Available online at www.sciencedirect.com
SCIENCE DIRECT®
European Journal of Operational Research 165 (2005) 610–624
EUROPEAN JOURNAL OF OPERATIONAL RESEARCH
www.elsevier.com/locate/dsw

Discrete Optimization

Distribution network design: New problems and related models

Daniela Ambrosino [a,*], Maria Grazia Scutellà [b]

[a] Dip. di Economia e Metodi Quantitativi, Università di Genova, 5, Via Vivaldi, 16126 Genova, Italy
[b] Dipartimento di Informatica, Università di Pisa, Via Buonarroti, 56127 Pisa, Italy

Received 11 June 2002; accepted 1 April 2003
Available online 8 April 2004

Abstract

We study some complex distribution network design problems, which involve facility location, warehousing, transportation and inventory decisions. Several realistic scenarios are investigated. Two kinds of mathematical programming formulations are proposed for all the introduced problems, together with a proof of their correctness. Some formulations extend models proposed by Perl and Daskin (1985) for some warehouse location-routing problems; other formulations are based on flow variables and constraints.

Keywords: Distribution; Location-routing; Integer linear programming models

1. Introduction

All companies that aim to be competitive on the market have to pay attention to their organization related to the entire supply chain. In particular, companies have to analyze the supply chain in order to improve the customer service level without an uncontrolled growth of costs. In few words, companies have to increase the efficiency of their logistics operations. It is therefore of fundamental importance to optimize the flows of goods (and also of information) among the actors of the supply chain, that is suppliers, producers, distributors, and customers.

The Efficient Customer Response (ECR) study [5] pointed out that vertical as well as horizontal integrations are required for the optimization of the flows in the supply chain, and for the optimization of all related activities. That implies agreements among subjects that operate at different levels of the supply chain (vertical integration), and among actors of the same level (horizontal integration).

In this work, we focus on some internal organization issues of the supply chain; in particular, we analyze some network design problems which are typical of producer companies (although they are significant for

* Corresponding author. Tel.: +39-010-2095494; fax: +39-010-2095243.
E-mail address: ambrosin@economia.unige.it (D. Ambrosino).

0377-2217/$ - see front matter
doi:10.1016/j.ejor.2003.04.009

other types of companies, like the distribution companies). The aim is to optimize the flows of goods through the producer network, also called in the literature *distribution network*, from the plants where the goods are produced (supply points) to the demand points, which are essentially distributors such as wholesalers or retailers.

While analyzing a distribution network, we can generally distinguish between:

- the optimization of the flows of goods: in this case we consider an existing distribution network, and we want to optimize the flows of goods through the network;
- the improvement of the existing network: in this case we want to choose the best configuration of the facilities in the network in order to satisfy the goals of the company, while minimizing the overall costs.

Distribution network design problems involve both kinds of analysis. More precisely, these problems consist of determining the best way to transfer goods from the supply to the demand points by choosing the structure of the network (layers, different kinds of facilities operating at different layers, their number and their location), while minimizing the overall costs.

Distribution network design problems involve strategic decisions which influence tactical and operational decisions [4]. In particular, they involve facility location, transportation and inventory decisions, which affect the cost of the distribution system and the quality of the customer service level. So, they are core problems for each company.

As emerged from the above discussion, distribution network design problems involve a lot of integrated decisions, which are difficult to consider all together. Generally, some simplifying assumptions have been adopted in the literature, and only some aspects related to the complex network decisions have been modeled. For instance, in the past some authors dealt with distribution network design problems as pure location problems, without trying to address and integrate the different types of strategic decisions.

Webb [24] and, more recently, Salhi and Rand [21] recognized the error introduced into location problems by ignoring the interdependence between routing and location decisions. Since then, some papers focused on the relationships between facilities and transportation costs, stressing that location of distribution facilities and routing of vehicles from facilities are interdependent decisions. In particular, in recent years, some location–routing problems (LRP) arising in the context of distribution network design problems have been investigated. In these problems, the facility location and the vehicle routing aspects are solved simultaneously [9]. Given a set of candidate depot sites and customer requirements, in its simplest form LRP consists of determining the location of the depots and the routes of the vehicles for serving the customers, in such a way that some constraints, generally related to depot and vehicle capacity, route lengths and durations, and all the customer requirements are satisfied, while minimizing an objective function involving routing costs, vehicle fixed costs, depot fixed costs and depot operating costs.

In [9], the distribution network design problems have been classified according to the number of layers in the distribution network, and to the type of routes between layers. In particular, Laporte introduced the terminology 'route of type R' (for *replenishment*), if the route connects a pair of nodes of two different layers (for instance, a depot is connected to a customer), and 'route of type T' (for *tour*), if it is a tour connecting a node in a layer with more nodes belonging to other layers (for instance, a depot is connected via a tour to a certain number of customers served by the same vehicle). Laporte observed that a distribution network design problem can be formulated as a location–routing problem if and only if routes of type T are allowed, and location decisions arise at least at one layer.

In the last two decades, many LRP models have been proposed in the literature to formulate and solve distribution network design problems. Most of them are related to a simple network with two layers (depots and customers), where routes of type T are allowed. Each model is characterized by the number of depots to locate (single depot or multi-depot), by the presence of capacity constraints (depot capacity and vehicle capacity) and other route constraints, and by the form of the objective function.

In [9], some mathematical models have been proposed by distinguishing between three-index and two-index formulations. Two-index formulations were used for: the single depot LRP, solved in [10] via an exact approach; the multi-depot LRP, which is an extension of the single depot LRP [12]; the multi-depot capacitated LRP [11], solved in an exact way; and some asymmetric versions [13].

Some interesting three-index formulations were proposed in [17] for solving the problem of locating regional blood banks to serve hospitals, and in [19] for evaluating the design of the division's distribution system, with regard to the number, size, and locations of central depots, in the area of Missouri, Oklahoma and Western Kansas. Hansen et al. [6] modified the integer linear programming formulation of Perl and Daskin [19] in order to provide an improved formulation, based on flow variables and flow constraints. They used the model for helping a Danish company operating in the chemical industry to choose the plant where to start a new production. This is also used as a long term strategical decision tool. Another classification of location–routing problems can be found in [15].

