Inverse CDF Sampling

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1 Introduction

Inverse CDF sampling is a method for obtaining samples from both discrete and continuous probability distributions that requires the CDF to be invertable. The method generates a CDF value from a Uniform random variable on [0, 1] that is then used as input into the inverted CDF to generate a sample with the desired distribution. Here examples for both cases are discussed. For the continuous case a proof is given that demonstrates the samples produced have the expected distribution.

2 Sampling Discrete Distributions

A discrete probability distribution consisting of a finite set of N probability values is defined by,

$$\{p_1, p_2, \ldots, p_N\}$$

with $p_i \geq 0, \forall i$ and $\sum_{i=1}^{N} p_i = 1$.

The CDF specifies the probability that $i \leq n$ and is given by,

$$P(i \leq n) = P(n) = \sum_{i=1}^{n} p_i, \tag{1}$$

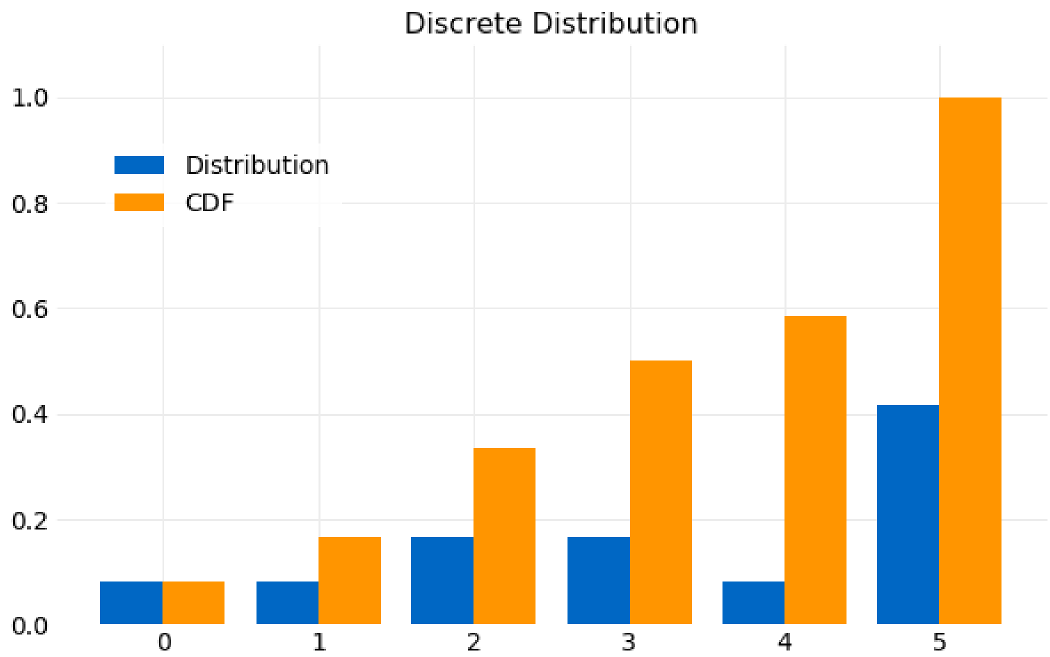
where $P(N) = 1$.

For a given generated CDF value, u, equation (1) can always be inverted by evaluating it for each n and searching for the value of n that satisfies, $P(n) \geq u$. It can be seen that the generated samples will have distribution $\{p_n\}$ since the intervals $P(n) - P(n-1) = p_n$ are Uniformly sampled.

Consider the distribution,

$$\left\{\frac{1}{12}, \frac{1}{12}, \frac{1}{6}, \frac{1}{6}, \frac{1}{12}, \frac{5}{12}\right\} \tag{2}$$

It is shown in the following plot with its CDF.



