The Asymptotic Z-Transform as a Field Isomorphism

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Abstract

The Z-Transform, also known as the method of generating functions, is a complex valued power series that is used in a wide range of fields from signal processing to combinatorics. In signal processing, a Z-Transform takes an input sequence as a signal and transforms it into a complex-valued function. In this project, causal sequences are considered as inputs to the Z-Transform, which are transformed to analytic functions. This research aims to establish a field isomorphism between absolutely summable causal sequences, under the multiplication of convolution, and analytic functions. This isomorphism is then broadened to include all causal sequences which are mapped to meromorphic functions under the asymptotic Z-Transform.

Keywords: Z-Transform, Causal Sequences, Convolution

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1 Introduction

The Z-transform is a complex valued function that is used primarily in connecting discrete mathematics (particularly sequences) with continuous mathematics. The Z-transform, also known as the method of generating functions, has been used classically to solve difference equations and in combinatorics to solve enumeration problems. This paper aims to develop a sense of these classical uses of the Z-transform in Section 2. In Section 3, the Z-transform relates the algebra structures of sets of sequences (causal sequences are considered in this paper) to analytic functions centered at the origin. In Section 4, notions of convergence will be expanded to allow for divergent series to be included in the algebra structure.

2 Generating Functions

Generating functions were first used in 1940 to solve linear, constant-coefficient difference equations and were later used in signal processing to transform a sample sequence [2]. Generating functions are also a powerful tool used in combinatorics for certain enumeration problems. The purpose of using generating functions is to transform a sequence into a complex valued function. This allows the use known analytical methods to determine properties of interest in the sequence.
The next two sections of this introduction focus on the classical uses of generating functions in combinatorics and their use in solving difference equations.

2.1 Combinatorics

The method of generating functions is a technique used to solve combinatorial problems involving selection with repetition. Generating functions will be used to answer the following type of questions:

1. The number of ways to select r objects from a collection of n distinct types of objects.
2. The number of ways to distribute r identical objects into n distinct boxes.
3. The number of non-negative integer values solutions to

$$e_1 + e_2 + \cdots + e_n = r.$$

While question number 3. could be interpreted as a Number Theoretic question, [7] includes this in the section on generating functions.

Definition 2.1. A **generating function** for a sequence a_r, which represents the number of ways to select r objects in a certain procedure, is given by

$$g(x) = a_0 + a_1x + a_2x^2 + \cdots + a_rx^r + \cdots + a_nx^n$$

which could be finite or infinite.

The value of generating functions does not lie in computing values of $g(x)$, but rather in the known algebraic manipulations of functions that we can use to compute the coefficients of $g(x)$.

Example 1. (Modifying question 17 from [7] on page 246.)
Let r be the number of candy bars chosen from 9 different types of bars if each type of bar comes in packets of 4. Solve for $g_r(x)$, the corresponding generating function, and the coefficients of a_r for $r = 4, 24, 256$.

Let us create $g_r(x)$ using the above information and interpret this question as asking, "what are the non-negative integer valued solutions to $e_1 + e_2 + \cdots + e_9 = r$ with $e_i = 4$ and how many are there?" Then,

$$g_r(x) = (1 + x^4 + x^8 + x^{12} + \dots)^9.$$

This is verified by noting that each type of candy bar comes in packets of 4 so that means for each type of bar there are $0, 4, 8, 12, \dots$ bars. Since there are 9 types of bars, all following the same constraints, $g_r(x)$ is verified.
To solve for a_r, we employ methods from [7]. Note the following two equations.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots \tag{2.1}$$

$$\frac{1}{(1-x)^n} = 1 + \binom{1+n-1}{1}x + \binom{2+n-1}{2}x^2 + \cdots + \binom{r+n-1}{r}x^r + \dots \tag{2.2}$$

Making the substitution $y = x^4$ into $g_r(x)$ yields $g_r(x) = (1 + y + y^2 + y^3 + \dots)^9$. Now using Equation 2.1, $g_r(x) = \left(\frac{1}{1-y}\right)^9$. Thus by Equation 2.2 and back substitution,

$$g_r(x) = 1 + \binom{12}{4}x^4 + \binom{16}{8}x^8 + \cdots + \binom{r+9-1}{r}x^{4r} \dots .$$

Solving for a_4, a_{24}, a_{256} yields $a_4 = \binom{12}{4} = 495$, a_{24} is the coefficient of x^{96} which is $\binom{24+9-1}{24} = 10,518,300$, and finally $a_{256} = \binom{256+9-1}{256} = 525,783,425,977,953$, which is the coefficient of x^{1024}.
To interpret a_r in terms of the original problem: a_r is the number of ways to select r candy bars given the constraints of the problem, so there are 495 ways to select 4 candy bars, 10,518,300 ways to select 24 candy bars, and over 500 trillion ways to select 256 candy bars!

Combinatorics relies on the use of generating functions because of the power to obtain any desired solution to any problem surrounding the given model, i.e. the coefficients are built into the model for any arbitrary value. The next example dictates for an even more robust model.

Example 2. Exponential Generating Functions.
(Consider question 5 on page 266 from [7].)
Begin with the following definition:

Definition 2.2. An **exponential generating function** for a sequence a_r is given by the following power series expansion:

$$g(x) = a_0 + a_1 x + a_2\frac{x^2}{2!} + a_3\frac{x^3}{3!} + \cdots + a_r\frac{x^r}{r!} + \dots .$$

Similar to Example 1, a model is developed and then known methods for solving for various coefficients are employed.
An exponential generating function is obtained for the number of ways to deal a sequence of 13 cards (from a standard deck) if the suits are ignored (i.e. only the face value of the card is of value) in the following way:
Since the standard deck is used, it is assume that there are 4 of each type of card, with 13 total different types of cards. This yields:

$$g(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)^{13},$$

and the coefficient of $\frac{x^{13}}{13!}$ is desired. Obtained by Mathematica, there are $\frac{245,437,033}{5184}$ ways to deal a sequence of 13 cards.

The previous two examples illustrated using generating functions as a tool in the realm of combinatorics. Similar to their use in solving enumeration problems, generating functions can be used as a tool for solving linear, constant-coefficient difference equations.

2.2 Difference Equations

A difference equation, or recurrence relation, is a sequence of numbers in which each number is determined by previous values in the sequence. An example would be the sequence $\{a_n\}$ given by $a_n = 2a_{n-1}$ with the initial condition $a_0 = 1$. The question is: what is the general n^{th}-term? For this example, each new term is twice the previous term, therefore the explicit formula is given by $a_n = 2^n$. Generating functions are used as a powerful tool for solving more complicated difference equations, like the Fibonacci difference equation.

2.2.1 Fibonacci Sequences

Fibonacci sequences date back to the beginning of the thirteenth century. Leonardo Pisano, an Italian mathematician, is credited with introducing the original Fibonacci numbers in his book *Liber abaci*, which earned him the name "Fibonacci." The following problem from *Liber abaci* gives rise to the Fibonacci sequence:

> "A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair with from the second month on becomes productive?"

Define a to be the sequence representing the number of pairs of rabbits present inside the walls, then $a = (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots)$. The computation for the future terms of a becomes a difference equation of the form: $a_n = a_{n-1} + a_{n-2}$ with $a_1 = a_2 = 1$. This computation becomes cumbersome rather quickly; given a large value of n, it is not readily available to what a_n is. In this section of the paper, generating functions are used to determine a_n for a Generalized Fibonacci Sequence.

