilde{b}/dz \;=\; \lim_{\Delta\tilde{b}\to 0} \frac{\Delta\tilde{b}}{\Delta\overline{z}\left(\tilde{b}\right)}, \tag{4}$$

so that they would publish diapycnal diffusivities calculated according to

$$\text{diapynal diffusivity} \;=\; \frac{\widetilde{\Gamma\varepsilon}}{\tilde{b}_z} \;=\; \lim_{\Delta\tilde{b}\to 0} \frac{\Delta\overline{z}\left(\tilde{b}\right)}{\Delta\tilde{b}}\,\widetilde{\Gamma\varepsilon}\,. \tag{5}$$

If the microstructure community delivered this diapycnal diffusivity, then we would not have to do any Squeeze correction in (1) an inverse model or budget study, (2) a layered-coordinate model or (3) a z-coordinate model. Moreover, the Squeeze that the microstructure community would have included is the Squeezing that occurs in the internal wave and inertial bands and may not have included the Squeezing that occurs in the mesoscale band. The TRM takes care of that mesoscale band of Squeezing.

What you would not want to do is to have the microstructure community deliver a diffusivity according to Eqn. (5), and then go a Squeeze again on entering an ocean model; that would be accounting for Squeeze twice.

As you can see in the above, I have considered quite separately (a) the construction of a diapycnal diffusivity estimate by a microscale mixing obervationalist from (b) the use of a diffusivity in a modelling context. I hope that is useful.

Anyway, I have certainly learnt a lot from thinking about this, in particular in reminding myself that a non-eddy permitting model that has GM in it has already done thickness-weighted averaging; this is the only consistent way to interpret the model variables; and this is the direct opposite of what I said in my review about z-coordinate averaging, for which I apologise.

young Trevor