Elementary Complex Kleinian Groups

Taller de Geometría y Sistemas Dinámicos

Mauricio Toledo-Acosta

Departamento de Matemáticas
Universidad de Sonora

- Toledo-Acosta, M. The Dynamics of Solvable Subgroups of $\mathrm{PSL}(3,\mathbb{C})$. Bulletin of the Brazilian Mathematical Society, New Series 53, 127–171 (2022). https://doi.org/10.1007/s00574-021-00254-9
- Toledo-Acosta, M. Representations of Solvable Subgroups of $\mathrm{PSL}(3,\mathbb{C})$. Bulletin of the Brazilian Mathematical Society, New Series 54, (2023). https://doi.org/10.1007/s00574-023-00372-6
- Ongoing collaboration with Angel Cano, IMATE Cuernavaca.

Orígenes: Introducidos por José Seade & Alberto Verjovsky en 1998.

Table of Contents

The Complex Projective Space

The complex projective space $\mathbb{CP}^n$ is defined as

$$\mathbb{CP}^n = \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \mathbb{C}^*,$$

where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ acts by the usual scalar multiplication. Let

$$[\,] : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$$

be the quotient map. We denote the projectivization of the point $x = (x_1, \ldots, x_{n+1}) \in \mathbb{C}^{n+1}$ by $[x] = [x_1 : \ldots : x_{n+1}]$. We denote by $e_1, \ldots, e_n$ the projectivization of the canonical base of $\mathbb{C}^{n+1}$.

For every $p \in \mathbb{CP}^n$, we denote by $\mathbf{p}$ to any of its preimages of the projectivization.

The Complex Projective Space

- For $n = 1$, we have the complex projective line

$$\mathbb{CP}^1 \cong \hat{\mathbb{C}} \cong \mathbb{C} \cup \{\infty\}$$

  the Riemann Sphere.

- For $n = 2$, we have the complex projective plane

$$\mathbb{CP}^2 \cong \mathbb{C}^2 \cup \mathbb{CP}^1$$

In general, we can think of $\mathbb{CP}^n$ as the union of $\mathbb{C}^n$ and the *hyperplane at infinity*

$$\mathbb{CP}^n \cong \mathbb{C}^n \cup \mathbb{CP}^{n-1}.$$

Projective Supspaces

Projective Subspace

A nonempty subset $H \subset \mathbb{CP}^n$ is said to be a projective subspace of dimension k if there is a $\mathbb{C}$-linear subspace $\hat{H} \subset \mathbb{C}^{n+1}$ of dimension $k+1$ such that $[\hat{H}] = H$.

Preliminaries

The complex projective plane $\mathbb{CP}^2$ is defined as

$$\mathbb{CP}^2 = \left(\mathbb{C}^3 \setminus \{0\}\right) / \mathbb{C}^*,$$

where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ acts by the usual scalar multiplication. Let

$$[\,] : \mathbb{C}^3 \setminus \{0\} \to \mathbb{CP}$$

be the quotient map. We denote the projectivization of the point $x = (x_1, x_2, x_3) \in \mathbb{C}^3$ by $[x] = [x_1 : x_2 : x_3]$. We denote by e_1, e_2, e_3 the projectivization of the canonical base of $\mathbb{C}^3$.

The Complex Projective Plane

Projective Subspace

A nonempty subset $H \subset \mathbb{CP}^2$ is said to be a projective subspace of dimension k if there is a $\mathbb{C}$-linear subspace $\hat{H} \subset \mathbb{C}^3$ of dimension $k+1$ such that $[\hat{H}] = H$.

Given $p, q \in \mathbb{CP}^2$, we denote the projective line passing through p, q by

$$\overleftrightarrow{p, q} = [\langle \mathbf{p}, \mathbf{q} \rangle]$$

If ℓ_1, ℓ_2 are different complex lines in $\mathbb{CP}^2$, then $\ell_1 \cap \ell_2$ consists of exactly one point.

Preliminaries

Let $\mathrm{GL}(3,\mathbb{C}) \subset \mathcal{M}_3(\mathbb{C})$ be the subgroup of matrices with determinant not equal to 0. The group of biholomorphic automorphisms of $\mathbb{CP}^2$ is given by

$$\mathrm{PSL}(3,\mathbb{C}) := \mathrm{GL}(3,\mathbb{C}) / \{\text{scalar matrices}\}.$$

$\mathrm{PSL}(3,\mathbb{C})$ acts transitively on $\mathbb{CP}^2$, taking projective lines into projective lines.

Elements of PSL$(3,\mathbb{C})$

We denote the projectivization of $(a_{ij}) \in \mathrm{GL}(3,\mathbb{C})$ by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

We denote the upper triangular subgroup of PSL$(3,\mathbb{C})$ by

$$U_+ = \left\{ \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \middle| \, a_{11}a_{22}a_{33} = 1, \;\; a_{ij} \in \mathbb{C} \right\}.$$

Discrete subgroups of $PSL(3,\mathbb{C})$

$PSL(3,\mathbb{C})$ is a topological group.

Discrete subgroups of $PSL(3,\mathbb{C})$

A discrete subgroup of $PSL(3,\mathbb{C})$ is a subgroup which is discrete with respect to the topology of $PSL(3,\mathbb{C})$.

Classification of Elements

We classify the elements of $\mathrm{PSL}(3, \mathbb{C})$ in three classes according to their dynamic behaviour:

$$\begin{cases} \text{Elliptic} \\ \text{Parabolic} \\ \text{Loxodromic} \end{cases}$$

There are several subclasses in each class. We now give a quick summary of some of these subclasses.