Theorem 2.1. *Generalized Fibonacci Sequences.*
Let $a = (a_0, a_1, a_2, a_3, a_4, a_5, \dots)$ where a_0 and a_1 are given and

$$a_n = \alpha a_{n-1} + \beta a_{n-2}$$

where $\alpha, \beta \in \mathbb{C}$. Then the closed form for the general a_n is

$$a_n = \frac{-c}{z_0^{n+1}} - \frac{d}{z_1^{n+1}} \tag{2.3}$$

where

$$c = \frac{z_0(\alpha a_0 - a_1) - a_0}{\beta(z_0 - z_1)}, \quad d = \frac{z_1(\alpha a_0 - a_1) - a_0}{\beta(z_1 - z_0)}, \quad z_0 = \frac{-\alpha + \sqrt{\alpha^2 + 4\beta}}{2\beta}, \quad z_1 = \frac{-\alpha - \sqrt{\alpha^2 + 4\beta}}{2\beta}.$$

Proof. Let $g(z) := \sum_{n=0}^{\infty} a_n z^n$ be the generating function for the sequence a, then:

$$\begin{aligned}
g(z) &= \sum_{n=0}^{\infty}(\alpha a_{n-1} + \beta a_{n-2})z^n \\
&= a_0 + a_1 z + \sum_{n=2}^{\infty} \alpha a_{n-1} z^n + \sum_{n=2}^{\infty} \beta a_{n-2} z^n \\
g(z) - a_0 - a_1 z &= \alpha z \sum_{n=2}^{\infty} a_{n-1} z^{n-1} + \beta z^2 \sum_{n=2}^{\infty} a_{n-2} z^{n-2} \\
&= \alpha z \sum_{m=1}^{\infty} a_m z^m + \beta z^2 \sum_{i=0}^{\infty} a_i z^i \\
&= \alpha z[g(z) - a_0] + \beta z^2 g(z) \\
\alpha a_0 z - a_0 - a_1 z &= \alpha z g(z) + \beta z^2 g(z) - g(z) \\
g(z) &= \frac{z(\alpha a_0 - a_1) - a_0}{\beta z^2 + \alpha z - 1}.
\end{aligned}$$

Factoring the denominator using the Quadratic Formula yields

$$z = \frac{-\alpha \pm \sqrt{\alpha^2 + 4\beta}}{2\beta}.$$

Defining

$$z_0 := \frac{-\alpha + \sqrt{\alpha^2 + 4\beta}}{2\beta} \quad \text{and} \quad z_1 := \frac{-\alpha - \sqrt{\alpha^2 + 4\beta}}{2\beta}$$

yields

$$g(z) = \frac{z(\alpha a_0 - a_1) - a_0}{(z - z_0)(z - z_1)},$$

using the method of partial fraction decomposition,

$$g(z) = \frac{z(\alpha a_0 - a_1) - a_0}{(z - z_0)(z - z_1)} = \frac{c}{(z - z_0)} + \frac{d}{(z - z_1)}.$$

Solving the above system gives:

$$c := \frac{z_0(\alpha a_0 - a_1) - a_0}{\beta(z_0 - z_1)} \quad \text{and} \quad d := \frac{z_1(\alpha a_0 - a_1) - a_0}{\beta(z_1 - z_0)}.$$

To compute the n^{th} term in our sequence, use the Geometric Series $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ as follows:

$$g(z) = \frac{\frac{-c}{z}}{1 - \frac{z}{z_0}} - \frac{\frac{d}{z}}{1 - \frac{z}{z_1}} = \frac{-c}{z_0} \sum_{n=1}^{\infty} \frac{z^n}{z_0^n} - \frac{d}{z_1} \sum_{n=1}^{\infty} \frac{z^n}{z_1^n} = \sum_{n=1}^{\infty} \left(\frac{-c}{z_0^{n+1}} - \frac{d}{z_1^{n+1}} \right) z^n.$$

It is now clear to see that

$$a_n = \frac{-c}{z_0^{n+1}} - \frac{d}{z_1^{n+1}}.$$

□

Theorem 2.1 gives a rather simple way to compute a_n in the Fibonacci sequence for all values of n. Consider the following two examples.

Example 3. The Rabbit Population.
In Fibonacci's original problem, $a_0 = a_1 = \alpha = \beta = 1$, thus by Theorem 2.1,

$$a_n = \frac{1}{\sqrt{5}}\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)^{n+1} - \Big(\frac{1-\sqrt{5}}{2}\Big)^{n+1}\Big], \tag{2.4}$$

which follows from computing and reduction through algebra.
To expand the original problem proposed by Fibonacci:
Suppose the produced rabbits never die and of interest is computing how long it would take for there to be enough rabbits to fill the observable universe. According to Wikipedia, the current size of the observable universe is 4.21×10^{32} cubic light years and each light year is 4.05×10^{107} cubic meters. Thus an estimate for the size of the universe is 1.705×10^{140} cubic meters. Assuming that each pair of rabbits needs only one cubic meter of space then we could fit 3.41×10^{140} rabbits in the universe. Now, we seek to find *when* this filling of the universe would occur. Using Equation 2.4 and noting that the second part of the equation is negligible for large values of n, we can use

$$\widetilde{a}_n = \frac{1}{\sqrt{5}}\Big(\frac{1+\sqrt{5}}{2}\Big)^{n+1}$$

as an approximation for a_n. To obtain a value for when the filling of the universe would occur, we must solve the following equation:

$$\widetilde{a}_n = 3.42 \times 10^{140} = \frac{1}{\sqrt{5}}\Big(\frac{1+\sqrt{5}}{2}\Big)^{n+1}.$$

Then,

$$\begin{aligned} \ln(3.42 \times 10^{140}) &= \ln\Big(\frac{1}{\sqrt{5}}\Big) + (n+1)\ln\Big(\frac{1+\sqrt{5}}{2}\Big) \\ n &= \frac{\ln(3.42 \times 10^{140}) - \ln\left(\frac{1}{\sqrt{5}}\right)}{\ln\left(\frac{1+\sqrt{5}}{2}\right)} - 1 \\ n &\approx 673.124 \quad \text{months.} \end{aligned}$$

Thus, if rabbits could live forever, it would only take 56.09 years to fill the observable universe with rabbits!

Example 4. Fibonacci Numbers, Lucas Numbers, and Nature.
As shown in the last example, there exists a formula for the general n^{th}-term in the original Fibonacci sequence. Another important set of numbers is the Lucas numbers [8]; this sequence is given by the same formula in Theorem 2.1 with $a_0 = \alpha = \beta = 1$ and $a_1 = 3$. Thus,

$$a_L = (1, 3, 4, 7, 11, 18, 29, 47, 76, \dots).$$

Using Theorem 2.1, the general n^{th}-term in a_L can be computed with the following equation:

$$a_{L_n} = \Big(\frac{1+\sqrt{5}}{2}\Big)^{n+1} + \Big(\frac{1-\sqrt{5}}{2}\Big)^{n+1}. \tag{2.5}$$

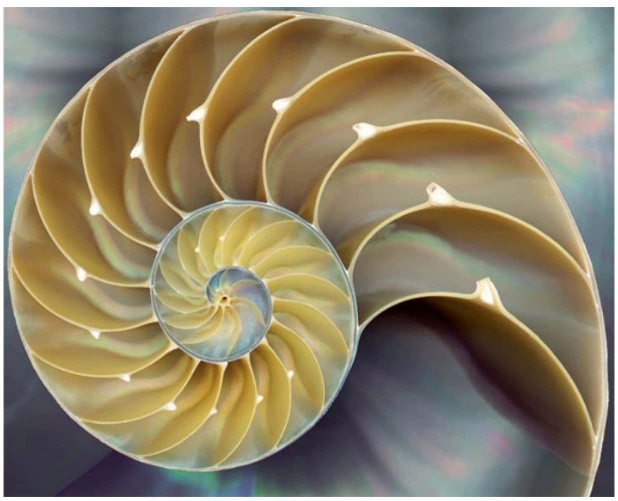
Both Lucas numbers and Fibonacci numbers appear throughout nature; this is no coincidence. Computing the ratio of successive values of a_{F_n} and a_{L_n}, we find that this ratio approaches a constant. For both sequences,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{(n+1)+1} \pm \left(\frac{1-\sqrt{5}}{2}\right)^{(n+1)+1}}{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} \pm \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}} = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{(n+1)+1}}{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}} = \left(\frac{1+\sqrt{5}}{2}\right) := \phi.$$

The constant ϕ is known as "The Golden Ratio" and appears in the shell of a nautilus, the shape of a hurricane, and the spiral of a galaxy. The Lucas and Fibonacci numbers appear in the number of petals on flowers, the number of spirals on a pine cone, and the number of branches on a tree (pictured below).

The Fibonacci numbers, Lucas numbers, and ϕ illustrate how our imperfect world does its best to imitate mathematical perfection as seen below.

2.2.2 Fibonacci Images



insteading.com

gofiguremath.org

wallpapercave.com

pinterest.com

3 The Z-Transform

The Z-Transform is a tool used in signal processing to transform a sequence into a complex-valued function (see Section 3.2.1). For this paper, the Z-Transform is used to perform a field isomorphism in the following section. This introduction will serve as a reminder for the rules and manipulations of power series and will also display some notable properties of the Z-Transform. To begin let us define the Z-Transform.

Definition 3.1. The **Z-Transform** of a sequence a is given by

$$Z(a) = \sum_{k=\infty}^{\infty} a_k z^{-k}.$$

This is, by definition, the *bilateral* Z-Transform; in this paper, we consider the *unilateral* Z-Transform, specifically for $k \geq 0$, which yields a new definition of the Z-Transform:

$$Z(a) = \sum_{k=0}^{\infty} a_k z^k. \tag{3.1}$$

Note that in Definition 3.1 $Z(a)$ is actually a *Laurent series* and by a simple substitution of $z := \frac{1}{z}$ equation 3.1 is obtained. This distinction between the bilateral and unilateral Z-Transform is crucial as this allows us to define a *Radius of Convergence* centered around $0 \in \mathbb{C}$. Consider the following theorem from [2].

