El Plano Proyectivo Complejo $\mathbb{CP}^2$

Seminario de Estructuras Geométricas y Combinatorias

Mauricio Toledo-Acosta

Departamento de Matemáticas
Universidad de Sonora

Table of Contents

The Complex Projetive Plane

The complex projective plane $\mathbb{CP}^2$ is defined as

$$\mathbb{CP}^2 = \left(\mathbb{C}^3 \setminus \{0\}\right)/\mathbb{C}^*,$$

where $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ acts by the usual scalar multiplication. Let

$$[\,] : \mathbb{C}^3 \setminus \{0\} \to \mathbb{CP}^2$$

be the quotient map. We denote by e_1, e_2, e_3 the projectivization of the canonical base of $\mathbb{C}^3$.

Decomposition of $\mathbb{CP}^2$

$$\boxed{\mathbb{CP}^2 = \mathbb{C}^2 \cup \mathbb{CP}^1}$$

- $\mathbb{C}^2 \cong \{[z_0 : z_1 : z_2] \mid z_0 \neq 0\}$, affine chart.
- $\mathbb{CP}^1 \cong \{[0 : z_1 : z_2]\}$, line at infinity.

Observation: This generalizes $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$

Complex Projective Lines

El Plano Proyectivo Complejo $\mathbb{CP}^2$

The Complex Projetive Plane

Lines in General Position

El plano proyectivo de un anillo

Given $p, q \in \mathbb{CP}^2$, we denote the complex projective line passing through p, q by

$$\overleftrightarrow{p, q} = [\langle \mathbf{p}, \mathbf{q} \rangle] \cong \hat{\mathbb{C}}$$

If ℓ_1, ℓ_2 are different complex lines in $\mathbb{CP}^2$, then $\ell_1 \cap \ell_2$ consists of exactly one point.

Complex Lines

Given $p, q \in \mathbb{C}^2$, the complex line passing through p, q is

$$\ell(p,q) = \{p + t(q-p) : t \in \mathbb{C}\} \cong \mathbb{C}$$

If ℓ_1, ℓ_2 are different complex lines in $\mathbb{C}^2$, then $\ell_1 \cap \ell_2$ consists of either:

- exactly one point (if they intersect), or
- the empty set (if they are parallel)

Given $p, w \in \mathbb{C}^2$ we also write

$$\ell(t) = \{p + tw \mid t \in \mathbb{C}\}$$

the complex line passing through p with complex direction w.

Relationship between Complex Lines

Let $p, q \in \mathbb{C}^2 \subset \mathbb{CP}^2$. Then:

$$\overleftrightarrow{p, q} = \ell(p, q) \cup \{\text{point at infinity}\}$$

More precisely:

- The complex projective line $\overleftrightarrow{p, q} \cong \hat{\mathbb{C}}$ is the compactification of the affine line $\ell(p, q) \cong \mathbb{C}$
- Conversely, $\ell(p, q) = \overleftrightarrow{p, q} \cap \mathbb{C}^2$ is the affine part of $\overleftrightarrow{p, q}$

Parallel lines in $\mathbb{C}^2$ meet at their common point at infinity in $\mathbb{CP}^2$. Furthermore, each pair of complex lines with the same complex direction determine a point in $\mathbb{CP}^1$ (their intersection).

$\mathbb{CP}^2$ as a Homogeneous Space

$\mathbb{CP}^2$ is a homogeneous space: it "looks the same" at every point.

The group $\mathrm{PSL}(3, \mathbb{C})$ acts transitively on $\mathbb{CP}^2$:

- For any two points $p, q \in \mathbb{CP}^2$, there exists $g \in \mathrm{PSL}(3, \mathbb{C})$ such that $g \cdot p = q$
- No intrinsic way to distinguish points: all points are equivalent under the symmetry group

This is analogous to how S^2 is homogeneous under rotations $SO(3)$.

Homogeneity and Structure

A space is homogeneous with respect to a geometric structure if the group of automorphisms of that structure acts transitively.