Classification of elements of $\mathrm{PSL}(3,\mathbb{C})$

An element $g \in \mathrm{PSL}(3,\mathbb{C})$ is said to be:

- Elliptic if it has a diagonalizable lift in $\mathrm{SL}(3,\mathbb{C})$ such that every eigenvalue has norm 1.

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$$

  $|\alpha_1| = |\alpha_2| = |\alpha_3| = 1.$

- Parabolic if it has a non-diagonalizable lift in $\mathrm{SL}(3,\mathbb{C})$ such that every eigenvalue has norm 1.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^{-2} \end{pmatrix}$$

Classification of elements of $PSL(3,\mathbb{C})$

- Loxodromic if it has a lift in $SL(3,\mathbb{C})$ with an eigenvalue of norm distinct of 1. Furthermore, we say that g is:
  - Loxo-parabolic

$$\mathbf{h} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix}, \ |\lambda| \neq 1.$$

  - A complex homothety, $\mathbf{h} = \mathrm{Diag}(\lambda, \lambda, \lambda^{-2})$, $|\lambda| \neq 1$.
  - A rational (resp. irrational) screw, $\mathbf{h} = \mathrm{Diag}(\lambda_1, \lambda_2, \lambda_3)$, $|\lambda_1| = |\lambda_2| \neq |\lambda_3|$ and $\lambda_1 \lambda_2^{-1} = e^{2\pi i \theta}$ with $\theta \in \mathbb{Q}$ (resp. $\theta \in \mathbb{R} \setminus \mathbb{Q}$).
  - Strongly loxodromic, $\mathbf{h} = \mathrm{Diag}(\lambda_1, \lambda_2, \lambda_3)$, where $\{|\lambda_1|, |\lambda_2|, |\lambda_3|\}$ are pairwise different.

Restrictions

Dynamic and Algebraic Classification

• Parabólicos	• Loxodrómicos	• Elípticos	
Hay un levant. no diagonalizable con eigenvalores de norma 1.	Levantamiento con eigenvalor de norma $\neq 1$.	Hay un levant. diagonalizable en $SL(3, \mathbb{C})$ tq cada eigenvalor es de norma 1.	Alg.
$\gamma \in PSL(3, \mathbb{C})$ es parabólico si hay una familia de esferas γ-invariantes $\mathcal{F}$ y $z_\gamma \in \mathbb{CP}^2$ tq (i) $\forall\, T_1, T_2 \in \mathcal{F}$ $T_1 \cap T_2 = \{z_\gamma\}$ (ii) El conj. $\overline{\bigcup \mathcal{F}}$ es una esfera	$\gamma \in PSL(3, \mathbb{C})$ es loxodrómico si hay una esfera $T \subset \mathbb{CP}^2$ tq $\gamma(T \cup X_i) \subset X_i$ para algún $i = 1, 2$ donde X_1, X_2 son las componentes conexas de $\mathbb{CP}^2 \setminus T$	Transformaciones que preservan cada hoja de la foliación. $\gamma \in PSL(3, \mathbb{C})$ elip $\Updownarrow$ $h^{-1} \gamma h (T(r)) = T(r)$ $\forall\, r > 0$ p.a. $h \in PSL(3, \mathbb{C})$ (detalles)	Dinámica.

Proper and discontinuous actions

Properly and discontinous action

Let Γ be a subgroup of $\mathrm{PSL}(n+1,\mathbb{C})$ and let Ω be a Γ-invariant open subset of $\mathbb{CP}^n$. The action of Γ on Ω is properly discontinuous if for every compact set $K \subset \Omega$ we have that K intersects at most a finite number of copies of its Γ-orbit.

Complex Kleinian Group

A discrete subgroup Γ of $\mathrm{PSL}(n+1,\mathbb{C})$ is complex Kleinian if it acts properly discontinuously on some open subset of $\mathbb{CP}^n$.

The Goal

We want to study the dynamics of discrete subgroups of $\mathrm{PSL}(3,\mathbb{C})$ acting on $\mathbb{CP}^2$.

- Description of accumulation points of orbits.
- Uniformization of complex surfaces.
- Construction of a Sullivan's Dictionary linking the theory of Complex Kleinian groups and the study of iterates of holomorphic functions of $\mathbb{CP}^n$.

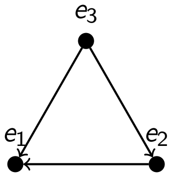
Example

$$\gamma = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}$$

$$|\alpha_1| > |\alpha_2| > |\alpha_3|$$

$$\Gamma = \langle \gamma \rangle$$

- e_1 es un atractor, e_2 punto silla y e_3 repulsor.
- $\overleftrightarrow{e_1, e_2}$, $\overleftrightarrow{e_2, e_3}$ son invariantes.
- Las órbitas de puntos en $\overleftrightarrow{e_2, e_3}$ se acumulan en e_2 (hacia adelante) y en e_3 (hacia atrás).
- Análogamente con $\overleftrightarrow{e_1, e_2}$.
- La órbita de los puntos en $\mathbb{CP}^2 \setminus (\overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_2, e_3})$ se acumulan en $\{e_1, e_3\}$.



Proper and discontinuous actions

Properly and discontinous action

Let Γ be a subgroup of $\mathrm{PSL}(n+1,\mathbb{C})$ and let Ω be a Γ-invariant open subset of $\mathbb{CP}^n$. The action of Γ on Ω is properly discontinuous if for every compact set $K \subset \Omega$ we have that K intersects at most a finite number of copies of its Γ-orbit.

