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A relax-and-price heuristic for the inventory-location-routing problem

W. J. Guerrero[a,c,d], C. Prodhon[a], N. Velasco[b,c] and C. A. Amaya[c]

[a] *ICD-LOSI, UMR CNRS STMR, Université de Technologie de Troyes, 12 Rue Marie Curie, CS 42060, 10004 Troyes CEDEX, France*

[b] *Invited Professor, ICD-LOSI, UMR CNRS, Université de Technologie de Troyes, France*

[c] *PyLO, Depto. de Ingeniería Industrial, School of Engineering, Universidad de los Andes, Cra 1 Este No 19A – 40, Bogotá, Colombia*

[d] *Facultad de Ingeniería Industrial, Escuela Colombiana de Ingeniería Julio Garavito, AK.45 No.205-59, Bogotá, Colombia*

E-mail: wi-guerr@uniandes.edu.co [Guerrero]; caroline.prodhon@utt.fr [Prodhon]; nvelasco@uniandes.edu.co [Velasco]; ca.amaya@uniandes.edu.co [Amaya]

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Abstract

This paper considers the problem of designing a supply chain assuming routing decisions. The objective is to select a subset of depots to open from a set of candidates, the inventory policies for a two-echelon system, and the set of routes to perform distribution from the upper echelon to the next by a homogeneous fleet of vehicles over a finite planning horizon considering deterministic demand. To solve the problem, a partition is proposed using a Dantzig–Wolfe formulation on the routing variables. A hybridization between column generation, Lagrangian relaxation, and local search is presented within a heuristic procedure. Results demonstrate the capability of the algorithm to compute high quality solutions and empirically estimate the improvement in the cost function of the proposed model at up to 9% compared to the sequential approach. Furthermore, the suggested pricing problem is a new variant of the shortest path problem with applications in urban transportation and telecommunications.

Keywords: inventory-location-routing problem; Lagrangian relaxation; column generation; supply chain design; vehicle routing problem

1. Introduction

Most supply chain design problems (SCDPs), being a strategic-level decision, consider the link between facilities at different levels but not the links between those at a common level. In the case of SCDP models, in order to identify the optimal subset of plants and their location such that logistic costs are minimal, the model graph is restricted to be incomplete by forbidding links between facilities at the same level (e.g., edges connecting two retailers are not allowed). A specific hierarchical order of echelons must be respected (Melo et al., 2009) and routing decisions are neglected. Nonetheless, when vehicles have enough capacity to deliver to more than one retailer per route, this assumption is not suitable.

A variant of traditional SCDP is presented here. The inventory-location-routing problem (ILRP) extends the SCDP by taking into account routing decisions, inventory management policies, and their mutual interactions over a multiperiod planning horizon. An integrated approach is proposed, given the fact that decomposing the problems of locating facilities, designing inventory policies and finding the optimal set of routes to visit the clients is often suboptimal (Daskin et al., 2005). This is due to the fact that the search space is truncated by the lack of information sharing between steps when optimizing independently (or sequentially) the location, inventory, and routing decisions.

Manzini (2012) recently presented an example of this top-down approach proposing a series of mathematical models integrated under a single framework to provide solutions to supply chain design and management problems. This technique is also known as hierarchical optimization since location decisions are optimized at the highest level. Inventory policies are optimized for the given location-allocation scheme afterwards. Finally, routing is solved using a cluster-first route-second approach at the very last step.

More to the point, if the location decision is based on the minimization of the sum of distances (or maximum) between depots and retailers, when vehicles are not performing single-visit tours, optimality is not guaranteed. This statement is verified by Salhi and Rand (1989) by testing the effects of ignoring routing decisions when locating depots. Hence, location-routing problems (LRPs) propose to simultaneously optimize location and routing decisions. Examples are presented by Prins et al. (2006, 2007) and Belenguer et al. (2011), and a recent survey on the LRP is presented by Prodhon and Prins (2014).

Considering deterministic demand, Ambrosino and Scutellà (2005) proposed a linear programming model combining simultaneously depot location, vehicle routing, and inventory control policies on a multiperiod setting but provided feasible solutions on 12 single-period instances (LRP) with up to 13 depots, 95 retailers, showing that commercial mixed integer programming (MIP) solvers are not able to prove optimality within 25 hours for most instances exposing empirically the difficulty of the problem.

On the stochastic demand setting, most of the research addressing this issue assume constant demand (Wilson model) and propose an EOQ-like cost in the objective function to include the inventory management component. Papers presented by Javid and Azad (2010), Sajjadi and Cheraghi (2011) and Shen and Qi (2007) seek to solve a nonlinear LRP. Their proposal is to include in the objective function the annual expected cost of ordering plus holding stock for a random demand that is the nonlinear term. On the contrary, the research presented by Liu and Lee (2003) and Liu and Lin (2005) proposed to fix the lot sizes to be equal to expected quantities to deliver in the supply chain for a single period and optimize location-routing decisions from that point of depart. This last approach loses a global perspective as it optimizes sequentially the components of the problem.

An interesting application of the aggregated decision making under incertitude is exposed recently by Mete and Zabinsky (2010). Their research aims to locate emergency stock of medicines and the routes to perform distribution in case of a catastrophe. Their solution method is also hierarchical. After generating several scenarios for the demand of each hospital and the availability of the highways, they use stochastic programming to decide the optimal emergency inventory levels at each opened depot. Based on this information, they use MIP to solve the allocation of hospitals (clients) to depots. In a final step, a set covering problem is solved using MIP to select the routes to distribute product on a single-period setting.

This paper deals with the problem of integrating location, inventory, and routing decisions. Consider a two-echelon supply chain, assuming deterministic demand, and the routing resolution to be made for a discrete and finite planning horizon. In the open literature, it is often discussed how strategic-level decisions, such as location, should not be integrated with tactical/operational planning. Nonetheless, examples are provided next of situations where integrating routing and inventory management decisions when making location planning is beneficial.

In the first place, the model presented is suitable when location decisions are not made for the long run. It is the case for companies that strategically decide to lease depots and pay rent, signing a rental contract for specific periods of time. The benefit for them is the flexibility to change locations periodically as needed. It is also the case on humanitarian logistics. When a catastrophe happens, emergency response teams set in place facilities to distribute water, medicines, and other relief inventories (Balcik et al., 2010). Often this location requires an investment and it is not supposed to be used to satisfy permanent needs. On the contrary, it is expected to be temporary until the situation is normalized. Finally, it is also the case for military logistics. On the battlefield, bases must be located to store ammunition, supplies or to provide medical support. Military tactics often require these bases to be strongly protected and its location to be changed to minimize the risk of being attacked.

In the second place, the model is also suitable for companies requiring to make better approximations of their operational costs in the long run when locating facilities. It is the case for supply chain designs allowing different frequencies of replenishment at retailers and distribution to be performed by vehicles capable of visiting more than one retailer per route. Industries concerned by integrating these decisions usually face high inventory and distribution costs in the long run when compared to the fixed cost of locating depots and making the assumption of performing single-period routing is not realistic enough. Still, depot opening costs should be scaled on the modeled horizon to be in balance with the operational costs.

