# Structure-Preserving Discretizations I

## Differential geometry, De Rham sequence and other scary things

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W24-09 Geometric Mechanics Formulations and Structure Preserving Discretizations: An Introductory Course, 2024

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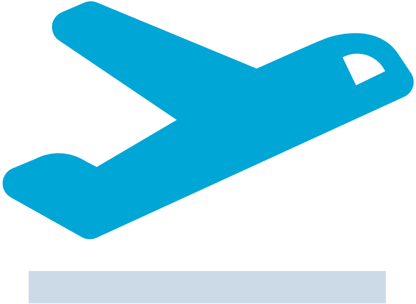

# Outline

In this part we will look at:

- What are differential forms
- The exterior derivative
- De Rham complex
- Discrete representation
- Incidence matrices
- Duality
- Hodge matrices

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# 01

# Differential forms

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According to Harley Flanders[1] differential forms are "the things which occur under the integral sign".



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For example, a line integral

$$\int A(x, y, z)\,\mathrm{d}x + B(x, y, z)\,\mathrm{d}y + C(x, y, z)\,\mathrm{d}z$$

gives the 1-form

$$\omega^{(1)}(x, y, z) = A(x, y, z)\,\mathrm{d}x + B(x, y, z)\,\mathrm{d}y + C(x, y, z)\,\mathrm{d}z$$



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# Differential forms

A surface integral

$$\iint P(x,y,z)\,\mathrm{d}y\mathrm{d}z + Q(x,y,z)\,\mathrm{d}z\mathrm{d}x + R(x,y,z)\,\mathrm{d}x\mathrm{d}y$$

gives the 2-form

$$\alpha^{(2)}(x,y,z) = P(x,y,z)\,\mathrm{d}y\mathrm{d}z + Q(x,y,z)\,\mathrm{d}z\mathrm{d}x + R(x,y,z)\,\mathrm{d}x\mathrm{d}y$$

# Differential forms

A volume integral

$$\iiint H(x, y, z)\, \mathrm{dxdydz}$$

gives the 3-form

$$\lambda^{(3)}(x, y, z) = H(x, y, z)\, \mathrm{dxdydz}$$

# Differential forms

Although not strictly related to integration we define a 0-form as a function $f(x, y, z)$, i.e. without any d$x$'s, d$y$'s or d$z$'s.

$$\beta^{(0)}(x, y, z) = f(x, y, z)$$

If we restrict ourselves to smooth functions, then we can say that a 0-form can be integrated over a point by which we mean, evaluated at a point.

# Differential forms

When we restrict $(x, y, z) \in \Omega \subset \mathbb{R}^3$, then the space $\Lambda^k(\Omega)$ denotes all smooth $k$-forms on the domain $\Omega$.

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In vector calculus we distinguish between scalar functions and vector valued functions. Note that in differential geometry we make a more subtle distinction. 0-forms consist of scalar function defined in points, while 3-forms consist of scalar functions to be integrated over volumes (densities).

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$$\omega^{(1)}(x,y,z) = \underbrace{A(x,y,z)}_{\text{vector proxy}} \, \mathrm{d}x + \underbrace{B(x,y,z)}_{\text{vector proxy}} \, \mathrm{d}y + \underbrace{C(x,y,z)}_{\text{vector proxy}} \, \mathrm{d}z$$

$$\alpha^{(2)}(x,y,z) = \underbrace{P(x,y,z)}_{\text{vector proxy}} \, \mathrm{d}y\mathrm{d}z + \underbrace{Q(x,y,z)}_{\text{vector proxy}} \, \mathrm{d}z\mathrm{d}x + \underbrace{R(x,y,z)}_{\text{vector proxy}} \, \mathrm{d}x\mathrm{d}y$$

1-forms are vectors (3 components functions) to be integrated over curves/lines, while 2-forms also consist vectors to be integrated over surfaces.

# Differential forms

When we restrict $(x, y, z) \in \Omega \subset \mathbb{R}^3$, then the space $\Lambda^k(\Omega)$ denotes all smooth $k$-forms on the domain $\Omega$.

Adding a 0-form to a 3-form is not defined in differential geometry, even though they both describe scalar functions. In vector calculus, this is not a problem and this operation is often performed (with disastrous consequences).

# The wedge product

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- Scalar multiplication: $a\alpha^{(k)} \wedge \beta^{(m)} = \alpha^{(k)} \wedge a\beta^{(m)} = a\left(\alpha^{(k)} \wedge \beta^{(m)}\right)$;
- Skew-symmetry: $\alpha^{(k)} \wedge \beta^{(m)} = (-1)^{km}\beta^{(m)} \wedge \alpha^{(k)}$.

# Summary

So far we introduced differential $k$-forms, which are the natural objects to integrate.

In addition, we introduced the wedge product which allows one to generate higher order differential forms from elementary differential forms as we will see.

# 02

## The exterior derivative

# Connecting the spaces

The exterior derivative maps $k$-forms to $(k+1)$-forms and we can set up the following sequence

$$\mathbb{R} \hookrightarrow \Lambda^0(\Omega) \xrightarrow{d} \Lambda^1(\Omega) \xrightarrow{d} \Lambda^2(\Omega) \xrightarrow{d} \Lambda^3(\Omega) \longrightarrow 0$$

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- $d\left(\alpha^{(k)} \wedge \beta^{(m)}\right) = d\alpha^{(k)} \wedge \beta^{(m)} + (-1)^k \alpha^{(k)} \wedge d\beta^{(m)}$.
- $d^2\alpha^{(k)} = d(d\alpha^{(k)}) \equiv 0$ for all $k$-forms $\alpha^{(k)}$.

# The exterior derivative of a 0 form

The second rule for differential forms reads: $d\alpha^{(0)}$ is the usual differential of the function $\alpha^{(0)}$. Remember that a 0-form is just a function, so $\alpha^{(0)}(x, y, z) = \alpha(x, y, z)$.

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$$\mathrm{d}\alpha^{(0)} = \frac{\partial \alpha}{\partial x}\,\mathrm{d}x + \frac{\partial \alpha}{\partial y}\,\mathrm{d}y + \frac{\partial \alpha}{\partial z}\,\mathrm{d}z$$

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$$\mathrm{d}\alpha^{(0)} = \frac{\partial \alpha}{\partial x}\,\mathrm{d}x + \frac{\partial \alpha}{\partial y}\,\mathrm{d}y + \frac{\partial \alpha}{\partial z}\,\mathrm{d}z$$

If we want to associate this operation to well-known operations in vector calculus, we could say that the exterior derivative applied to a 0-form is the gradient, where $dx$, $dy$ and $dz$ act as basis forms/vectors.

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$$\begin{aligned}\mathrm{d}\beta^{(1)} &= \mathrm{d}\left(P \wedge \mathrm{d}x + Q \wedge \mathrm{d}y + R \wedge \mathrm{d}z\right)\\ &= \mathrm{d}\left(P \wedge \mathrm{d}x\right) + \mathrm{d}\left(Q \wedge \mathrm{d}y\right) + \mathrm{d}\left(R \wedge \mathrm{d}z\right)\end{aligned}$$

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&= \mathrm{d}\left(P \wedge \mathrm{d}x\right) + \mathrm{d}\left(Q \wedge \mathrm{d}y\right) + \mathrm{d}\left(R \wedge \mathrm{d}z\right) \\
&= \mathrm{d}P \wedge \mathrm{d}x + P \wedge \mathrm{dd}x + \mathrm{d}Q \wedge \mathrm{d}y + Q \wedge \mathrm{dd}y + \mathrm{d}R \wedge \mathrm{d}z + R \wedge \mathrm{dd}z
\end{aligned}$$

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&= \left(\frac{\partial P}{\partial x}\mathrm{d}x + \frac{\partial P}{\partial y}\mathrm{d}y + \frac{\partial P}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}x + \left(\frac{\partial Q}{\partial x}\mathrm{d}x + \frac{\partial Q}{\partial y}\mathrm{d}y + \frac{\partial Q}{\partial z}\mathrm{d}z\right) \wedge \mathrm{d}y \\
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\end{aligned}$$

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&= \mathrm{d}P \wedge \mathrm{d}x + P \wedge \mathrm{d}\mathrm{d}x + \mathrm{d}Q \wedge \mathrm{d}y + Q \wedge \mathrm{d}\mathrm{d}y + \mathrm{d}R \wedge \mathrm{d}z + R \wedge \mathrm{d}\mathrm{d}z \\
&= \left(\frac{\partial P}{\partial x}\,\mathrm{d}x + \frac{\partial P}{\partial y}\,\mathrm{d}y + \frac{\partial P}{\partial z}\,\mathrm{d}z\right) \wedge \mathrm{d}x + \left(\frac{\partial Q}{\partial x}\,\mathrm{d}x + \frac{\partial Q}{\partial y}\,\mathrm{d}y + \frac{\partial Q}{\partial z}\,\mathrm{d}z\right) \wedge \mathrm{d}y \\
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\end{aligned}
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This resembles the curl

# The exterior derivative of a 2 form

Let $\delta^{(2)} = P\mathrm{d}y \wedge \mathrm{d}z + Q\mathrm{d}z \wedge \mathrm{d}x + R\mathrm{d}x \wedge \mathrm{d}y$, then

