

We write μ_δ to denote a Bernoulli distribution, where outcome 1 occurs with probability δ and 0 with probability $1 - \delta$ where $0 \leq \delta \leq 1$. Moreover, we use μ_δ^k to denote a probability distribution over k -tuples, where each bit of a k -tuple is drawn independently according to μ_δ . Finally, let $u \leftarrow \mu_\delta^k$ denote that a k -tuple u is chosen according to μ_δ^k .

The protocol execution between two probabilistic algorithms A and B is denoted by $\langle A, B \rangle$. The output of A in such a protocol execution is denoted by $\langle A, B \rangle_A$ and of B by $\langle A, B \rangle_B$. Finally, let $\langle A, B \rangle_{\text{trans}}$ denote the transcript of communication between $\langle A, B \rangle_{\text{trans}}$.

We define a *two phase algorithm* $A := (A_1, A_2)$ as an algorithm where in the first phase an algorithm A_1 is executed and in the second phase an algorithm A_2 .

We say that an event happens *almost surely* or with *high probability* if its has probability at least $1 - 2^{-n \text{poly}(n)}$.

Definition 1.1 (Dynamic weakly verifiable puzzle.) A *dynamic weakly verifiable puzzle* (DWVP) is defined by a probabilistic algorithm P called a *problem poser*. A *problem solver* $S := (S_1, S_2)$ for P is a probabilistic two phase algorithm. We write $P_n(\pi)$ to denote the execution of P with the randomness fixed to $\pi \in \{0, 1\}^n$, and $(S_1, S_2)(\rho)$ to denote the execution of both S_1 and S_2 with the randomness fixed to $\rho \in \{0, 1\}^*$.

In the first phase, the poser $P_n(\pi)$ and the solver $S_1(\rho)$ interact. As the result of the interaction $P_n(\pi)$ outputs a verification circuit Γ_V and a hint circuit Γ_H . The algorithm $S_1(\rho)$ produces no output. The circuit Γ_V takes as input $q \in Q$, an answer $y \in \{0, 1\}^*$, and outputs a bit. We say that an answer (q, y) is a *correct solution* if and only if $\Gamma_V(q, y) = 1$. The circuit Γ_H on input $q \in Q$ outputs a hint such that $\Gamma_V(q, \Gamma_H(q)) = 1$.

In the second phase, S_2 takes as input $x := \langle P_n(\pi), S_1(\rho) \rangle_{\text{trans}}$, and has oracle access to Γ_V and Γ_H . The execution of S_2 with the input x and the randomness fixed to ρ is denoted by $S_2(x, \rho)$. The queries of S_2 to Γ_V and Γ_H are called *verification queries* and *hint queries* respectively. The algorithm S_2 asks at most h hint queries, v verification queries, and succeeds if and only if it makes a verification query (q, y) such that $\Gamma_V(q, y) = 1$, and it has not previously asked for a hint query on q .

Definition 1.2 (k -wise direct-product of DWVPs.) Let $g : \{0, 1\}^k \rightarrow \{0, 1\}$ be a monotone function and $P^{(1)}$ a problem poser as in Definition 1.1. The k -wise direct product of $P^{(1)}$ is a DWVP defined by a probabilistic algorithm $P^{(g)}$. We write $P_{kn}^{(g)}(\pi^{(k)})$ to denote the execution of $P^{(g)}$ with the randomness fixed to $\pi^{(k)} := (\pi_1, \dots, \pi_k)$ where for each $1 \leq i \leq k : \pi_i \in \{0, 1\}^n$. Let $(S_1, S_2)(\rho)$ be a solver for $P^{(g)}$ as in Definition 1.1. In the first phase, the algorithm $S_1(\rho)$ sequentially interacts in k rounds with $P_{kn}^{(g)}(\pi^{(k)})$. In the i -th round $S_1(\rho)$ interacts with $P_n^{(1)}(\pi_i)$, and as the result $P_n^{(1)}(\pi_1)$ generates circuits Γ_V^i, Γ_H^i . Finally, after k rounds $P_{kn}^{(g)}(\pi^{(k)})$ outputs a verification circuit

$$\Gamma_V^{(g)}(q, y_1, \dots, y_k) := g(\Gamma_V^1(q, y_1), \dots, \Gamma_V^k(q, y_k))$$

and a hint circuit

$$\Gamma_H^{(k)}(q) := (\Gamma_H^1(q), \dots, \Gamma_H^k(q)).$$

If it is clear from a context we omit the parameter n and write $P(\pi)$ instead of $P_n(\pi)$ where $\pi \in \{0, 1\}^n$.

A verification query (q, y) of a solver S for which a hint query on this q has been asked before can not be a verification query that succeeds. Therefore, without loss of generality, we make the assumption that S does not ask verification queries on q for which a hint query has been asked before. Furthermore, we assume that once S asked a verification query that succeeds, it does not ask any further hint or verification queries.

Let C be a circuit that corresponds to a solver S as in Definition 1.1. Similarly as for a two phase algorithm, we write $C(\rho) := (C_1, C_2)(\rho)$ to denote that C in the first phase uses a circuit

C_1 and in the second phase a circuit C_2 . Additionally, the randomness in both phases is fixed to $\rho \in \{0, 1\}^*$.

Experiment $Success^{P,C}(\pi, \rho)$

Oracle: A problem poser P , a solver circuit $C = (C_1, C_2)$.

Input: Bitstrings $\pi \in \{0, 1\}^n$, $\rho \in \{0, 1\}^*$.

Output: A bit $b \in \{0, 1\}$.

run $\langle P(\pi), C_1(\rho) \rangle$

$(\Gamma_V, \Gamma_H) := \langle P(\pi), C_1(\rho) \rangle_P$

$x := \langle P(\pi), C_1(\rho) \rangle_{\text{trans}}$

run $C_2^{\Gamma_V, \Gamma_H}(x, \rho)$

if $C_2^{\Gamma_V, \Gamma_H}(x, \rho)$ asks a verification query (q, y) such that $\Gamma_V(q, y) = 1$ **then**

return 1

return 0

We define the *success probability* of C in solving a puzzle defined by P as

$$\Pr_{\pi, \rho}[Success^{P,C}(\pi, \rho) = 1]. \quad (0.0.1)$$

Furthermore, we say that C succeeds for π, ρ if $Success^{P,C}(\pi, \rho) = 1$.

Theorem 1.3 (Security amplification for a dynamic weakly verifiable puzzle.) *Let $P^{(1)}$ be a fixed problem poser as in Definition 1.1, and $P^{(g)}$ be a poser for the k -wise direct product of $P^{(1)}$. There exists a probabilistic algorithm Gen with oracle access to: a solver circuit C for $P^{(g)}$, a monotone function $g : \{0, 1\}^k \rightarrow \{0, 1\}$ and $P^{(1)}$. Additionally, Gen takes as input parameters ε, δ, n , the number of verification queries v and hint queries h asked by C , and outputs a solver circuit D for $P^{(1)}$ as in Definition 1.1 such that the following holds: If C is such that*

$$\Pr_{\substack{\pi^{(k)} \in \{0, 1\}^{kn} \\ \rho \in \{0, 1\}^*}} \left[Success^{P^{(g)}, C}(\pi^{(k)}, \rho) = 1 \right] \geq 8(h + v) \left(\Pr_{u \leftarrow \mu_\delta^k} [g(u) = 1] + \varepsilon \right)$$

then D satisfies almost surely

$$\Pr_{\substack{\pi \in \{0, 1\}^n \\ \rho \in \{0, 1\}^*}} \left[Success^{P^{(1)}, D}(\pi, \rho) = 1 \right] \geq (\delta + \frac{\varepsilon}{6k}).$$

Additionally, D requires oracle access to $g, P^{(1)}, C$, and asks at most $\frac{6k}{\varepsilon} \log(\frac{6k}{\varepsilon}) h$ hint queries and one verification query. Finally, $Size(D) \leq Size(C) \cdot \frac{6k}{\varepsilon}$ and $Time(Gen) = poly(k, \frac{1}{\varepsilon}, n, v, h)$.