Theorem 3.1. *Define the limit of a sequence a to be:*

$$L := \lim_{k\to\infty} \sqrt[k]{|a_k|}.$$

i. If $L = 0$, then $\sum_{k=0}^{\infty} a_k z^k$ converges absolutely for all z.

ii. If $L = \infty$, then $\sum_{k=0}^{\infty} a_k z^k$ diverges for all $z \neq 0$.

iii. If $0 < L < \infty$, then define $R := \frac{1}{L}$ to be the Radius of Convergence of $\sum_{k=0}^{\infty} a_k z^k$; thus $\sum_{k=0}^{\infty} a_k z^k$ converges absolutely for $|z| < R$ and diverges for $|z| > R$.

iv. If $f(z) = \sum_{k=0}^{\infty} a_k z^k$ converges absolutely for $|z| < R$, then $\sum_{k=1}^{\infty} k a_k z^{k-1}$ converges absolutely for $|z| < R$, f is differentiable, and $f'(z) := \sum_{k=1}^{\infty} k a_k z^{k-1}$ for all $|z| < R$.

Proof.

i. If $\lim_{k\to\infty} \sqrt[k]{|a_k|} = 0$, then for all $z \in \mathbb{C}$, $\lim_{k\to\infty} \sqrt[k]{|a_k|}|z| = 0$. Thus for any z there exists an N such that $\lim_{k\to\infty} \sqrt[k]{|a_k|}|z| \leq \frac{1}{2}$ for all $k > N$. Then $|a_k z^k| \leq \frac{1}{2^k}$ for all $k > N$. By the comparison test, $\sum_{k=0}^{\infty} a_k z^k$ converges absolutely for all $z \in \mathbb{C}$.

ii. If $\lim_{k\to\infty} \sqrt[k]{|a_k|} = \infty$, then for all $z \neq 0$, $|a_k|^{\frac{1}{k}} \geq \frac{1}{|z|}$ for infinitely many values of k. Then $|a_k z^k| \geq 1$ for infinitely many values of k and thus the sequence of partial sums of the series do not approach 0 and therefore the series diverges for all $z \neq 0$.

iii. Define $R := \frac{1}{L}$, where $0 < L < \infty$. Assume that $|z| < R$, then there exists $0 < \delta < \frac{1}{2}$ such that $|z| = R(1-2\delta) = \frac{1}{L}(1-2\delta)$. Then $\lim_{k\to\infty} \sqrt[k]{|a_k|}|z| = (1-2\delta)$, so $|a_k|^{\frac{1}{k}} < 1-\delta$ which implies $|a_k| \leq (1-\delta)^k$ for large enough k. Using the comparison test, $\sum_{k=0}^{\infty} a_k z^k$ converges absolutely. If $|z| > R$, then $L|z| > 1$ so $|a_k z^k| > 1$ for infinitely many values of k and thus $\sum_{k=0}^{\infty} a_k z^k$ diverges.

iv. Assume $R = \infty$. Then $f(z) = \sum_{k=0}^{\infty} a_k z^k$ converges for all $z \in \mathbb{C}$ and the derivative of f is given by,

$$\frac{f(z+h)-f(z)}{h} = \sum_{k=0}^{\infty} \frac{a_k[(z+h)^k - z^k]}{h} = \sum_{k=0}^{\infty} k a_k z^{k-1} + \sum_{k=2}^{\infty} a_k b_k \tag{3.2}$$

where

$$b_k = \frac{(z+h)^k - z^k}{h} - kz^{k-1} = \sum_{n=2}^{k} \binom{k}{n} h^{n-1} z^{k-n} \leq |h| \sum_{n=2}^{k} \binom{k}{n} |z|^{k-n} = |h|(|z|+1)^n \text{ for } |h| \leq 1.$$

Therefore, for $|h| \leq 1$, we have

$$\left|\frac{f(z+h)-f(z)}{h} - \sum_{k=0}^{\infty} k a_k z^{k-1}\right| \leq |h| \sum_{k=0}^{\infty} |a_k|(|z|+1)^k \leq A|h| \text{ for some } A < \infty,$$

since $\sum_{k=0}^{\infty} |a_k|(z+1)^k$ converges for all $z+1 > 0$. Thus if $h \to 0$, then $f'(z) = \sum_{k=0}^{\infty} k a_k z^{k-1}$.
Now assume $0 < R < \infty$. For $|z| < R$ choose $\delta > 0$ such that $|z| = R - 2\delta$. Let $|h| < \delta$. Then $|z+h| \leq |z| + |h| \leq R - 2\delta + \delta = R - \delta < R$, thus Equation 3.2 holds with

$$b_k = \sum_{n=2}^{k} \binom{k}{n} h^{n-1} z^{k-n}.$$

If $z = 0$ and $b_k = h^{k-1}$ the proof follows. Otherwise we can estimate b_k as follows: Note that

$$\binom{k}{n} \leq k^2 \binom{k}{n-2}, \quad \forall n \geq 2.$$

Thus, for $z \neq 0$,

$$|b_k| \leq \frac{k^2|h|}{|z|^2} \sum_{n=2}^{k} \binom{k}{n-2} |h|^{n-2} |z|^{k-(n-2)} \leq \frac{k^2|h|}{|z|^2} \sum_{j=0}^{k} \binom{k}{j} |h|^j |z|^{k-j}$$
$$= \frac{k^2|h|}{|z|^2}(|z|+|h|)^k \leq \frac{k^2|h|}{|z|^2}(R-\delta)^k$$

and

$$\left|\frac{f(z+h)-f(z)}{h} - \sum_{k=0}^{\infty} k a_k z^{k-1}\right| \leq \frac{|h|}{|z|^2} \sum_{k=0}^{\infty} k^2 |a_k| (R-\delta)^k \leq A|h|,$$

since $z \neq 0$ is fixed and $\sum_{k=0}^{\infty} k^2 |a_k| z^k$ also converges for $|z| < R$. Letting $h \to 0$ yields $f'(z) = \sum_{k=0}^{\infty} k a_k z^{k-1}$.

□

Having now established some fundamental properties of power series which will become important in the later part of this section, properties of the Z-Transform will be established.

Proposition 3.1. *The Z-Transform is:*

i. ***Linear:*** *Given sequences x_k and y_k with Z-Transforms $X(z)$ and $Y(z)$, respectively, and constants $\alpha, \beta \in \mathbb{C}$, then*

$$Z(\alpha x_k + \beta y_k) = \alpha X(z) + \beta Y(z).$$

ii. ***Time Invariant:*** *Given a sequence x_k with a corresponding Z-Transform $X(z)$, then*

$$Z(x_{k-n}) = z^n X(z).$$

Proof.

i. Let Z be the Z-Transform of $\alpha x_k + \beta y_k$. Thus $Z(\alpha x_k + \beta y_k) = \sum_{k=-\infty}^{\infty} (\alpha x_k + \beta y_k) z^k$. Then

$$\sum_{k=-\infty}^{\infty} (\alpha x_k + \beta y_k) z^k = \sum_{k=-\infty}^{\infty} \alpha x_k z^k + \sum_{k=-\infty}^{\infty} \beta y_k z^k = \alpha \sum_{k=-\infty}^{\infty} x_k z^k + \beta \sum_{k=-\infty}^{\infty} y_k z^k = \alpha X(z) + \beta Y(z).$$

ii. Let Z be the Z-Transform of x_{k-n}. Thus $Z(x_{k-n}) = \sum_{k=-\infty}^{\infty} x_{k-n} z^k$ by equation 3.1. Then,

$$\sum_{k=-\infty}^{\infty} x_{k-n} z^k = z^n \sum_{k=-\infty}^{\infty} x_{k-n} z^{k-n} = z^n \sum_{m=-\infty}^{\infty} x_m z^m = z^n X(z).$$

Note in the second to last step we used a change of variables with $m = n - k$.

□

Proposition 3.1 gives very useful facts about the Z-Transform that are applicable to the field of signal processing. For instance, if it is a Friday afternoon and one is eager to leave work but their boss insists that they transform a given signal for him that minute, you can reassure him that if his request is completed on Monday it will have the same output as if it were done right then and there (just cite Proposition 3.1 part ii.). They could also assure their boss that scaling a signal and then transforming it would result in the output being scaled too by part i. of Proposition 3.1. For more on this see Section 3.2, [2], and [3].