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\begin{aligned}
\mathrm{d}\delta^{(2)} &= \mathrm{d}\left(P \wedge \mathrm{d}y \wedge \mathrm{d}z + Q \wedge \mathrm{d}z \wedge \mathrm{d}x + R \wedge \mathrm{d}x \wedge \mathrm{d}y\right) \\
&= \mathrm{d}\left(P \wedge \mathrm{d}y \wedge \mathrm{d}z\right) + \mathrm{d}\left(Q \wedge \mathrm{d}z \wedge \mathrm{d}x\right) + \mathrm{d}\left(R \wedge \mathrm{d}x \wedge \mathrm{d}y\right) \\
&= \left(\frac{\partial P}{\partial x}\mathrm{d}x + \frac{\partial P}{\partial y}\mathrm{d}y + \frac{\partial P}{\partial z}\right) \wedge \mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial Q}{\partial x}\mathrm{d}x + \frac{\partial Q}{\partial y}\mathrm{d}y + \frac{\partial Q}{\partial z}\right) \wedge \mathrm{d}z \wedge \mathrm{d}x + \\
&\quad \left(\frac{\partial R}{\partial x}\mathrm{d}x + \frac{\partial R}{\partial y}\mathrm{d}y + \frac{\partial R}{\partial z}\right) \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z
\end{aligned}
$$

This resembles the divergence

TU Delft

# Small or not small?

Suppose we introduce a coordinate system, i.e. we associate to each point in the domain a triple of three functions $(x, y, z)$, then the exterior derivative of these three functions is $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$.

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So $\mathrm{d}x$, $\mathrm{d}y$ and $\mathrm{d}z$ are 1-forms. In engineering these quantities are interpreted as infinitesimally small increments in the $x$, $y$ and $z$ direction.

# De Rham sequence

$$\mathbb{R} \hookrightarrow \Lambda^0(\Omega) \xrightarrow[\nabla]{\mathrm{d}} \Lambda^1(\Omega) \xrightarrow[\nabla\times]{\mathrm{d}} \Lambda^2(\Omega) \xrightarrow[\nabla\cdot]{\mathrm{d}} \Lambda^3(\Omega) \longrightarrow 0$$

https://en.wikipedia.org/wiki/De_Rham_cohomology

In a finite element context, one can find:

$$\mathbb{R} \hookrightarrow H^1(\Omega) = H(\mathrm{grad};\Omega) \xrightarrow[\nabla]{\mathrm{d}} H(\mathrm{curl};\Omega) \xrightarrow[\nabla\times]{\mathrm{d}} H(\mathrm{div};\Omega) \xrightarrow[\nabla\cdot]{\mathrm{d}} L^2(\Omega) \longrightarrow 0$$

# Notes about the exterior derivative

- Although the operations resemble well-known vector operations (grad, curl and div), these operators only have this form in vector calculus in a Cartesian coordinate system, while these operations on differential forms <u>always</u> have this form;

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- Most important property: The exterior derivative can be discretized without approximation on any grid.

# The Mother of all Equations

Now that we know what differential forms are and how to compute the exterior derivative, we can present the generalized Stokes Theorem, which we referred to in the past as The Mother of all Equations.

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Let $\Omega^{k+1}$ be a smooth manifold with smooth boundary $\partial\Omega^{k+1}$ then for $\alpha^{(k)} \in \Lambda^k(\partial\Omega^{k+1})$ we have

$$\boxed{\int_{\Omega^{k+1}} \mathrm{d}\alpha^{(k)} = \int_{\partial\Omega^{k+1}} \alpha^{(k)}}$$

# Summary

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- The exterior derivative commutes with mappings;
- The generalized Stokes Theorem equals the gradient theorem for $k=0$, the Stokes' theorem for $k=1$ and the divergence theorem or Gauss' theorem for $k=2$.

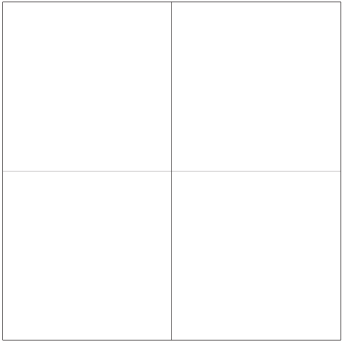
# 03

# Playing with meshes

# Playing with meshes

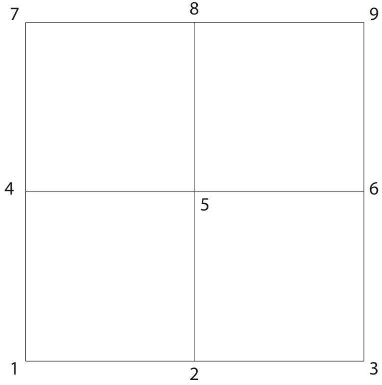
# A simple 2D mesh

We take a small break from all the calculus and look at a simple mesh as usually employed in numerical calculations. This grid consists of 9 vertices, 12 edges and 4 surfaces.



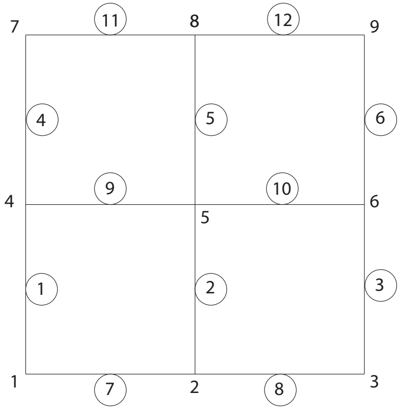
# A simple 2D mesh

First we number all the vertices in our mesh.



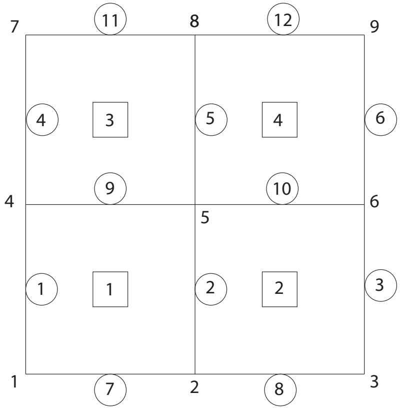
# A simple 2D mesh

Then we number all the edges in the grid



# A simple 2D mesh

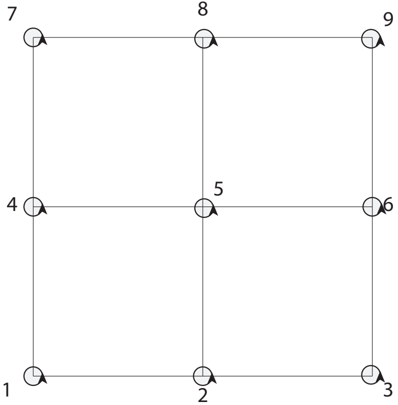
Finally we number all the surfaces (or 2D volumes) in the mesh.



# A simple 2D mesh

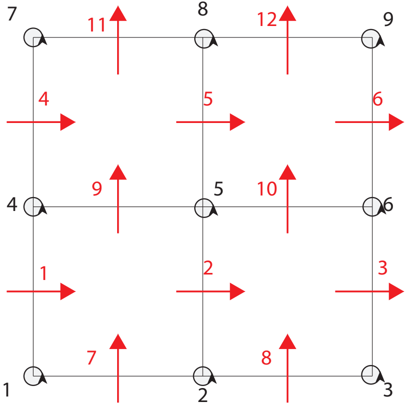
Next we need to assign an |alertorientation to all these geometric elements, which make up the grid. In this particular case, the orientations on the grid will be outer-oriented.

First we are going to assign an orientation to the vertices.



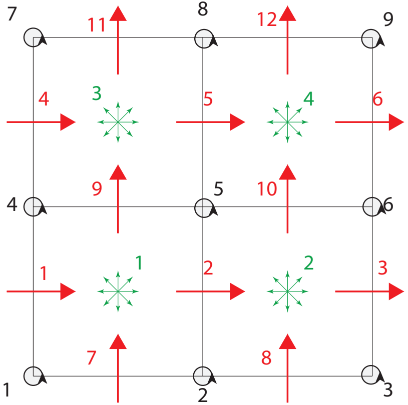
# A simple 2D mesh

Next we are going to equip the edges with an outer orientation.



# A simple 2D mesh

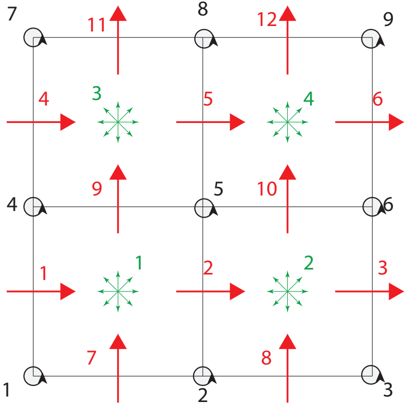
Finally, we orient the surfaces is the mesh as shown.



# Incidence matrix $\mathbb{E}^{1,0}$

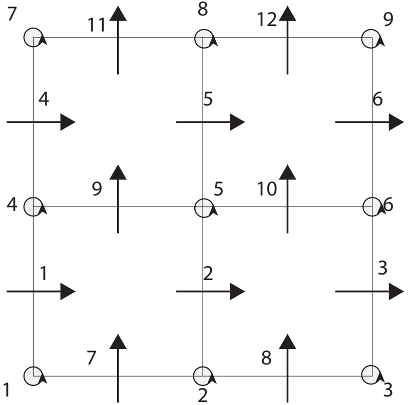
On this oriented mesh we are going to set up 2 matrices, which we will call incidence matrices.

