Persistent Homology

Análisis Topológico de Datos (2264)

Mauricio Toledo-Acosta

October 28, 2025

Topological Data Analysis

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The initial motivation is to study the shape of data. TDA combines algebraic topology and other tools from pure mathematics to allow mathematically rigorous study of "shape". One of its main tools is persistent homology, which has been applied to many types of data across many fields.

Introduction

- Persistent Homology is a method used in topological data analysis to study qualitative features of data that persist across multiple scales. It was introduced in 2002.

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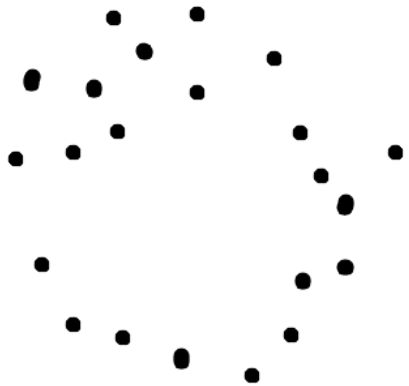

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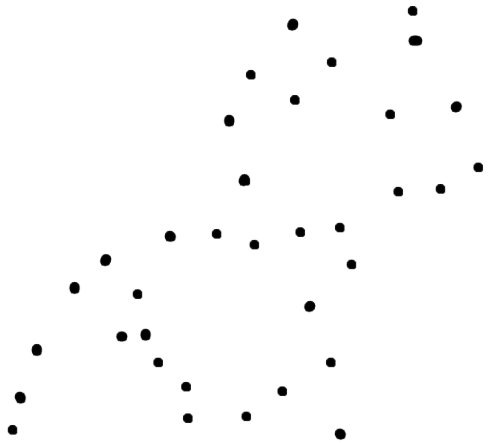

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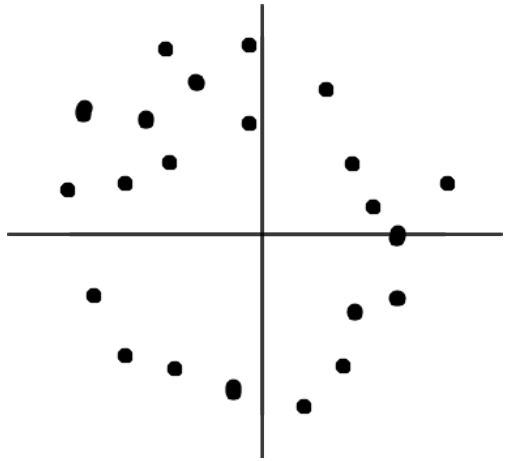

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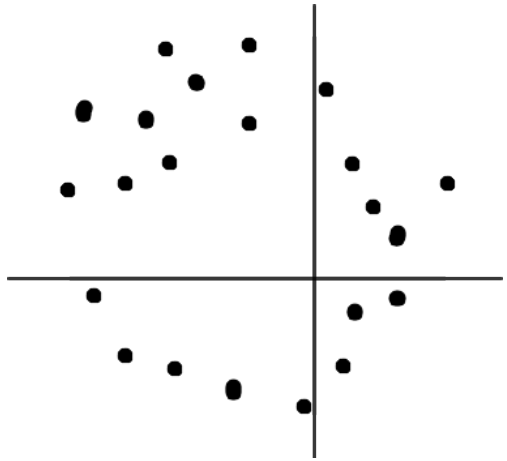

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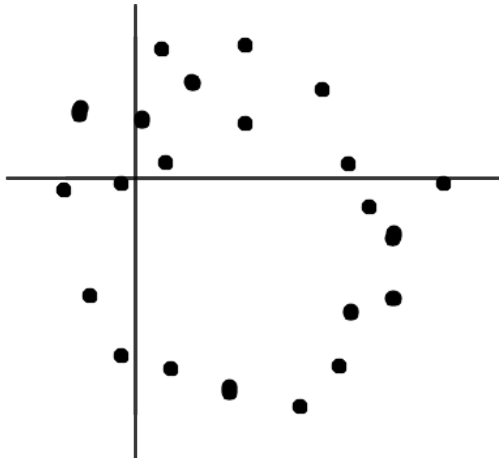

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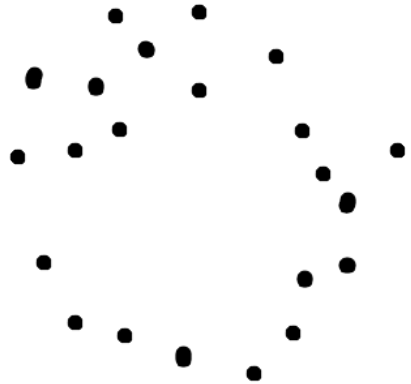

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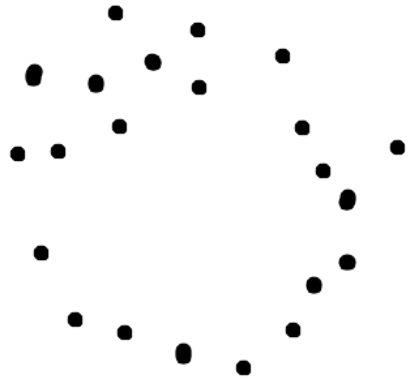

