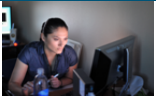
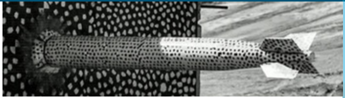
Geometric Mechanics Formulations





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Shallow Water Equations

In terms of fluid height h and velocity $\mathbf{u}$:

$$\begin{aligned} \frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{u}) &= 0 \qquad (1) \\ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla (gh) &= 0 \qquad (2) \end{aligned}$$

Alternatively in terms of momentum $\mathbf{m} = h\mathbf{u}$:

$$\frac{\partial \mathbf{m}}{\partial t} + \nabla \cdot (h\mathbf{u}\,\mathbf{u}) + \nabla (g\frac{h^2}{2}) = 0 \qquad (3)$$

In (2) can write $\mathbf{u} \cdot \nabla \mathbf{u} = (\nabla^T \cdot \mathbf{u})\mathbf{u}^T + \nabla (\frac{\mathbf{u}\cdot\mathbf{u}}{2})$ and in (3) can write $\nabla (g\frac{h^2}{2}) = h\nabla (gh)$.

Mathematical Preliminaries

Functionals

- **Functional**: Mapping from a space X to $\mathbb{R}$; $\mathscr{F}[\mathbf{x}]$ for $\mathbf{x} \in X$
- If $\mathscr{F}$ is independent of derivatives of $\mathbf{x}$, it is *hyperlocal*
- In our case, we have $X = \chi(\Omega) \times \Lambda^k(\Omega)$ ie $\mathscr{F}[\mathbf{u}, a]$
- Examples if $a = s, \mathbf{b}$ ie $\mathscr{F}[\mathbf{u}, s, \mathbf{b}]$
  - $\mathscr{F}[\mathbf{u}, s, \mathbf{b}] = \int \mathbf{u} \cdot \mathbf{u}$ (hyperlocal)
  - $\mathscr{F}[\mathbf{u}, s, \mathbf{b}] = \int s^2$ (hyperlocal)
  - $\mathscr{F}[\mathbf{u}, s, \mathbf{b}] = \int \nabla s \cdot (f \nabla s)$
  - $\mathscr{F}[\mathbf{u}, s, \mathbf{b}] = \int s \nabla \cdot \mathbf{b}$

  where f is some (time-independent) function.

Functional Derivatives I

Consistent with assumed duality pairings, we can define functional derivatives

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\mathscr{A}[\mathbf{u}+\varepsilon\delta\mathbf{u},a] = \int_\Omega \frac{\delta\mathscr{A}}{\delta\mathbf{u}}\cdot\delta\mathbf{u}\,dx = \left\langle\frac{\delta\mathscr{A}}{\delta\mathbf{u}},\delta\mathbf{u}\right\rangle \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\mathscr{A}[\mathbf{u},a+\varepsilon\delta a] = \int_\Omega \frac{\delta\mathscr{A}}{\delta a}\delta a dx = \left\langle\frac{\delta\mathscr{A}}{\delta a},\delta a\right\rangle \tag{5}$$

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\mathscr{A}[\mathbf{u},\mathbf{a}+\varepsilon\delta\mathbf{a}] = \int_\Omega \frac{\delta\mathscr{A}}{\delta\mathbf{a}}\cdot\delta\mathbf{a} dx = \left\langle\frac{\delta\mathscr{A}}{\delta\mathbf{a}},\delta\mathbf{a}\right\rangle \tag{6}$$

Note that $\frac{\delta\mathscr{A}}{\delta\mathbf{u}}\cdot\hat{\mathbf{n}} = 0$ on $\partial\Omega$, consistent with $\mathbf{u}\cdot\hat{\mathbf{n}} = 0$ For an arbitrary functional such a functional derivative may or may not exist.

Strictly speaking, $\frac{\delta\mathscr{A}}{\delta\mathbf{x}}$ is a functional, and can be identified with an element of X through Riesz representation theorem.

Functional Derivatives II

- Examples for $\mathscr{F}[\mathbf{u}, s, \mathbf{b}]$
  - $\mathscr{F}[\mathbf{u}, s, \mathbf{b}] = \int \mathbf{u} \cdot \mathbf{u}$ (hyperlocal)

$$\frac{\delta \mathscr{F}}{\delta \mathbf{u}} = 2\mathbf{u} \qquad \frac{\delta \mathscr{F}}{\delta s} = 0 \qquad \frac{\delta \mathscr{F}}{\delta \mathbf{b}} = 0 \tag{7}$$

  - $\mathscr{F}[\mathbf{u}, s, \mathbf{b}] = \int s^2$ (hyperlocal)

$$\frac{\delta \mathscr{F}}{\delta \mathbf{u}} = 0 \qquad \frac{\delta \mathscr{F}}{\delta s} = 2s \qquad \frac{\delta \mathscr{F}}{\delta \mathbf{b}} = 0 \tag{8}$$

  - $\mathscr{F}[\mathbf{u}, s, \mathbf{b}] = \int \nabla s \cdot (f \nabla s)$

$$\frac{\delta \mathscr{F}}{\delta \mathbf{u}} = 0 \qquad \frac{\delta \mathscr{F}}{\delta s} = -2\nabla \cdot (f \nabla s) \qquad \frac{\delta \mathscr{F}}{\delta \mathbf{b}} = 0 \tag{9}$$