A Dantzig–Wolfe formulation is proposed on the routing variables, allowing taking apart subtour elimination constraints. Nonetheless, this formulation still contains an exponential set of constraints to force a link between routing and inventory management decisions. These constraints are tackled with Lagrangian relaxation. By doing so, a decomposition in subproblems is possible.

Previous research in vehicle routing problems (VRPs) and other combinatorial problems integrating column generation and Lagrangian relaxation techniques show improved computation times and results. Afterall, there is an important link between these two techniques. Geoffrion (1974) states that the Lagrangian dual is equivalent to the dual of the continuous relaxation of the Dantzig-Wolfe reformulation. Both examples of such hybridizations are presented by Kallehauge et al. (2006) who were able to solve with proven optimality two Homberger VRP instances with time windows (VRPTW) with 400 and 1000 customers, the largest to be solved to date.

Also, Nishi et al. (2011) presented their integrated approach using column generation and Lagrangian relaxation for a flow shop scheduling problem. The execution time is reduced with their technique to about 25% for instances with 50 jobs and three stages when compared to a pure column generation framework. Large-scale problems were solved faster with the hybrid version of the algorithm than with the pure benchmark one. Their conclusions show a high sensitivity

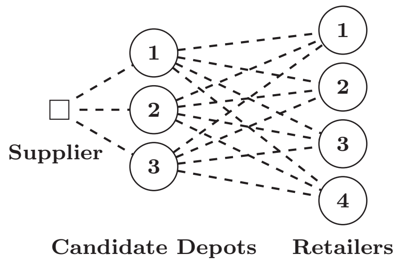

Fig. 1. SCDP graph.

between Lagrangian multipliers and the performance of the column generation. In this sense, lots of unnecessary columns are generated if Lagrangian multipliers are far from the optimum.

This research is on a location-routing problem that includes inventory management decisions. The mathematical formulation has two dependent constraint sets with an exponential nature. One set is tackled through Lagrangian relaxation while the VRP constraints are handled with a column generation technique. A heuristic procedure is proposed based on these techniques in order to have better control on computational times.

Section 2 presents the problem definition. Section 3 is dedicated to the partition principle and heuristic methodology. The computational experiments are detailed in Section 4. Analysis and conclusions are given in Section 5.

2. Problem definition

The considered problem is to decide the location of depots from which a set of routes will depart in order to serve a set of retailers facing deterministic demand over a finite planning horizon. In other words, the issue is to design a two-echelon supply chain comprising the depot location decision, the assignment of each retailer to an open depot, the lot sizes over the planning horizon for each facility (depots and retailers), and the routes per period to perform the distribution activities (dedicated routes to depots, nondedicated to retailers). The costs include the opening costs, the delivery costs, and the inventory costs, including an obsolescence penalty cost.

Let J be a set of n retailers, I the set of m candidate depots, and H the set of p periods in the planning horizon. Retailers face a deterministic nonconstant demand $d_{jt}, \forall j \in J, \forall t \in H$. The ILRP is defined on a weighted and directed graph $G = (V, A, C)$. $V = \{J \cup I\}$ is the set of nodes in the graph. C is the cost matrix c_{ij} associated to the traveling cost from node i to node j in the set of arcs A. Figures 1 and 2 present the associated graphs for a traditional SCDP and for the ILRP in a small example with three candidate depots and four retailers. Note that the set of arcs A in G includes the arcs linking every pair of retailers for the ILRP in Fig. 2 whereas the graph for the SCDP forbids the links between two retailers (Fig. 1).

Each node $j \in V$ is associated to a storage capacity W_j. Each depot $i \in I$ is associated to an opening cost o_i and an ordering cost s_i (dedicated route from the factory).

To tackle inventory management, let $\tilde{G}_i$ be defined as an auxiliary bipartite graph to model inventory decisions for each particular facility $i \in V$. The sets of source nodes $H_0 = \{0\} \cup H$ and

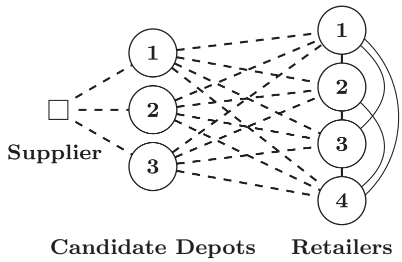

Fig. 2. ILRP graph.

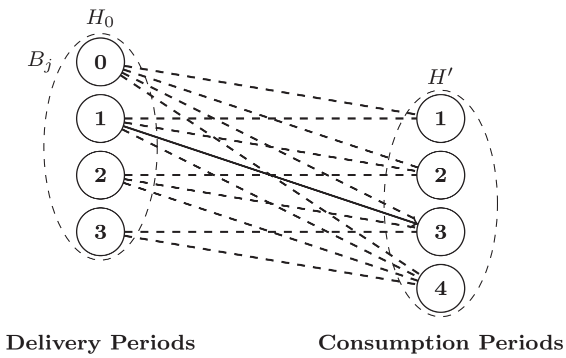

Fig. 3. Auxiliary graph $\tilde{G}_i$ for inventory features at facility $i \in V$.

destination nodes $H' = H \cup \{p+1\}$ are included to model initial and final conditions on stock levels. Figure 3 presents an example of $\tilde{G}_i$ considering a three-period planning horizon ($p = 3$).

A solution to the inventory management plan at facility i is represented by the flow of product in $\tilde{G}_i$ from every source node $t \in H_0$ to every destination node $l \in H'$. Thus, the flow from t to l is interpreted as stock kept at facility i from period t until period l. $\tilde{G}_i$ is a bipartite graph to permit splitting of the inflow demanded at destination nodes in contrast to the graph proposed by Wagner and Whitin (1958).

The flow leaving node 0 must be equal to the initial inventory at i, denoted by B_i, and the flow incoming to node $p+1$ represents the residual stock at the end of the horizon. The edge connecting a source node t to a destination node l is associated to a cost q_{jtl} denoting the holding plus obsolescence penalty cost for one unit of product kept at facility $j \in V$ from period t until period l.

In addition, $\tilde{G}_i$ is incomplete because the edges $\{(t, l),\ \forall t \in H_0, \forall l \in H' | t > l\}$ are not included to forbid backlogging. To illustrate the graph with an example, consider that the flow from delivery node 1 to consumption node 3 to be the quantity kept on stock at j from period 1 to 3 at cost q_{j13}.

To proceed with the routing aspect, consider an unlimited fleet of vehicles with capacity v_{cap} and b as the cost of using a vehicle at least once in the planning horizon. Consider Ω as the set of all feasible routes (sequences of retailers $j \in J \subseteq J$). For each route $r \in \Omega$ and retailer $j \in J$, the associated parameters a_{jr} is equal to 1 iff route r visits retailer j and $\hat{c}_{ir}$ is the cost of the tour r plus the best insertion cost of depot i into r.