As far as the solution methods are concerned, due to the complexity of LRP exact methods have been limited to small sized instances, and to two-index formulations. Three-index formulations, more versatile but more complex, have not been solved exactly until now.

For solving larger problems and real instances, the only helpful methods have been heuristics (see, for example, [8,14], where the practical use of LRP for designing a newspaper distribution system is illustrated). The major part of the heuristic approaches are based on the decomposition of the problem into subproblems which are then solved sequentially, in order to address interdependencies. Subproblems are usually solved in an approximate way [6,19]. Another popular approach, also used within the decomposition methods, is the saving method [6,22].

A new approach has been presented in [23]. The authors proposed a two-phase tabù search approach which integrates facility location and routing decisions. They also compared the performance of alternative LRP heuristics, by comparing their approach with one of the algorithms proposed in [22], and by furnishing a set of test problems. Finally, we want to mention an interesting set-partitioning formulation of some LRP problems proposed in [3].

Generally, no LRP studied in the literature includes inventories, except for two cases. In [20], a mathematical model for explicitly representing the trade off among facility, transportation and inventory costs is proposed; this integrated model differs from existing models only in the form of the objective function. In [16], the authors try to estimate the inventory costs and include them in the fixed costs related to facilities.

In this work, we will address more complex distribution network design problems, which have so far received limited attention, and which involve facility location, transportation and inventory decisions. We will refer to these problems as the *integrated distribution network design problems*. More precisely, we will consider distribution networks made up of four layers (plants, central depots, regional depots and customers (demand points)), with the aim of defining the number and the location of the different types of facilities for designing a new distribution network or for improving an existing network. The analysis will take into account facility, warehousing, transportation and inventory costs. Realistic scenarios will be investigated.

Dynamic versions of the integrated problems will be discussed as well, where the distribution network is analyzed over a certain time horizon rather than in a fixed period of time.

The main contribution of this paper is the statement of two kinds of mathematical programming formulations for all the introduced problems. Some formulations extend models proposed in [19] for some warehouse location–routing problems; other formulations are based on flow variables and constraints. These formulations have to be viewed as the first step towards the definition of solution methods for complex distribution network design problems. In fact, especially when defining heuristic solution methods, in order to estimate the performance of the algorithms it is generally required to compute the optimum solutions (at least for some small sized instances) and/or suitable lower bounds (in the case of minimization). In both cases, the statement of mathematical programming formulations, to be solved exactly via an existing general-purpose code, or to be relaxed to generate suitable lower bounds, generally constitutes the starting point for the algorithm definition and evaluation. Furthermore, some hints can be obtained about the practical complexity of the problems under investigation. This aspect will be illustrated in Section 4 for some random instances, as well as for some instances related to a real case study.

The paper is organized as follows. In Section 2 we introduce the integrated distribution network design problems under investigation. Section 3 presents models to mathematically formulate the problems introduced in Section 2. Section 4 discusses some preliminary computational results. Finally, Section 5 contains some conclusions and future research development.

2. The integrated distribution network design problems

Here we introduce some distribution network design problems involving several kinds of nodes at different layers of the distribution network, and more possibilities for supplying depots and for serving customers. According to Laporte's classification [9], in their general form the problems can be categorized as 4/R/T/T, that means distribution networks made up of four layers, with routes of type replenishment and of type tour. The goal of the analysis is to determine the best distribution system in order to minimize facility, warehousing, transportation and inventory costs, and to grant a certain customer service level.

Let us start by defining the nodes of the distribution network, and the kinds of available routes.

The nodes of the distribution network are partitioned into *plants* (or supply points, *P*), *central depots* (*CD*), *regional depots* (*RD*), *transit points* (*TP*), *clients* (*C*) and *big clients* (*BC*). The nodes are grouped into four layers as follows: supply points (P), central depots (CD), regional facilities (RD and TP) and demand points (BC and C).

In the network, plants send goods to CD; CD transfer them to TP and RD, and they may serve clients C which, however, are usually served by TP and RD; BC are served directly by CD because of their large demand. TP are similar to RD, but they do not maintain inventories; in fact, TP receive goods and immediately deliver them to the clients.

We assume to know the location and the demand of each client and big client, whose demands are specified in units of a single representative commodity. Also the location and the capacity of each potential facility are known. In particular, the capacity of CD and RD is the maximum inventory level that they are able to maintain, while the capacity of a transit point is the maximum quantity of commodity that may enter the transit point. Furthermore, we assume to know the maximum number of vehicles available for the whole distribution network and the capacity of each vehicle (the fleet is non-homogeneous).

Finally, as far as the connection among the nodes of the network is concerned, we admit the following kinds of routes: direct replenishment from the plants to the CD (routes of type R); tours from CD to RD and TP (including exactly one CD); delivery tours from CD to BC (including exactly one CD); mixed tours from CD to RD, TP and BC (including exactly one CD); and delivery tours from CD, RD or TP to the clients C (including exactly one facility among CD, RD and TP).

We can now define the integrated distribution network design problem, in its general form, as the problem of determining:

- (*location decisions*) where to locate CD, RD, TP, and the number of each type of facility;
- (*allocation decisions*) how to allocate the clients to the open facilities (CD, RD or TP); how to allocate the big clients to the open CD; how to allocate the open RD and TP to the open CD for their supply; how to allocate the open CD to the plants;
- (*routing decisions*) the vehicle routes for serving the clients, starting from a CD, a RD or a TP (we will refer to this routing as *routing of the second level*), and the mixed tours for visiting RD, TP and BC, starting from a CD (we will refer to this routing as *routing of the first level*), in such a way each client and big

client, as well as each open facility (in the routing of the first level), be served by exactly one vehicle, and by satisfying capacity constraints related to the vehicles;
- (*inventory decisions*) the quantity of goods which must be shipped from the plants to the CD, and from the CD to the RD and the TP, and the inventory level at CD and RD, in such a way to satisfy the capacity constraints at the facilities;

in order to guarantee a certain customer service level, and to minimize the sum of the facility, warehousing, transportation and inventory costs. The customer service level is expressed as a minimum stock level which has to be maintained at each open CD and RD.