  - $\mathscr{F}[\mathbf{u}, s, \mathbf{b}] = \int s \nabla \cdot \mathbf{b}$

$$\frac{\delta \mathscr{F}}{\delta \mathbf{u}} = 0 \qquad \frac{\delta \mathscr{F}}{\delta s} = \nabla \cdot \mathbf{b} \qquad \frac{\delta \mathscr{F}}{\delta \mathbf{b}} = -\nabla s \tag{10}$$

Careful: non-hyperlocal functionals generally have boundary terms in functional derivatives when non-material boundary conditions are used

Lagrangian Formulations

Lagrangian Formulation

Start with *Lagrangian* $\mathscr{L}[\mathbf{u}, h]$, which is *kinetic energy* minus *internal energy*:

$$\mathscr{L}[\mathbf{u}, h] = \int_\Omega h\frac{\mathbf{u}\cdot\mathbf{u}}{2} - g\frac{h^2}{2} \tag{11}$$

The functional derivatives of (11) are

$$\frac{\delta\mathscr{L}}{\delta\mathbf{u}} = h\mathbf{u} \qquad\qquad \frac{\delta\mathscr{L}}{\delta h} = \frac{\mathbf{u}\cdot\mathbf{u}}{2} - gh \tag{12}$$

Variational Formulation: Variational Principle

The equations of motion come from taking variations of the action $\mathscr{S} = \int_{t_1}^{t_2} \mathscr{L}[\mathbf{u}, h]$

$$\delta \mathscr{S} = \delta \int_{t_1}^{t_2} \mathscr{L}[\mathbf{u}, h] = 0 \tag{13}$$

subject to the constraints

$$\begin{aligned} \delta \mathbf{u} &= \partial_t \zeta + [\mathbf{u}, \zeta] = \partial_t \zeta + \mathbf{u} \cdot \nabla \zeta - \nabla \mathbf{u} \cdot \zeta & (14) \\ \delta h &= -\nabla \cdot (h \zeta) & (15) \end{aligned}$$

for arbitrary variations ζ that satisfy $\zeta(t_1) = \zeta(t_2) = 0$ and $\zeta \cdot \hat{\mathbf{n}} = 0$ on $\partial\Omega$. This is an example of an *Euler-Poincaré variational* principle. These constraints (known as *Lin constraints*) enforce the transport equation (1).

Variational Formulation: Euler-Poincaré equations

Introducing the notation $\mathbf{m} = \frac{\delta \mathscr{L}}{\delta \mathbf{u}}$, the variations of (13) give

$$\int_{t_1}^{t_2} \left[\langle \mathbf{m}, \delta \mathbf{u} \rangle + \left\langle \frac{\delta \mathscr{L}}{\delta h}, \delta h \right\rangle \right] = 0 \tag{16}$$

$$\int_{t_1}^{t_2} \left[\langle \mathbf{m}, \partial_t \zeta + [\mathbf{u}, \zeta] \rangle + \left\langle \frac{\delta \mathscr{L}}{\delta h}, -\nabla \cdot (\zeta h) \right\rangle \right] = 0 \tag{17}$$

$$\int_{t_1}^{t_2} \left[-\langle \partial_t \mathbf{m}, \zeta \rangle - \langle \mathrm{L}_{\mathbf{u}} \mathbf{m}, \zeta \rangle + \left\langle h \nabla \frac{\delta \mathscr{L}}{\delta h}, \zeta \right\rangle \right] = 0 \tag{18}$$

Since ζ is arbitrary, this gives the *Euler-Poincaré equations*

$$\frac{\partial}{\partial t} \mathbf{m} + \mathrm{L}_{\mathbf{u}} \mathbf{m} - h \nabla \frac{\delta \mathscr{L}}{\delta h} = 0 \tag{19}$$

where $\mathrm{L}_{\mathbf{u}} \mathbf{m} = \nabla \mathbf{m} \cdot \mathbf{u} + \mathbf{m} \cdot \nabla \mathbf{u} + \mathbf{m} \nabla \cdot \mathbf{u}$ is the Lie derivative for 1-form densities (covector-valued volume forms). Inserting functional derivatives (12) into (19) yields (3).

Kelvin Noether Form

Now define the velocity $\mathbf{v} = \frac{1}{h}\frac{\delta \mathscr{L}}{\delta \mathbf{u}} = \frac{\mathbf{m}}{h}$. Thus we have

$$\left(\frac{\partial}{\partial t} + \mathrm{L}_{\mathbf{u}}\right)(h\mathbf{v}) - h\nabla \frac{\delta \mathscr{L}}{\delta h} = 0 \tag{20}$$

Since $\left(\frac{\partial}{\partial t} + \mathrm{L}_{\mathbf{u}}\right) h = 0$ (see (1)) and $\mathrm{L}_{\mathbf{u}}(h\mathbf{v}) = \mathbf{v}\,\mathrm{L}_{\mathbf{u}}\,h + h\,\mathrm{L}_{\mathbf{u}}\,\mathbf{v}$, this gives

$$h\left(\frac{\partial}{\partial t} + \mathrm{L}_{\mathbf{u}}\right)(\mathbf{v}) - h\nabla \frac{\delta \mathscr{L}}{\delta h} = 0 \tag{21}$$

leading to the *Kelvin-Noether* form of the Euler-Poincaré equations:

$$\frac{\partial}{\partial t}\mathbf{v} + \mathrm{L}_{\mathbf{u}}\,\mathbf{v} - \nabla \frac{\delta \mathscr{L}}{\delta h} = 0 \tag{22}$$

where $\mathrm{L}_{\mathbf{u}}\,\mathbf{v} = (\nabla^T \cdot \mathbf{v})\,\mathbf{u}^T + \nabla(\mathbf{u}\cdot\mathbf{v})$ is the Lie derivative for 1-forms (circulations). Inserting functional derivatives (12) into (22) yields (2).

