Algunos subgrupos solubles de $\mathrm{PSL}(3,\mathbb{C})$: Estructura y dinámica

IX Taller de Estructuras Geométricas y Combinatorias

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- Ongoing collaboration with Angel Cano, IMATE Cuernavaca.

Orígenes: Introducidos por José Seade & Alberto Verjovsky en 1998.

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Preliminaries

The complex projective plane $\mathbb{CP}^2$ is defined as

$$\mathbb{CP}^2 = \left(\mathbb{C}^3 \setminus \{0\}\right) / \mathbb{C}^*,$$

where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ acts by the usual scalar multiplication. Let

$$[\,] : \mathbb{C}^3 \setminus \{0\} \to \mathbb{CP}$$

be the quotient map. We denote the projectivization of the point $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ by $[x] = [x_1 : x_2 : x_3]$. We denote by e_1, e_2, e_3 the projectivization of the canonical base of $\mathbb{C}^3$.

The Complex Projective Plane

Given $p, q \in \mathbb{CP}^2$, we denote the projective line passing through p, q by

$$\overleftrightarrow{p, q} = [\langle \mathbf{p}, \mathbf{q} \rangle]$$

If ℓ_1, ℓ_2 are different complex lines in $\mathbb{CP}^2$, then $\ell_1 \cap \ell_2$ consists of exactly one point.

Preliminaries

Let $\mathrm{GL}(3, \mathbb{C}) \subset \mathcal{M}_3(\mathbb{C})$ be the subgroup of matrices with determinant not equal to 0. The group of biholomorphic automorphisms of $\mathbb{CP}^2$ is given by

$$\mathrm{PSL}(3, \mathbb{C}) := \mathrm{GL}(3, \mathbb{C}) / \{\text{scalar matrices}\}.$$

$\mathrm{PSL}(3, \mathbb{C})$ acts transitively on $\mathbb{CP}^2$, taking projective lines into projective lines.

Elements of $\mathrm{PSL}(3,\mathbb{C})$

We denote the projectivization of $(a_{ij}) \in \mathrm{GL}(3,\mathbb{C})$ by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We denote the upper triangular subgroup of $\mathrm{PSL}(3,\mathbb{C})$ by

$$U_+ = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \;\middle|\; a_{11}a_{22}a_{33} = 1,\ a_{ij} \in \mathbb{C} \right\}.$$

Discrete subgroups of $\mathrm{PSL}(3,\mathbb{C})$

$\mathrm{PSL}(3,\mathbb{C})$ is a topological group.

Discrete subgroups of $\mathrm{PSL}(3,\mathbb{C})$

A discrete subgroup of $\mathrm{PSL}(3,\mathbb{C})$ is a subgroup which is discrete with respect to the topology of $\mathrm{PSL}(3,\mathbb{C})$.

Classification of Elements

We classify the elements of PSL $(3,\mathbb{C})$ in three classes according to their dynamic behaviour:

$$\begin{cases} \text{Elliptic} \\ \text{Parabolic} \\ \text{Loxodromic} \end{cases}$$

There are several subclasses in each class. We now give a quick summary of some of these subclasses.

Classification of elements of $PSL(3,\mathbb{C})$

An element $g \in PSL(3,\mathbb{C})$ is said to be:

- Elliptic if it has a diagonalizable lift in $SL(3,\mathbb{C})$ such that every eigenvalue has norm 1.

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$$

  $|\alpha_1| = |\alpha_2| = |\alpha_3| = 1.$

- Parabolic if it has a non-diagonalizable lift in $SL(3,\mathbb{C})$ such that every eigenvalue has norm 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^{-2} \end{pmatrix}$$

Classification of elements of $PSL(3, \mathbb{C})$

- Loxodromic if it has a lift in $SL(3, \mathbb{C})$ with an eigenvalue of norm distinct of 1. Furthermore, we say that g is:
  - Loxo-parabolic

$$\mathbf{h} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}, \ |\lambda| \neq 1.$$