3.1 Field of Causal Sequences

As mentioned in the introduction to Section 3, the Z-Transform takes a sequence (or signal) as an input and returns a complex-valued function. For the rest of this paper, causal sequences are considered.

Definition 3.2. A **causal sequence** is a sequence that takes the following form:

$$a = (\ldots, 0, 0, a_k, a_{k+1}, a_{k+2}, a_{k+3}, \ldots),$$

where each $a_i \in \mathbb{C}$.

An example of such a sequence would be if a biologist decided to sample bacteria levels from a local lake; since there will be no recorded levels from before their initial sample, the samples they take will form a sequence preceded by zeros.

If for all $j < k$, $a_j = 0$, then call j the lower index of a and denote it by $ind_l(a)$. Likewise, if for all $k < m$, $a_m = 0$, then call m the upper index of a, denoted $ind_u(a)$. Note that $ind_u(a) = \infty$ is allowed if a is an infinite sequence.

The goal of this section is to create a field algebra structure using causal sequences. Addition and scalar multiplication is be accomplished accomplished point-wise. This immediately implies that

$$m := \{\text{all causal sequences}\}$$

is a vector space. The following section defines a multiplication on m.

3.1.1 Convolution

Multiplying two causal sequences together yields the convolution of those sequences.

Definition 3.3. The **convolution**, or Cauchy product, of two sequences a and b is denoted by $*$ and defined to be:

$$(a * b) := c_k = \sum_{i=-\infty}^{\infty} a_{k-i} b_i = \sum_{i+j=k} a_j b_i.$$

Appendix I in the back of this paper provides a Mathematica code for computing the convolution of two sequences. The following propositions yield important facts regrading convolution.

Proposition 3.2.

i. *If $a, b \in m$ with $k_{0_a} = ind_l(a)$ and $k_{0_b} = ind_l(b)$, then $ind_l(a*b) = ind_l(a)+ind_l(b) = k_{0_a}+k_{0_b}$.*

ii. *If $a, b \in m$ such that a and b are finite with $ind_u(a) < \infty$ and $ind_u(b) < \infty$, then $ind_u(a*b) = ind_u(a) + ind_u(b)$.*

Proof.

i. Applying the final equality of Definition 3.3, define $k := ind_l(c)$. In order to satisfy $i+j = k$, define $j := k_{0_a}$ and $i := k_{0_b}$ which completes the proof since all $a_j, b_i < 0$ for all $j < k_{0_a}, k_{0_b}$.

ii. Define $k := ind_u(c)$. To satisfy $i + j = k$ and since a and b are finite, there exists some $n, i \in \mathbb{N}$ with $a_{k_0+n} = ind_u(a)$ and $b_{k_0+i} = ind_u(b)$. Applying the last equality of Definition 3.3 implies that $ind_u(c) = k = a_{k_0+n} + b_{k_0+i} = ind_u(a) + ind_u(b)$.

□

Proposition 3.3. *Let $a, b \in m$ with $a = (\dots, 0, 0, a_{k_0}, a_{k_0+1}, a_{k_0+2}, \dots)$ and $b = (\dots, 0, 0, b_{k_0}, b_{k_0+1}, b_{k_0+2}, \dots)$, then*

$$(a * b) = (\dots, 0, 0, a_{k_0}b_{k_0}, a_{k_0}b_{k_0+1} + a_{k_0+1}b_{k_0}, a_{k_0}b_{k_0+2} + a_{k_0+1}b_{k_0+1} + a_{k_0+2}b_{k_0}, \dots) \tag{3.3}$$

with $a_{k_0}b_{k_0}$ in the $(k_0 + k_0)^{th}$ position. Also, if $(a * b) = 0$, then $a = 0$ or $b = 0$.

Proof. The proof follows directly from Definition 3.3. To prove the second part of the proposition, let $a \neq 0$ and $b \neq 0$, then there exists some $a_k \neq 0$ and $b_m \neq 0$ in the $(k+m)^{th}$ position by that is $\neq 0$ by equation 3.3, thus $(a * b) \neq 0$. □

Convolution arises naturally in probability theory during the summation of independent discrete random variables; consider some examples of such summations.

Example 5. A classic problem in an introductory probability course involves determining the probability distribution of the summation of two fair die. If we denote d_k to be the probability of rolling a k, the probability distribution of all possible outcomes can be represented as:

$$d = (\dots, 0, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, 0, \dots)$$

with our sequence beginning at d_1.
To determine the probability distribution of $d + d$, use the final equality in Definition 3.3 and Proposition 3.2. Clearly, the minimum sum of two fair die role is 2 (two 1's) and the maximum is 12 (two 6's); this is verified by using Proposition 3.2: $ind_l(d) = 1$ thus $ind_l(d*d) = ind_l(d)+ind_l(d) = 2$ and $ind_u(d) = 6$ so $ind_u(d*d) = ind_u(d)+ind_u(d) = 12$. Using Definition 3.3, compute the various values of $(d*d)_k$ by using the fact that $i + j = k$, thus $(d*d)_2 = \sum_{i+j=2} d_i d_j = d_1 d_1 = \frac{1}{36}$ which is the first term in the resulting sequence. The computation for next few terms is given below:

$$
\begin{aligned}
(d*d)_3 &= \sum_{i+j=3} d_i d_j = d_1 d_2 + d_2 d_1 = \frac{2}{36} \\
(d*d)_4 &= \sum_{i+j=4} d_i d_j = d_1 d_3 + d_2 d_2 + d_3 d_1 = \frac{3}{36} \\
(d*d)_5 &= \sum_{i+j=5} d_i d_j = d_1 d_4 + d_2 d_3 + d_3 d_2 + d_4 d_1 = \frac{4}{36} \\
&\vdots \\
(d*d)_{12} &= \sum_{i+j=12} d_i d_j = d_6 d_6 = \frac{1}{36}
\end{aligned}
$$

Thus the convolution of two random variables representing die rolls yields the random variable that represents the sum of the two rolls:

$$(d*d) = (\dots, 0, 0, \overbrace{\frac{1}{36}}^{(d*d)_1}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36}, 0, 0, \dots)$$

Example 6. Another problem that is often handled in an introductory probability is deriving the Poisson Distribution and some of its properties.
Recall that a Bernoulli Trial is an event with probability of success equal to p and probability of failure equal to $1 - p$. The Binomial Distribution results from taking n Bernoulli trials, each with probability of success equal to p, and has the following probability mass function (pmf):

$$P(K = k) = \binom{n}{k} p^k (1-p)^{n-k} \quad \text{for } k = 0, 1, 2, \dots, n.$$

Modeling the Binomial probability distribution as a causal sequence like so,

$$b(n,p) = (\dots, 0, 0, \binom{n}{0}(1-p)^n, \binom{n}{1} p(1-p)^{n-1}, \binom{n}{2} p^2 (1-p)^{n-2}, \dots, \binom{n}{n} p^n, 0, 0, \dots)$$

with $\binom{n}{0}(1-p)^n$ occurring where $P(k=0)$ (i.e. in the 0^{th} position). To get the Poisson Distribution, let $n \to \infty$ and $p \to 0$ at a constant rate, λ, such that $\lambda = np$. Some algebra yields that the pmf for a Poisson random variable is,

$$P(K=k) = \frac{\lambda^k}{k!}e^{-\lambda} \quad \text{for } \ k \in \{0,1,2,\dots\}$$

which modeled as a causal sequence of the form,

$$p(\lambda) = e^{-\lambda}(\dots,0,0,1,\lambda,\frac{\lambda^2}{2!},\frac{\lambda^3}{3!},\frac{\lambda^4}{4!},\dots).$$

It will now be shown that the multiplication of two Poisson random variables is equivalent to the convolution of those variables:

$$p(\lambda)*p(\mu) = \sum_{i=0}^{k} p(\lambda)_i p(\mu)_{k-i} = \sum_{i=0}^{k} e^{-\lambda}\frac{\lambda^i}{i!}e^{-\mu}\frac{\mu^{k-i}}{(k-i)!} = e^{-(\lambda+\mu)}\frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}\lambda^i\mu^{k-i}$$

$$= e^{-(\lambda+\mu)}\frac{1}{k!}(\lambda+\mu)^k = p(\lambda+\mu).$$

The previous two examples dealt with probability distributions so $\sum\limits_k a_k = 1$ and $a_k \geq 0$ for all $k \in \mathbb{Z}$ by definition. Let us define a norm on causal sequences and establish a triangle inequality.

