Biomedical Engineering Summer School, Wilhelmshaven 2018

Instrumentation, Acquisition and Signal Processing for Biosignals

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Gerardo Marx Chávez-Campos
Mehmet Yüksekkaya

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BAŞKENT UNIVERSITY

JADE HOCHSCHULE
Wilhelmshaven Oldenburg Elsfleth

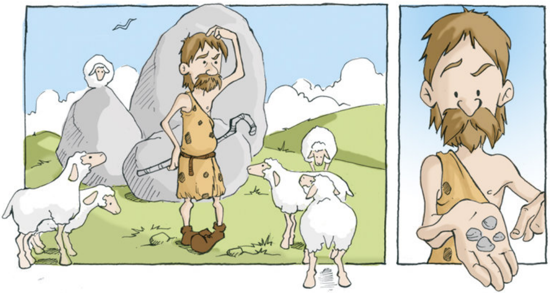
ITM
TÉCNICA, PROGRESO DE MÉXICO

Contest

Sequence of numbers

Sequence of numbers

- Numbers have been used to represent things and count it.
- The counting activity allows to know “How many?”



- However, human beings observe that numbers are in nature in specific sequences[1]...


5
8
13
21
34
55

Let’s take a look to this sequence:

- 1
- 2
- 3
- 5
- 8
- 13
- 21
- 34
- 55
- ... and so on

Can you find some way to relate this numbers?

How many spirals can you count?

The same sequence...

The Fibonacci sequence

The FIBONACCI sequence is widely found in most nature phenomena. The sequence is easily to create it by the sum of the previos terms:

$$
\begin{aligned}
u_0 &= 0 \\
u_1 &= 1 \\
u_2 = u_0 + u_1 &= 0 + 1 = 1 \\
u_3 = u_1 + u_2 &= 1 + 1 = 2 \\
u_4 = u_2 + u_3 &= 2 + 1 = 3 \\
&\vdots \\
u_n &= u_{n-1} + u_{n-2}
\end{aligned}
$$

What is a sequence?

Sequence definition

A **sequence** is a list of things (usually numbers) that are in order:

$$\underbrace{3}_{\text{1st term}}, \underbrace{5}_{\text{2nd term}}, \underbrace{7}_{\text{3rd term}}, \underbrace{9}_{\text{4th term}}, \ldots, \underbrace{n}_{\text{nth term}} \tag{1}$$

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A sequence is usually defined by a RULE, this is a way or equation to find each term[3]. Thus, in order to be able of determine (u_n, nth term) the RULE is written as a formula, where n is any term.

Then, the RULE for the sequence $\{3, 5, 7, 9, \ldots, \infty\}$ is:

n	**Term**	**Test rule**
1	3	$2 \times 1 + 1 = 3$
2	5	$2 \times 2 + 1 = 5$
3	7	$2 \times 3 + 1 = 7$

Special sequences

- Arithmetic sequence has a constant value between one term and other e.g. $\{1, 4, 7, 10, 13, \ldots\}$, write as an equation: $a_n = a_1 + (n-1)d$.

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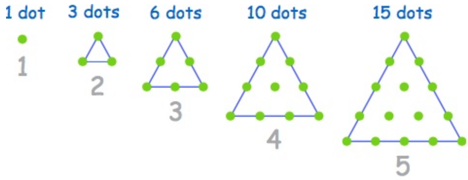

Matlab time...

Series

NETFLIX

No, no this kind of series...

Finite and infinite series

Finite Series

Let u_n be a sequence. Then the finite sum (partial sum) order is:

$$S_k = \sum_{n=1}^{k} = u_1 + u_2 + u_3 + \ldots + u_k \tag{2}$$

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Infinite Series

Let u_n be a sequence. Then the Infinite sum order is:

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \ldots \tag{3}$$

The n_{th} term theorem

If the partial sums S_k converges to L as $k \longrightarrow \infty$, we can say that the infinite series converges and the sum tends to L [4].

