Midterm 2

Questions

0.0.1. Determine the value of

$$\int_0^{\pi}\int_{-1}^{1} x^4 e^{x^2+y^2}\sin(y)dydx.$$

Do not use integration by parts.

0.0.2. **Changing the Order of Integration**
The volume, V, of the solid bounded by

$$z=0, \quad x=y^2, \quad x+z=1$$

can be found using the triple integral

$$V=\int_0^1\int_{-\sqrt{x}}^{\sqrt{x}}\int_0^{1-x} dz\ dy\ dx=\frac{8}{15}.$$

By simply changing the order of integration, we can find five other ways of expressing the volume of the solid. Find three of them.

Note that for this question, you do not need to perform any integration. Simply set up three other triple integrals that represent the volume of the solid. *Hint: you may however want to check that your integrals represent the same volume with WolframAlpha.*

Solutions

0.0.1. The integral can be written as

$$\int_0^\pi \int_{-1}^1 e^{x^2+y^2}\sin(y)dydx = \int_0^\pi \int_{-1}^1 e^{x^2}e^{y^2}\sin(y)dydx = \left(\int_0^\pi e^{x^2}dx\right)\left(\int_{-1}^1 e^{y^2}\sin(y)dy\right)$$

The second term is the integral of an odd function over a symmetrical interval, and so is equal to zero. Therefore,

$$\int_0^\pi \int_{-1}^1 e^{x^2+y^2}\sin(y)dydx = 0.$$

0.0.2. **Changing the Order of Integration**
Noting that the limits of integration are not dependent on y, the z and the y integrals can be interchanged:

$$\begin{aligned} V &= \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \int_0^{1-x} dz\ dy\ dx \\ &= \int_0^1 \left(\int_{-\sqrt{x}}^{\sqrt{x}} dy\right)\left(\int_0^{1-x} dz\right)dx \\ &= \int_0^1 \left(\int_0^{1-x} dz\right)\left(\int_{-\sqrt{x}}^{\sqrt{x}} dy\right)dx \\ &= \int_0^1 \int_0^{1-x} \int_{-\sqrt{x}}^{\sqrt{x}} dy\ dz\ dx \end{aligned}$$

We can also integrate y last. We are given that the solid can be expressed as the region

$$S = \{(x,y,z) \mid 0 \le x \le 1, -\sqrt{x} \le y \le \sqrt{x}, 0 \le z \le 1-x\}.$$

We can re-write this as

$$S = \{(x,y,z) \mid y^2 \le x \le 1, -1 \le y \le 1, 0 \le z \le 1-x\}.$$

This gives us the integral

$$V = \int_{-1}^1 \int_{y^2}^1 \int_0^{1-x} dz\ dx\ dy$$

We can also define the region in this way ***wrong***

$$S = \{(x,y,z) \mid 0 \le x \le 1-z, -1 \le y \le 1, 0 \le z \le 1\}.$$

This gives us the integral

$$V = \int_{-1}^1 \int_0^1 \int_0^{1-z} dx\ dz\ dy$$