Let the decision variables $y_i = 1$ iff depot $i \in I$ is opened. $f_{ij} = 1$ iff retailer $j \in J$ is assigned to depot $i \in I$, $\theta_{rti} = 1$ iff route $r \in \Omega$ is assigned to depot i for period $t \in H$, and r_i be the maximum number of vehicles used from depot $i \in I$ over H. Inventory decisions at echelon e are denoted by the variable w_e. The quantity replenished from depot i to retailer j on period t to satisfy the demand on period l is denoted by w_{2ijtl}. The quantity of product used from initial stock at retailer j to satisfy demand in period $t \in H'$ is denoted by w_{2j0t}. At the first echelon, $z_{li} = 1$ iff depot $i \in I$ is replenished in period $l \in H$, 0 otherwise. The quantity to replenish in depot $i \in I$ that is delivered in period $t \in H$ to satisfy the demand in period $l \in H'$ is w_{1itl}. Therefore, the ILRP objective function can be stated as:

$$\min \sum_{i\in I}\left(o_i y_i + b r_i + \sum_{l\in H} s_i z_{li}\right) + \sum_{i\in I}\sum_{t\in H_0}\sum_{l=t|l>0}^{p+1} q_{itl} w_{1itl} \\ + \sum_{j\in J}\sum_{t\in H'} q_{j0t} w_{2j0t} + \sum_{i\in I}\sum_{j\in J}\sum_{t\in H}\sum_{l=t}^{p+1} q_{jtl} w_{2ijtl} + \sum_{i\in I}\sum_{t\in H}\sum_{r\in\Omega} \hat{c}_{ir}\theta_{rti}. \tag{1}$$

The objective function (1) sums, in this order, the opening costs, the costs of using a vehicle at least once, and ordering costs for every depot in the first term. Holding costs at depots are added in the second term while third and fourth terms add holding costs at retailers. The last term in the objective function sums the distribution costs (the sum of the costs of the selected routes).

Let Ψ be a set of subsets of retailers (all feasible combinations of retailers) indexed to k. Each subset $k \in \Psi$ is associated to a set $S_k \subseteq J$, where the parameter β_{rk} is binary indicating whether route $r \in \Omega$ visits any customer $j \in S_k$. Then, the ILRP is subject to the following constraints:

$$\sum_{i\in I}\sum_{l=1}^{t} w_{2ijlt} + w_{2j0t} = d_{jt} \quad \forall j \in J, \forall t \in H \tag{2}$$

$$\sum_{l=1}^{t} w_{2ijlt} \le f_{ij} d_{jt} \quad \forall i \in I,\ \forall j \in J, \forall t \in H \tag{3}$$

$$\sum_{i\in I} f_{ij} = 1, \quad \forall j \in J \tag{4}$$

$$f_{ij} \le y_i, \quad \forall j \in J,\ \forall i \in I \tag{5}$$

$$\sum_{l=0}^{t} w_{1ilt} = \sum_{l=t}^{p+1}\sum_{j\in J} w_{2ijtl} \quad \forall i \in I, \forall t \in H \tag{6}$$

$$\sum_{t\in H'} w_{2j0t} = B_j, \quad \forall j \in J \tag{7}$$

$$\sum_{t \in H'} w_{1i0t} = B_i \cdot y_i, \quad \forall i \in I \tag{8}$$

$$\sum_{r=0}^{t} \sum_{l=t}^{p+1} w_{1irl} \leq W_i \cdot y_i \quad \forall i \in I, \ \forall t \in H \tag{9}$$

$$\sum_{l=t}^{p+1} \left(w_{2j0l} + \sum_{r=1}^{t} w_{2ijrl} \right) \leq W_j, \quad \forall i \in I, \ \forall j \in J, \ \forall t \in H \tag{10}$$

$$\sum_{l=t}^{P+1} w_{1itl} \leq W_i \cdot z_{ti} \quad \forall i \in I, \ \forall t \in H \tag{11}$$

$$\sum_{l=t}^{p+1} \sum_{j \in S_k} w_{2ijtl} \leq v_{cap} \sum_{r \in \Omega} \beta_{rk} \theta_{rti} \quad \forall i \in I, \ \forall t \in H, S_k \subseteq J, \ \forall k \in \Psi. \tag{12}$$

$$\sum_{r \in \Omega} \theta_{rti} \leq r_i \quad \forall t \in H, \ \forall i \in I \tag{13}$$

$$\sum_{i \in I} \sum_{r \in \Omega} \theta_{rti} a_{jr} \leq 1 \quad \forall t \in H, \ \forall j \in J \tag{14}$$

$$\sum_{r \in \Omega} \theta_{rti} a_{jr} \leq f_{ij} \quad \forall t \in H, \ \forall i \in I, \ \forall j \in J \tag{15}$$

$$y_i \in \{0, 1\} \quad \forall i \in I \tag{16}$$

$$z_{it} \in \{0, 1\} \quad \forall i \in I, \ \forall t \in H \tag{17}$$

$$f_{ij} \in \{0, 1\} \quad \forall i \in I, \ \forall j \in J \tag{18}$$

$$\theta_{rti} \in \{0, 1\}, \ r \in \Omega, \quad \forall t \in H, \ \forall i \in I \tag{19}$$

$$r_i \in \mathbb{N} \quad \forall i \in I \tag{20}$$

$$w_{2ijtl} \in \mathbb{R}^+ \quad \forall i \in I, \ \forall j \in J, \ \forall t \in H, \ \forall l \in H' | l \geq t \tag{21}$$

$$w_{2j0t} \in \mathbb{R}^+ \quad \forall j \in J, \ \forall t \in H' \tag{22}$$

$$w_{1itl} \in \mathbb{R}^+ \quad \forall i \in I, \ \forall t \in H_0, \ \forall l \in H | l \geq t. \tag{23}$$

Constraints (2) force to satisfy the demand. Each retailer must be assigned and replenished from a single opened depot as stated by constraints (3)–(5). Inventory flow conservation is forced by constraints (6). The sum over the horizon of the quantity kept on stock from period one up to period $p + 1$ is equal to the initial stock as stated by constraints (7) and (8). Capacity for depots and retailers is guaranteed by (9) and (10). Ordering decisions at depots are forced by constraints (11). If a retailer is replenished on period t, it must be visited accordingly by any route departing from the assigned depot. Constraints (12) force this statement along with the limited vehicle capacity. To explain further, these constraints state that the total quantity delivered to a predefined cluster of retailers $k \in \Psi$ at period t, must be at the most v_{cap} times the number of routes that visit that cluster. Next, Equations (13) link the cost of using vehicles with the routing decisions. Equations (14) state that each retailer is visited once per period at the most. This constraint could be reinforced by Equations (15). Finally, constraints (16)–(23) state the nature of the decision variables. Further, Theorem 1 presents a set of valid inequalities for the presented problem.

