Written Assignment 5

The following questions are related to sections 2.5, 2.7 and parts of 4.6 of *Vector Calculus* by Michael Corral.

Questions

0.0.1. Use the second derivative test to classify the local extrema of the following functions:

(a) $f(x, y) = x^2 + 2x - xy + y^2$

(b) $h(x, y) = \sin(xy)$

0.0.2. Find the extreme values of

(a) the function $f(x, y) = e^{-(x^2+y^2)}$ along the curve $x = y^2$

(b) the function $g(x, y) = e^{x-y^2}$ along the boundary of the ellipse $\frac{x^2}{4} + y + y^2 = 1$

0.0.3. Consider the function $f(x, y, z) = \begin{bmatrix} x^2 + y^2 \\ y^2 + z^2 \\ z^2 + x^2 \end{bmatrix}$. Find its divergence $\nabla \cdot f$ and curl $\nabla \times f$.

0.0.4. Find the point on the sphere $x^2 + y^2 + z^2 = 1$ that is closest to the plane $x + 2y + 2z = 5$.

0.0.5. **Application: Cost Optimization**

Suppose you are given a budget of \$500 to build a large glass triangular prism. Each rectangular side must have equal dimensions and the two triangular sides must also have equal dimensions. You can purchase glass for the rectangular sides at a cost of $\$8/ft^2$ and for the triangular sides at a cost of $\$10/ft^2$.

What is the volume of the prism of maximum surface area that you can build under these constraints?

0.0.6. **Application: Labor and Capital**

Solutions

0.0.1. Local extreme values of a function occur at points where the gradient of the function is the zero vector.

(a) First we find the critical points:

$$
\begin{aligned}
\nabla f(x,y) &= \begin{bmatrix}0\\0\end{bmatrix}\\
\begin{bmatrix}\dfrac{\partial}{\partial x}(x^2+2x-xy+y^2)\\ \dfrac{\partial}{\partial y}(x^2+2x-xy+y^2)\end{bmatrix} &= \begin{bmatrix}0\\0\end{bmatrix}\\
\begin{bmatrix}2x+2\\ -x+2y\end{bmatrix} &= \begin{bmatrix}0\\0\end{bmatrix}\\
x &= -1\\
y &= -\frac{1}{2}
\end{aligned}
$$

To categorize this point $(-1,-\frac{1}{2})$ as a local minimum or maximum we apply the second partial derivative test:

$$
\begin{aligned}
D(x,y) &= \begin{vmatrix}\dfrac{\partial^2 f(x,y)}{\partial x^2} & \dfrac{\partial^2 f(x,y)}{\partial y\partial x}\\ \dfrac{\partial^2 f(x,y)}{\partial x\partial y} & \dfrac{\partial^2 f(x,y)}{\partial y^2}\end{vmatrix}\\
&= \begin{vmatrix}\dfrac{\partial^2}{\partial x^2}(x^2+2x-xy+y^2) & \dfrac{\partial^2}{\partial y\partial x}(x^2+2x-xy+y^2)\\ \dfrac{\partial^2}{\partial x\partial y}(x^2+2x-xy+y^2) & \dfrac{\partial^2}{\partial y^2}(x^2+2x-xy+y^2)\end{vmatrix}\\
&= \begin{vmatrix}2 & 0\\ -1 & 2\end{vmatrix}\\
&= 4\\
&> 0
\end{aligned}
$$

for all values of x, y. Additionally, $\dfrac{\partial^2 f(x,y)}{\partial x^2} = 2 > 0$, so that the point $(-1,-\frac{1}{2})$ is a local minimum.

(b) Again, we look for the critical points by setting the gradient to zero.

$$
\begin{aligned}
\nabla h(x,y) &= \begin{bmatrix}0\\0\end{bmatrix}\\
\begin{bmatrix}y\cos(xy)\\ x\cos(xy)\end{bmatrix} &= \begin{bmatrix}0\\0\end{bmatrix}
\end{aligned}
$$

The solutions (and therefore the critical points) are the point $(0,0)$ and the family of curves $xy = \pi/2 + k\pi$, where $k \in \mathbb{Z}$.

To categorize the points, again compute the matrix

$$\begin{aligned}
D(x,y) &= \begin{vmatrix} \frac{\partial^2 h(x,y)}{\partial x^2} & \frac{\partial^2 h(x,y)}{\partial y \partial x} \\ \frac{\partial^2 h(x,y)}{\partial x \partial y} & \frac{\partial^2 h(x,y)}{\partial y^2} \end{vmatrix} \\
&= \begin{vmatrix} \frac{\partial^2}{\partial x^2}(\sin(xy)) & \frac{\partial^2}{\partial y \partial x}(\sin(xy)) \\ \frac{\partial^2}{\partial x \partial y}(\sin(xy)) & \frac{\partial^2}{\partial y^2}(\sin(xy)) \end{vmatrix} \\
&= \begin{vmatrix} -y^2 \sin(xy) & \cos(xy) - xy\sin(xy) \\ \cos(xy) - xy\sin(xy) & -x^2\sin(xy) \end{vmatrix} \\
D(0,0) &= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\
&= -1
\end{aligned}$$