Theorem

If

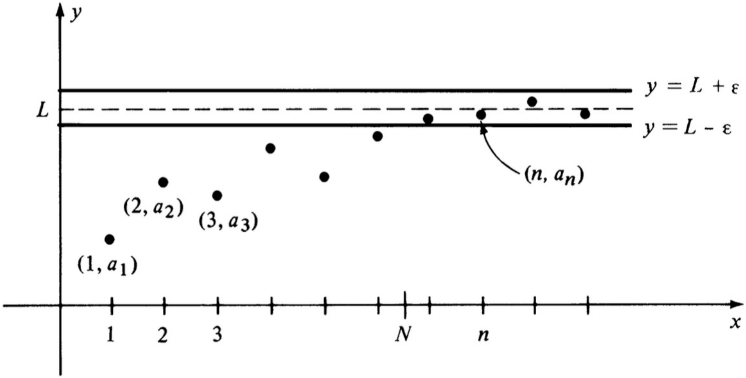
$$\lim_{n \to +\infty} U_n = 0 \tag{4}$$

the infinity series $\sum_{n=1}^{+\infty} U_n$ *is* CONVERGENT

If

$$\lim_{n \to +\infty} U_n \neq 0 \tag{5}$$

the infinity series $\sum_{n=1}^{+\infty} U_n$ *is* DIVERGENT



A convergent series

Does the harmonic series converges?

Note that the Theorem (4) not always is true. In other words, it is possible to have a DIVERGENT series for which $\lim_{n \to +\infty} U_n = 0$. An example of such a series is the one known as the harmonic, which is

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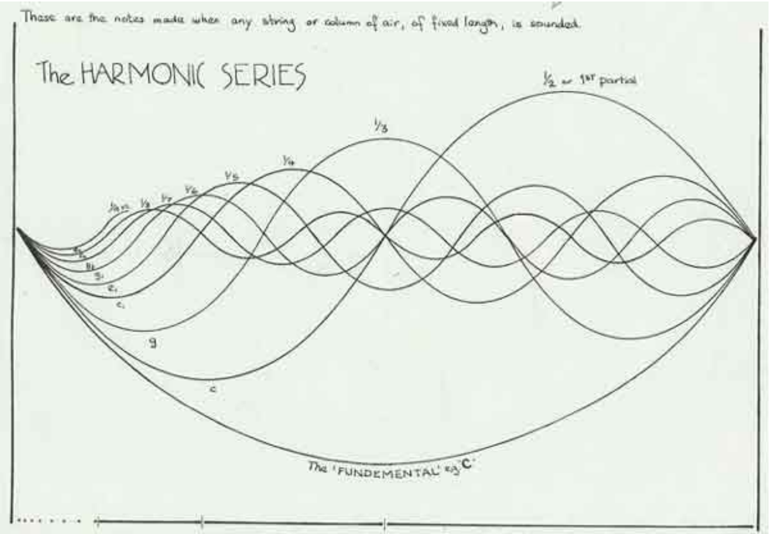
$$\sum_{n=1}^{+\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \cdots + \frac{1}{n} \cdots \tag{6}$$



The harmonic series

Infinite series of constant terms

Giving the infinite series :

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

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$$\begin{aligned} s_1 &= u_1 = \frac{1}{1 \cdot 2} = \frac{1}{2} \\ s_2 &= s_1 + u_2 = \frac{1}{2} + \frac{1}{2 \cdot 3} = \frac{2}{3} \\ s_3 &= s_2 + u_3 = \frac{2}{3} + \frac{1}{3 \cdot 4} = \frac{3}{4} \end{aligned}$$

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$$s_4 = s_3 + u_3 = \frac{3}{4} + \frac{1}{4\cdot 5} = \frac{4}{5}$$

using partial fractions, it is more evident to see that

$$u_k = \frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}$$

Therefore

$$\begin{aligned} s_n = & \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots \\ & + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) \end{aligned}$$

Removing the opposite terms, we have

$$s_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

Do your own code for a series..

More test are required

Therefore, considering the possibilities in sequences, hence in series, more ways to know if a series CONVERGES or DIVERGES are needed.