The introduced problem involves many strategic decisions which arise in designing of several kinds of distribution networks. This problem is complex for several reasons. One is that the location decisions are related to two different kinds of facilities, which have a different role in the distribution network, that is CD from one hand, and RD and TP from the other hand. Their location influences all the available routes.

The problem under investigation differs from the classical warehouse location–routing problem (WLRP), defined as the problem of a company which has to ship goods from some supply sources (plants) to a certain number of depots via truck loads, and then has to deliver the goods from the depots to some customers (see [18,19]), in many aspects, like:

- there are different types of facilities to locate and, consequently, the distribution network is more complex;
- there are different types of clients (clients and big clients);
- the routing decisions involve more layers of the distribution network;
- there are inventory decisions at CD and RD.

In order to formally state the problem, let us introduce the following notation, which will be used throughout the paper:

$C = \{1, \ldots, n\}$ set of clients;
$BC = \{1, \ldots, n'\}$ set of the big clients;
$NIW = \{1, \ldots, m\}$ set of the potential transit points (the acronym stands for No Inventory Warehouse);
$IW = \{1, \ldots, m'\}$ set of the potential RD;
$W = NIW \cup IW$ set of the potential regional facilities;
$CD = \{1, \ldots, h\}$ set of the potential CD;
$P = \{1, \ldots, p\}$ set of the existing plants;
$V = \{1, \ldots, v\}$ set of vehicles.

Then, let us introduce a directed graph $G = (N, A)$ to model the distribution network, where:

$N = BC \cup C \cup W \cup CD$ is the set of the nodes, which does not include the plants;
$N_1 = BC \cup W \cup CD$ denotes the set of nodes involved by the routing decisions of the first level;
$N_2 = C \cup W \cup CD$ denotes the set of nodes involved by the routing decisions of the second level;
A is the set of the arcs (that is the available links between nodes of the distribution network).

Furthermore, define:

D_i demand of client i, $\forall i \in C \cup BC$;
C_{gh} distance (in kilometers) between nodes g and h, $\forall g, h \in N$;

K_k capacity of vehicle k, $\forall k \in V$;
WC_j capacity of facility j, $\forall j \in W \cup CD$;
VC_j warehousing cost per unit throughput at j, $\forall j \in W \cup CD$ (only for clients and big clients);
DC_k transportation cost per kilometer of vehicle k, $\forall k \in V$;
TC_k fixed cost for the usage of vehicle k, $\forall k \in V$;
FC_j fixed cost of establishing the facility j, $\forall j \in W \cup CD$;
CP_{uj} unit shipping cost for transferring goods from the plant u to CD j, $\forall u \in P$, $\forall j \in CD$;
IC_j inventory cost for unit of good at depot j, $\forall j \in IW \cup CD$;
s_j initial stock level at depot j, $\forall j \in IW \cup CD$;
bs_j minimum stock level to keep at depot j (if opened), $\forall j \in IW \cup CD$.

3. The models

Starting from the general problem introduced in Section 2, here we will discuss both static and dynamic scenarios. In both cases, mathematical programming formulations will be proposed: someones extend the three-index formulation of WLRP proposed by Perl and Daskin [19]; others use flow variables and constraints, and are inspired by models proposed in [6].

3.1. The static scenario

Let us consider a special case of the integrated distribution network design problem, where only one plant is assumed to be operative: it represents the only supply point of the entire distribution network.

Furthermore, inventories are not maintained. Therefore, there is no distinction between RD and TP (we will refer to TP when mentioning potential regional facilities), and there are no minimum stock level considerations.

In this scenario, the distribution network has the following structure:

1° layer: it contains the only existing plant. We assume that there are no capacity constraints related to the quantity of throughput of the plant. The plant supplies all the opened CD (facilities of the second layer). Due to the presence of a single plant, the problem does not involve the allocation of the CD to the plant.
2° layer: it contains the potential CD. The CD supply all the opened TP (facilities of the third layer), serve the BC and they may supply some clients (nodes of the fourth layer). There is the problem to allocate both the opened TP and the big clients BC to the open CD.
3° layer: it includes the potential transit points, which supply the clients.
4° layer: it is formed by clients and big clients.

Fig. 1 shows a feasible solution to the static scenario.

By the above discussion, this scenario specializes the integrated distribution network design problem as follows. It is the problem of determining:

- (*location decisions*) where to locate the CD and the TP, and their number;
- (*allocation decisions*) how to allocate the big clients and the open TP to the open CD, and how to allocate the clients to the open facilities (CD or TP);
- (*routing decisions*) the vehicle routes for serving the clients, starting from an open CD or from an open TP (*routing of the second level*), and the mixed tours for visiting the open TP and the BC, starting from an open CD (*routing of the first level*), in such a way each client and big client, as well as each open TP (in

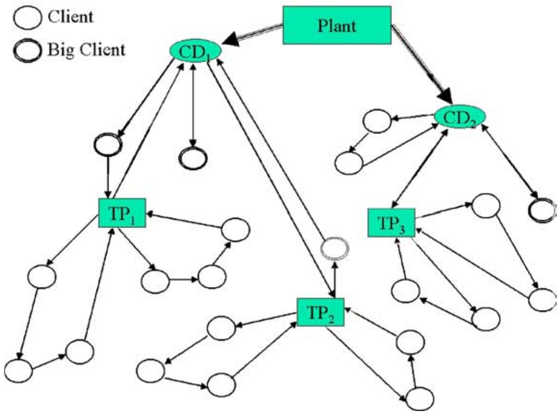

Fig. 1. A feasible solution to the static scenario.

the routing of the first level), be served by exactly one vehicle, and by satisfying the capacity constraints related to the vehicles;

- (*inventory decisions*) the quantity of good which must be shipped from the existing plant to the open CD, and the quantity to transfer from the open CD to the open TP, in such a way to satisfy the capacity constraints at the CD and at the TP;

in order to minimize the overall costs.