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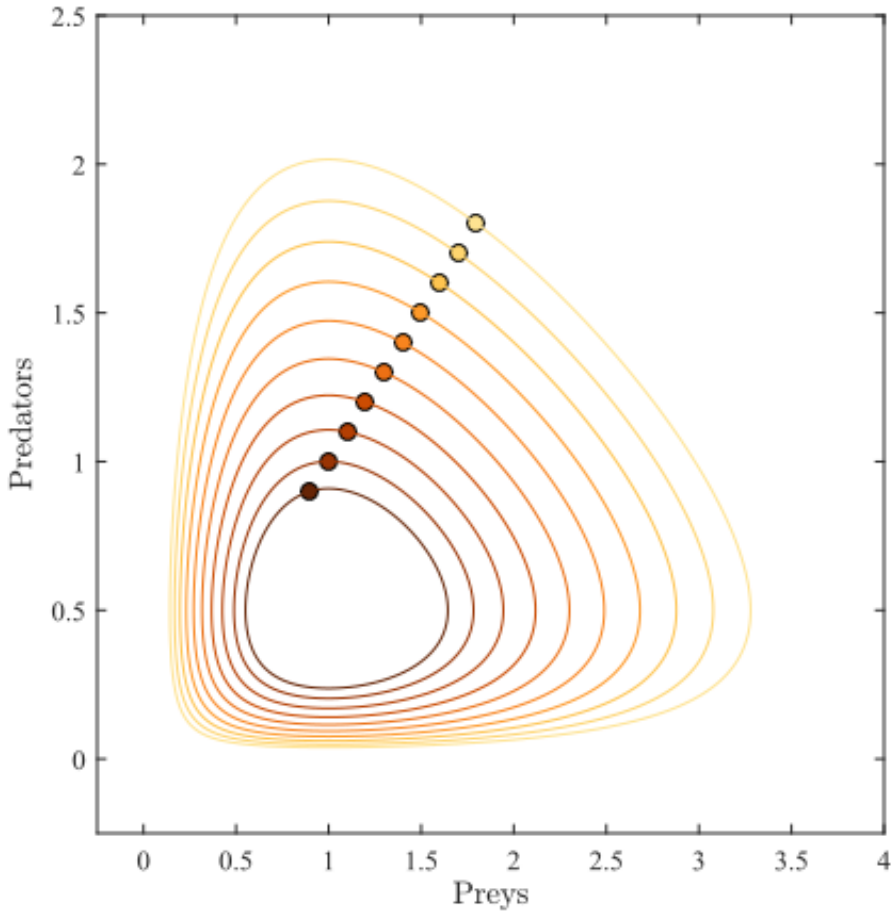
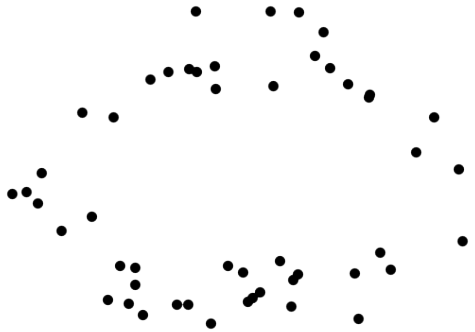
- Persistent Homology accounts for local geometry and global topology of a dataset.
- It has many applications: Image analysis, NLP, Flows, input features for machine learning algorithms, mathematics results.

An example: The Lotka-Volterra equations



The shape of a cloud of points

We have a set of points $X = \{x_1, ..., x_n\} \subset \mathbb{R}^D$, we want to find out the *shape* of X.



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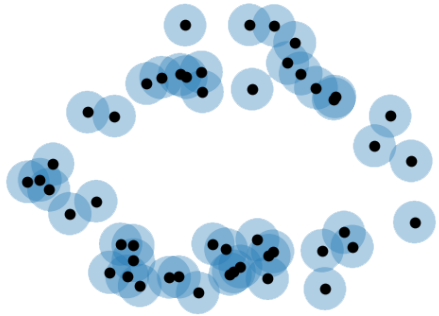

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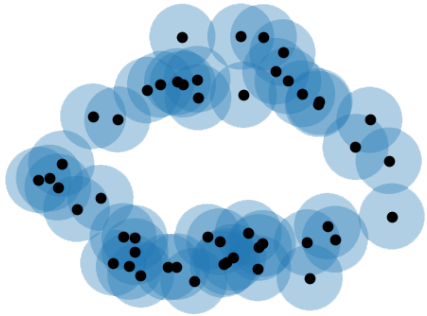

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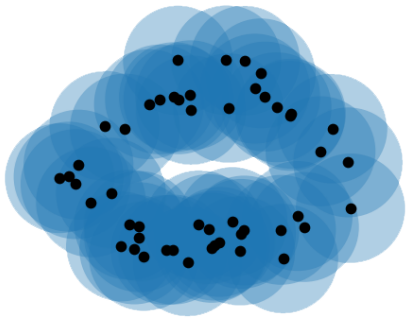

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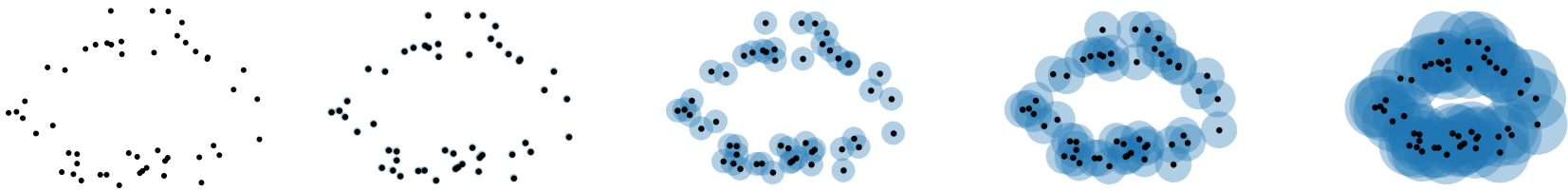

What is the *right* radius?

