An Exploration of the Z-Transform

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Outline

What is Transform Theory?

Definition and Translation

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Transform Theory is the idea of changing the domain of a problem (e.g. calculus to algebra)

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Translation
Basically makes hard problems "easier"

Some Common Transforms

1. Fourier Transform (e.g. time domain to frequency domain)

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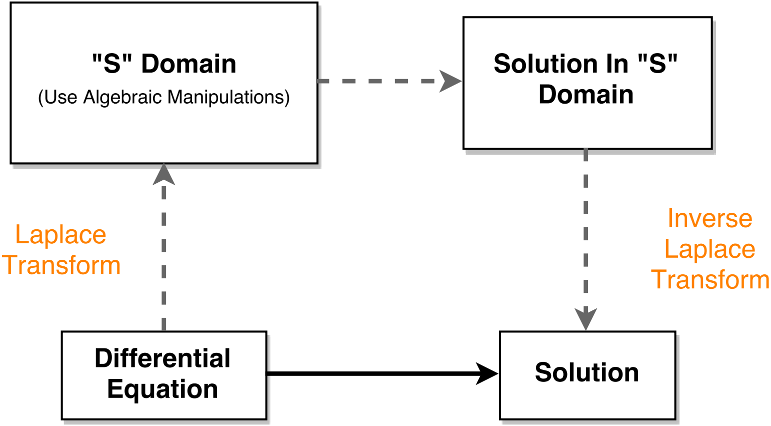
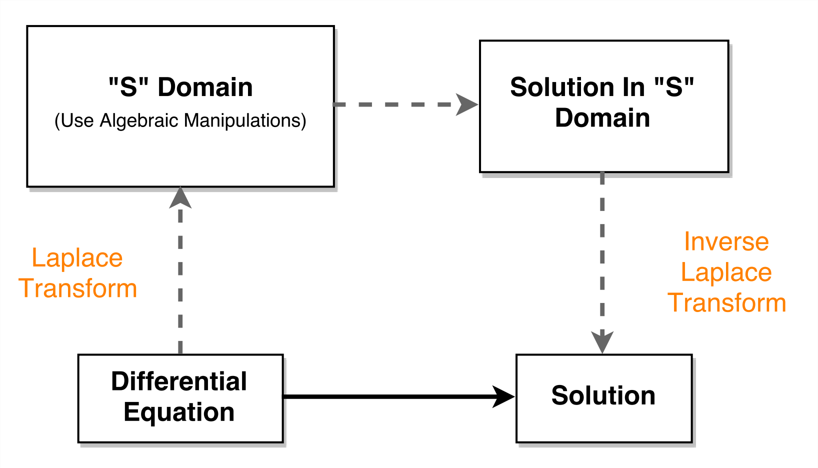
Some Common Transforms

1. Fourier Transform
2. **Laplace Transform**
3. Z-Transform

The Laplace Transform



The Laplace Transform



- Used to solve linear constant-coefficient differential equations

Motivation for Z-Transform

Use Laplace Transform:

$$\alpha y'' + \beta y' + \gamma y = \delta(t)$$

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Use Z-Transform:

$$\alpha a_{n-2} + \beta a_{n-1} + \gamma a_n = \delta(n)$$

Motivation for Z-Transform

Fibonacci Sequence:

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

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$$a_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Motivation for Z-Transform

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Proof.

$\square$

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Proof.
Use the Z-Transform! $\square$

Classical Z-Transform

Question
How does your radio equalizer work?

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Answer
Linear Time Invariant Systems

Signal Processing

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Linear Time Invariant Systems

$$\boxed{x_n} \longrightarrow \boxed{h_n} \longrightarrow \boxed{y_n}$$

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$$(x * h)_n = y_n$$

Example 1

$$(x * h)_n = ?$$

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- Take $x = (2764.6z, \overbrace{-3.52 \times 10^9 z^2}^{x_1}, 1.35 \times 10^{15} z^3, -2.45 \times 10^{20} z^4, \ldots)$

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- Apply $h = (1, \overbrace{-3.82 \times 10^6 z}^{h_1}, 2.43 \times 10^{12} z^2, -6.2 \times 10^{17} z^3, \ldots)$

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HARD...