Let $hash : Q \rightarrow \{0, 1, \dots, 2(h + v) - 1\}$, the idea is to partition Q such that the set of preimages of 0 for $hash$ contains $q \in Q$ on which C is not allowed to ask hint queries, and the first successful verification query (q, y) of C is such that $hash(q) = 0$. Therefore, if C makes a verification query (q, y) such that $hash(q) = 0$, then we know that no hint query is ever asked on this q .

We denote the i -th query of C by q_i if it is a hint query, and by (q_i, y_i) if it is a verification query. We define now an experiment $CanonicalSuccess$ in which we partition Q using a function $hash$. We say that a solver circuit C *succeeds* in the experiment $CanonicalSuccess$ if it asks a successful verification query (q_j, y_j) such that $hash(q_j) = 0$, and no hint query q_i is asked before (q_j, y_j) such that $hash(q_i) = 0$.

Experiment $\text{CanonicalSuccess}^{P,C,\text{hash}}(\pi, \rho)$

Oracle: A problem poser P , a solver circuit $C = (C_1, C_2)$,
a function $\text{hash} : Q \rightarrow \{0, \dots, 2(h+v) - 1\}$.

Input: Bitstrings $\pi \in \{0, 1\}^n$, $\rho \in \{0, 1\}^*$.

Output: A bit $b \in \{0, 1\}$.

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run  $\langle P(\pi), C_1(\rho) \rangle$ 
       $(\Gamma_V, \Gamma_H) := \langle P(\pi), C_1(\rho) \rangle_P$ 
       $x := \langle P(\pi), C_1(\rho) \rangle_{\text{trans}}$ 

run  $C_2^{\Gamma_V, \Gamma_H}(x, \rho)$ 
      if  $C_2$  does not succeed for any verification query then
        return 0
      Let  $(q_j, y_j)$  be the first verification query of  $C_2$  such that  $\Gamma_v(q_j, y_j) = 1$ .

If  $(\forall i < j : \text{hash}(q_i) \neq 0)$  and  $(\text{hash}(q_j) = 0)$  then
  return 1
else
  return 0

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We define the *canonical success probability* of a solver C for P with respect to a function hash as

$$\Pr_{\pi, \rho}[\text{CanonicalSuccess}^{P,C,\text{hash}}(\pi, \rho) = 1]. \quad (0.0.2)$$

For fixed hash and a problem poser P a *canonical success* of C for π, ρ is a situation where $\text{CanonicalSuccess}^{P,C,\text{hash}}(\pi, \rho) = 1$.

We show that if a solver circuit C for $P^{(g)}$ often succeeds in the experiment *Success*, then it also often succeeds in the experiment *CanonicalSuccess*. Let \mathcal{H} be the family of pairwise independent functions $Q \rightarrow \{0, 1, \dots, 2(h+v) - 1\}$. We write $\text{hash} \leftarrow \mathcal{H}$ to denote that hash is chosen from \mathcal{H} uniformly at random.¹

Lemma 1.4 (Success probability in solving a k -wise direct product of $P^{(1)}$ with respect to a function hash .) For fixed P let C be a solver for P with the success probability at least γ , asking at most h hint queries and v verification queries. There exists a probabilistic algorithm *FindHash* that takes as input: parameters γ , n , the number of verification queries v and hint queries h , and has oracle access to C and P . Furthermore, *FindHash* runs in time $O((h+v)^4/\gamma^4)$, and with high probability outputs a function $\text{hash} \in \mathcal{H}$ such that the canonical success probability of C with respect to hash is at least $\frac{\gamma}{16(h+v)}$.

Proof. We fix a problem poser P and a solver C for P in the whole proof of Lemma 1.4. For all $m, n \in \{1, \dots, (h+v)\}$ and $k, l \in \{0, 1, \dots, 2(h+v) - 1\}$ by the pairwise independence property of \mathcal{H} , we have

$$\forall q_m, q_n \in Q, q_m \neq q_n : \Pr_{\text{hash} \leftarrow \mathcal{H}}[\text{hash}(q_m) = k \mid \text{hash}(q_n) = l] = \Pr_{\text{hash} \leftarrow \mathcal{H}}[\text{hash}(q_m) = k] = \frac{1}{2(h+v)}. \quad (0.0.3)$$

Let $\mathcal{P}_{\text{Success}}$ be a set containing all (π, ρ) for which $\text{Success}^{P,C}(\pi, \rho) = 1$. We fix $(\pi^*, \rho^*) \in \mathcal{P}_{\text{Success}}$, and are interested in the probability over a choice of function hash of the event

¹It is possible to implement a random function hash efficiently by for example building its function table on the fly.

$\text{CanonicalSuccess}^{P,C,\text{hash}}(\pi^*, \rho^*) = 1$. Let (q_j, y_j) denote the first query such that $\Gamma_V(q_j, y_j) = 1$. We have

$$\begin{aligned}
& \Pr_{\text{hash} \leftarrow \mathcal{H}} [\text{CanonicalSuccess}^{P,C,\text{hash}}(\pi^*, \rho^*) = 1] \\
&= \Pr_{\text{hash} \leftarrow \mathcal{H}} [\text{hash}(q_j) = 0 \wedge (\forall i < j : \text{hash}(q_i) \neq 0)] \\
&= \Pr_{\text{hash} \leftarrow \mathcal{H}} [\forall i < j : \text{hash}(q_i) \neq 0 \mid \text{hash}(q_j) = 0] \Pr_{\text{hash} \leftarrow \mathcal{H}} [\text{hash}(q_j) = 0] \\
&\stackrel{(0.0.3)}{=} \frac{1}{2(h+v)} \left(1 - \Pr_{\text{hash} \leftarrow \mathcal{H}} [\exists i < j : \text{hash}(q_i) = 0 \mid \text{hash}(q_j) = 0] \right) \\
&\stackrel{(0.0.3)}{=} \frac{1}{2(h+v)} \left(1 - \Pr_{\text{hash} \leftarrow \mathcal{H}} [\exists i < j : \text{hash}(q_i) = 0] \right) \\
&\stackrel{(\text{u.b.})}{\geq} \frac{1}{2(h+v)} \left(1 - \sum_{i < j} \Pr_{\text{hash} \leftarrow \mathcal{H}} [\text{hash}(q_i) = 0] \right) \\
&\stackrel{(0.0.3)}{\geq} \frac{1}{4(h+v)}. \tag{0.0.4}
\end{aligned}$$

We denote the set of those (π, ρ) for which $\text{CanonicalSuccess}^{P,C,\text{hash}}(\pi, \rho) = 1$ by $\mathcal{P}_{\text{Canonical}}$. If for π, ρ the circuit C succeeds canonically, then for the same π, ρ we also have $\text{Success}^{P,C}(\pi, \rho) = 1$. Hence, $\mathcal{P}_{\text{Canonical}} \subseteq \mathcal{P}_{\text{Success}}$, and we conclude

$$\begin{aligned}
& \Pr_{\substack{\text{hash} \leftarrow \mathcal{H} \\ \pi, \rho}} [\text{CanonicalSuccess}^{P,C,\text{hash}}(\pi, \rho) = 1] \\
&= \Pr_{\substack{\text{hash} \leftarrow \mathcal{H} \\ (\pi, \rho) \in \mathcal{P}_{\text{Success}}}} [\text{hash}(q_j) = 0 \wedge (\forall i < j : \text{hash}(q_i) \neq 0)] \\
&= \mathbb{E}_{(\pi, \rho) \in \mathcal{P}_{\text{Success}}} \left[\Pr_{\text{hash} \leftarrow \mathcal{H}} [\text{hash}(q_j) = 0 \wedge (\forall i < j : \text{hash}(q_i) \neq 0)] \right] \\
&\stackrel{(0.0.4)}{\geq} \frac{\gamma}{4(h+v)}. \tag{0.0.5}
\end{aligned}$$

Algorithm: FindHash(γ, n, h, v)

Oracle: A problem poser P , a solver circuit C for P .

Input: Parameters γ, n . The number of h hint and v verification queries.

