Structure preserving discretizations II

Application to the shallow waters equations

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@WCCM24 – W24-09 Geometric Mechanics Formulations and Structure Preserving Discretizations: An Introductory course

21/07/2024

TU Delft

Overview

- Intro to shallow waters equations

  Recap on the equations. Arakawa-Lamb grids and different formulations.

- Hamiltonian formulation of shallow waters equations

  Curl-form Hamiltonian formulation. Conservation properties.

- Spatial discretization of shallow waters equations

  Discrete function spaces. Semi-discretization of the shallow waters equations. Discrete conservation properties.

- Temporal discretization of shallow waters equations

  Midpoint rule. Poisson integrator. Full discretization of the shallow waters equations.

Overview

- Hands on work

  Implementing structure preserving discretization for shallow waters with Firedrake.

Intro shallow waters equations

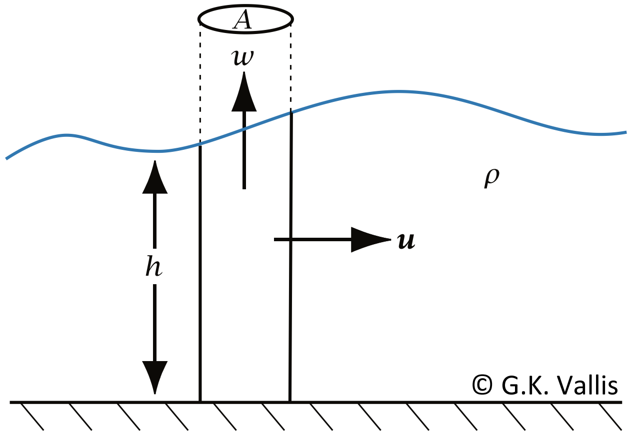
Recap

Intro shallow waters equations

The equations and possible discretizations

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\boldsymbol{u}) = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla (hg) = 0$$



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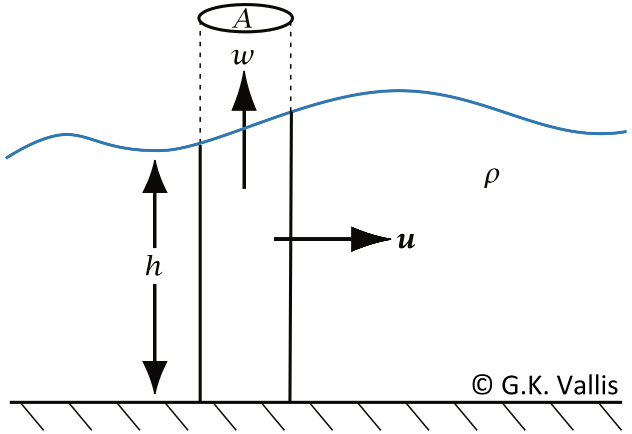

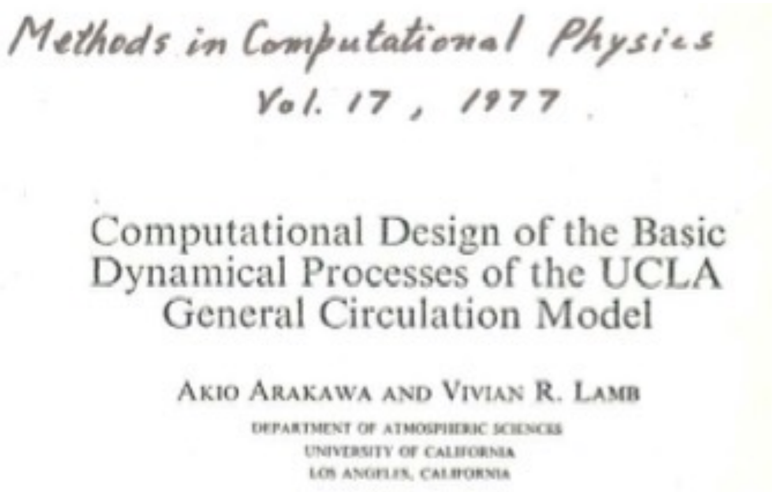

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Computational Design of the Basic Dynamical Processes of the UCLA General Circulation Model

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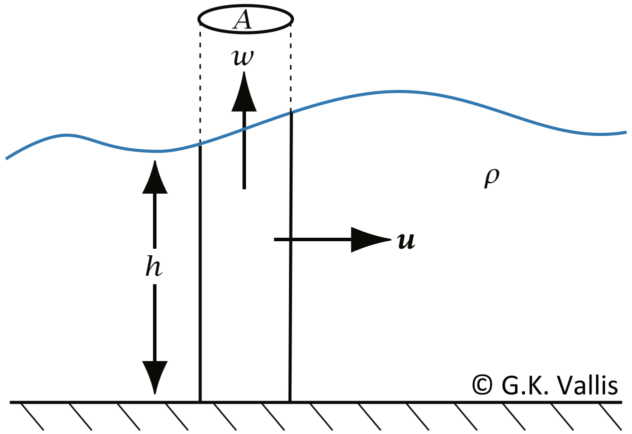

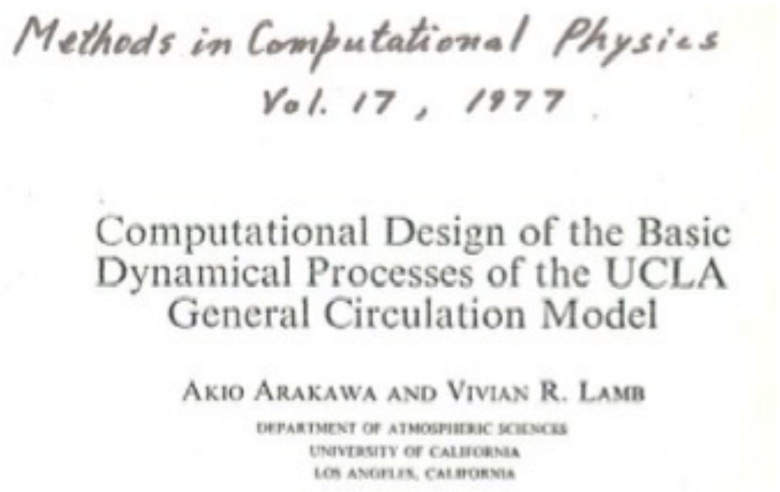
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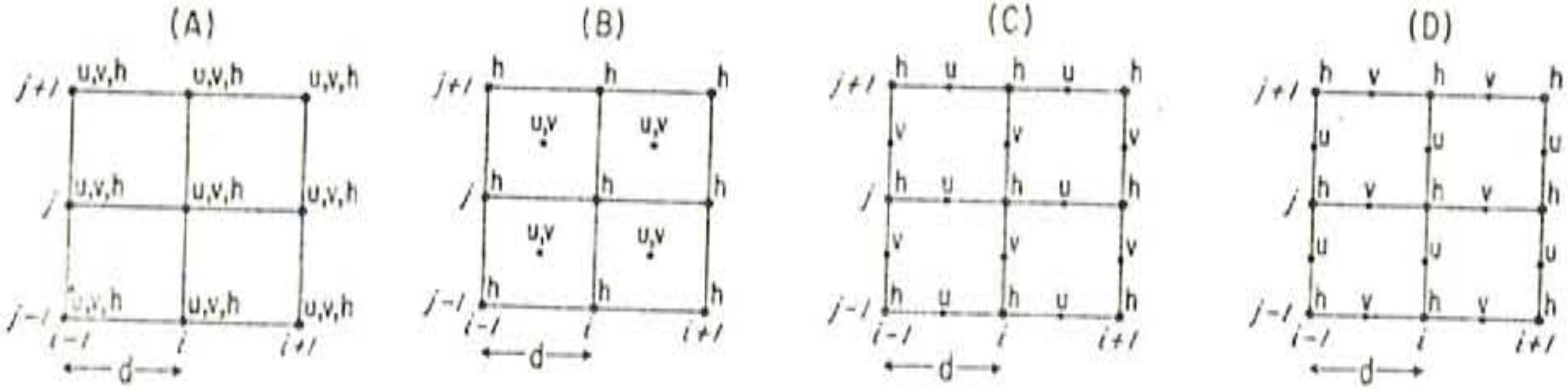

All finite differences but unknowns placed at different locations

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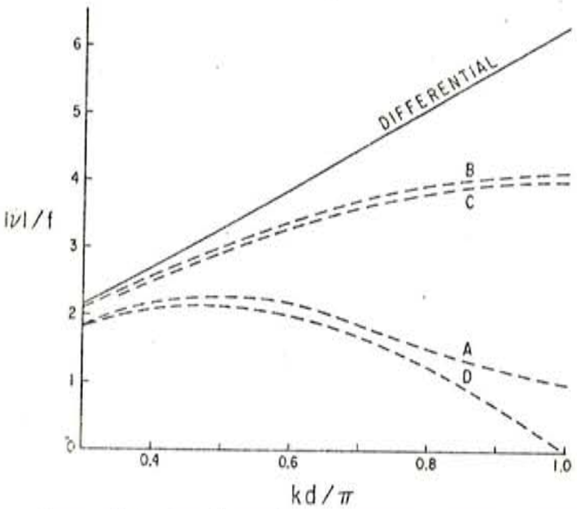
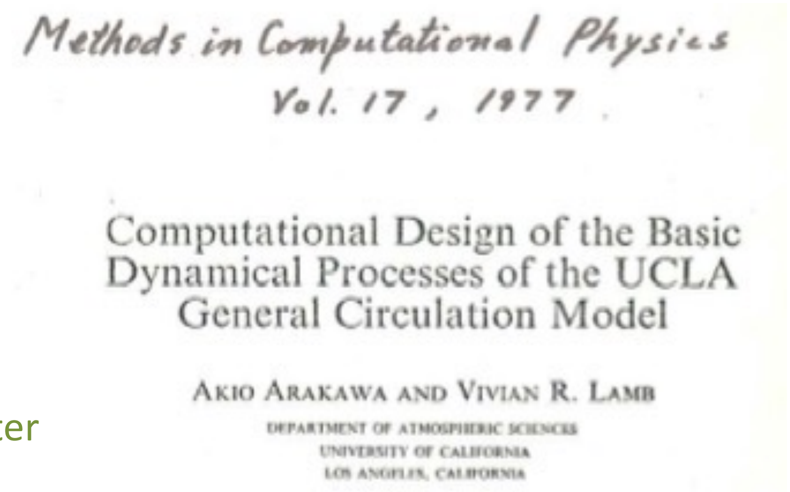


B, C
Better

A, D
Worse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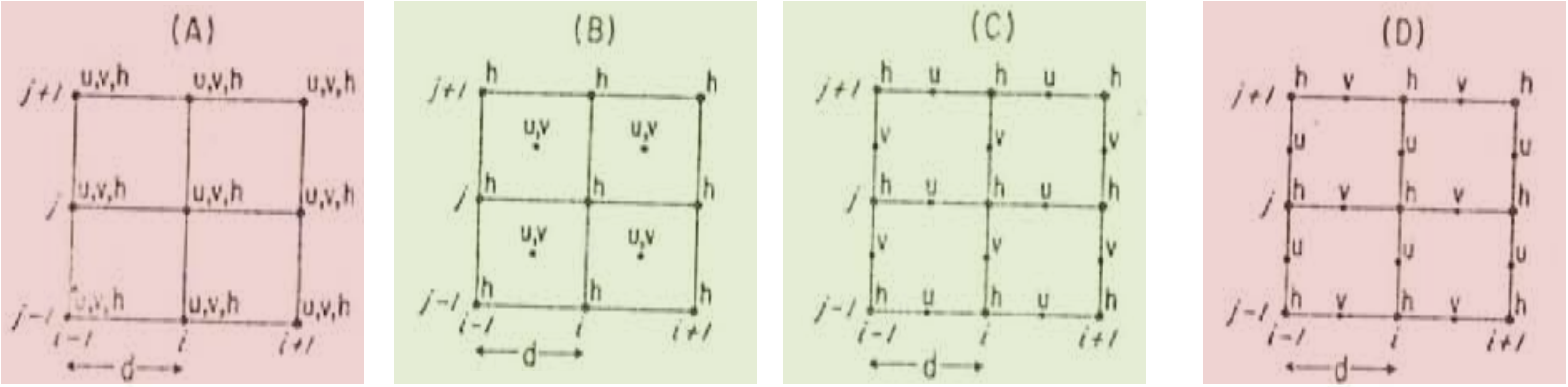

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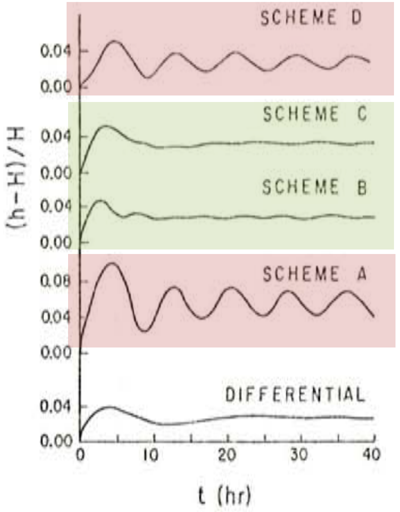
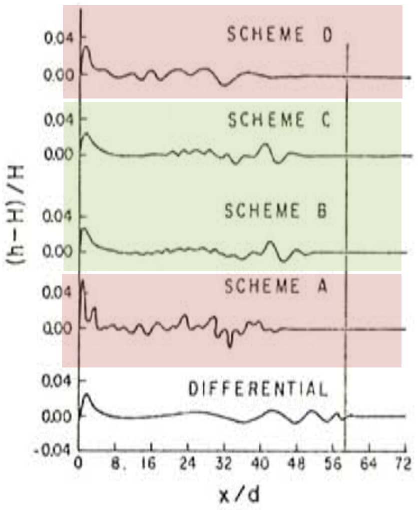
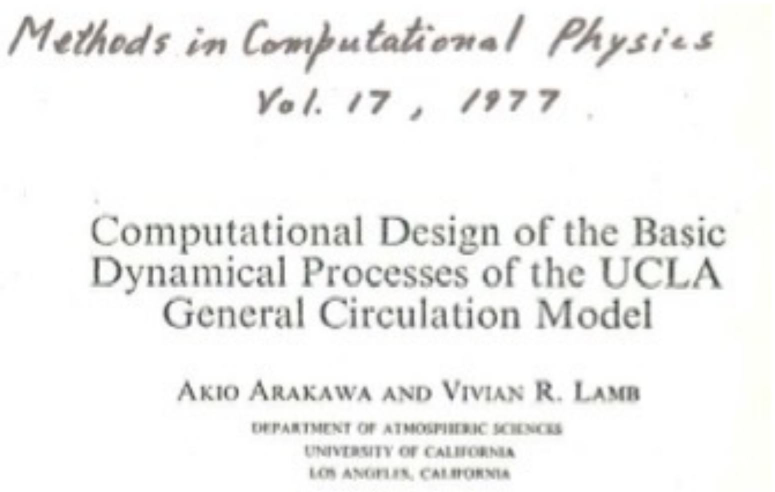
Time evolution



Spatial at an instant

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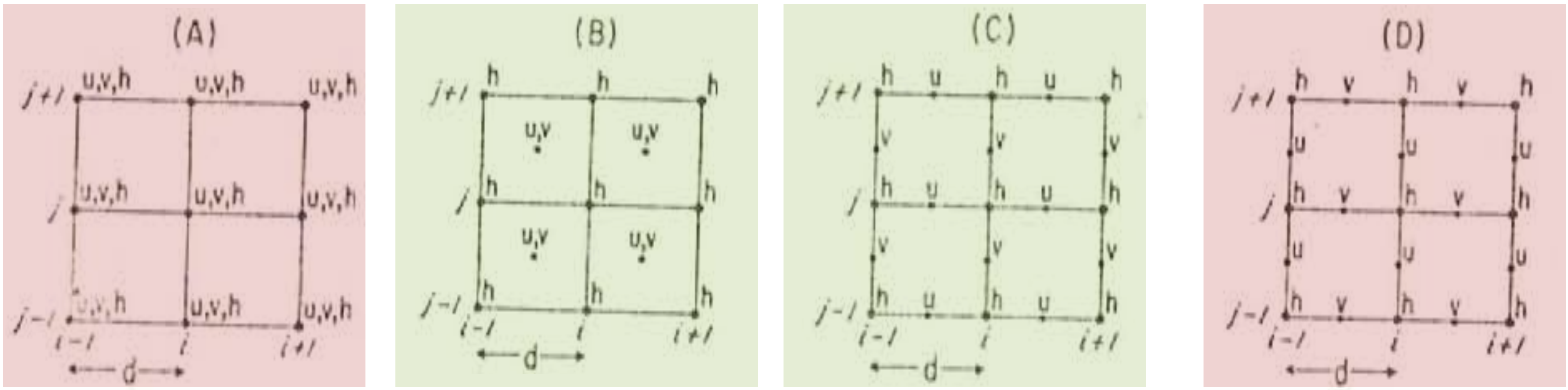

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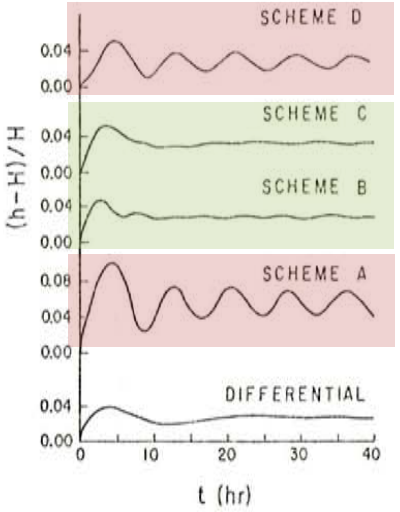

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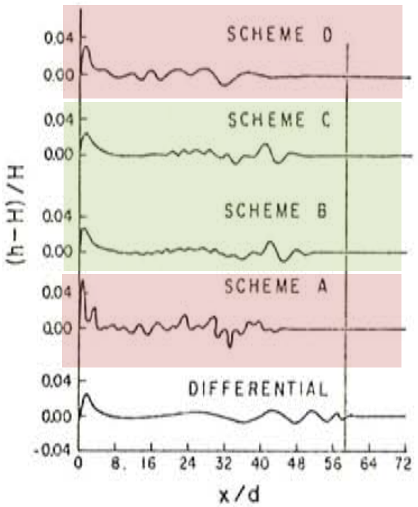

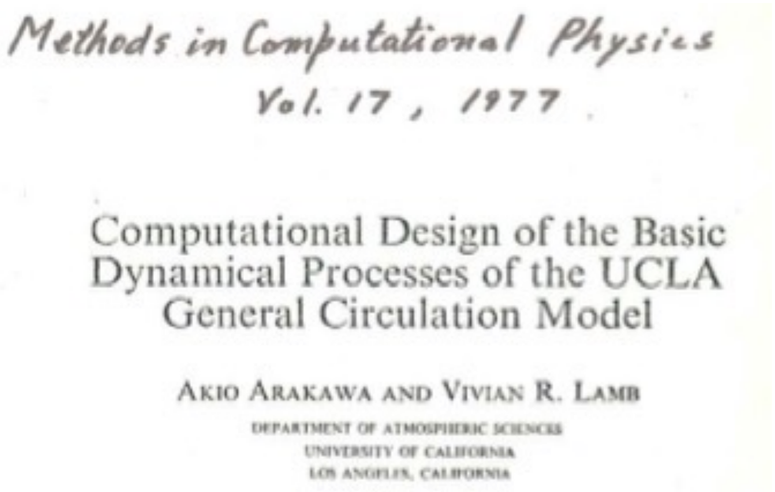
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Quality of numerical results depends on placement of DOFs

Some perform better than others

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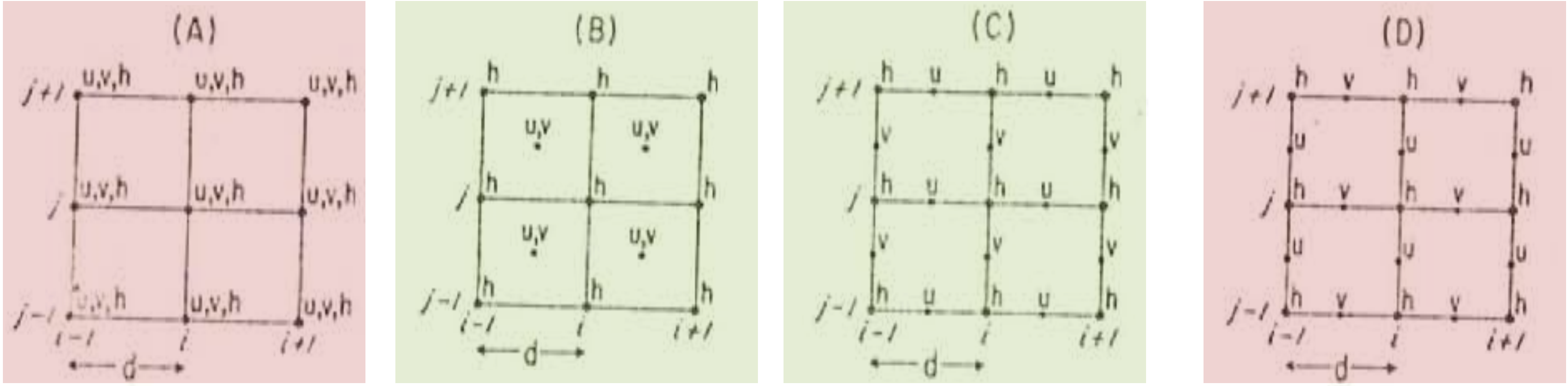