The just introduced scenario differs from WLRP [19] essentially for the routing decisions and for the presence of a fixed cost for the usage of the vehicles (that may imply a better usage of the fleet). On the other hand, there is no limit to the maximum length of the routes. Furthermore, the scenario addresses two types of customers: clients and big clients.

Let us propose now a three-index formulation for the scenario by using the notation introduced in Section 2, where $W = NIW$ denotes the set of the potential transit points, and p represents the only existing plant. The formulation is based on the following set of variables, where a distinction is made between variables defining the routing of the first level, and variables defining the routing of the second level:

$$x_{ghk} = \begin{cases} 1 & \text{if } g \text{ precedes } h \text{ in route of vehicle } k \text{ in routing of second level}, \quad \forall g,h \in N_2,\ \forall k \in V, \\ 0 & \text{otherwise,} \end{cases}$$

$$r_{ghk} = \begin{cases} 1 & \text{if } g \text{ precedes } h \text{ in route of vehicle } k \text{ in routing of first level}, \quad \forall g,h \in N_1,\ \forall k \in V, \\ 0 & \text{otherwise,} \end{cases}$$

$$y_{ij} = \begin{cases} 1 & \text{if client } i \text{ is assigned to facility } j, \quad \forall i \in C \cup BC,\ \forall j \in W \cup CD, \\ 0 & \text{otherwise,} \end{cases}$$

$$z_j = \begin{cases} 1 & \text{if the facility } j \text{ is opened}, \quad \forall j \in W \cup CD, \\ 0 & \text{otherwise,} \end{cases}$$

$$v_k^1 = \begin{cases} 1 & \text{if vehicle } k \text{ is used in a tour of the first level}, \quad \forall k \in V, \\ 0 & \text{otherwise,} \end{cases}$$

$$v_k^2 = \begin{cases} 1 & \text{if vehicle } k \text{ is used in a tour of the second level}, \quad \forall k \in V, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, let us introduce the following variables:

$fp_j \geqslant 0$: it is the quantity of good shipped from the plant p to the CD j, $\forall j \in CD$,
$f_{ljk} \geqslant 0$: it is the quantity of good shipped from the CD l to the TP j with the vehicle k, $\forall l \in CD$, $\forall j \in W, \forall k \in V$.

The formulation is the following:

$$\text{(S)} \quad \text{Min} \sum_{j \in W \cup CD} FC_j * z_j + \sum_{j \in W \cup CD} VC_j \left(\sum_{i \in C \cup BC} D_i * y_{ij} \right) + \sum_{j \in CD} CP_{pj} * fp_j + \sum_{k \in V} TC_k * (v_{k_1} + v_{k_2}) + \sum_{k \in V} \sum_{g \in N_2} \sum_{h \in N_2} C_{gh} * DC_k * x_{ghk} + \sum_{k \in V} \sum_{g \in N_1} \sum_{h \in N_1} C_{gh} * DC_k * r_{ghk}, \tag{1}$$

$$\sum_{k \in V} \sum_{h \in N_2} x_{ihk} = 1, \quad \forall i \in C, \tag{2}$$

$$\sum_{g \in N_2} x_{ghk} - \sum_{g \in N_2} x_{hgk} = 0, \quad \forall h \in N_2, \ \forall k \in V, \tag{3}$$

$$\sum_{g \in S} \sum_{h \in \overline{S}} \sum_{k \in V} x_{ghk} \geqslant 1, \quad \forall S \subset N_2, \text{ such that } W \cup CD \subseteq S, \tag{4}$$

$$\sum_{i \in N_2} \sum_{j \in W \cup CD} x_{ijk} \leqslant 1, \quad \forall k \in V, \tag{5}$$

$$\sum_{k \in V} \sum_{h \in N_1} r_{ihk} = 1, \quad \forall i \in BC, \tag{6}$$

$$\sum_{k \in V} \sum_{h \in N_1} r_{jhk} = z_j, \quad \forall j \in W, \tag{7}$$

$$\sum_{g \in N_1} r_{ghk} - \sum_{g \in N_1} r_{hgk} = 0, \quad \forall h \in N_1, \ \forall k \in V, \tag{8}$$

$$\sum_{g \in S} \sum_{h \in \overline{S}} \sum_{k \in V} r_{ghk} \geqslant 1, \quad \forall S \subset N_1, \text{ such that } CD \subseteq S, \quad \overline{S} \cap BC \neq \emptyset, \tag{9}$$

$$\sum_{g \in S} \sum_{h \in \overline{S}} \sum_{k \in V} r_{ghk} \geqslant z_j, \quad \forall j \in W, \ \forall S \subset N_1, \text{ such that } CD \subseteq S, \quad \overline{S} \cap \{j\} \neq \emptyset, \tag{10}$$

$$\sum_{h \in N_1} \sum_{j \in CD} r_{hjk} \leqslant 1, \quad \forall k \in V, \tag{11}$$

$$K_k * \sum_{h \in N_1} r_{jhk} - f_{ljk} \geqslant 0, \quad \forall k \in V, \ \forall j \in W, \ \forall l \in CD, \tag{12}$$

$$K_k * \sum_{h \in N_1} r_{lhk} - f_{ljk} \geqslant 0, \quad \forall k \in V, \ \forall j \in W, \ \forall l \in CD, \tag{13}$$

$$\sum_{i \in C} D_i \sum_{h \in N_2} x_{ihk} \leqslant K_k * v_k^2, \quad \forall k \in V, \tag{14}$$

$$\sum_{l \in CD} \sum_{j \in W} f_{ljk} + \sum_{i \in BC} D_i \sum_{h \in N_1} r_{ihk} \leqslant K_k * v_k^1, \quad \forall k \in V, \tag{15}$$

$$v_k^1 + v_k^2 \leqslant 1, \quad \forall k \in V, \tag{16}$$

$$\sum_{l \in CD} \sum_{k \in V} f_{ljk} - WC_j * z_j \leqslant 0, \quad \forall j \in W, \tag{17}$$