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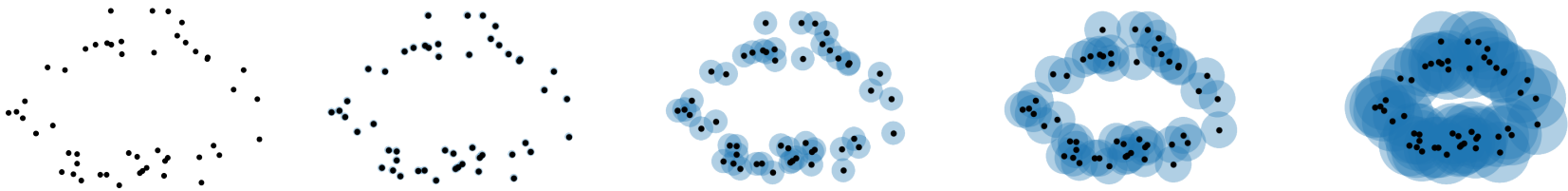
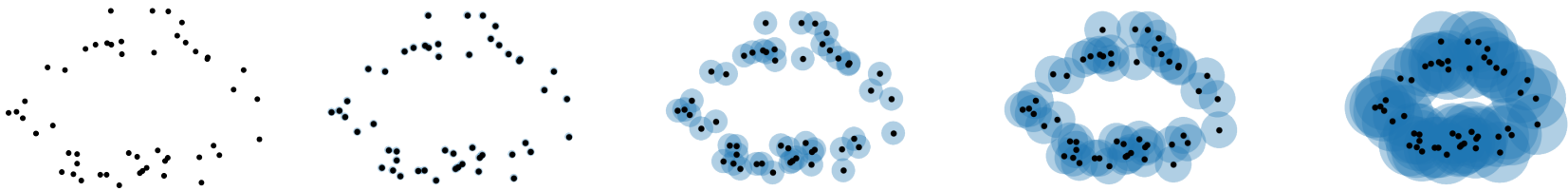


What is the *right* radius? We consider all the possible radii.

Filtrations

- We consider the family of these spaces for each radius $r > 0$.



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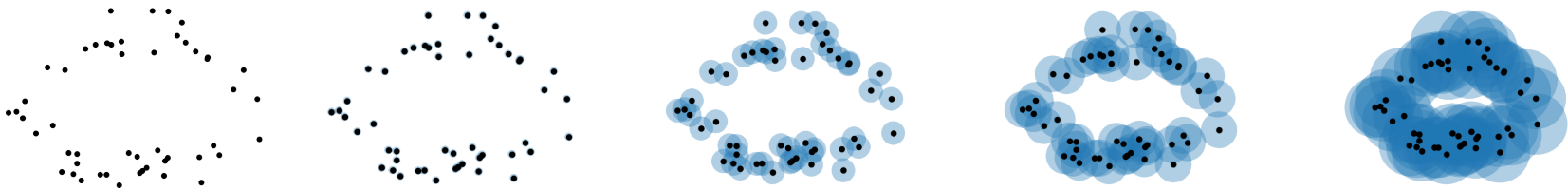

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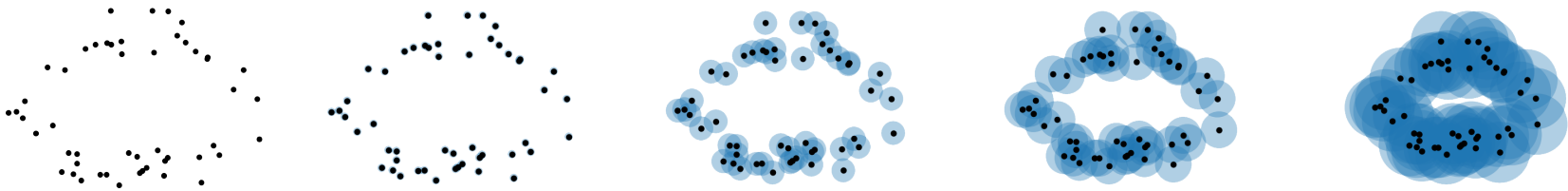
- Each of these spaces represent the figure hiding behind the set of points.
- Each space is contained in the next one. This is called a **filtration**.



Counting holes

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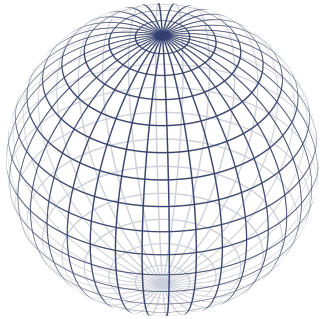

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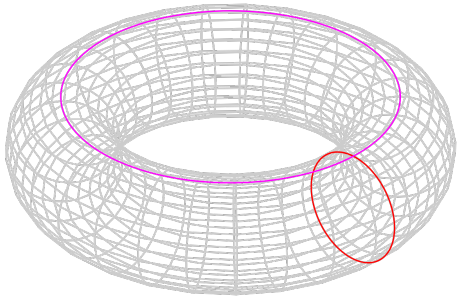

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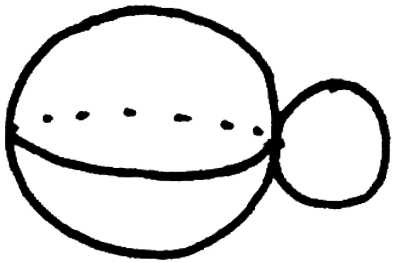

$= S^1 \vee S^2$

This sequence of numbers are the **Betti numbers**.