Kelvin Circulation Theorem

Now consider the quantity

$$I = \oint_{\gamma_t} \mathbf{v} \tag{23}$$

where γ_t is an *advected loop* moving with the fluid velocity $\mathbf{u}$. The *Leibniz integral* (and actually the definition of $\mathrm{L}_{\mathbf{u}}$) says that

$$\frac{\mathrm{d}}{\mathrm{d}t} \oint_{\gamma_t} \mathbf{x} = \oint_{\gamma_t} \left(\frac{\partial}{\partial t} + \mathrm{L}_{\mathbf{u}}\right)\mathbf{x} \tag{24}$$

for any vector $\mathbf{x}$. Therefore we have

$$\frac{\mathrm{d}}{\mathrm{d}t} I = \frac{\mathrm{d}}{\mathrm{d}t} \oint_{\gamma_t} \mathbf{v} = \oint_{\gamma_t} \left(\frac{\partial}{\partial t} + \mathrm{L}_{\mathbf{u}}\right)\mathbf{v} = \oint_{\gamma_t} \nabla \frac{\delta \mathscr{L}}{\delta h} = 0 \tag{25}$$

which is the celebrated *Kelvin Circulation Theorem*.

Hamiltonian Formulations

Hamiltonian

Start with the Hamiltonian $\mathbb{H}[\mathbf{m}, h]$, which is kinetic energy plus internal energy:

$$\mathbb{H}[\mathbf{m}, h] = \int_{\Omega} \frac{\mathbf{m} \cdot \mathbf{m}}{2h} + g\frac{h^2}{2} \tag{26}$$

The functional derivatives of (26) are

$$\frac{\delta \mathbb{H}}{\delta \mathbf{m}} = \frac{\mathbf{m}}{h} \qquad \frac{\delta \mathbb{H}}{\delta h} = -\frac{\mathbf{m} \cdot \mathbf{m}}{2h} + gh \tag{27}$$

Lie-Poisson Variational Principle

The *action* associated with the Hamiltonian $\mathbb{H}[\mathbf{m}, h]$ is

$$\mathscr{S} = \int_{t_1}^{t_2} \langle \mathbf{m}, \mathbf{u} \rangle - \mathbb{H} \tag{28}$$

where $\mathbf{u} := \frac{\delta \mathbb{H}}{\delta \mathbf{m}}$. Get *equations of motion* by taking (constrained) *variations* of the action (28)

$$\delta \int_{t_1}^{t_2} \langle \mathbf{m}, \mathbf{u} \rangle - \mathbb{H} = 0 \tag{29}$$

subject to the constraints

$$\begin{aligned} \delta \mathbf{u} &= \partial_t \zeta + \nabla \zeta \cdot \mathbf{u} - \nabla \mathbf{u} \cdot \zeta = \partial_t \zeta + [\mathbf{u}, \zeta] && (30) \\ \delta h &= -\nabla \cdot (\zeta h) && (31) \end{aligned}$$

with $\delta \mathbf{m}$ free, for arbitrary variations ζ that satisfy $\zeta(t_1) = \zeta(t_2) = 0$ and $\zeta \cdot \hat{\mathbf{n}} = 0$ on $\partial\Omega$. These constraints enforce the transport equations (1).

Lie-Poisson Equations

The variations of (28) give

$$\int_{t_1}^{t_2}\left[\langle\delta\mathbf{m},\mathbf{u}\rangle+\langle\mathbf{m},\delta\mathbf{u}\rangle-\langle\mathbf{u},\delta\mathbf{m}\rangle-\left\langle\frac{\delta\mathbb{H}}{\delta h},\delta h\right\rangle\right] = 0 \tag{32}$$

$$\int_{t_1}^{t_2}\left[\langle\mathbf{m},\partial_t\zeta+[\mathbf{u},\zeta]\rangle-\left\langle\frac{\delta\mathbb{H}}{\delta h},-\nabla\cdot(\zeta h)\right\rangle\right] = 0 \tag{33}$$

$$\int_{t_1}^{t_2}\left[-\langle\partial_t\mathbf{m},\zeta\rangle-\langle\mathsf{L}_{\mathbf{u}}\mathbf{m},\zeta\rangle-\left\langle h\nabla\frac{\delta\mathbb{H}}{\delta h},\zeta\right\rangle\right] = 0 \tag{34}$$

Since ζ is arbitrary, this gives the *Lie-Poisson momentum equation*

$$\frac{\partial}{\partial t}\mathbf{m}+\mathsf{L}_{\mathbf{u}}\mathbf{m}+h\nabla\frac{\delta\mathbb{H}}{\delta h}=0 \tag{35}$$

recalling $\mathbf{u}:=\frac{\delta\mathbb{H}}{\delta\mathbf{m}}$. Inserting the specific expressions for functional derivatives (27) into (35) yields (3), as expected.