Use Z-Transform!

Example 1

$$(x * h)_n = ?$$

- $x_n = (-1)^n \cdot \frac{(440 \cdot 2\pi)^{2n+1} z^{n+1}}{(2n+1)!}$
- $h_n = (-1)^n \cdot \frac{(440 \cdot 2\pi)^{2n} z^n}{(2n)!}$

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Definition
The Z-Transform of a sequence a is given by

$$\mathcal{Z}(a) = \sum_{n=0}^{\infty} a_n z^n$$

Example 1

$$(x * h)_n = ?$$

$$Z(x) = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(440 \cdot 2\pi)^{2n+1} z^{n+1}}{(2n+1)!} z^n$$

Example 1

$$(x * h)_n = ?$$

$$\begin{aligned} Z(x) &= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(440 \cdot 2\pi)^{2n+1} z^{n+1}}{(2n+1)!} z^n \\ &= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(440 \cdot 2\pi)^{2n+1} z^{2n+1}}{(2n+1)!} \end{aligned}$$

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- $\mathcal{Z}(x) = \sin(880\pi z)$
- $\mathcal{Z}(h) = \cos(880\pi z)$

$$Z(x * h) \equiv Z(x) \cdot Z(h)$$

$$\Rightarrow Z(x) \cdot Z(h) = \sin(880\pi z)\cos(880\pi z) = \frac{1}{2}\sin(1760\pi z)$$

Example 1

$$(x * h)_n = ?$$

Applying the inverse Z-Transform yields:

$$y_n = \frac{1}{2}(-1)^n \frac{(1760\pi)^{2n+1} z^{n+1}}{(2n+1)!}$$

In Practice

- Taking 1000 samples per second yields:

$\boxed{\text{Play: } x_n}$ $\boxed{\text{Play: } x_n * h_n}$

Example 2

$$(? * h)_n = y_n$$

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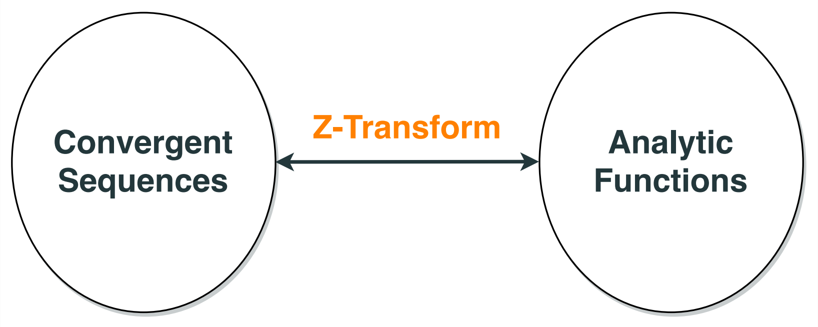
$$(? * h)_n = y_n$$

- Multiply both sides by h^{-1} to get $?_n = (y * h^{-1})_n$

How Is This Possible?

How Is This Possible?

- The Z-Transform, $Z : \widetilde{m} \to \widetilde{M}$, is a **field isomorphism**
- This means an algebra structure is preserved



How Is This Possible?

Proof.

$$Z(a)Z(b) = \Big(\sum_{k=0}^{\infty} a_k z^k\Big)\Big(\sum_{k=0}^{\infty} b_k z^k\Big)$$

□

How Is This Possible?