Complex Kleinian Group

A discrete subgroup Γ of $\mathrm{PSL}(n+1,\mathbb{C})$ is complex Kleinian if it acts properly discontinuously on some open subset of $\mathbb{CP}^n$.

Table of Contents

Background

Action on $\mathbb{CP}^2$

Background

Action on $\mathbb{CP}^2$

Irreducible

Action on $\mathbb{CP}^2$

Irreducible ✓

Background

Action on $\mathbb{CP}^2$

Irreducible ✓

Reducible action $\Bigg\{$

Background

Action on $\mathbb{CP}^2$

Irreducible ✓

$$\text{Reducible action} \begin{cases} \text{Solvable} \\ \\ \end{cases}$$

Action on $\mathbb{CP}^2$

Irreducible ✓

$$\text{Reducible action} \begin{cases} \text{Solvable} \\ \text{Non-solvable} \end{cases}$$

Background

Action on $\mathbb{CP}^2$

Irreducible ✓

$$\text{Reducible action} \begin{cases} \text{Solvable} \\ \text{Non-solvable} \end{cases}$$

Background

Action on $\mathbb{CP}^2$

Irreducible ✓

$$\text{Reducible action} \begin{cases} \text{Solvable} \\ \text{Non-solvable} \end{cases}$$

- W. Barrera, A. Cano, J. P. Navarrete, Subgroups of $\mathrm{PSL}(3,\mathbb{C})$ with four lines in general position in its limit set, Conformal geometry and dynamics 15 (2011), 160-176.
- W. Barrera, A. Cano, J. P. Navarrete, One line complex Kleinian groups, Pacific J. Mathematics 272 (2014), 275-303.
- W. Barrera, A. Cano, J. P. Navarrete, On the number of lines in the limit set for discrete subgroups of $\mathrm{PSL}(3,\mathbb{C})$, Pacific J. Mathematics 281 (2016), 17-49.
- W. Barrera, A. Cano, J. P. Navarrete, J. Seade, Discrete parabolic groups in $\mathrm{PSL}(3,\mathbb{C})$, Linear Algebra and its Applications, Vol. 653, 15 November 2022, Pages 430-500.

Limit sets

There are several types of possible *limit sets* for the action of Complex Kleinian Groups:

1. The complements of maximal regions of discontinuity.
2. The complement of the region of equicontinuity.
3. The closure of the fixed points of loxodromic elements.
4. The limit set of Kulkarni.
5. The limit set of Conze-Guivarch.

In complex dimension 1, all of them coincide.

Example

$$\gamma = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix}$$

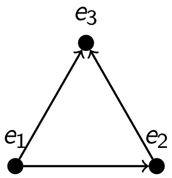
$$|\alpha_1| < |\alpha_2| < |\alpha_3|$$

$$\Gamma = \langle \gamma \rangle$$

$$\Omega_1 = \mathbb{CP}^2 \setminus \left(\overleftrightarrow{e_1, e_2} \cup \{e_3\} \right)$$

$$\Omega_2 = \mathbb{CP}^2 \setminus \left(\overleftrightarrow{e_2, e_3} \cup \{e_1\} \right)$$

Ω_1 and Ω_2 are maximal open sets where the action of Γ is properly discontinuous.



The Kulkarni Limit Set

From now on, let $\Gamma \subset \mathrm{PSL}(3,\mathbb{C})$ be a discrete subgroup acting on $\mathbb{CP}^2$.

Definition

- *Let $L_0(\Gamma)$ be the closure of the set of points in $\mathbb{CP}^n$ with infinite isotropy group.*
- *Let $L_1(\Gamma)$ be the closure of the set of cluster points of orbits of points in $\mathbb{CP}^n \setminus L_0(\Gamma)$.*
- *Let $L_2(\Gamma)$ be the closure of the set of cluster points of compact sets of $\mathbb{CP}^n \setminus (L_0(\Gamma) \cup L_1(\Gamma))$.*

$$\Lambda_{Kul}(\Gamma) = \overline{L_0(\Gamma) \cup L_1(\Gamma) \cup L_2(\Gamma)}, \quad \Omega_{Kul}(\Gamma) = \mathbb{CP}^n \setminus \Lambda_{Kul}(\Gamma).$$

Limit sets for complex Kleinian groups

The Kulkarni limit set $\Lambda_{\mathrm{Kul}}(\Gamma)$ in $\mathbb{CP}^2$ is made up of points and complex projective lines. It contains 1, 2, 3, 4 or ∞ lines in general position.

The equicontinuity region for Γ, denoted $\mathrm{Eq}(\Gamma)$, is defined to be the set of points $z \in \mathbb{CP}^n$ for which there is an open neighborhood U of z such that Γ restricted to U is a normal family. Γ acts properly and discontinuously on $\mathrm{Eq}(\Gamma)$, and

$$\mathrm{Eq}(\Gamma) \subset \Omega_{\mathrm{Kul}}(\Gamma).$$

Limit sets for complex Kleinian groups

The Kulkarni limit set $\Lambda_{\mathrm{Kul}}(\Gamma)$ in $\mathbb{CP}^2$ is made up of points and complex projective lines. It contains 1, 2, 3, 4 or ∞ lines in general position.

The equicontinuity region for Γ, denoted $\mathrm{Eq}(\Gamma)$, is defined to be the set of points $z \in \mathbb{CP}^n$ for which there is an open neighborhood U of z such that Γ restricted to U is a normal family. Γ acts properly and discontinuously on $\mathrm{Eq}(\Gamma)$, and

$$\mathrm{Eq}(\Gamma) \subset \Omega_{\mathrm{Kul}}(\Gamma).$$

Limit sets for complex Kleinian groups

The Kulkarni limit set $\Lambda_{\mathrm{Kul}}(\Gamma)$ in $\mathbb{CP}^2$ is made up of points and complex projective lines. It contains 1, 2, 3, 4 or ∞ lines in general position.