Lie Poisson Brackets and Equations

Associated with the Lie-Poisson variational principle are the *Lie-Poisson brackets* (which come from semi-direct product theory)

$$\{\mathbb{A},\mathbb{B}\}_{LP} = \left\langle \mathbf{m}, [\frac{\delta \mathbb{A}}{\delta \mathbf{m}}, \frac{\delta \mathbb{B}}{\delta \mathbf{m}}] \right\rangle + \left\langle h, \nabla \cdot (\frac{\delta \mathbb{A}}{\delta \mathbf{m}} \frac{\delta \mathbb{B}}{\delta h}) - \nabla \cdot (\frac{\delta \mathbb{B}}{\delta \mathbf{m}} \frac{\delta \mathbb{A}}{\delta h}) \right\rangle \tag{36}$$

which yield (after integration by parts, and recalling $\frac{\mathrm{d}\mathbb{F}}{\mathrm{d}t} = \{\mathbb{F},\mathbb{H}\}$) the (full) *Lie-Poisson equations*

$$\frac{\partial}{\partial t}\mathbf{m} + \mathrm{L}_{\frac{\delta \mathbb{H}}{\delta \mathbf{m}}}\mathbf{m} + h\nabla \frac{\delta \mathbb{H}}{\delta h} \quad = \quad 0 \tag{37}$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\frac{\delta \mathbb{H}}{\delta \mathbf{m}}) \quad = \quad 0 \tag{38}$$

Symplectic Operator

Associated with the Lie-Poisson equations is the *symplectic operator* $\mathbb{J}(\mathbf{m}, h)$ is given by

$$\mathbb{J}(\mathbf{m}, h) = -\begin{bmatrix} \mathrm{L}_{\square}\,\mathbf{m} & h\nabla\square \\ \nabla\cdot(h\square) & 0 \end{bmatrix} \tag{39}$$

The $\square$'s indicate where the functional derivatives $\frac{\delta\mathbb{A}}{\delta\mathbf{x}}$ acted on by $\mathbb{J}(\mathbf{m}, h)$ should go. For example $\mathbb{J}\frac{\delta\mathbb{A}}{\delta\mathbf{x}}$ would put $\frac{\delta\mathbb{A}}{\delta\mathbf{m}}$ in the first column and $\frac{\delta\mathbb{A}}{\delta h}$ in the second column. The evolution equations (37) - (38) can be written in terms of $\mathbb{J}(\mathbf{m}, h)$ as

$$\begin{bmatrix} \frac{\partial\mathbf{m}}{\partial t} \\ \frac{\partial h}{\partial t} \end{bmatrix} = \mathbb{J}(\mathbf{m}, h)\begin{bmatrix} \frac{\delta\mathbb{H}}{\delta\mathbf{m}} \\ \frac{\delta\mathbb{H}}{\delta h} \end{bmatrix} \tag{40}$$

Lie-Poisson Poisson Brackets Uniqueness

When there are no boundaries, or if material boundary conditions ($\frac{\delta \mathbb{A}}{\delta \mathbf{m}} \cdot \hat{\mathbf{n}} = 0$ and $\mathbf{u} \cdot \hat{\mathbf{n}} = 0$) are assumed, the Poisson brackets are not unique, and many different forms can be obtained through integration by parts. The main alternative form is *symplectic operator* Poisson bracket, given by

$$\{\mathbb{A}, \mathbb{B}\}_{SYMP} = \left\langle \frac{\delta \mathbb{A}}{\delta \mathbf{x}}, \mathbb{J}(\mathbf{m}, h) \frac{\delta \mathbb{B}}{\delta \mathbf{x}} \right\rangle \tag{41}$$

Specifically, the symplectic operator Poisson brackets corresponding to $\mathbb{J}(\mathbf{m}, a)$ are

$$\{\mathbb{A}, \mathbb{B}\}_{SYMP} = -\left\langle \frac{\delta \mathbb{A}}{\delta \mathbf{m}}, \mathsf{L}_{\frac{\delta \mathbb{B}}{\delta \mathbf{m}}} \mathbf{m} \right\rangle - \left\langle \frac{\delta \mathbb{A}}{\delta h}, \nabla \cdot (\frac{\delta \mathbb{B}}{\delta \mathbf{m}} h) \right\rangle - \left\langle \frac{\delta \mathbb{A}}{\delta \mathbf{m}}, h \nabla \frac{\delta \mathbb{B}}{\delta h} \right\rangle \tag{42}$$

Connecting Lagrangian and Hamiltonian Formulations

Lagrangian and Hamiltonian formulations can be connected through the *Legendre transform*:

$$\mathbb{H} = \langle \mathbf{m}, \mathbf{u} \rangle - \mathscr{L} \tag{43}$$

which exists if and only if the Lagrangian (or Hamiltonian) is *nondegenerate* (i.e. the Legendre transform is invertible).
Based on this, the functional derivatives can be related as

$$\frac{\delta \mathbb{H}}{\delta \mathbf{m}} = \mathbf{u} \qquad \frac{\delta \mathscr{L}}{\delta \mathbf{u}} = \mathbf{m} \qquad \frac{\delta \mathbb{H}}{\delta h} = -\frac{\delta \mathscr{L}}{\delta h} \tag{44}$$

Degenerate Lagrangians or Hamiltonians lead to the theory of *Dirac brackets* and *Morse families*.