  - A complex homothety, $\mathbf{h} = \mathrm{Diag}(\lambda, \lambda, \lambda^{-2})$, $|\lambda| \neq 1$.
  - A rational (resp. irrational) screw, $\mathbf{h} = \mathrm{Diag}(\lambda_1, \lambda_2, \lambda_3)$, $|\lambda_1| = |\lambda_2| \neq |\lambda_3|$ and $\lambda_1 \lambda_2^{-1} = e^{2\pi i \theta}$ with $\theta \in \mathbb{Q}$ (resp. $\theta \in \mathbb{R} \setminus \mathbb{Q}$).
  - Strongly loxodromic, $\mathbf{h} = \mathrm{Diag}(\lambda_1, \lambda_2, \lambda_3)$, where $\{|\lambda_1|, |\lambda_2|, |\lambda_3|\}$ are pairwise different.

Proper and discontinuous actions

Properly and discontinous action

Let Γ be a subgroup of $\mathrm{PSL}\,(n+1,\mathbb{C})$ and let Ω be a Γ-invariant open subset of $\mathbb{CP}^n$. The action of Γ on Ω is properly discontinuous if for every compact set $K \subset \Omega$ we have that K intersects at most a finite number of copies of its Γ-orbit.

Complex Kleinian Group

A discrete subgroup Γ of $\mathrm{PSL}\,(n+1,\mathbb{C})$ is complex Kleinian if it acts properly discontinuously on some open subset of $\mathbb{CP}^n$.

The Goal

We want to study the dynamics of discrete subgroups of $\mathrm{PSL}(3,\mathbb{C})$ acting on $\mathbb{CP}^2$.

- Description of accumulation points of orbits.
- Uniformization of complex surfaces.
- Construction of a Sullivan's Dictionary linking the theory of Complex Kleinian groups and the study of iterates of holomorphic functions of $\mathbb{CP}^n$.

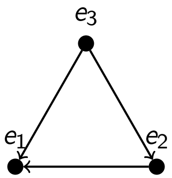
Example: A cyclic group

$$\gamma = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}$$

$$|\alpha_1| > |\alpha_2| > |\alpha_3|$$

$$\Gamma = \langle \gamma \rangle$$

- e_1 es un atractor, e_2 punto silla y e_3 repulsor.
- $\overleftrightarrow{e_1, e_2}$, $\overleftrightarrow{e_2, e_3}$ son invariantes.
- Las órbitas de puntos en $\overleftrightarrow{e_2, e_3}$ se acumulan en e_2 (hacia adelante) y en e_3 (hacia atrás).
- Análogamente con $\overleftrightarrow{e_1, e_2}$.
- La órbita de los puntos en $\mathbb{CP}^2 \setminus (\overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_2, e_3})$ se acumulan en $\{e_1, e_3\}$.



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Background

Action on $\mathbb{CP}^2$

Irreducible ✓

$$\text{Reducible action} \begin{cases} \text{Solvable} \\ \text{Non-solvable} \end{cases}$$

Background

Action on $\mathbb{CP}^2$

$$\begin{array}{l} \text{Irreducible } \checkmark \\ \text{Reducible action} \begin{cases} \text{Solvable} \\ \text{Non-solvable} \end{cases} \end{array}$$

- W. Barrera, A. Cano, J. P. Navarrete, Subgroups of $\mathrm{PSL}(3, \mathbb{C})$ with four lines in general position in its limit set, Conformal geometry and dynamics 15 (2011).
- W. Barrera, A. Cano, J. P. Navarrete, One line complex Kleinian groups, Pacific J. Mathematics 272 (2014).
- W. Barrera, A. Cano, J. P. Navarrete, On the number of lines in the limit set for discrete subgroups of $\mathrm{PSL}(3, \mathbb{C})$, Pacific J. Mathematics 281 (2016).
- W. Barrera, A. Cano, J. P. Navarrete, J. Seade, Discrete parabolic groups in $\mathrm{PSL}(3, \mathbb{C})$, Linear Algebra and its Applications, Vol. 653.