The equicontinuity region for Γ, denoted $\mathrm{Eq}(\Gamma)$, is defined to be the set of points $z \in \mathbb{CP}^n$ for which there is an open neighborhood U of z such that Γ restricted to U is a normal family. Γ acts properly and discontinuously on $\mathrm{Eq}(\Gamma)$, and

$$\mathrm{Eq}(\Gamma) \subset \Omega_{\mathrm{Kul}}(\Gamma).$$

Limit sets for complex Kleinian groups

The Kulkarni limit set $\Lambda_{\mathrm{Kul}}(\Gamma)$ in $\mathbb{CP}^2$ is made up of points and complex projective lines. It contains 1, 2, 3, 4 or ∞ lines in general position.

The equicontinuity region for Γ, denoted $\mathrm{Eq}(\Gamma)$, is defined to be the set of points $z \in \mathbb{CP}^n$ for which there is an open neighborhood U of z such that Γ restricted to U is a normal family. Γ acts properly and discontinuously on $\mathrm{Eq}(\Gamma)$, and

$$\mathrm{Eq}(\Gamma) \subset \Omega_{\mathrm{Kul}}(\Gamma).$$

Elementary Complex Kleinian Group

Elementary Complex Kleinian Groups are groups exhibihiting simple dynamics.

How can we define *elementary* complex Kleinian groups?

- Discrete subgroups of $\mathrm{PSL}(3, \mathbb{C})$ such that its Kulkarni limit set contains a finite number of lines.

In complex dimension 1 all of them coincide.

Elementary Complex Kleinian Group

Elementary Complex Kleinian Groups are groups exhibihiting simple dynamics.

How can we define *elementary* complex Kleinian groups?

- Discrete subgroups of $\mathrm{PSL}(3, \mathbb{C})$ such that its Kulkarni limit set contains a finite number of lines.
- Discrete subgroups of $\mathrm{PSL}(3, \mathbb{C})$ such that its Kulkarni limit set contains a finite number of lines in general position.

In complex dimension 1 all of them coincide.

Elementary Complex Kleinian Group

Elementary Complex Kleinian Groups are groups exhibihiting simple dynamics.

How can we define *elementary* complex Kleinian groups?

- Discrete subgroups of $\mathrm{PSL}(3,\mathbb{C})$ such that its Kulkarni limit set contains a finite number of lines.
- Discrete subgroups of $\mathrm{PSL}(3,\mathbb{C})$ such that its Kulkarni limit set contains a finite number of lines in general position.
- Discrete subgroups of $\mathrm{PSL}(3,\mathbb{C})$ with reducible action.

In complex dimension 1 all of them coincide.

Elementary Complex Kleinian Group

Elementary Complex Kleinian Groups are groups exhibihiting simple dynamics.

How can we define *elementary* complex Kleinian groups?

- Discrete subgroups of $PSL(3,\mathbb{C})$ such that its Kulkarni limit set contains a finite number of lines.
- Discrete subgroups of $PSL(3,\mathbb{C})$ such that its Kulkarni limit set contains a finite number of lines in general position.
- Discrete subgroups of $PSL(3,\mathbb{C})$ with reducible action.
- Discrete solvable subgroups of $PSL(3,\mathbb{C})$.

In complex dimension 1 all of them coincide.

Table of Contents

Solvable groups

- If $g, h \in G$, we define the commutator as $[g, h] = g^{-1}h^{-1}gh$.
- We define the commutator subgroup as

$$[G, G] = \{[g, h] \mid g, h \in G\}.$$

- The derived series of G is given by

$$G^{(0)} = G, \quad G^{(i+1)} = \left[G^{(i)}, G^{(i)}\right].$$

- We say that G is solvable if, for some $n \geq 0$, we have $G^{(n)} = \{id\}$.

Solvable groups

- If $g, h \in G$, we define the commutator as $[g,h] = g^{-1}h^{-1}gh$.
- We define the commutator subgroup as

$$[G,G] = \{[g,h] \mid g,h \in G\}.$$

- The derived series of G is given by

$$G^{(0)} = G, \quad G^{(i+1)} = \left[G^{(i)}, G^{(i)}\right].$$

- We say that G is solvable if, for some $n \geq 0$, we have $G^{(n)} = \{id\}$.

Solvable groups

- If $g, h \in G$, we define the commutator as $[g, h] = g^{-1}h^{-1}gh$.
- We define the commutator subgroup as

$$[G, G] = \{[g, h] \mid g, h \in G\}.$$

- The derived series of G is given by

$$G^{(0)} = G, \quad G^{(i+1)} = \left[G^{(i)}, G^{(i)}\right].$$

- We say that G is solvable if, for some $n \geq 0$, we have $G^{(n)} = \{id\}$.

Solvable groups

- If $g, h \in G$, we define the commutator as $[g, h] = g^{-1}h^{-1}gh$.
- We define the commutator subgroup as

$$[G, G] = \{[g, h] \mid g, h \in G\}.$$

- The derived series of G is given by

$$G^{(0)} = G, \quad G^{(i+1)} = \left[G^{(i)}, G^{(i)}\right].$$

- We say that G is solvable if, for some $n \geq 0$, we have $G^{(n)} = \{id\}$.