The integral test

Theorem

Let f be a function which is continuous, decreasing, and positive valued for all $x >= 1$, then the infinite series

$$\sum_{n=1}^{+\infty} f(n) = f(1) + f(2) + \cdots + f(n) + \cdots$$

is CONVERGENT *if the improper integral*

$$\int_1^{+\infty} f(x)dx$$

exists, and is DIVERGENT *if the improper integral increase without bound.*

Theorem

Let $\sum a_n$ be an infinite series of nonzero terms and let L to be calculated by (7)

$$\lim_{n \to +\infty} \left| \frac{a_{n+1}}{a_n} \right| = L \tag{7}$$

thus:

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$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \tag{7}$$

thus:

- *If $L < 1$, the series is absolutely convergent.*
- *If $L > 1$, or $L \to \infty$, the series is divergent.*
- *If $L = 1$, the series may be absolutely convergent, conditionally convergent, or divergent.*

Power series

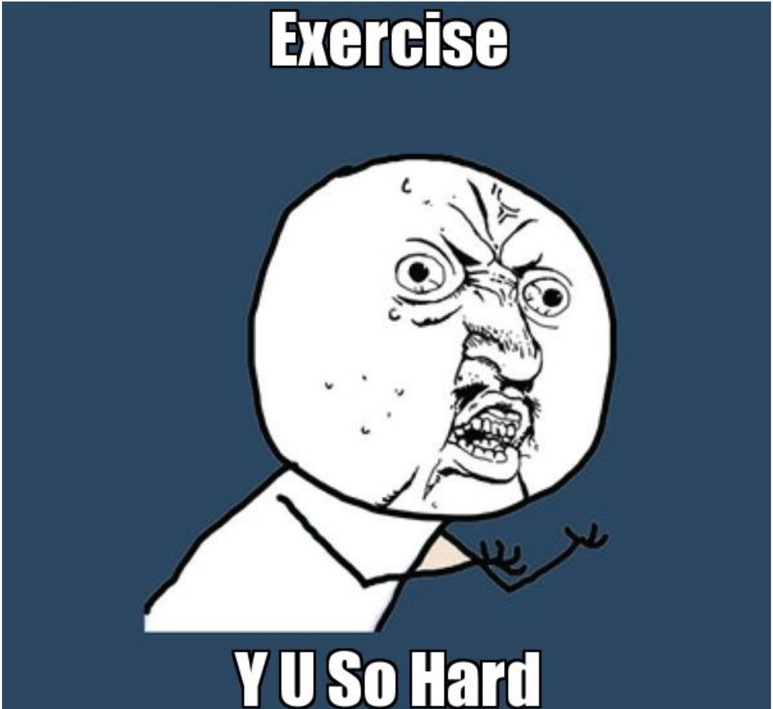
Power series

Theorem

Let x be a variable. A POWER SERIES IN x *is a series of the form:*

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \tag{8}$$

where each a_n is a real number[5].

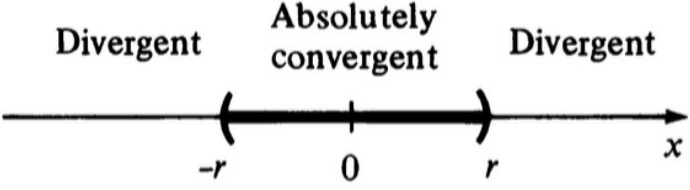

Exercise
Y U So Hard

Exercise:

Find all values of x for which the following power series is absolutely convergent:

$$1 + \frac{1}{5}x + \frac{2}{5^2}x^2 + \cdots + \frac{n}{5^n}x^n + \cdots \tag{9}$$

USE THE RATIO TEST!



convergency radius

Power series convergency theorem

Theorem

If $\sum a_n x^n$ *is a power series, then precisely one of the following is true*

- *The series converges only if* $x = 0$*,*
- *The series is absolutely convergent for all* x*,*
- *There is a positive number* r *such that the series is absolutely convergent if* $|x| < r$ *and divergent if* $|x| > r$