Proof.

$$
\begin{aligned}
Z(a)Z(b) &= \left(\sum_{k=0}^{\infty} a_k z^k\right)\left(\sum_{k=0}^{\infty} b_k z^k\right) \\
&= (a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots)(b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \dots)
\end{aligned}
$$

$\square$

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&= (a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots)(b_0 + b_1 z + b_2 z^2 + b_3 z^3 + \dots) \\
&= (a_0 b_0) + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \dots
\end{aligned}
$$

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&= (a_0 b_0) + (a_0 b_1 + a_1 b_0)z + (a_0 b_2 + a_1 b_1 + a_2 b_0)z^2 + \dots \\
&= Z(a * b)
\end{aligned}
$$

$\square$

Asymptotic Analysis

Asymptotic Ideas Explored

1. "Little-oh" notation $\rightarrow$ "Equivalence"

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2. Ritt's Theorem $\rightarrow$ "Mapping tool"

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2. Ritt's Theorem $\rightarrow$ "Mapping tool"
3. Computations with divergent series $\rightarrow$ "Practical use"

Asymptotic Ideas Explored

1. "Little-oh" notation
2. Ritt's Theorem
3. **Computations with divergent series**

Computations With Divergent Series

$$F(x) = xe^{x^2}\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} xe^{x^2} \int_x^\infty e^{-t^2}\,dt$$

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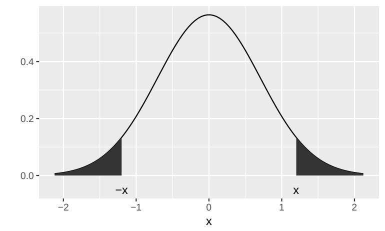

Computations With Divergent Series

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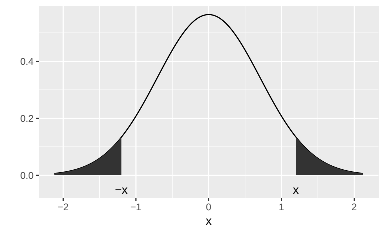

$$F(x) \approx \frac{1}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-1)^i (2i+1)!!}{(2x^2)^i}$$

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Computations With Divergent Series

$$S_n := \frac{1}{\sqrt{\pi}} \sum_{i=0}^{n} \frac{(-1)^i (2i+1)!!}{(2x^2)^i}$$

x	$F(x)$	S_1	S_2	S_3	S_4	S_5	S_{10}	S_n
1	0.42758	-0.2821	1.8336	-5.57	27.75	-155.5	6.92×10^{6}	6.7×10^{18}
2	0.51079	0.3526	0.4849	0.3691	0.4993	0.3203	5.697	5.3×10^{6}
5	0.55352	0.5303	0.5337	0.5332	0.5333	0.5333	0.5333	0.5400
10	0.5614	0.5557	0.5559	0.5559	0.5559	0.5559	0.5559	0.5587

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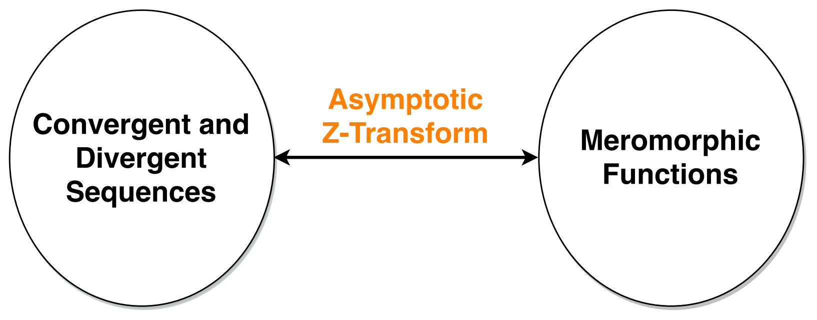
10	**0.5614**	0.5557	0.5559	0.5559	0.5559	0.5559	0.5559	0.5587

- We used a divergent series to compute these!

How Is This Possible?

- The Asymptotic Z-Transform, $Z_{as} : m \to M$, is also a **field isomorphism**



Conclusion

Future Work

- Deepen complex analysis understanding

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- Explore more complicated LTI systems

Future Work

- Deepen complex analysis understanding
- Explore more complicated LTI systems
- Use asymptotic methods on other divergent series

References

- Bleistein, Norman; Handelsman, Richard. "Asymptotic Expansion of Integrals" (1986). Dover Publications, INC.
- Champagne, Scott Jude. "The asymptotic Z-transform" (2005). LSU Master's Theses. 3680.
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THANK YOU FOR COMING

QUESTIONS?