Definition 3.4. Let $a \in m$ and denote the **1-norm** of an absolutely summable sequence to be

$$\|a\|_1 := \sum_{k=-\infty}^{\infty} |a_k|.$$

Proposition 3.4. *Triangle Inequality for 1-Norm on Causal Sequences.*
If $a, b \in m$ are absolutely summable, then $a*b$ is absolutely summable and

$$\|a*b\|_1 \leq \|a\|_1\|b\|_1 \tag{3.4}$$

where equality occurs if $a_k, b_k \geq 0$.

Proof. By Definition's 3.4 and 3.3, $\|a*b\|_1 = \sum\limits_{k=-\infty}^{\infty}|\sum\limits_{i+j=k} a_ib_j| \leq \sum\limits_{k=-\infty}^{\infty}\sum\limits_{i+j=k}|a_i||b_j| = \|a\|_1\|b\|_1$. Now, let a and b be sequences with $a_k, b_k \geq 0$ for all k. Define $\widetilde{a} := \frac{a}{\|a\|_1}$ and $\widetilde{b} := \frac{b}{\|b\|_1}$, then $\|\widetilde{a}\|_1 = 1$ and $\|\widetilde{b}\|_1 = 1$. Using Definition 3.3,

$$\|\widetilde{a}*\widetilde{b}\|_1 = \sum_{k=-\infty}^{\infty}\big(\sum_{i+j=k}\widetilde{a}_i\widetilde{b}_j\big) = \sum_{i=-\infty}^{\infty}\widetilde{a}_i\sum_{k=-\infty}^{\infty}\widetilde{b}_k = \sum_{k=-\infty}^{\infty}\sum_{i=-\infty}^{\infty}\widetilde{a}_{k-i}\widetilde{b}_i$$

$$= \sum_{i=-\infty}^{\infty}\sum_{k=-\infty}^{\infty}\widetilde{a}_{k-i}\widetilde{b}_i = \sum_{i=-\infty}^{\infty}\widetilde{b}_i\sum_{k=-\infty}^{\infty}\widetilde{a}_{k-i} = 1.$$

Hence, $\|\widetilde{a}*\widetilde{b}\|_1 = \|\widetilde{a}\|_1\|\widetilde{b}\|_1$ so $\|a*b\|_1 = \|a\|_1\|b\|_1$. ∎

The following proposition and theorem establishes m as a field with the operations of addition and convolution.

Proposition 3.5. *The operation of convolution is:*

i. commutative,

ii. associative,

iii. and distributive over addition.

Proof. (Also see [2] and [5])

i. By Definition 3.3,

$$c_k = (a * b)_k = \sum_{i=-\infty}^{\infty} a_{k-i} b_i \quad \text{letting } j = k - i \Rightarrow \quad \sum_{j=-\infty}^{\infty} a_j b_{k-j} = \sum_{j=-\infty}^{\infty} b_{k-j} a_j = (b * a)_k.$$

ii. Without loss of generality, it will be shown that $(a * b * c)_k = (a * (b * c))_k$

$$\begin{aligned}
(a * b * c)_k &= \sum_{l=0}^{k} \sum_{i+j=l} a_{k-i-j} b_i c_j = \sum_{l=0}^{k} \sum_{j=0}^{t} a_{k-l} b_{l-j} c_j \\
&= \sum_{t=0}^{k} a_{k-l} \sum_{j=0}^{t} b_{l-j} h_j = \sum_{l=0}^{k} a_{k-l} (b * c)_k \\
&= (a * (b * c))_k.
\end{aligned}$$

iii. Again, applying Definition 3.3,

$$(a * (b + c)_i)_k = \sum_{i=-\infty}^{\infty} (a_{k-i}(b_i + c_i)) = \sum_{i=-\infty}^{\infty} (a_{k-i} b_i + a_{k-i} c_i)$$

$$= \sum_{i=-\infty}^{\infty} a_{k-i} b_i + \sum_{i=-\infty}^{\infty} a_{k-i} c_i = (a * b)_k + (a * c)_k.$$

□

Theorem 3.2. *The set m, equipped with the operations of addition and convolution, $(m, +, *)$, is a field with multiplication identity $I := e_0$.*

Proof. Consider the following sequence,

$$e_k := (\dots, 0, 0, 1, 0, 0, \dots)$$

where 1 is in the k^{th} position. We begin by showing that e_0 is the multiplicative identity, $(a * e_0)_k = \sum\limits_{i+j=k} a_j (e_0)_i = a_k$ since

$$(e_0)_j = \begin{cases} 1 & j = 0 \\ 0 & j \neq 0 \end{cases}.$$

The existence of a multiplicative identity coupled with Proposition 3.5 establishes $(m, +, *)$ as a ring.

To prove the existence and uniqueness of multiplicative inverses, by Proposition 3.3 use $a * b = e_0$ where

$$b = a^{-1} = (\dots, 0, 0, b_{-k}, b_{-k+1}, b_{-k+2}, \dots)$$

such that

$$e_0 = (a * b) = (\dots, 0, 0, a_k b_{-k}, a_{k+1} b_{-k} + a_k b_{-k+1}, a_{k+2} b_{-k} + a_{k+1} b_{-k+1} + a_k b_{-k+2}, \dots).$$

Equating and solving for b_k's yields that, in general for $m \in \mathbb{N}$,

$$b_{-k+m} = -b_{-k} \sum_{i=1}^{m} a_{k-i+m+1} b_{-k+i-1}. \tag{3.5}$$

Thus the first few terms are:

$$\begin{aligned}
b_{-k} &= \frac{1}{a_k} \\
b_{-k+1} &= \frac{-1}{a_k}(a_{k+1} b_{-k}) \\
b_{-k+2} &= \frac{-1}{a_k}(a_{k+2} b_{-k} + a_{k+1} b_{-k+1}) \\
&\vdots
\end{aligned}$$

Appendix II contains an updated version of the Mathematica code created in [2] for computing the inverse of a causal sequence.
To show uniqueness of these inverses, note that $(m, +, *)$ is an integral domain so proceeding for sake of contradiction assume there exists $b_1, b_2 \in m$ such that $a * b_1 = e_0$ and $a * b_2 = e_0$ for some $a \neq 0 \in m$. But then $a * (b_1 - b_2) = 0$ (since $(m, +, *)$ is an integral domain) and since $a \neq 0$ it must be that $b_1 - b_2 = 0$ and therefore $b_1 = b_2$. $\square$

The remainder of this section is dedicated to proving that $\widetilde{m} \subset m$, defined to be:

$$\widetilde{m} := \{\text{causal sequences with } L = \lim_{k \to \infty} \sqrt[k]{a} < \infty\}$$

is a field and the Z-Transform is a field isomorphism between $\widetilde{m}$ and $\widetilde{M}$, defined to be the set of analytic functions defined on a punctured disk centered at the origin. Section 4 handles a broader isomorphism involving m. Let us begin by establishing that $\widetilde{M}$ is a field.

Definition 3.5. Let D be an open set in $\mathbb{C}$. Then a function $f : D \to \mathbb{C}$ is **analytic** if, for all $z \in D$,

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists.

Definition 3.6. Denote $U_\epsilon(0)$ to be the **disk of radius** $\epsilon > 0$ centered at the origin in the complex plane.

Theorem 3.3. *Let $f : U_\epsilon(0) \to \mathbb{C}$. Then the following are equivalent:*

i. f is analytic.

ii. There exists $b_n \in \mathbb{C}$ such that $f(z) = \sum_{n=0}^{\infty} b_n z^n$ for all $z \in U_\epsilon(0)$.

Also, if (ii.) is true, f is infinitely often differentiable on $U_\epsilon(0)$ and $b_n = \frac{f^{(n)}(0)}{n!}$.

Theorem 3.3 and Definition's 3.5 and 3.6 imply that an analytic function is given by a convergent power series defined for a certain radius of convergence. Modifying Definition 3.6, denote $U_\epsilon^*(0)$ to be the punctured disk (i.e. $U_\epsilon(0) - \{0\}$) of radius $\epsilon > 0$.

Definition 3.7. Denote $\widetilde{M}$ to be:

$$\widetilde{M} := \Big\{ f : \exists a \in \widetilde{m} \quad \text{such that} \quad f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad \forall z \in U^*_{R_a}(0) \Big\}.$$

Thus $\widetilde{M}$ is the set of complex-valued function defined by a convergent power series with the radius of convergence given by Theorem 3.1.

Lemma 3.1. *$f \in \widetilde{M}$ if and only if $\exists k \in \mathbb{Z}$ such that the function $g(z) := z^{-k} f(z)$ satisfies:*

i. $\exists b \in \widetilde{m}$ such that $g(z) = \sum\limits_{k=o}^{\infty}$ for all $z \in U_{R_b}(0)$ and

ii. $g(0) = b_0 \neq 0$.