Solvable groups: Examples

- The infinite dihedral group is solvable

$$\mathrm{Dih}_\infty = \langle \mathrm{Rot}_\infty,\ z \mapsto -z \rangle.$$

- Any triangular group is solvable, with solvability length at most 3.
- Cyclic groups, abelian groups.
- The special orthogonal group is **not** solvable,

$$\left\{ \begin{bmatrix} a & -c \\ c & \overline{a} \end{bmatrix} \middle| \, |a|^2 + |b|^2 = 1 \right\} \subset \mathrm{PSL}(2, \mathbb{C})$$

Solvable groups: Examples

- The infinite dihedral group is solvable

$$\mathrm{Dih}_\infty = \langle \mathrm{Rot}_\infty,\ z \mapsto -z \rangle.$$

- Any triangular group is solvable, with solvability length at most 3.
- Cyclic groups, abelian groups.
- The special orthogonal group is **not** solvable,

$$\left\{ \begin{bmatrix} a & -c \\ c & \bar{a} \end{bmatrix} \middle| |a|^2 + |b|^2 = 1 \right\} \subset \mathrm{PSL}(2, \mathbb{C})$$

Solvable groups: Examples

- The infinite dihedral group is solvable

$$\mathrm{Dih}_\infty = \langle \mathrm{Rot}_\infty,\ z \mapsto -z \rangle.$$

- Any triangular group is solvable, with solvability length at most 3.
- Cyclic groups, abelian groups.
- The special orthogonal group is **not** solvable,

$$\left\{ \begin{bmatrix} a & -c \\ c & \bar{a} \end{bmatrix} \middle| |a|^2 + |b|^2 = 1 \right\} \subset \mathrm{PSL}\,(2, \mathbb{C})$$

Solvable groups: Examples

- The infinite dihedral group is solvable

$$\mathrm{Dih}_\infty = \langle \mathrm{Rot}_\infty,\ z \mapsto -z \rangle.$$

- Any triangular group is solvable, with solvability length at most 3.
- Cyclic groups, abelian groups.
- The special orthogonal group is **not** solvable,

$$\left\{ \begin{bmatrix} a & -c \\ c & \bar{a} \end{bmatrix} \middle| \, |a|^2 + |b|^2 = 1 \right\} \subset \mathrm{PSL}\,(2, \mathbb{C})$$

Why Solvable Subgroups of PSL $(3, \mathbb{C})$?

Tit's Alternative

Let G be a finitely generated linear group over a field. Then two following possibilities occur:

- either G is virtually solvable (i.e., has a solvable subgroup of finite index).
- or it contains a nonabelian free group.

If $\Gamma \subset \mathrm{PSL}(3, \mathbb{C})$ is a discrete solvable subgroup, then Γ is virtually triangularizable.

Therefore, we can focus on discrete subgroups of U_+.

Table of Contents

Main Result

Theorem

Let $\Gamma \subset PSL(3,\mathbb{C})$ be a triangularizable complex Kleinian group such that its Kulkarni limit set does not consist of exactly four lines in general position. Then there exists a non-empty open region $\Omega_\Gamma \subset \mathbb{CP}^2$ such that

1. Ω_Γ *is the maximal open set where the action is proper and discontinuous.*
2. Ω_Γ *is homeomorphic to one of the following regions:* $\mathbb{C}^2$, $\mathbb{C}^2 \setminus \{0\}$, $\mathbb{C} \times (\mathbb{H}^+ \cup \mathbb{H}^-)$ *or* $\mathbb{C} \times \mathbb{C}^*$.
3. Γ *is finitely generated and* $\mathrm{rank}(\Gamma) \leq 4$.

Main Result

Theorem

Let $\Gamma \subset PSL(3, \mathbb{C})$ be a triangularizable complex Kleinian group such that its Kulkarni limit set does not consist of exactly four lines in general position. Then

(iv) *The group Γ can be written as*

$$\Gamma = \Gamma_p \rtimes \underbrace{\langle \eta_1 \rangle \rtimes \ldots \rtimes \langle \eta_m \rangle}_{\text{loxo-parabolic}} \rtimes \underbrace{\langle \gamma_1 \rangle \rtimes \ldots \rtimes \langle \gamma_n \rangle}_{\text{strongly loxodromic}}$$

where Γ_p is the subgroup of Γ consisting of all the parabolic elements of Γ.

(v) *The group Γ leaves a full flag invariant.*

Main Result

Theorem

Let $\Gamma \subset PSL(3, \mathbb{C})$ be a solvable complex Kleinian group such that its Kulkarni limit set does not consist of exactly four lines in general position. Then

- (iv) *The group Γ can be written as*

$$\Gamma = \Gamma_p \rtimes \underbrace{\langle \eta_1 \rangle \rtimes \ldots \rtimes \langle \eta_m \rangle}_{\textit{loxo-parabolic}} \rtimes \underbrace{\langle \gamma_1 \rangle \rtimes \ldots \rtimes \langle \gamma_n \rangle}_{\textit{strongly loxodromic}}$$

  where Γ_p is the subgroup of Γ consisting of all the parabolic elements of Γ.

- (v) *The group Γ leaves a full flag invariant.*

Table of Contents

Ideas behind the proof

The parabolic part is described in

Barrera, W., Cano, A., Navarrete, J. P., & Seade, J. (2022). Discrete parabolic groups in PSL (3, C). Linear Algebra and its Applications, 653, 430-500.

(v) Invariant Flag

Borel fixed point theorem: Let G be a connected solvable group acting morphically on a non-empty complete variety V. Then G has a fixed point in V. Morphical action

Applying this theorem to the Zariski closure of Γ yields that Γ is virtually triangularizable. Namely, Γ has a finite index subgroup such that, up to conjugation, is upper triangular.