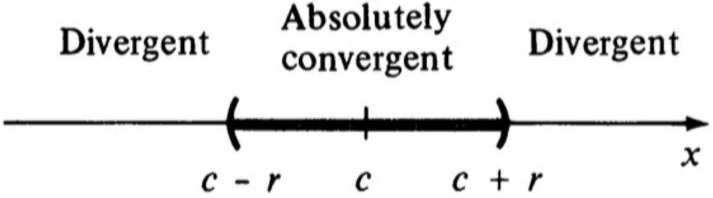
Power series $(x - c)$

Theorem

Let c be a real number and x a variable. A POWER SERIES IN *$(x - c)$ is a series of the form*

$$\sum_{n=0}^{+\infty} a_n(x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots + a_n(x-c)^n + \cdots \quad (10)$$

where each a_n is a real number.



convergency radius

Power series representations of functions

A POWER SERIES $\sum a_n x^n$ can be used to define a function of $f(x)$ whose domain is the interval of convergence of the series. Specifically, for each x in this interval we let $f(x)$ equal the sum of the series, that is

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \tag{11}$$

If a function $f(x)$ is defined in this way we say that $\sum a_n x^n$ is A POWER SERIES REPRESENTATIVE FOR $f(x)$.

THIS ALLOWS US TO FIND VALUES IN A NEW WAY. Specifically, if c is the interval of convergence, thus $f(c)$ can be found or approximated ($\simeq$) be the series.

Derivatives

Theorem

Suppose a power series $\sum a_n x^n$ *has a nonzero radius of convergence* r *and let the function* f *be defined by*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{12}$$

for every x *in the interval of convergence. If* $-r < x < r$*, then:*

$$\begin{aligned} f'(x) &= \sum_{n=0}^{\infty} D_x(a_n x^n) = \sum_{n=1}^{\infty} n a_n x^{(n-1)} \qquad (13) \\ &= a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{(n-1)} + \cdots \end{aligned}$$

Integral

Theorem

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \int_0^x (a_n t^n)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \tag{14}$$
$$= a_0 x + \frac{1}{2} a_1 x^2 + \frac{1}{3} a_2 x^3 + \cdots + \frac{1}{n+1} a_n x^{n+1} + \cdots$$

TO CONSIDER

as can be shown in Equations (13) and (14) the convergency radius remains equal to (12).

Tylor and Maclaurin Series

Tylor and Maclaurin Series

Suppose a function f is represented by a power series in $x - c$, such that

$$f(x) = \sum_{n=0}^{+\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots \tag{15}$$

where the domain of f is an open interval containing c

$$
\begin{aligned}
f'(x) &= \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} && (16) \\
&= a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + 4a_4(x-c)^3 + \cdots \\
f''(x) &= \sum_{n=2}^{\infty} n(n-1) a_n (x-c)^{n-2} && (17) \\
&= 2a_2 + (3 \cdot 2)a_3(x-c) + (4 \cdot 3)a_4(x-c)^2 + \cdots \\
f'''(x) &= \sum_{n=3}^{\infty} n(n-1)(n-2) a_n (x-c)^{n-3} && (18) \\
&= (3 \cdot 2)a_3 + (4 \cdot 3 \cdot 2)a_4(x-c) + \cdots
\end{aligned}
$$

and, for every positive integer k,

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-c)^{n-k} \tag{19}$$

Moreover, each series obtained by differentiation has the same radius of convergence as the original series. Substituting c for x in each of these series representation, we obtain

$$f(c) = a_0 \tag{20}$$

$$f'(c) = a_1 \tag{21}$$

$$f''(c) = 2a_2 \tag{22}$$

$$f'''(c) = (3 \cdot 2)a_3 \tag{23}$$

$$f^{(n)}(c) = n!a_n \longrightarrow a_n = \frac{f^{(n)}(c)}{n!} \tag{24}$$

Taylor series

Theorem

If f is a function and

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n \tag{25}$$

for all x in an open interval containing c, then

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \tag{26}$$

Maclaurin

Corollary

If f is a function and $f(x) = \sum a_n x^n$ for all x in an open interval $(-r, r)$, then

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \qquad (27)$$

References

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