Proof. By Definition 3.7, $f \in \widetilde{M}$ if and only if there exists $a \in \widetilde{m}$ such that

$$f(z) = a_k z^k + a_{k+1} z^{k+1} + a_{k+2} z^{k+2} + \cdots = z^k g(z)$$

where $g(z) = a_k + a_{k+1} z + a_{k+2} z^2 + \ldots$ and clearly $g(0) = a_k \neq 0$. $\square$

Theorem 3.4. *$(\widetilde{M}, +, \cdot)$ is a field.*

Proof. Clearly $\widetilde{M}$ is closed under addition and scalar multiplication. Note that additive inverses are also apparent, since addition is accomplished point-wise. So it must be proven

i. that $g_1, g_2 \in \widetilde{M}$ if $f_1, f_2 \in \widetilde{M}$ and

ii. $\frac{1}{f} \in \widetilde{M}$ if $f \in \widetilde{M}$.

To show (i.) use Lemma 3.1 and let $f_1(z) = z^{-k_1} g_1(z)$ and $f_2(z) = z^{-k_2} g_2(z)$. Then $(f_1 \cdot f_2)(z) = z^{-(k_1+k_2)} g_1(z) g_2(z) = z^{-(k_1+k_2)} g(z)$ where

$$g(z) = g_1(z) g_2(z) = (\sum_{k=0}^{\infty} b^1_k z^k)(\sum_{k=0}^{\infty} b^2_k z^k) = \sum_{k=0}^{\infty}(\sum_{i=0}^{k} b^1_{k-i} b^2_i) z^k = \sum_{k=0}^{\infty} c_k z^k.$$

By Definition 3.3, $c = b^1 * b^2 \in \widetilde{m}$ and $g(0) = c_0 = b^1_0 b^2_0 \neq 0$ thus $f_1, f_2 \in \widetilde{M}$.
To show (ii.) use Lemma 3.1 and let $f(z) = z^k g(z)$. Then, defining $h(z) := \frac{1}{g(z)}$, note $\frac{1}{f(z)} = z^{-k} h(z)$. Since $g(z)$ is analytic on $U_\epsilon(0)$ for some $\epsilon > 0$ and $g(0) \neq 0$, by supposition, there exists $0 < \epsilon_1 \leq \epsilon$ such that $g(z) \neq 0$ for $z \in U_{\epsilon_1}(0)$ and $h(0) \neq 0$. Since $\sum\limits_{k=0}^{\infty} b_k z^k$ exists for some $z \neq 0$ so, by Theorem 3.1, $b \in \widetilde{m}$. $\square$

Theorem 3.5. *$(\widetilde{m}, +, *)$ is a field and the Z-Transform, $Z : \widetilde{m} \to \widetilde{M}$, is a field isomorphism.*

Proof. (Also see [2] and [3]) Notice that $\widetilde{m}$ is closed under addition and convolution in the field $(m, +, *)$. To show addition, let $a, b \in m$ with $R_a = \frac{1}{L_a}, R_b = \frac{1}{L_a}$, respectively, such that for $|z| < R_a$ and $|z| < R_b$, $\sum\limits_{k=-\infty}^{\infty} a_k z^k$ and $\sum\limits_{k=-\infty}^{\infty} b_k z^k$ converge absolutely. Then $\sum\limits_{k=-\infty}^{\infty} (a_k + b_k) z^k$ converges absolutely for $|z| < \min(R_a, R_b)$ and has radius of convergence $R_{a+b} \geq \min(R_a, R_b)$. Now, if we let $c = a * b$ and let $|z| < \min(R_a, R_b)$ then $\sum\limits_{k=-\infty}^{\infty} c_k z^k$ exists since

$$\sum_{n=0}^{N} |\sum_{k=0}^{n} a_{n-k} b_k| |z|^n \leq \sum_{n=0}^{N} (\sum_{k=0}^{n} |a_{n-k}| |b_k|) |z|^n \leq (\sum_{n=0}^{N} |a_n| |z|^n)(\sum_{n=0}^{N} |b_n| |z|^n).$$

This completes the proof that $\widetilde{m}$ is closed under addition and convolution.
Now, $Z : \widetilde{m} \to \widetilde{M}$ is linear (see Proposition 3.1) so it will be shown that $Z : \widetilde{m} \to \widetilde{M}$ is a multiplicative bijection. Let $c = a * b \in \widetilde{m}$, then by above, $\sum_{n=0}^{\infty} c_n z^n$ converges for $|z| < \min(R_a, R_b)$. Now let $\epsilon > 0$,

$$\begin{aligned}
|\Big(\sum_{n=0}^{k} a_n z^n\Big)\Big(\sum_{n=0}^{k} b_n z^n\Big) - \sum_{n=0}^{\infty} c_n z^n| &= |a_0 b_0 + (a_1 b_0 + a_0 b_1)z + \dots \\
&\quad + (a_{n+2}b_0 + a_{n+1}b_1 + \dots + a_0 b_{n+2})z^{n+2} \\
&\quad - [a_0 b_0 + (a_1 b_0 + a_0 b_1)z + \dots]| \\
&= |\sum_{n=k+1}^{2k} (\sum_{j+s=n} a_j b_s) z^n + \sum_{n=2k+1}^{\infty} c_n z^n| \qquad (s, j \geq k+1) \\
&\leq \sum_{n=k+1}^{\infty} \Big(\sum_{s=0}^{n} |a_{n-s}||b_s|\Big)|z|^n \\
&< \epsilon.
\end{aligned}$$

Hence,

$$Z(a * b) = Z(c) = Z(a) \cdot Z(b)$$

which shows that $Z : \widetilde{m} \to \widetilde{M}$ is multiplicative. By definition Z is onto, to show that Z is one-to-one. Let $Z(a) = f = 0$ then $f(z) = a_k z^k + a_{k+1} z^{k+1} + \dots = 0$ for all $z \in U^*_{R_a}(0)$ and by Lemma 3.1, $g(z) := z^{-k} f(z) = a_k + a_{k+1} z + a_{k+2} z^2 + \dots = 0$ for all $z \in U^*_{R_a}(0)$ and thus for all $z \in U_{R_a}(0)$ since $g(z)$ is continuous on $U_{R_a}(0)$. But then $g(0) = a_k = 0$ which implies that $a = 0$ and hence Z is one-to-one. Thus Z is a linear multiplicative bijection and since $\widetilde{M}$ is a field, $\widetilde{m}$ is a field. □

The use of this isomorphism is essential to the area of signal processing; in particular, the Z-Transform is used to derive error bounds when computing the inverse of finite sequences, i.e. sequences where $ind_u < \infty$. For more on this see the next section and [3].

3.2 The Inverse Z-Transform and Application to Signal Processing

The inverse Z-Transform is given by

$$x_n = \frac{1}{2\pi i} \oint_C X(z) z^{n-1} dz \quad \text{for } n = 0, 1, 2, \dots$$

where C is a simple closed path which encircles the origin. In practice, computing this value can be rather difficult. In this paper, the Residue Theorem is given as a method for computing the inverse Z-Transform. As a note, the multiplicative property identified in Theorem 3.5 can also be used to compute $Z^{-1}(a) = Z(a^{-1})$.

Theorem 3.6. *The Residue Theorem [4].*
Let f be analytic on a simply connected domain $D \subset \mathbb{C}$ except for a finite number of isolated singularities as points $z_1, \dots, z_n$ of D. Let C be a smooth positively oriented closed curve in D that does not pass through any of the points $z_1, \dots, z_n$, then

$$\oint_C f(z) dz = 2\pi i \sum_{z_k \text{ inside } C} Res(f, z_k) \tag{3.6}$$

where the sum is taken over all of those singularities z_k of f inside C.

Proof. The proof relies on Cauchy's Integral Formula and the Cauchy-Gourset Theorem. For a full proof, see [4]. $\square$

To compute these residues in practice a more functional formula will be used, if $f(z)$ has poles of order m at z_0, compute:

$$Res(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[\frac{d^{m-1}}{dz^{m-1}} \left[(z - z_0)^m f(z) \right] \right].$$

Example 7. Let $X(z) = \frac{2z}{z^2-1}$ which has poles at $z = -1, 1$. Using the Residue Theorem, compute each of the x_n as follows: assume C is a positively oriented path that encircles $z = -1, 0, 1$, then

$$\begin{aligned}
x_0 &= \frac{1}{2\pi i} \oint_C \frac{2z}{z^2-1} \cdot z^{-1} dz = Res(X, -1) + Res(X, 0) + Res(X, 1) = -1 + 0 + 1 = 0 \\
x_1 &= \frac{1}{2\pi i} \oint_C \frac{2z}{z^2-1} dz = Res(X, -1) + Res(X, 1) = 1 + 1 = 2 \\
x_2 &= \frac{1}{2\pi i} \oint_C \frac{2z}{z^2-1} \cdot z^{-2} dz = Res(X, -1) + Res(X, 0) + Res(X, 1) = -1 + 0 + 1 = 0 \\
&\vdots
\end{aligned}$$

Thus for n even, $x_n = 0$ and for n odd, $x_n = 2$ so for the original $x \in \widetilde{m}$,
$x = (\dots, \overbrace{0}^{x_0}, 2, 0, 2, 0, 2, \dots)$.