This proves (v).

(v) Invariant Flag

Borel fixed point theorem: Let G be a connected solvable group acting morphically on a non-empty complete variety V. Then G has a fixed point in V. Morphical action

Applying this theorem to the Zariski closure of Γ yields that Γ is virtually triangularizable. Namely, Γ has a finite index subgroup such that, up to conjugation, is upper triangular.

This proves (v).

Conclusions (i)-(iv) are proved together

The proof is divided into two cases:

$$\begin{cases} \Gamma \text{ is not commutative} \\ \\ \Gamma \text{ is commutative} \end{cases}$$

Restrictions on the Elements of a Non-commutative Group

Proposition

Let $\Gamma \subset U_+$ be a discrete subgroup. Let $\gamma \in \Gamma$ be an irrational screw $\gamma = Diag(\beta^{-2}e^{-6\pi i\theta}, \beta e^{4\pi i\theta}, \beta e^{2\pi i\theta})$, for some $|\beta| \neq 1$ and $\theta \in \mathbb{R} \setminus \mathbb{Q}$, then Γ is commutative.

Proposition

Let $\Gamma \subset U_+$ be a non-commutative, torsion-free discrete subgroup, then Γ cannot contain a type I complex homothety.

CH and IS

The Core of a Group

The core of Γ is an important purely parabolic subgroup of a complex Kleinian group Γ which determines the dynamics of Γ.

Proposition

The elements of Core(Γ) have the form

$$g_{x,y} = \begin{bmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

for some $x, y \in \mathbb{C}$.

The Core of a Group

It is straightforward to verify that

$$\Lambda_{\mathrm{Kul}}\left(\mathrm{Core}(\Gamma)\right) = \bigcup_{g_{x,y} \in \mathrm{Core}(\Gamma)} \overleftrightarrow{e_1, [0 : -y : x]}$$

We denote this pencil of lines by $\mathcal{C}(\Gamma) = \Lambda_{\mathrm{Kul}}\left(\mathrm{Core}(\Gamma)\right)$.

Proposition

Let $\Gamma \subset U_+$ be a discrete group, then every element of Γ leaves $\mathcal{C}(\Gamma)$ invariant.

Decomposition of Non-Commutative Triangular Groups

Theorem

Let $\Gamma \subset U_+$ be a non-commutative, torsion free, complex Kleinian group, then

$$\Gamma = Core(\Gamma) \rtimes \langle \xi_1 \rangle \rtimes \ldots \rtimes \langle \xi_r \rangle \rtimes \\ \rtimes \langle \eta_1 \rangle \rtimes \ldots \rtimes \langle \eta_m \rangle \rtimes \langle \gamma_1 \rangle \rtimes \ldots \rtimes \langle \gamma_n \rangle.$$

Furthermore, if $k = rank\,(Core(\Gamma))$ then $k + r + m + n \leq 4$.

$$\underbrace{\begin{bmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}}_{\text{Parabolic}} \quad \underbrace{\begin{bmatrix} \alpha & x & y \\ 0 & \beta & z \\ 0 & 0 & \beta \end{bmatrix} \quad \begin{bmatrix} \alpha & x & y \\ 0 & \beta & z \\ 0 & 0 & \gamma \end{bmatrix}}_{\text{Loxodromic}}$$

$z \neq 0$ $\qquad$ $\alpha \neq \beta,\ z \neq 0$ $\qquad$ $\beta \neq \gamma$

Loxo-parabolic $\qquad$ Strongly loxodromic

$\mathrm{Core}(\Gamma)$ $\qquad$ $A \setminus \mathrm{Ker}(\Gamma)$ $\qquad$ $\mathrm{Ker}(\lambda_{23}) \setminus A$

Classification of elements

Morphisms λ

Let $\lambda_{12}, \lambda_{23}, \lambda_{13} : (U_+, \cdot) \to (\mathbb{C}^*, \cdot)$ be the group morphisms given by

$$\lambda_{12}([\alpha_{ij}]) = \alpha_{11}\alpha_{22}^{-1}$$
$$\lambda_{23}([\alpha_{ij}]) = \alpha_{22}\alpha_{33}^{-1}$$
$$\lambda_{13}([\alpha_{ij}]) = \alpha_{11}\alpha_{33}^{-1}.$$

Strategy of the proof:

- Decomposition of Γ in terms of Ker(λ_{23}).
- Decompose Ker(λ_{23}) in terms of Ker(λ_{12}).
- Decompose $A = \text{Ker}(\lambda_{12}) \cap \text{Ker}(\lambda_{23})$ in terms of Ker(Γ)

Rank

Theorem (Bestvina, Kapovich, Kleiner)

Let Γ be a group acting properly and discontinuously on a contractible manifold of dimension m, then obdim$(\Gamma) \leq m$.

Rank

Theorem (Bestvina, Kapovich, Kleiner)

Let Γ be a group acting properly and discontinuously on a contractible manifold of dimension m, then obdim$(\Gamma) \leq m$.

In our case, it can be re-stated as:

Theorem

Let $\Gamma \subset U_+$ be a non-commutative, torsion free, complex Kleinian group acting properly and discontinuously on a simply connected domain $\Omega \subset \mathbb{CP}^2$, then $k + r + m + n \leq 4$.

Rank

Theorem (Bestvina, Kapovich, Kleiner)

Let Γ be a group acting properly and discontinuously on a contractible manifold of dimension m, then obdim$(\Gamma) \leq m$.