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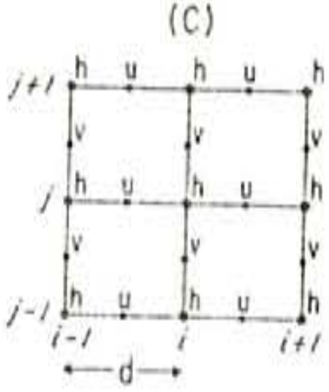
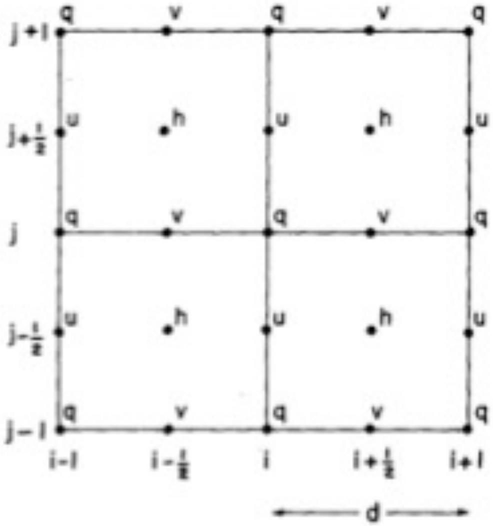


New grid based on C-grid



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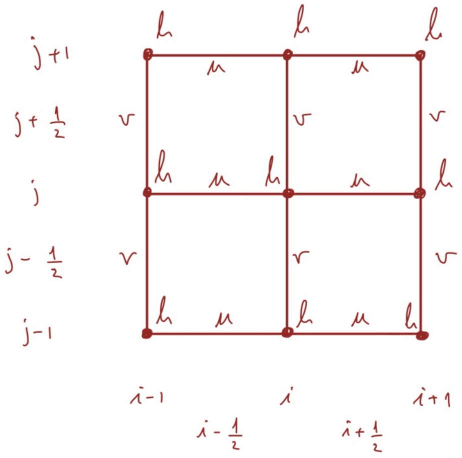
Original C-grid



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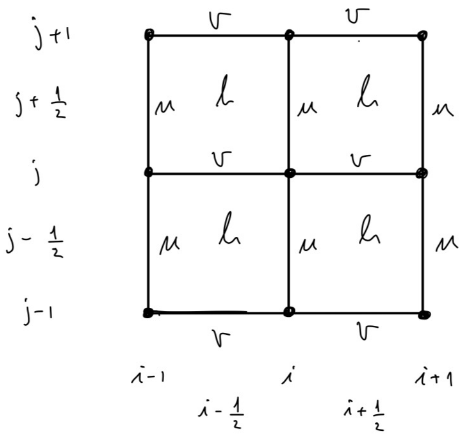

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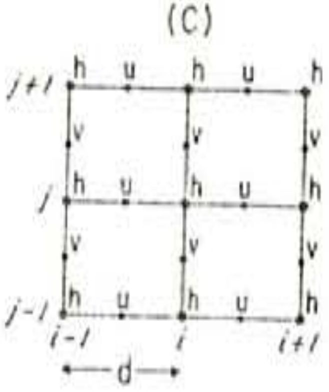

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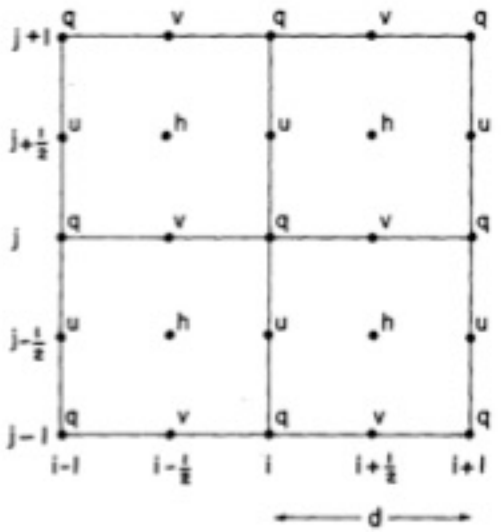

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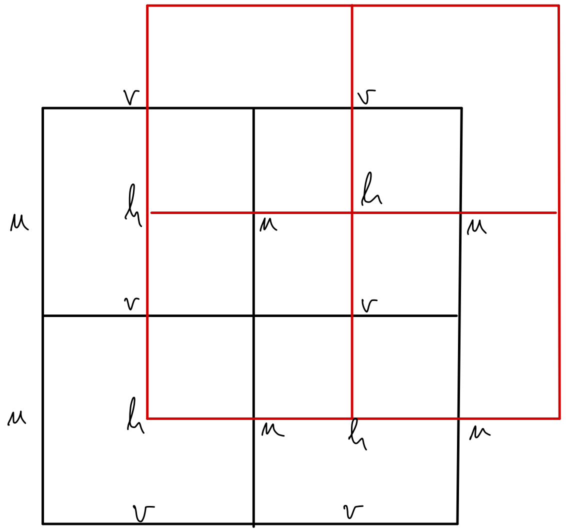

The relation between DOFs is the *same**

Velocity dofs go from line integrals (circulations) to flux integrals

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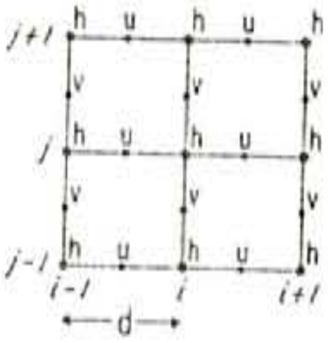
(C)



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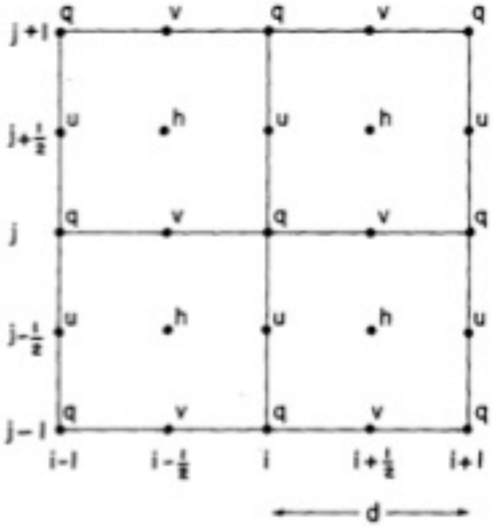

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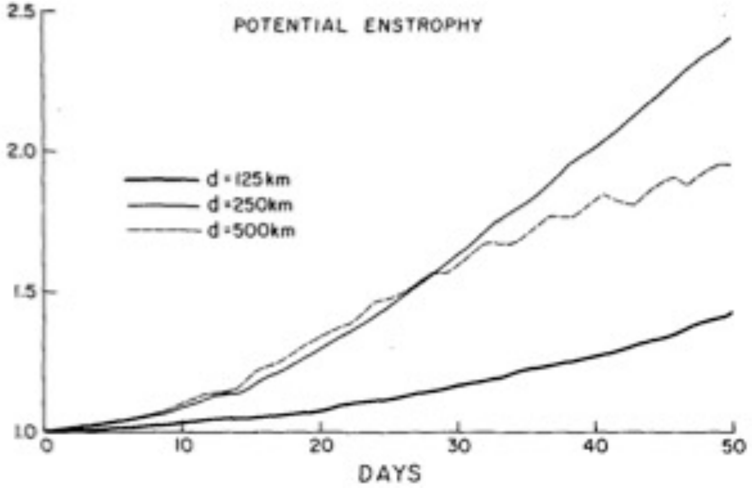
Non-conservative methods



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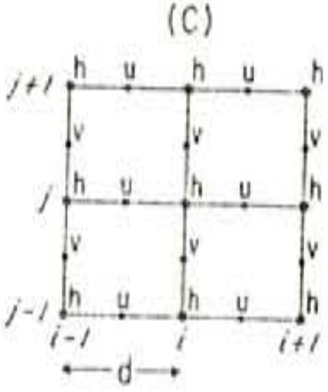

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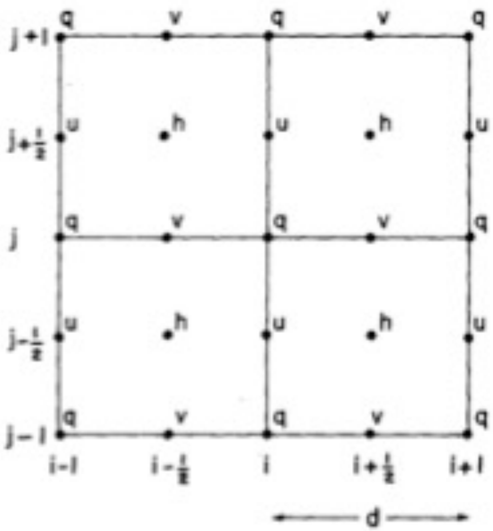

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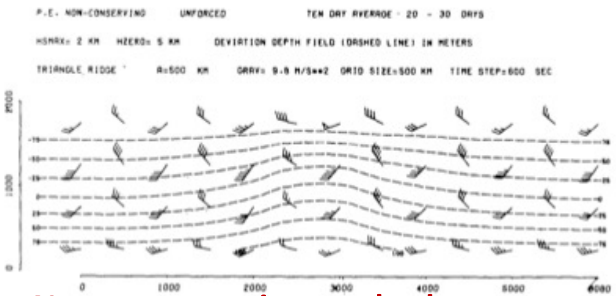
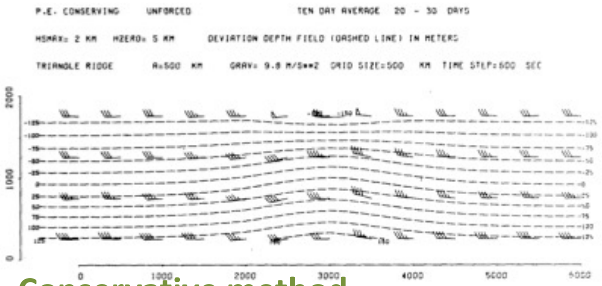
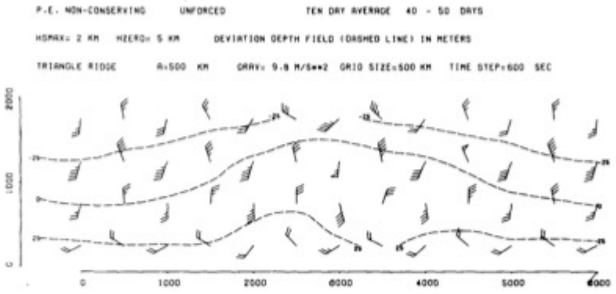
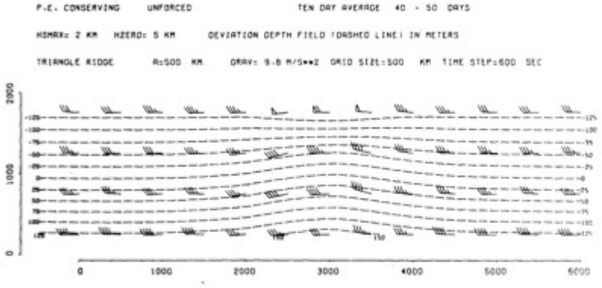




Non-conservative method

Conservative method





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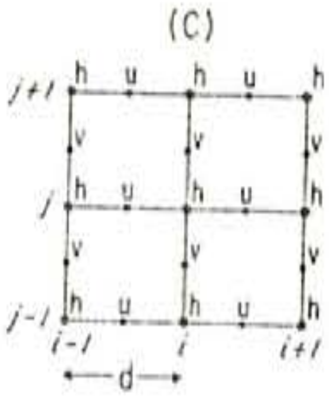

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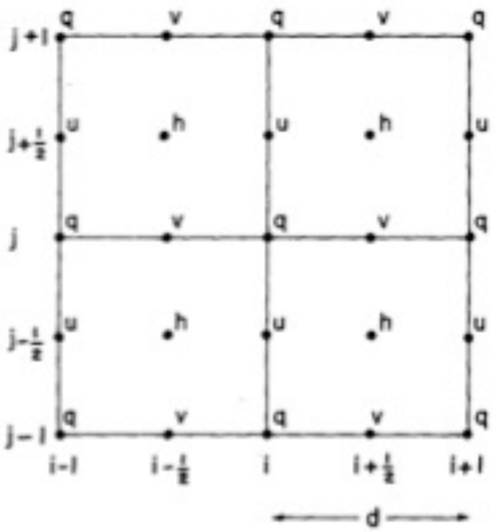

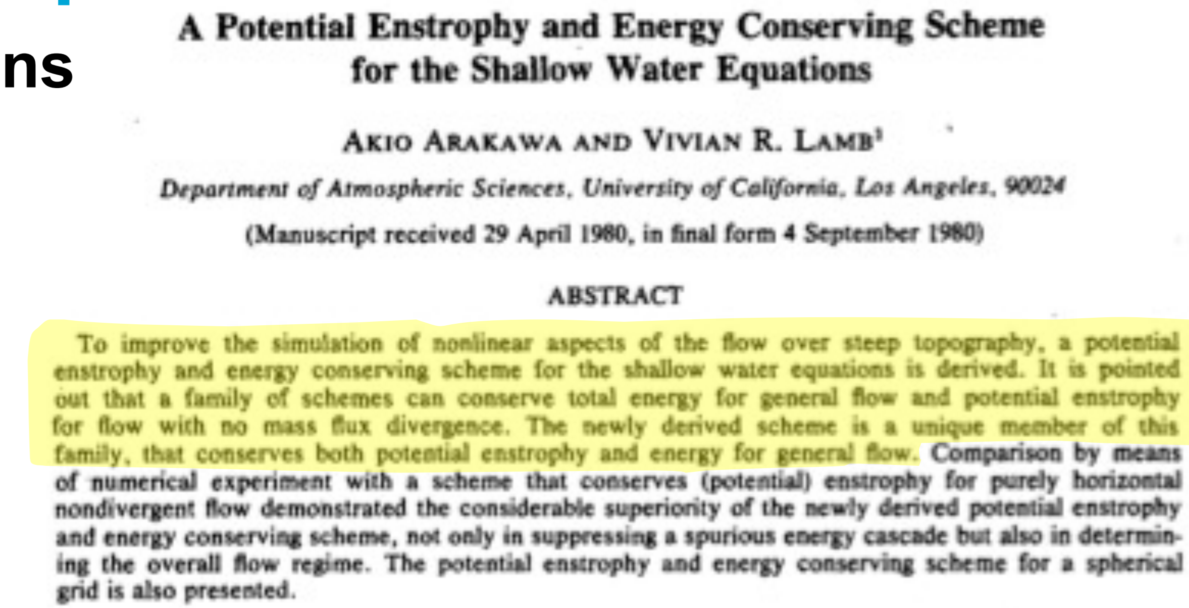

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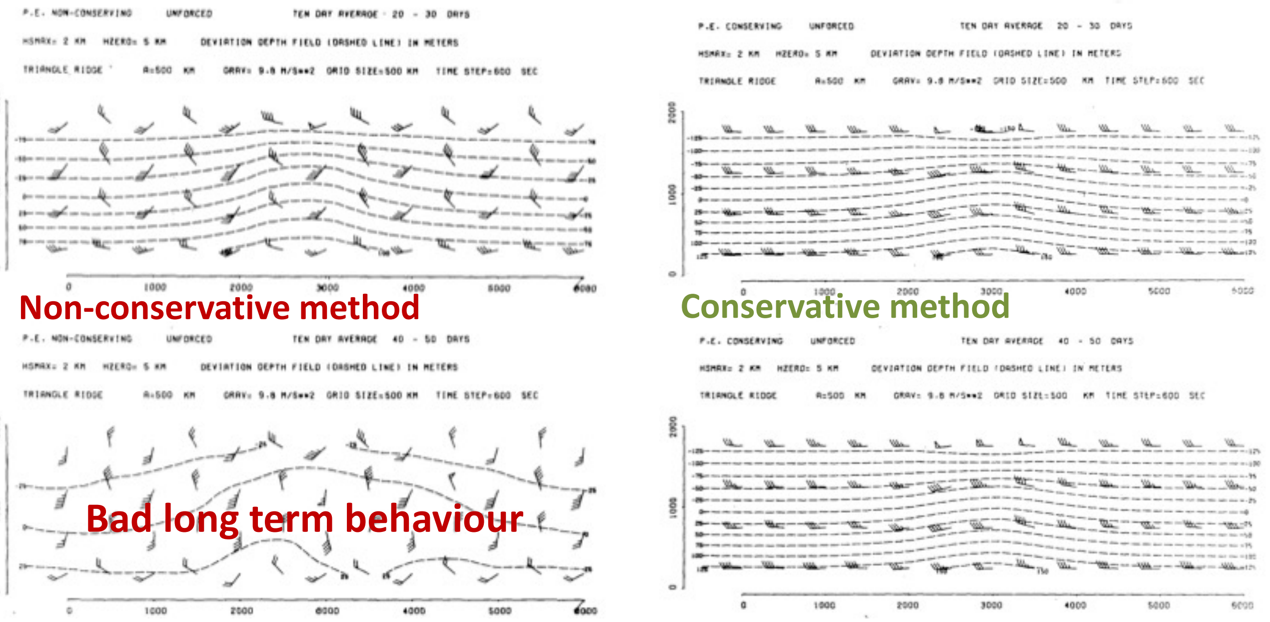

Intro shallow waters equations

Key take-aways

1. Different placement of DOFs leads to numerical methods with different properties. Even when using the same method (FD, FV, FEM). Some are better, others are worse.

2. Different formulations of a PDE lead to different numerical methods. Even when using the same numerical discretization and similar/same placement of DOFs. Some are better, others are worse.

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How to obtain the better numerical methods in a systematic way?

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Key take-aways

1. Different placement of DOFs leads to numerical methods with different properties. Even when using the same method (FD, FV, FEM). Some are better, others are worse.

2. Different formulations of a PDE lead to different numerical methods. Even when using the same numerical discretization and similar/same placement of DOFs. Some are better, others are worse.

How to obtain the better numerical methods in a systematic way?

Disclaimer: hopefully, these are not the final words on the matter, it is one direction.