Theorem 1. *The set of constraints* (24) *are valid inequalities for the ILRP.*

$$\sum_{r\in\Omega}\sum_{l=1}^{t}\sum_{i\in I}\theta_{rli}a_{jr} \geq \left\lceil \frac{\sum_{l=1}^{t} d_{jl} - B_j}{v_{cap}} \right\rceil, \quad \forall\, j \in J,\ \forall\, t \in H. \tag{24}$$

Proof. Consider the quantity $\sum_{l=1}^{t} d_{jl} - B_j$ as the total demand faced by retailer j up to period $t \in H$ that cannot be satisfied with the initial inventory at j (B_j). In order to satisfy constraints (2), every retailer $j \in J$ must be visited at least $\lceil \frac{\sum_{l=1}^{t} d_{jl} - B_j}{v_{cap}} \rceil$ times up to period t (right hand). Then, the number of vehicles visiting retailer j up to period t from any depot is defined by $\sum_{r\in\Omega}\sum_{l=1}^{t}\sum_{i\in I}\theta_{rli}a_{jr}$ (left hand) and must be larger, the right size as stated by (24). $\square$

3. Solution method

The problem combines two well known NP-hard problems: the SCDP and VRP. A problem partition is proposed without performing hierarchical or sequential optimization. The suggested pattern combines exact and heuristic procedures leading to a heuristic defined as a matheuristic (Raidl and Puchinger, 2008) that will be called as a relax-and-price algorithm. In this context, the relaxation of the set of constraints (12) in a Lagrangian scheme allows to decompose the problem into two subproblems, which are solvable using column generation. Based on this idea, a heuristic procedure is developed. The partition principle to tackle the problem is explained in Section 3.1, followed by the solution algorithm and its components detailed in Sections 3.2–3.5.

3.1. Partition principle

The ILRP formulation presented in Section 2, has an exponential number of θ variables in addition to the exponential number of constraints in Equations (12). In practice, this makes the model very hard to solve even for small instances.

The set of constraints (12) link the distribution activities (routing) with the flow of stock through the supply chain, and force to respect the limited vehicle capacity. If these constraints are relaxed, ignoring the reinforcement provided by constraints (15), a relaxed ILRP (RILRP) is obtained. RILRP can be optimized by solving independently an SCDP and a VRP. This follows from the structure, where there is no constraint linking variables θ and w_2 other than Equations (12). The RILRP objective function is expressed as follows:

$$\begin{aligned}\min \sum_{i\in I}\left(o_i y_i + br_i + \sum_{l\in H} s_i z_{li}\right) + \sum_{i\in I}\sum_{t\in H_0}\sum_{l=t|l>0}^{p+1} q_{itl} w_{1itl} \\ + \sum_{j\in J}\sum_{t\in H'} q_{j0t} w_{2j0t} + \sum_{i\in I}\sum_{t\in H}\sum_{l=t}^{p+1}\left(\sum_{j\in J} w_{2ijtl} q_{jtl} + \sum_{k\in\Psi}\sum_{j\in S_k} w_{2ijtl}\mu_{itk}\right) \\ + \sum_{i\in I}\sum_{t\in H}\sum_{r\in\Omega}\theta_{rti}\left(\hat{c}_{ir} - \sum_{k\in\Psi} v_{cap}\beta_{kr}\mu_{itk}\right).\end{aligned} \tag{25}$$

Subject to: (2)–(11), (13), (14), (16)–(23).

μ_{itk} are the Lagrangian multipliers associated to the set of constraints (12). As explained before, given that variables θ and w_2 are independent in RILRP, the problem can be decomposed into two MIPs. Subproblem 1 handles inventory-location decisions, denoted as ILP1; and subproblem 2 makes the routing decisions, denoted as VRP2. ILP1 and VRP2 are formulated as follows:

$$\begin{aligned}ILP1: \min \sum_{i\in I}\left(o_i y_i + \sum_{l\in H} s_i z_{li}\right) + \sum_{i\in I}\sum_{t\in H_0}\sum_{l=t|l>0}^{p+1} q_{itl} w_{1itl} \\ + \sum_{j\in J}\sum_{t\in H'} q_{j0t} w_{2j0t} + \sum_{i\in I}\sum_{t\in H}\sum_{l=t}^{p+1}\left(\sum_{j\in J} w_{2ijtl} q_{jtl} + \sum_{k\in\Psi}\sum_{j\in S_k} w_{2ijtl}\mu_{itk}\right)\end{aligned} \tag{26}$$

subject to: (2)–(11), (16)–(18), (21)–(23).

$$VRP2: \min \sum_{i\in I} br_i + \sum_{i\in I}\sum_{t\in H}\sum_{r\in\Omega}\theta_{rti}\left(\hat{c}_{ir} - \sum_{k\in\Psi} v_{cap}\beta_{kr}\mu_{itk}\right) \tag{27}$$

subject to: (13), (14), (19), and (20).

The distinctive challenge of the problem arises from two main differences between VRP formulations based on column generation and the presented one: (1) In the ILRP, a periodic routing problem is tackled where quantities to deliver are decision variables and therefore, the master problem is not modeled as a set covering problem where the solution is forced to visit retailers with fixed frequencies; (in the ILRP, the vehicle capacity constraint is not considered in the subproblem of generating columns with negative reduced cost (Feillet, 2010). The violation of these constraints is penalized in the master problem objective function.

Two issues arise from this decomposition. The first is how to compute the objective functions (26) and (27), given that Ψ is a set that grows exponentially with the number of retailers $|J|$. Then, estimating the Lagrangian multiplier μ_{itk} for every subset $k \in \Psi$ is a complex task. The second issue to solve is how to handle the set Ω for subproblem VRP2.

To solve these issues, it is required to consider that the set Ω is the set of feasible permutations of a subset of retailers and the set Ψ is the set of the corresponding combinations of subsets of retailers, therefore, $|\Psi| \leq |\Omega|$.

The proposed decomposition method is based on column generation. Thus, the ILRP can be restrained to consider only a subset of feasible routes $\Omega' \subseteq \Omega$. Given this subset of permutations of retailers Ω', the corresponding subset of combinations $\Psi' \subseteq \Psi$ is computed. This restrained version will be denoted as Rv-ILRP. Further, it would be easy to show that limiting the set of constraints (12) to the set Ψ' for Rv-ILRP provides a feasible solution if each retailer is visited at least by a single route in Ω'. Then, it is proven that the set of constraints (12) corresponding to the set $\Psi \setminus \Psi'$ is dominated by the set of constraints (12) corresponding to the set Ψ'.

Similarly, by restraining the RILRP to consider only the sets Ψ' and Ω' instead of Ψ and Ω, respectively, a restrained-version RILRP (Rv-RILRP) is obtained and the optimal solution is a lower bound on Rv-ILRP. Moreover, considering that $|\Psi'| \leq |\Omega'|$, the computation of Equations (26) and (27) becomes easier.

It is proposed to start by solving Rv-RILRP with elementary routes on the pool of routes Ω'. Interesting columns (routes) are going to be dynamically added to Ω'. The corresponding combination of retailers is going to be added to Ψ' to maintain the principle exposed previously. In the following section, this principle is used to develop the proposed relax-and-price heuristic.