In our case, it can be re-stated as:

Theorem

Let $\Gamma \subset U_+$ be a non-commutative, torsion free, complex Kleinian group acting properly and discontinuously on a simply connected domain $\Omega \subset \mathbb{CP}^2$, then $k + r + m + n \leq 4$.

Find a simply connected domain $\Omega \subset \mathbb{CP}^2$ where Γ acts properly and discontinuously, and then apply the theorem. In some cases, we write the explicit decomposition of Γ and verify that rank$(\Gamma) \leq 4$.

Some cases

Denote $\Sigma = \Pi(\Gamma)$. If Σ is discrete and Ker(Γ) is finite. If $|\Lambda(\Sigma)| \neq 2$, let

$$\Omega = \left(\bigcup_{z \in \Omega(\Sigma)} \overleftrightarrow{e_1, z} \right) \setminus \{e_1\}.$$

We know that Γ acts properly and discontinuously on Ω. If $|\Lambda(\Sigma)| = 0, 1$ or ∞, then each connected component of Ω is simply connected, since they are respectively homeomorphic to $\mathbb{CP}^2$, $\mathbb{C}^2$ or $\mathbb{C} \times \mathbb{H}$. By the theorem, it follows $k + r + m + n \leq 4$

For non-commutative Γ, using these ideas, we have constructed an open subset $\Omega_\Gamma \subset \mathbb{CP}^2$ such that the orbits of every compact set $K \subset \Omega_\Gamma$ accumulate on $\mathbb{CP}^2 \setminus \Omega_\Gamma$. Thus we can define a limit set for the action of Γ by $\Lambda_\Gamma := \mathbb{CP}^2 \setminus \Omega_\Gamma$. This limit set describes the dynamics of Γ, and the open region Ω_Γ satisfies (i) and (ii).

Also, we prove that $\text{rank}(\Gamma) \leq 4$. This verifies (iii).

Commutative groups

Theorem (Barrera, Cano, Navarrete, Seade)

Let $\Gamma \subset U_+$ be a commutative group, then Γ is conjugate in $PSL(3, \mathbb{C})$ to a subgroup of one of the following Abelian Lie Groups:

- $$C_1 = \left\{ \left(\begin{array}{ccc} \alpha^{-2} & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & 0 & \alpha \end{array} \right) \middle| \alpha \in \mathbb{C}^*, \beta \in \mathbb{C} \right\}.$$

- $$C_2 = \left\{ Diag\left(\alpha, \beta, \alpha^{-1}\beta^{-1}\right) \,\middle|\, \alpha, \beta \in \mathbb{C}^* \right\}.$$

- $$C_3 = \left\{ \left(\begin{array}{ccc} 1 & 0 & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 1 \end{array} \right) \middle| \beta, \gamma \in \mathbb{C} \right\}.$$

Commutative groups

Theorem (Barrera, Cano, Navarrete, Seade)

- $$C_4 = \left\{ \left(\begin{array}{ccc} 1 & \beta & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) \middle| \beta, \gamma \in \mathbb{C} \right\}.$$
- $$C_5 = \left\{ \left(\begin{array}{ccc} 1 & \beta & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{array} \right) \middle| \beta, \gamma \in \mathbb{C} \right\}.$$

Case 1: Form

Proposition

Let $\Gamma \subset U_+$ be a commutative subgroup such that each element of Γ has the form C_1. Then there exists an additive subgroup $W \subset (\mathbb{C}, +)$, and a group morphism $\mu : (W, +) \to (\mathbb{C}^*, \cdot)$ such that

$$\Gamma = \Gamma_{W,\mu} = \left\{ \begin{bmatrix} \mu(w)^{-2} & 0 & 0 \\ 0 & \mu(w) & w\mu(w) \\ 0 & 0 & \mu(w) \end{bmatrix} \middle| w \in W \right\}.$$

Case 1: Discreteness and Rank

Proposition

Let $\Gamma = \Gamma_{W,\mu} \subset U_+$ be a group as described in previous proposition. Γ is discrete if and only if rank$(W) \leq 3$ and the morphism μ satisfies the following condition:

- *Whenever we have a sequence $\{w_k\} \in W$ of distinct elements such that $w_k \to 0$, either $\mu(w_k) \to 0$ or $\mu(w_k) \to \infty$.*

Case	Conditions
C1.1	$\mu(W)$ has rational rotations and W is discrete.
C1.2	$\mu(W)$ has rational rotations and W is not discrete.
C1.3	$\mu(W)$ has no rational rotations but has irrational rotations, and W is discrete.
C1.4	$\mu(W)$ has no rational or irrational rotations, and W is discrete.
C1.5	$\mu(W)$ has no rational rotations but has irrational rotations, and W is not discrete.
C1.6	$\mu(W)$ has no rational or irrational rotations, and W is not discrete.

Case 1: Kulkarni Limit Set

Theorem

Let $\Gamma \subset PSL(3,\mathbb{C})$ be a commutative discrete group having the form given previous proposition, then

$$\Lambda_{Kul}(\Gamma) = \begin{cases} \overleftrightarrow{e_1, e_2}, & \begin{cases} \text{Cases C1.3 or C1.4, with condition} \\ \text{Case C1.1} \end{cases} \\ \{e_1\} \cup \overleftrightarrow{e_2, e_3}, & \text{Cases C1.5 or C1.6 no condition } \textbf{(F)}. \\ \overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_2, e_3}, & \begin{cases} \text{Cases C1.5 or C1.6, with condition} \\ \text{Case C1.2} \end{cases} \end{cases}$$

Case 2: Form

Proposition

Let $\Gamma \subset U_+$ be a commutative subgroup such that each element of Γ has the form $Diag(\alpha, \beta, \alpha^{-1}\beta^{-1})$. Then there exist two multiplicative subgroups $W_1, W_2 \subset (\mathbb{C}^*, \cdot)$ such that

$$\Gamma = \Gamma_{W_1,W_2} = \{Diag(w_1, w_2, 1) \,|\, w_1 \in W_1,\ w_2 \in W_2\}.$$

Case 2: Rank

Proposition

Let $\Gamma \subset U_+$ be a diagonal discrete group such that every element has the form $\gamma = Diag(w_1, w_2, 1)$. Then $rank(\Gamma) \leq 2$.