Limit sets

There are several types of possible *limit sets* for the action of Complex Kleinian Groups:

1. The complements of maximal regions of discontinuity.
2. The complement of the region of equicontinuity.
3. The closure of the fixed points of loxodromic elements.
4. The limit set of Kulkarni.
5. The limit set of Conze-Guivarch.

In complex dimension 1, all of them coincide.

Example

$$\gamma = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}$$

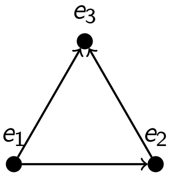
$$|\alpha_1| < |\alpha_2| < |\alpha_3|$$

$$\Gamma = \langle \gamma \rangle$$

$$\Omega_1 = \mathbb{CP}^2 \setminus \left(\overleftrightarrow{e_1, e_2} \cup \{e_3\}\right)$$

$$\Omega_2 = \mathbb{CP}^2 \setminus \left(\overleftrightarrow{e_2, e_3} \cup \{e_1\}\right)$$

Ω_1 and Ω_2 are maximal open sets where the action of Γ is properly discontinuous.



The Kulkarni Limit Set

From now on, let $\Gamma \subset PSL(3,\mathbb{C})$ be a discrete subgroup acting on $\mathbb{CP}^2$.

Definition

- *Let $L_0(\Gamma)$ be the closure of the set of points in $\mathbb{CP}^n$ with infinite isotropy group.*
- *Let $L_1(\Gamma)$ be the closure of the set of cluster points of orbits of points in $\mathbb{CP}^n \setminus L_0(\Gamma)$.*
- *Let $L_2(\Gamma)$ be the closure of the set of cluster points of compact sets of $\mathbb{CP}^n \setminus (L_0(\Gamma) \cup L_1(\Gamma))$.*

$$\Lambda_{Kul}(\Gamma) = \overline{L_0(\Gamma) \cup L_1(\Gamma) \cup L_2(\Gamma)}, \quad \Omega_{Kul}(\Gamma) = \mathbb{CP}^n \setminus \Lambda_{Kul}(\Gamma).$$

Limit sets for complex Kleinian groups

The Kulkarni limit set $\Lambda_{\mathsf{Kul}}(\Gamma)$ in $\mathbb{CP}^2$ is made up of points and complex projective lines. It contains 1, 2, 3, 4 or ∞ lines in general position.

This limit set is a set of points and lines where the Γ-orbits of points and compact sets accumulates.

Elementary Complex Kleinian Group

Elementary Complex Kleinian Groups are groups exhibihiting simple dynamics.

How can we define *elementary* complex Kleinian groups?

- Discrete subgroups of $\mathrm{PSL}\,(3, \mathbb{C})$ such that its Kulkarni limit set contains a finite number of lines.

In complex dimension 1 all of them coincide.

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- Discrete subgroups of $PSL(3,\mathbb{C})$ such that its Kulkarni limit set contains a finite number of lines in general position.
- Discrete subgroups of $PSL(3,\mathbb{C})$ with reducible action.

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- Discrete subgroups of $\mathrm{PSL}(3, \mathbb{C})$ such that its Kulkarni limit set contains a finite number of lines in general position.
- Discrete subgroups of $\mathrm{PSL}(3, \mathbb{C})$ with reducible action.
- Discrete solvable subgroups of $\mathrm{PSL}(3, \mathbb{C})$.

In complex dimension 1 all of them coincide.

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Solvable groups

- If $g, h \in G$, we define the commutator as $[g, h] = g^{-1}h^{-1}gh$.
- We define the commutator subgroup as

$$[G, G] = \{[g, h] \mid g, h \in G\}.$$

- The derived series of G is given by

$$G^{(0)} = G, \quad G^{(i+1)} = \left[G^{(i)}, G^{(i)}\right].$$

- We say that G is solvable if, for some $n \geq 0$, we have $G^{(n)} = \{id\}$.