The first incidence matrix will be denoted by $\mathbb{E}^{1,0}$ and it is a matrix which will relate the edges in our mesh to the vertices in our mesh.



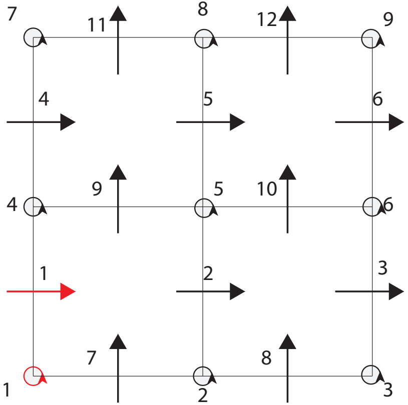
# Incidence matrix $\mathbb{E}^{1,0}$

$$
\mathbb{E}^{1,0} = \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
$$



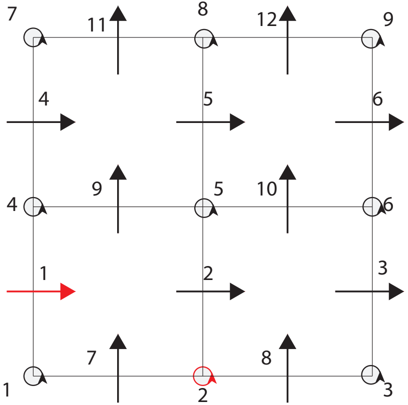
# Incidence matrix $\mathbb{E}^{1,0}$

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



# Incidence matrix $\mathbb{E}^{1,0}$

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



# Incidence matrix $\mathbb{E}^{1,0}$

$$
\mathbb{E}^{1,0} = \begin{pmatrix}
-1 & 0 & {\color{red}0} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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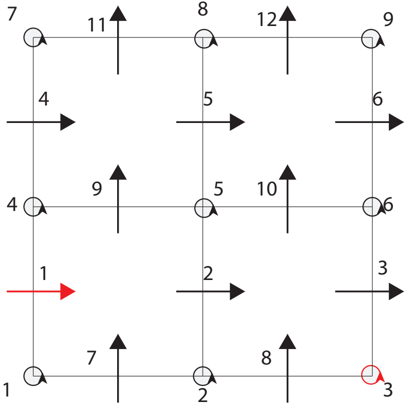
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
$$



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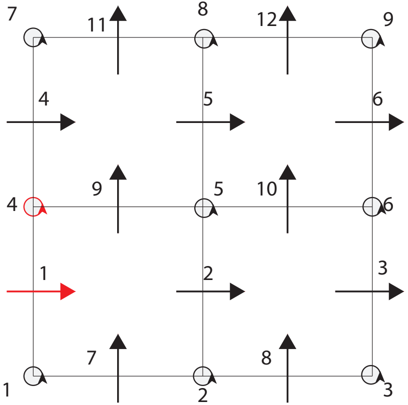
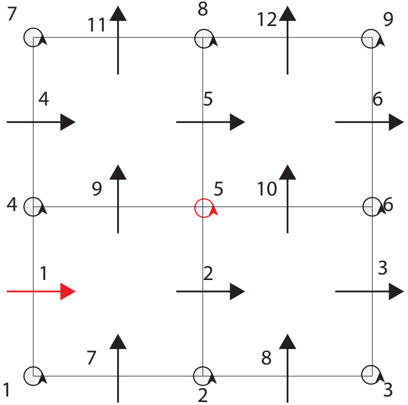


# Incidence matrix $\mathbb{E}^{1,0}$

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 0 & 1 & \color{red}{0} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



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-1 & 0 & 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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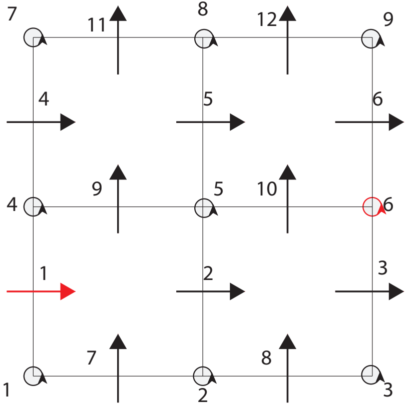
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
$$



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-1 & 0 & 0 & 1 & 0 & 0 & \color{red}{0} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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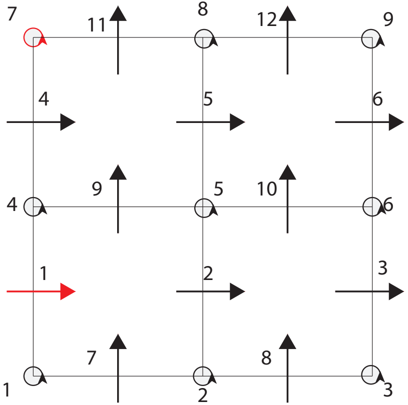

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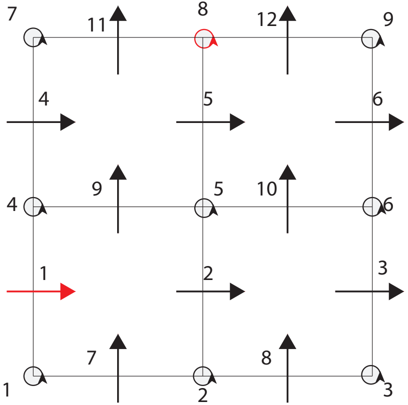
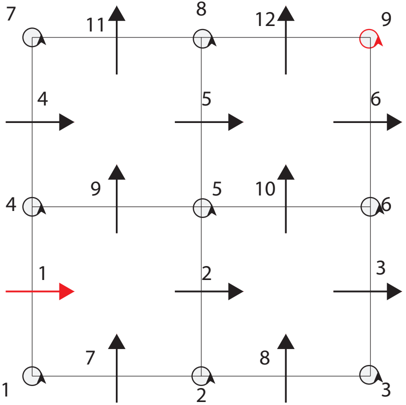


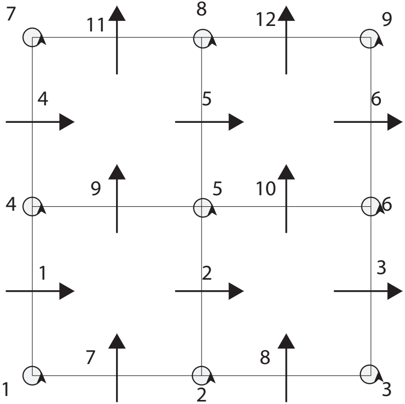
# Incidence matrix $\mathbb{E}^{1,0}$

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \color{red}{0} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



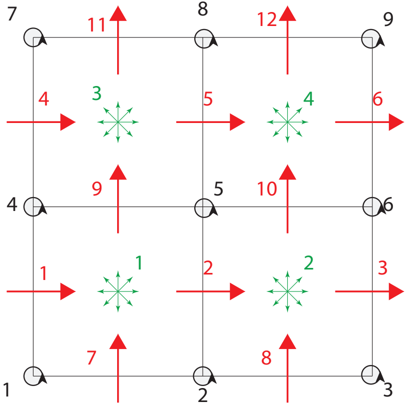
# Incidence matrix $\mathbb{E}^{1,0}$

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$



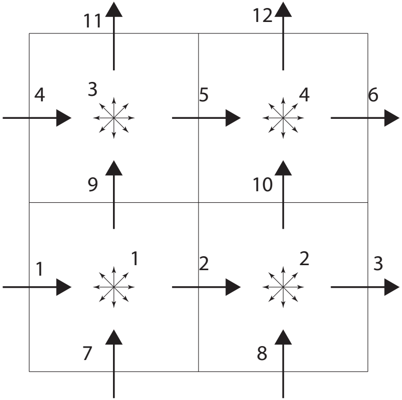
# Incidence matrix $\mathbb{E}^{2,1}$

The next matrix that we will set up is referred to as $\mathbb{E}^{2,1}$ and it will relate geometrically the edge in the mesh to the surface/2D volumes in the mesh