$$fp_j - WC_j * z_j \leqslant 0, \quad \forall j \in CD, \tag{18}$$

$$\sum_{h \in N_2} x_{ihk} + \sum_{h \in N_2} x_{jhk} - y_{ij} \leqslant 1, \quad \forall i \in C, \ \forall k \in V, \ \forall j \in W \cup CD, \tag{19}$$

$$\sum_{h \in N_1} r_{ihk} + \sum_{h \in N_1} r_{jhk} - y_{ij} \leqslant 1, \quad \forall i \in BC, \ \forall k \in V, \ \forall j \in CD, \tag{20}$$

$$\sum_{l \in CD} \sum_{k \in V} f_{ljk} - \sum_{i \in C} D_i * y_{ij} = 0, \quad \forall j \in W, \tag{21}$$

$$\sum_{h \in W} \sum_{k \in V} f_{jhk} + \sum_{i \in C \cup BC} D_i * y_{ij} = fp_j, \quad \forall j \in CD, \tag{22}$$

$$\begin{aligned} & x_{ghk} \in \{0,1\}, \quad \forall g,h \in N_2, \ \forall k \in V, \\ & r_{ghk} \in \{0,1\}, \quad \forall g,h \in N_1, \ \forall k \in V, \\ & y_{ij} \in \{0,1\}, \quad \forall i \in C \cup BC, \ \forall j \in W \cup CD, \\ & z_j \in \{0,1\}, \quad \forall j \in W \cup CD, \\ & v_k^1, v_k^2 \in \{0,1\}, \quad \forall k \in V, \\ & f_{ljk} \geqslant 0, \quad \forall l \in CD, \ \forall j \in W, \ \forall k \in V, \\ & fp_j \geqslant 0, \quad \forall j \in CD. \end{aligned} \tag{23}$$

Model (S) is a correct formulation of the scenario under investigation. In fact, the constraints related to the routing of the second level (2, 3, 4, 5) define tours which include all the clients, and exactly one facility. Similarly, the constraints related to the routing of the first level (6, 7, 8, 9, 10, 11) define tours which include all the big clients and all the open TP, and exactly one CD. However, to have feasible tours, it is necessary to guarantee that the unique facility in the tours of the second level, and the unique CD in the tours of the first level, be open. This is implied by the constraints linking the routing of the second level to the one of the first level, (19) and (21), together with the capacity constraints (17), and by the constraints linking the routing of the first level to the shipment of good from the plant to the CD, (22) and (20), together with the capacity constraints (18). In fact, the first kind of constraints state that the overall demand of the clients allocated to a TP j must be equal to the amount of flow received by j, through the unique tour of the first level serving j: from (17), this quantity can be positive only if j is an open TP. Similarly, the second kind of constraints state that the overall amount of good sent by a CD j to the TP and the big clients allocated to it must be equal to the quantity of good received by the plant p: from (18), this quantity can be positive only if j is an open CD. Observe that (12) and (13) guarantee that $f_{ljk} \geqslant 0$ only if both the CD l and the depot j are visited by the same vehicle k.

The remaining constraints are related to the capacity of the vehicles. These are different for the routing of the two levels: in the routing of the first level they sum the quantities of good shipped to TP and the ones delivered to BC, while in the routing of the second level they sum the amounts of good delivered to the clients. This distinction is made possible in the model via two kinds of variables, v_k^1 and v_k^2, which specify whether a vehicle k is used for the routing of the first or of the second level, respectively. Such a distinction imposes the presence of the constraints (16), which limit the usage of each vehicle to at most one kind of routing (either of the first or of the second level).

The problem under investigation can be alternatively formulated using flow variables and flow constraints, along the lines suggested by Hansen et al. [6]. At this end, let us introduce the following set of variables, which model the flow of good along the distribution network, by distinguishing the flow of good in the routing of the first level from the one of the second level:

$fx_{gik} \geqslant 0$: quantity of good shipped through the arc (g, i) with the vehicle k, $\forall g, i \in N_2$, $\forall k \in V$ (routing of the second level);

$fr_{gik} \geqslant 0$: quantity of good shipped through the arc (g, i) with the vehicle k, $\forall g, i \in N_1$, $\forall k \in V$ (routing of the first level).

The flow formulation, referred to as (F), differs from (S) for the presence of flow constraints which replace the subtour elimination constraints. More precisely, in the routing of the second level constraints (4) are replaced by:

$$fx_{gik} \leqslant x_{gik} K_k, \quad \forall g, i \in N_2, \ \forall k \in V, \tag{24}$$

$$\sum_{g \in N_2} fx_{gik} - \sum_{g \in N_2} fx_{igk} = \sum_{h \in N_2} x_{ihk} * D_i, \quad \forall i \in C, \ \forall k \in V, \tag{25}$$

$$\sum_{g \in N_2} fx_{gjk} - \sum_{g \in N_2} fx_{jgk} = -\left[\sum_{i \in C}\left[\sum_{h \in N_2} x_{ihk} * D_i\right]\right], \quad \forall j \in W \cup CD, \ \forall k \in V. \tag{26}$$

Constraints (9) and (10) are replaced by:

$$fr_{gik} \leqslant r_{gik} * K_k, \quad \forall g, i \in N_1, \ \forall k \in V, \tag{27}$$

$$\sum_{g \in N_1} fr_{gik} - \sum_{g \in N_1} fr_{igk} = \sum_{h \in N_1} r_{ihk} * D_i, \quad \forall i \in BC, \forall k \in V, \tag{28}$$

$$\sum_{g \in N_1} fr_{gjk} - \sum_{g \in N_1} fr_{jgk} = \sum_{l \in CD} f_{ljk}, \quad \forall j \in W, \forall k \in V, \tag{29}$$

$$\sum_{g \in N_1} fr_{gCDk} - \sum_{g \in N_1} fr_{CDgk} = -\left[\sum_{j \in W}\sum_{l \in CD} f_{ljk} + \sum_{i \in BC}\left[\sum_{h \in N_1} r_{ihk} * D_i\right]\right], \quad \forall k \in V. \tag{30}$$

The flow conservation constraints (25) and (26), together with (24), guarantee that subtours formed only by clients are not feasible, since each client requires a positive amount of flow. A source of flow, that is a facility, must therefore belong to each feasible tour of the second level. The same reasoning holds true for the routing of the first level.