Hamiltonian formulation

Curl-form Hamiltonian formulation of shallow waters

Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: starting point

$$\mathcal{H}[\boldsymbol{u}, h] := \int_\Omega \left(\frac{1}{2}h\boldsymbol{u}\cdot\boldsymbol{u} + \frac{1}{2}gh^2\right)\mathrm{d}\Omega$$

Hamiltonian

Hamiltonian formulation shallow waters

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$$\{A,B\}_{\mathrm{Symp}} := \int_\Omega \begin{bmatrix} \frac{\delta A}{\delta \boldsymbol{u}} & \frac{\delta A}{\delta h} \end{bmatrix} \mathbb{J} \begin{bmatrix} \frac{\delta B}{\delta \boldsymbol{u}} \\ \frac{\delta B}{\delta h} \end{bmatrix} \mathrm{d}\Omega$$

Poisson bracket (the symplectic)

$$\mathbb{J} := \begin{bmatrix} -\boldsymbol{q}\times & -\nabla \\ -\nabla\cdot & 0 \end{bmatrix} \qquad \boldsymbol{q} := \frac{\nabla\times\boldsymbol{u}}{h}$$

Symplectic (tensor) operator

Hamiltonian formulation shallow waters

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$$\frac{\mathrm{d}F}{\mathrm{d}t}[\boldsymbol{u}, h] = \langle \frac{\delta F}{\delta \boldsymbol{u}}, \frac{\partial u}{\partial t} \rangle + \langle \frac{\delta F}{\delta \boldsymbol{h}}, \frac{\partial h}{\partial t} \rangle = \{F, \mathcal{H}\}$$

Evolution equation for functional $F[\boldsymbol{u}, h]$

Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: starting point

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Both the Hamiltonian and the Poisson bracket define the dynamics

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Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: velocity evolution equations (weak form)

1. $F[\boldsymbol{u}, h] = \langle \boldsymbol{\sigma}, \boldsymbol{u} \rangle := \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{u} \, \mathrm{d}\Omega$ Arbitrary $\boldsymbol{\sigma}$ such that $\boldsymbol{\sigma}, \boldsymbol{u} \in U$

Hamiltonian formulation shallow waters

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Hamiltonian formulation shallow waters

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Hamiltonian formulation shallow waters

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Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: velocity evolution equations (weak form)

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Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: velocity evolution equations (weak form)

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$$\frac{\mathrm{d}F}{\mathrm{d}t}[\boldsymbol{u}, h] = \langle \boldsymbol{\sigma}, \frac{\partial \boldsymbol{u}}{\partial t} \rangle = \{F, \mathcal{H}\} = \int_{\Omega} \left[-\boldsymbol{\sigma} \cdot \boldsymbol{q} \times \frac{\delta \mathcal{H}}{\delta \boldsymbol{u}} - \boldsymbol{\sigma} \cdot \nabla \frac{\delta \mathcal{H}}{\delta h} \right] \mathrm{d}\Omega$$

Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: velocity evolution equations (weak form)

1. $F[\boldsymbol{u},h] = \langle \boldsymbol{\sigma}, \boldsymbol{u} \rangle := \int_\Omega \boldsymbol{\sigma}\cdot\boldsymbol{u}\,\mathrm{d}\Omega$ Arbitrary $\boldsymbol{\sigma}$ such that $\boldsymbol{\sigma},\boldsymbol{u}\in U$

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Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: velocity evolution equations (weak form)

1. $F[\boldsymbol{u},h] = \langle \boldsymbol{\sigma}, \boldsymbol{u} \rangle := \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{u} \,\mathrm{d}\Omega$

Arbitrary $\boldsymbol{\sigma}$ such that $\boldsymbol{\sigma}, \boldsymbol{u} \in U$

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Weak form

Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: height evolution equations (weak form)

2. $F[\boldsymbol{u}, h] = \langle \theta, h \rangle := \int_{\Omega} \theta h \, \mathrm{d}\Omega$ Arbitrary θ such that $\theta, h \in Q$

Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: height evolution equations (weak form)

2. $F[\boldsymbol{u}, h] = \langle \theta, h \rangle := \int_\Omega \theta h \,\mathrm{d}\Omega$

Arbitrary θ such that $\theta, h \in Q$

$$\frac{\mathrm{d}F}{\mathrm{d}t}[\boldsymbol{u}, h] = \langle \frac{\delta F}{\delta \boldsymbol{u}}, \frac{\partial \boldsymbol{u}}{\partial t} \rangle + \langle \frac{\delta F}{\delta \boldsymbol{h}}, \frac{\partial h}{\partial t} \rangle = \{F, \mathcal{H}\}$$

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Weak form

Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: evolution equations (weak form)

1. $F[\boldsymbol{u}, h] = \langle \boldsymbol{\sigma}, \boldsymbol{u} \rangle := \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{u} \, \mathrm{d}\Omega$ Arbitrary $\boldsymbol{\sigma}$ such that $\boldsymbol{\sigma}, \boldsymbol{u} \in U$

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Hamiltonian formulation shallow waters

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$$\begin{cases} \langle \boldsymbol{\sigma}, \dfrac{\partial \boldsymbol{u}}{\partial t} \rangle = -\langle \boldsymbol{\sigma}, \boldsymbol{q} \times \dfrac{\delta \mathcal{H}}{\delta \boldsymbol{u}} \rangle - \langle \boldsymbol{\sigma}, \nabla \dfrac{\delta \mathcal{H}}{\delta h} \rangle, & \forall \boldsymbol{\sigma} \in U \\ \langle \theta, \dfrac{\partial h}{\partial t} \rangle = -\langle \theta, \nabla \cdot \dfrac{\delta \mathcal{H}}{\delta \boldsymbol{u}} \rangle, & \forall \theta \in Q \end{cases}$$

Weak form of
shallow waters equations

Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: evolution equations (weak form)

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Weak form of shallow waters equations

$$\mathcal{H}[\boldsymbol{u}, h] := \int_\Omega \left(\frac{1}{2} h \boldsymbol{u} \cdot \boldsymbol{u} + \frac{1}{2} g h^2 \right) \mathrm{d}\Omega \quad \Rightarrow \quad \frac{\delta \mathcal{H}}{\delta \boldsymbol{u}} = h \boldsymbol{u} \,, \qquad \frac{\delta \mathcal{H}}{\delta h} = \frac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u} + g h$$

Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: evolution equations (weak form)

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$$
\begin{cases}
\langle \boldsymbol{\sigma}, \dfrac{\partial \boldsymbol{u}}{\partial t} \rangle = -\langle \boldsymbol{\sigma}, \boldsymbol{q} \times (h\boldsymbol{u}) \rangle - \langle \boldsymbol{\sigma}, \nabla \left(\dfrac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u} + gh \right) \rangle, & \forall \boldsymbol{\sigma} \in U \\
\langle \theta, \dfrac{\partial h}{\partial t} \rangle = -\langle \theta, \nabla \cdot (h\boldsymbol{u}) \rangle, & \forall \theta \in Q
\end{cases}
$$

Weak form of shallow waters equations

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\mathcal{H}[\boldsymbol{u}, h] := \int_\Omega \left(\frac{1}{2} h \boldsymbol{u} \cdot \boldsymbol{u} + \frac{1}{2} g h^2 \right) \mathrm{d}\Omega \quad \Rightarrow \quad \frac{\delta \mathcal{H}}{\delta \boldsymbol{u}} = h\boldsymbol{u}\,, \qquad \frac{\delta \mathcal{H}}{\delta h} = \frac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u} + gh
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Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: evolution equations (weak form)

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Weak form of shallow waters equations

Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: evolution equations (weak form)

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Weak form of shallow waters equations

Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: evolution equations (weak form)

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Weak form of shallow waters equations

$$\mathbb{J} := \begin{bmatrix} -\boldsymbol{q} \times & -\nabla \\ -\nabla \cdot & 0 \end{bmatrix}$$

Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: evolution equations (weak form)

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$\Rightarrow$ Weak form of shallow waters equations

Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: evolution equations (weak form)

1. $F[\boldsymbol{u}, h] = \langle \boldsymbol{\sigma}, \boldsymbol{u} \rangle := \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{u} \, \mathrm{d}\Omega$ Arbitrary $\boldsymbol{\sigma}$ such that $\boldsymbol{\sigma}, \boldsymbol{u} \in U$
2. $F[\boldsymbol{u}, h] = \langle \theta, h \rangle := \int_\Omega \theta h \, \mathrm{d}\Omega$ Arbitrary θ such that $\theta, h \in Q$

$$\begin{bmatrix} \langle \boldsymbol{\sigma}, \dfrac{\partial \boldsymbol{u}}{\partial t} \rangle \\ \langle \theta, \dfrac{\partial h}{\partial t} \rangle \end{bmatrix} = \begin{bmatrix} -\langle \boldsymbol{\sigma}, \boldsymbol{q} \times \square \rangle & -\langle \boldsymbol{\sigma}, \nabla \square \rangle \\ -\langle \theta, \nabla \cdot \square \rangle & 0 \end{bmatrix} \begin{bmatrix} \dfrac{\delta \mathcal{H}}{\delta \boldsymbol{u}} \\ \dfrac{\delta \mathcal{H}}{\delta h} \end{bmatrix}$$

Weak form of shallow waters equations

Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: evolution equations (strong form)

1. $F[\boldsymbol{u}, h] = \langle \boldsymbol{\sigma}, \boldsymbol{u} \rangle := \int_\Omega \boldsymbol{\sigma} \cdot \boldsymbol{u} \, \mathrm{d}\Omega$ Arbitrary $\boldsymbol{\sigma}$ such that $\boldsymbol{\sigma}, \boldsymbol{u} \in U$

2. $F[\boldsymbol{u}, h] = \langle \theta, h \rangle := \int_\Omega \theta h \, \mathrm{d}\Omega$ Arbitrary θ such that $\theta, h \in Q$

$$\begin{cases} \dfrac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{q} \times (h\boldsymbol{u}) + \nabla \left(\dfrac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u} + gh \right) = 0 \,, \\ \dfrac{\partial h}{\partial t} + \nabla \cdot (h\boldsymbol{u}) = 0 \,. \end{cases}$$

Strong form of shallow waters equations from Hamiltonian formulation

$$\begin{cases} \dfrac{\partial \mathbf{v}}{\partial t} + q\mathbf{k} \times \mathbf{v}^* + \boldsymbol{\nabla}(K + \Phi) = 0, & (2.1) \\ \dfrac{\partial h}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{v}^* = 0. & (2.2) \end{cases}$$

Strong form of shallow waters equations from Arakawa and Lamb (1980)

Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: Function spaces

$$\begin{cases} \langle \boldsymbol{\sigma}, \dfrac{\partial \boldsymbol{u}}{\partial t} \rangle = -\langle \boldsymbol{\sigma}, \boldsymbol{q} \times (h\boldsymbol{u}) \rangle - \langle \boldsymbol{\sigma}, \nabla \left(\dfrac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u} + gh \right) \rangle, & \forall \boldsymbol{\sigma} \in U \\ \langle \theta, \dfrac{\partial h}{\partial t} \rangle = -\langle \theta, \nabla \cdot (h\boldsymbol{u}) \rangle, & \forall \theta \in Q \end{cases}$$

Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: Function spaces

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Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: Function spaces

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?

In 2D we have two de Rham complexes:

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla"] & H(\nabla\times, \Omega) \arrow[r, "\nabla\times"] \arrow[l, bend left, "\widetilde{\nabla\cdot}"] & L^2(\Omega) \arrow[l, bend left, "\widetilde{\nabla}^{\perp}"] \end{tikzcd}$$

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Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: Function spaces

$$\begin{cases} \langle \boldsymbol{\sigma}, \dfrac{\partial \boldsymbol{u}}{\partial t}\rangle = -\langle \boldsymbol{\sigma}, \boldsymbol{q}\times(h\boldsymbol{u})\rangle - \langle\boldsymbol{\sigma}, \nabla\left(\dfrac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}+gh\right)\rangle, & \forall\boldsymbol{\sigma}\in U \\ \langle \theta, \dfrac{\partial h}{\partial t}\rangle = -\langle \theta, \nabla\cdot(h\boldsymbol{u})\rangle, & \forall\theta\in Q \end{cases}$$

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$$\begin{tikzcd} H^1(\Theta_0,\Omega) \arrow[r, "\nabla"] & H(\nabla\times,\Theta_1,\Omega) \arrow[l, bend left, "\widetilde{\nabla\cdot}"] \arrow[r, "\nabla\times"] & H(\nabla\cdot,\Theta_2,\Omega) \arrow[l, bend left, "\widetilde{\nabla\times}"] \arrow[r, "\nabla\cdot"] & L^2(\Theta_3,\Omega) \arrow[l, bend left, "\widetilde{\nabla}"] \end{tikzcd}$$

3D Complex

Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: Function spaces

$$
\begin{cases}
\langle \boldsymbol{\sigma}, \dfrac{\partial \boldsymbol{u}}{\partial t} \rangle = -\langle \boldsymbol{\sigma}, \boldsymbol{q} \times (h\boldsymbol{u}) \rangle - \langle \boldsymbol{\sigma}, \nabla \left(\dfrac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u} + gh \right) \rangle, & \forall \boldsymbol{\sigma} \in U \\[2ex]
\langle \theta, \dfrac{\partial h}{\partial t} \rangle = -\langle \theta, \nabla \cdot (h\boldsymbol{u}) \rangle, & \forall \theta \in Q
\end{cases}
$$

?

In 2D we have two de Rham complexes:

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3D Complex

Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: Function spaces

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3D Complex

Hamiltonian formulation shallow waters

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Hamiltonian formulation shallow waters

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Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: Function spaces

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Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: Function spaces

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Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: Function spaces

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Hamiltonian formulation shallow waters

Curl-form Hamiltonian formulation: Function spaces

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Follows directly from the de Rham complex

In 2D we have two de Rham complexes:

$$\begin{array}{ccccc} H^1(\Omega) & \xrightarrow{\nabla} & H(\nabla\times, \Omega) & \xrightarrow{\nabla\times} & L^2(\Omega) \\ & \xleftarrow{\widetilde{\nabla\cdot}} & & \xleftarrow{\widetilde{\nabla}^{\perp}} & \end{array}$$

2 options to discretize

$$\begin{array}{ccccc} H^1(\Omega) & \xrightarrow{\nabla^{\perp}} & H(\nabla\cdot, \Omega) & \xrightarrow{\nabla\cdot} & L^2(\Omega) \\ & \xleftarrow{\widetilde{\nabla\times}} & & \xleftarrow{\widetilde{\nabla}} & \end{array}$$

$$\boldsymbol{u} \in H(\nabla\times, \Omega), \quad h \in H^1(\Omega)$$
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Hamiltonian formulation shallow waters

Conservation properties: Energy

Hamiltonian (Energy)

$$\mathcal{H}[\boldsymbol{u}, h] := \int_\Omega \left(\frac{1}{2} h \boldsymbol{u} \cdot \boldsymbol{u} + \frac{1}{2} g h^2 \right) \mathrm{d}\Omega$$

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}[\boldsymbol{u}, h] = \langle \frac{\delta \mathcal{H}}{\delta \boldsymbol{u}}, \frac{\partial u}{\partial t} \rangle + \langle \frac{\delta \mathcal{H}}{\delta \boldsymbol{h}}, \frac{\partial h}{\partial t} \rangle = \{\mathcal{H}, \mathcal{H}\}$$

Poisson bracket

Skew symmetric $\longleftarrow$ $\{A, B\}_{\text{Symp}} := \int_\Omega \begin{bmatrix} \frac{\delta A}{\delta \boldsymbol{u}} & \frac{\delta A}{\delta h} \end{bmatrix} \mathbb{J} \begin{bmatrix} \frac{\delta B}{\delta \boldsymbol{u}} \\ \frac{\delta B}{\delta h} \end{bmatrix} \mathrm{d}\Omega$

Symplectic (tensor) operator

$$\mathbb{J} := \begin{bmatrix} -\boldsymbol{q} \times & -\nabla \\ -\nabla \cdot & 0 \end{bmatrix}$$

Evolution equation for functional

$$\frac{\mathrm{d}F}{\mathrm{d}t}[\boldsymbol{u}, h] = \langle \frac{\delta F}{\delta \boldsymbol{u}}, \frac{\partial u}{\partial t} \rangle + \langle \frac{\delta F}{\delta \boldsymbol{h}}, \frac{\partial h}{\partial t} \rangle = \{F, \mathcal{H}\}$$

Hamiltonian formulation shallow waters

Conservation properties: Energy

Hamiltonian (Energy)

$$\mathcal{H}[\boldsymbol{u}, h] := \int_\Omega \left(\frac{1}{2} h\boldsymbol{u}\cdot\boldsymbol{u} + \frac{1}{2} g h^2\right) \mathrm{d}\Omega$$

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}[\boldsymbol{u}, h] = \langle \frac{\delta\mathcal{H}}{\delta\boldsymbol{u}}, \frac{\partial u}{\partial t}\rangle + \langle \frac{\delta\mathcal{H}}{\delta\boldsymbol{h}}, \frac{\partial h}{\partial t}\rangle = \{\mathcal{H},\mathcal{H}\} = -\{\mathcal{H},\mathcal{H}\} = 0$$

Skew symmetric

Poisson bracket

$$\{A, B\}_{\text{Symp}} := \int_\Omega \begin{bmatrix} \frac{\delta A}{\delta \boldsymbol{u}} & \frac{\delta A}{\delta h} \end{bmatrix} \mathbb{J} \begin{bmatrix} \frac{\delta B}{\delta \boldsymbol{u}} \\ \frac{\delta B}{\delta h} \end{bmatrix} \mathrm{d}\Omega$$

Symplectic (tensor) operator

$$\mathbb{J} := \begin{bmatrix} -\boldsymbol{q}\times & -\nabla \\ -\nabla\cdot & 0 \end{bmatrix}$$

Evolution equation for functional

$$\frac{\mathrm{d}F}{\mathrm{d}t}[\boldsymbol{u}, h] = \langle \frac{\delta F}{\delta\boldsymbol{u}}, \frac{\partial u}{\partial t}\rangle + \langle \frac{\delta F}{\delta\boldsymbol{h}}, \frac{\partial h}{\partial t}\rangle = \{F,\mathcal{H}\}$$

Hamiltonian formulation shallow waters

Conservation properties: Energy

Hamiltonian (Energy)

$$\mathcal{H}[\boldsymbol{u},h] := \int_\Omega \left(\frac{1}{2}h\boldsymbol{u}\cdot\boldsymbol{u} + \frac{1}{2}gh^2\right)\mathrm{d}\Omega$$

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We can look into the inner details

$$\begin{bmatrix} \langle\boldsymbol{\sigma},\frac{\partial\boldsymbol{u}}{\partial t}\rangle \\ \langle\theta,\frac{\partial h}{\partial t}\rangle \end{bmatrix} = \begin{bmatrix} -\langle\boldsymbol{\sigma},\boldsymbol{q}\times\square\rangle & -\langle\boldsymbol{\sigma},\nabla\square\rangle \\ -\langle\theta,\nabla\cdot\square\rangle & 0 \end{bmatrix} \begin{bmatrix} \frac{\delta\mathcal{H}}{\delta\boldsymbol{u}} \\ \frac{\delta\mathcal{H}}{\delta h}\end{bmatrix}, \qquad \forall(\boldsymbol{\sigma},\theta)\in(U,Q)$$

$\Rightarrow$ Weak form of shallow waters equations

Hamiltonian formulation shallow waters

Conservation properties: Energy

Hamiltonian (Energy)

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➡ Weak form of shallow waters equations

$$\boldsymbol{\sigma} = \frac{\delta\mathcal{H}}{\delta\boldsymbol{u}}, \quad \theta = \frac{\delta\mathcal{H}}{\delta h}$$

Hamiltonian formulation shallow waters

Conservation properties: Energy

Hamiltonian (Energy)

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We can look into the inner details

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Weak form of
shallow waters equations

Hamiltonian formulation shallow waters

Conservation properties: Energy

Hamiltonian (Energy)

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We can look into the inner details

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➡ Weak form of shallow waters equations

$$\boldsymbol{\sigma} = \frac{\delta\mathcal{H}}{\delta\boldsymbol{u}}, \quad \theta = \frac{\delta\mathcal{H}}{\delta h}$$

If we sum the two rows

Hamiltonian formulation shallow waters

Conservation properties: Energy

Poisson bracket

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Hamiltonian formulation shallow waters

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Weak
Symplectic (tensor) operator
Skew-symmetric

Hamiltonian formulation of shallow waters

Key take-aways

1. Arakawa and Lamb **(better)** formulation is retrieved from the Hamiltonian formulation. Both strong and weak form.

2. Function space associated to each field is clearly defined. But two options exist. Choice depends on objectives.

3. Energy conservation (and other conserved properties) built in from the Poisson bracket.

Hamiltonian formulation of shallow waters

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3. Energy conservation (and other conserved properties) built in from the Poisson bracket.

How to discretize?