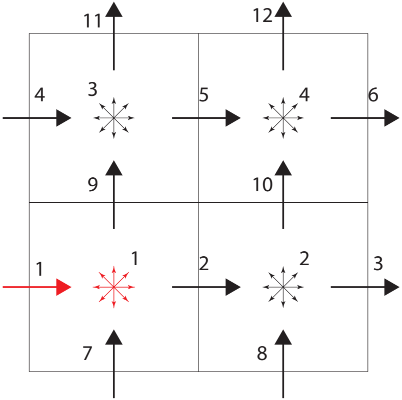
# Incidence matrix $\mathbb{E}^{2,1}$

$$\mathbb{E}^{2,1} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



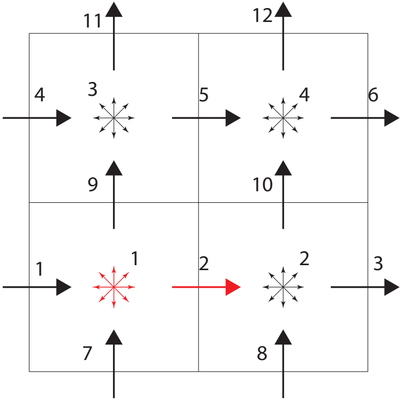
# Incidence matrix $\mathbb{E}^{2,1}$

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



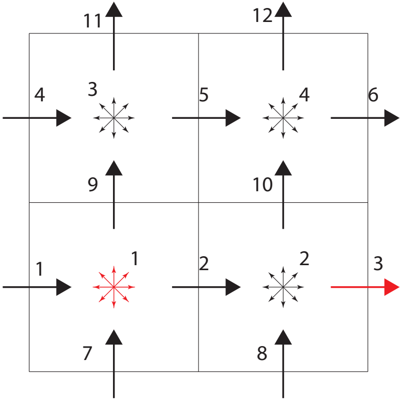
# Incidence matrix $\mathbb{E}^{2,1}$

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



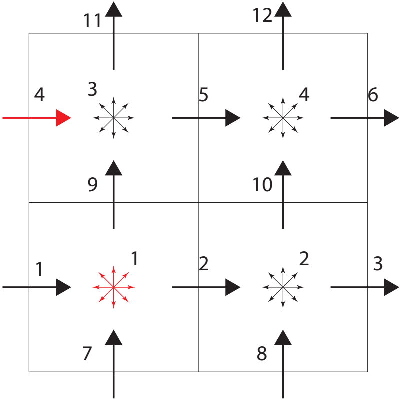
# Incidence matrix $\mathbb{E}^{2,1}$

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & {\color{red}0} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



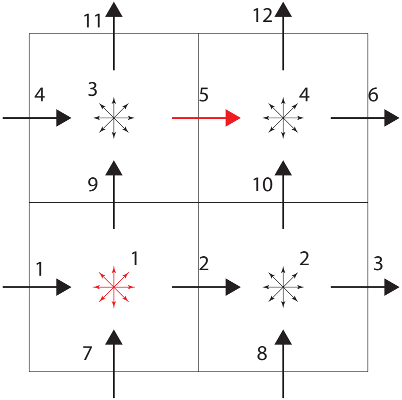
# Incidence matrix $\mathbb{E}^{2,1}$

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 0 & \color{red}{0} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



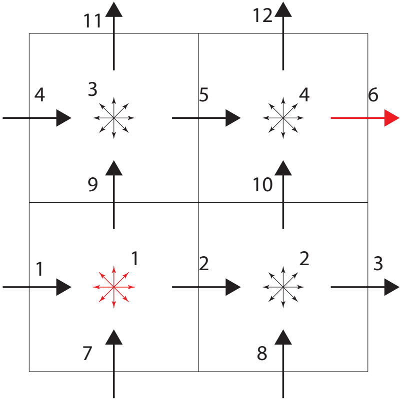
# Incidence matrix $\mathbb{E}^{2,1}$

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 0 & 0 & \color{red}{0} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



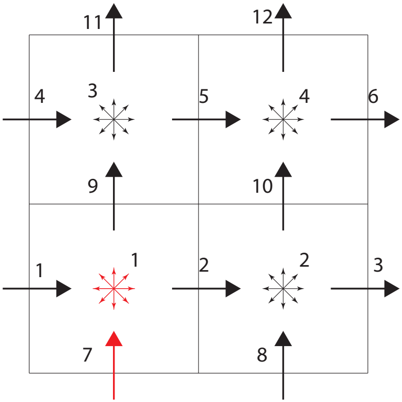
# Incidence matrix $\mathbb{E}^{2,1}$

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & \color{red}{0} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



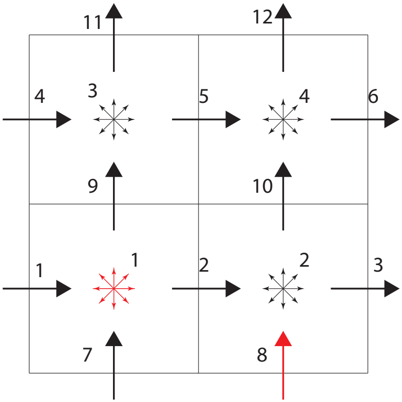
# Incidence matrix $\mathbb{E}^{2,1}$

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



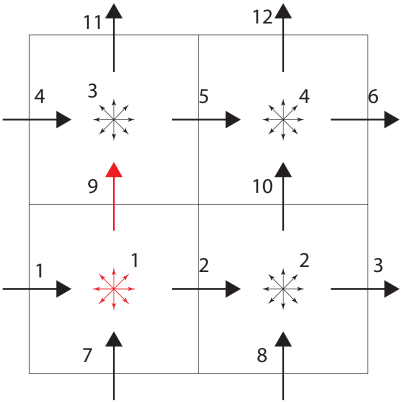
# Incidence matrix $\mathbb{E}^{2,1}$

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & -1 & \color{red}{0} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



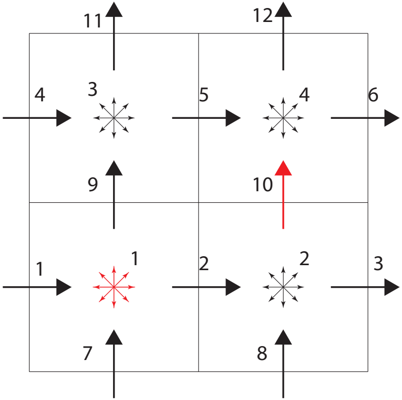
# Incidence matrix $\mathbb{E}^{2,1}$

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & \color{red}{1} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



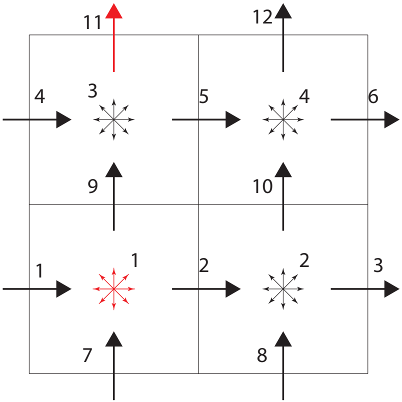
# Incidence matrix $\mathbb{E}^{2,1}$

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & \color{red}{0} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



# Incidence matrix $\mathbb{E}^{2,1}$

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & \color{red}{0} & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



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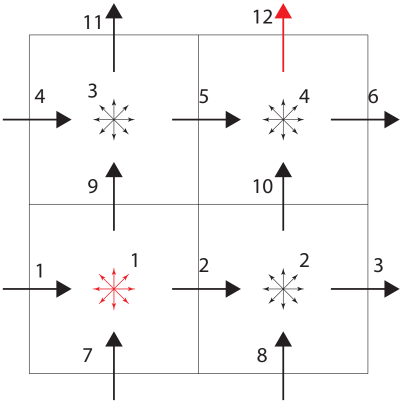
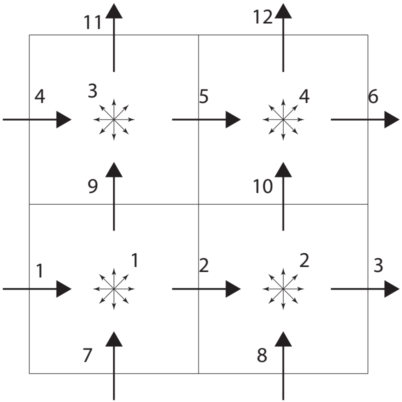
$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & {\color{red}0} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$



# Incidence matrix $\mathbb{E}^{2,1}$

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$



# Incidence matrices

We have now set up two incidence matrices, $\mathbb{E}^{1,0}$ and $\mathbb{E}^{2,1}$

$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

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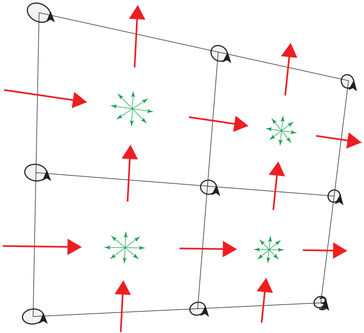
$$\mathbb{E}^{1,0} = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$\mathbb{E}^{2,1} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}$$

What is the product $\mathbb{E}^{2,1} \cdot \mathbb{E}^{1,0}$?

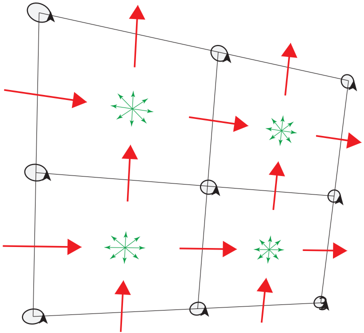
# Properties of incidence matrices

- The incidence matrices are independent of the size of mesh;



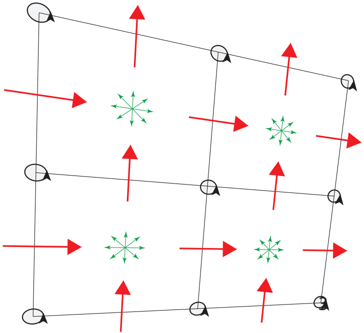
# Properties of incidence matrices

- The incidence matrices are independent of the size of mesh;
- We can distort the mesh as much as we want and the incidence matrices will remain the same;



# Properties of incidence matrices

- The incidence matrices are independent of the size of mesh;
- We can distort the mesh as much as we want and the incidence matrices will remain the same;
- Incidence matrices are very sparse and the non-zero entries, —1 and 1, can be represented by short integers.