Output: A function $\text{hash} : Q \rightarrow \{0, 1, \dots, 2(h+v) - 1\}$.

for $i = 1$ **to** $32(h+v)^2/\gamma^2$ **do:**

$\text{hash} \leftarrow \mathcal{H}$

$\text{count} := 0$

for $j := 1$ **to** $32(h+v)^2/\gamma^2$ **do:**

$\pi \xleftarrow{\$} \{0, 1\}^n$

$\rho \xleftarrow{\$} \{0, 1\}^*$

if $\text{CanonicalSuccess}^{P,C,\text{hash}}(\pi, \rho) = 1$ **then**

$\text{count} := \text{count} + 1$

if $\text{count} \geq \frac{\gamma}{12(h+v)} \frac{32(h+v)^2}{\gamma^2}$ **then**

return hash

return \perp

We show that **FindHash** chooses $hash \in \mathcal{H}$ such that the canonical success probability of C with respect to $hash$ is at least $\frac{\gamma}{16(h+v)}$ almost surely. Let \mathcal{H}_{Good} denote a family of functions $hash \in \mathcal{H}$ for which

$$\Pr_{\pi, \rho} [CanonicalSuccess^{P, C, hash}(\pi, \rho) = 1] \geq \frac{\gamma}{8(h+v)}, \quad (0.0.6)$$

and \mathcal{H}_{Bad} be the family of functions $hash \in \mathcal{H}$ such that

$$\Pr_{\pi, \rho} [CanonicalSuccess^{P, C, hash}(\pi, \rho) = 1] \leq \frac{\gamma}{16(h+v)}. \quad (0.0.7)$$

Let N denote the number of iterations of the inner loop of **FindHash**. For a fixed $hash$, we define independent, identically distributed binary random variables X_1, \dots, X_N such that

$$X_i = \begin{cases} 1 & \text{if in the } i\text{-th iteration of the inner loop } count \text{ is increased} \\ 0 & \text{otherwise.} \end{cases}$$

We show now that **FindHash** is unlikely to return $hash \in \mathcal{H}_{Bad}$. For $hash \in \mathcal{H}_{Bad}$ by (0.0.7) we have $\mathbb{E}_{\pi, \rho}[X_i] \leq \frac{\gamma}{16(h+v)}$. Therefore, for any fixed $hash \in \mathcal{H}_{Bad}$ using the Chernoff bound we get ²

$$\Pr_{\pi, \rho} \left[\frac{1}{N} \sum_{i=1}^N X_i \geq \frac{\gamma}{12(h+v)} \right] \leq \Pr_{\pi, \rho} \left[\frac{1}{N} \sum_{i=1}^N X_i \geq (1 + \frac{1}{3}) \mathbb{E}[X_i] \right] \leq e^{-\frac{\gamma}{16(h+v)} N/27} \leq e^{-\frac{2}{27} \frac{(h+v)}{\gamma}}.$$

The probability that $hash \in \mathcal{H}_{Good}$, when picked, is not returned amounts

$$\Pr_{\pi, \rho} \left[\frac{1}{N} \sum_{i=1}^N X_i \leq \frac{\gamma}{12(h+v)} \right] \leq \Pr_{\pi, \rho} \left[\frac{1}{N} \sum_{i=1}^N X_i \leq (1 - \frac{1}{3}) \mathbb{E}[X_i] \right] \leq e^{-\frac{\gamma}{8(h+v)} N/18} \leq e^{-\frac{2}{9} \frac{(h+v)}{\gamma}},$$

where we once more used the Chernoff bound. We show now that the probability of picking a $hash \in \mathcal{H}_{Good}$ is at least $\frac{\gamma}{8(h+v)}$. We prove this statement by contradiction. Let assume us that

$$\Pr_{hash \leftarrow \mathcal{H}} [hash \in \mathcal{H}_{Good}] < \frac{\gamma}{8(g+v)}. \quad (0.0.8)$$

We have

$$\begin{aligned} & \Pr_{\substack{hash \leftarrow \mathcal{H} \\ \pi, \rho}} [CanonicalSuccess^{P, C, hash}(\pi, \rho) = 1] \\ &= \Pr_{\substack{hash \leftarrow \mathcal{H} \\ \pi, \rho}} [CanonicalSuccess^{P, C, hash}(\pi, \rho) = 1 \mid hash \in \mathcal{H}_{Good}] \Pr_{hash \leftarrow \mathcal{H}} [hash \in \mathcal{H}_{Good}] \\ & \quad + \Pr_{\substack{hash \leftarrow \mathcal{H} \\ \pi, \rho}} [CanonicalSuccess^{P, C, hash}(\pi, \rho) = 1 \mid hash \notin \mathcal{H}_{Good}] \Pr_{hash \leftarrow \mathcal{H}} [hash \notin \mathcal{H}_{Good}] \\ &\leq \Pr_{hash \leftarrow \mathcal{H}} [hash \in \mathcal{H}_{Good}] + \Pr_{\substack{hash \leftarrow \mathcal{H} \\ \pi, \rho}} [CanonicalSuccess^{P, C, hash}(\pi, \rho) = 1 \mid hash \notin \mathcal{H}_{Good}] \\ &\stackrel{(0.0.6)}{<} \stackrel{(0.0.8)}{<} \frac{\gamma}{8(h+v)} + \frac{\gamma}{8(h+v)} = \frac{\gamma}{4(h+v)}, \end{aligned}$$

but this contradicts (0.0.5). Therefore, we know that the probability of choosing a $hash \in \mathcal{H}_{Good}$ amounts at least $\frac{\gamma}{8(h+v)}$ where the probability is taken over a choice of $hash$.

²For $X = \sum_{i=1}^N X_i$ and $0 < \delta \leq 1$ we use the Chernoff bounds in the form $\Pr[X \geq (1 + \delta)\mathbb{E}[X]] \leq e^{-\mathbb{E}[X]\delta^2/3}$ and $\Pr[X \leq (1 - \delta)\mathbb{E}[X]] \leq e^{-\mathbb{E}[X]\delta^2/2}$.

We show that *FindHash* picks in one of its iteration $hash \in \mathcal{H}_{Good}$ almost surely. Let K be the number of iterations of the outer loop of *FindHash* and Y_i be a random variable for the event that in the i -th iteration of the outer loop $hash \notin \mathcal{H}_{Good}$ is picked. We conclude using $\Pr_{hash \leftarrow \mathcal{H}}[hash \in \mathcal{H}_{Good}] \geq \frac{\gamma}{8(g+v)}$ and $K \leq \frac{32(h+v)^2}{\gamma^2}$ that

$$\Pr_{hash \leftarrow \mathcal{H}} \left[\bigcap_{1 \leq i \leq K} Y_i \right] \leq \left(1 - \frac{\gamma}{8(h+v)} \right)^K \leq e^{-\frac{\gamma}{8(h+v)}K} \leq e^{-\frac{4(h+v)}{\gamma}}. \quad \square$$

Circuit $\tilde{C}_2^{\Gamma_H, C_2, hash}(x, \rho)$

Oracle: A hint circuit Γ_H , a circuit C_2 ,
a function $hash : Q \rightarrow \{0, 1, \dots, 2(h+v) - 1\}$.

Input: Bitstrings $x \in \{0, 1\}^*$, $\rho \in \{0, 1\}^*$.

Output: A tuple (q, y) .

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run  $C_2^{(\cdot, \cdot)}(x, \rho)$ 
  if  $C_2^{(\cdot, \cdot)}(x, \rho)$  asks a hint query on  $q$  then
    if  $hash(q) = 0$  then
      return  $\perp$ 
    else
      answer the query of  $C_2^{(\cdot, \cdot)}(x, \rho)$  using  $\Gamma_H(q)$ 

  if  $C_2^{(\cdot, \cdot)}(x, \rho)$  asks a verification query  $(q, y)$  then
    if  $hash(q) = 0$  then
      return  $(q, y)$ 
    else
      answer the verification query of  $C_2^{(\cdot, \cdot)}(x, \rho)$  with 0

return  $\perp$ 

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Given $C = (C_1, C_2)$ we define a circuit $\tilde{C} = (C_1, \tilde{C}_2)$. Every hint query q asked by \tilde{C} is such that $hash(q) \neq 0$. Furthermore, \tilde{C} asks no verification queries, instead it returns \perp or (q, y) such that $hash(q) = 0$.