Spatial discretization

Semi-discretization of shallow waters

Spatial discretization shallow waters

Replicating the continuous structures

$$\begin{cases} \langle \boldsymbol{\sigma}, \dfrac{\partial \boldsymbol{u}}{\partial t} \rangle = -\langle \boldsymbol{\sigma}, \boldsymbol{q} \times (h\boldsymbol{u}) \rangle - \langle \boldsymbol{\sigma}, \nabla \left(\dfrac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u} + gh \right) \rangle, & \forall \boldsymbol{\sigma} \in U \\ \langle \theta, \dfrac{\partial h}{\partial t} \rangle = -\langle \theta, \nabla \cdot (h\boldsymbol{u}) \rangle, & \forall \theta \in Q \end{cases}$$

Weak form from Hamiltonian formulation

Spatial discretization shallow waters

Replicating the continuous structures

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Weak form from Hamiltonian formulation

In 2D we have two de Rham complexes:

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla"] & H(\nabla\times,\Omega) \arrow[r, "\nabla\times"] \arrow[l, bend left, "\widetilde{\nabla\cdot}"] & L^2(\Omega) \arrow[l, bend left, "\widetilde{\nabla}^{\perp}"] \end{tikzcd}$$

$$\boldsymbol{u} \in H(\nabla\times,\Omega), \quad h \in H^1(\Omega)$$
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Spatial discretization shallow waters

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Replicating the continuous structures

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Spatial discretization shallow waters

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Spatial discretization shallow waters

Replicating the continuous structures

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla"] & H(\nabla\times,\Omega) \arrow[r, "\nabla\times"] \arrow[l, bend left, "\widetilde{\nabla}\cdot"] & L^2(\Omega) \arrow[l, bend left, "\tilde{\nabla}^{\perp}"] \end{tikzcd}$$

$$\boldsymbol{u} \in H(\nabla\times,\Omega), \quad h \in H^1(\Omega)$$
$$h\boldsymbol{u} \in H(\nabla\times,\Omega), \quad \boldsymbol{q} \in L^2(\Omega), \quad \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} \in H^1(\Omega)$$

$$\begin{cases} \langle \boldsymbol{\sigma}, \dfrac{\partial \boldsymbol{u}}{\partial t}\rangle = -\langle \boldsymbol{\sigma}, \boldsymbol{q} \times (h\boldsymbol{u})\rangle - \langle \boldsymbol{\sigma}, \nabla\left(\dfrac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + gh\right)\rangle, & \forall \boldsymbol{\sigma} \in H(\nabla\times,\Omega) \\ \langle \theta, \dfrac{\partial h}{\partial t}\rangle = \underbrace{\langle \nabla\theta, h\boldsymbol{u}\rangle}_{\text{Integration by parts}}, & \forall \theta \in H^1(\Omega) \end{cases}$$

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla^{\perp}"] & H(\nabla\cdot,\Omega) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & L^2(\Omega) \arrow[l, bend left, "\tilde{\nabla}"] \end{tikzcd}$$

$$\boldsymbol{u} \in H(\nabla\cdot,\Omega), \quad h \in L^2(\Omega)$$
$$h\boldsymbol{u} \in H(\nabla\cdot,\Omega), \quad \boldsymbol{q} \in H^1(\Omega), \quad \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} \in L^2(\Omega)$$

$$\begin{cases} \langle \boldsymbol{\sigma}, \dfrac{\partial \boldsymbol{u}}{\partial t}\rangle = -\langle \boldsymbol{\sigma}, \boldsymbol{q} \times (h\boldsymbol{u})\rangle + \overbrace{\langle \nabla\cdot\boldsymbol{\sigma}, \dfrac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + gh\rangle}^{\text{Integration by parts}}, & \forall \boldsymbol{\sigma} \in H(\nabla\cdot,\Omega) \\ \langle \theta, \dfrac{\partial h}{\partial t}\rangle = -\langle \theta, \nabla\cdot(h\boldsymbol{u})\rangle, & \forall \theta \in L^2(\Omega) \end{cases}$$

Spatial discretization shallow waters

Replicating the continuous structures

$$
\begin{tikzcd}
H^1(\Omega) \arrow[r, "\nabla"] & H(\nabla\times,\Omega) \arrow[l, bend left, "\widetilde{\nabla}\cdot"] \arrow[r, "\nabla\times"] & L^2(\Omega) \arrow[l, bend left, "\widetilde{\nabla}^\perp"]
\end{tikzcd}
$$

$$
\boldsymbol{u} \in H(\nabla\times,\Omega), \quad h \in H^1(\Omega)
$$
$$
h\boldsymbol{u} \in H(\nabla\times,\Omega), \quad \boldsymbol{q} \in L^2(\Omega), \quad \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} \in H^1(\Omega)
$$

$$
\begin{cases}
\langle \boldsymbol{\sigma}, \frac{\partial \boldsymbol{u}}{\partial t}\rangle = -\langle \boldsymbol{\sigma}, \boldsymbol{q}\times(h\boldsymbol{u})\rangle - \langle \boldsymbol{\sigma}, \nabla\left(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + gh\right)\rangle, & \forall \boldsymbol{\sigma} \in H(\nabla\times,\Omega) \\
\langle \theta, \frac{\partial h}{\partial t}\rangle = \underbrace{\langle \nabla\theta, h\boldsymbol{u}\rangle}_{\text{Integration by parts}}, & \forall \theta \in H^1(\Omega)
\end{cases}
$$

$$
\begin{bmatrix} \langle \boldsymbol{\sigma}, \frac{\partial \boldsymbol{u}}{\partial t}\rangle \\ \langle \theta, \frac{\partial h}{\partial t}\rangle \end{bmatrix} = \underbrace{\begin{bmatrix} -\langle \boldsymbol{\sigma}, \boldsymbol{q}\times\square\rangle & -\langle \boldsymbol{\sigma}, \nabla\square\rangle \\ -\langle \theta, \widetilde{\nabla}\cdot\square\rangle & 0 \end{bmatrix}}_{\mathbb{J}} \begin{bmatrix} \frac{\delta\mathcal{H}}{\delta\boldsymbol{u}} \\ \frac{\delta\mathcal{H}}{\delta h} \end{bmatrix},
$$

$$
\begin{tikzcd}
H^1(\Omega) \arrow[r, "\nabla^\perp"] & H(\nabla\cdot,\Omega) \arrow[l, bend left, "\widetilde{\nabla\times}"] \arrow[r, "\nabla\cdot"] & L^2(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"]
\end{tikzcd}
$$

$$
\boldsymbol{u} \in H(\nabla\cdot,\Omega), \quad h \in L^2(\Omega)
$$
$$
h\boldsymbol{u} \in H(\nabla\cdot,\Omega), \quad \boldsymbol{q} \in H^1(\Omega), \quad \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} \in L^2(\Omega)
$$

$$
\begin{cases}
\langle \boldsymbol{\sigma}, \frac{\partial \boldsymbol{u}}{\partial t}\rangle = -\langle \boldsymbol{\sigma}, \boldsymbol{q}\times(h\boldsymbol{u})\rangle + \overbrace{\langle \nabla\cdot\boldsymbol{\sigma}, \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + gh\rangle}^{\text{Integration by parts}}, & \forall \boldsymbol{\sigma} \in H(\nabla\cdot,\Omega) \\
\langle \theta, \frac{\partial h}{\partial t}\rangle = -\langle \theta, \nabla\cdot(h\boldsymbol{u})\rangle, & \forall \theta \in L^2(\Omega)
\end{cases}
$$

$$
\begin{bmatrix} \langle \boldsymbol{\sigma}, \frac{\partial \boldsymbol{u}}{\partial t}\rangle \\ \langle \theta, \frac{\partial h}{\partial t}\rangle \end{bmatrix} = \underbrace{\begin{bmatrix} -\langle \boldsymbol{\sigma}, \boldsymbol{q}\times\square\rangle & -\langle \boldsymbol{\sigma}, \widetilde{\nabla}\square\rangle \\ -\langle \theta, \nabla\cdot\square\rangle & 0 \end{bmatrix}}_{\mathbb{J}} \begin{bmatrix} \frac{\delta\mathcal{H}}{\delta\boldsymbol{u}} \\ \frac{\delta\mathcal{H}}{\delta h} \end{bmatrix},
$$

Spatial discretization shallow waters

Replicating the continuous structures: de Rham complex

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla"] & H(\nabla\times,\Omega) \arrow[r, "\nabla\times"] \arrow[l, bend left, "\widetilde{\nabla\cdot}"] & L^2(\Omega) \arrow[l, bend left, "\widetilde{\nabla}^{\perp}"] \end{tikzcd}$$

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla^{\perp}"] & H(\nabla\cdot,\Omega) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & L^2(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"] \end{tikzcd}$$

Spatial discretization shallow waters

Replicating the continuous structures: de Rham complex

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla"] & H(\nabla\times,\Omega) \arrow[r, "\nabla\times"] \arrow[l, bend left, "\widetilde{\nabla\cdot}"] & L^2(\Omega) \arrow[l, bend left, "\widetilde{\nabla}^{\perp}"] \end{tikzcd}$$

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla^{\perp}"] & H(\nabla\cdot,\Omega) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & L^2(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"] \end{tikzcd}$$

$$\begin{tikzcd} G_p(\Omega_h) \arrow[r, "\nabla"] & C_p(\Omega_h) \arrow[r, "\nabla\times"] \arrow[l, bend left, "\widetilde{\nabla\cdot}"] & S_p(\Omega) \arrow[l, bend left, "\widetilde{\nabla}^{\perp}"] \end{tikzcd}$$

Discrete spaces constitute a de Rham complex

$$\begin{tikzcd} G_p(\Omega_h) \arrow[r, "\nabla^{\perp}"] & D_p(\Omega_h) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & S_p(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"] \end{tikzcd}$$

Spatial discretization shallow waters

Replicating the continuous structures: de Rham complex

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla"] & H(\nabla\times,\Omega) \arrow[r, "\nabla\times"] \arrow[l, bend left, "\widetilde{\nabla}\cdot"] & L^2(\Omega) \arrow[l, bend left, "\widetilde{\nabla}^{\perp}"] \end{tikzcd}$$

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla^{\perp}"] & H(\nabla\cdot,\Omega) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla}\times"] & L^2(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"] \end{tikzcd}$$

$$\begin{tikzcd} G_p(\Omega_h) \arrow[r, "\nabla"] & C_p(\Omega_h) \arrow[r, "\nabla\times"] \arrow[l, bend left, "\widetilde{\nabla}\cdot"] & S_p(\Omega) \arrow[l, bend left, "\widetilde{\nabla}^{\perp}"] \end{tikzcd}$$

Discrete spaces constitute a de Rham complex

$$\begin{tikzcd} G_p(\Omega_h) \arrow[r, "\nabla^{\perp}"] & D_p(\Omega_h) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla}\times"] & S_p(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"] \end{tikzcd}$$

Quadrilaterals (and Hexahedra in 3D)

Tensor product spaces of polynomials of degree (e.g., mimetic spectral elements):

$$(p,p) \xrightarrow{\nabla} (p-1,p) \times (p,p-1) \xrightarrow{\nabla\times} (p-1,p-1)$$

$$(p,p) \xrightarrow{\nabla^{\perp}} (p,p-1) \times (p-1,p) \xrightarrow{\nabla\cdot} (p-1,p-1)$$

Spatial discretization shallow waters

Replicating the continuous structures: de Rham complex

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla"] & H(\nabla\times,\Omega) \arrow[r, "\nabla\times"] \arrow[l, bend left, "\widetilde{\nabla\cdot}"] & L^2(\Omega) \arrow[l, bend left, "\widetilde{\nabla}^{\perp}"] \end{tikzcd}$$

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla^{\perp}"] & H(\nabla\cdot,\Omega) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & L^2(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"] \end{tikzcd}$$

$$\begin{tikzcd} G_p(\Omega_h) \arrow[r, "\nabla"] & C_p(\Omega_h) \arrow[r, "\nabla\times"] \arrow[l, bend left, "\widetilde{\nabla\cdot}"] & S_p(\Omega) \arrow[l, bend left, "\widetilde{\nabla}^{\perp}"] \end{tikzcd}$$

Discrete spaces constitute a de Rham complex

$$\begin{tikzcd} G_p(\Omega_h) \arrow[r, "\nabla^{\perp}"] & D_p(\Omega_h) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & S_p(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"] \end{tikzcd}$$

Quadrilaterals (and Hexahedra in 3D) (Firedrake nomenclature)

Tensor product spaces of polynomials of degree (e.g., mimetic spectral elements):

$$\mathtt{CG}_p \xrightarrow{\nabla} \mathtt{RTCE}_p \xrightarrow{\nabla\times} (\mathtt{DQ\ \ L2})_{p-1}$$

$$\mathtt{CG}_p \xrightarrow{\nabla^{\perp}} \mathtt{RTCF}_p \xrightarrow{\nabla\cdot} (\mathtt{DQ\ \ L2})_{p-1}$$

Spatial discretization shallow waters

Replicating the continuous structures: de Rham complex

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla"] & H(\nabla\times,\Omega) \arrow[r, "\nabla\times"] \arrow[l, bend left, "\widetilde{\nabla}\cdot"] & L^2(\Omega) \arrow[l, bend left, "\widetilde{\nabla}^{\perp}"] \end{tikzcd}$$

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla^{\perp}"] & H(\nabla\cdot,\Omega) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & L^2(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"] \end{tikzcd}$$

$$\begin{tikzcd} G_p(\Omega_h) \arrow[r, "\nabla"] & C_p(\Omega_h) \arrow[r, "\nabla\times"] \arrow[l, bend left, "\widetilde{\nabla}\cdot"] & S_p(\Omega) \arrow[l, bend left, "\widetilde{\nabla}^{\perp}"] \end{tikzcd}$$

Discrete spaces constitute a de Rham complex

$$\begin{tikzcd} G_p(\Omega_h) \arrow[r, "\nabla^{\perp}"] & D_p(\Omega_h) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & S_p(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"] \end{tikzcd}$$

Quadrilaterals (and Hexahedra in 3D) (Firedrake nomenclature)

Tensor product spaces of polynomials of degree (e.g., mimetic spectral elements):

$$\mathtt{CG}_p \xrightarrow{\nabla} \mathtt{RTCE}_p \xrightarrow{\nabla\times} (\mathtt{DQ\ \ L2})_{p-1}$$

$$\mathtt{CG}_p \xrightarrow{\nabla^{\perp}} \mathtt{RTCF}_p \xrightarrow{\nabla\cdot} (\mathtt{DQ\ \ L2})_{p-1}$$

Simplices

More complex choices but the idea is the same:

$$(p,p) \xrightarrow{\nabla} (p-1,p)\times(p,p-1) \xrightarrow{\nabla\times} (p-1,p-1)$$

$$(p,p) \xrightarrow{\nabla^{\perp}} (p,p-1)\times(p-1,p) \xrightarrow{\nabla\cdot} (p-1,p-1)$$

Spatial discretization shallow waters

Replicating the continuous structures: de Rham complex

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla"] & H(\nabla\times,\Omega) \arrow[r, "\nabla\times"] \arrow[l, bend left, "\widetilde{\nabla\cdot}"] & L^2(\Omega) \arrow[l, bend left, "\widetilde{\nabla}^{\perp}"] \end{tikzcd}$$

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla^{\perp}"] & H(\nabla\cdot,\Omega) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & L^2(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"] \end{tikzcd}$$

$$\begin{tikzcd} G_p(\Omega_h) \arrow[r, "\nabla"] & C_p(\Omega_h) \arrow[r, "\nabla\times"] \arrow[l, bend left, "\widetilde{\nabla\cdot}"] & S_p(\Omega) \arrow[l, bend left, "\widetilde{\nabla}^{\perp}"] \end{tikzcd}$$

Discrete spaces constitute a de Rham complex

$$\begin{tikzcd} G_p(\Omega_h) \arrow[r, "\nabla^{\perp}"] & D_p(\Omega_h) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & S_p(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"] \end{tikzcd}$$

Quadrilaterals (and Hexahedra in 3D) (Firedrake nomenclature)

Tensor product spaces of polynomials of degree (e.g., mimetic spectral elements):

$$\mathtt{CG}_p \xrightarrow{\nabla} \mathtt{RTCE}_p \xrightarrow{\nabla\times} (\mathtt{DQ\ \ L2})_{p-1}$$

$$\mathtt{CG}_p \xrightarrow{\nabla^{\perp}} \mathtt{RTCF}_p \xrightarrow{\nabla\cdot} (\mathtt{DQ\ \ L2})_{p-1}$$

Simplices (Firedrake nomenclature)

More complex choices but the idea is the same:

$$\mathtt{CG}_p \xrightarrow{\nabla} \mathtt{NED}_p \xrightarrow{\nabla\times} \mathtt{DG}_{p-1}$$

$$\mathtt{CG}_p \xrightarrow{\nabla^{\perp}} \mathtt{RT}_p \xrightarrow{\nabla\cdot} \mathtt{DG}_{p-1}$$

Spatial discretization shallow waters

Replicating the continuous structures: de Rham complex

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla"] & H(\nabla\times,\Omega) \arrow[r, "\nabla\times"] \arrow[l, bend left, "\widetilde{\nabla\cdot}"] & L^2(\Omega) \arrow[l, bend left, "\widetilde{\nabla}^{\perp}"] \end{tikzcd}$$

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla^{\perp}"] & H(\nabla\cdot,\Omega) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & L^2(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"] \end{tikzcd}$$

$$\begin{tikzcd} G_p(\Omega_h) \arrow[r, "\nabla"] & C_p(\Omega_h) \arrow[r, "\nabla\times"] \arrow[l, bend left, "\widetilde{\nabla\cdot}"] & S_p(\Omega) \arrow[l, bend left, "\widetilde{\nabla}^{\perp}"] \end{tikzcd}$$

Discrete spaces constitute a de Rham complex

$$\begin{tikzcd} G_p(\Omega_h) \arrow[r, "\nabla^{\perp}"] & D_p(\Omega_h) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & S_p(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"] \end{tikzcd}$$

Quadrilaterals (and Hexahedra in 3D) (Firedrake nomenclature)

Tensor product spaces of polynomials of degree (e.g., mimetic spectral elements):

$$\begin{tikzcd} \mathtt{CG}_p \arrow[r, "\nabla"] & \mathtt{RTCE}_p \arrow[r, "\nabla\times"] \arrow[l, bend left, "\widetilde{\nabla\cdot}"] & (\mathtt{DQ\ L2})_{p-1} \arrow[l, bend left, "\widetilde{\nabla}^{\perp}"] \end{tikzcd}$$

Integration by parts

$$\begin{tikzcd} \mathtt{CG}_p \arrow[r, "\nabla^{\perp}"] & \mathtt{RTCF}_p \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & (\mathtt{DQ\ L2})_{p-1} \arrow[l, bend left, "\widetilde{\nabla}"] \end{tikzcd}$$

Simplices (Firedrake nomenclature)

More complex choices but the idea is the same:

$$\begin{tikzcd} \mathtt{CG}_p \arrow[r, "\nabla"] & \mathtt{NED}_p \arrow[r, "\nabla\times"] \arrow[l, bend left, "\widetilde{\nabla\cdot}"] & \mathtt{DG}_{p-1} \arrow[l, bend left, "\widetilde{\nabla}^{\perp}"] \end{tikzcd}$$

Integration by parts

$$\begin{tikzcd} \mathtt{CG}_p \arrow[r, "\nabla^{\perp}"] & \mathtt{RT}_p \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & \mathtt{DG}_{p-1} \arrow[l, bend left, "\widetilde{\nabla}"] \end{tikzcd}$$

Spatial discretization shallow waters

Replicating the continuous structures: de Rham complex

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla"] & H(\nabla\times,\Omega) \arrow[l, bend left, "\widetilde{\nabla\cdot}"] \arrow[r, "\nabla\times"] & L^2(\Omega) \arrow[l, bend left, "\tilde{\nabla}^{\perp}"] \end{tikzcd}$$

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla^{\perp}"] & H(\nabla\cdot,\Omega) \arrow[l, bend left, "\widetilde{\nabla\times}"] \arrow[r, "\nabla\cdot"] & L^2(\Omega) \arrow[l, bend left, "\tilde{\nabla}"] \end{tikzcd}$$

Continuous

$$\boldsymbol{u} \in H(\nabla\times,\Omega), \quad h \in H^1(\Omega)$$
$$h\boldsymbol{u} \in H(\nabla\times,\Omega), \quad \boldsymbol{q} \in L^2(\Omega), \quad \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} \in H^1(\Omega)$$

$$\boldsymbol{u} \in H(\nabla\cdot,\Omega), \quad h \in L^2(\Omega)$$
$$h\boldsymbol{u} \in H(\nabla\cdot,\Omega), \quad \boldsymbol{q} \in H^1(\Omega), \quad \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} \in L^2(\Omega)$$

Spatial discretization shallow waters

Replicating the continuous structures: de Rham complex

$$H^1(\Omega) \xrightarrow{\nabla} H(\nabla\times,\Omega) \xrightarrow{\nabla\times} L^2(\Omega)$$
$$H^1(\Omega) \xleftarrow{\widetilde{\nabla\cdot}} H(\nabla\times,\Omega) \xleftarrow{\widetilde{\nabla}^\perp} L^2(\Omega)$$

$$H^1(\Omega) \xrightarrow{\nabla^\perp} H(\nabla\cdot,\Omega) \xrightarrow{\nabla\cdot} L^2(\Omega)$$
$$H^1(\Omega) \xleftarrow{\widetilde{\nabla\times}} H(\nabla\cdot,\Omega) \xleftarrow{\widetilde{\nabla}} L^2(\Omega)$$

Continuous

$$\boldsymbol{u} \in H(\nabla\times,\Omega), \quad h \in H^1(\Omega)$$
$$h\boldsymbol{u} \in H(\nabla\times,\Omega), \quad \boldsymbol{q} \in L^2(\Omega), \quad \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} \in H^1(\Omega)$$

$$\boldsymbol{u} \in H(\nabla\cdot,\Omega), \quad h \in L^2(\Omega)$$
$$h\boldsymbol{u} \in H(\nabla\cdot,\Omega), \quad \boldsymbol{q} \in H^1(\Omega), \quad \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} \in L^2(\Omega)$$

$$G_p(\Omega_h) \xrightarrow{\nabla} C_p(\Omega_h) \xrightarrow{\nabla\times} S_p(\Omega)$$
$$G_p(\Omega_h) \xleftarrow{\widetilde{\nabla\cdot}} C_p(\Omega_h) \xleftarrow{\widetilde{\nabla}^\perp} S_p(\Omega)$$

$$\boldsymbol{u}_h \in C_p(\Omega_h), \quad h_h \in G_p(\Omega_h)$$
$$(h\boldsymbol{u})_h \in C_p(\Omega_h), \quad \boldsymbol{q}_h \in S_p(\Omega_h), \quad \left(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)_h \in C_p(\Omega_h)$$

$$G_p(\Omega_h) \xrightarrow{\nabla^\perp} D_p(\Omega_h) \xrightarrow{\nabla\cdot} S_p(\Omega)$$
$$G_p(\Omega_h) \xleftarrow{\widetilde{\nabla\times}} D_p(\Omega_h) \xleftarrow{\widetilde{\nabla}} S_p(\Omega)$$

$$\boldsymbol{u}_h \in D_p(\Omega_h), \quad h_h \in S_p(\Omega_h)$$
$$(h\boldsymbol{u})_h \in D_p(\Omega_h), \quad \boldsymbol{q}_h \in G_p(\Omega_h), \quad \left(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)_p \in S_p(\Omega_h)$$

Discrete

Spatial discretization shallow waters

Replicating the continuous structures: de Rham complex

$$H^1(\Omega) \xrightarrow{\nabla} H(\nabla\times,\Omega) \xrightarrow{\nabla\times} L^2(\Omega)$$
$$\widetilde{\nabla\cdot} \qquad \widetilde{\nabla}^{\perp}$$

$$H^1(\Omega) \xrightarrow{\nabla^{\perp}} H(\nabla\cdot,\Omega) \xrightarrow{\nabla\cdot} L^2(\Omega)$$
$$\widetilde{\nabla\times} \qquad \widetilde{\nabla}$$

