Design and Analysis of Algorithms
CS575 Spring 2025

Theory Assignment 1

Due on 2/27/2025 (Thursday)

Remember to include the following statement at the start of your answers with a signature by the side. “I have done this assignment completely on my own. I have not copied it, nor have I given my solution to anyone else. I understand that if I am involved in plagiarism or cheating I will have to sign an official form that I have cheated and that this form will be stored in my official university record. I also understand that I will receive a grade of 0 for the involved assignment for my first offense and that I will receive a grade of “F” for the course for any additional offense.”

Please handwrite or type your answer to each question, scan or save your answers into a pdf file (with a vertical orientation so that we do not need to rotate your file to grade), and upload it to the homework submission site.

1. [20 points] Fill in all the missing values. For column A you need to compute the sums. For column B (the last two rows) you need to guess a function that does not contradict any of the yes/no answers already in the next three columns. Fill in each empty entry in the last three columns with either a yes/no answer.

Function	Function	O	Ω	Θ
A	B	A = O(B)	A = Ω(B)	A = Θ(B)
n^4	$n^3 \lg n$			
$n\sqrt{n}$	n^2			
$(n+1)!$	$n!$			
$lg\ n$	n^k where $k > 0$			
$\sum_{i=1}^{n}(i+1) = ?$				yes
$\sum_{i=0}^{n-1} 3^i = ?$		no		

2. [20 points] Prove the following using the original definitions of O, Ω, θ, o, $and\ \omega$.

   (a) $4n^3 + 57n^2 + 4n - 9 \in O(n^3)$

   (b) $1007n^3 \in \Omega(n^2)$

   (c) $12n^3 + 5n^2 \in \omega(n^2)$

   (d) $35n^3 \in o(n^4)$

   (e) $n^2 + 3n - 10 \in \Theta(n^2)$

3. [15 points] Prove the following using limits.

   (a) $n^{1/n} \in \Theta(1)$ [Hint: you can use $x=e^{\ln x}$]

   (b) $3^n \in \omega(n^k)$

   (c) $lg^3 n \in o(n^{0.5})$

4. [10 points] Order the functions below by increasing growth rates (no justification required):

n^n, n, $n\ln n$, $n^{1/2}$, $2^{\lg n}$, $\ln n$, 10, $n^{1/n}$, $\sqrt{2}^{lgn}$, $n!$, $lg(n^{10})$, 2^n

Let $g_i(n)$ be the ith function from the left after the ordering (the leftmost function has the slowest growth rate). In the order, $g_i(n)$ should satisfy $g_i(n) \in O(g_{i+1}(n))$. If two or more functions are equivalent (in terms of Θ), put them in [ ] separated by comma (e.g., [n^2, $5n^2$]).

5. [20 points] Let $f(n)$ and $g(n)$ be asymptotically positive functions. For each of the following conjectures, either prove it is true or provide a counter example to show it is not true.

   a. $(f(n) + g(n)) \in \Theta(\max(f(n), g(n)))$.
   b. $f(n) \in O(g(n))$ implies $2^{f(n)} \in O(2^{g(n)})$.
   c. $f(n) \in O(g(n))$ implies $g(n) \in \Omega(f(n))$.
   d. $f(n) \in O(g(n))$ implies $\lg(f(n)) \in O(\lg(g(n)))$, where $\lg(g(n)) \geq 1$ and $f(n) \geq 1$ for sufficiently large n.

6. [10 points] Prove that for all integers n>0,

$$\left(\sum_{i=1}^{n} i\right)^2 = \sum_{i=1}^{n} i^3.$$

by mathematical induction. Divide your proof into the three required parts: Induction Base, Induction Hypothesis, and Induction Steps.

7. [15 points] Consider the following algorithm:

```
for(i = 2; i <= n; i++) {
    for(j = 0; j <= n) {
        cout << i << j;
        j = j + ⌊n/4⌋;
    }
}
```

(a) What is the output when n=4?
(b) What is the time complexity T(n). You may assume that n is divisible by 4.

8. [10 points] What is the time complexity T(n) of the nested loops below? For simplicity, you may assume that n is a power of 2. That is, $n = 2^k$ for some positive integer k. Give some justification for your answer.

```
for (i = 1; i <= n; i++){
    j = n;
    while (j >= 1){
        < body of the while loop>    //Needs Θ(1).
        j = ⌊j/2⌋;
    }
}
```