Counting holes

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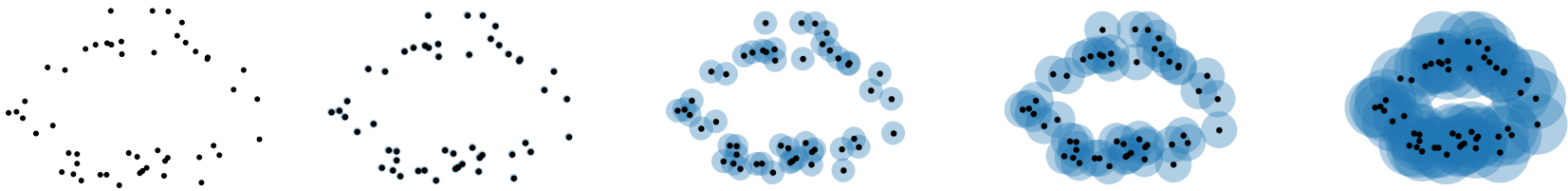
This sequence of numbers are the **Betti numbers**. They can be regarded as:

- b_0: number of connected components.
- b_1: number of one-dimensional or "circular" holes.
- b_2: number of two-dimensional "voids" or "cavities.

Tracking holes

The goal is to keep track of the appearing and disappearing holes as we go through the filtration.



The time each hole (cycle) persists is an indicative of the presence of said cycle in the figure.

Čech Complex and Vietoris-Rips Complex

- Computing the Betti numbers (rank of the homology groups) in the previous filtration is hard.

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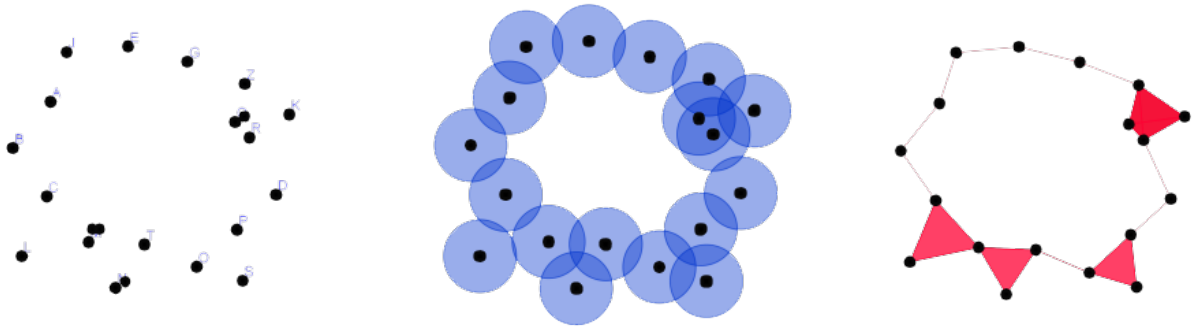
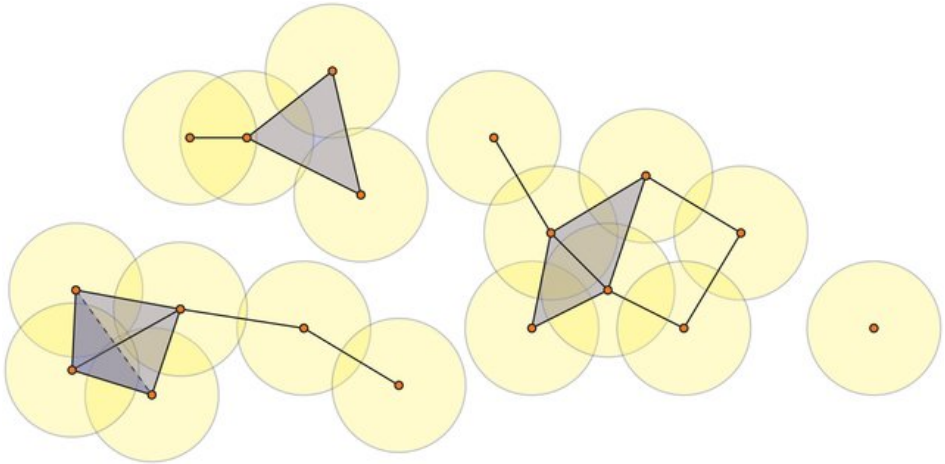
- We construct a simplicial complex:



Čech Complex and Vietoris-Rips Complex

There are two ways of constructing the simplicial complex:

- Vietoris-Rips Complex:



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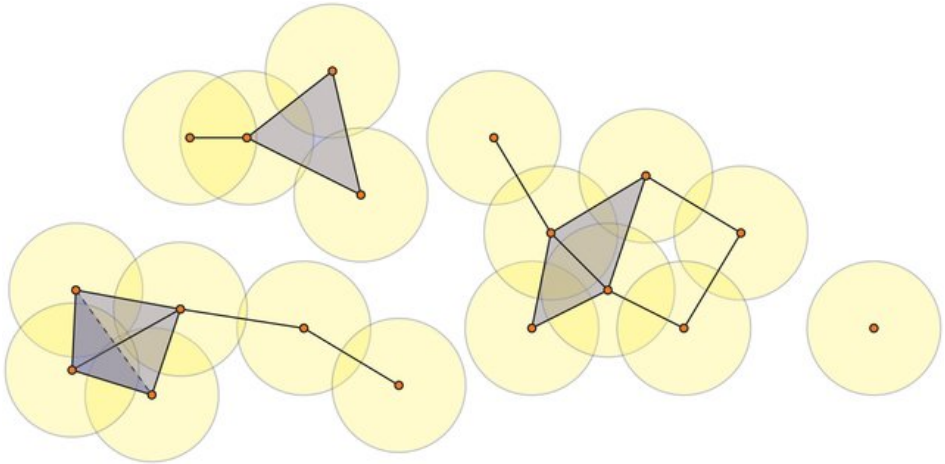
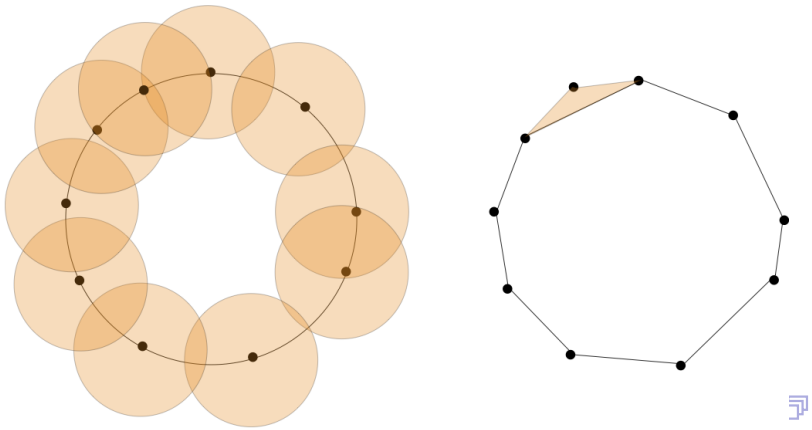


- Čech Complex:



Čech Complex and Vietoris-Rips Complex: Differences

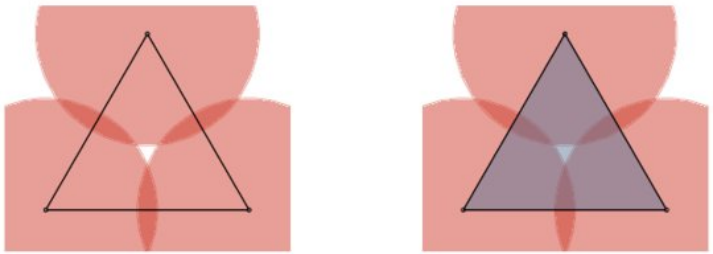


Figure: Čech (left), Vietoris-Rips (right)

Čech complex	**Vietoris-Rips complex**
Smaller	Bigger
Computationally expensive	Less expensive
More accurate	Less accurate

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The filtration is infinite, but there is only a finite number of different complexes.

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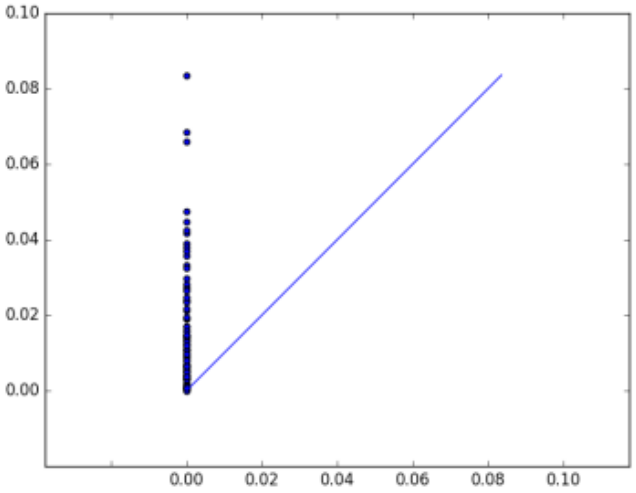
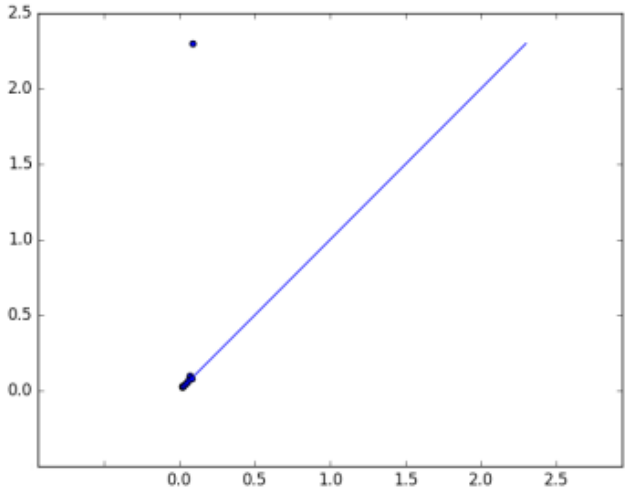
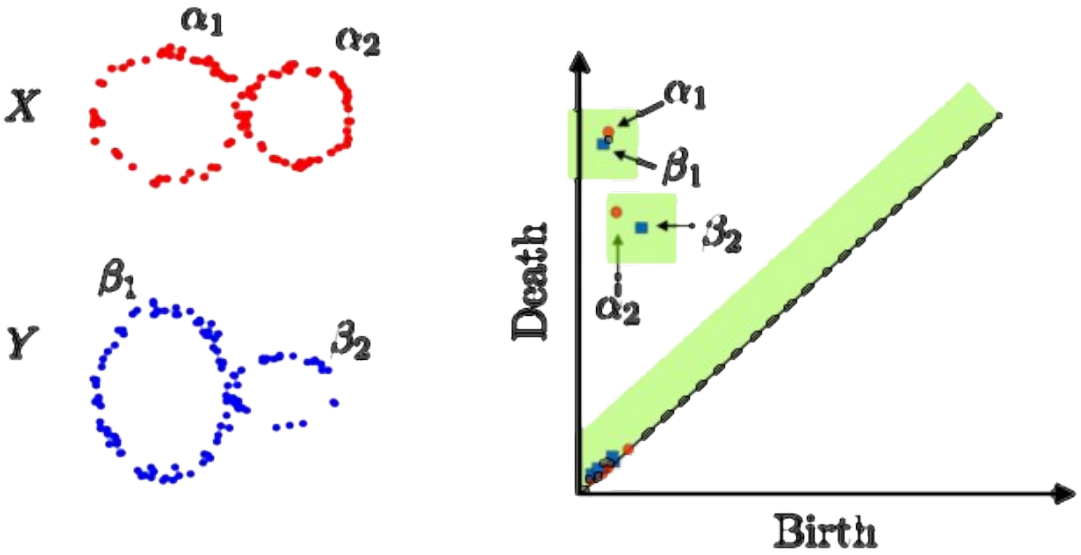
We track the appearing and disappearing holes as we go through each complex of the filtration. The time each cycle persists is an indicative of the presence of said cycle in the figure.