It is easy to prove that constraints (12), (14) and (15) of (S) are redundant, and therefore can be eliminated. Also variables v_k^1 and v_k^2, and constraints (16), can be removed, provided that the following constraints are inserted to (F):

$$\sum_{i \in N_2} \sum_{j \in W \cup CD} x_{ijk} + \sum_{i \in N_1} \sum_{j \in CD} r_{ijk} \leqslant 1, \quad \forall k \in V. \tag{31}$$

Finally, it is possible to remove constraints (19)–(22), and therefore the allocation variables y_{ij}, by introducing additional flow conservation constraints for each transit point and for each central depot, stating that the total flow entering the transit point (or the central depot) must be equal to total outgoing flow. Clearly, the objective function of (F) must be rewritten, by substituting y_{ij} via the Eqs. (21) and (22), and by suitably substituting the terms $(v_k^1 + v_k^2)$ by means of the design variables x_{gik} and r_{gik}.

In conclusion, observe that a key difference between model (F) and model (S) is given by the number of the constraints: in (F) such a number is polynomial with respect to the problem size; that makes (F) tractable in practice, when solution approaches are investigated. As far as the number of the variables is concerned, (F) uses the flow variables, which are not present in (S); on the other hand, it does not use the variables v_k^1, v_k^2 and y_{ij}. So, depending on the number of the links, of the clients and of the potential facilities in the input distribution network, and depending on the size of the fleet of vehicles, (F) may be also more compact than (S) in the number of the variables.

3.1.1. A special scenario

Sometimes producer companies prefer to outsource some logistics activities to some specialized operators. For instance, a common usage is to commit some transportation activities, like the delivery operations to the clients. This is the case, for example, of several producer companies operating in the Italian market.

In this case, the integrated distribution network design problem does not involve routing decisions for serving the clients (that is the elements of C) but, on the other hand, it involves a (simpler) assignment problem of the clients to the facilities. The assignment costs represent now the costs of the transportation service.

The above company choice can be easily formulated by replacing the set of the constraints related to the routing of the second level, in model (S), by a set of assignment constraints. More precisely, (2)–(5) and (19) are replaced by: $\sum_{j \in W \cup CD} y_{ij} = 1$, $\forall i \in C$.

Also constraints (14) can be suitably rewritten, by using the assignment variables y_{ij}. In the objective function, the term $\sum_{k \in V} \sum_{g \in N_2} \sum_{h \in N_2} C_{gh} * DC_k * x_{ghk}$ is replaced by $\sum_{i \in C} \sum_{j \in W \cup CD} a_{ij} * y_{ij} * D_i$. That is, the cost for serving the clients is now given in terms of the costs a_{ij}, where a_{ij} is the cost, for unit of good, payed to the operator for a delivery to client i from depot j, $\forall i \in C$, $\forall j \in W \cup CD$.

Similarly, the flow formulation (F) can be suitably specialized in order to deal with the commitment of the delivery operations to the clients.

3.2. Some dynamic models

In the context of distribution network design problems, it may be interesting to analyze the distribution network during a time horizon $T = \{1, 2, \ldots, \tau\}$, split into τ equal periods of time, rather that in a single period of time, as assumed in the static scenarios. In this case, it is particularly relevant to take into account inventory decisions.

Here we will extend the previous scenario over the time, assuming that each location decision is taken at the beginning of the time horizon T, and it can not be changed anymore. In order to present the model, let us extend the notation introduced in Section 2 as follows:

D_{it} is the demand of client i in period t, $\forall i \in C \cup BC$, $\forall t \in T$;
$\tilde{C}_t$ is the set of the clients with a positive demand in period t, $\forall t \in T$;
$\widetilde{BC}_t$ is the set of the big clients with a positive demand in period t, $\forall t \in T$.

In the proposed extension, inventories are admitted both at CD and at the depots of the regional layer. Therefore, the location decisions now include the choice of the kind of facility to open (RD or TP) at the regional layer. At each CD and at each RD, inventories are maintained in such a way to grant a certain service level; that is, we impose that the stock of good in each period of time be higher than a minimum safety stock level.

Usually, the minimum safety stock level depends on the lead time for receiving the goods, and on the overall customer demand allocated to the depot. For the sake of the simplicity, we will assume here that the minimum safety stock level to maintain at depot j (if opened), $\forall j \in IW \cup CD$, denoted by bs_j in Section 2, be an input data.

Now, we can easily extend model (S), by modifying the variables of (S) as follows, in order to take into account the time horizon T:

$$_{ghkt} = \begin{cases} 1 & \text{if } g \text{ precedes } h \text{ in the route of vehicle } k \text{ in the routing of the second level,} \\ & \quad \text{in period } t, \forall g, h \in N_2,\ \forall k \in V,\ \forall t \in T, \\ 0 & \text{otherwise,} \end{cases}$$

$$r_{ghkt} = \begin{cases} 1 & \text{if } g \text{ precedes } h \text{ in the route of vehicle } k \text{ in the routing of the first level,} \\ & \quad \text{in period } t, \forall g, h \in N_1,\ \forall k \in V,\ \forall t \in T, \\ 0 & \text{otherwise,} \end{cases}$$

$$y_{ijt} = \begin{cases} 1 & \text{if client } i \text{ is allocated to facility } j, \text{ in period } t, \forall i \in C,\ \forall j \in W \cup CD,\ \forall t \in T, \\ 0 & \text{otherwise,} \end{cases}$$

$$v^1_{kt} = \begin{cases} 1 & \text{if vehicle } k \text{ is used in a tour of the first level, in period } t, \forall k \in V,\ \forall t \in T, \\ 0 & \text{otherwise,} \end{cases}$$

$$v^2_{kt} = \begin{cases} 1 & \text{if vehicle } k \text{ is used in a tour of the second level, in period } t, \forall k \in V,\ \forall t \in T, \\ 0 & \text{otherwise.} \end{cases}$$

$f_{ljkt} \geqslant 0$: quantity of good shipped from the CD l to the depot j with vehicle k in period t, $\forall l \in CD$, $\forall j \in W$, $\forall k \in V$, $\forall t \in T$.