Curl-Form Hamiltonian Formulation: Hamiltonian

It is often useful to consider a change of variables from momentum $\mathbf{m}$ to velocity $\mathbf{v} = \frac{\mathbf{m}}{h}$ (which for the shallow water equations happens to be $\mathbf{u}$). This is known as *curl-form* (also as *vector-invariant* or *Carter-Licnerowicz*). Writing $\mathbb{H}[\mathbf{m}, h] = \mathscr{H}[\mathbf{v}, h]$ and using the *chain rule* for functional derivatives gives

$$\frac{\delta \mathbb{H}}{\delta \mathbf{m}} = \frac{1}{h} \frac{\delta \mathscr{H}}{\delta \mathbf{v}} \qquad \frac{\delta \mathbb{H}}{\delta h} = \frac{\delta \mathscr{H}}{\delta h} - \frac{\mathbf{v}}{h} \cdot \frac{\delta \mathscr{H}}{\delta \mathbf{v}} \tag{45}$$

Using the shallow water Hamiltonian (26) these are

$$\frac{\delta \mathscr{H}}{\delta \mathbf{v}} = h\mathbf{u} \qquad \frac{\delta \mathscr{H}}{\delta h} = \frac{\mathbf{v} \cdot \mathbf{v}}{2} + gh \tag{46}$$

recalling $\mathbf{v} = \mathbf{u}$ in this case.

Curl-Form Hamiltonian Formulation: Equations

The new functional derivatives (46) yield new equations of motion:

$$
\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} + q \frac{\delta \mathscr{H}}{\delta \mathbf{v}}^{T} + \nabla \frac{\delta \mathscr{H}}{\delta h} &= 0 \qquad (47) \\
\frac{\partial h}{\partial t} + \nabla \cdot \frac{\delta \mathscr{H}}{\delta \mathbf{v}} &= 0 \qquad (48)
\end{aligned}
$$

with *potential vorticity* $q = \frac{\nabla^{T} \cdot \mathbf{v}}{h}$. Inserting (46) into (47) yields (2), as expected.

Curl-Form Hamiltonian Formulation: Symplectic Operator

The new symplectic operator $\mathbb{J}(\mathbf{v}, h)$ is given by

$$\mathbb{J}(\mathbf{v}, h) = -\begin{bmatrix} q\square^{T} & \nabla\square \\ \nabla\cdot\square & 0 \end{bmatrix} \tag{49}$$

The $\square$'s indicate where the functional derivatives $\frac{\delta\mathscr{A}}{\delta\mathbf{x}}$ acted on by $\mathbb{J}(\mathbf{v}, h)$ should go. For example $\mathbb{J}\frac{\delta\mathscr{A}}{\delta\mathbf{x}}$ would put $\frac{\delta\mathscr{A}}{\delta\mathbf{v}}$ in the first column and $\frac{\delta\mathscr{A}}{\delta h}$ in the second column. The associated symplectic Poisson brackets $\{\mathscr{A}, \mathscr{B}\}$ will be given by

$$\left\langle \frac{\delta\mathscr{A}}{\delta\mathbf{x}}, \mathbb{J}(\mathbf{v}, h)\frac{\delta\mathscr{B}}{\delta\mathbf{x}} \right\rangle \tag{50}$$

Curl-Form Hamiltonian Formulation: Poisson Brackets

Specifically, the Poisson brackets $\{\mathscr{A},\mathscr{B}\}_{SYMP}$ corresponding to (49) are

$$\{\mathscr{A},\mathscr{B}\}_{SYMP} = \left\langle \frac{\delta\mathscr{A}}{\delta\mathbf{v}}, q\frac{\delta\mathscr{B}}{\delta\mathbf{v}}^{T} \right\rangle - \left\langle \frac{\delta\mathscr{A}}{\delta h}, \nabla\cdot\frac{\delta\mathscr{B}}{\delta\mathbf{v}} \right\rangle + \left\langle \frac{\delta\mathscr{A}}{\delta\mathbf{v}}, \nabla\frac{\delta\mathscr{B}}{\delta h} \right\rangle \quad (51)$$

As before, when there are no boundaries, or if material boundary conditions ($\frac{\delta\mathscr{A}}{\delta\mathbf{v}}\cdot\hat{\mathbf{n}} = 0$ and $\mathbf{u}\cdot\hat{\mathbf{n}} = 0$) are assumed, the Poisson brackets are not unique and many different forms can be obtained through integration by parts.

Generalized Conservation Form

Generalized Conservation Form

Can we write the geometric mechanics form of the equations in *(generalized) conservation form*?

- Write momentum equation **m** using tensor divergence instead of Lie derivative L :

$$\frac{\partial \mathbf{m}}{\partial t} + \nabla \cdot \mathbf{M} + \nabla p = S \tag{52}$$

  in terms of a momentum flux **M**, a (generalized) pressure p and a source term S. We could also write $\nabla p = \nabla \cdot (p\mathbb{I})$.
- The equation for h is already in conservation form.