We say that for a fixed $\pi, \rho, hash$ the circuit \tilde{C} *succeeds* if for $x := \langle P(\pi), C_1(\rho) \rangle_{trans}$, $(\Gamma_V, \Gamma_H) := \langle P(\pi), C_1(\rho) \rangle_P$, we have

$$\Gamma_V(\tilde{C}_2^{\Gamma_H, C_2, hash}(x, \rho)) = 1.$$

Lemma 1.5 *For fixed P, C and $hash$ the following statement is true*

$$\Pr_{\pi, \rho}[CanonicalSuccess^{P, C, hash}(\pi, \rho) = 1] \leq \Pr_{\substack{\pi, \rho \\ x := \langle P(\pi), C_1(\rho) \rangle_{trans} \\ (\Gamma_V, \Gamma_H) := \langle P(\pi), C_1(\rho) \rangle_P}} [\Gamma_V(\tilde{C}_2^{\Gamma_H, C_2, hash}(x, \rho)) = 1]$$

Proof. If for some fixed π , ρ and $hash$ the circuit C succeeds canonically, then for the same π , ρ and $hash$ also \tilde{C} succeeds. Using this observation, we conclude that

$$\begin{aligned}
& \Pr_{\pi, \rho} \left[CanonicalSuccess^{P, C, hash}(\pi, \rho) = 1 \right] \\
&= \mathbb{E}_{\pi, \rho} \left[CanonicalSuccess^{P, C, hash}(\pi, \rho) = 1 \right] \\
&\leq \mathbb{E}_{\pi, \rho} \left[\Gamma_V(\tilde{C}_2^{\Gamma_H, C_2, hash}(x, \rho)) = 1 \right] \\
&\quad x := \langle P(\pi), C_1(\rho) \rangle_{trans} \\
&\quad (\Gamma_V, \Gamma_H) := \langle P(\pi), C_1(\rho) \rangle_P \\
&= \Pr_{\pi, \rho} \left[\Gamma_V(\tilde{C}_2^{\Gamma_H, C_2, hash}(x, \rho)) = 1 \right] \\
&\quad x := \langle P(\pi), C_1(\rho) \rangle_{trans} \\
&\quad (\Gamma_V, \Gamma_H) := \langle P(\pi), C_1(\rho) \rangle_P
\end{aligned}$$

□

Lemma 1.6 (Security amplification of a dynamic weakly verifiable puzzle with respect to hash.) For fixed $P^{(1)}$ there exists an algorithm Gen that takes as input parameters $\varepsilon, \delta, n, k$ has oracle access to $P^{(1)}, P^{(g)}, \tilde{C}$, $hash : Q \rightarrow \{0, 1, \dots, 2(h + v - 1)\}$, $g : \{0, 1\}^k \rightarrow \{0, 1\}$, and outputs a circuit $D := (D_1, D_2)$ such that the following holds: If $\tilde{C} := (C_1, \tilde{C}_2)$ with oracle access to a solver circuit $C := (C_1, C_2)$ for $P^{(g)}$, which asks at most h hint queries and v verification queries, is such that

$$\Pr_{\substack{\pi^{(k)} \in \{0, 1\}^{kn}, \rho \in \{0, 1\}^* \\ x := \langle P^{(g)}(\pi^{(k)}), C_1(\rho) \rangle_{trans} \\ (\Gamma_H^{(k)}, \Gamma_V^{(g)}) := \langle P^{(g)}(\pi^{(k)}), C_1(\rho) \rangle_{P^{(g)}}}} \left[\Gamma_V^{(g)}(\tilde{C}_2^{\Gamma_H^{(k)}, C_2, hash}(x, \rho)) = 1 \right] \geq \Pr_{u \leftarrow \mu_\delta^k} [g(u) = 1] + \varepsilon,$$

then D satisfies almost surely

$$\Pr_{\substack{\pi \in \{0, 1\}^n, \rho \in \{0, 1\}^* \\ x := \langle P^{(1)}(\pi), D_1^{P^{(1)}, \tilde{C}}(\rho) \rangle_{trans} \\ (\Gamma_H, \Gamma_V) := \langle P^{(1)}(\pi), D_1^{P^{(1)}, \tilde{C}}(\rho) \rangle_{P^{(1)}}}} \left[\Gamma_V(D_2^{\Gamma_V, \Gamma_H, P^{(1)}, \tilde{C}, hash, g}(x, \rho)) = 1 \right] \geq (\delta + \frac{\varepsilon}{6k}).$$

Furthermore, D has oracle access to hint and verification circuits, $P^{(1)}, \tilde{C}$, $hash$, g , and asks at most $\frac{6k}{\varepsilon} \log(\frac{6k}{\varepsilon}) h$ hint queries and no verification queries. Finally, $Size(D) \leq Size(C) \frac{6k}{\varepsilon}$ and $Time(Gen) = poly(k, \frac{1}{\varepsilon}, n, v, h)$.

Before proving Lemma 1.6 we define additional algorithms that are later used by Gen . First, we are interested in the probability that for $u \leftarrow \mu_\delta^k$ and a bit $b \in \{0, 1\}$ the function g with the first input bit set to b takes value 1. The estimate of this probability is calculated by the algorithm *EstimateFunctionProbability*.

Algorithm: EstimateFunctionProbability^g($b, k, \varepsilon, \delta$)

Oracle: A function $g : \{0, 1\}^k \rightarrow \{0, 1\}$.

Input: A bit $b \in \{0, 1\}$, parameters k, ε, δ .

Output: An estimate \tilde{g} of $\Pr_{u \leftarrow \mu_\delta^k} [g(b, u_2, \dots, u_k) = 1]$.

for $i := 1$ **to** $\frac{64^2}{\varepsilon^2} n$ **do:**

$u \leftarrow \mu_\delta^{(k)}$

$$g_i := g(b, u_2, \dots, u_k)$$

$$\textbf{return } \frac{\varepsilon^2}{64k^2n} \sum_{i=1}^{\frac{64k^2}{\varepsilon^2}n} g_i$$

Lemma 1.7 *The algorithm **EstimateFunctionProbability** ^{g} ($b, k, \varepsilon, \delta$) outputs an estimate \tilde{g} of $\Pr_{u \leftarrow \mu_\delta^k}[g(b, u_2, \dots, u_k) = 1]$ where $b \in \{0, 1\}$ such that $|\tilde{g} - \Pr_{u \leftarrow \mu_\delta^k}[g(b, u_2, \dots, u_k) = 1]| \leq \frac{\varepsilon}{8k}$ almost surely.*

Proof. We define independent, identically distributed binary random variables $K_1, K_2, \dots, K_{\frac{64k^2}{\varepsilon^2}n}$ such that for each $1 \leq i \leq \frac{64k^2}{\varepsilon^2}n$ the random variable K_i takes value g_i . We use the Chernoff bound to obtain ³

$$\Pr \left[\left| \left(\frac{\varepsilon^2}{64k^2n} \sum_{i=1}^{\frac{64k^2}{\varepsilon^2}n} K_i \right) - \mathbb{E}[K_i] \right| \geq \frac{\varepsilon}{8k} \right] \leq 2 \cdot e^{-n/3}. \quad \square$$

The next algorithm **EvaluatePuzzles** ^{$P^{(1)}, P^{(g)}, \tilde{C}, hash$} ($\pi^{(k)}, \rho$) evaluates which of k puzzles of the k -wise direct product defined by $P^{(g)}$ are solved successfully by $\tilde{C}(\rho) := (C_1, \tilde{C}_2)(\rho)$. To decide whether the i -th puzzle of the k -wise direct product is solved successfully we need to gain access to the verification oracle for the puzzle generated in the i -th round of the interaction between $P^{(g)}$ and \tilde{C} . Therefore, in the algorithm **EvaluatePuzzles**, we use $P^{(1)}$, and invoke it k times to simulate the interaction with $C_1(\rho)$. Let us introduce some additional notation. We denote by $\langle P^{(1)}(\pi_i), C_1(\rho) \rangle^i$ the execution of the i -th round of the simulation, and by $\langle P^{(1)}(\pi_i), C_1(\rho) \rangle_{P^{(1)}}^i$ the output of $P^{(1)}(\pi_i)$ in the i -th round. Furthermore, we write $\langle P^{(1)}(\pi_i), C_1(\rho) \rangle_{trans}^i$ to denote a transcript of communication in the i -th round.