$fp_{jt} \geqslant 0$: quantity of good shipped from the existing plant p to the CD j in period t, $\forall j \in CD$, $\forall t \in T$.

Furthermore, let us introduce the following new set of variables:

s_{jt}: stock level at depot j at the end of period t, $\forall j \in IW \cup CD$, $\forall t \in T$.

The dynamic model differs from (S) in the objective function. In fact, now costs are referred to the whole time horizon (except for the fixed costs for opening facilities), and there is a new term taking into account the inventory costs:

$$\sum_{j \in IW \cup CD} \sum_{t \in T} IC_j * s_{jt}.$$

As far as the set of constraints is concerned, let us put the emphasis on the constraints related to RD and CD, since they involve stock level considerations. In fact, whereas the constraints related to TP are similar to those presented in model (S), when suitably extended to the time horizon T, the ones related to RD and CD must take into account the stock level, and the satisfaction of the customer service level:

$$s_{jt-1} + \sum_{l \in CD} \sum_{k \in V} f_{ljkt} - \sum_{i \in \tilde{C}_t} D_{it} * y_{ijt} = s_{jt}, \quad \forall j \in IW, \ \forall t \in T, \tag{32}$$

$$s_{jt-1} + fp_{jt} - \sum_{h \in W} \sum_{k \in V} f_{jhkt} - \sum_{i \in \tilde{C}_t \cup \tilde{B}C_t} D_{it} * y_{ijt} = s_{jt}, \quad \forall j \in CD, \ \forall t \in T, \tag{33}$$

$$s_{jt} - WC_j * z_j \leqslant 0, \quad \forall j \in IW \cup CD, \ \forall t \in T, \tag{34}$$

$$s_{jt} \geqslant bs_j, \quad \forall j \in IW \cup CD, \ \forall t \in T. \tag{35}$$

Observe that, in each period of time t, only the clients and the big clients having a positive demand in t are taken into consideration. Moreover, the same client can be allocated to different facilities in different periods of time.

Therefore, the subtour elimination constraints must be modified accordingly. For additional details about the dynamic models, the interested reader is referred to [2].

4. Some computational results

In order to investigate the practical complexity of the proposed scenarios, we used, as a case study, the distribution network design problem described in Section 3.1.1. More precisely, we solved the corresponding flow formulation, on a limited set of instances, via a general-purpose code, with the aim of computing the optimum solution (whenever possible) and the linear programming lower bound. As already observed, mathematical models are often used in this way for having a hint about the practical "difficulty" of hard combinatorial optimization problems and, at the same time, to gather useful information, which can then be used for the definition and the mathematical analysis of specific algorithmic approaches.

In our study, we considered 12 instances, whose main characteristics are reported in Table 1. Instances 11 and 12 describe an Italian real case study, and they differ only in the cardinality of the fleet of vehicles. The others have been randomly generated, in such a way to reflect the main characteristics of the case study (in terms of proportion between number of clients and number of potential facilities, and in terms of form of the involved costs). The random instances, of increasing size, differ for the number of potential facilities to locate (#CD and #TP), for the number of clients to serve (#BC and #C), and for the cardinality of the fleet (#V). Given the number of clients and big clients (ranging from 25 to 100 and from 5 to 30, respectively), given the number of potential CD and TP (ranging from 2 to 5 and from 5 to 20, respectively),

Table 1
The instances

Instance	# *CD*	# *TP*	# *BC*	# *C*	# *V*	# Variables	Integer	# Constraints
1	2	5	5	25	5	1.614	842	894
2	2	5	10	50	10	6.069	3.077	3.319
3	3	10	10	50	10	11.316	5.723	5.856
4	3	10	20	75	17	37.966	18.940	19.783
5	3	15	20	75	17	50.586	25.270	26.003
6	3	15	30	100	22	107.736	53.706	55.404
7	4	15	20	75	17	53.519	26.638	27.348
8	4	15	30	100	22	112.622	56.015	57.683
9	4	20	30	100	22	137.162	68.250	69.773
10	5	20	30	100	22	142.715	70.835	72.328
11	5	23	31	104	30	213.485	105.600	108.214
12	5	23	31	104	41	290.683	143.242	147.814

Table 2
Computational results

Instance	Best integer		MIP best bound	LP bound		Gap %
	CPU time (seconds)	Cost		CPU time (seconds)	Cost	
1	982.6	10,838,332	10,838,332	0.67	9,849,963	0.00
2	38921.0	20,364,917	19,713,518	13.42	17,755,396	3.20
3	90000.0	33,746,464	29,287,291	91.31	27,868,779	13.21
4	90000.0	59,717,705	45,556,479	294.18	42,127,147	23.71
5	314652.8	74,294,000	54,736,000	671.49	48,603,764	26.33
6	403530.2	108,890,000	76,743,000	4389.29	72,930,351	29.52
7	176484.5	91,788,000	61,501,000	721.66	53,656,490	33.00
8	470558.3	116,170,000	77,164,000	13,302.85	71,410,007	33.58
9	651130.9	120,730,000	74,059,000	34,602.36	68,171,245	38.66
10	647877.2	142,130,000	88,132,000	21,074.54	81,791,019	37.99
11	822156.3	82,544,000	46,320,000	59,140.86	42,898,988	43.89
12	925552.8	88,770,000	47,120,000	122,837.47	42,898,988	46.92

and given the cardinality of the fleet, the demand of the clients (C and BC) and the capacity of CD and TP have been randomly generated as follows.