Deriving Conservation Form

Start by defining $\mathscr{L} = \int_\Omega \mathfrak{l}(\mathbf{u}, h, \mathbf{x})$, which gives

$$\nabla \mathfrak{l} = \nabla \mathbf{u} \cdot \frac{\delta \mathscr{L}}{\delta \mathbf{u}} + \frac{\delta \mathscr{L}}{\delta h} \nabla h + \frac{\partial \mathfrak{l}}{\partial \mathbf{x}} \tag{53}$$

noting that $\frac{\partial \mathfrak{l}}{\partial \mathbf{u}} \neq \frac{\delta \mathscr{L}}{\delta \mathbf{u}}$ and $\frac{\partial \mathfrak{l}}{\partial h} \neq \frac{\delta \mathscr{L}}{\delta h}$ in general. When this is true, have a *hyperregular* Lagrangian, ie one that does not depend on derivatives of $\mathbf{u}$ or h. Then lots of algebra yields

$$\frac{\partial \mathbf{m}}{\partial t} + \nabla \cdot (\mathbf{u}\mathbf{m}) + \nabla\left(\mathfrak{l} - h \frac{\delta \mathscr{L}}{\delta h}\right) = -\frac{\partial \mathfrak{l}}{\partial \mathbf{x}} \tag{54}$$

Thus by inspection we see that

$$\mathbf{M} = \mathbf{u}\mathbf{m} \qquad p = \mathfrak{l} - h\frac{\delta \mathscr{L}}{\delta h} \qquad S = -\frac{\partial \mathfrak{l}}{\partial \mathbf{x}} \tag{55}$$

Conservation Form based on Hamiltonian

It is straightforward to rewrite $\mathbf{M}$ and p in terms of $\mathbb{H} = \int_\Omega \mathfrak{h}(\mathbf{m}, h, \mathbf{x})$ instead of $\mathscr{L}$ as

$$\mathbf{M} = \mathbf{u}\,\mathbf{m} \qquad p = \mathbf{u}\cdot\mathbf{m} + h\frac{\delta\mathscr{H}}{\delta h} - \mathfrak{h} \qquad S = \frac{\partial \mathfrak{h}}{\partial \mathbf{x}} \tag{56}$$

Conserved Quantities

Conserved Quantities in Lagrangian Formulation

- Noether Terms (discarded integration by parts terms from Euler-Poincaré calculations):

$$\delta \mathscr{L} = \frac{\partial(\mathbf{m}\cdot\zeta)}{\partial t} + \nabla\cdot[(\mathbf{m}\cdot\zeta)\mathbf{u} + (p-\mathfrak{l})\zeta] \tag{57}$$

  Note $p-\mathfrak{l} = h\frac{\delta\mathscr{L}}{\delta h}$.

- Now assume there is a continuous symmetry ζ such that $\mathscr{L}$ is invariant under this symmetry:

$$\delta_\zeta \mathscr{L} = 0 \tag{58}$$

- This gives two expressions for $\delta\mathscr{L}$: one from Noether terms, one from symmetry expression
- Equate them to obtain a conserved quantity and flux (Noether charges/Noether currents) and therefore associated conservation law

This is *Noether's theorem*: 1-1 relationship between continuous symmetries and conservation law; independent of the specific Lagrangian!

Lagrangian Formulation- Energy

Distinguish between $L(\mathbf{x},t)$ and $\mathfrak{l}(\mathbf{u},h,\mathbf{x};t)$, recalling $\mathscr{L} = \int L(\mathbf{x},t) = \int \mathfrak{l}(\mathbf{u},h,\mathbf{x};t)$
Time symmetry: $\zeta = -\tau\mathbf{u}$ for infinitesimal τ yields

$$\delta_\zeta \mathscr{L} = -\tau \frac{\partial L}{\partial t} + \frac{\partial \mathfrak{l}}{\partial t} \tag{59}$$

Insert ζ expression into Noether terms and equate with (59) to yield

$$\frac{\partial E}{\partial t} + \nabla \cdot [(E+p)\mathbf{u}] + \frac{\partial l}{\partial t} = 0 \tag{60}$$

where $E = \mathbf{m}\cdot\mathbf{u} - \mathfrak{l}$ is the energy density. Thus energy is conserved if $\frac{\partial \mathfrak{l}}{\partial t}$!

Lagrangian Formulation- (Linear) Momentum

Translational symmetry: ζ is constant yields (for hyperlocal Lagrangian)

$$\delta_{\zeta}\mathscr{L} = -\nabla L \cdot \zeta + \frac{\partial l}{\partial \mathbf{x}} \cdot \zeta \tag{61}$$

Insert ζ expression into Noether terms and equate with (61) (assuming a hyperlocal Lagrangian) to yield

$$\frac{\partial \mathbf{m}}{\partial t} + \nabla \cdot \mathbf{M} + \nabla p - \frac{\partial l}{\partial \mathbf{x}} = 0 \tag{62}$$

Much easier way to derive momentum equation than manipulations above!

Lagrangian Formulation- Relabelling (Casimirs)

Consider a symmetry ζ such that

$$\delta \mathbf{u} = [\mathbf{u}, \zeta] = 0 \qquad \delta h = -\mathsf{L}_{\zeta} h = 0 \tag{63}$$

These are called *relabelling symmetries*, since they correspond to a change of Lagrangian labels. The associated conserved quantity is called a *Casimir*.