Continuous

$$\boldsymbol{u} \in H(\nabla\times,\Omega), \quad h \in H^1(\Omega)$$
$$h\boldsymbol{u} \in H(\nabla\times,\Omega), \quad \boldsymbol{q} \in L^2(\Omega), \quad \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} \in H^1(\Omega)$$

$$\boldsymbol{u} \in H(\nabla\cdot,\Omega), \quad h \in L^2(\Omega)$$
$$h\boldsymbol{u} \in H(\nabla\cdot,\Omega), \quad \boldsymbol{q} \in H^1(\Omega), \quad \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} \in L^2(\Omega)$$

$$G_p(\Omega_h) \xrightarrow{\nabla} C_p(\Omega_h) \xrightarrow{\nabla\times} S_p(\Omega)$$
$$\widetilde{\nabla\cdot} \qquad \widetilde{\nabla}^{\perp}$$

$$G_p(\Omega_h) \xrightarrow{\nabla^{\perp}} D_p(\Omega_h) \xrightarrow{\nabla\cdot} S_p(\Omega)$$
$$\widetilde{\nabla\times} \qquad \widetilde{\nabla}$$

Discrete

$$\boldsymbol{u}_h \in C_p(\Omega_h), \quad h_h \in G_p(\Omega_h)$$
$$(h\boldsymbol{u})_h \in C_p(\Omega_h), \quad \boldsymbol{q}_h \in S_p(\Omega_h), \quad \left(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)_h \in C_p(\Omega_h)$$

$$\boldsymbol{u}_h \in D_p(\Omega_h), \quad h_h \in S_p(\Omega_h)$$
$$(h\boldsymbol{u})_h \in D_p(\Omega_h), \quad \boldsymbol{q}_h \in G_p(\Omega_h), \quad \left(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)_p \in S_p(\Omega_h)$$

Spatial discretization shallow waters

Replicating the continuous structures: de Rham complex

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla"] & H(\nabla\times,\Omega) \arrow[r, "\nabla\times"] \arrow[l, bend left, "\widetilde{\nabla\cdot}"] & L^2(\Omega) \arrow[l, bend left, "\widetilde{\nabla}^{\perp}"] \end{tikzcd}$$

$$\begin{tikzcd} H^1(\Omega) \arrow[r, "\nabla^{\perp}"] & H(\nabla\cdot,\Omega) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & L^2(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"] \end{tikzcd}$$

Continuous

$$\boldsymbol{u} \in H(\nabla\times,\Omega), \quad h \in H^1(\Omega)$$
$$h\boldsymbol{u} \in H(\nabla\times,\Omega), \quad \boldsymbol{q} \in L^2(\Omega), \quad \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} \in H^1(\Omega)$$

$$\boldsymbol{u} \in H(\nabla\cdot,\Omega), \quad h \in L^2(\Omega)$$
$$h\boldsymbol{u} \in H(\nabla\cdot,\Omega), \quad \boldsymbol{q} \in H^1(\Omega), \quad \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} \in L^2(\Omega)$$

Focus on this!

$$\begin{tikzcd} G_p(\Omega_h) \arrow[r, "\nabla"] & C_p(\Omega_h) \arrow[r, "\nabla\times"] \arrow[l, bend left, "\widetilde{\nabla\cdot}"] & S_p(\Omega) \arrow[l, bend left, "\widetilde{\nabla}^{\perp}"] \end{tikzcd}$$

$$\begin{tikzcd} G_p(\Omega_h) \arrow[r, "\nabla^{\perp}"] & D_p(\Omega_h) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & S_p(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"] \end{tikzcd}$$

Discrete

$$\boldsymbol{u}_h \in C_p(\Omega_h), \quad h_h \in G_p(\Omega_h)$$
$$(h\boldsymbol{u})_h \in C_p(\Omega_h), \quad \boldsymbol{q}_h \in S_p(\Omega_h), \quad \left(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)_h \in C_p(\Omega_h)$$

$$\boldsymbol{u}_h \in D_p(\Omega_h), \quad h_h \in S_p(\Omega_h)$$
$$(h\boldsymbol{u})_h \in D_p(\Omega_h), \quad \boldsymbol{q}_h \in G_p(\Omega_h), \quad \left(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)_p \in S_p(\Omega_h)$$

Spatial discretization shallow waters

Replicating the continuous structures: symplectic (tensor) operator

$$
\begin{cases}
\langle \boldsymbol{\sigma}, \dfrac{\partial \boldsymbol{u}}{\partial t} \rangle = -\langle \boldsymbol{\sigma}, \boldsymbol{q} \times (h\boldsymbol{u}) \rangle + \underbrace{\langle \nabla \cdot \boldsymbol{\sigma}, \frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u} + gh \rangle}^{\text{Integration by parts}}, & \forall \boldsymbol{\sigma} \in H(\nabla \cdot, \Omega) \\
\langle \theta, \dfrac{\partial h}{\partial t} \rangle = -\langle \theta, \nabla \cdot (h\boldsymbol{u}) \rangle, & \forall \theta \in L^2(\Omega)
\end{cases}
$$

$$
\begin{tikzcd}
G_p(\Omega_h) \arrow[r, "\nabla^\perp"] & D_p(\Omega_h) \arrow[r, "\nabla \cdot"] \arrow[l, bend left, "\widetilde{\nabla \times}"] & S_p(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"]
\end{tikzcd}
$$

$$\boldsymbol{u}_h \in D_p(\Omega_h), \quad h_h \in S_p(\Omega_h)$$

$$\boxed{(h\boldsymbol{u})_h \in D_p(\Omega_h)}, \quad \boldsymbol{q}_h \in G_p(\Omega_h), \quad \boxed{\left(\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}\right)_p \in S_p(\Omega_h)}$$

Focus on this formulation
the steps are identical

Spatial discretization shallow waters

Replicating the continuous structures: symplectic (tensor) operator

$$\begin{cases} \langle \boldsymbol{\sigma}_h, \dfrac{\partial \boldsymbol{u}_h}{\partial t} \rangle = -\langle \boldsymbol{\sigma}_h, \boldsymbol{q}_h \times (h\boldsymbol{u})_h \rangle + \langle \nabla \cdot \boldsymbol{\sigma}_h, \left(\dfrac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u} \right)_h + g h_h \rangle, & \forall \boldsymbol{\sigma}_h \in D_p(\Omega_h) \\ \langle \theta_h, \dfrac{\partial h_h}{\partial t} \rangle = -\langle \theta_h, \nabla \cdot (h\boldsymbol{u})_h \rangle, & \forall \theta_h \in S_p(\Omega_h) \end{cases}$$

$$\begin{tikzcd} G_p(\Omega_h) \arrow[r, "\nabla^\perp"] & D_p(\Omega_h) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & S_p(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"] \end{tikzcd}$$

$$\boldsymbol{u}_h \in D_p(\Omega_h), \quad h_h \in S_p(\Omega_h)$$

$$\boxed{(h\boldsymbol{u})_h \in D_p(\Omega_h)}, \quad \boldsymbol{q}_h \in G_p(\Omega_h), \quad \boxed{\left(\frac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u} \right)_p \in S_p(\Omega_h)}$$

Miss definitions for these *diagnostic fields*

Spatial discretization shallow waters

Replicating the continuous structures: symplectic (tensor) operator

$$
\begin{cases}
\langle \boldsymbol{\sigma}_h, \dfrac{\partial \boldsymbol{u}_h}{\partial t} \rangle = -\langle \boldsymbol{\sigma}_h, \boldsymbol{q}_h \times \boldsymbol{F}_h \rangle + \langle \nabla \cdot \boldsymbol{\sigma}_h, K_h + g h_h \rangle, & \forall \boldsymbol{\sigma}_h \in D_p(\Omega_h) \\
\langle \theta_h, \dfrac{\partial h_h}{\partial t} \rangle = -\langle \theta_h, \nabla \cdot \boldsymbol{F}_h \rangle, & \forall \theta_h \in S_p(\Omega_h) \\
\langle \boldsymbol{\sigma}_h, \boldsymbol{F}_h \rangle = \langle \boldsymbol{\sigma}_h, h_h \boldsymbol{u}_h \rangle, & \forall \boldsymbol{\sigma}_h \in D_p(\Omega_h) \\
\langle \theta_h, K_h \rangle = \langle \theta_h, \dfrac{1}{2} \boldsymbol{u}_h \cdot \boldsymbol{u}_h \rangle, & \forall \theta_h \in S_p(\Omega_h) \\
\langle \boldsymbol{\xi}_h, h_h \boldsymbol{q}_h \rangle = \langle \nabla \times \boldsymbol{\xi}_h, \boldsymbol{u}_h \rangle, & \forall \boldsymbol{\xi}_h \in G_p(\Omega_h)
\end{cases}
$$

Prognostic equations

Diagnostic equations

$$
\begin{tikzcd}
G_p(\Omega_h) \arrow[r, "\nabla^\perp"] & D_p(\Omega_h) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & S_p(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"]
\end{tikzcd}
$$

$$
\boldsymbol{u}_h \in D_p(\Omega_h), \quad h_h \in S_p(\Omega_h)
$$

$$
\boxed{(h\boldsymbol{u})_h \in D_p(\Omega_h)}, \quad \boldsymbol{q}_h \in G_p(\Omega_h), \quad \boxed{\left(\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}\right)_p \in S_p(\Omega_h)}
$$

Spatial discretization shallow waters

Replicating the continuous structures: symplectic (tensor) operator

$$
\begin{cases}
\langle \boldsymbol{\sigma}_h, \dfrac{\partial \boldsymbol{u}_h}{\partial t}\rangle = -\langle \boldsymbol{\sigma}_h, \boldsymbol{q}_h \times \boldsymbol{F}_h\rangle + \langle \nabla\cdot\boldsymbol{\sigma}_h, K_h + g h_h\rangle, & \forall \boldsymbol{\sigma}_h \in D_p(\Omega_h) \\
\langle \theta_h, \dfrac{\partial h_h}{\partial t}\rangle = -\langle \theta_h, \nabla\cdot\boldsymbol{F}_h\rangle, & \forall \theta_h \in S_p(\Omega_h) \\
\langle \boldsymbol{\sigma}_h, \boldsymbol{F}_h\rangle = \langle \boldsymbol{\sigma}_h, h_h \boldsymbol{u}_h\rangle, & \forall \boldsymbol{\sigma}_h \in D_p(\Omega_h) \\
\langle \theta_h, K_h\rangle = \langle \theta_h, \dfrac{1}{2}\boldsymbol{u}_h\cdot\boldsymbol{u}_h\rangle, & \forall \theta_h \in S_p(\Omega_h) \\
\langle \boldsymbol{\xi}_h, h_h \boldsymbol{q}_h\rangle = \langle \nabla\times\boldsymbol{\xi}_h, \boldsymbol{u}_h\rangle, & \forall \boldsymbol{\xi}_h \in G_p(\Omega_h)
\end{cases}
$$

Prognostic equations

Diagnostic equations

$$
\begin{tikzcd}
G_p(\Omega_h) \arrow[r, "\nabla^{\perp}"] & D_p(\Omega_h) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & S_p(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"]
\end{tikzcd}
$$

$$\boldsymbol{u}_h \in D_p(\Omega_h), \quad h_h \in S_p(\Omega_h)$$

$$(h\boldsymbol{u})_h \in D_p(\Omega_h), \quad \boldsymbol{q}_h \in G_p(\Omega_h), \quad \left(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)_p \in S_p(\Omega_h)$$

Standard FEM procedure

$$G_p(\Omega_h) = \mathrm{span}\left\{\boldsymbol{\epsilon}_i^G(x,y)\right\}_{i=1}^{n_G}$$

$$D_p(\Omega_h) = \mathrm{span}\left\{\boldsymbol{\epsilon}_i^D(x,y)\right\}_{i=1}^{n_D}$$

$$S_p(\Omega_h) = \mathrm{span}\left\{\epsilon_i^S(x,y)\right\}_{i=1}^{n_S}$$

Spatial discretization shallow waters

Replicating the continuous structures: symplectic (tensor) operator

$$
\begin{cases}
\langle \boldsymbol{\sigma}_h, \dfrac{\partial \boldsymbol{u}_h}{\partial t}\rangle = -\langle \boldsymbol{\sigma}_h, \boldsymbol{q}_h \times \boldsymbol{F}_h\rangle + \langle \nabla\cdot\boldsymbol{\sigma}_h, K_h + g h_h\rangle, & \forall \boldsymbol{\sigma}_h \in D_p(\Omega_h) \\
\langle \theta_h, \dfrac{\partial h_h}{\partial t}\rangle = -\langle \theta_h, \nabla\cdot\boldsymbol{F}_h\rangle, & \forall \theta_h \in S_p(\Omega_h) \\
\langle \boldsymbol{\sigma}_h, \boldsymbol{F}_h\rangle = \langle \boldsymbol{\sigma}_h, h_h \boldsymbol{u}_h\rangle, & \forall \boldsymbol{\sigma}_h \in D_p(\Omega_h) \\
\langle \theta_h, K_h\rangle = \langle \theta_h, \dfrac{1}{2}\boldsymbol{u}_h\cdot\boldsymbol{u}_h\rangle, & \forall \theta_h \in S_p(\Omega_h) \\
\langle \boldsymbol{\xi}_h, h_h \boldsymbol{q}_h\rangle = \langle \nabla\times\boldsymbol{\xi}_h, \boldsymbol{u}_h\rangle, & \forall \boldsymbol{\xi}_h \in G_p(\Omega_h)
\end{cases}
$$

Prognostic equations

Diagnostic equations

$$
\begin{tikzcd}
G_p(\Omega_h) \arrow[r, "\nabla^{\perp}"] & D_p(\Omega_h) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & S_p(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"]
\end{tikzcd}
$$

$$\boldsymbol{u}_h \in D_p(\Omega_h), \quad h_h \in S_p(\Omega_h)$$

$$(h\boldsymbol{u})_h \in D_p(\Omega_h), \quad \boldsymbol{q}_h \in G_p(\Omega_h), \quad \left(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)_p \in S_p(\Omega_h)$$

Standard FEM procedure

$$G_p(\Omega_h) = \text{span}\left\{\boldsymbol{\epsilon}_i^G(x,y)\right\}_{i=1}^{n_G}$$

$$\boldsymbol{q}_h = \sum_{i=1}^{n_G} q_i \boldsymbol{\epsilon}_i^G(x,y)$$

$$D_p(\Omega_h) = \text{span}\left\{\boldsymbol{\epsilon}_i^D(x,y)\right\}_{i=1}^{n_D}$$

$$\boldsymbol{u}_h = \sum_{i=1}^{n_D} u_i \boldsymbol{\epsilon}_i^D(x,y)$$

$$\boldsymbol{F}_h = \sum_{i=1}^{n_D} F_i \boldsymbol{\epsilon}_i^D(x,y)$$

$$S_p(\Omega_h) = \text{span}\left\{\epsilon_i^S(x,y)\right\}_{i=1}^{n_S}$$

$$h_h = \sum_{i=1}^{n_S} h_i \epsilon_i^S(x,y)$$

$$K_h = \sum_{i=1}^{n_S} K_i \epsilon_i^S(x,y)$$

Spatial discretization shallow waters

Replicating the continuous structures: symplectic (tensor) operator

$$
\begin{cases}
\displaystyle\sum_{j=1}^{n_D} \frac{\mathrm{d}u_j}{\mathrm{d}t}\langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^D\rangle = -\sum_{j=1}^{n_D} F_j \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h \times \boldsymbol{\epsilon}_j^D\rangle + \sum_{j=1}^{n_S} (K_j + g h_j)\,\langle \nabla\cdot\boldsymbol{\epsilon}_i^D, \epsilon_j^S\rangle, & i = 1,\dots,n_D \\
\displaystyle\sum_{j=1}^{n_S} \frac{\mathrm{d}h_j}{\mathrm{d}t}\langle \epsilon_i^S, \epsilon_j^D\rangle = -\sum_{j=1}^{n_D} F_j \langle \epsilon_i^S, \nabla\cdot\boldsymbol{\epsilon}_j^D\rangle, & i = 1,\dots,n_S \\
\displaystyle\sum_{i=1}^{n_D} F_j \langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^D\rangle = \langle \boldsymbol{\epsilon}_i^D, h_h \boldsymbol{u}_h\rangle, & i = 1,\dots,n_D \\
\displaystyle\sum_{j=1}^{n_S} K_j \langle \epsilon_i^S, \epsilon_j^S\rangle = \langle \epsilon_i^S, \frac{1}{2}\boldsymbol{u}_h\cdot\boldsymbol{u}_h\rangle, & i = 1,\dots,n_S \\
\displaystyle\sum_{j=1}^{n_G} q_j \langle \boldsymbol{\epsilon}_i^G, h_h \epsilon_j^G\rangle = \langle \nabla\times\boldsymbol{\epsilon}_i^G, \boldsymbol{u}_h\rangle, & i = 1,\dots,n_G
\end{cases}
$$

Prognostic equations

Diagnostic equations

$$
\begin{tikzcd}
G_p(\Omega_h) \arrow[r, "\nabla^\perp"] & D_p(\Omega_h) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & S_p(\Omega) \arrow[l, bend left, "\widetilde{\nabla}"]
\end{tikzcd}
$$

$$\boldsymbol{u}_h \in D_p(\Omega_h), \quad h_h \in S_p(\Omega_h)$$

$$(h\boldsymbol{u})_h \in D_p(\Omega_h), \quad \boldsymbol{q}_h \in G_p(\Omega_h), \quad \left(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)_p \in S_p(\Omega_h)$$

$$G_p(\Omega_h) = \mathrm{span}\left\{\boldsymbol{\epsilon}_i^G(x,y)\right\}_{i=1}^{n_G}$$

$$\boldsymbol{q}_h = \sum_{i=1}^{n_G} q_i \boldsymbol{\epsilon}_i^G(x,y)$$

$$D_p(\Omega_h) = \mathrm{span}\left\{\boldsymbol{\epsilon}_i^D(x,y)\right\}_{i=1}^{n_D}$$

$$\boldsymbol{u}_h = \sum_{i=1}^{n_D} u_i \boldsymbol{\epsilon}_i^D(x,y)$$

$$\boldsymbol{F}_h = \sum_{i=1}^{n_D} F_i \boldsymbol{\epsilon}_i^D(x,y)$$

$$S_p(\Omega_h) = \mathrm{span}\left\{\epsilon_i^S(x,y)\right\}_{i=1}^{n_S}$$

$$h_h = \sum_{i=1}^{n_S} h_i \epsilon_i^S(x,y)$$

$$K_h = \sum_{i=1}^{n_S} K_i \epsilon_i^S(x,y)$$

Spatial discretization shallow waters

Replicating the continuous structures: symplectic (tensor) operator

$$
\begin{cases}
\displaystyle\sum_{j=1}^{n_D} \frac{\mathrm{d}u_j}{\mathrm{d}t}\langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^D\rangle = -\sum_{j=1}^{n_D} F_j \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h \times \boldsymbol{\epsilon}_j^D\rangle + \sum_{j=1}^{n_S} (K_j + g h_j)\,\langle \nabla\cdot\boldsymbol{\epsilon}_i^D, \epsilon_j^S\rangle, & i = 1,\dots,n_D \\
\displaystyle\sum_{j=1}^{n_S} \frac{\mathrm{d}h_j}{\mathrm{d}t}\langle \epsilon_i^S, \epsilon_j^D\rangle = -\sum_{j=1}^{n_D} F_j \langle \epsilon_i^S, \nabla\cdot\boldsymbol{\epsilon}_j^D\rangle, & i = 1,\dots,n_S \\
\displaystyle\sum_{i=1}^{n_D} F_j \langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^D\rangle = \langle \boldsymbol{\epsilon}_i^D, h_h \boldsymbol{u}_h\rangle, & i = 1,\dots,n_D \\
\displaystyle\sum_{j=1}^{n_S} K_j \langle \epsilon_i^S, \epsilon_j^S\rangle = \langle \epsilon_i^S, \frac{1}{2}\boldsymbol{u}_h\cdot\boldsymbol{u}_h\rangle, & i = 1,\dots,n_S \\
\displaystyle\sum_{j=1}^{n_G} q_j \langle \boldsymbol{\epsilon}_i^G, h_h \epsilon_j^G\rangle = \langle \nabla\times\boldsymbol{\epsilon}_i^G, \boldsymbol{u}_h\rangle, & i = 1,\dots,n_G
\end{cases}
$$