Solvable groups: Examples

- Any triangular group is solvable, with solvability length at most 3.
- Cyclic groups, abelian groups.
- The special orthogonal group is **not** solvable,

$$\left\{ \begin{bmatrix} a & -c \\ c & \bar{a} \end{bmatrix} \middle| \, |a|^2 + |b|^2 = 1 \right\} \subset \mathrm{PSL}\,(2, \mathbb{C})$$

Why Solvable Subgroups of $\mathrm{PSL}(3,\mathbb{C})$?

Tit's Alternative

Let G be a finitely generated linear group over a field. Then two following possibilities occur:

- either G is virtually solvable (i.e., has a solvable subgroup of finite index).
- or it contains a non-abelian free group.

If $\Gamma \subset \mathrm{PSL}(3,\mathbb{C})$ is a discrete solvable subgroup, then Γ is virtually triangularizable.

Therefore, we can focus on discrete subgroups of U_+.

Why Solvable Subgroups of PSL $(3, \mathbb{C})$?

Theorem (Borel fixed point theorem)

Let G be a connected solvable group acting morphically on a non-empty complete variety V. Then G has a fixed point in V.

Applying this theorem to the Zariski closure of Γ yields that Γ is virtually triangularizable. Namely, Γ has a finite index subgroup such that, up to conjugation, is upper triangular.

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Main Result

Theorem (MT)

Let $\Gamma \subset PSL(3, \mathbb{C})$ be a triangularizable complex Kleinian group such that its Kulkarni limit set does not consist of exactly four lines in general position. Then there exists a non-empty open region $\Omega_\Gamma \subset \mathbb{CP}^2$ such that

1. *Ω_Γ is the maximal open set where the action is proper and discontinuous.*
2. *Ω_Γ is homeomorphic to one of the following regions: $\mathbb{C}^2$, $\mathbb{C}^2 \setminus \{0\}$, $\mathbb{C} \times (\mathbb{H}^+ \cup \mathbb{H}^-)$ or $\mathbb{C} \times \mathbb{C}^*$.*
3. *Γ is finitely generated and rank$(\Gamma) \leq 4$.*

Main Result

Theorem (MT)

Let $\Gamma \subset PSL(3,\mathbb{C})$ be a triangularizable complex Kleinian group such that its Kulkarni limit set does not consist of exactly four lines in general position. Then

- (iv) *The group Γ can be written as*

$$\Gamma = \Gamma_p \rtimes \underbrace{\langle \eta_1 \rangle \rtimes \ldots \rtimes \langle \eta_m \rangle}_{\text{loxo-parabolic}} \rtimes \underbrace{\langle \gamma_1 \rangle \rtimes \ldots \rtimes \langle \gamma_n \rangle}_{\text{strongly loxodromic}}$$

  where Γ_p is the subgroup of Γ consisting of all the parabolic elements of Γ.

- (v) *The group Γ leaves a full flag invariant.*

Main Result

Theorem (MT)

Let $\Gamma \subset PSL(3,\mathbb{C})$ be a solvable *complex Kleinian group such that its Kulkarni limit set does not consist of exactly four lines in general position. Then*

(iv) *The group Γ can be written as*

$$\Gamma = \Gamma_p \rtimes \underbrace{\langle \eta_1 \rangle \rtimes \ldots \rtimes \langle \eta_m \rangle}_{\textit{loxo-parabolic}} \rtimes \underbrace{\langle \gamma_1 \rangle \rtimes \ldots \rtimes \langle \gamma_n \rangle}_{\textit{strongly loxodromic}}$$

where Γ_p is the subgroup of Γ consisting of all the parabolic elements of Γ.

(v) *The group Γ leaves a full flag invariant.*

The parabolic part

The parabolic part is described in

Barrera, W., Cano, A., Navarrete, J. P., & Seade, J. (2022). Discrete parabolic groups in $\mathrm{PSL}(3,\mathbb{C})$. Linear Algebra and its Applications, 653, 430-500.