So $(0,0)$ is a saddle point.

$$\begin{aligned}
D(x, \frac{\pi/2 + k\pi}{x}) &= \begin{vmatrix} -\frac{\pi/2 + k\pi}{x}^2 \sin(\pi/2 + k\pi) & \cos(\pi/2 + k\pi) - (\pi/2 + k\pi)\sin(\pi/2 + k\pi) \\ \cos(\pi/2 + k\pi) - (\pi/2 + k\pi)\sin(\pi/2 + k\pi) & -x^2 \sin(\pi/2 + k\pi) \end{vmatrix} \\
&= \begin{vmatrix} -(\frac{\pi/2 + k\pi}{x})^2(-1)^k & -(\pi/2 + k\pi)(-1)^k \\ -(\pi/2 + k\pi)(-1)^k & -x^2(-1)^k \end{vmatrix} \\
&= (\pi/2 + k\pi)^2 - (\pi/2 + k\pi)^2 \\
&= 0
\end{aligned}$$

So the test is inconclusive.

0.0.2. We use the method of Lagrange Multipliers. To find the extreme values of the function $f(x,y)$ subject to the constraint $g(x,y) = c$, we define a new function $F(x,y,t) = f(x,y) + t(g(x,y) - c)$ and solve the equation $\nabla F(x,y,t) = 0$. The solutions (x,y) are the candidate solutions, which we test by plugging into the original equation f.

(a) Define $F(x,y,t) = e^{-(x^2+y^2)} - t(x - y^2)$. Then

$$\begin{aligned}
\nabla F(x,y,t) &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -2xe^{-(x^2+y^2)} - t \\ -2ye^{-(x^2+y^2)} + 2yt \\ -(x - y^2) \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (1)
\end{aligned}$$

$(0,0)$ is one solution. When $x \neq 0$, by the third component of (1) we see that $y \neq 0$, so that we get $t = e^{-(x^2+y^2)}$ from the second component, which in turn by the first component shows that $x = -1/2$. However, there is no y that satisfies the third component of (1) when $x = -1/2$. Thus $(0,0)$ is the only extreme value.

(b) Define $F(x,y,t) = e^{x-y^2} - t(\frac{x^2}{4} + y + y^2 - 1)$. Then

$$\begin{aligned}
\nabla F(x,y,t) &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} e^{x-y^2} - tx/2 \\ -2ye^{x-y^2} - t + 2yt \\ -(\frac{x^2}{4} + y + y^2 - 1) \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (2)
\end{aligned}$$

From the first two components we see that

$$\begin{aligned} e^{x-y^2} &= tx/2 \\ e^{x-y^2} &= \frac{t-2yt}{-2y}. \end{aligned}$$

This yields the equation $y = \dfrac{1}{2-x}$, so that the extreme values of f fall on the intersection of the ellipse $\frac{x^2}{4} + y + y^2 = 1$ and the curve $y = \dfrac{1}{2-x}$. Plugging y into the third component of (2) gives the equation $\frac{x^2}{4} + \frac{1}{2-x} + (\frac{1}{2-x})^2 = 1$.

0.0.3. The divergence is given by the equation

$$\nabla \cdot f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

and the curl by the equation

$$\nabla \times f = \begin{bmatrix} \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{bmatrix}.$$

Applying these equations yields the solutions:

$$\begin{aligned} \nabla \cdot f &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \\ &= \frac{\partial}{\partial x}(x^2+y^2) + \frac{\partial}{\partial y}(y^2+z^2) + \frac{\partial}{\partial z}(z^2+x^2) \\ &= 2x + 2y + 2z \\ \nabla \times f &= \begin{bmatrix} \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} -2z \\ -2x \\ -2y \end{bmatrix}. \end{aligned}$$

0.0.4. It is possible to solve this problem using Lagrangian optimization; however, the closest point on the sphere to the plane will occur at the point on the sphere where its normal vector is parallel to the plane's normal vector.

Knowing this, we first find the sphere's normal vector by computing the gradient of the function $F(x,y,z) = x^2+y^2+z^2-1$, which by a straightforward calculation is the vector $\nabla F = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$.

Now, by rearranging the equation $x + 2y + 2z = 5$ into the form $\begin{bmatrix} x-3 \\ y-2 \\ z+1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 0$, we see that $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is a vector normal to the plane.

We want to find a point on the sphere where these two normal vectors are parallel; equivalently, we want to find a point on the sphere such that $\begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ for some k. We can plug $x = k/2$, $y = k$, $z = k$ into the equation $x^2+y^2+z^2 = 1$, which yields the solution $k = 2/3$. Thus $(1/3, 2/3, 2/3)$ is a point on the sphere where the sphere's normal vector is parallel to the plane's, and therefore is the point closest to the plane.