The following sections serves as an application of the Z-Transform to signal processing.

3.2.1 Linear Time Invariant Systems

Proposition 3.1 displayed the fundamental properties of linearity and time invariance of the Z-Transform. This section will serve as an example of the application of the Z-Transform. In signal processing, a discrete linear time invariant system takes an input sequence, applies a filter, and returns an output sequence. The following diagram illustrates this procedure.

$$\boxed{x_n} \longrightarrow \boxed{\;h_n\;} \longrightarrow \boxed{y_n}$$

Note that h_n is a filter in this scenario. LTI systems are crucial for the field of signal processing; in particular, only knowing h_n and y_n allows for the derivation x_n within a certain degree of accuracy. Let $x, h, y \in m$ such that $x * h = y$ (i.e. these sequences represent the LTI system pictured above). Given that the filter sequence and output sequence are known (h and y), estimating x becomes the question. Using Theorem 3.2, $x = h^{-1} * y$. If h^{-1} is infinite, compute the first N terms, $h_N^{-1} = (\dots, 0, h_0^{-1}, h_1^{-1}, \dots, h_N^{-1}, 0, \dots)$ and define $x_N = h_N^{-1} * y$, now to see how the approximation of the inverse filter sequence affects the received signal, compute:

$$\|x - x_N\|_1 = \|h^{-1} * y - h_N^{-1} * y\|_1 \leq \|y\|_1 \|h^{-1} - h_N^{-1}\|_1 \leq \|x\|_1 \|h\|_1 \|h^{-1} - h_N^{-1}\|_1.$$

Thus the main question becomes approximating

$$\|h^{-1} - h_N^{-1}\|_1 = \sum_{k=N+1}^{\infty} |h_k^{-1}|.$$

According to [2], if $h = (\ldots, 0, 0, -b, 1, 0, 0, \ldots)$ then $h^{-1} = (\ldots, 0, \frac{-1}{b}, \frac{-1}{b^2}, \frac{-1}{b^3}, \ldots)$ thus

$$\|h^{-1} - h_N^{-1}\|_1 = \sum_{k=N+1}^{\infty} \frac{1}{|b|^{k+1}} = \frac{1}{|b|^{N+2}} \sum_{k=0}^{\infty} \frac{1}{|b|^k} = \frac{1}{|b|^{N+2}} \frac{1}{1 - \frac{1}{|b|}} = \frac{1}{|b|^{N+1}} \|h^{-1}\|_1.$$

Hence

$$\|x - x_n\|_1 \leq \|x\|_1 \frac{|b| + 1}{|b| - 1} \frac{1}{|b|^{N+1}}$$

so for b large and N small or $|b| \approx 1$ and N large, x can be computed to a give precision if the size of x is known.

4 The Asymptotic Z-Transform

The Asymptotic Z-Transform allows us to consider all $a \in m$ that do not necessarily converge according to Theorem 3.1. As proved in Theorem 3.5, the Z-Transform serves as a field isomorphism from $\widetilde{m}$ to $\widetilde{M}$. Since $\widetilde{m} \subset m$ it would seem there exists a $M \supset \widetilde{M}$ that all of m can be mapped to. In this section, the Asymptotic Z-Transform, Z_{as}, will be given and used to prove an isomorphism from m to M.

Definition 4.1. Define a **sectorial region** of the complex plane to be:

$$S := \{z : 0 < |z| < R, |\arg z| < \theta\}$$

with $R > 0$ and $\theta \geq 0$.

Definition 4.2. Note $a \in m$ represents $f(z)$ asymptotically if, for all $n \geq ind_l(a)$,

$$\lim_{z\to 0} \frac{f(z) - \sum_{i=-\infty}^{n} a_i z^i}{z^n} \to 0$$

we denote this $f \approx a$.

One can see that Definition 4.2 relies on "little-o" notation from asymptotic analysis. This allows us to create an equivalence class of functions that are asymptotically equivalent to $0 \in m$, i.e. $f \approx 0$ at $0 \in \mathbb{C}$ if for $z \in S$,

$$\lim_{z\to 0} \frac{f(z)}{z^n} \to 0 \quad \forall n \in \mathbb{Z}.$$

Denote this equivalence class to be

$$[0] := \{f : f \approx 0\}.$$

Example 8. An example of a function which is asymptotically equivalent to 0 at $0 \in \mathbb{C}$ is the function $f(z) = e^{\frac{-1}{z}}$ on $S = \{z \neq 0,\ \ |\arg z| = \theta < \frac{\pi}{2}\}$ since $|f(z)| \leq e^{\frac{-1}{|z|}\cos\theta}$ thus

$$\lim_{z\to\infty} \frac{f(z)}{z^k} = 0.$$

Now M can be defined as follows.

Definition 4.3. If f is a complex-valued function defined on some sectorial region S, then $[f] := f + [0]$ and

$$M := \{[f] : f \approx a \quad \text{on a sectorial region } S \text{ for some } a \in m\}.$$

The next theorem provides a way to find f such that $f \approx a$, we begin with a lemma.

Lemma 4.1. *Let $a \in m$ and $j \in \mathbb{Z}$, then $f \approx a$ if and only if $z^j f(z) \approx e_j * a$.*

Proof. Let $f \approx a$, then

$$\lim_{z\to 0} \frac{f(z) - \sum_{i=-\infty}^{k} a_i z^i}{z^k} = \lim_{z\to 0} \frac{f(z) - \sum_{i=-\infty}^{k-j} a_i z^i}{z^{k-j}} = \lim_{z\to 0} \frac{f(z) - \sum_{i=-\infty}^{k} a_{i-j} z^{i-j}}{z^{k-j}} = \lim_{z\to 0} \frac{z^j f(z) - \sum_{i=-\infty}^{k} a_{i-j} z^i}{z^k}.$$

Noting that $a_{i-j} = e_j * a$ implies $z^j f(z) \approx e_j * a$. Following the equality signs backward completes the proof. $\square$

The following theorem is known as Ritt's Theorem; see [2].

Theorem 4.1. *Ritt's Theorem. Let $a \in m$, then there exists f such that $f \approx a$.*

Proof. Let $a \in m$, by Lemma 4.1 assume that

$$a = (\dots, 0, 0, a_0, a_1, a_2, \dots)$$

where a_0 is in the 0^{th} position. Assume $a \in m - \widetilde{m}$ (i.e. a is divergent), then to construct $f(z)$ such that $f \approx a$, we need to replace the divergent series with a convergent series that is uniformly convergent and shares the same asymptotic property at 0 of the original. Define

$$\alpha_k(z) := 1 - e^{\left(\frac{-b_k}{z^\beta}\right)} \text{ with } b_k > 0 \text{ and } 0 < \beta < 1$$

thus the desired function is of the form $f(z) = \sum_{k=0}^{\infty} a_k \alpha_k(z) z^k$. The following fact is needed for the proof:

$$|1 - e^z| = |z \int_0^1 e^{tz} dt| \leq |z| \text{ for } \Re(z) \leq 0.$$

Using above, $|a_k \alpha_k(z) z^k| \leq |a_k||b_k||z|^{k-\beta}$, thus defining $b_k = |a_k|^{-1}$ for $a_k \neq 0$ and $b_k = 0$ for $a_k = 0$, then

$$f(z) := \sum_{k=1}^{\infty} a_k \alpha_k(z) z^k \leq \sum_{k=1}^{\infty} |z|^{k-\beta}$$

which converges uniformly for $|z| \leq |z_0| < 1$. We must now show that $f \approx a$. Thus,

$$\frac{f(z) - \sum_{k=0}^{\infty} a_k z^k}{z^m} = -\sum_{k=0}^{m} a_k \exp\left(\frac{-b_k}{z^\beta}\right) z^{-(m-k)} + \sum_{k=m+1}^{\infty} a_k \alpha_k(z) z^{k-m},$$

the first term tends to 0 as $z \to 0$, and for $|z| \leq z_0$,

$$|\sum_{k=m+1}^{\infty} a_k \alpha_k(z) z^{k-m}| \leq \sum_{k=m+1}^{\infty} |z|^{k-m-\beta} < \frac{|z|^{1-\beta}}{1 - |z|} \to 0 \text{ as } z \to 0.$$

For $z_0 \geq 1$, choose $b_k = |a_k z_0^k|^{-1}$ and the proof holds. □

Ritt's Theorem guarantees that the following definition holds.