Prognostic equations

Diagnostic equations

$$
\begin{tikzcd}
G_p(\Omega_h) \arrow[r, "\nabla^\perp"] & D_p(\Omega_h) \arrow[r, "\nabla\cdot"] \arrow[l, bend left, "\widetilde{\nabla\times}"] & S_p(\Omega) \arrow[l, bend left, "\tilde{\nabla}"]
\end{tikzcd}
$$

$$\boldsymbol{u}_h \in D_p(\Omega_h), \quad h_h \in S_p(\Omega_h)$$

$$\boxed{(h\boldsymbol{u})_h \in D_p(\Omega_h)}, \quad \boldsymbol{q}_h \in G_p(\Omega_h), \quad \boxed{\left(\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}\right)_p \in S_p(\Omega_h)}$$

$$G_p(\Omega_h) = \operatorname{span}\left\{\boldsymbol{\epsilon}_i^G(x,y)\right\}_{i=1}^{n_G}$$

$$\boldsymbol{q}_h = \sum_{i=1}^{n_G} q_i \boldsymbol{\epsilon}_i^G(x,y)$$

$$D_p(\Omega_h) = \operatorname{span}\left\{\boldsymbol{\epsilon}_i^D(x,y)\right\}_{i=1}^{n_D}$$

$$\boldsymbol{u}_h = \sum_{i=1}^{n_D} u_i \boldsymbol{\epsilon}_i^D(x,y)$$

$$\boldsymbol{F}_h = \sum_{i=1}^{n_D} F_i \boldsymbol{\epsilon}_i^D(x,y)$$

$$S_p(\Omega_h) = \operatorname{span}\left\{\epsilon_i^S(x,y)\right\}_{i=1}^{n_S}$$

$$h_h = \sum_{i=1}^{n_S} h_i \epsilon_i^S(x,y)$$

$$K_h = \sum_{i=1}^{n_S} K_i \epsilon_i^S(x,y)$$

Spatial discretization shallow waters

Replicating the continuous structures: symplectic (tensor) operator

$$\left.\begin{aligned}
&\sum_{j=1}^{n_D} \frac{\mathrm{d}u_j}{\mathrm{d}t} \langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^D \rangle = -\sum_{j=1}^{n_D} F_j \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h \times \boldsymbol{\epsilon}_j^D \rangle + \sum_{j=1}^{n_S} (K_j + g h_j) \langle \nabla \cdot \boldsymbol{\epsilon}_i^D, \epsilon_j^S \rangle, \quad && i = 1, \ldots, n_D \\
&\sum_{j=1}^{n_S} \frac{\mathrm{d}h_j}{\mathrm{d}t} \langle \epsilon_i^S, \epsilon_j^D \rangle = -\sum_{j=1}^{n_D} F_j \langle \epsilon_i^S, \nabla \cdot \boldsymbol{\epsilon}_j^D \rangle, && i = 1, \ldots, n_S
\end{aligned}\right\} \textit{Prognostic}\text{ equations}$$

$$\begin{bmatrix} \langle \boldsymbol{\epsilon}_j^D, \dfrac{\partial \boldsymbol{u}_h}{\partial t} \rangle \\ \langle \epsilon_j^S, \dfrac{\partial h_h}{\partial t} \rangle \end{bmatrix} = \underbrace{\begin{bmatrix} -\langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h \times \square \rangle & -\langle \boldsymbol{\epsilon}_j^D, \widetilde{\nabla} \square \rangle \\ -\langle \epsilon_j^S, \nabla \cdot \square \rangle & 0 \end{bmatrix}}_{\mathbb{J}_h} \begin{bmatrix} \boldsymbol{F}_h \\ K_h + g h_h \end{bmatrix},$$

Discrete

Spatial discretization shallow waters

Replicating the continuous structures: symplectic (tensor) operator

$$\left.\begin{aligned}
&\sum_{j=1}^{n_D}\frac{\mathrm{d}u_j}{\mathrm{d}t}\langle\boldsymbol{\epsilon}_i^D,\boldsymbol{\epsilon}_j^D\rangle = -\sum_{j=1}^{n_D}F_j\langle\boldsymbol{\epsilon}_j^D,\boldsymbol{q}_h\times\boldsymbol{\epsilon}_j^D\rangle + \sum_{j=1}^{n_S}(K_j+gh_j)\,\langle\nabla\cdot\boldsymbol{\epsilon}_i^D,\epsilon_j^S\rangle, \quad i=1,\dots,n_D\\
&\sum_{j=1}^{n_S}\frac{\mathrm{d}h_j}{\mathrm{d}t}\langle\epsilon_i^S,\epsilon_j^D\rangle = -\sum_{j=1}^{n_D}F_j\langle\epsilon_i^S,\nabla\cdot\boldsymbol{\epsilon}_j^D\rangle, \quad i=1,\dots,n_S
\end{aligned}\right\}$$

Prognostic equations

$$\begin{bmatrix}\langle\boldsymbol{\epsilon}_j^D,\dfrac{\partial\boldsymbol{u}_h}{\partial t}\rangle\\ \langle\epsilon_j^S,\dfrac{\partial h_h}{\partial t}\rangle\end{bmatrix} = \underbrace{\begin{bmatrix}-\langle\boldsymbol{\epsilon}_j^D,\boldsymbol{q}_h\times\square\rangle & -\langle\boldsymbol{\epsilon}_j^D,\widetilde{\nabla}\square\rangle\\ -\langle\epsilon_j^S,\nabla\cdot\square\rangle & 0\end{bmatrix}}_{\mathbb{J}_h}\begin{bmatrix}\boldsymbol{F}_h\\ K_h+gh_h\end{bmatrix},$$

Discrete

The same structure but at the discrete level

$$\begin{bmatrix}\langle\boldsymbol{\sigma},\dfrac{\partial\boldsymbol{u}}{\partial t}\rangle\\ \langle\theta,\dfrac{\partial h}{\partial t}\rangle\end{bmatrix} = \underbrace{\begin{bmatrix}-\langle\boldsymbol{\sigma},\boldsymbol{q}\times\square\rangle & -\langle\boldsymbol{\sigma},\widetilde{\nabla}\square\rangle\\ -\langle\theta,\nabla\cdot\square\rangle & 0\end{bmatrix}}_{\mathbb{J}}\begin{bmatrix}\dfrac{\delta\mathcal{H}}{\delta\boldsymbol{u}}\\ \dfrac{\delta\mathcal{H}}{\delta h}\end{bmatrix},$$

Continuous

Spatial discretization shallow waters

Replicating the continuous structures: Semi-discrete energy conservation

$$\left.\begin{aligned}
&\sum_{j=1}^{n_D}\frac{\mathrm{d}u_j}{\mathrm{d}t}\langle\boldsymbol{\epsilon}_i^D,\boldsymbol{\epsilon}_j^D\rangle = -\sum_{j=1}^{n_D}F_j\langle\boldsymbol{\epsilon}_j^D,\boldsymbol{q}_h\times\boldsymbol{\epsilon}_j^D\rangle + \sum_{j=1}^{n_S}(K_j+gh_j)\,\langle\nabla\cdot\boldsymbol{\epsilon}_i^D,\epsilon_j^S\rangle, \quad i=1,\dots,n_D\\
&\sum_{j=1}^{n_S}\frac{\mathrm{d}h_j}{\mathrm{d}t}\langle\epsilon_i^S,\epsilon_j^D\rangle = -\sum_{j=1}^{n_D}F_j\langle\epsilon_i^S,\nabla\cdot\boldsymbol{\epsilon}_j^D\rangle, \quad i=1,\dots,n_S
\end{aligned}\right\}\ \textit{Prognostic}\text{ equations}$$

$$\begin{bmatrix}\langle\boldsymbol{\epsilon}_j^D,\dfrac{\partial\boldsymbol{u}_h}{\partial t}\rangle\\[2ex]\langle\epsilon_j^S,\dfrac{\partial h_h}{\partial t}\rangle\end{bmatrix} = \underbrace{\begin{bmatrix}-\langle\boldsymbol{\epsilon}_j^D,\boldsymbol{q}_h\times\Box\rangle & -\langle\boldsymbol{\epsilon}_j^D,\widetilde{\nabla}\Box\rangle\\[2ex] -\langle\epsilon_j^S,\nabla\cdot\Box\rangle & 0\end{bmatrix}}_{\mathbb{J}_h}\begin{bmatrix}\boldsymbol{F}_h\\[2ex] K_h+gh_h\end{bmatrix},$$

Spatial discretization shallow waters

Replicating the continuous structures: Semi-discrete energy conservation

$$\left.\begin{aligned}
&\sum_{j=1}^{n_D} \frac{\mathrm{d}u_j}{\mathrm{d}t} \langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^D \rangle = -\sum_{j=1}^{n_D} F_j \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h \times \boldsymbol{\epsilon}_j^D \rangle + \sum_{j=1}^{n_S} (K_j + g h_j) \langle \nabla \cdot \boldsymbol{\epsilon}_i^D, \epsilon_j^S \rangle, \quad && i = 1, \dots, n_D \\
&\sum_{j=1}^{n_S} \frac{\mathrm{d}h_j}{\mathrm{d}t} \langle \epsilon_i^S, \epsilon_j^D \rangle = -\sum_{j=1}^{n_D} F_j \langle \epsilon_i^S, \nabla \cdot \boldsymbol{\epsilon}_j^D \rangle, && i = 1, \dots, n_S
\end{aligned}\right\} \textit{Prognostic} \text{ equations}$$

$$\underbrace{\begin{bmatrix} \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{\epsilon}_i^D \rangle & 0 \\ 0 & \langle \epsilon_j^S, \epsilon_i^S \rangle \end{bmatrix}}_{\mathbb{M}_h} \begin{bmatrix} \frac{\mathrm{d}u_i}{\mathrm{d}t} \\ \frac{\mathrm{d}h_i}{\mathrm{d}t} \end{bmatrix} = \underbrace{\begin{bmatrix} -\langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h \times \boldsymbol{\epsilon}_i^D \rangle & -\langle \boldsymbol{\epsilon}_j^D, \widetilde{\nabla} \epsilon_i^S \rangle \\ -\langle \epsilon_j^S, \nabla \cdot \boldsymbol{\epsilon}_i^D \rangle & 0 \end{bmatrix}}_{\mathbb{J}_h} \begin{bmatrix} \boldsymbol{F}_i \\ K_i + g h_i \end{bmatrix},$$

Spatial discretization shallow waters

Replicating the continuous structures: Semi-discrete energy conservation

$$
\left.
\begin{aligned}
&\sum_{j=1}^{n_D} \frac{\mathrm{d}u_j}{\mathrm{d}t} \langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^D \rangle = -\sum_{j=1}^{n_D} F_j \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h \times \boldsymbol{\epsilon}_j^D \rangle + \sum_{j=1}^{n_S} (K_j + g h_j) \langle \nabla \cdot \boldsymbol{\epsilon}_i^D, \epsilon_j^S \rangle, \quad && i = 1, \dots, n_D \\
&\sum_{j=1}^{n_S} \frac{\mathrm{d}h_j}{\mathrm{d}t} \langle \epsilon_i^S, \epsilon_j^D \rangle = -\sum_{j=1}^{n_D} F_j \langle \epsilon_i^S, \nabla \cdot \boldsymbol{\epsilon}_j^D \rangle, && i = 1, \dots, n_S
\end{aligned}
\right\}
$$

Prognostic equations

$$
\underbrace{\begin{bmatrix} \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{\epsilon}_i^D \rangle & 0 \\ 0 & \langle \epsilon_j^S, \epsilon_i^S \rangle \end{bmatrix}}_{\mathbb{M}_h}
\begin{bmatrix} \dfrac{\mathrm{d}u_i}{\mathrm{d}t} \\ \dfrac{\mathrm{d}h_i}{\mathrm{d}t} \end{bmatrix}
=
\underbrace{\begin{bmatrix} -\langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h \times \boldsymbol{\epsilon}_i^D \rangle & -\langle \boldsymbol{\epsilon}_j^D, \widetilde{\nabla} \epsilon_i^S \rangle \\ -\langle \epsilon_j^S, \nabla \cdot \boldsymbol{\epsilon}_i^D \rangle & 0 \end{bmatrix}}_{\mathbb{J}_h}
\begin{bmatrix} \boldsymbol{F}_i \\ K_i + g h_i \end{bmatrix},
$$

$$
\mathcal{H}[\boldsymbol{u}, h] := \int_\Omega \left(\frac{1}{2} h \boldsymbol{u} \cdot \boldsymbol{u} + \frac{1}{2} g h^2 \right) \mathrm{d}\Omega
$$

Spatial discretization shallow waters

Replicating the continuous structures: Semi-discrete energy conservation

$$\left.\begin{aligned}
&\sum_{j=1}^{n_D} \frac{\mathrm{d}u_j}{\mathrm{d}t} \langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^D \rangle = -\sum_{j=1}^{n_D} F_j \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h \times \boldsymbol{\epsilon}_j^D \rangle + \sum_{j=1}^{n_S} (K_j + g h_j) \langle \nabla \cdot \boldsymbol{\epsilon}_i^D, \epsilon_j^S \rangle, \quad i = 1, \dots, n_D \\
&\sum_{j=1}^{n_S} \frac{\mathrm{d}h_j}{\mathrm{d}t} \langle \epsilon_i^S, \epsilon_j^D \rangle = -\sum_{j=1}^{n_D} F_j \langle \epsilon_i^S, \nabla \cdot \boldsymbol{\epsilon}_j^D \rangle, \quad i = 1, \dots, n_S
\end{aligned}\right\}$$

Prognostic equations

$$\underbrace{\begin{bmatrix} \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{\epsilon}_i^D \rangle & 0 \\ 0 & \langle \epsilon_j^S, \epsilon_i^S \rangle \end{bmatrix}}_{\mathbb{M}_h} \begin{bmatrix} \frac{\mathrm{d}u_i}{\mathrm{d}t} \\ \frac{\mathrm{d}h_i}{\mathrm{d}t} \end{bmatrix} = \underbrace{\begin{bmatrix} -\langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h \times \boldsymbol{\epsilon}_i^D \rangle & -\langle \boldsymbol{\epsilon}_j^D, \widetilde{\nabla} \epsilon_i^S \rangle \\ -\langle \epsilon_j^S, \nabla \cdot \boldsymbol{\epsilon}_i^D \rangle & 0 \end{bmatrix}}_{\mathbb{J}_h} \begin{bmatrix} \boldsymbol{F}_i \\ K_i + g h_i \end{bmatrix},$$

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}[\boldsymbol{u}_h, h_h] := \int_\Omega \underbrace{(h\boldsymbol{u})_h}_{\boldsymbol{F}_h} \cdot \frac{\mathrm{d}\boldsymbol{u}_h}{\mathrm{d}t} \mathrm{d}\Omega + \int_\Omega \frac{\mathrm{d}h_h}{\mathrm{d}t} \underbrace{\left(\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}\right)_h}_{K_h} \mathrm{d}\Omega + \int_\Omega \frac{1}{2} g h_h \frac{\mathrm{d}h_h}{\mathrm{d}t} \mathrm{d}\Omega \longleftarrow \mathcal{H}[\boldsymbol{u}, h] := \int_\Omega \left(\frac{1}{2} h \boldsymbol{u} \cdot \boldsymbol{u} + \frac{1}{2} g h^2\right) \mathrm{d}\Omega$$

Spatial discretization shallow waters

Replicating the continuous structures: Semi-discrete energy conservation

$$\left.\begin{aligned}
&\sum_{j=1}^{n_D} \frac{\mathrm{d}u_j}{\mathrm{d}t}\langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^D\rangle = -\sum_{j=1}^{n_D} F_j \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h \times \boldsymbol{\epsilon}_j^D\rangle + \sum_{j=1}^{n_S} (K_j + g h_j)\, \langle \nabla\cdot\boldsymbol{\epsilon}_i^D, \epsilon_j^S\rangle, \quad && i = 1,\dots,n_D \\
&\sum_{j=1}^{n_S} \frac{\mathrm{d}h_j}{\mathrm{d}t}\langle \epsilon_i^S, \epsilon_j^D\rangle = -\sum_{j=1}^{n_D} F_j \langle \epsilon_i^S, \nabla\cdot\boldsymbol{\epsilon}_j^D\rangle, && i = 1,\dots,n_S
\end{aligned}\right\}\ \textit{Prognostic}\ \text{equations}$$

$$\underbrace{\begin{bmatrix} \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{\epsilon}_i^D\rangle & 0 \\ 0 & \langle \epsilon_j^S, \epsilon_i^S\rangle \end{bmatrix}}_{\mathbb{M}_h} \begin{bmatrix} \frac{\mathrm{d}u_i}{\mathrm{d}t} \\ \frac{\mathrm{d}h_i}{\mathrm{d}t} \end{bmatrix} = \underbrace{\begin{bmatrix} -\langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h \times \boldsymbol{\epsilon}_i^D\rangle & -\langle \boldsymbol{\epsilon}_j^D, \widetilde{\nabla}\epsilon_i^S\rangle \\ -\langle \epsilon_j^S, \nabla\cdot\boldsymbol{\epsilon}_i^D\rangle & 0 \end{bmatrix}}_{\mathbb{J}_h} \begin{bmatrix} \boldsymbol{F}_i \\ K_i + g h_i \end{bmatrix},$$

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}[\boldsymbol{u}_h, h_h] := \int_\Omega \boldsymbol{F}_h \cdot \frac{\mathrm{d}\boldsymbol{u}_h}{\mathrm{d}t}\mathrm{d}\Omega + \int_\Omega \left(K_h + \frac{1}{2} g h_h\right)\frac{\mathrm{d}h_h}{\mathrm{d}t}\mathrm{d}\Omega \longleftarrow \mathcal{H}[\boldsymbol{u}, h] := \int_\Omega \left(\frac{1}{2} h\boldsymbol{u}\cdot\boldsymbol{u} + \frac{1}{2} g h^2\right)\mathrm{d}\Omega$$

Spatial discretization shallow waters

Replicating the continuous structures: Semi-discrete energy conservation

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&\sum_{j=1}^{n_S} \frac{\mathrm{d}h_j}{\mathrm{d}t}\langle \epsilon_i^S, \epsilon_j^D\rangle = -\sum_{j=1}^{n_D} F_j \langle \epsilon_i^S, \nabla \cdot \boldsymbol{\epsilon}_j^D\rangle, \quad i = 1, \dots, n_S
\end{aligned}\right\}$$

Prognostic equations

$$\begin{bmatrix} F_i & K_i + gh_i \end{bmatrix} \underbrace{\begin{bmatrix} \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{\epsilon}_i^D\rangle & 0 \\ 0 & \langle \epsilon_j^S, \epsilon_i^S\rangle \end{bmatrix}}_{\mathbb{M}_h} \begin{bmatrix} \frac{\mathrm{d}u_i}{\mathrm{d}t} \\ \frac{\mathrm{d}h_i}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} F_i & K_i + gh_i \end{bmatrix} \underbrace{\begin{bmatrix} -\langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h \times \boldsymbol{\epsilon}_i^D\rangle & -\langle \boldsymbol{\epsilon}_j^D, \widetilde{\nabla} \epsilon_i^S\rangle \\ -\langle \epsilon_j^S, \nabla \cdot \boldsymbol{\epsilon}_i^D\rangle & 0 \end{bmatrix}}_{\mathbb{J}_h} \begin{bmatrix} F_i \\ K_i + gh_i \end{bmatrix}$$

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}[\boldsymbol{u}_h, h_h] := \int_\Omega \boldsymbol{F}_h \cdot \frac{\mathrm{d}\boldsymbol{u}_h}{\mathrm{d}t}\mathrm{d}\Omega + \int_\Omega \left(K_h + \frac{1}{2} g h_h\right) \frac{\mathrm{d}h_h}{\mathrm{d}t}\mathrm{d}\Omega \longleftarrow \mathcal{H}[\boldsymbol{u}, h] := \int_\Omega \left(\frac{1}{2} h \boldsymbol{u} \cdot \boldsymbol{u} + \frac{1}{2} g h^2\right) \mathrm{d}\Omega$$

Spatial discretization shallow waters

Replicating the continuous structures: Semi-discrete energy conservation

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\end{aligned}\right\}\ \textit{Prognostic}\text{ equations}$$

$$\begin{bmatrix} F_i & K_i + g h_i \end{bmatrix} \underbrace{\begin{bmatrix} \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{\epsilon}_i^D\rangle & 0 \\ 0 & \langle \epsilon_j^S, \epsilon_i^S\rangle \end{bmatrix}}_{\mathbb{M}_h} \begin{bmatrix} \dfrac{\mathrm{d}u_i}{\mathrm{d}t} \\ \dfrac{\mathrm{d}h_i}{\mathrm{d}t} \end{bmatrix} = \begin{bmatrix} F_i & K_i + g h_i \end{bmatrix} \underbrace{\begin{bmatrix} -\langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h \times \boldsymbol{\epsilon}_i^D\rangle & -\langle \boldsymbol{\epsilon}_j^D, \widetilde{\nabla} \epsilon_i^S\rangle \\ -\langle \epsilon_j^S, \nabla \cdot \boldsymbol{\epsilon}_i^D\rangle & 0 \end{bmatrix}}_{\mathbb{J}_h} \begin{bmatrix} F_i \\ K_i + g h_i \end{bmatrix} = 0$$

From the skew-symmetry

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}[\boldsymbol{u}_h, h_h] := \int_\Omega \boldsymbol{F}_h \cdot \frac{\mathrm{d}\boldsymbol{u}_h}{\mathrm{d}t}\mathrm{d}\Omega + \int_\Omega \left(K_h + \frac{1}{2} g h_h\right) \frac{\mathrm{d}h_h}{\mathrm{d}t}\mathrm{d}\Omega \longleftarrow \mathcal{H}[\boldsymbol{u}, h] := \int_\Omega \left(\frac{1}{2} h \boldsymbol{u} \cdot \boldsymbol{u} + \frac{1}{2} g h^2\right) \mathrm{d}\Omega$$

Spatial discretization shallow waters

How does this compare to Arakawa and Lamb?