Algorithm: EvaluatePuzzles ^{$P^{(1)}, P^{(g)}, \tilde{C}, hash$} ($\pi^{(k)}, \rho$)

Oracle: Problem posers $P^{(1)}, P^{(g)}$, a circuit $\tilde{C} = (C_1, \tilde{C}_2)$,
a function $hash : Q \rightarrow \{0, 1, \dots, 2(h + v) - 1\}$.

Input: Bitstrings $\pi^{(k)} \in \{0, 1\}^{kn}$, $\rho \in \{0, 1\}^*$.

Output: A tuple $(c_1, \dots, c_k) \in \{0, 1\}^k$.

for $i := 1$ **to** k **do:** //simulate k rounds of interaction

$(\Gamma_V^i, \Gamma_H^i) := \langle P^{(1)}(\pi_i), C_1(\rho) \rangle_{P^{(1)}}^i$

$x_i := \langle P^{(g)}(\pi^{(k)}), C_1(\rho) \rangle_{trans}^i$

$x := (x_1, \dots, x_k)$

$\Gamma_H^{(k)} := (\Gamma_H^1, \dots, \Gamma_H^k)$

$(q, y_1, \dots, y_k) := \tilde{C}_2^{\Gamma_H^{(k)}, C_2, hash}(x, \rho)$

$(c_1, \dots, c_k) := (\Gamma_V^1(q, y_1), \dots, \Gamma_V^k(q, y_k))$

return (c_1, \dots, c_k)

All puzzles used by the algorithm **EvaluatePuzzles** are generated internally. Thus, the algorithm can answer itself all queries of \tilde{C}_2 to the hint oracle.

For fixed $\pi^{(k)}, \rho$ let $(\Gamma_V^{(g)}, \Gamma_H^{(k)}) := \langle P^{(g)}(\pi^{(k)}), C_1(\rho) \rangle_{P^{(g)}}$ and $x := \langle P^{(g)}(\pi^{(k)}), C_1(\rho) \rangle_{trans}$. Additionally, we denote by (Γ_V^i, Γ_H^i) the verification and hint circuits generated by $P^{(1)}(\pi_i)$ in

³For independent Bernoulli distributed random variables X_1, \dots, X_n with $X := \sum_{i=1}^n X_i$ and $0 \leq \delta \leq 1$ we use the Chernoff bound in the form $\Pr[|X - \mathbb{E}[X]| \geq \delta \mathbb{E}[X]] \leq 2e^{-\mathbb{E}[X]\delta^2/3}$.

the i -th round of the simulated interaction with $C_1(\rho)$. Finally, for $(q, y_1, \dots, y_k) := \tilde{C}_2(x, \rho)$ we denote the output of $\Gamma_V^i(q, y_i)$ by c_i .

We are interested in the success probability of \tilde{C} with the bitstring π_1 fixed to π^* when the fact whether \tilde{C} succeeds in solving the first puzzle defined by $P^{(1)}(\pi_1)$ is neglected, and instead the bit $b \in \{0, 1\}$ is used as the input on the first position to g . More formally, we define the surplus $S_{\pi^*, b}$ as

$$S_{\pi^*, b} = \Pr_{\pi^{(k)}, \rho} [g(b, c_2, \dots, c_k) = 1 \mid \pi_1 = \pi^*] - \Pr_{u \leftarrow \mu^{(k)}} [g(b, u_2, \dots, u_k) = 1]. \quad (0.0.9)$$

The algorithm **EstimateSurplus** returns an estimate $\tilde{S}_{\pi^*, b}$ for $S_{\pi^*, b}$.

EstimateSurplus ^{$P^{(1)}, P^{(g)}, \tilde{C}, g, hash$} ($\pi^*, b, k, \varepsilon, \delta$)

Oracle: Problem posers $P^{(1)}, P^{(g)}$, a circuit \tilde{C} , a function $g : \{0, 1\}^k \rightarrow \{0, 1\}$
a function $hash : Q \rightarrow \{0, 1, \dots, 2(h + v) - 1\}$.

Input: A bistring $\pi^* \in \{0, 1\}^n$, a bit $b \in \{0, 1\}$, parameters k, ε, δ .

Output: An estimate $\tilde{S}_{\pi^*, b}$ for $S_{\pi^*, b}$.

$\tilde{g}_b := \mathbf{EstimateFunctionProbability}^g(b, k, \varepsilon, \delta)$

for $i := 1$ **to** $\frac{64k^2}{\varepsilon^2}n$ **do:**

$(\pi_2, \dots, \pi_k) \xleftarrow{\$} \{0, 1\}^{(k-1)n}$

$\rho \xleftarrow{\$} \{0, 1\}^*$

$(c_1, \dots, c_k) := \mathbf{EvaluatePuzzles}^{P^{(1)}, P^{(g)}, \tilde{C}, hash}(\pi^*, \pi_2, \dots, \pi_k, \rho)$

$\tilde{s}_{\pi^*, b}^i := g(b, c_2, \dots, c_k)$

return $\left(\frac{\varepsilon^2 n}{64k^2} \sum_{i=1}^{\frac{64k^2}{\varepsilon^2}n} \tilde{s}_{\pi^*, b}^i \right) - \tilde{g}_b$

Lemma 1.8 *The estimate $\tilde{S}_{\pi^*, b}$ returned by **EstimateSurplus** differs from $S_{\pi^*, b}$ by at most $\frac{\varepsilon}{4k}$ almost surely.*

Proof. We use the union bound and similar argument as in Lemma 1.7 which yields that

$\frac{\varepsilon^2}{16k^2n} \sum_{i=1}^{\frac{16k^2}{\varepsilon^2}n} \tilde{s}_{\pi^*, b}^i$ differs from $\mathbb{E}[g(b, c_2, \dots, c_k)]$ by at most $\frac{\varepsilon}{8k}$ almost surely. Together, with Lemma 1.7 we conclude that the surplus estimate returned by **EstimateSurplus** differs from $S_{\pi^*, b}$ by at most $\frac{\varepsilon}{4k}$ almost surely. \square

We define a circuit $C' = (C'_1, \tilde{C}'_2)$ build using the circuit C to which C' has oracle access. However, to make the notation easier we omit the oracle signature, and write C' instead of C^C .

Circuit $C'_1(\rho)$

Input: A bitstring $\rho \in \{0, 1\}^*$

Hard-coded: A bitstring $\pi^* \in \{0, 1\}^n$

Simulate $\langle P^{(1)}(\pi), C_1(\rho) \rangle^1$

Use $C_1(\rho)$ for remaining $k - 1$ rounds of interaction.

Circuit $\tilde{C}'_2(x^{(k-1)}, \rho)$

Oracle: A hint oracle $\Gamma_H^{(k-1)} := (\Gamma_H^2, \dots, \Gamma_H^k)$.

Input: A tuple $x^{(k-1)} := (x_2, \dots, x_k) \in \{0, 1\}^*$, a bitstring $\rho \in \{0, 1\}^*$

Hard-coded: A bitstring $\pi^* \in \{0, 1\}^n$

Simulate $\langle P^{(1)}(\pi^*), C_1(\rho) \rangle^1$
 $(\Gamma_H^*, \Gamma_V^*) := \langle P^{(1)}(\pi^*), C_1(\rho) \rangle_{P^{(1)}}^1$
 $x^* := \langle P^{(1)}(\pi^*), C_1(\rho) \rangle_{\text{trans}}^1$

Let $\Gamma_H^{(k)} := (\Gamma_H^*, \Gamma_H^2, \dots, \Gamma_H^k)$

Let $x^{(k)} := (x^*, x_2, \dots, x_k)$

$(q, y_1, \dots, y_k) := \tilde{C}_2^{\Gamma_H^{(k)}, \text{hash}}(x^{(k)}, \rho)$

return (q, y_2, \dots, y_k)

We are ready to define the circuit $D^{P^{(1)}, P^{(g)}, \tilde{C}, \text{hash}, g} = (D_1^{P^{(1)}, \tilde{C}}, D_2^{P^{(1)}, P^{(g)}, \text{hash}, g})$ and the algorithm *Gen*.