Definition 4.4. The **Asymptotic Z-Transform**, $Z_{as} : m \to M$ is given by:

$$Z_{as}(a) = \hat{a} = [f]$$

where $f \approx a$.

Proposition 4.1. *Operational Properties in M.*

a. *If $[f] \cap [g] \neq \emptyset$, then $[f] = [g]$.*
b. $[f \cdot g] = [f][g]$.
c. $[f] + [g] = [f + g]$.
d. *Let $a, b \in m$ with $f \approx a$ and $f \approx b$, then $f \cdot g \approx a * b$.*

For a proof, see [2].

The following theorem is the culmination of the propositions in this section and provides a way to compute values of functions defined by divergent series.

Theorem 4.2. *$(M, +, \cdot)$ is a field and $Z_{as} : m \to M$ is a field isomorphism.*

Proof. Let $a, b \in m$ with $Z_{as}(a) = [f]$ and $Z_{as}(b) = [g]$, by the proposition above,

$$Z_{as}(a)Z_{as}(b) = [f][g] = [fg] = Z_{as}(a * b)$$

so $Z_{as} : m \to M$ is multiplicative and also additive and linear. To show Z_{as} is one-to-one, proceed for sake of a contradiction, suppose $Z_{as} = [0]$ with $a \neq 0$, then

$$\frac{0 - \sum\limits_{i=-\infty}^{k} a_i z^i}{z^k} \to 0 \quad \text{for all } k \geq ind_l(a).$$

Let $k_0 := ind_l(a)$ so $a_{k_0} \neq 0$ and

$$\frac{0 - \sum\limits_{i=-\infty}^{k_0} a_i z^i}{z^k} = a_{k_0} \to 0. \Rightarrow\Leftarrow$$

By definition Z_{as} is onto thus Z_{as} is an isomorphism and since m is a field then M is a field. $\square$

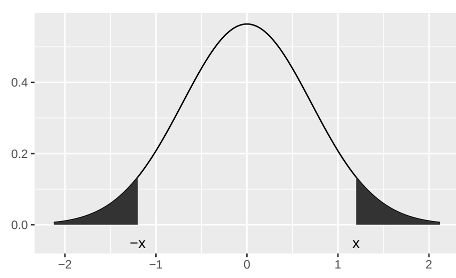
We conclude this paper by giving an example of using Theorem 4.2 on a well known function: the complimentary error function.

4.1 The Complimentary Error Function

The Complimentary Error Function is given by:

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

and is equivalent to $1 - \operatorname{erf}(x) = 1 - \frac{2}{\sqrt{\pi}} \int\limits_0^x e^{-t^2} dt$. This function plays an important role in probability theory as its values represent the area under the two tails of a random variable $Y \sim N(0, \frac{1}{2})$ [6], this is pictured in the graph below:



Unfortunately, this integral cannot be solved analytically; as such Theorem 4.2 is employed. First make the change of variable, $t^2 = \tau$, so the integral becomes,

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x^2}^{\infty} \frac{1}{2} \frac{e^{-\tau}}{\sqrt{\tau}} d\tau = \frac{1}{\sqrt{\pi}} \int_{x^2}^{\infty} \frac{e^{-\tau}}{\sqrt{\tau}} d\tau.$$

Performing integration by parts yields the following infinite series:

$$\mathrm{erfc}(x) = \frac{1}{\sqrt{\pi}} \Big(\frac{e^{-x^2}}{x} - \frac{e^{-x^2}}{2x^3} + \frac{3 \cdot e^{-x^2}}{2^2 x^5} - \cdots + (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1) e^{-x^2}}{2^n x^{2n+1}} + \dots \Big).$$

Introducing the notation,

$$(2n-1)!! = (2n-1) \cdot (2n-3) \cdot \cdots \cdot 5 \cdot 3 \cdot 1$$

(i.e. the factorial of only odd numbers), then, integration by parts $N+1$ times will yield,

$$\mathrm{erfc}(x) \approx \frac{e^{-x^2}}{\pi x} \Big(1 - \frac{1}{2x^2} + \frac{1 \cdot 3}{(2x^2)^2} - \cdots + (-1)^N \frac{(2N-1)!!}{(2x^2)^N} + (-1)^{N+1} \frac{(2N+1)!!}{(2x^2)^{N+1}} \int_{x^2}^{\infty} \frac{e^{-\tau}}{\tau^{N+1}} d\tau \Big).$$

Define $R_N(x) = (-1)^{N+1} \frac{(2N+1)!!}{(2x^2)^{N+1}} \int\limits_{x^2}^{\infty} \frac{e^{-\tau}}{\tau^{N+1}} d\tau$ then,

$$|R_N(x)| = |\frac{(2N+1)!!}{(2x^2)^{N+1}} \int\limits_{x^2}^{\infty} \frac{e^{-\tau}}{\tau^{N+1}} d\tau|.$$

Defining

$$f(x) = x e^{x^2} \mathrm{erfc}(x)$$

yields the following asymptotic expansion for f,

$$f(x) \approx \frac{1}{\pi} \Big(1 + \sum_{i=1}^{N} \frac{(-1)^i (2i-1)!!}{(2x^2)^i} + x e^{x^2} R_N(x) \Big).$$

Using Theorem 3.2, the sum in f diverges for all x if $N \to \infty$.

5 Appendix I: Computing the Convolution of Sequences with Mathematica

The following Mathematica code can be used to compute $a * b$ for $a, b \in m$ with

$$|ind_u(a) - ind_l(a)| < 100 \quad \text{and} \quad |ind_u(b) - ind_l(b)| < 100,$$

i.e. the number of terms in a, b must be less than 100. The code takes the upper and lower index of a and b, as well as the terms (in order) of a and b. The output is the lower and upper index of $a * b$ as well as the sequence $a * b$. This code is novel in that the user is able to specify the relevant sequences involved in the convolution.

```
lowerindexa = Input["Enter lower index of sequence a."];
upperindexa = Input["Enter upper index of sequence a."];
a = Table[i * 0, {i, 1, 100}];
Do[a[[i]] = Input["Enter next number in sequence a:"],
 {i, 1, upperindexa - lowerindexa + 1}]
Print["Starting position of sequence a: ", lowerindexa]
Print[a]
lowerindexb = Input["Enter lower index of sequence b."];
upperindexb = Input["Enter upper index of sequence b."];
b = Table[i * 0, {i, 1, 100}];
Do[b[[i]] = Input["Enter next number in sequence b: "],
 {i, 1, upperindexb - lowerindexb + 1}]
Print["Starting position of sequence b: ", lowerindexb]
Print[b]
c = Table[i * 0, {i, 1, 100}];
Print["Starting position of sequence c=(a*b): ", lowerindexa + lowerindexb]
Print["Upper index of c=(a*b): ", upperindexa + upperindexb]
Do[Sow[c[[k]] = Sum[(a[[k - i]] * b[[i]]), {i, 1, k-1}]], {k, 1, 99}] // Reap // Last // Flatten
```

6 Appendix II: Computing the Inverse Convolution of Finite Sequences with Mathematica

The following Mathematica code can be used to compute the first 100 terms in a^{-1} for a given $a \in m$ such that
$$|ind_l(a) - ind_u(a)| < 100.$$

The code takes the lower and upper indices of a and the terms, in order, of a. The output is the lower index of a^{-1} and the first 100 terms of a^{-1}. Note that this code is an updated version of the code given in [2].

```
lowerindex = Input["Enter lower index of sequence."];
upperindex = Input["Enter upper index of sequence."];
a = Table[i * 0, {i, 1, 100}];
Do[a[[i]] = Input["Enter next number in sequence:"], {i, 1, upperindex - lowerindex + 1}]
Print["Starting position of sequence: ", lowerindex]
Print[a]
ainvlowerindex = -lowerindex;
Print["Starting position of inverse: ", ainvlowerindex]
ainv = Table[i * 0, {i, 1, 100}];
ainv[[1]] = 1 / a[[1]];
Prepend[
 Do[ainv[[1 + m]] = Sow[-ainv[[1]] * Sum[a[[1 - i + m + 1]] * ainv[[1 + i - 1]], {i, 1, m}]], {m, 1, 98}] // Reap //
   Last // Flatten, ainv[[1]]]
```

7 References

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