Spatial discretization shallow waters

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Arakawa and Lamb

$$\frac{\partial h}{\partial t} + \nabla \cdot \boldsymbol{F} = 0$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + q\boldsymbol{e}_z \times \boldsymbol{F} + \nabla\left(K + hg\right) = 0$$

$$K := \frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}$$

$$q := \frac{\omega}{h}$$

$$\omega := \boldsymbol{e}_z \cdot \nabla \times \boldsymbol{u}$$

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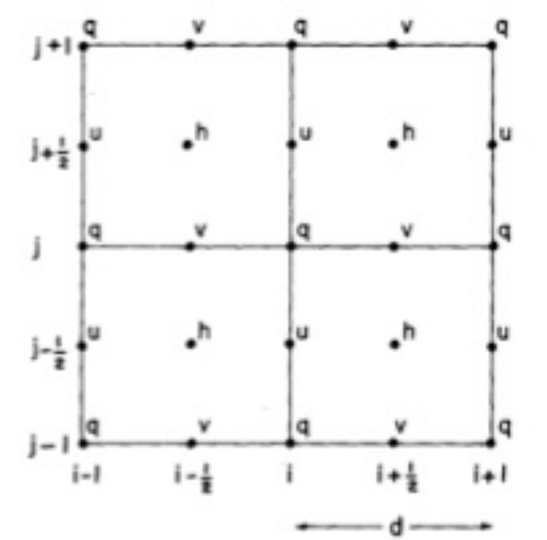


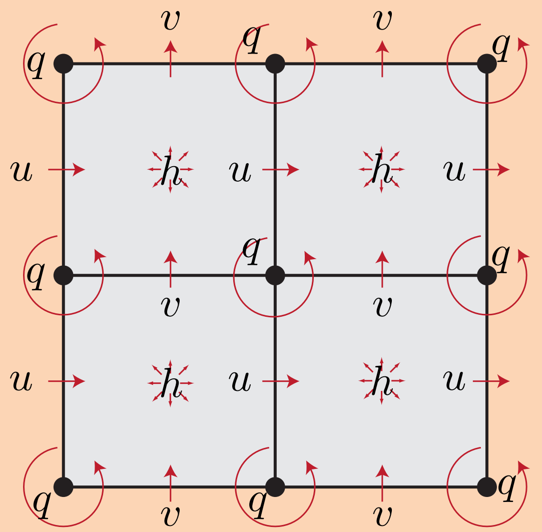
e.g., FEM

Hamiltonian formulation + discrete structure preserving

$$\begin{cases} \langle \boldsymbol{\sigma}_h, \dfrac{\partial \boldsymbol{u}_h}{\partial t}\rangle = -\langle \boldsymbol{\sigma}_h, \boldsymbol{q}_h \times \boldsymbol{F}_h\rangle + \langle \nabla\cdot\boldsymbol{\sigma}_h, K_h + gh_h\rangle, & \forall \boldsymbol{\sigma}_h \in D_p(\Omega_h) \\ \langle \theta_h, \dfrac{\partial h_h}{\partial t}\rangle = -\langle \theta_h, \nabla\cdot\boldsymbol{F}_h\rangle, & \forall \theta_h \in S_p(\Omega_h) \\ \langle \boldsymbol{\sigma}_h, \boldsymbol{F}_h\rangle = \langle \boldsymbol{\sigma}_h, h_h\boldsymbol{u}_h\rangle, & \forall \boldsymbol{\sigma}_h \in D_p(\Omega_h) \\ \langle \theta_h, K_h\rangle = \langle \theta_h, \dfrac{1}{2}\boldsymbol{u}_h\cdot\boldsymbol{u}_h\rangle, & \forall \theta_h \in S_p(\Omega_h) \\ \langle \boldsymbol{\xi}_h, h_h\boldsymbol{q}_h\rangle = \langle \nabla\times\boldsymbol{\xi}_h, \boldsymbol{u}_h\rangle, & \forall \boldsymbol{\xi}_h \in G_p(\Omega_h) \end{cases}$$

Lowest order

p = 1



Spatial discretization shallow waters

How does this compare to Arakawa and Lamb?

Arakawa and Lamb

$$\frac{\partial h}{\partial t} + \nabla \cdot \boldsymbol{F} = 0$$

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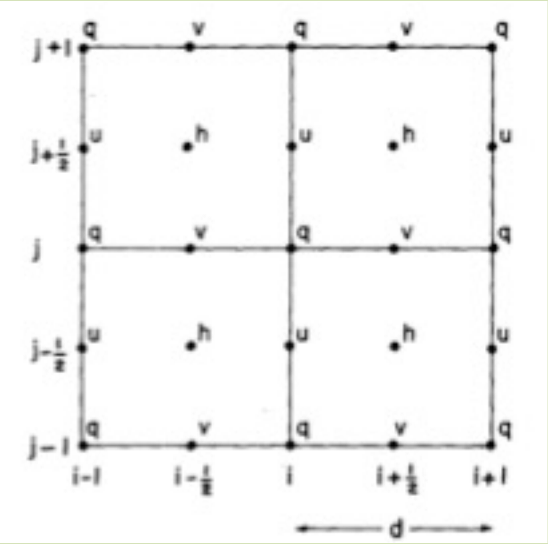
All pointwise values



e.g., FEM

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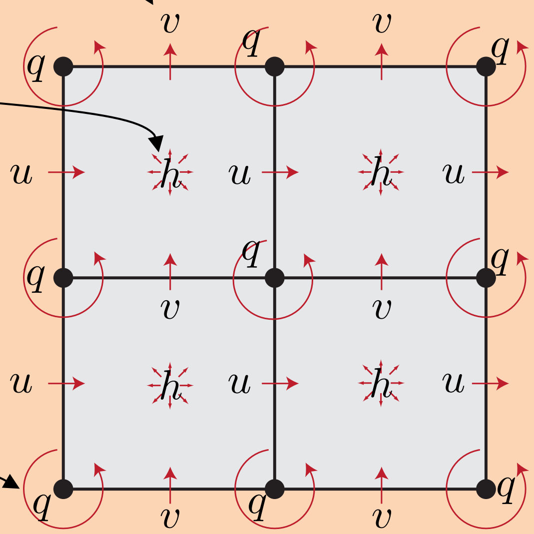
\langle \theta_h, \dfrac{\partial h_h}{\partial t}\rangle = -\langle \theta_h, \nabla\cdot\boldsymbol{F}_h\rangle, & \forall \theta_h \in S_p(\Omega_h) \\
\langle \boldsymbol{\sigma}_h, \boldsymbol{F}_h\rangle = \langle \boldsymbol{\sigma}_h, h_h\boldsymbol{u}_h\rangle, & \forall \boldsymbol{\sigma}_h \in D_p(\Omega_h) \\
\langle \theta_h, K_h\rangle = \langle \theta_h, \dfrac{1}{2}\boldsymbol{u}_h\cdot\boldsymbol{u}_h\rangle, & \forall \theta_h \in S_p(\Omega_h) \\
\langle \boldsymbol{\xi}_h, h_h\boldsymbol{q}_h\rangle = \langle \nabla\times\boldsymbol{\xi}_h, \boldsymbol{u}_h\rangle, & \forall \boldsymbol{\xi}_h \in G_p(\Omega_h)
\end{cases}$$

Surface integrals

Fluxes

Pointwise values



Spatial discretization shallow waters

How does this compare to Arakawa and Lamb?

e.g., FEM

Arakawa and Lamb

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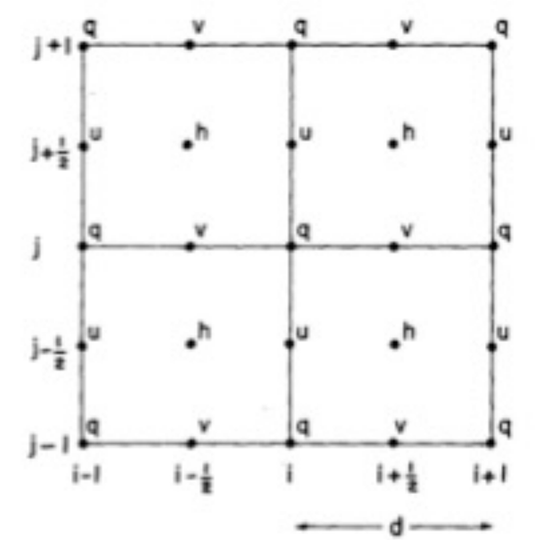
All pointwise values



Hamiltonian formulation + discrete structure preserving

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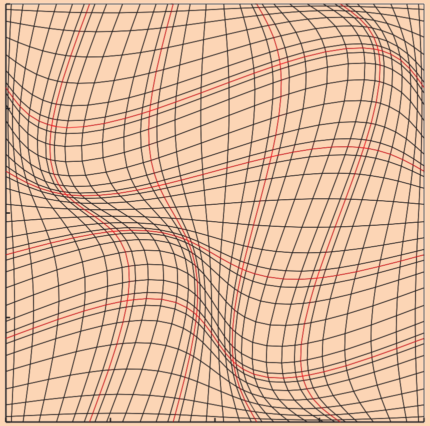
\end{cases}$$

Directly obtain a high order (arbitrary p) approximation



4x4 elements of degree 7

Spatial discretization shallow waters

How does this compare to Arakawa and Lamb?

e.g., FEM

Arakawa and Lamb

$$\frac{\partial h}{\partial t} + \nabla \cdot \boldsymbol{F} = 0$$

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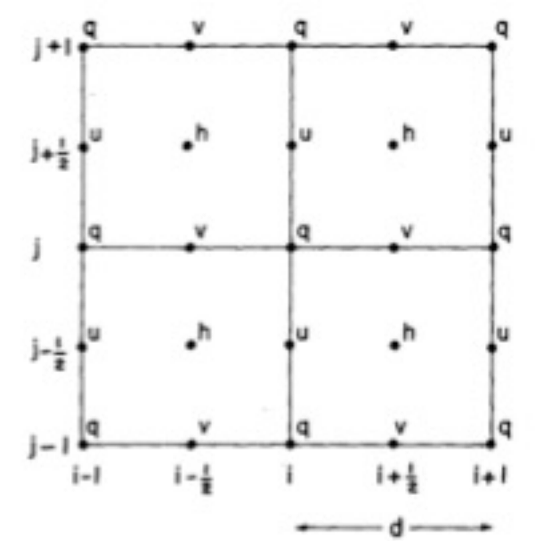

Hamiltonian formulation + discrete structure preserving

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\langle \theta_h, \dfrac{\partial h_h}{\partial t}\rangle = -\langle \theta_h, \nabla\cdot\boldsymbol{F}_h\rangle, & \forall \theta_h \in S_p(\Omega_h) \\
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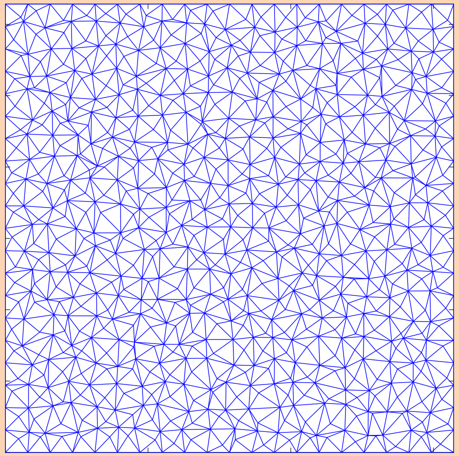
\langle \boldsymbol{\xi}_h, h_h\boldsymbol{q}_h\rangle = \langle \nabla\times\boldsymbol{\xi}_h, \boldsymbol{u}_h\rangle, & \forall \boldsymbol{\xi}_h \in G_p(\Omega_h)
\end{cases}$$

Use either structured or unstructured grids



Spatial discretization shallow waters

Key take-aways

1. Spatial discretization follows directly from the continuous weak form and the discrete de Rham complex.

2. Arbitrary order approximation on structured and unstructured grids.

3. Prognostic variables are directly represented by their discrete counterparts.

4. Diagnostic variables (typically nonlinear combination of the prognostic ones) need to be explicitly constructed by introducting additional unknown fields.

5. Obtain semi-discrete conservation properties.

Spatial discretization shallow waters

Key take-aways

1. Spatial discretization follows directly from the continuous weak form and the discrete de Rham complex.

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4. Diagnostic variables (typically nonlinear combination of the prognostic ones) need to be explicitly constructed by introducting additional unknown fields.

5. Obtain semi-discrete conservation properties.

How to preserve these properties time?

Temporal discretization

Full discretization of shallow waters

Temporal discretization shallow waters

Midpoint rule time integrator (lowest order symplectic Gauss-Lobatto integrator)

$$\left.\begin{aligned}
&\sum_{j=1}^{n_D} \frac{\mathrm{d}u_j}{\mathrm{d}t} \langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^D \rangle = -\sum_{j=1}^{n_D} F_j \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h \times \boldsymbol{\epsilon}_j^D \rangle + \sum_{j=1}^{n_S} (K_j + g h_j) \langle \nabla \cdot \boldsymbol{\epsilon}_i^D, \epsilon_j^S \rangle, \quad i = 1, \ldots, n_D \\
&\sum_{j=1}^{n_S} \frac{\mathrm{d}h_j}{\mathrm{d}t} \langle \epsilon_i^S, \epsilon_j^D \rangle = -\sum_{j=1}^{n_D} F_j \langle \epsilon_i^S, \nabla \cdot \boldsymbol{\epsilon}_j^D \rangle, \qquad\qquad\qquad\qquad\qquad\qquad i = 1, \ldots, n_S
\end{aligned}\right\} \textit{Prognostic}\text{ equations}$$

Temporal discretization shallow waters

Midpoint rule time integrator (lowest order symplectic Gauss-Lobatto integrator)

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&\sum_{j=1}^{n_D}\frac{\mathrm{d}u_j}{\mathrm{d}t}\langle\boldsymbol{\epsilon}_i^D,\boldsymbol{\epsilon}_j^D\rangle = -\sum_{j=1}^{n_D}F_j\langle\boldsymbol{\epsilon}_j^D,\boldsymbol{q}_h\times\boldsymbol{\epsilon}_j^D\rangle + \sum_{j=1}^{n_S}(K_j+gh_j)\,\langle\nabla\cdot\boldsymbol{\epsilon}_i^D,\epsilon_j^S\rangle, \quad i=1,\dots,n_D\\
&\sum_{j=1}^{n_S}\frac{\mathrm{d}h_j}{\mathrm{d}t}\langle\epsilon_i^S,\epsilon_j^D\rangle = -\sum_{j=1}^{n_D}F_j\langle\epsilon_i^S,\nabla\cdot\boldsymbol{\epsilon}_j^D\rangle, \quad i=1,\dots,n_S
\end{aligned}\right\} \text{ \textit{Prognostic} equations}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x(t),t)$$

Integral form

$$x(t+\Delta t) - x(t) = \int_t^{t+\Delta t} f(x(t),t)\,\mathrm{d}t$$

Temporal discretization shallow waters

Midpoint rule time integrator (lowest order symplectic Gauss-Lobatto integrator)

$$\sum_{j=1}^{n_D} \frac{\mathrm{d}u_j}{\mathrm{d}t} \langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^D \rangle = -\sum_{j=1}^{n_D} F_j \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h \times \boldsymbol{\epsilon}_j^D \rangle + \sum_{j=1}^{n_S} (K_j + g h_j) \langle \nabla \cdot \boldsymbol{\epsilon}_i^D, \epsilon_j^S \rangle, \quad i = 1, \ldots, n_D$$

$$\sum_{j=1}^{n_S} \frac{\mathrm{d}h_j}{\mathrm{d}t} \langle \epsilon_i^S, \epsilon_j^D \rangle = -\sum_{j=1}^{n_D} F_j \langle \epsilon_i^S, \nabla \cdot \boldsymbol{\epsilon}_j^D \rangle, \quad i = 1, \ldots, n_S$$

Prognostic equations

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x(t), t)$$

Integral form

$$x(t + \Delta t) - x(t) = \int_t^{t+\Delta t} f(x(t), t)\,\mathrm{d}t$$

Midpoint rule

$$x(t + \Delta t) - x(t) = f(x(t + \frac{\Delta t}{2}), t + \frac{\Delta t}{2})\Delta t$$

First approximation

Temporal discretization shallow waters

Midpoint rule time integrator (lowest order symplectic Gauss-Lobatto integrator)

$$\left.\begin{aligned}
&\sum_{j=1}^{n_D}\frac{\mathrm{d}u_j}{\mathrm{d}t}\langle\boldsymbol{\epsilon}_i^D,\boldsymbol{\epsilon}_j^D\rangle = -\sum_{j=1}^{n_D}F_j\langle\boldsymbol{\epsilon}_j^D,\boldsymbol{q}_h\times\boldsymbol{\epsilon}_j^D\rangle+\sum_{j=1}^{n_S}(K_j+gh_j)\,\langle\nabla\cdot\boldsymbol{\epsilon}_i^D,\epsilon_j^S\rangle, \quad && i=1,\dots,n_D\\
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\end{aligned}\right\}$$

Prognostic equations

$$\frac{\mathrm{d}x}{\mathrm{d}t}=f(x(t),t)$$

Integral form

$$x(t+\Delta t)-x(t)=\int_t^{t+\Delta t}f(x(t),t)\,\mathrm{d}t$$

Midpoint rule

$$x(t+\Delta t)-x(t)=f(\frac{x(t)+x(t+\Delta t)}{2},t+\frac{\Delta t}{2})\Delta t$$

Second approximation

Temporal discretization shallow waters

Midpoint rule time integrator (lowest order symplectic Gauss-Lobatto integrator)

$$\sum_{j=1}^{n_D}\left(u_j^{k+1}-u_j^k\right)\langle\boldsymbol{\epsilon}_i^D,\boldsymbol{\epsilon}_j^D\rangle = -\sum_{j=1}^{n_D}F_j^{k+\frac{1}{2}}\langle\boldsymbol{\epsilon}_j^D,\boldsymbol{q}_h^{k+\frac{1}{2}}\times\boldsymbol{\epsilon}_j^D\rangle+\sum_{j=1}^{n_S}\left(K_j^{k+\frac{1}{2}}+g\frac{h_j^{k+1}+h_j^k}{2}\right)\langle\nabla\cdot\boldsymbol{\epsilon}_i^D,\epsilon_j^S\rangle,\quad i=1,\dots,n_D$$

$$\sum_{j=1}^{n_S}\left(h_j^{k+1}-h_j^k\right)\langle\epsilon_i^S,\epsilon_j^D\rangle = -\sum_{j=1}^{n_D}F_j^{k+\frac{1}{2}}\langle\epsilon_i^S,\nabla\cdot\boldsymbol{\epsilon}_j^D\rangle,\qquad i=1,\dots,n_S$$

Prognostic equations

$$\frac{\mathrm{d}x}{\mathrm{d}t}=f(x(t),t)$$

Integral form

$$x(t+\Delta t)-x(t)=\int_t^{t+\Delta t}f(x(t),t)\,\mathrm{d}t$$

Midpoint rule

$$x(t+\Delta t)-x(t)=f\left(\frac{x(t)+x(t+\Delta t)}{2},t+\frac{\Delta t}{2}\right)\Delta t$$

Second approximation

Temporal discretization shallow waters

Midpoint rule time integrator (lowest order symplectic Gauss-Lobatto integrator)

$$\sum_{j=1}^{n_D} (u_j^{k+1} - u_j^k)\, \langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^D \rangle = -\sum_{j=1}^{n_D} F_j^{k+\frac{1}{2}} \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h^{k+\frac{1}{2}} \times \boldsymbol{\epsilon}_j^D \rangle + \sum_{j=1}^{n_S} \left(K_j^{k+\frac{1}{2}} + g \frac{h_j^{k+1} + h_j^k}{2} \right) \langle \nabla \cdot \boldsymbol{\epsilon}_i^D, \epsilon_j^S \rangle, \quad i = 1, \ldots, n_D$$

$$\sum_{j=1}^{n_S} (h_j^{k+1} - h_j^k)\, \langle \epsilon_i^S, \epsilon_j^D \rangle = -\sum_{j=1}^{n_D} F_j^{k+\frac{1}{2}} \langle \epsilon_i^S, \nabla \cdot \boldsymbol{\epsilon}_j^D \rangle, \qquad i = 1, \ldots, n_S$$

Prognostic equations

$$\sum_{i=1}^{n_D} F_j \langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^S \rangle = \langle \boldsymbol{\epsilon}_i^D, h_h \boldsymbol{u}_h \rangle, \qquad i = 1, \ldots, n_D$$

$$\sum_{j=1}^{n_S} K_j \langle \epsilon_i^S, \epsilon_j^S \rangle = \langle \epsilon_i^S, \frac{1}{2} \boldsymbol{u}_h \cdot \boldsymbol{u}_h \rangle, \qquad i = 1, \ldots, n_S$$

$$\sum_{j=1}^{n_G} q_j \langle \boldsymbol{\epsilon}_i^G, h_h \epsilon_j^G \rangle = \langle \nabla \times \boldsymbol{\epsilon}_i^G, \boldsymbol{u}_h \rangle, \qquad i = 1, \ldots, n_G$$