Case 2

If $\alpha^n = \beta^m$ for some $n, m \in \mathbb{Z}$:

[D1] $L_0(\Gamma) \cup L_1(\Gamma) = \overleftrightarrow{e_1, e_2} \cup \{e_3\}$, if $|\alpha| > 1 > |\beta|$ or $|\alpha| < 1 < |\beta|$.

[D2] $L_0(\Gamma) \cup L_1(\Gamma) = \overleftrightarrow{e_1, e_2} \cup \{e_3\}$, if $|\alpha| > |\beta| > 1$ or $|\alpha| < |\beta| < 1$.

If there are no integers n, m such that $\alpha^n = \beta^m$:

[D3] $L_0(\Gamma) \cup L_1(\Gamma) = \{e_1, e_2, e_3\}$, if $|\alpha| > 1 > |\beta|$ or $|\alpha| < 1 < |\beta|$.

[D4] $L_0(\Gamma) \cup L_1(\Gamma) = \{e_1, e_2, e_3\}$, if $|\alpha| > |\beta| > 1$ or $|\alpha| < |\beta| < 1$.

[D5] $L_0(\Gamma) \cup L_1(\Gamma) = \overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_2, e_3}$, if β is an irrational rotation.

Case 2: Kulkarni Limit Set

Theorem

Let $\Gamma_{\alpha,\beta} \subset U_+$ be a discrete group containing loxodromic elements, then

1. $\Lambda_{Kul}(\Gamma) = \overleftrightarrow{e_1, e_2} \cup \{e_3\}$ *in Cases* **[D1]** *and* **[D2]**.
2. $\Lambda_{Kul}(\Gamma) = \{e_1, e_2, e_3\}$ *in Cases* **[D3]** *and* **[D4]**.
3. $\Lambda_{Kul}(\Gamma) = \overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_2, e_3}$ *in Case* **[D5]**.

Commutative case: Proof of the Main Theorem

If Γ is commutative, it is conjugate to a sugroup of the Lie groups C_1 or C_2. In this setting, the region $\Omega_{\mathrm{Kul}}(\Gamma)$ satisfies conclusions (i) and (ii) as a consequence of the previous heorems. Again, $\mathrm{rank}(\Gamma) \leq 4$, this proves conclusion (iii).

On the other hand, $\Gamma \cong \mathbb{Z}^r$ with $r = \mathrm{rank}(\Gamma)$, and then we can write Γ as a trivial semidirect product of copies of $\mathbb{Z}$, thus verifying conclusion (iv).

Table of Contents

A First Generalization ✓

Theorem

Let $\Gamma \subset PSL(3, \mathbb{C})$ be a solvable complex Kleinian group such that its Kulkarni limit set does not consist of exactly four lines in general position. Let $\Gamma_0 \subset \Gamma$ be a virtually triangularizable finite index subgroup. If Γ_0 is commutative then there exists a non-empty open region $\Omega_\Gamma \subset \mathbb{CP}^2$ such that

1. *Ω_Γ is the maximal open set where the action is proper and discontinuous.*
2. *Ω_Γ is homeomorphic to one of the following regions: $\mathbb{C}^2$, $\mathbb{C}^2 \setminus \{0\}$, $\mathbb{C} \times (\mathbb{H}^+ \cup \mathbb{H}^-)$ or $\mathbb{C} \times \mathbb{C}^*$.*
3. *Up to a finite index subgroup, the group Γ leaves a full flag invariant.*

Thank you

Table of Contents

A Group Acting Morphically

Definition

Let G be an algebraic group, V a variety, and let $\alpha : G \times V \to V$ be an action of the group G in V, $(g, x) \mapsto gx = \alpha(g, x)$. One says that G acts morphically on V if the action α satisfies the following axioms:

1. $\alpha(e, x) = x$, *for any $x \in V$, where $e \in G$ is the identity element.*
2. $\alpha(g, hx) = \alpha(gh, x)$ *for any $g, h \in G$ and $x \in V$.*

Solvable groups are virtually triangular

Kernel of a Group

Consider a subgroup $\Gamma \subset \mathrm{PSL}(3,\mathbb{C})$ acting on $\mathbb{CP}^2$ with a global fixed point $p \in \mathbb{CP}^2$. Let $\ell \subset \mathbb{CP}^2 \setminus \{p\}$ be a projective complex line. We define the projection $\pi = \pi_{p,\ell} : \mathbb{CP}^2 \to \ell$ given by $\pi(x) = \ell \cap \overleftrightarrow{p,x}$. This function is holomorphic, and it determines the group homomorphism

$$\Pi = \Pi_{p,\ell} : \mathrm{PSL}(3,\mathbb{C}) \to \mathrm{Bihol}(\ell) \cong \mathrm{PSL}(2,\mathbb{C})$$

given by $\Pi(g)(x) = \pi(g(x))$ for $g \in \Gamma$. We write Ker(Γ) instead of $\mathrm{Ker}(\Pi) \cap \Gamma$.

The control group of Γ is $\Pi(\Gamma)$.