04

# Mimetic spectral element discretization

# 1D example

In one dimension we only have two differential $k$-forms, namely 0-forms and 1-forms. Let's see how we discretize these on a 1D mesh. Let $\Omega = [-1, 1]$ and define a grid $-1 = \xi_0 < \xi_1 < \ldots < \xi_{N-1} < \xi_N = 1$

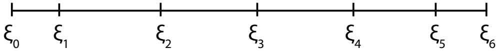



Figure: 1D mesh with $N = 6$

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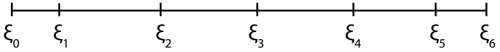


Figure: 1D mesh with $N = 6$

Remember Flanders: 0-forms are naturally evaluated at point, so we are going to sample a 0-form in the points. We cal this process reduction, denoted by $\mathcal{R}(\alpha^{(0)}) = (\alpha_0, \ldots, \alpha_N)$

# Reduction and reconstruction of a 0-form $\alpha^{(0)}$

Let $\alpha^{(0)}(\xi) = \sin(\pi\xi) + 0.4\sin(3\pi\xi) + 2\sin(5\pi\xi/4) + 1.7\sin(7\pi\xi + 1.4)$

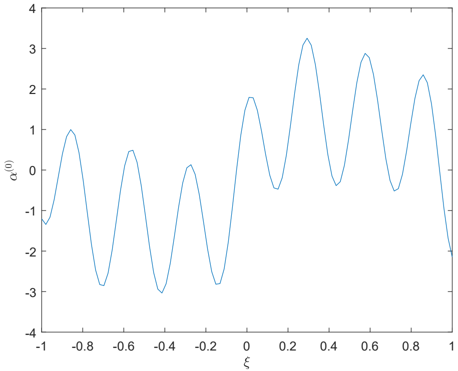


Figure: 0-form $\alpha^{(0)}$

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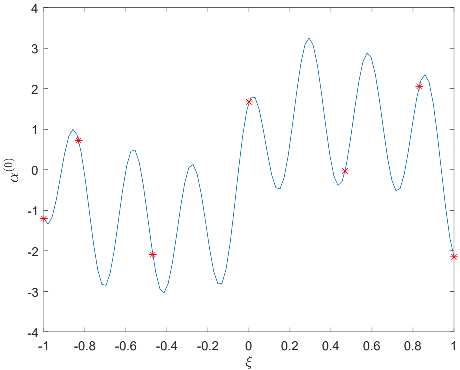


Figure: Evaluation of $\alpha^{(0)}$ in the grid points

# Reduction and reconstruction of a 0-form $\alpha^{(0)}$

Let $\alpha^{(0)}(\xi) = \sin(\pi\xi) + 0.4\sin(3\pi\xi) + 2\sin(5\pi\xi/4) + 1.7\sin(7\pi\xi + 1.4)$

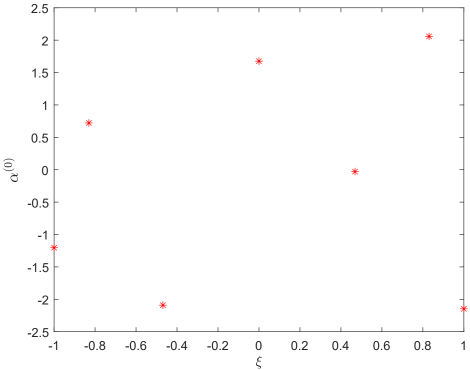


Figure: The sampled values, 0-cochains

# Reduction and reconstruction of a 0-form $\alpha^{(0)}$

Let $\alpha^{(0)}(\xi) = \sin(\pi\xi) + 0.4\sin(3\pi\xi) + 2\sin(5\pi\xi/4) + 1.7\sin(7\pi\xi + 1.4)$

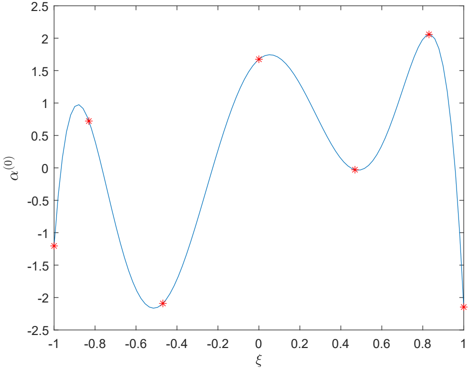


Figure: Reconstruction or cochain interpolation

# Reduction and reconstruction of a 0-form $\alpha^{(0)}$

Let $\alpha^{(0)}(\xi) = \sin(\pi\xi) + 0.4\sin(3\pi\xi) + 2\sin(5\pi\xi/4) + 1.7\sin(7\pi\xi + 1.4)$

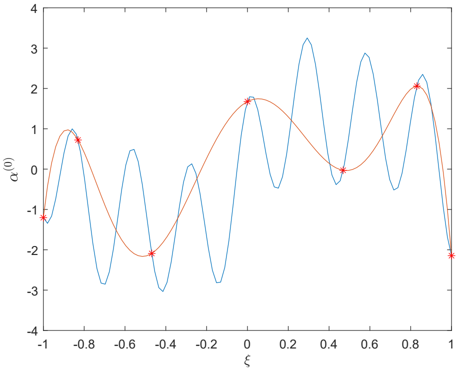


Figure: Comparison original 0-form and reconstructed 0-form

# Reconstruction of a 0-form

If we have sampled the 0-form $\alpha^{(0)}$ on the grid $-1 = \xi_0 < \xi_1 < \ldots < \xi_{N-1} < \xi_N = 1$ and obtained the 0-cochain $(\alpha_0, \alpha_1, \ldots, \alpha_N)$ then we can reconstruct the 0-form using Lagrange polynomials

$$h_i(\xi) = \prod_{k=0,\ k\neq i}^{N} \frac{\xi - \xi_k}{\xi_i - \xi_k}$$

These polynomials have the property that

$$h_i(\xi_j) = \delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

# Reconstruction of a 0-form

If we have sampled the 0-form $\alpha^{(0)}$ on the grid
$-1 = \xi_0 < \xi_1 < \ldots < \xi_{N-1} < \xi_N = 1$ and obtained the 0-cochain $(\alpha_0, \alpha_1, \ldots, \alpha_N)$
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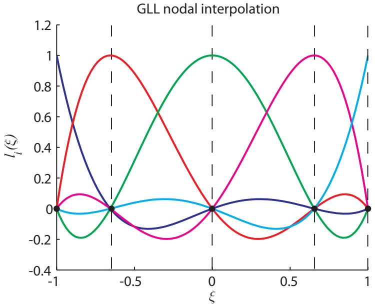


Figure: Lagrange polynomials for $N = 4$.

# Reconstruction of a 0-form

The reconstruction is then given by

$$\mathcal{I}(\alpha_0, \alpha_1, \ldots, \alpha_N)(\xi) = \sum_{i=1}^{N} \alpha_i h_i(\xi)$$

# Reconstruction of a 0-form

The reconstruction is then given by

$$\mathcal{I}(\alpha_0, \alpha_1, \ldots, \alpha_N)(\xi) = \sum_{i=1}^{N} \alpha_i h_i(\xi)$$

$$\mathcal{I}(\alpha_0, \alpha_1, \ldots, \alpha_N)(\xi_j) = \alpha_i$$

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Note that the Lagrange polynomials are not the only functions to interpolate the 0-cochain. We can always add a function $f(\xi)\prod_{k=0}^{N}(\xi - \xi_k)$ to the Lagrange polynomials and this will also give a nodal interpolation.

# Reconstruction of a 0-form

Whatever the choice of reconstruction operator $\mathcal{I}$, it should satisfy the two conditions

$$\boxed{\mathcal{R}\mathcal{I} = \mathbb{I}}$$

and

$$\boxed{\pi := \mathcal{I}\mathcal{R} = \mathbb{I} + \mathcal{O}(h^p)}$$

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The conditions are standard in finite element methods. The first condition is the unisolvency condition and the second the approximation property

# Reduction and reconstruction of a 1-form $\beta^{(1)}$

$$\beta^{(1)}(\xi) = \pi\cos(\pi\xi)\mathrm{d}\xi + 1.2\pi\cos(3\pi\xi)\mathrm{d}\xi + 2.5\pi\cos(5\pi\xi/4)\mathrm{d}\xi + 11.9\pi\cos(7\pi\xi + 1.4)\mathrm{d}\xi$$

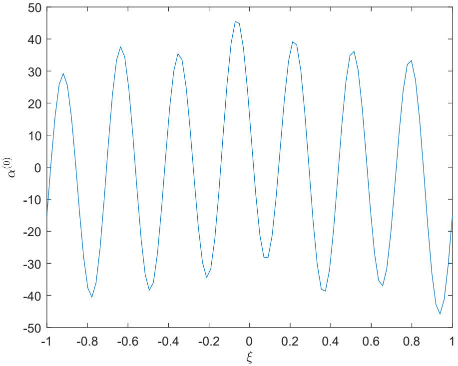


Figure: 1-form $\beta^{(1)}$

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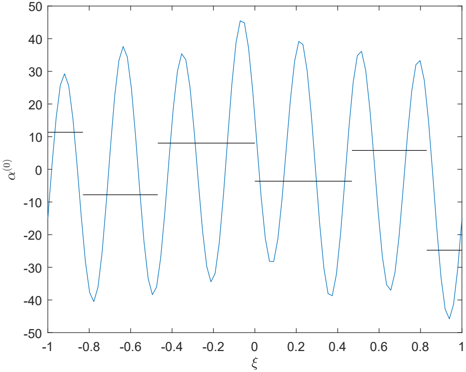