Diagnostic equations

Temporal discretization shallow waters

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Prognostic equations

$$\sum_{i=1}^{n_D} F_j^{k+\frac{1}{2}} \langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^S \rangle = \langle \boldsymbol{\epsilon}_i^D, \frac{h_h^{k+1}\boldsymbol{u}_h^{k+1} + h_h^k \boldsymbol{u}_h^k}{2} \rangle, \quad i = 1, \ldots, n_D$$

$$\sum_{j=1}^{n_S} K_j^{k+\frac{1}{2}} \langle \epsilon_i^S, \epsilon_j^S \rangle = \langle \epsilon_i^S, \frac{\boldsymbol{u}_h^{k+1} \cdot \boldsymbol{u}_h^{k+1} + \boldsymbol{u}_h^k \cdot \boldsymbol{u}_h^k}{4} \rangle, \quad i = 1, \ldots, n_S$$

$$\sum_{j=1}^{n_G} q_j^{k+\frac{1}{2}} \langle \boldsymbol{\epsilon}_i^G, \frac{h_h^{k+1} + h_h^k}{2} \epsilon_j^G \rangle = \langle \nabla \times \boldsymbol{\epsilon}_i^G, \frac{\boldsymbol{u}_h^{k+1} + \boldsymbol{u}_h^k}{2} \rangle, \quad i = 1, \ldots, n_G$$

Computed at fractional time steps

Diagnostic equations

Temporal discretization shallow waters

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$$\sum_{j=1}^{n_D}(u_j^{k+1}-u_j^k)\,\langle\boldsymbol{\epsilon}_i^D,\boldsymbol{\epsilon}_j^D\rangle = -\sum_{j=1}^{n_D}F_j^{k+\frac{1}{2}}\langle\boldsymbol{\epsilon}_j^D,\boldsymbol{q}_h^{k+\frac{1}{2}}\times\boldsymbol{\epsilon}_j^D\rangle + \sum_{j=1}^{n_S}\left(K_j^{k+\frac{1}{2}}+g\frac{h_j^{k+1}+h_j^k}{2}\right)\langle\nabla\cdot\boldsymbol{\epsilon}_i^D,\epsilon_j^S\rangle,\quad i=1,\dots,n_D$$

$$\sum_{j=1}^{n_S}(h_j^{k+1}-h_j^k)\,\langle\epsilon_i^S,\epsilon_j^D\rangle = -\sum_{j=1}^{n_D}F_j^{k+\frac{1}{2}}\langle\epsilon_i^S,\nabla\cdot\boldsymbol{\epsilon}_j^D\rangle,\quad i=1,\dots,n_S$$

Prognostic equations

$$\sum_{i=1}^{n_D}F_j^{k+\frac{1}{2}}\langle\boldsymbol{\epsilon}_i^D,\boldsymbol{\epsilon}_j^S\rangle = \langle\boldsymbol{\epsilon}_i^D,\frac{h_h^{k+1}\boldsymbol{u}_h^{k+1}+h_h^k\boldsymbol{u}_h^k}{2}\rangle,\quad i=1,\dots,n_D$$

$$\sum_{j=1}^{n_S}K_j^{k+\frac{1}{2}}\langle\epsilon_i^S,\epsilon_j^S\rangle = \langle\epsilon_i^S,\frac{\boldsymbol{u}_h^{k+1}\cdot\boldsymbol{u}_h^{k+1}+\boldsymbol{u}_h^k\cdot\boldsymbol{u}_h^k}{4}\rangle,\quad i=1,\dots,n_S$$

$$\sum_{j=1}^{n_G}q_j^{k+\frac{1}{2}}\langle\boldsymbol{\epsilon}_i^G,\frac{h_h^{k+1}+h_h^k}{2}\epsilon_j^G\rangle = \langle\nabla\times\boldsymbol{\epsilon}_i^G,\frac{\boldsymbol{u}_h^{k+1}+\boldsymbol{u}_h^k}{2}\rangle,\quad i=1,\dots,n_G$$

Computed at fractional time steps

Diagnostic equations

Problem: midpoint rule is a symplectic integrator ➡ **Only conserves quadratic invariants**

Only total mass and total potential vorticity (linear)

Temporal discretization shallow waters

Poisson time integrator

$$\sum_{j=1}^{n_D} (u_j^{k+1} - u_j^k) \langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^D \rangle = -\sum_{j=1}^{n_D} F_j^{k+\frac{1}{2}} \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h^{k+\frac{1}{2}} \times \boldsymbol{\epsilon}_j^D \rangle + \sum_{j=1}^{n_S} \left(K_j^{k+\frac{1}{2}} + g \frac{h_j^{k+1} + h_j^k}{2} \right) \langle \nabla \cdot \boldsymbol{\epsilon}_i^D, \epsilon_j^S \rangle, \quad i = 1, \ldots, n_D$$

$$\sum_{j=1}^{n_S} (h_j^{k+1} - h_j^k) \langle \epsilon_i^S, \epsilon_j^D \rangle = -\sum_{j=1}^{n_D} F_j^{k+\frac{1}{2}} \langle \epsilon_i^S, \nabla \cdot \boldsymbol{\epsilon}_j^D \rangle, \quad i = 1, \ldots, n_S$$

Prognostic equations

Same idea, but…

$$\sum_{i=1}^{n_D} F_j^{k+\frac{1}{2}} \langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^S \rangle = \langle \boldsymbol{\epsilon}_i^D, \frac{h_h^{k+1} \boldsymbol{u}_h^{k+1} + h_h^k \boldsymbol{u}_h^k}{2} \rangle, \quad i = 1, \ldots, n_D$$

$$\sum_{j=1}^{n_S} K_j^{k+\frac{1}{2}} \langle \epsilon_i^S, \epsilon_j^S \rangle = \langle \epsilon_i^S, \frac{\boldsymbol{u}_h^{k+1} \cdot \boldsymbol{u}_h^{k+1} + \boldsymbol{u}_h^k \cdot \boldsymbol{u}_h^k}{4} \rangle, \quad i = 1, \ldots, n_S$$

$$\sum_{j=1}^{n_G} q_j^{k+\frac{1}{2}} \langle \boldsymbol{\epsilon}_i^G, \frac{h_h^{k+1} + h_h^k}{2} \epsilon_j^G \rangle = \langle \nabla \times \boldsymbol{\epsilon}_i^G, \frac{\boldsymbol{u}_h^{k+1} + \boldsymbol{u}_h^k}{2} \rangle, \quad i = 1, \ldots, n_G$$

Computed at fractional time steps

Diagnostic equations

Temporal discretization shallow waters

Poisson time integrator

$$\sum_{j=1}^{n_D} \left(u_j^{k+1} - u_j^k\right) \langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^D \rangle = -\sum_{j=1}^{n_D} F_j^{k+\frac{1}{2}} \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h^{k+\frac{1}{2}} \times \boldsymbol{\epsilon}_j^D \rangle + \sum_{j=1}^{n_S} \left(K_j^{k+\frac{1}{2}} + g \frac{h_j^{k+1} + h_j^k}{2} \right) \langle \nabla \cdot \boldsymbol{\epsilon}_i^D, \epsilon_j^S \rangle, \quad i = 1, \dots, n_D$$

$$\sum_{j=1}^{n_S} \left(h_j^{k+1} - h_j^k\right) \langle \epsilon_i^S, \epsilon_j^D \rangle = -\sum_{j=1}^{n_D} F_j^{k+\frac{1}{2}} \langle \epsilon_i^S, \nabla \cdot \boldsymbol{\epsilon}_j^D \rangle, \quad i = 1, \dots, n_S$$

Prognostic equations

Same idea, but…

$$\sum_{i=1}^{n_D} F_j^{k+\frac{1}{2}} \langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^S \rangle = \langle \boldsymbol{\epsilon}_i^D, \boxed{\frac{h_h^{k+1}\boldsymbol{u}_h^{k+1} + h_h^k \boldsymbol{u}_h^k}{2}} \rangle, \quad i = 1, \dots, n_D$$

$$\sum_{j=1}^{n_S} K_j^{k+\frac{1}{2}} \langle \epsilon_i^S, \epsilon_j^S \rangle = \langle \epsilon_i^S, \boxed{\frac{\boldsymbol{u}_h^{k+1} \cdot \boldsymbol{u}_h^{k+1} + \boldsymbol{u}_h^k \cdot \boldsymbol{u}_h^k}{4}} \rangle, \quad i = 1, \dots, n_S$$

$$\sum_{j=1}^{n_G} q_j^{k+\frac{1}{2}} \langle \boldsymbol{\epsilon}_i^G, \frac{h_h^{k+1} + h_h^k}{2} \epsilon_j^G \rangle = \langle \nabla \times \boldsymbol{\epsilon}_i^G, \frac{\boldsymbol{u}_h^{k+1} + \boldsymbol{u}_h^k}{2} \rangle, \quad i = 1, \dots, n_G$$

Diagnostic equations

This is one way of approximating the value at the midpoint for nonlinear terms

It is not symmetric!

Temporal discretization shallow waters

Poisson time integrator

$$\sum_{j=1}^{n_D} (u_j^{k+1} - u_j^k) \langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^D \rangle = - \sum_{j=1}^{n_D} F_j^{k+\frac{1}{2}} \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h^{k+\frac{1}{2}} \times \boldsymbol{\epsilon}_j^D \rangle + \sum_{j=1}^{n_S} \left(K_j^{k+\frac{1}{2}} + g \frac{h_j^{k+1} + h_j^k}{2} \right) \langle \nabla \cdot \boldsymbol{\epsilon}_i^D, \epsilon_j^S \rangle, \quad i = 1, \ldots, n_D$$

$$\sum_{j=1}^{n_S} (h_j^{k+1} - h_j^k) \langle \epsilon_i^S, \epsilon_j^D \rangle = - \sum_{j=1}^{n_D} F_j^{k+\frac{1}{2}} \langle \epsilon_i^S, \nabla \cdot \boldsymbol{\epsilon}_j^D \rangle, \quad i = 1, \ldots, n_S$$

Prognostic equations

Same idea, but…

$$\sum_{i=1}^{n_D} F_j^{k+\frac{1}{2}} \langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^S \rangle = \langle \boldsymbol{\epsilon}_i^D, \boxed{\frac{h_h^{k+1} \boldsymbol{u}_h^{k+1} + h_h^k \boldsymbol{u}_h^k}{2}} \rangle, \quad i = 1, \ldots, n_D$$

$$\sum_{j=1}^{n_S} K_j^{k+\frac{1}{2}} \langle \epsilon_i^S, \epsilon_j^S \rangle = \langle \epsilon_i^S, \boxed{\frac{\boldsymbol{u}_h^{k+1} \cdot \boldsymbol{u}_h^{k+1} + \boldsymbol{u}_h^k \cdot \boldsymbol{u}_h^k}{4}} \rangle, \quad i = 1, \ldots, n_S$$

$$\sum_{j=1}^{n_G} q_j^{k+\frac{1}{2}} \langle \boldsymbol{\epsilon}_i^G, \frac{h_h^{k+1} + h_h^k}{2} \epsilon_j^G \rangle = \langle \nabla \times \boldsymbol{\epsilon}_i^G, \frac{\boldsymbol{u}_h^{k+1} + \boldsymbol{u}_h^k}{2} \rangle, \quad i = 1, \ldots, n_G$$

Diagnostic equations

$$\frac{\boldsymbol{u}_h^{k+1} \cdot \boldsymbol{u}_h^{k+1} + \boldsymbol{u}_h^k \cdot \boldsymbol{u}_h^k}{2} \longrightarrow \frac{\boldsymbol{u}_h^{k+1} \cdot \boldsymbol{u}_h^{k+1} + \boldsymbol{u}_h^k \cdot \boldsymbol{u}_h^k + \boldsymbol{u}_h^{k+1} \cdot \boldsymbol{u}_h^k}{3}$$

Temporal discretization shallow waters

Poisson time integrator

$$\sum_{j=1}^{n_D} (u_j^{k+1} - u_j^k)\,\langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^D \rangle = -\sum_{j=1}^{n_D} F_j^{k+\frac{1}{2}} \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h^{k+\frac{1}{2}} \times \boldsymbol{\epsilon}_j^D \rangle + \sum_{j=1}^{n_S} \left(K_j^{k+\frac{1}{2}} + g\,\frac{h_j^{k+1} + h_j^k}{2} \right) \langle \nabla \cdot \boldsymbol{\epsilon}_i^D, \epsilon_j^S \rangle, \quad i = 1, \ldots, n_D$$

$$\sum_{j=1}^{n_S} (h_j^{k+1} - h_j^k)\,\langle \epsilon_i^S, \epsilon_j^D \rangle = -\sum_{j=1}^{n_D} F_j^{k+\frac{1}{2}} \langle \epsilon_i^S, \nabla \cdot \boldsymbol{\epsilon}_j^D \rangle, \quad i = 1, \ldots, n_S$$

Prognostic equations

Same idea, but…

$$\sum_{i=1}^{n_D} F_j^{k+\frac{1}{2}} \langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^S \rangle = \langle \boldsymbol{\epsilon}_i^D, \boxed{\frac{h_h^{k+1}\boldsymbol{u}_h^{k+1} + h_h^k \boldsymbol{u}_h^k}{2}} \rangle, \quad i = 1, \ldots, n_D$$

$$\sum_{j=1}^{n_S} K_j^{k+\frac{1}{2}} \langle \epsilon_i^S, \epsilon_j^S \rangle = \langle \epsilon_i^S, \boxed{\frac{\boldsymbol{u}_h^{k+1} \cdot \boldsymbol{u}_h^{k+1} + \boldsymbol{u}_h^k \cdot \boldsymbol{u}_h^k}{4}} \rangle, \quad i = 1, \ldots, n_S$$

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Diagnostic equations

$$\frac{\boldsymbol{u}_h^{k+1} \cdot \boldsymbol{u}_h^{k+1} + \boldsymbol{u}_h^k \cdot \boldsymbol{u}_h^k}{2} \longrightarrow \frac{\boldsymbol{u}_h^{k+1} \cdot \boldsymbol{u}_h^{k+1} + \boldsymbol{u}_h^k \cdot \boldsymbol{u}_h^k + \boldsymbol{u}_h^{k+1} \cdot \boldsymbol{u}_h^k}{3}$$

$$\frac{\boldsymbol{h}_h^{k+1} \cdot \boldsymbol{u}_h^{k+1} + \boldsymbol{h}_h^k \cdot \boldsymbol{u}_h^k}{2} \longrightarrow \frac{\boldsymbol{h}_h^{k+1} \cdot \boldsymbol{u}_h^{k+1} + \boldsymbol{h}_h^k \cdot \boldsymbol{u}_h^k + \frac{1}{2} h_h^{k+1} \boldsymbol{u}_h^k + \frac{1}{2} h_h^k \boldsymbol{u}_h^{k+1}}{3}$$

Temporal discretization shallow waters

Poisson time integrator

$$\sum_{j=1}^{n_D} \left(u_j^{k+1} - u_j^k\right) \langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^D \rangle = -\sum_{j=1}^{n_D} F_j^{k+\frac{1}{2}} \langle \boldsymbol{\epsilon}_j^D, \boldsymbol{q}_h^{k+\frac{1}{2}} \times \boldsymbol{\epsilon}_j^D \rangle + \sum_{j=1}^{n_S} \left(K_j^{k+\frac{1}{2}} + g\frac{h_j^{k+1} + h_j^k}{2} \right) \langle \nabla \cdot \boldsymbol{\epsilon}_i^D, \epsilon_j^S \rangle, \quad i = 1, \ldots, n_D$$

$$\sum_{j=1}^{n_S} \left(h_j^{k+1} - h_j^k\right) \langle \epsilon_i^S, \epsilon_j^D \rangle = -\sum_{j=1}^{n_D} F_j^{k+\frac{1}{2}} \langle \epsilon_i^S, \nabla \cdot \boldsymbol{\epsilon}_j^D \rangle, \quad i = 1, \ldots, n_S$$

Prognostic equations

Same idea, but…

$$\sum_{i=1}^{n_D} F_j^{k+\frac{1}{2}} \langle \boldsymbol{\epsilon}_i^D, \boldsymbol{\epsilon}_j^S \rangle = \langle \boldsymbol{\epsilon}_i^D, \frac{\boldsymbol{h}_h^{k+1} \cdot \boldsymbol{u}_h^{k+1} + \boldsymbol{h}_h^k \cdot \boldsymbol{u}_h^k + \frac{1}{2} h_h^{k+1} \boldsymbol{u}_h^k + \frac{1}{2} h_h^k \boldsymbol{u}_h^{k+1}}{3} \rangle, \quad i = 1, \ldots, n_D$$

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$$\sum_{j=1}^{n_G} q_j^{k+\frac{1}{2}} \langle \boldsymbol{\epsilon}_i^G, \frac{h_h^{k+1} + h_h^k}{2} \epsilon_j^G \rangle = \langle \nabla \times \boldsymbol{\epsilon}_i^G, \frac{\boldsymbol{u}_h^{k+1} + \boldsymbol{u}_h^k}{2} \rangle, \quad i = 1, \ldots, n_G$$

Diagnostic equations

$$f[\boldsymbol{u}_h, h_h] \longrightarrow f^{k+\frac{1}{2}} = \int_0^1 f\left[\boldsymbol{u}_h^k + s\left(\boldsymbol{u}_h^{k+1} - \boldsymbol{u}_h^k\right), h_h^k + s\left(h_h^{k+1} - h_h^k\right)\right] \mathrm{d}s$$

General rule

Temporal discretization shallow waters

Poisson time integrator

$$\sum_{j=1}^{n_D}(u_j^{k+1}-u_j^k)\langle\boldsymbol{\epsilon}_i^D,\boldsymbol{\epsilon}_j^D\rangle=-\sum_{j=1}^{n_D}F_j^{k+\frac{1}{2}}\langle\boldsymbol{\epsilon}_j^D,\boldsymbol{q}_h^{k+\frac{1}{2}}\times\boldsymbol{\epsilon}_j^D\rangle+\sum_{j=1}^{n_S}\left(K_j^{k+\frac{1}{2}}+g\frac{h_j^{k+1}+h_j^k}{2}\right)\langle\nabla\cdot\boldsymbol{\epsilon}_i^D,\epsilon_j^S\rangle,\quad i=1,\ldots,n_D$$

$$\sum_{j=1}^{n_S}(h_j^{k+1}-h_j^k)\langle\epsilon_i^S,\epsilon_j^D\rangle=-\sum_{j=1}^{n_D}F_j^{k+\frac{1}{2}}\langle\epsilon_i^S,\nabla\cdot\boldsymbol{\epsilon}_j^D\rangle,\quad i=1,\ldots,n_S$$

Prognostic equations

Same idea, but…

$$\sum_{i=1}^{n_D}F_j^{k+\frac{1}{2}}\langle\boldsymbol{\epsilon}_i^D,\boldsymbol{\epsilon}_j^S\rangle=\langle\boldsymbol{\epsilon}_i^D,\frac{\boldsymbol{h}_h^{k+1}\cdot\boldsymbol{u}_h^{k+1}+\boldsymbol{h}_h^k\cdot\boldsymbol{u}_h^k+\frac{1}{2}h_h^{k+1}\boldsymbol{u}_h^k+\frac{1}{2}h_h^k\boldsymbol{u}_h^{k+1}}{3}\rangle,\quad i=1,\ldots,n_D$$

$$\sum_{j=1}^{n_S}K_j^{k+\frac{1}{2}}\langle\epsilon_i^S,\epsilon_j^S\rangle=\langle\epsilon_i^S,\frac{\boldsymbol{u}_h^{k+1}\cdot\boldsymbol{u}_h^{k+1}+\boldsymbol{u}_h^k\cdot\boldsymbol{u}_h^k+\boldsymbol{u}_h^{k+1}\cdot\boldsymbol{u}_h^k}{6}\rangle,\quad i=1,\ldots,n_S$$

$$\sum_{j=1}^{n_G}q_j^{k+\frac{1}{2}}\langle\boldsymbol{\epsilon}_i^G,\frac{h_h^{k+1}+h_h^k}{2}\epsilon_j^G\rangle=\langle\nabla\times\boldsymbol{\epsilon}_i^G,\frac{\boldsymbol{u}_h^{k+1}+\boldsymbol{u}_h^k}{2}\rangle,\quad i=1,\ldots,n_G$$

Diagnostic equations

Conserves cubic invariants!

Temporal discretization shallow waters

Key take-aways

1. Time integrators conserve only some of the invariants.

2. Quadratic invariants are conserved by midpoint rule integrator.

3. Higher order invariants require Poisson integrators.

Temporal discretization shallow waters

Key take-aways

1. Time integrators conserve only some of the invariants.
2. Quadratic invariants are conserved by midpoint rule integrator.
3. Higher order invariants require Poisson integrators.

How to implement this?

Hands on work

Implementation in Firedrake

Hands on work: Firedrake

Look at the Jupyter notebook

Thank you