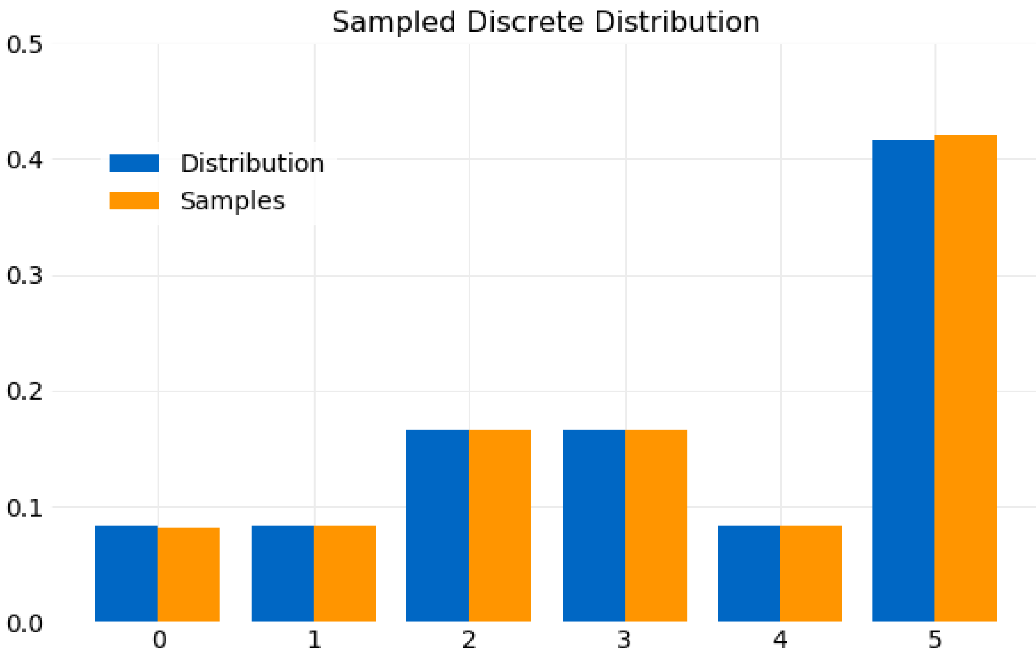
A sampler using the Inverse CDF method can be implemented in Python in a few lines of code,

```
import numpy

n = 10000
df = numpy.array([1/12, 1/12, 1/6, 1/6, 1/12, 5/12])
cdf = numpy.cumsum(df)

cdf_star = numpy.random.rand(n)
samples = [numpy.flatnonzero(cdf >= cdf_star[i])[0] for i in range(n)]
```

The figure below favoably compares samples generated by the Inverse CDF sampler and distribution (2),



It is also possible to directly sample $\{p_n\}$ using the `multinomial` sampler from `numpy`,

```
import numpy

n = 10000
df = numpy.array([1/12, 1/12, 1/6, 1/6, 1/12, 5/12])
samples = numpy.random.multinomial(n, df, size=1)/n
```

3 Sampling Continuous Distributions

A continuous probability distribution is defined by the PDF,

$$f_X(x),$$

where $f_X(x) \geq 0, \forall x$ and $\int f_X(x)dx = 1$.
The CDF is a monotonically increasing function that specifies probability that $X \leq x$,

$$P(X \leq x) = F_X(x) = \int^{x} f_X(w)dw \tag{3}$$

To prove that Inverse CDF sampling works is is necessary to show that,

$$P[F_X^{-1}(u) \le x] = F_X(x),$$

where $F_X^{-1}(x)$ is the inverse of $F_X(x)$ and u is Uniform on [0, 1]. Since $F_X(x)$ is monitonically increasing,

$$\begin{aligned} P[F_X^{-1}(u) \le x] &= P[F_X(F_X^{-1}(u)) \le F_X(x)] \\ &= P[u \le F_X(x)] \\ &= \int_0^{F_X(x)} dw \\ &= F_X(x) \end{aligned}$$

and the desired result is obtained.