Circuit $D_1^{P^{(1)}, \tilde{C}}(r)$

Oracle: A circuit $\tilde{C} = (C_1, \tilde{C}_2)$, a poser $P^{(1)}$.

Input: A tuple $r := (\rho, \sigma)$ where $\rho \in \{0, 1\}^*$ and $\sigma \in \{0, 1\}^*$.

Interact with the problem poser $\langle P^{(1)}, C_1(\rho) \rangle^1$.

Circuit $D_2^{P^{(1)}, P^{(g)}, \tilde{C}, \text{hash}, g, \Gamma_V^*, \Gamma_H^*}(x^*, r)$

Oracle: A poser $P^{(1)}$, a solver circuit $\tilde{C} = (C_1, \tilde{C}_2)$,
functions $\text{hash} : Q \rightarrow \{0, 1, \dots, 2(h+v)-1\}$, $g : \{0, 1\}^k \rightarrow \{0, 1\}$,
verification and hint circuits Γ_V^*, Γ_H^* for $P^{(1)}$.

Input: Bitstrings $x^* \in \{0, 1\}^*$, $r := (\rho, \sigma)$ such that $\rho \in \{0, 1\}^*$ and $\sigma \in \{0, 1\}^*$

Output: A tuple (q, y^*) .

for at most $\frac{6k}{\epsilon} \log(\frac{6k}{\epsilon})$ iterations **do:**

$(\pi_2, \dots, \pi_k) \leftarrow$ read next $(k-1) \cdot n$ bits from ρ

for $i := 2$ **to** k **do:**

run $\langle P^{(1)}(\pi_i), C_1(\rho) \rangle^i$

$(\Gamma_V^i, \Gamma_H^i) := \langle P^{(1)}(\pi_i), C_1(\rho) \rangle_{P^{(1)}}^i$

$x_i := \langle P^{(1)}(\pi_i), C_1(\rho) \rangle_{\text{trans}}^i$

$\Gamma_V^{(g)}(q, y_1, \dots, y_k) := g(\Gamma_V^*(q, y_1), \Gamma_V^2(q, y_2), \dots, \Gamma_V^k(q, y_k))$

$\Gamma_H^{(k)}(q) := (\Gamma_H^*(q), \Gamma_H^2(q), \dots, \Gamma_H^k(q))$

$(q, y^*, y_2, \dots, y_k) := \tilde{C}_2^{\Gamma_H^{(k)}, \text{hash}}((x^*, x_2, \dots, x_k), \rho)$

$(c^*, c_2, \dots, c_k) := (\Gamma_V^*(q, y^*), \Gamma_V^2(q, y_2), \dots, \Gamma_V^k(q, y_k))$

if $g(1, c_2, \dots, c_k) = 1$ **and** $g(0, c_2, \dots, c_k) = 0$ **then**

return (q, y^*)

return \perp

Algorithm $Gen^{P^{(1)}, P^{(g)}, \tilde{C}, g, hash}(\varepsilon, \delta, n, v, h, k)$

Oracle: Posers $P^{(1)}, P^{(g)}$, circuit \tilde{C} , functions $g : \{0, 1\}^k \rightarrow \{0, 1\}$,
 $hash : Q \rightarrow \{0, 1, \dots, 2(h + v) - 1\}$.

Input: Parameters $\varepsilon, \delta, n, k$, the number of verification v and hint h queries.

Output: A circuit D .

```

for  $i := 1$  to  $\frac{6k}{\varepsilon}n$  do:
     $\pi^* \xleftarrow{\$} \{0, 1\}^n$ 
     $\tilde{S}_{\pi^*, 0} := \mathbf{EstimateSurplus}^{P^{(1)}, P^{(g)}, \tilde{C}, g, hash}(\pi^*, 0)$ 
     $\tilde{S}_{\pi^*, 1} := \mathbf{EstimateSurplus}^{P^{(1)}, P^{(g)}, \tilde{C}, g, hash}(\pi^*, 1)$ 
    if  $\exists b \in \{0, 1\} : \tilde{S}_{\pi^*, b} \geq (1 - \frac{3}{4k})\varepsilon$  then
        Let  $C'_1$  use  $C_1$  and have hard-coded  $\pi^*$ 
        Let  $\tilde{C}'_2$  use  $\tilde{C}_2$  and have hard-coded  $\pi^*$ .
         $\tilde{C}' := (C'_1, \tilde{C}'_2)$ 
         $g'(b_2, \dots, b_k) := g(b, b_2, \dots, b_k)$ 
        return  $Gen^{P^{(1)}, P^{(g)}, \tilde{C}', g', hash}(\varepsilon, \delta, n, v, h, k - 1)$ 
// all estimates are lower than  $(1 - \frac{3}{4k})\varepsilon$ 
return  $D^{P^{(1)}, P^{(g)}, \tilde{C}, hash, g}$ 

```

Proof (Lemma 1.6). First let us consider the case where $k = 1$. The function $g : \{0, 1\} \rightarrow \{0, 1\}$ is either the identity or a constant function. If g is the identity function, then the circuit D returned by Gen directly uses \tilde{C} to find a solution. From the assumptions of Lemma 1.6 we know that \tilde{C} succeeds with probability at least $\delta + \varepsilon$. Hence, D trivially satisfies the statement. If g is a constant function the statement is vacuously true.

The general case is more involved. We distinguish two possibilities. If Gen manages to find in one of the iterations π^* such that an estimate $\tilde{S}_{\pi^*, b} \geq (1 - \frac{3}{4k})\varepsilon$, then we define a new monotone function $g'(b_2, \dots, b_k) := g(b, b_2, \dots, b_k)$ and a circuit $\tilde{C}' = (C'_1, \tilde{C}'_2)$, where C'_1 first internally simulates the interaction between C_1 and $P^{(1)}(\pi^*)$, and use C_1 to interact with $P^{(g')}$. The circuit \tilde{C}'_2 uses \tilde{C}_2 , but returns a $(k - 1)$ -tuple. In this case the surplus estimate is greater or equal $1 - \frac{3}{4k}\varepsilon$, and using Lemma 1.8 we conclude that $S_{\pi^*, b} \geq \tilde{S}_{\pi^*, b} - \frac{\varepsilon}{4k} \geq 1 - \frac{\varepsilon}{k}$ almost surely. The circuit \tilde{C} succeeds in solving the $(k - 1)$ -wise direct product of puzzles with probability at least $\Pr_{u \leftarrow \mu_\delta^{k-1}}[g'(u_1, \dots, u_{k-1})] + \varepsilon$. We see that in this case \tilde{C}' satisfies the conditions of Lemma 1.6 for the $(k - 1)$ -wise direct product of puzzles, and we recurse using g' and \tilde{C}' .

If all estimates are less than $(1 - \frac{3}{4k})\varepsilon$, then intuitively C does not succeeds on the remaining $k - 1$ puzzles with the higher probability than an algorithm that solves each puzzle independently with probability δ . However, from the assumption we know that on all k puzzles the success probability of \tilde{C} is higher. Therefore, it is likely that the first puzzle is correctly solved unusual often. It remains to prove that this intuition is indeed correct. Let $\mathcal{G}_b := \{b_1, b_2, \dots, b_k : g(b, b_2, \dots, b_k) = 1\}$. We observe that

$$\begin{aligned}
 \Pr_{u \leftarrow \mu_\delta^k} [u \in G_b] &= \Pr_{u \leftarrow \mu_\delta^k} [g(b, u_2, \dots, u_k) = 1] \\
 \Pr_{\pi^{(k)}, \rho} [c \in G_b] &= \Pr_{\pi^{(k)}, \rho} [g(b, c_2, \dots, c_k) = 1].
 \end{aligned} \tag{0.0.10}$$

We fix π^* and use (0.0.9), (0.0.10) to obtain

$$\Pr_{u \leftarrow \mu_\delta^k} [u \in G_1] - \Pr_{u \leftarrow \mu_\delta^k} [u \in G_0] = \Pr_{\pi^{(k)}, \rho} [c \in G_1 \mid \pi_1 = \pi^*] - \Pr_{\pi^{(k)}, \rho} [c \in G_0 \mid \pi_1 = \pi^*] - (S_{\pi^*, 1} - S_{\pi^*, 0}) \tag{0.0.11}$$