The demand of the clients is generated in the interval $[1, 33]$, while the demand of the big clients is generated in the interval $(33, 54]$. Moreover, the capacity of each TP is no greater than 110, while the capacity of each CD is generated in an interval $[500, b]$, where b depends on the global demand of C and BC. In particular, the capacities of the CD are generated in such a way to guarantee the existence of a feasible solution.

The mathematical formulations corresponding to the considered instances are characterized by a large number of variables and constraint (see the last three columns of Table 1). We tried to solve them to optimality using the commercial code CPLEX 7.0 [7], within a time limit of 25 hours for the small instances, and of some days for the largest instances.

For each instance, in Table 2 we report the CPU time (in seconds) and the cost of the best integer solution found by CPLEX, the best lower bound provided by CPLEX (MIP bound), the CPU time and the cost of the linear programming bound, and the gap (in percentage) between the cost of the best integer solution and the higher lower bound.

Table 2 shows that the optimum solution has been obtained only for the first instance; for the others, the best integer solution is generally far from the best lower bound, which is the one provided by CPLEX. In particular, our computational investigation showed that, when solving large instances, CPLEX spends the majority of its time computing the first integer solution, and this time grows with the size of the instances. These results suggest that the integrated distribution network design problems can be very difficult to solve (at least, using the proposed formulations). Also the computational time for solving the linear programming relaxation is considerable; for instance, the LP bound of the instances 11 and 12 required 59,140.9 and 122,837.5 seconds, respectively.

In order to close the gap, the study of some heuristic approaches seems thus a promising avenue. A local search algorithm is under development, which is based both on classical neighborhood structures, and also on multi-exchange neighborhood structures (see [1]). This heuristic approach will be the subject of a future work.

5. Conclusions

In this work, we have studied some complex distribution network design problems, which involve facility, warehousing, transportation and inventory decisions. Some realistic scenarios have been investigated. Dynamic versions have been discussed as well, where the distribution network is analyzed over a certain time horizon rather than in a fixed period of time.

Two kinds of mathematical programming formulations have been proposed. Some formulations extend models proposed by Perl and Daskin [19] for some warehouse location–routing problems; other formulations are based on flow variables and constraints. These formulations constitute the first step towards the study of distribution network design problems which are more complex with respect to the ones studied in the literature so far. In particular, these formulations can be used both for evaluating the practical "difficulty" of some benchmark instances, and as a mathematical basis to design efficient solution methods. The research is under development, and it will be the subject of a future work.

References

[1] R.K. Ahuja, J.B. Orlin, D. Sharma, Very large-scale neighborhood search, International Transactions in Operational Research 7 (2000) 301–317.
[2] D. Ambrosino, M.G. Scutellà, Distribution network design: New problems and related models, TR 01-5, Dipartimento di Informatica, Università di Pisa, 2001.
[3] R.T. Berger, Location–routing models for distribution system design, UMI, Dissertation Information Service, 1997.
[4] T.G. Crainic, G. Laporte, Planning models for freight transportation, European Journal of Operational Research 97 (1997) 409–438.
[5] ECR Italia: Riprogettazione della Supply Chain, SDA Bocconi, Aprile 1995.
[6] P.H. Hansen, B. Hegedahl, S. Hjortkjaer, B. Obel, A heuristic solution to the warehouse location–routing problem, European Journal of Operational Research 76 (1994) 111–127.
[7] ILOG, CPLEX 7, Manuals, 2000.
[8] S.K. Jacobsen, O.B.G. Madsen, A comparative study of heuristics for a two-level location–routing problem, European Journal of Operational Research 5 (1980) 378–387.
[9] G. Laporte, Location–routing problems, in: B.L. Golden, A.A. Assad (Eds.), Vehicle Routing: Methods and Studies, North-Holland, Amsterdam, 1988, pp. 163–198.
[10] G. Laporte, Y. Nobert, An exact algorithm for minimizing routing and operating cost in depot location, European Journal of Operational Research 6 (1981) 224–226.
[11] G. Laporte, Y. Nobert, D. Arpin, An exact algorithm for solving a capacitated location–routing problem, Annals of Operations Research 6 (1986) 293–310.
[12] G. Laporte, Y. Nobert, P. Pelletier, Hamiltonian location problems, European Journal of Operational Research 12 (1983) 82–89.
[13] G. Laporte, Y. Nobert, S. Taillefer, Solving a family of multi-depot vehicle routing and location–routing problems, Transportation Science 22 (1988) 161–172.
[14] O.B.G. Madsen, Methods for solving combined two level location–routing problems of realistic dimensions, European Journal of Operational Research 12 (1983) 295–301.
[15] H. Min, V. Jayaraman, R. Srivastava, Combined location–routing problems: A synthesis and future research directions, European Journal of Operational Research 108 (1998) 1–15.
[16] L.K. Nozick, M.A. Turnquist, Integrating inventory impacts into a fixed-charge model for location distribution centers, Transportation Research E 34/3 (1998) 173–186.
[17] I. Or, W.P. Pierskalla, A transportation location-allocation model for regional blood banking, AIIE Transactions 11 (2) (1979) 86–95.
[18] J. Perl, A unified warehouse location–routing analysis, UMI, Dissertation Information Service, 1983.
[19] J. Perl, M.S. Daskin, A warehouse location–routing problem, Transportation Research B 19B (1985) 381–396.
[20] J. Perl, Sirisoponsilps, Distribution networks: Facility location, transportation and inventory, International Journal of Physical Distribution and Materials Management 18 (6) (1988) 18–26.
[21] S. Salhi, G.K. Rand, The effect of ignoring routes when locating depots, European Journal of Operational Research 39 (1989) 150–156.
[22] R. Srivastava, Alternative solution procedures for the location–routing problem, Omega International Journal of Management Science 21/4 (1993) 497–506.
[23] D. Tuzun, L.I. Burke, A two-phase tabu search approach to the location routing problem, European Journal of Operational Research 116 (1999) 87–99.
[24] M.H.T. Webb, Cost functions in the location of depots for multiple delivery journeys, Operational Research Quarterly 19 (1968) 311–328.