Decomposition of Non-Commutative Triangular Groups

Theorem (MT)

Let $\Gamma \subset U_+$ be a non-commutative, torsion free, complex Kleinian group, then

$$\Gamma = Core(\Gamma) \rtimes \langle \xi_1 \rangle \rtimes \ldots \rtimes \langle \xi_r \rangle \rtimes \rtimes \langle \eta_1 \rangle \rtimes \ldots \rtimes \langle \eta_m \rangle \rtimes \langle \gamma_1 \rangle \rtimes \ldots \rtimes \langle \gamma_n \rangle .$$

Furthermore, if $k = rank\,(Core(\Gamma))$ then $k + r + m + n \leq 4$.

$$
\overbrace{\begin{bmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \underset{z \neq 0}{\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}}}^{\text{Parabolic}}
\quad
\overbrace{\underset{\text{Loxo-parabolic}}{\underset{\alpha \neq \beta,\ z \neq 0}{\begin{bmatrix} \alpha & x & y \\ 0 & \beta & z \\ 0 & 0 & \beta \end{bmatrix}}} \quad \underset{\text{Strongly loxodromic}}{\underset{\beta \neq \gamma}{\begin{bmatrix} \alpha & x & y \\ 0 & \beta & z \\ 0 & 0 & \gamma \end{bmatrix}}}}^{\text{Loxodromic}}
$$

Core(Γ)

Classification of elements

Commutative groups

Theorem (Barrera, Cano, Navarrete, Seade)

Let $\Gamma \subset U_+$ be a commutative group, then Γ is conjugate in $PSL(3,\mathbb{C})$ to a subgroup of one of the following Abelian Lie Groups:

- $$C_1 = \left\{ \left(\begin{array}{ccc} \alpha^{-2} & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{array} \right) \middle| \alpha \in \mathbb{C}^*, \beta \in \mathbb{C} \right\}.$$

- $$C_2 = \left\{ Diag\left(\alpha, \beta, \alpha^{-1}\beta^{-1}\right) \middle| \alpha, \beta \in \mathbb{C}^* \right\}.$$

- $$C_3 = \left\{ \left(\begin{array}{ccc} 1 & 0 & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{array} \right) \middle| \beta, \gamma \in \mathbb{C} \right\}.$$

Commutative groups

Theorem (Barrera, Cano, Navarrete, Seade)

- $$C_4 = \left\{ \left(\begin{array}{ccc} 1 & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \middle| \beta, \gamma \in \mathbb{C} \right\}.$$

- $$C_5 = \left\{ \left(\begin{array}{ccc} 1 & \beta & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{array} \right) \middle| \beta, \gamma \in \mathbb{C} \right\}.$$

Case 1: Form

Proposition (MT)

Let $\Gamma \subset U_+$ be a commutative subgroup such that each element of Γ has the form C_1. Then there exists an additive subgroup $W \subset (\mathbb{C},+)$, and a group morphism $\mu : (W,+) \to (\mathbb{C}^*,\cdot)$ such that

$$\Gamma = \Gamma_{W,\mu} = \left\{ \begin{bmatrix} \mu(w)^{-2} & 0 & 0 \\ 0 & \mu(w) & w\mu(w) \\ 0 & 0 & \mu(w) \end{bmatrix} \middle| w \in W \right\}.$$

Case 1: Discreteness and Rank

Proposition (MT)

Let $\Gamma = \Gamma_{W,\mu} \subset U_+$ be a group as described in previous proposition. Γ is discrete if and only if rank$(W) \leq 3$ and the morphism μ satisfies the following condition:

- *Whenever we have a sequence $\{w_k\} \in W$ of distinct elements such that $w_k \to 0$, either $\mu(w_k) \to 0$ or $\mu(w_k) \to \infty$.*

Condition **(F)**: there is sequence $\{w_k\} \subset W$ such that $w_k \to \infty$, $\mu(w_k) \to 0$ and $w_k\mu(w_k)^3 \to b \in \mathbb{C}^*$.