Since g is monotone, we have that $\mathcal{G}_0 \subseteq \mathcal{G}_1$, and can write (0.0.11) as

$$\Pr_{u \leftarrow \mu_\delta^k} [u \in \mathcal{G}_1 \setminus \mathcal{G}_0] = \Pr_{\pi^{(k)}, \rho} [c \in \mathcal{G}_1 \setminus \mathcal{G}_0 \mid \pi_1 = \pi^*] - (S_{\pi^*, 1} - S_{\pi^*, 0}). \quad (0.0.12)$$

Still fixing $\pi_1 = \pi^*$ we multiply both sides of (0.0.12) by

$$\frac{\Pr_{\rho} [\Gamma_V(D_2(x, \rho)) = 1]}{x := \langle P^{(1)}(\pi^*), D_1(\rho) \rangle_{\text{trans}} \quad (\Gamma_V, \Gamma_H) := \langle P^{(1)}(\pi^*), D_1(\rho) \rangle_{P^{(1)}}} \Pr_{u \leftarrow \mu_\delta^k} [u \in \mathcal{G}_1 \setminus \mathcal{G}_0].$$

which yields

$$\begin{aligned} & \Pr_{\rho} [\Gamma_V(D_2(x, \rho)) = 1] \\ & \frac{x := \langle P^{(1)}(\pi^*), D_1(\rho) \rangle_{\text{trans}} \quad (\Gamma_V, \Gamma_H) := \langle P^{(1)}(\pi^*), D_1(\rho) \rangle_{P^{(1)}}}{= \Pr_{\rho} [\Gamma_V(D_2(x, \rho)) = 1] \Pr_{\pi^{(k)}, \rho} [c \in \mathcal{G}_1 \setminus \mathcal{G}_0 \mid \pi = \pi^*]} \frac{1}{\Pr_{u \leftarrow \mu_\delta^k} [u \in \mathcal{G}_1 \setminus \mathcal{G}_0]} \\ & - \Pr_{\rho} [\Gamma_V(D_2(x, \rho)) = 1] (S_{\pi^*, 1} - S_{\pi^*, 0}) \frac{1}{\Pr_{u \leftarrow \mu_\delta^k} [u \in \mathcal{G}_1 \setminus \mathcal{G}_0]} \end{aligned} \quad (0.0.13)$$

We make use of the fact that the event $c \in \mathcal{G}_1 \setminus \mathcal{G}_0$ implies $D_2(x^*, \rho) \neq \perp$, and write the first summand of (0.0.13) as

$$\begin{aligned} & \Pr_{\rho} [\Gamma_V(D_2(x, \rho)) = 1] \Pr_{\pi^{(k)}, \rho} [c \in \mathcal{G}_1 \setminus \mathcal{G}_0 \mid \pi_1 = \pi^*] \\ & \frac{x := \langle P^{(1)}(\pi^*), D_1(\rho) \rangle_{\text{trans}} \quad (\Gamma_V, \Gamma_H) := \langle P^{(1)}(\pi^*), D_1(\rho) \rangle_{P^{(1)}}}{= \Pr_{\rho} [D_2(x, \rho) \neq \perp] \Pr_{\pi^{(k)}, \rho} [c_1 = 1 \mid c \in \mathcal{G}_1 \setminus \mathcal{G}_0, \pi_1 = \pi^*]} \Pr_{\pi^{(k)}, \rho} [c \in \mathcal{G}_1 \setminus \mathcal{G}_0 \mid \pi_1 = \pi^*] \end{aligned} \quad (0.0.14)$$

Now we consider two cases: if $\Pr_{\pi^{(k)}, \rho} [c \in \mathcal{G}_1 \setminus \mathcal{G}_0 \mid \pi_1 = \pi^*] \leq \frac{\varepsilon}{6k}$ then

$$\Pr_{\pi^{(k)}} [c_1 = 1 \mid c \in \mathcal{G}_1 \setminus \mathcal{G}_0, \pi_1 = \pi^*] \Pr_{\pi^{(k)}} [c \in \mathcal{G}_1 \setminus \mathcal{G}_0 \mid \pi_1 = \pi^*] \leq \frac{\varepsilon}{6k}, \quad (0.0.15)$$

for $\Pr_{\pi^{(k)}, \rho} [c \in \mathcal{G}_1 \setminus \mathcal{G}_0] > \frac{\varepsilon}{6k}$ the circuit D_2 outputs \perp if and only if it fails in all $\frac{6k}{\varepsilon} \log(\frac{6k}{\varepsilon})$ iterations to find $\pi^{(k)}$ such that $g(1, c_2, \dots, c_k) = 1 \wedge g(0, c_2, \dots, c_k) = 0$ (i.e. in none of the iterations $c \in \mathcal{G}_1 \setminus \mathcal{G}_0$) which happens with probability

$$\Pr_{\rho} [D_2(x, \rho) = \perp] \leq (1 - \frac{\varepsilon}{6k})^{\frac{6k}{\varepsilon} \log(\frac{6k}{\varepsilon})} \leq \frac{\varepsilon}{6k}. \quad (0.0.16)$$

We conclude that in both cases:

$$\begin{aligned}
& \Pr_{\rho} [D_2(x, \rho) \neq \perp] \Pr_{\pi^{(k)}, \rho} [c_1 = 1 \mid c \in \mathcal{G}_1 \setminus \mathcal{G}_0, \pi_1 = \pi^*] \Pr_{\pi^{(k)}, \rho} [c \in \mathcal{G}_1 \setminus \mathcal{G}_0 \mid \pi_1 = \pi^*] \\
& \geq \Pr_{\pi^{(k)}, \rho} [c_1 = 1 \mid c \in \mathcal{G}_1 \setminus \mathcal{G}_0, \pi_1 = \pi^*] \Pr_{\pi^{(k)}, \rho} [c \in \mathcal{G}_1 \setminus \mathcal{G}_0 \mid \pi_1 = \pi^*] - \frac{\varepsilon}{6k} \\
& \geq \Pr_{\pi^{(k)}, \rho} [c_1 = 1 \wedge c \in \mathcal{G}_0 \setminus \mathcal{G}_1 \mid \pi_1 = \pi^*] - \frac{\varepsilon}{6k} \\
& = \Pr_{\pi^{(k)}, \rho} [g(c_1, c_2, \dots, c_k) = 1 \wedge g(0, c_2, \dots, c_k) = 0 \mid \pi_1 = \pi^*] - \frac{\varepsilon}{6k} \\
& = \Pr_{\pi^{(k)}, \rho} [g(c) = 1 \mid \pi_1 = \pi^*] - \Pr_{\pi^{(k)}, \rho} [c \in \mathcal{G}_0 \mid \pi_1 = \pi^*] - \frac{\varepsilon}{6k} \\
& \stackrel{(0.0.9)}{=} \Pr_{\pi^{(k)}, \rho} [g(c_1, c_2, \dots, c_k) = 1 \mid \pi_1 = \pi^*] - \Pr_{u \leftarrow \mu_{\delta}^{(k)}} [u \in \mathcal{G}_0] - S_{\pi^*, 0} - \frac{\varepsilon}{6k}. \quad (0.0.17)
\end{aligned}$$

For the second summand of (0.0.13) we show that if we do not recurse, then the majority of the estimates is low almost surely. Let us assume that

$$\Pr_{\pi, \rho} \left[\left(S_{\pi, 0} \leq (1 - \frac{1}{2k})\varepsilon \right) \wedge \left(S_{\pi, 1} \leq (1 - \frac{1}{2k})\varepsilon \right) \right] < 1 - \frac{\varepsilon}{6k}, \quad (0.0.18)$$

then the algorithm recurses almost surely. Therefore, under the assumption that *Gen* does not recurse, we have almost surely

$$\Pr_{\pi, \rho} \left[\left(S_{\pi, 0} \leq (1 - \frac{1}{2k})\varepsilon \right) \wedge \left(S_{\pi, 1} \leq (1 - \frac{1}{2k})\varepsilon \right) \right] \geq 1 - \frac{\varepsilon}{6k}. \quad (0.0.19)$$