3.2. *Relax-and-price algorithm*

Algorithm 1 presents the general procedure proposed to find solutions to the ILRP. For notation, consider that S^* represents the best found solution, S the incumbent solution, and $\hat{S}$ an unfeasible solution. The operator C($\cdot$) returns the cost of a solution, C($\emptyset$) returns a very large number, and C(0) returns zero. At line 1 and 2, the algorithm is initialized with subset $\Omega' \subset \Omega$ of dedicated routes to retailers. Subset $\Psi' \subset \Psi$ is also generated considering the combinations of retailers for every route in Ω'. The Lagrangian multipliers μ_{itk} are initialized to zero.

Algorithm 1 Main Algorithm (Overview)

1: $S^* \leftarrow \emptyset$; $\mu \leftarrow 0$; //Initialization
2: $\Omega' \leftarrow$ ELEMENTARY_ROUTES;
3: **for** $(i = 0;\ i < N_1;\ i++)$ **do**
4: $\quad \hat{S} \leftarrow 0$;
5: $\quad \Omega' \leftarrow \Omega' \cup$ GENERATE_COLUMNS(μ) //Solve the pricing problem
6: $\quad$ **while**(C($\hat{S}$) < C(S^*) or N_2 iterations) **do**
7: $\quad\quad \hat{S} \leftarrow$ SOLVE_RV_RILRP(μ); //Get unfeasible solution
8: $\quad\quad S \leftarrow$ CORRECTION_PROCEDURE($\hat{S}$); //Repair solution
9: $\quad\quad S \leftarrow$ LOCALSEARCH(S);
10: $\quad\quad$ **if** (C(S) < C(S^*)) **then**
11: $\quad\quad\quad S^* \leftarrow S$;
12: $\quad\quad$ **end if**
13: $\quad\quad \mu \leftarrow$ SUBGRADIENTMETHOD(μ, $\hat{S}$, S^*); //Upgrade multipliers
14: $\quad$ **end while**
15: $\quad$ ADD_RANDOM_CUTS;
16: **end for**

For a fixed number of iterations, in lines 3 to 16, the algorithm starts by adding to the set Ω' new columns in line 5. The procedure GENERATE_COLUMNS iterates by solving the LP relaxation of VRP2, to obtain the optimal values of dual variables associated to constraints (13), (14), and (24). These are required in the pricing procedure to compute new columns to be added. More details are given in Section 3.3 GENERATE_COLUMNS iterates until no further columns with negative reduced cost are found.

In line 7, once new promising routes are included into Ω' along with their associated decisions variables θ, Rv-RILRP described in Section 3.1 is solved using an MIP solver providing an unfeasible solution $\hat{S}$. In line 8, $\hat{S}$ is repaired by the procedure CORRECTION_PROCEDURE to guarantee the feasibility in the solution S. This procedure will be described in Section 3.4 while Section 3.5 details the local search operator that improves S at line 9. Lines 10–12 store the best found solution.

The subgradient method is proposed to update Lagrangian multipliers in line 13 through Equations (28) to (30) as in Beasley (1993). The method computes the gap between the best feasible solution C(S^*) and the current lower bound C($\hat{S}$) to set a step size $\delta^{(p)}$ at the iteration p by Equation (29). The direction of the multipliers $\mu_j, \forall j \in J$ is also corrected with Equation (30) that penalizes the relaxed constraints (12). The new μ_j coefficients are computed by Equation (28).

$$\mu_{itk}^{(p)} = \max\left\{0, \mu_{itk}^{(p-1)} + \delta_{i,t,k}^{(p)} \nu_{i,t,k}^{(p)}\right\} \quad \forall i \in I, \ \ \forall t \in H, \ \ \forall k \in \Psi' \tag{28}$$

$$\delta^{(p)} = \frac{(C(S^*) - C(\hat{S}))}{\left\| \nu^{(p)} \right\|} \tag{29}$$

$$\nu_{i,t,k}^{(p)} = \sum_{l=t}^{p+1} \sum_{j \in S_k} w_{2ijtl} - v_{cap} \sum_{r \in \Omega'} \beta_{rk} \theta_{rti}. \tag{30}$$

The cost of $\hat{S}$ works as a lower bound to Rv-ILRP, while the cost of the best found solution S^* is the best upper bound. Lines 6 to 14 are repeated until $C(\hat{S})$ is larger than $C(S^*)$, in which case it is required to add more routes to Rv-RILRP. Finally, in line 15, random cuts are included in the MIP formulation of Rv-RILRP. Two random retailers $R_1, R_2 \in J$ are forced to be assigned to their corresponding closest depot D_{R_1} and $D_{R_2} \in I$ ($f_{R_1, D_{R_1}} = 1, f_{R_2, D_{R_2}} = 1$). These cuts are deleted after the Lagrangian multipliers are updated $\mu_j \forall j \in J$ if no improvement is found in line 13. By perturbing μ_j for some iterations, some diversification is induced in the search.

3.3. Pricing problem

In order to identify interesting routes to be dynamically included into Ω', the following dual problem to the continuous relaxation of VRP2 is formulated. λ_{1it}, λ_{2jt}, and λ_{3jt} are defined as the dual variables associated to constraints (13), (14), and (24), respectively

$$\max \sum_{j \in J} \sum_{t \in H} (\bar{b}_{jt} \lambda_{3jt} - \lambda_{2jt}) \tag{31}$$

Subject to:

$$\sum_{t \in H} \lambda_{1it} \leq b, \quad \forall\ i \in I \tag{32}$$

$$\sum_{j \in J} \left[a_{jr} \left(-\lambda_{2jt} + \sum_{l=t}^{p} \lambda_{3jl} \right) \right] - \lambda_{1it} \leq \bar{c}_{rti}, \quad \forall r \in \Omega,\ \forall\ t \in H,\ \forall\ i \in I \tag{33}$$

$$\lambda_{1it} \geq 0, \lambda_{2jt} \geq 0, \lambda_{3jt} \geq 0, \quad \forall i \in I, \forall j \in J, \forall t \in H, \tag{34}$$

where $\bar{b}_{jt}$ is equal to the right-hand size of Equations (24) and $\bar{c}_{rti}$ is the cost coefficient of θ_{rti} in Equation (27): $(\hat{c}_{ir} - \sum_{k \in \Psi'} v_{cap} \beta_{kr} \mu_{itk})$. Ideally, it is required to add as many columns with negative reduced cost as possible to solve to optimality the master problem (VRP2). New columns can be generated by solving the associated pricing problem.

The presented formulation of the pricing problem is interpreted as a generalized elementary shortest path problem (GESPP) for each period $t \in H$ and depot $i \in I$. Further explanation is provided next.

It is natural to state that $a_{jr} = \sum_{u \in J \cup \{i\}} x_{ujr}$ into Equation (33). Therefore, the reduced cost of a route $r \in \Omega$ for period t assigned to depot i is equal to:

$$\sum_{u \in J \cup \{i\}} \sum_{v \in J \cup \{i\}} x_{uvr} \cdot \left[c_{uv} + \lambda_{2jt} - \sum_{l=t}^{p} \lambda_{3vl} \right] + \lambda_{1it} - \sum_{k \in \Psi'} v_{cap} \beta_{kr} \mu_{itk}. \tag{35}$$

Now, recall that vehicle capacity constraints (12) were relaxed and therefore, the pricing problem has no resource constraints. The only difference between the shortest path problem objective function and Equation (35) is the last term. It is interpreted as a profit obtained if the selected path visits the predefined clusters $k \in \Psi'$.