Case	Conditions
C1.1	$\mu(W)$ has rational rotations and W is discrete.
C1.2	$\mu(W)$ has rational rotations and W is not discrete.
C1.3	$\mu(W)$ has no rational rotations but has irrational rotations, and W is discrete.
C1.4	$\mu(W)$ has no rational or irrational rotations, and W is discrete.
C1.5	$\mu(W)$ has no rational rotations but has irrational rotations, and W is not discrete.
C1.6	$\mu(W)$ has no rational or irrational rotations, and W is not discrete.

Case 1: Kulkarni Limit Set

Theorem (MT)

Let $\Gamma \subset PSL(3,\mathbb{C})$ be a commutative discrete group having the form given previous proposition, then

$$\Lambda_{Kul}(\Gamma) = \begin{cases} \{e_1, e_2\}, & \text{Cases C1.3 or C1.4 no } \textbf{(F)}. \\ \overleftrightarrow{e_1, e_2}, & \begin{cases} \text{Cases C1.3 or C1.4, with } \textbf{(F)} \\ \text{Case C1.1} \end{cases} \\ \{e_1\} \cup \overleftrightarrow{e_2, e_3}, & \text{Cases C1.5 or C1.6 no } \textbf{(F)}. \\ \overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_2, e_3}, & \begin{cases} \text{Cases C1.5 or C1.6, with } \textbf{(F)} \\ \text{Case C1.2} \end{cases} \end{cases}$$

Example 1

$W = \langle e_1, e_2 \rangle \subset \mathbb{C}$, $\mu : W \to \mathbb{C}^*$ dado por $\mu(w) = e^w$.

$$\Gamma = \left\{ \begin{bmatrix} e^{-2w} & 0 & 0 \\ 0 & e^w & we^w \\ 0 & 0 & e^w \end{bmatrix} \middle| w = m + ni, \ m, n \in \mathbb{Z} \right\}$$

- $\mu(W)$ has irrational rotations and no rational rotations, $e^w \in \mathbb{S}^1 \Leftrightarrow w = ni, \ n \in \mathbb{Z}$.
- Γ satisfies condition **(C)** and not **(F)**, then $\Lambda_{\mathrm{Kul}}(\Gamma) = \{e_1, e_2\}$
- $\Gamma \cong W \cong \mathbb{Z} \times \mathbb{Z}$
- Γ fixes e_1 and its projection on $\overleftrightarrow{e_2, e_3}$ is parabolic.

Example 1

$$W = \langle e_1, e_2 \rangle \subset \mathbb{C},\ \mu : W \to \mathbb{C}^* \text{ dado por } \mu(w) = e^w.$$

$$\Gamma = \left\{ \left[\left[\begin{array}{ccc} e^{-2w} & 0 & 0 \\ 0 & e^w & we^w \\ 0 & 0 & e^w \end{array} \right]\right] \middle| \, w = m + ni,\ \ m, n \in \mathbb{Z} \right\}$$

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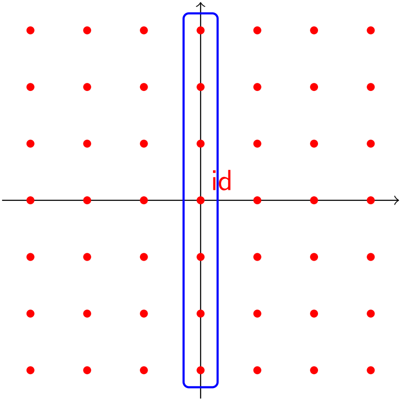
- $\Gamma \cong W \cong \mathbb{Z} \times \mathbb{Z}$
- Γ fixes e_1 and its projection on $\overleftrightarrow{e_2, e_3}$ is parabolic.

- Elements $\gamma \in \Gamma$ with $\mathrm{Re}(w) = 0$ are parabolic.
- Elements $\gamma \in \Gamma$ with $\mathrm{Re}(w) \neq 0$ are loxodromic.



Dual Torus Groups

$$\mathcal{T}^*(W) = \left\{ \begin{bmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| (x, y) \in W \right\}$$

where $W \subset \mathbb{C}^2$ is a discrete additive subgroup with $r(W) \leq 2$.