Figure: Integration of $\beta^{(1)}$ over the edges $[\xi_{i-1}, \xi_i]$, $i = 1, \ldots, N$

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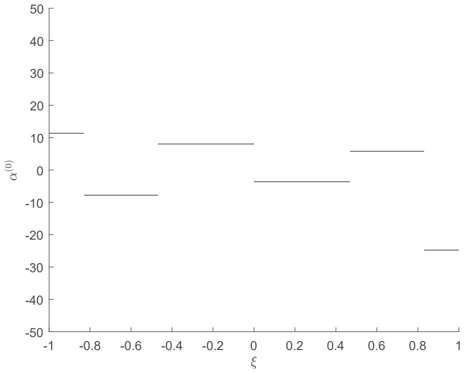




Figure: The sampled values, 1-cochains

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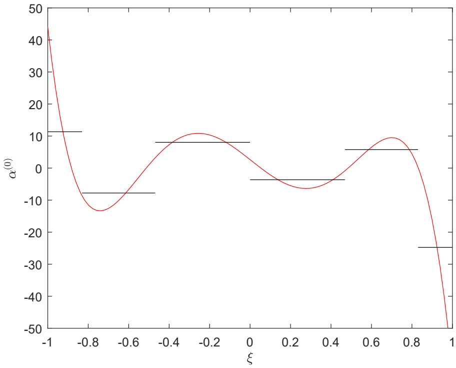


Figure: Reconstruction or cochain interpolation

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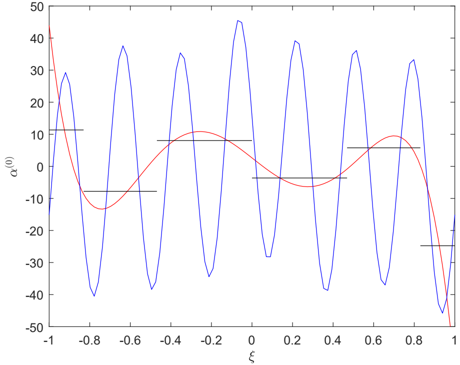


Figure: Comparison original 1-form and reconstructed 1-form

# Reconstruction of a 1-form

The 1-cochains or the sampling of 1-forms are obtained by

$$\beta_i = \int_{\xi_{i-1}}^{\xi_i} \beta^{(1)}\,, \qquad i = 1, \ldots, N$$

To interpolate these discrete values, we use edge functions given by

$$e_i(\xi) = -\sum_{k=0}^{i-1} \mathrm{d}h_k(\xi)$$

These polynomials have the property that

$$\int_{\xi_{j-1}}^{\xi_j} e_i(\xi) = \delta_{i,j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

# Reconstruction of a 1-form

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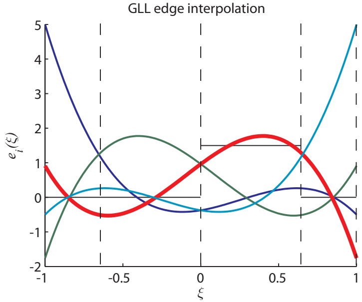


Figure: The edge polynomials for $N = 4$.

# Reconstruction of a 1-form

With these edge functions, the discrete 1-form samples, i.e. the 1-cochains

$$\beta_i = \int_{\xi_{i-1}}^{\xi_i} \beta^{(1)}\,, \quad i = 1, \ldots, N$$

can be reconstructed

$$\mathcal{I}(\beta_1, \ldots, \beta_N)(\xi) = \sum_{i=1}^{N} \beta_i e_i(\xi)$$

# Reconstruction of a 1-form

Once again, these reduction and reconstruction operators satisfy the two conditions

$$\boxed{\mathcal{R}\mathcal{I} = \mathbb{I}}$$

and

$$\boxed{\pi := \mathcal{I}\mathcal{R} = \mathbb{I} + \mathcal{O}(h^p)}$$

$\pi$ is the interpolation or discretization operator.

# The exterior derivative of a 0-form

We know that the exterior derivative of a 0-form gives a 1-form. In the finite diemensional setting this gives:

Let $\alpha^{(0)}(\xi) = \sum_{i=0}^{N} \alpha_i h_i(\xi)$ then

$$\mathrm{d}\alpha^{(0)}(\xi) = \sum_{i=1}^{N} (\alpha_i - \alpha_{i-1})\, e_i(\xi)$$

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# Commuting diagram

$$\begin{tikzcd}
\alpha^{(0)} \arrow[r, "\mathrm{d}"] \arrow[d, "\mathcal{R}"] & \mathrm{d}\alpha^{(0)} \arrow[d, "\mathcal{R}"] \\
\alpha_i \arrow[r, "\mathbb{E}"] & (\alpha_i - \alpha_{i-1})
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Because

$$\mathcal{R}(\mathrm{d}\alpha^{(0)}) = \int_{\xi_{i-1}}^{\xi_i} \mathrm{d}\alpha^{(0)} = \alpha^{(0)}(\xi_i) - \alpha^{(0)}(\xi_{i-1}) = \alpha_i - \alpha_{i-1}$$

# Commuting diagram

$$
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\alpha^{(0)} \arrow[r, "\mathrm{d}"] \arrow[d, "\mathcal{R}"] & \mathrm{d}\alpha^{(0)} \arrow[d, "\mathcal{R}"] \\
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\end{tikzcd}
$$

Here

$$
\mathbb{E} = \begin{pmatrix} -1 & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{pmatrix}
$$

# Commuting diagram

The same holds for the discretization

$$\begin{tikzcd}
\alpha^{(0)} \arrow[rr, "\mathrm{d}"] \arrow[d, "\pi"] & & \mathrm{d}\alpha^{(0)} \arrow[d, "\pi"] \\
\sum_i \alpha_i h_i(\xi) \arrow[r, "\mathrm{d}"] & \sum_i (\alpha_i - \alpha_{i-1})\, e_i(\xi) &
\end{tikzcd}$$

# De Rham sequence in 1D

In this 1D example we have the following short De Rham sequence

$$
\begin{tikzcd}
\mathbb{R} \arrow[r] & \Lambda^{(0)}(\Omega) \arrow[r, "\mathrm{d}"] \arrow[d, "\pi"] & \Lambda^{(1)}(\Omega) \arrow[r] \arrow[d, "\pi"] & 0 \\
\mathbb{R} \arrow[r] & \Lambda_h^{(0)}(\Omega) \arrow[r, "\mathrm{d}"] & \Lambda_h^{(1)}(\Omega) \arrow[r] & 0
\end{tikzcd}
$$

where the space $\Lambda_h^{(0)}$ is spanned by the Lagrange polynomials, $h_i(\xi)$ and the space $\Lambda_h^{(1)}$ is spanned by the edge polynomials, $e_i(\xi)$

# Discretization in multi-D

In the 1D example we saw that we assigned values to points and we assigned (integral) values to edges.

In higher dimensions we can assign values to vertices in the mesh, to edges in the mesh,, to faces and volumes in the mesh. Here we will focus on the 2D case, see also the paper[2]. We will use an outer-oriented grid[3][4]



[2] A. Palha and M. Gerritsma (2017). "A mass, energy, enstrophy and vorticity conserving (MEEVC) mimetic spectral element discretization for the 2D incompressible Navier-Stokes equations". In: *Journal of Computational Physics* 328, pp. 200–220

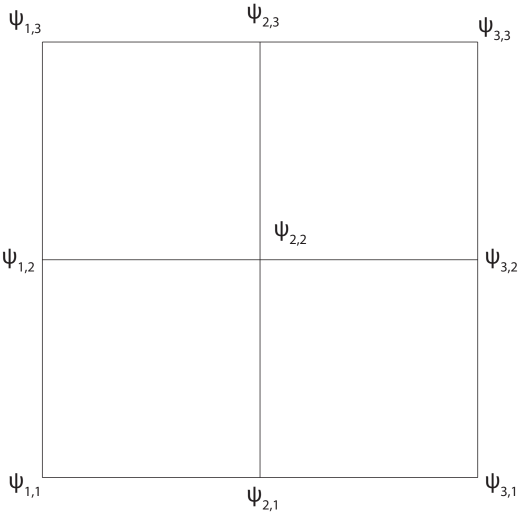
[3] A. Bossavit (1998). *Computational Electromagnetism*. Academic Press

[4] E. Tonti (2013). *The Mathematical Structure of Classical and Relativistic Physics*. Springer