- Vector space structure: linear transformations $\mathrm{GL}\,(3,\mathbb{C})$
- Affine structure: affine transformations (linear + translations)
- Projective structure: projective transformations $\mathrm{PSL}\,(3,\mathbb{C})$
- Metric structure: isometries

Examples:

- $\mathbb{C}^2$ as a vector space: NOT homogeneous under $\mathrm{GL}\,(2,\mathbb{C})$ (origin is fixed)
- $\mathbb{C}^2$ as an affine space: homogeneous under $\mathrm{Aff}(2,\mathbb{C})$
- $\mathbb{CP}^2$ as a projective space: homogeneous under $\mathrm{PSL}\,(3,\mathbb{C})$

Foliation of $\mathbb{CP}^2 \setminus \overleftrightarrow{e_i, e_j} \cup \{e_k\}$

For $r > 0$, let

$$T_1(r) = \{[z_1 : z_2 : z_3] \in \mathbb{CP}^2 \mid |z_2|^2 + |z_3|^2 = r|z_1|^2\},$$
$$T_2(r) = \{[z_1 : z_2 : z_3] \in \mathbb{CP}^2 \mid |z_1|^2 + |z_3|^2 = r|z_2|^2\},$$
$$T_3(r) = \{[z_1 : z_2 : z_3] \in \mathbb{CP}^2 \mid |z_1|^2 + |z_2|^2 = r|z_3|^2\}.$$

The family $\{T_i(r)\}_{r>0}$ is a foliation of $\mathbb{CP}^2 \setminus \overleftrightarrow{e_{i-1}, e_{i+1}} \cup \{e_i\}$.

Observe that $T_i(1) \cong S^3 \subset \mathbb{C}^2$.

(Singular) Foliation of S^3 by tori

Consider the unit sphere $T_3(1)$, and consider the intersections

$$C_r := T_3(1) \cap T_1(r)$$

for $r > 0$. We have three cases:

- If $r < 1$, $C_r = \emptyset$.
- If $r = 1$,

$$\begin{aligned} C_r &= \{[z_1 : z_2 : 1] \,|\, |z_1| = 1,\ z_2 = 0\} \\ &\cong S^1. \end{aligned}$$

- If $r > 1$,

$$\begin{aligned} C_r &= \left\{ [z_1 : z_2 : 1] \,\middle|\, |z_1| = \sqrt{\frac{2}{r+1}},\ |z_2| = \sqrt{\frac{r-1}{r+1}} \right\} \\ &\cong S^1 \times S^1. \end{aligned}$$

Table of Contents

Lines in General Position

A collection of complex projective lines $\{\ell_1, \ell_2, \ldots, \ell_n\}$ in $\mathbb{CP}^2$ is in general position if:

No three lines meet at a common point

Equivalently:

- Any two lines ℓ_i, ℓ_j intersect at exactly one point
- For any three distinct lines ℓ_i, ℓ_j, ℓ_k, the intersection points $\ell_i \cap \ell_j$, $\ell_i \cap \ell_k$, and $\ell_j \cap \ell_k$ are all different

For n lines in general position, there are exactly $\binom{n}{2}$ intersection points.

The space of arrays of 5 CPL in GP

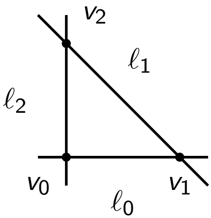
Ultimate Goal: Define a measure supported on the limit set for the action of a discrete subgroup of $\mathrm{PSL}(3, \mathbb{C})$.

- We want to describe the space of arrays of 5 complex projective lines in general position in $\mathbb{CP}^2$.
- Each of these configurations of lines determines a domain in $\mathbb{CP}^2$: The complement in $\mathbb{CP}^2$ of the array.
- The existence of the measure ultimately depends on whether the entropy volume of the Kobayashi metric is finite on this domain.

The space of arrays of 5 CPL in GP

The group $\mathrm{PSL}(3,\mathbb{C})$ acts sharply 4-transitively on the space of complex projective lines in general position in $\mathbb{CP}^2$.

Given an array of 5 complex projective lines in general position, we can arbitrarily pick 4 of them and describe the parameter space as the parameter space of configurations of 4 complex lines in general position.



The space of arrays of 5 CPL in GP

The space of arrays of 4 complex lines in general position in $\mathbb{C}^2$ is given by

$$\mathcal{P} = \left\{ (\zeta_1, \zeta_2) \in \mathbb{C}^2 \,\middle|\, \zeta_1 \in \mathbb{C} \setminus \{0,1\}\,,\ \zeta_2 \in \mathbb{C} \setminus \{0,1,\zeta_1\} \right\}.$$

The set $\mathcal{P} \subset \mathbb{C}^2$ can be regarded as

$$(\mathbb{C} \setminus \{0,1\}) \times (\mathbb{C} \setminus \{0,1\}) \ \setminus \ \ell$$

where ℓ is a complex line going through the origin.

Table of Contents

El plano proyectivo de un anillo