The names given to these families stem from the fact that they are subgroups of fundamental groups of such complex surfaces.

Example 2: The Setting

Theorem (MT)

Let $\tilde{\Gamma} \subset U_+$ be a discrete non-abelian group containing loxodromic elements. If the parabolic part of $\tilde{\Gamma}$ is conjugate to a dual torus group $\mathcal{T}^*(W)$ with $r(W) = 2$, then there exists a finite index subgroup $\Gamma \subset \tilde{\Gamma}$ conjugate to:

1. $$\mathcal{T}^*(W) \rtimes \left\langle \begin{bmatrix} p & \gamma_{12} & \gamma_{13} \\ 0 & 1 & \frac{q}{p} \\ 0 & 0 & 1 \end{bmatrix} \right\rangle,$$

   where $W \cong \langle (1,0), (0,1) \rangle$, $p \in \mathbb{Z} \setminus \{-1, 0, 1\}$, $q \in \mathbb{Z} \setminus \{0\}$ and $\gamma_{12}, \gamma_{13} \in \mathbb{C}$.

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2.

$$\mathcal{T}^*(W) \rtimes \langle \begin{bmatrix} p_1 & \gamma_{12} & \gamma_{13} \\ 0 & 1 & \frac{q_1}{p_1} \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} p_2 & \mu_{12} & \mu_{13} \\ 0 & 1 & \frac{q_2}{p_2} \\ 0 & 0 & 1 \end{bmatrix} \rangle,$$

where $W \cong \langle (1,0),(0,1) \rangle$, $p_1, p_2 \in \mathbb{Z} \setminus \{-1,0,1\}$, $q_1, q_2 \in \mathbb{Z} \setminus \{0\}$ and $\gamma_{12}, \gamma_{13}, \mu_{12}, \mu_{13} \in \mathbb{C}$ satisfying certain conditions.

Example 2

$$\Gamma = \mathcal{T}^*(W) \rtimes \left\langle \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \right\rangle$$

$$= \left\langle \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\rangle \rtimes \left\langle \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \right\rangle$$

$$= \left\{ \begin{bmatrix} 2^k & m & \frac{km}{2} + n \\ 0 & 1 & \frac{k}{2} \\ 0 & 0 & 1 \end{bmatrix} \middle| \, k, m, n \in \mathbb{Z} \right\}$$

- $r(\Gamma) = 3$
- $\Lambda_{\mathrm{Kul}}(\Gamma) = \overleftrightarrow{e_1, e_2}$ (Detalles, p. 21)

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Kernel of a Group

Consider a subgroup $\Gamma \subset \mathrm{PSL}(3, \mathbb{C})$ acting on $\mathbb{CP}^2$ with a global fixed point $p \in \mathbb{CP}^2$. Let $\ell \subset \mathbb{CP}^2 \setminus \{p\}$ be a projective complex line. We define the projection $\pi = \pi_{p,\ell} : \mathbb{CP}^2 \to \ell$ given by $\pi(x) = \ell \cap \overleftrightarrow{p, x}$. This function is holomorphic, and it determines the group homomorphism

$$\Pi = \Pi_{p,\ell} : \mathrm{PSL}(3, \mathbb{C}) \to \mathrm{Bihol}(\ell) \cong \mathrm{PSL}(2, \mathbb{C})$$

given by $\Pi(g)(x) = \pi(g(x))$ for $g \in \Gamma$. We write $\mathrm{Ker}(\Gamma)$ instead of $\mathrm{Ker}(\Pi) \cap \Gamma$.

The control group of Γ is $\Pi(\Gamma)$.

The Core of a Group

The core of Γ is an important purely parabolic subgroup of a complex Kleinian group Γ which determines the dynamics of Γ.

Proposition

The elements of $Core(\Gamma)$ *have the form*

$$g_{x,y} = \begin{bmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

for some $x, y \in \mathbb{C}$.