Unlike energy and momentum, the conserved quantities and flux (Noether charges/currents) and conservation law for this type of symmetry depend on the number and type of advected quantities. These calculations can get quite complicated. However, here we have 1 advected density h. The associated Casimir in this case is

$$\mathscr{C} = \int hf(q) \tag{64}$$

where $f(q)$ is an arbitrary function, with evolution equation

$$\frac{\partial hf(q)}{\partial t} + \nabla \cdot (hf(q)\mathbf{u}) = 0 \tag{65}$$

Conserved Quantities in Hamiltonian Formulation: Energy

The antisymmetry of $\mathbb{J}(\mathbf{x})$ (or equivalently of $\{\mathscr{A},\mathscr{B}\}$) gives energy conservation, since

$$\frac{d\mathscr{H}}{dt} = \{\mathscr{H},\mathscr{H}\} = -\{\mathscr{H},\mathscr{H}\} = 0 \tag{66}$$

A local conservation law for *Hamiltonian density* $h = \mathbf{u}\cdot\mathbf{m} - \mathfrak{l}$, where $l(\mathbf{u},h)$ is the *Lagrangian density*, also holds

$$\frac{\partial \mathfrak{h}}{\partial t} + \nabla\cdot[\mathbf{u}(\mathfrak{h}+p)] = 0 \tag{67}$$

where $p = \mathfrak{l} - h\frac{\delta\mathscr{L}}{\delta h} = \mathbf{u}\cdot\mathbf{m} + h\frac{\delta\mathbb{H}}{\delta h} - \mathfrak{h} = h\frac{\delta\mathscr{H}}{\delta h} - \mathfrak{h}$ is the generalized pressure. Functionals and densities are related by $\mathscr{L}[\mathbf{u},h] = \int \mathfrak{l}(\mathbf{u},h)$ and $\mathscr{H} = \mathbb{H} = \int \mathfrak{h}$.

Conserved Quantities in Hamiltonian Formulation: Casimirs

The shallow water equations conserve the Casimirs

$$\mathbb{C}[\mathbf{m}, h] = \mathscr{C}[\mathbf{v}, h] = \int hf(q) \tag{68}$$

where $f(q)$ is an arbitrary function of potential vorticity $q = \frac{\nabla^T \cdot \mathbf{v}}{h}$. The functional derivatives of (68) are given by

$$\frac{\delta \mathscr{C}}{\delta \mathbf{v}} = \nabla^T (\frac{\partial f}{\partial q}) \qquad \frac{\delta \mathscr{C}}{\delta h} = f - q\frac{\partial f}{\partial q} \tag{69}$$

and

$$\frac{\delta \mathbb{C}}{\delta \mathbf{m}} = \frac{1}{h}\nabla^T (\frac{\partial f}{\partial q}) \qquad \frac{\delta \mathbb{C}}{\delta h} = f - q\frac{\partial f}{\partial q} - \frac{\mathbf{m}}{h^2} \cdot \nabla^T (\frac{\partial f}{\partial q}) \tag{70}$$

It is a simple exercise to show that

$$\{\mathscr{A}, \mathscr{C}\} = \{\mathbb{A}, \mathbb{C}\} = 0 \qquad \forall \mathscr{A}, \mathbb{A} \tag{71}$$

Important examples are total potential vorticity ($\int hq$, $f = q$) and total mass ($\int h$, $f = 1$).

Conserved Quantities in Hamiltonian Formulation: Momenta

Just as in Lagrangian formulation, there are conserved quantities associated with coordinate transformation symmetries. For a continuous spatial coordinate transform η, these are functionals $\mathscr{M}$ that satisfy

$$\mathbb{J}(\mathbf{x}) \frac{\delta \mathscr{M}}{\delta \mathbf{x}} = \mathsf{L}_\eta \, \mathbf{x} \tag{72}$$

For the shallow water equations, they are

$$\mathscr{M} = \int \mathbf{m} \cdot \hat{\eta} \tag{73}$$

Energy is actually a conserved quantity of this type as well, associated with time symmetry:

$$\mathbb{J}(\mathbf{x}) \frac{\delta \mathscr{M}}{\delta \mathbf{x}} = \frac{\partial \mathbf{x}}{\partial t} \tag{74}$$

which is just $\mathscr{M} = \mathscr{H}$.

Summary and Outlook

Summary and Outlook

- Can write shallow water equations using GM formulations (Lagrangian, Hamiltonian)
- Strong connections with conservation properties (Hamiltonian/Energy, Casimirs, Noether Invariants) and also conservation form
- **Key Insight**: most of what was done is completely independent of the specific Lagrangian/Hamiltonian: it depended ONLY on the choice of advected quantities i.e. a density (h)

This suggests a natural **generalization**: what about an **arbitrary number and type (scalar, density, vectors, tensors, differential forms, etc.) of advected quantities**? Then with **specific choices of Lagrangian/Hamiltonian**, can we obtain other fluid models?

Yes- there is a general theory here!