The GESPP is studied by Guerrero et al. (2013b). Its purpose is to find the minimum reduced cost path for a fixed depot i and period t. Consider the set of retailers J to be aggregated in nondisjoint clusters where each cluster $k \in \Psi'$ is associated with a profit p_k to the cost function equal to the corresponding coefficient $v_{cap} \mu_{itk}$. The objective is to find the minimum cost path from a dummy node $\{0\}$ to a sink dummy node $\{n+1\}$ while visiting a subset of retailers. Both dummy nodes $\{0, n+1\}$ represent the depot i. The profit could be interpreted as a marginal decrease in the Lagrangian term in the objective function of Rv-RILRP. Additionally, let the GESPP be defined over a complete weighted graph with $\tilde{c}_{ij}$ the cost of connecting node j after node i in the path. $\tilde{c}_{ij}$ can be negative and the graph may contain negative cycles.

Figure 4 presents the graph for an example of the GESPP considering nine retailers. Nodes 0 and 10 are the source and sink of the problem to represent the depot. For the sake of simplicity, consider four clusters only (C_1 to C_4). $C_1 = \{1, 2, 4, 5\}$, $C_2 = \{2, 3\}$, $C_3 = \{7, 8\}$, and $C_4 = \{5, 6, 8, 9\}$. Each cluster k is associated to a profit p_k. Then, the path $\{0 - 2 - 4 - 7 - 10\}$ would have a cost equal to $\tilde{c}_{0,2} + \tilde{c}_{2,4} + \tilde{c}_{4,7} + \tilde{c}_{7,10} - p_1 - p_2 - p_3$. Since any retailer from cluster C_4 is visited, the profit p_4 is not included. Also, p_1 is added only once despite the fact that two retailers from cluster C_1 are visited (retailers 2 and 4).

To illustrate the mathematical formulation of GESPP, let x_{ij} be a binary decision variable indicating whether the arc (i, j) belongs to the shortest path; let y_k be a binary variable equal to 1

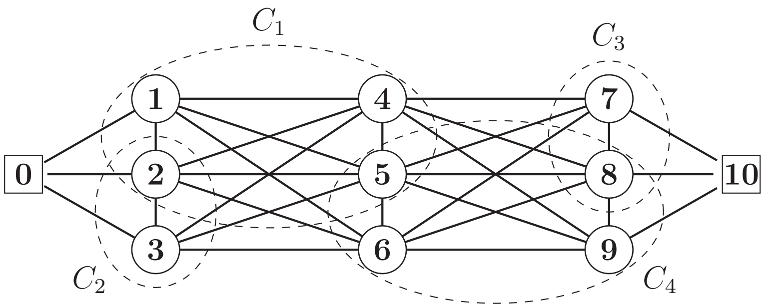

Fig. 4. Graph for the GESPP—an example.

iff cluster $k \in \Psi'$ is visited at least once. Let $\tilde{c}_{ij}$ be the cost of using arc (i, j) and p_k the profit for visiting cluster k. Let the GESPP be formulated as follows:

$$\text{GESPP:} \quad \min \sum_{i\in J\cup 0} \sum_{j\in J\cup n+1} \tilde{c}_{ij}x_{ij} - \sum_{k\in\Psi'} y_k p_k \tag{36}$$

Subject to:

$$\sum_{i\in J} x_{0,i} = 1 \tag{37}$$

$$\sum_{i\in J} x_{i,n+1} = 1 \tag{38}$$

$$\sum_{i\in J\cup 0} x_{ij} - \sum_{i\in J\cup\{n+1\}} x_{ji} = 0, \quad \forall j \in J \tag{39}$$

$$\sum_{i\in S}\sum_{j\in S} x_{ij} \leq |S| - 1, \quad \forall\, S \subseteq J \tag{40}$$

$$\sum_{i\in S_k} \sum_{j\in J\{S_k\}} x_{ij} \geq y_k, \quad \forall\, k \in \Psi' \tag{41}$$

$$x_{ij} \in \{0, 1\} \quad \forall\, i \in J \cup \{0\}, \forall\, j \in J \cup \{n+1\} \tag{42}$$

$$y_k \in \{0, 1\} \quad \forall\, k \in \Psi'. \tag{43}$$

Equation (36) presents the objective function. It is the minimization of the total path length value after subtracting the corresponding cluster profits. Constraints (37) and (38) force the path to start and end at nodes 0 and $n + 1$, respectively. Equations (39) force flow conservation while Equations (40) are traditional subtour elimination constraints. Additionally, the set of constrains (41) states that the cluster profits are obtained if the path visits any retailer belonging to the corresponding cluster.

To solve the GESPP, Guerrero et al. (2013b) present a truncated labeling heuristic algorithm also used in this relax-and-price procedure. The computation of a path requires each retailer j to be associated with a set of labels representing the nondominated paths from node 0 up to j. Each label keeps track of the visited clusters and visited retailers. This way, the algorithm enumerates the paths from 0 up to every node in the graph keeping only nondominated labels. To speed up the search, a limit of K labels per retailer is imposed, making the procedure to be heuristic. The algorithm stops when all the existing labels have been extended to unvisited retailers. A local search procedure is performed as postoptimization with the following traditional neighborhoods:

- Exchange: Modifies the position of a retailer in the path.
- Swap: Interchanges the position of two retailers in the path.
- 2-Opt: Erases two arcs in the path and reconnects it with two different arcs such that the path is still feasible.
- 3-Opt: Erases three arcs and reconnects the path with three different arcs in the best possible way.
- Insert: Insert an unvisited retailer into the path.

3.4. Repairing operator

As the constraints linking distribution and replenishment (12) were relaxed on Rv-RILRP, three cases might make a solution infeasible: (1) Retailers that are visited without being replenished; (2) replenished retailers without scheduled visit on a particular period (no vehicle visits the retailer); (3) overloaded routes. In the first case, the visit is simply removed. In the second case, a best-insertion procedure is performed. In the third case, the route is divided into two different routes. The point of division of the original route is computed as the point where the insertion of the depot is performed at the minimum cost. This procedure is repeated until no further routes violate the vehicle capacity constraint.

3.5. Local search

Once a feasible solution is found, a local search in the form of a variable neighborhood descent (VND) (Hansen and Mladenovic, 2003) is performed to intensify the search. Several neighborhoods are explored to improve routing, inventory costs, and location-allocation decisions, using a first improvement movement strategy, in the following order:

- *Routing neighborhoods:* These neighborhoods are limited to evaluate changes for scheduled visits to retailers sharing the same depot and period.
  - Move: The visit of a retailer is shifted from its current position to a different position.
  - Swap: The positions of two different retailers is exchanged.
  - 2-Opt: Two arcs are removed from the solution and new arcs are included to assure feasibility. The removed arcs might belong or not to the same route.
- *Inventory neighborhoods:* These neighborhoods are limited to evaluate changes for a single retailer at a time.