Representing Results: Persistence Diagrams

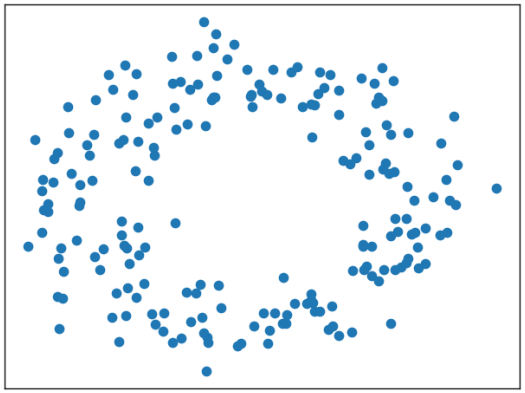


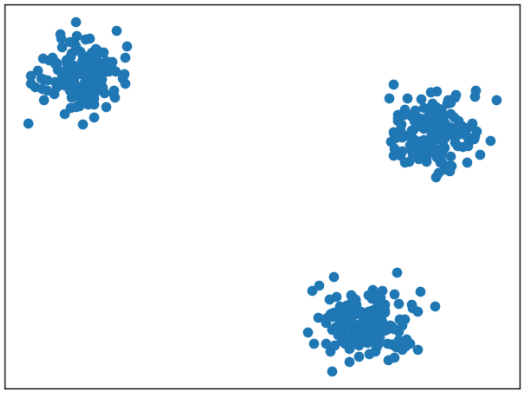


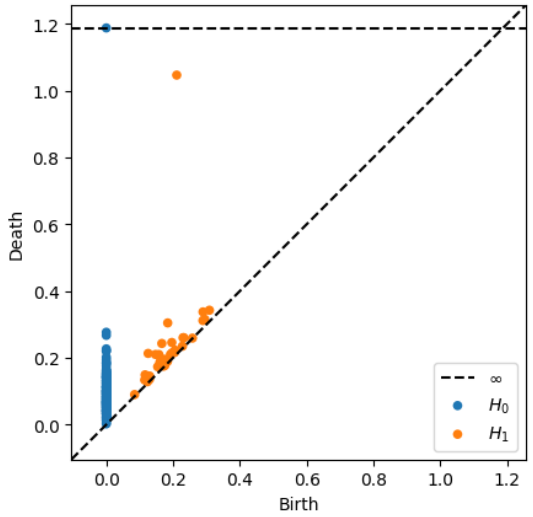
Visualizing Results: Persistence Diagrams

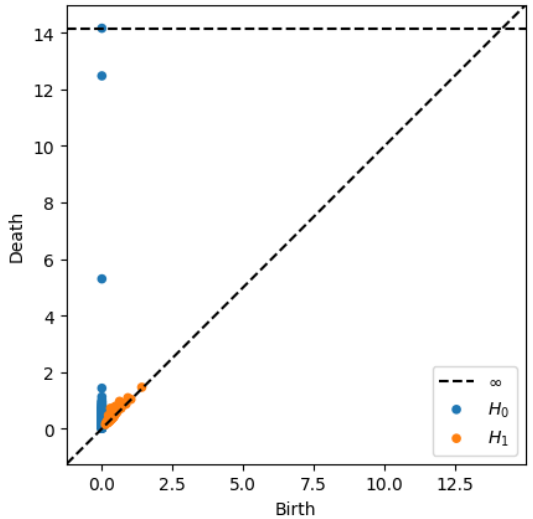


Examples






1.2
1.0
0.8
0.6
0.4
0.2
0.0
Death
0.0
0.2
0.4
0.6
0.8
1.0
1.2
Birth
∞
H_0
H_1


14
12
10
8
6
4
2
0
Death
0.0
2.5
5.0
7.5
10.0
12.5
Birth
∞
H_0
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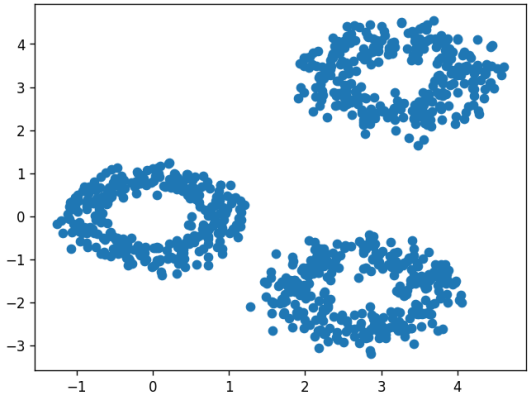