Let us define a set

$$\mathcal{W} = \left\{ \pi : \left(S_{\pi, 0} \leq (1 - \frac{1}{2k})\varepsilon \right) \wedge \left(S_{\pi, 1} \leq (1 - \frac{1}{2k})\varepsilon \right) \right\} \quad (0.0.20)$$

and use \mathcal{W}^c to denote the complement of \mathcal{W} . We bound the second summand in (0.0.13)

$$\begin{aligned}
& \mathbb{E}_{\pi^*} [S_{\pi^*, 0} + \Pr_{\rho} [\Gamma_V(D_2^{\tilde{C}}(x, \rho)) = 1] (S_{\pi^*, 1} - S_{\pi^*, 0})] \\
& \quad \substack{x := \langle P^{(1)}(\pi^*), D_1(\rho) \rangle_{\text{trans}} \\ (\Gamma_V, \Gamma_H) := \langle P^{(1)}(\pi^*), D_1(\rho) \rangle_{P(1)}} \\
& = \mathbb{E}_{\pi^* \in \mathcal{W}^c} [S_{\pi^*, 0} + \Pr_{\rho} [\Gamma_V(D_2^{\tilde{C}}(x, \rho)) = 1] (S_{\pi^*, 1} - S_{\pi^*, 0})] \\
& \quad \substack{x := \langle P^{(1)}(\pi^*), D_1(\rho) \rangle_{\text{trans}} \\ (\Gamma_V, \Gamma_H) := \langle P^{(1)}(\pi^*), D_1(\rho) \rangle_{P(1)}} \\
& \quad + \mathbb{E}_{\pi^* \in \mathcal{W}} [S_{\pi^*, 0} + \Pr_{\rho} [\Gamma_V(D_2^{\tilde{C}}(x, \rho)) = 1] (S_{\pi^*, 1} - S_{\pi^*, 0})] \\
& \quad \substack{x := \langle P^{(1)}(\pi^*), D_1(\rho) \rangle_{\text{trans}} \\ (\Gamma_V, \Gamma_H) := \langle P^{(1)}(\pi^*), D_1(\rho) \rangle_{P(1)}} \\
& \leq \frac{\varepsilon}{6k} + \mathbb{E}_{\pi^* \in \mathcal{W}^c} [S_{\pi^*, 0} + \Pr_{\rho} [\Gamma_V(D_2^{\tilde{C}}(x, \rho)) = 1] ((1 - \frac{1}{2k})\varepsilon - S_{\pi^*, 0})] \\
& \quad \substack{x := \langle P^{(1)}(\pi^*), D_1(\rho) \rangle_{\text{trans}} \\ (\Gamma_V, \Gamma_H) := \langle P^{(1)}(\pi^*), D_1(\rho) \rangle_{P(1)}} \\
& \leq \frac{\varepsilon}{6k} + 1 - \frac{\varepsilon}{2k} = 1 - \frac{\varepsilon}{3k} \quad (0.0.21)
\end{aligned}$$

We observe that

$$\begin{aligned}
& \Pr_{u \leftarrow \mu_{\delta}^k} [g(u) = 1] = \Pr[u \in \mathcal{G}_0 \vee (u \in \mathcal{G}_1 \setminus \mathcal{G}_0 \wedge u_1 = 1)] \\
& = \Pr[u \in \mathcal{G}_0] + \Pr[u \in \mathcal{G}_1 \setminus \mathcal{G}_0] \Pr[u_1 = 1]. \quad (0.0.22)
\end{aligned}$$

Finally, we insert (0.0.17) and (0.0.21) into equation (0.0.13) and use (0.0.22) to obtain

$$\Pr_{\substack{\rho \\ x:=\langle P^{(1)}(\pi^*), D_1(\rho) \rangle_{\text{trans}} \\ (\Gamma_V, \Gamma_H) := \langle P^{(1)}(\pi^*), D_1(\rho) \rangle_{P^{(1)}}}} [\Gamma_V(D_2^{\tilde{C}}(x, \rho)) = 1] \geq \mathbb{E}_{\pi^*} \left[\frac{\Pr_{\pi^{(k)}}[g(c) = 1 \mid \pi_1 = \pi^*] - \Pr_{u \leftarrow \mu_\delta^k}[u \in G_0] - (1 - \frac{1}{6k})\varepsilon}{\Pr_{u \leftarrow \mu_\delta^k}[u \in \mathcal{G}_1 \setminus \mathcal{G}_0]} \right].$$

Using the assumptions of Lemma 1.6, we get

$$\begin{aligned} \Pr_{\substack{\rho \\ x:=\langle P^{(1)}(\pi^*), D_1(\rho) \rangle_{\text{trans}} \\ (\Gamma_V, \Gamma_H) := \langle P^{(1)}(\pi^*), D_1(\rho) \rangle_{P^{(1)}}}} [\Gamma_V(D_2^{\tilde{C}}(x, \rho)) = 1] &\geq \frac{\Pr_{u \leftarrow \mu_\delta^k}[g(u) = 1] + \varepsilon + \Pr_{u \leftarrow \mu_\delta^k}[u \in \mathcal{G}_0] - (1 - \frac{1}{6k})\varepsilon}{\Pr_{u \leftarrow \mu_\delta^k}[u \in \mathcal{G}_1 \setminus \mathcal{G}_0]} \\ &\geq \frac{\varepsilon + \delta \Pr_{u \leftarrow \mu_\delta^k}[u \in \mathcal{G}_1 \setminus \mathcal{G}_0] - (1 - \frac{1}{6k})\varepsilon}{\Pr_{u \leftarrow \mu_\delta^k}[u \in \mathcal{G}_1 \setminus \mathcal{G}_0]} \geq \delta + \frac{\varepsilon}{6k} \end{aligned} \quad (0.0.23) \quad \square$$

Proof (Theorem 1.3). We show that Theorem 1.3 follows from Lemmas 1.4 and 1.6. First given a solver circuit $C = (C_1, C_2)$ such that

$$\Pr_{\pi^{(k)}, \rho} \left[\text{Success}^{P^{(g)}, C}(\pi^{(k)}, \rho) = 1 \right] \geq 8(h + v) \left(\Pr_{u \leftarrow \mu_\delta^k}[g(u) = 1] + \varepsilon \right)$$

we apply Lemma 1.4 to find a function *hash* such that

$$\Pr_{\pi^{(k)}, \rho} \left[\text{CanonicalSuccess}^{P^{(g)}, C, \text{hash}}(\pi^{(k)}, \rho) = 1 \right] \geq \Pr_{u \leftarrow \mu_\delta^k}[g(u) = 1] + \varepsilon.$$

By Lemma 1.5 we know that it is possible to create a circuit $\tilde{C} = (C_1, \tilde{C}_2)$ with oracle access to *hash* and C such that

$$\Pr_{\substack{\pi, \rho \\ x:=\langle P(\pi), C_1(\rho) \rangle_{\text{trans}} \\ (\Gamma_V^{(g)}, \Gamma_H^{(k)}) := \langle P(\pi), C_1(\rho) \rangle_P}} [\Gamma_V^{(g)}(\tilde{C}_2^{\Gamma_H^{(k)}, C_2, \text{hash}}(x, \rho)) = 1] \geq \Pr_{u \leftarrow \mu_\delta^k}[g(u) = 1] + \varepsilon$$

Now, we apply Lemma 1.6 for the function *hash* and the circuit \tilde{C} and obtain a circuit D such that

$$\Pr_{\substack{\pi, \rho \\ x:=\langle P^{(1)}(\pi), D_1(\rho) \rangle_{\text{trans}} \\ (\Gamma_H, \Gamma_V) := \langle P^{(1)}(\pi), D_1(\rho) \rangle_{P^{(1)}}}} [\Gamma_V(D_2(x, \rho)) = 1] \geq (\delta + \frac{\varepsilon}{6k}). \quad (0.0.24)$$

Finally, we create a circuit \tilde{D} that first runs the circuit D , and then make a verification query using (q, y) returned by D . We know that probability that the verification query (q, y) succeeds amounts at least $(\delta + \frac{\varepsilon}{6k})$. Therefore, we have

$$\Pr_{\pi, \rho} \left[\text{Success}^{P^{(1)}, \tilde{D}}(\pi, \rho) = 1 \right] \geq (\delta + \frac{\varepsilon}{6k}). \quad \square$$