– Deplete stock: Consider a particular retailer j, and two consecutive replenishments at periods t_0 and t_1 with quantities Q_{t_0} and Q_{t_1}, respectively. The stock level is reduced by reducing the quantity delivered Q_{t_0} such that the quantity Q_{t_1} is increased accordingly to guarantee demand fulfillment.
– Remove visit: Once more, consider a particular retailer j, and two consecutive replenishments at periods t_0 and t_1 with quantities Q_{t_0} and Q_{t_1}, respectively. The replenishment in period t_1 is removed if Q_{t_0} can be sufficiently raised (by at least Q_{t_1} units) to satisfy future demand. Routing costs decrease while inventory holding costs increase.

- *Location-allocation neighborhoods:* These neighborhoods consider the scheduled visits to remain unchanged. Routing and depot inventory policies are revisited.

– Depot reallocation: A retailer is reallocated to a different depot.
– Depot allocation swap: Two retailers are exchanged in their depot allocation.

4. Computational study

The algorithm is coded in C and the MIP model is solved with Xpress-IVE 7.0. Tests are performed on an Intel Xeon with 2.80 GHz processor and 12 GB of RAM. Twenty ILRP random instances were generated with the following size: $m : \{5\}$ depots, $n : \{5, 7\}$ retailers, $p : \{5, 7\}$ periods. They are labeled as $m - n - p - x$ and x is used to itemize.

Demand at retailer j for period t is $d_{jt} \sim N(\mu_j, \sigma_j)$, where $\mu_j \in [5, 15]$ and $\sigma_j \in [0, 5]$. The opening costs o_i is generated with a Normal distribution with parameters (μ_i,σ_i) chosen from the set of pairs $\{(1000, 20), (5000, 100), (8000, 300)\}$. s_i is chosen from the set $\{100, 500\}$. The coordinates (X_i, Y_i) for facility $i \in V$ are randomly generated in a square of size 100×100. Transportation cost c_{ij} is equal to the closest integer of a hundred times the Euclidean distance from i to j. v_{cap} is a random integer in the interval [15, 75]. The cost b is selected from the set $\{350, 1000, 5000\}$. Depot capacity W_i is randomly generated in the interval $[D/3, D]$, where $D = \sum_{j\in J}\sum_{t\in H} d_{jt}$. Retailer's capacity W_j are randomly generated in the interval $[g_j, 3 \cdot g_j]$ where $g_j = \max_t d_{jt}$. Initial inventories B_j were chosen from the set $\{0, d_{j1}\}$ for retailers and B_i from the set $\{0, 10 \cdot D/n\}$ for depots. Inventory holding costs for a single period $t \in Ho$ at retailers and depots $j \in V$, $q_{j,t,t+1}$ is generated in the interval [0.03,0.50]. The inventory holding costs for k periods as $q_{j,t,t+k} = \sum_{l=t}^{t+k-1} q_{j,l,l+1} + k \cdot \xi_2$, were $\xi_2 \sim U[0.01, 0.02]$ represents the obsolescence penalty cost per period.

Results of the presented heuristic are compared to the three alternative methodologies listed below.

SOLVER: A feasible solution is obtained by solving an MIP model using a commercial solver imposing a time limit of 2.5 hours for instances with five retailers and nine hours for instances with 15 retailers. This model is presented in Guerrero et al. (2013a).

H1: A constructive heuristic method with two phases following an intuitive idea is explored. This approach is based on hierarchical optimization where location decisions are fixed before optimizing inventory-routing decisions. In the first phase, an SCDP is solved using a commercial solver that fixes the location of depots and decides an initial allocation of retailers to depots together with

Table 1
Comparative performance on ILRP instances

Instance	BKS	SOLVER		Relax-and-price		H1		H2	
		gap^0	gap^1	gap	CPU	gap	CPU	gap	CPU
5-5-5-a	93,625.3	3.84	0.1	0	32.4	0	4.7	0	25.2
5-5-5-b	62,206.9	10.98	0.46	0	30.1	1.12	4	0	22.4
5-5-5-c	69,760.3	23.9	0	0.14	34.6	1.73	7.9	1.61	48
5-5-5-d	93,451.2	4.25	0.37	0.8	32.4	0	4.7	0	28.6
5-5-5-e	93,851	4.3	0	0.25	35.3	4.2	4.3	0.8	14.1
5-5-7-a	70,966.5	19.65	9.07	0	103.4	7.05	89.3	0	320.7
5-5-7-b	107,478.5	55.65	3.22	3.22	90.1	4.49	46.7	0	394.9
5-5-7-c	94,150.2	28.88	0	0	128.4	6.66	212	0	328.3
5-5-7-d	87,744.2	13.79	0	0	106.8	4.57	8.8	0	62.1
5-5-7-e	67,275.4	–	2.6	0	108.6	5.57	44.8	0	176.4
5-7-5-a	68,485.2	110.09	0	0.73	150.6	4.44	7.9	1.83	55
5-7-5-b	76,339.1	–	0	1.31	90.4	3.65	51	3.04	114.8
5-7-5-c	138,998	–	–	0.07	237.1	2.49	1409	0	4964.9
5-7-5-d	99,988.9	–	0	0.51	166.7	6.6	26.2	0.01	222.3
5-7-5-e	62,010.1	48.66	0	1.16	97.5	2.33	11.5	0.36	81.2
5-15-5-a	113,434.3	–	38.37	0	1502.8	9.65	101.8	0	1863.5
5-15-5-b	172,743.3	–	35.09	0.58	1524.1	2.87	443.6	0	2001.3
5-15-5-c	210,333	–	–	0.48	10240.1	1.2	4076.9	0	14301.9
5-15-5-d	165,939.7	–	142.99	0.05	1463.5	6.35	330.2	0	1530.7
5-15-5-e	219,634.4	–	–	0	1573.6	7.82	4880.7	4.02	22661.3
Average	102,599.8	29.45	13.66	0.47	887.43	4.14	588.3	0.58	2460.88

the inventory decisions. In the second stage, an iterative local search to tackle inventory-routing decisions is performed.

H2: A cooperative heuristic is also compared. H2 has two main components that share information. The first component solves the SCDP using a commercial solver just as in H1 but uses information of the estimated distribution costs computed by the second component. For a fixed number of iterations, the second components destroys and repairs inventory-routing decisions to obtain complete solutions based on the supply chain design fixed by the first component. By alternating between solution spaces, both components optimize simultaneously the different decisions. An iterative local search procedure is performed as postoptimization to intensify allocation decisions. This heuristic is proposed by Guerrero et al. (2013a).

To make a fair comparison between the proposed relax-and-price algorithm and H2, both are run in the same workstation, both using the same MIP solver. Additionally, a limit of 450 evaluations of the objective function or calls to the local search operator is imposed on both heuristics. Tunning test showed that the best results for the relax-and-price algorithm are obtained when $N_1 = 10$ and $N_2 = 45$.