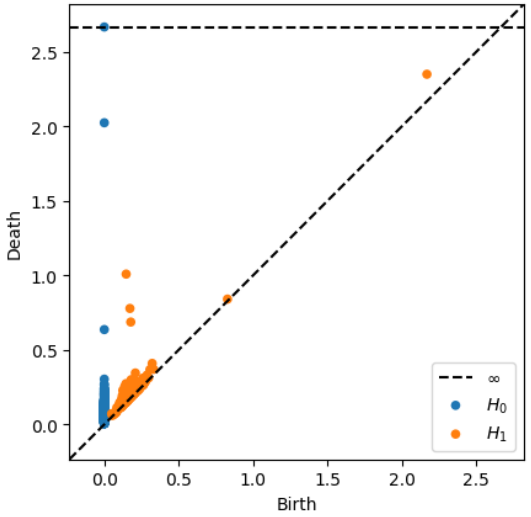

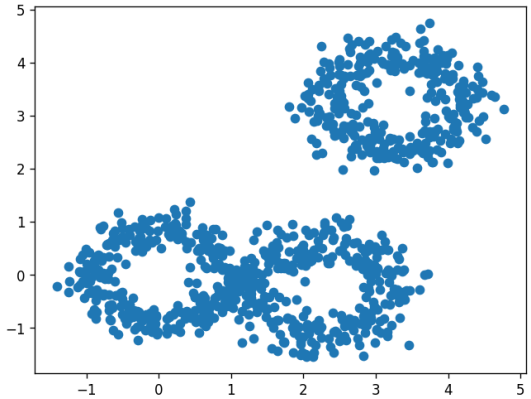

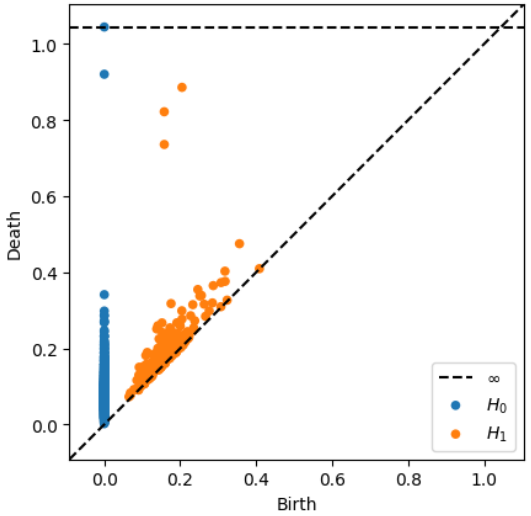

Local Geometry

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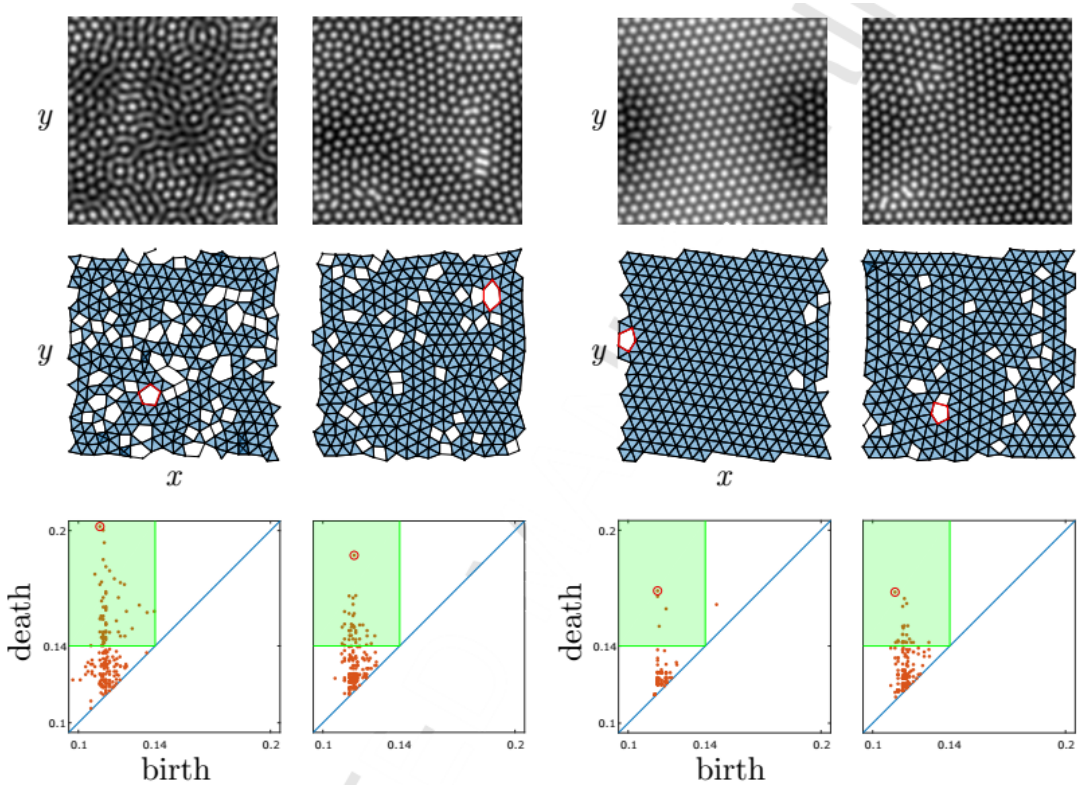
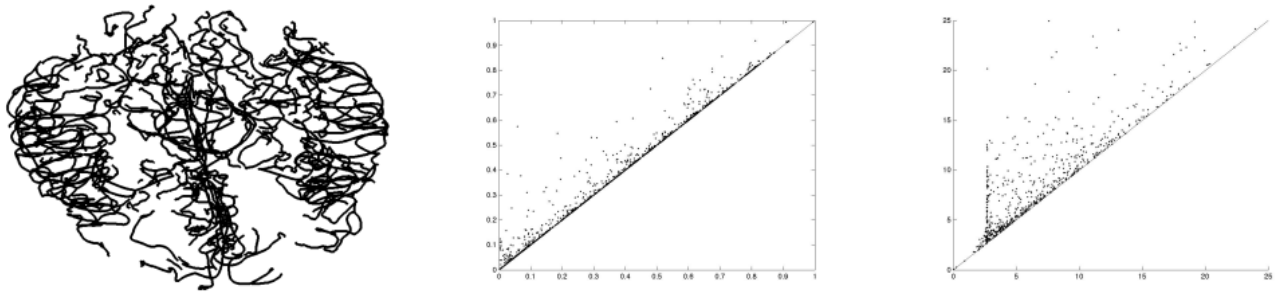