# Stream function

The stream function $\psi(x, y)$ is a 0-form and will therefore be represented in the nodes of the mesh. For the mesh shown on the right, this gives the 9 values $\psi_{i,j}$

$$\psi^h(x, y) = \sum_{i,j=0}^{N} \psi_{i,j} h_i(x) h_j(y)$$



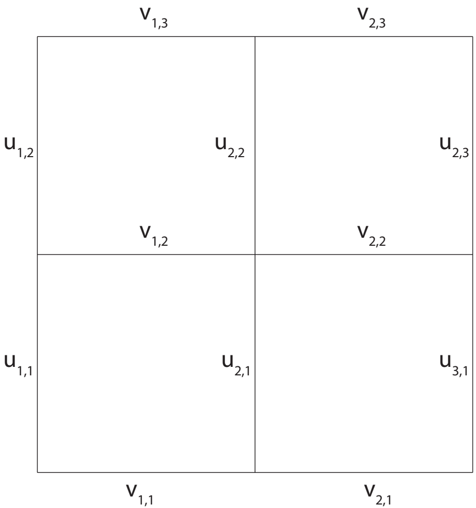
# Velocity

$$u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}$$

$$\psi^h(x,y) = \sum_{i,j=0}^{N} \psi_{i,j} h_i(x) h_j(y)$$

$$u^h(x,y) = \sum_{i=0}^{N} \sum_{j=1}^{N} (\psi_{i,j} - \psi_{i,j-1}) h_i(x) e_j(y)$$

$$v^h(x,y) = \sum_{i=1}^{N} \sum_{j=0}^{N} (\psi_{i-1,j} - \psi_{i,j}) e_i(x) h_j(y)$$



# Velocity-stream function

$$u^h(x,y) = \sum_{i=0}^{N}\sum_{j=1}^{N}(\psi_{i,j} - \psi_{i,j-1})h_i(x)e_j(y) = \sum_{i=0}^{N}\sum_{j=1}^{N}u_{i,j}h_i(x)e_j(y) \implies u_{i,j} = \psi_{i,j} - \psi_{i,j-1}$$

$$v^h(x,y) = \sum_{i=1}^{N}\sum_{j=0}^{N}(\psi_{i-1,j} - \psi_{i,j})e_i(x)h_j(y) = \sum_{i=1}^{N}\sum_{j=0}^{N}v_{i,j}e_i(x)h_j(y) \implies v_{i,j} = \psi_{i-1,j} - \psi_{i,j}$$

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$$\begin{pmatrix} u_{1,1}\\ u_{2,1}\\ u_{3,1}\\ u_{1,2}\\ u_{2,2}\\ u_{3,2}\\ v_{1,1}\\ v_{2,1}\\ v_{1,2}\\ v_{2,2}\\ v_{1,3}\\ v_{2,3} \end{pmatrix}=\begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1\\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} \psi_{1,1}\\ \psi_{2,1}\\ \psi_{3,1}\\ \psi_{1,2}\\ \psi_{2,2}\\ \psi_{3,2}\\ \psi_{1,3}\\ \psi_{2,3}\\ \psi_{3,3} \end{pmatrix}$$

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$$\boxed{\vec{u} = \nabla \times \psi \quad \Longleftrightarrow \quad \bar{\mathrm{u}} = \mathbb{E}^{1,0}\bar{\psi}}$$

# Velocity derivative

$$u^h(x,y) = \sum_{i=0}^{N}\sum_{j=1}^{N} u_{i,j} h_i(x) e_j(y)$$

$$v^h(x,y) = \sum_{i=1}^{N}\sum_{j=0}^{N} v_{i,j} e_i(x) h_j(y)$$

Then

$$\frac{\partial u^h}{\partial x} + \frac{\partial v^h}{\partial y} = \sum_{i,j=1}^{N} \left[ u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1} \right] e_i(x) e_j(y)$$

# Velocity derivative

If we want to solve $m = \nabla \cdot \vec{u}$, in fact $m^{(n)} = \mathrm{d}u^{(n-1)}$, then

$$\sum_{i,j=1}^{N} m_{i,j} e_i(x) e_j(y) = \sum_{i,j=1}^{N} \left[u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1}\right] e_i(x) e_j(y) \quad \Longrightarrow \quad m_{i,j} = \left[u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1}\right]$$

$$\begin{pmatrix} m_{1,1} \\ m_{2,1} \\ m_{1,2} \\ m_{2,2} \end{pmatrix} = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \\ u_{1,2} \\ u_{2,2} \\ u_{3,2} \\ v_{1,1} \\ v_{2,1} \\ v_{1,2} \\ v_{2,2} \\ v_{1,3} \\ v_{2,3} \end{pmatrix}$$

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$$\boxed{m = \nabla \cdot \vec{u} \quad \Longleftrightarrow \quad \bar{\mathrm{m}} = \mathbb{E}^{2,1} \bar{\mathrm{u}}}$$

# What does the discrete derivative mean?

We just saw that

$$\nabla \cdot \vec{u} = u_{i,j} - u_{i-1,j} + v_{i,j} - v_{i,j-1}$$

So we can express the degrees of freedom of the div $\vec{u}$ exactly in terms of the integrated velocity components over edges.

But the degrees of freedom of the div $\vec{u}$ actually represent the integral of div $\vec{u}$ over a 2D volume, so in fact this equation expresses

$$\int_{\Omega^k} \nabla \cdot \vec{u} \mathrm{d}\Omega^k = \int_{\partial\Omega^k} \vec{u} \cdot \vec{n} \mathrm{d}\Gamma^k$$

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So the incidence matrix encodes the divergence theorem, which is the Mother of all Equations for $k = n - 1$.

# Incidence matrices and vector operations

We just saw that the incidence matrices $\mathbb{E}^{1,0}$ and $\mathbb{E}^{2,1}$ represent the vector operations curl and divergence, respectively.

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- The stream function is defined in the points
- The velocity is defined over the edges

$$u_{i,j} = \int_{y_{j-1}}^{y_j} u(x_i, s)\,\mathrm{d}s\,, \quad i = 0, \dots, N\,,\ j = 1, \dots, N$$

$$v_{i,j} = \int_{x_{i-1}}^{x_i} v(s, y_j)\,\mathrm{d}s\,, \quad i = 1, \dots, N\,,\ j = 0, \dots, N$$

# Incidence matrices and vector operations[5]

- Note that this derivative is exact



[5] J. Kreeft, A. Palha, and M. Gerritsma (2011). “Mimetic framework on curvilinear quadrilaterals of arbitrary orde”. In: *arXiv*

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$$\mathbb{R} \xrightarrow{\mathrm{d}} \Lambda^{(0)}(\Omega) \xrightarrow[\mathbb{E}^{1,0}]{\mathrm{d}} \Lambda^{(1)}(\Omega) \xrightarrow[\mathbb{E}^{2,1}]{\mathrm{d}} \Lambda^{(2)} \xrightarrow{\mathrm{d}} 0$$



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05

# Dual mesh
# Dual operators

# Missing operators

In the previous section we discretized the the curl applied to scalar functions and the divergence operator in terms of incidence matrices $\mathbb{E}^{k+1,k}$.

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What are missing are the discrete representations of the gradient and the curl applied to vectors

# Missing operators

In the previous section we discretized the the curl applied to scalar functions and the divergence operator in terms of incidence matrices $\mathbb{E}^{k+1,k}$.

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It seems that if we go to 3D this problem is resolved. The $\mathbb{E}^{1,0}$ represents the gradient, The $\mathbb{E}^{2,1}$ represents the curl and The $\mathbb{E}^{3,2}$ represents the divergence, but that is not really the case. We need to define dual variables and dual operators as well.

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In order to keep things manageable we will outline the approach in 2D

# Different ways to include dual variables and operators

There are essentially 3 ways of including dual variables and dual operators

a. Construct explicitly a dual grid. This is commonly done in Discrete Exterior Calculus (DEC)[6] and most staggered finite volume schemes;



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a. Construct explicitly a dual grid. This is commonly done in Discrete Exterior Calculus (DEC)[6] and most staggered finite volume schemes;
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c. Define dual variables and dual operators using dual polynomials[8].



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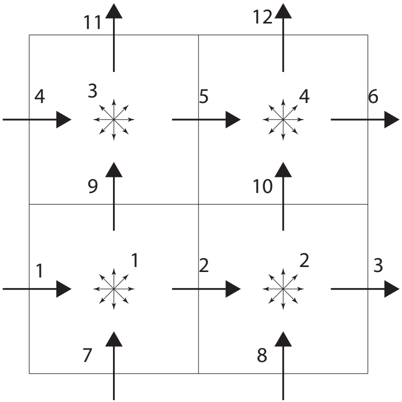
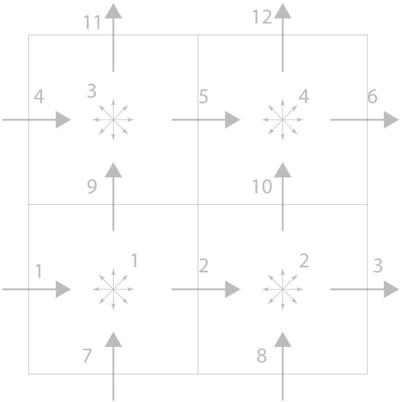
# Dual grid

In the dual grid approach, we explicitly construct a dual grid with as many vertices as primal surfaces, as many dual edges as primal edge and as many dual surfaces as primal vertices.



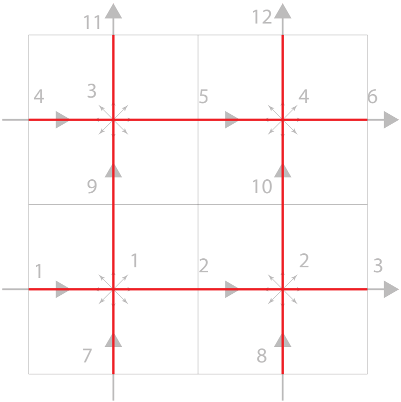
# Dual grid

In the general nD case we have as many $(n - k)$-dimensional geometric dual objects as $k$-dimensional geometric primal objects.