Exercises

Compressible Euler Equations

The compressible Euler equations are, in terms of specific entropy η, entropy density $S=\rho\eta$, density ρ, momentum $\mathbf{m}$ and velocity $\mathbf{u}$:

$$\begin{aligned}
\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\,\mathbf{u}) &= 0 && (75)\\
\frac{\partial S}{\partial t}+\nabla\cdot(S\mathbf{u}) &= 0 && (76)\\
\frac{\partial\eta}{\partial t}+\mathbf{u}\cdot\nabla\eta &= 0 && (77)\\
\frac{\partial\,\mathbf{m}}{\partial t}+\nabla\cdot(\rho\,\mathbf{u}\,\mathbf{u})+\nabla p &= 0 && (78)\\
\frac{\partial\,\mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\,\mathbf{u}+\alpha\nabla p &= 0 && (79)
\end{aligned}$$

Closed by $p=p(\alpha,\eta)=-\frac{\partial u}{\partial\alpha}(\alpha,\eta)$ where $\alpha=\frac{1}{\rho}$ is the specific volume and u the internal energy. In (79) can write $\mathbf{u}\cdot\nabla\,\mathbf{u}=\nabla\times\mathbf{u}\times\mathbf{u}+\nabla(\frac{\mathbf{u}\cdot\mathbf{u}}{2})$.

Lagrangian Formulation Exercises I

1. Derive the Euler-Poincar'e equations for the case of an arbitrary number of advected densities and scalars, where an advected scalar obeys the equation

$$\frac{\partial s}{\partial t} + \mathbf{u} \cdot \nabla s = 0 \tag{80}$$

and has constrained variations

$$\delta s = -\zeta \cdot \nabla s \tag{81}$$

2. Show that the compressible Euler equations (78), (75) and (76) can be derived using this variational principle with the Lagrangian

$$\mathscr{L}[\mathbf{u}, \rho, S] = \int_\Omega \rho \frac{\mathbf{u} \cdot \mathbf{u}}{2} - \rho U(\frac{1}{\rho}, \frac{S}{\rho}) \tag{82}$$

where ρ and S are advected densities

Lagrangian Formulation Exercises II

1. Show that compressible Euler equations (78), (75) and (77) can be derived using this variational principle with the Lagrangian

$$\mathscr{L}[\mathbf{u},\rho,\eta] = \int_\Omega \rho\frac{\mathbf{u}\cdot\mathbf{u}}{2} - \rho U(\frac{1}{\rho},\eta) \tag{83}$$

   where η is an advected scalar.

2. Derive a fluid model with 1 advected density (h) governed by a Lagrangian of the form

$$\mathscr{L}[\mathbf{u},h] = \int h\frac{\mathbf{u}\cdot\mathbf{u}}{2} + \frac{h^3}{6}(\nabla\cdot\mathbf{u})^2 - g\frac{h^2}{2} \tag{84}$$

   These are the *Green-Naghidi* equations, a type of layer model used in geophysical fluid dynamics.

Hamiltonian Formulation Exercises I

1. Derive the Lie-Poisson equations for the case of an arbitrary number of advected densities and scalars.
2. Show that the compressible Euler equations (78), (75) and (76) can be derived using this variational principle with the Hamiltonian

$$\mathbb{H}[\mathbf{m},\rho,S] = \int_\Omega \frac{\mathbf{m}\cdot\mathbf{m}}{2\rho} + \rho U(\frac{1}{\rho},\frac{S}{\rho}) \tag{85}$$

   where ρ and S are advected densities.
3. Show that compressible Euler equations (78), (75) and (77) can be derived using this variational principle with the Hamiltonian

$$\mathbb{H}[\mathbf{m},\rho,\eta] = \int_\Omega \frac{\mathbf{m}\cdot\mathbf{m}}{2\rho} + \rho U(\frac{1}{\rho},\eta) \tag{86}$$

   where η is an advected scalar.

Hamiltonian Formulation Exercises II

1. Derive a fluid model with two advected densities (ρ and S) where the Hamiltonian is given by

$$\mathbb{H}[\mathbf{m},\rho,S] = \int_\Omega \frac{(\mathbf{m}-\rho\mathbf{R})\cdot(\mathbf{m}-\rho\mathbf{R})}{2\rho} + \rho U(\frac{1}{\rho},\frac{S}{\rho}) + \rho\Phi \tag{87}$$

   where $\mathbf{R}$ is a *rotational velocity*, Φ is the geopotential and the Coriolis parameter $\mathbf{\Omega} = \nabla\times\mathbf{R}$. These are the *rotating compressible Euler equations*.

Extra Material

Clebsch Variational Principles

Is there a way to have purely free variations? ADD THIS

Lie Derivatives

Diamond Operators

General (Lagrangian) Picture for Reversible Dynamics

- Configuration space for system is a *Lie group* G with an associated Lie algebra $\mathfrak{g}$: usually G is the *diffeomorphism group*
- Lagrangian coordinate dynamics are the *Euler-Lagrange* equations on G, obtained for a Lagrangian L
- Eulerian coordinate dynamics are the *Euler-Poincaré* equations on the dual of Lie algebra $\mathfrak{g}^*$ for the reduced Lagrangian $\mathscr{L}$
  - These arise via *reduction by a symmetry* Q, assuming L is invariant to Q: the particle relabeling symmetry!
- Can extend general *matched pair dynamics* where $\mathrm{G} = \mathrm{G}_1 \bowtie \mathrm{G}_2$ where G_1 and G_2 are *compatible*, special case is *semi-direct product theory* where $\mathrm{G}_2 = V$ and $\mathrm{G} = \mathrm{G}_1 \ltimes V$

Hamiltonian Picture

ADD LOTS OF REFERENCES HERE (books and key papers)

Modern picture of fluids/solids with configuration space = Lie group ie matched pair dynamics (Lagrangian coordinates), reduction by symmetry to Lie algebra