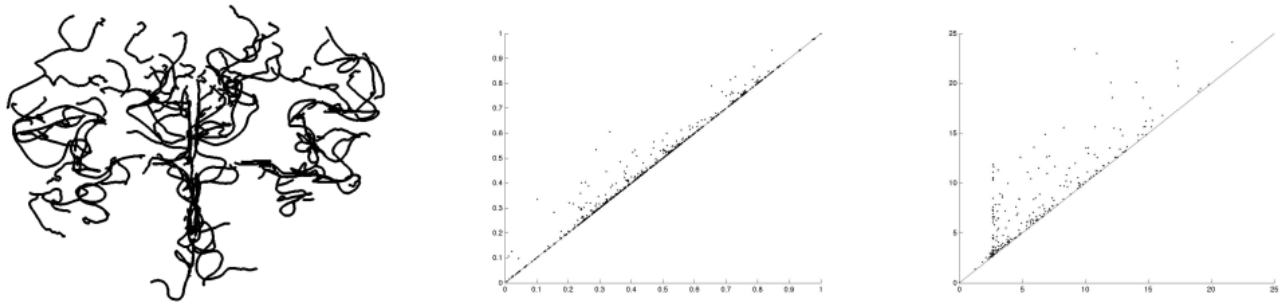
Motta, F. C., Neville, R., Shipman, P. D., Pearson, D. A., Bradley, R. M. (2018). **Measures of order for nearly hexagonal lattices**. Physica D: Nonlinear Phenomena, 380, 17-30.

When *noise* is the important feature



(a) Brain tree (b) Dgm_0 (c) Dgm_1

Figure 13: Persistent homology data objects from a 24-year old. Left: brain tree. Middle: zero-dimensional diagram. Right: one-dimensional diagram.



(a) Brain tree (b) Dgm_0 (c) Dgm_1

Figure 14: Persistent homology data objects from a 68-year old. Left: brain tree. Middle: zero-dimensional diagram. Right: one-dimensional diagram.

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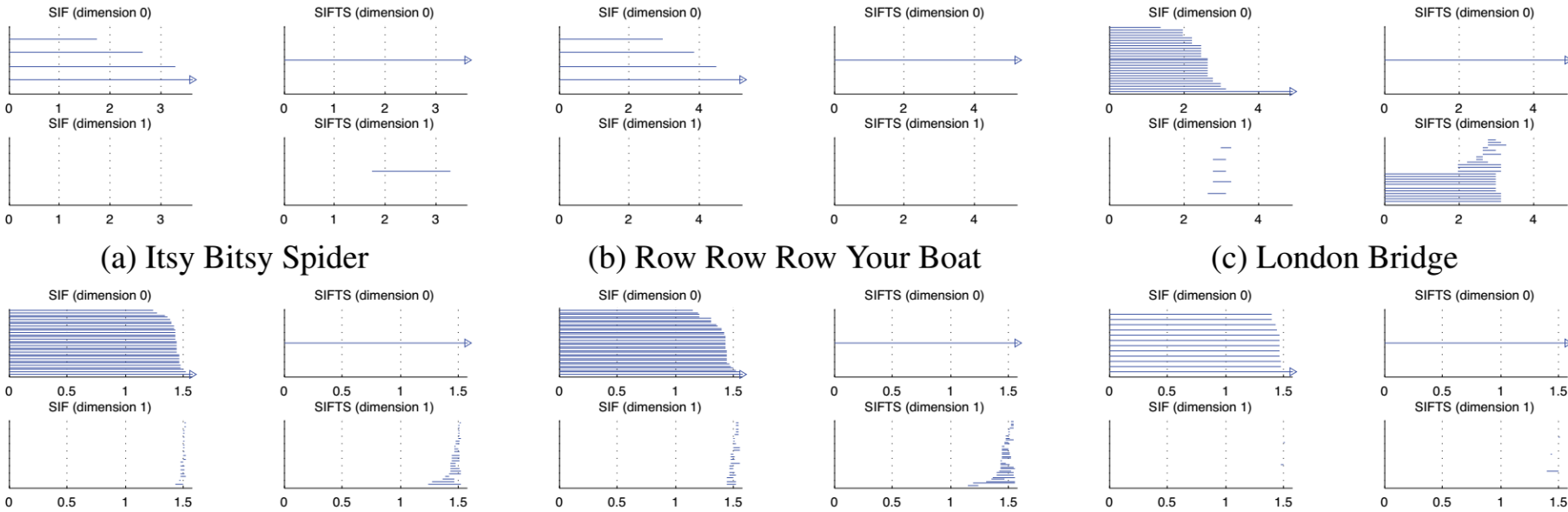

Figure 1: Persistent homology on nursery rhymes and other stories

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More examples

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