Table 1 presents the results of comparing the proposed methodology versus the three heuristics described earlier. The instance labels are presented in column one, and column two presents the value for the best know solutions (BKS) out of the four methods. So far, no instance has been solved to proven optimality. Columns three to ten present the solution quality of the corresponding

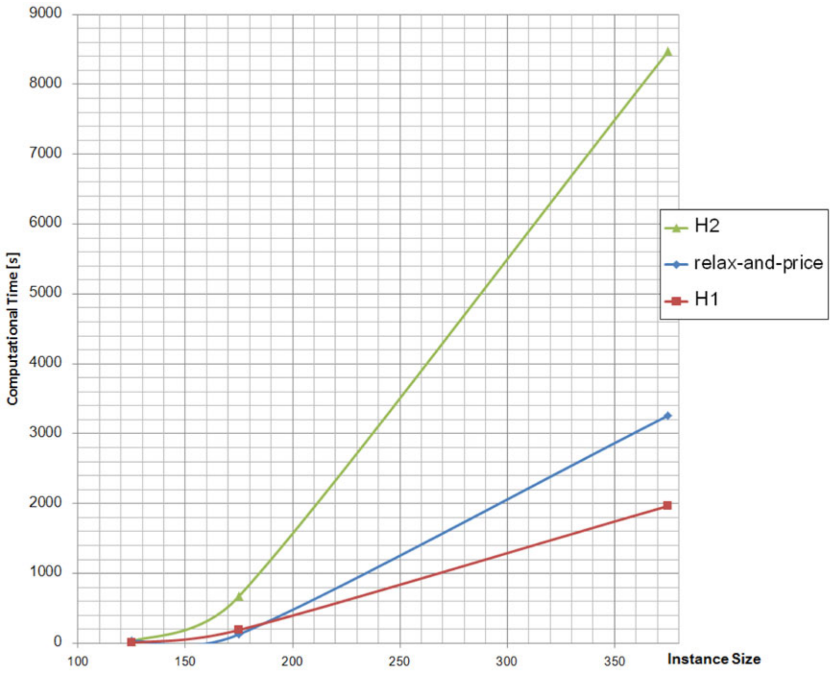

Fig. 5. Growth in computational times for the different solution approaches.

heuristic computed as $gap = 100 \cdot (Cost - BKS)/BKS$ and the computation time in seconds (CPU). Columns three and four present the gap between the best feasible solutions found by the solver at the first 60 seconds (gap^0) and after the imposed time limit (gap^1). For instances with $n = \{5, 7\}$, the time limit is 2.5 hours while for instances with $n = \{15\}$ the time limit is 9 hours. This comparison is provided since it is common for commercial solvers to find good solutions quickly and spend important computation time trying to prove optimality.

Columns five to ten present the average performance of the presented relax-and-price algorithm, H1, and H2 for three executions. Results show an average gap to BKS of 0.47% computed in 887 seconds while $H2$ computes solutions with a gap of 0.58% within more than 2000 seconds. Furthermore, H1 provides consistently good solutions. Eight of 20 BKS are found by the relax-and-price algorithm and 17 of 20 are below 1%.

The MIP solver is not capable of finding feasible solutions for two of the five instances with 15 retailers (5-15-5-c and 5-15-5-e) and five periods within nine hours. Similarly, the solver is incapable of finding a feasible solution for one instance with seven retailers and five periods (5-7-5-c) within 2.5 hours. Thus, the average gap to BKS is 25% within the first 60 seconds and 13.7% within the time limit.

On the other hand, H1 is the fastest heuristic but provides solutions with a gap of 4.14%. The largest instances with 15 retailers are computed with a gap of up to 9%. Hierarchical optimization, where location decisions are fixed before optimizing routing decisions, does not provide the best average solution quality.

Table 2
Impact on solution quality and CPU when removing components from Algorithm 1

Removed component	Avg gap (%)	CPU [s]	Gap incr. (%)	CPU difference (%)
Local search	42.35	974.36	41.68%	78.92%
Random cuts	12.87	1106.42	12.34%	98.49%
Local search on GESPP	24.46	1095.24	23.88%	96.48%

Figure 5 summarizes the average computational time for the relax-and-price, H1, and H2. Instances are classified according to their size, computed as $n \cdot m \cdot p$. As shown, the computational times required by H2 grow significantly faster than H1 and the relax-and-price procedure. Again, H1 is the method with the smallest average computational times.

Further, the impact of removing each component is evaluated. Three components are removable without affecting the stability of Algorithm 1: the local search procedure discussed in Section 3.5 performed in line 9, the inclusion of random cuts performed in line 15; and the local search for the pricing problem discussed in Section 3.3 when solving the GESPP at line 5. Table 2 presents in columns 2 and 3 the results for average gap to BKS and computational time in seconds for the different variants of Algorithm when removing a particular component. Additionally, columns 4 and 5 present the relative increase in quality and time from the original method (relax-and-price). Column 4 presents the relative increase in solution cost computed as $gap\ incr. = 100 \cdot (Cost - Cost_{R-\&-P})/Cost_{R-\&-P}$, where $Cost$ stands for the solution of the algorithm when the corresponding component is removed and $Cost_{R-\&-P}$ stands for the solution of the original method. Column 5 presents the increase in computational times when removing the components, computed as $100 \cdot (cpu - cpu_{R-\&-P})/cpu_{R-\&-P}$.

The operator with the largest impact in solution quality is the local search procedure. As a matter of fact, the solution cost is increased by about 42% when removing it. The convergence of the method is also disturbed since the average computational time is increased to 974 seconds. This is explained by the fact that Lagrangian multipliers are updated with slower convergence if the gap between the best feasible solution and the lower bound is larger.

Also, the random cuts are useful to help the procedure converge faster. By adding these cuts, the procedure SOLVE_RV_RILRP is solved faster since some decision variables are fixed. Nonetheless, the solution quality is decreased, on average, to about 12% while the computation time is significantly larger. Finally, the local search procedure that is applied when solving the pricing problem also has an important impact on solution quality and computational time. The quality of the columns in the pool Ω' is naturally decreased, resulting in solutions 24% more expensive than the best known solutions computed within 1095 seconds on average.

5. Conclusions

A Lagrangian relaxation based heuristic methodology to solve the Inventory-Location-Routing Problem is proposed, including the ideas of column generation for vehicle routing. The target is to simultaneously optimize a supply chain design considering inventory and routing decisions without decomposing the problem into subproblems in order to optimize globally.

The presented methodology combines a nontraditional column generation approach with a Lagrangian relaxation within a framework that is denoted as relax-and-price. The challenge is to coordinate the generation of new columns in the pricing problem and update of Lagrangian multipliers. In addition, to compute interesting routes, a shortest path problem with cluster profits, the generalized elementary shortest path problem, is solved through a two-phase algorithm.

Results for randomly generated instances show important cost savings over the traditional approach and efficient computation if compared to commercial solvers and other benchmark heuristics. The impact of each component of the algorithm is evaluated. Future research focuses on a multiobjective version of the problem and applying stochastic programming techniques in order to include the uncertainty associated to demand.

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