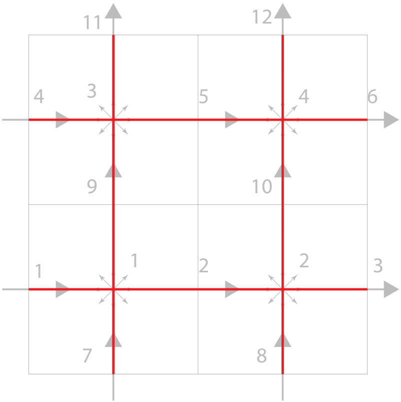
# Dual grid

A dual grid then looks something like this. Note that the numbering of the $(n-k)$-dimensional geometric dual objects equals the numbering of the $k$-dimensional geometric primal objects.



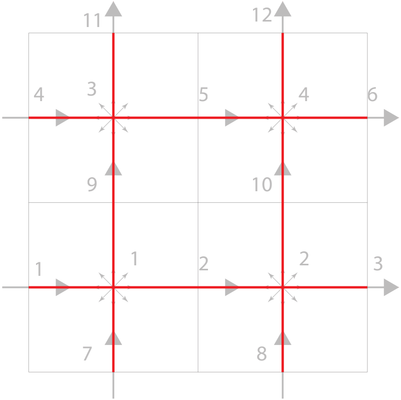
# Dual grid

But the type of orientation has changed. Where the primal grid was outer-oriented the corresponding dual grid is inner-oriented.



# Dual grid

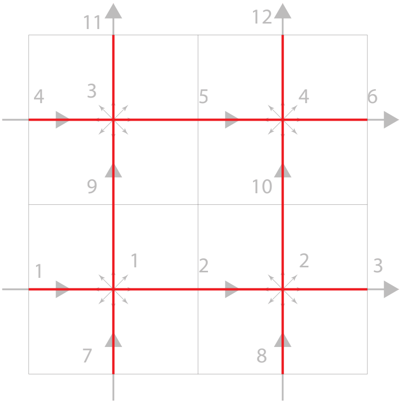
We can set up incidence matrices for this dual grid as well (exercise) and then you will see that $\widetilde{\mathbb{E}}^{1,0} = {\mathbb{E}^{2,1}}^{\mathsf{T}}$ and $\widetilde{\mathbb{E}}^{2,1} = {\mathbb{E}^{1,0}}^{\mathsf{T}}$.




Here the dual incidence matrices are denoted by a twiddle.

# Dual grid

Note that the dual grid 'does not have a boundary'. The edges extending towards the boundary do not end in a vertex and the surfaces near the boundary are not closed by an edge. This where boundary conditions can be inserted which is beyond the scope of this course.



# Dual grid approach

To use the dual grid approach, we need to have square, invertible matrices which transfer information from the primal mesh to the dual mesh. These matrices are known as Hodge matrices denoted by $\mathbb{H}^{k,n-k}$. These matrices complement what is known as the double De Rham complex

$$
\begin{tikzcd}
\mathbb{R} \arrow[r] & \Lambda_h^{(0)} \arrow[r, "\mathbb{E}^{1,0}", "\nabla^{\perp}"'] \arrow[d, dashed, shift right, "\mathbb{H}^{\tilde{2},0}"'] & \Lambda_h^{(1)} \arrow[r, "\mathbb{E}^{2,1}", "\nabla\cdot"'] \arrow[d, dashed, shift right, "\mathbb{H}^{\tilde{1},1}"'] & \Lambda_h^{(2)} \arrow[r] \arrow[d, dashed, shift right, "\mathbb{H}^{\tilde{0},2}"'] & 0 \\
0 & \widetilde{\Lambda}_h^{(2)} \arrow[l] \arrow[u, dashed, shift right, "\mathbb{H}^{0,\tilde{2}}"'] & \widetilde{\Lambda}_h^{(1)} \arrow[l, "\widetilde{\mathbb{E}}^{2,1}"', "\nabla\times"] \arrow[u, dashed, shift right, "\mathbb{H}^{1,\tilde{1}}"'] & \widetilde{\Lambda}_h^{(0)} \arrow[l, "\widetilde{\mathbb{E}}^{1,0}"', "-\nabla"] \arrow[u, dashed, shift right, "\mathbb{H}^{2,\tilde{0}}"'] & \mathbb{R} \arrow[l]
\end{tikzcd}
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\end{tikzcd}
$$

So, if we have a quantity defined over primal surfaces, i.e. expressed in the basis $e_i(x)e_j(y)$, then we can take its gradient by applying

$$-\mathbb{H}^{1,\tilde{1}}\tilde{\mathbb{E}}^{1,0}\mathbb{H}^{\tilde{0},2}$$

to its degrees of freedom, i.e. its 2-cochains.

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- Contain material constants of the physical model

06

# Finite element method

# Simple 2D example

Consider the functional

$$\mathcal{J}(\vec{u}, \phi; f) = \int_\Omega \frac{1}{2}|\vec{u}|^2 \mathrm{d}\Omega + \int_\Omega \phi \left(\nabla \cdot \vec{u} - f\right) \mathrm{d}\Omega$$

where $\Omega = [-1, 1]^2$.

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where $\Omega = [-1, 1]^2$.

We are looking for a stationary point of this functional. Calculus of variations then gives us

$$\int_\Omega (\vec{u}, \vec{v})\, \mathrm{d}\Omega + \int_\Omega \phi \nabla \cdot \vec{v} \mathrm{d}\Omega + \int_\Omega \varphi \left(\nabla \cdot \vec{u} - f\right) \mathrm{d}\Omega\,, \quad \forall \vec{v}, \varphi$$

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Let $\vec{u} = (u, v)$ and insert our expansions

$$u(x,y) = \sum_{i=0}^{N} \sum_{j=1}^{N} u_{i,j} h_i(x) e_j(y) \quad \text{and} \quad v(x,y) = \sum_{i=1}^{N} \sum_{j=0}^{N} v_{i,j} e_i(x) h_j(y)$$

and

$$\phi(x,y) = \sum_{i,j=1}^{N} \phi_{i,j} e_i(x) e_j(y)$$

and expand $\vec{v}$ and $\varphi$ in the same way.

# Simple 2D example

Then we obtain the following system of linear equation

$$\begin{pmatrix} \mathbb{M}^{(1)} & {\mathbb{E}^{2,1}}^{T}\mathbb{M}^{(2)} \\ \mathbb{M}^{(2)}\mathbb{E}^{2,1} & 0 \end{pmatrix}\begin{pmatrix} \bar{\mathrm{u}} \\ \bar{\phi} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbb{M}^{(2)}\bar{f} \end{pmatrix}$$

The matrices $\mathbb{M}^{(1)}$ and $\mathbb{M}^{(2)}$ are called mass matrices which are obtained by directly from the basis functions.

This equation approximates the Poisson equation in mixed form given by

$$\begin{cases} \vec{u} & -\nabla\phi & = & 0 \\ \nabla\cdot\vec{u} & & = & f \end{cases}$$

# Simple 2D example

By rewriting the linear system slightly we see

$$\begin{pmatrix} \mathbb{I} & \mathbb{M}^{(1)^{-1}} \mathbb{E}^{2,1^{T}} \mathbb{M}^{(2)} \\ \mathbb{E}^{2,1} & 0 \end{pmatrix} \begin{pmatrix} \bar{\mathrm{u}} \\ \bar{\phi} \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{f} \end{pmatrix}$$

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$$\begin{cases} \vec{u} & -\nabla \phi & = & 0 \\ \nabla \cdot \vec{u} & & = & f \end{cases}$$

We saw that minus the gradient, using a dual mesh was given by

$$\mathbb{H}^{1,\tilde{1}} \mathbb{E}^{2,1^{T}} \mathbb{H}^{\tilde{0},2}$$

while in this formulation by

$$\mathbb{M}^{(1)^{-1}} \mathbb{E}^{2,1^{T}} \mathbb{M}^{(2)}$$

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# Simple 2D example

By comparing

$$\mathbb{H}^{1,\tilde{1}}\mathbb{E}^{2,1^{\mathsf{T}}}\mathbb{H}^{\tilde{0},2}$$

with

$$\mathbb{M}^{(1)^{-1}}\mathbb{E}^{2,1^{\mathsf{T}}}\mathbb{M}^{(2)}$$

we actually see that in a finite element formulation the Hodge matrices are represented by mass matrices.

So whenever you see a mass matrix in a finite element formulation, you can interpret this as a switch between a primal representation to a dual representation[9].



[9] V. Jain, Y. Zhang, A. Palha, and M. Gerritsma (2021). "Construction and application of algebraic dual polynomial representations for finite element methods on quadrilateral and hexahedral meshes". In: *Computers & Mathematics with Applications*

07

# Summary

# Summary I

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- The exterior derivative resembles well-known vector operations which cab be discretized exactly. We call this a topological operator, i.e. independent of size or shape of the mesh.
- In 2D we have 4 different spaces of differential forms and the connecting exterior derivative gives us a De Rham sequence
- We set up the a discrete representation of the exterior derivative purely in terms of geometry
- We looked at reduction and reconstruction of forms on a mesh. This also gave us the basis functions

# Summary II

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- The discrete degrees of freedom, cochains, are integral quantities over the geometric elements that make up our mesh
- To complement the framework, we introduced a dual grid and described the role of Hodge matrices to switch between the primal and dual representations

# Thank you for your attention

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