Written Assignment 3

Questions in this assignment are based on

- Section 1.8 of *Vector Calculus* by Michael Corral, and
- Section 8.1 of *College Algebra* by Carl Stitz and Jeff Zeager.

Determinants are covered in Section 1.4 of *Vector Calculus* by Michael Corral.

Questions

1. Find all values of x that satisfy the following equations.

   (a) $\begin{vmatrix} 3 & 2 & 0 \\ 1 & x & 0 \\ 7 & -3 & 4 \end{vmatrix} = 4.$

   (b) $\begin{vmatrix} x & 1 \\ 4 & 4x \end{vmatrix} = \begin{vmatrix} -x & -2 \\ 2 & 2x+8 \end{vmatrix}.$

2. For A and $\mathbf{b}$ below, solve the linear system $A\mathbf{x} = \mathbf{b}$, if possible:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & -1 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 4 & 0 & 3 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 10 \end{bmatrix}.$$

   If it isn't possible to solve this system, explain why.

3. For the systems below,

   - Compute the determinant of matrix A, if possible. If it is not possible to do so, explain why.
   - Solve the linear system $A\mathbf{x} = \mathbf{b}$, if possible.
   - State whether the system has no solution, infinitely many solutions, or a unique solution.

   (a) $A = \begin{bmatrix} -3 & 1 & 2 & 4 \\ 2 & -1 & 2 & 3 \\ 1 & 0 & 4 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 0 \\ 4 & 9 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

4. Find the values of t and the points on the curve

$$\mathbf{r}(t) = (1 + t^2)\mathbf{i} + t\mathbf{j}, \quad t \in \mathbb{R}$$

   where

   (a) $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are perpendicular,

   (b) $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ have the same direction, and

   (c) $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ have opposite directions.

5. Consider the system of simultaneous linear equations

$$\begin{aligned} x + 2y - z &= 2 \\ 2x + ay - 2z &= b \\ 3x + 2y &= 1 \end{aligned}$$

   where x, y, z are unknown.

   (a) Find all values of a and b such that the above system has

i. exactly one solution;

ii. no solutions;

iii. infinitely many solutions.

(b) For those values of a and b from 5(a)i, what is the unique solution?

(c) For those values of a and b from 5(a)iii, parameterize the set of all solutions.

6. **Application to Mechanics, Part II**

Recall from a previous assignment that we defined torque as $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. Now that we have introduced the concepts of vector-valued functions and their derivatives, let's consider the more general case when $\boldsymbol{\tau}$, $\mathbf{r}$, and $\mathbf{F}$ are all functions of time:

$$\boldsymbol{\tau}(t) = \mathbf{r}(t) \times \mathbf{F}(t).$$

Moreover, using the relation $\mathbf{F}(t) = m\mathbf{a}(t)$ (Newton's second law), we can write our definition of torque as

$$\begin{aligned}\boldsymbol{\tau}(t) &= \mathbf{r}(t) \times \mathbf{F}(t)\\ &= \mathbf{r}(t) \times \big(m\mathbf{a}(t)\big)\\ &= m\big(\mathbf{r}(t) \times \mathbf{r}''(t)\big).\end{aligned}$$

These alternate forms for the torque vector may be helpful in solving the following problems.

(a) If the position of a particle with mass m is given by the position vector $\mathbf{r}(t)$, then its angular momentum is a vector defined as $\mathbf{L}(t) = m\mathbf{r}(t) \times \mathbf{r}'(t)$. Show that $\mathbf{L}'(t) = \boldsymbol{\tau}(t)$.

(b) Show that if the torque is a zero vector for all t, then the angular momentum of the particle is constant for all t. This is what is known as the **law of conservation of angular momentum**.

7. **Application to Polynomial Interpolation**

In many areas of engineering, experimental data is collected that must be analyzed to extract parameters that tell us something about a physical process. Suppose we have measured a set of experimental data that are represented in the xy-plane. An **interpreting polynomial** for the measured data is a polynomial that passes through every measured point. We can use this polynomial, for example, to estimate values between the measured data points.

Suppose for example that we have measured the data points (0,-6), (1,-2), (2,4), (3,10). To find an interpreting polynomial of order 2 for these data, we would try to find a polynomial of the form $p(x) = a_0 + a_1 x + a_2 x^2$ that passes through all four measured points. In other words, we need to find the unknown constants a_0, a_1, a_2 that satisfy the equations

$$\begin{aligned}p(0) &= a_0 + a_1(0) + a_2(0)^2 = -6\\ p(1) &= a_0 + a_1(1) + a_2(1)^2 = -2\\ p(2) &= a_0 + a_1(2) + a_2(2)^2 = 4\\ p(3) &= a_0 + a_1(3) + a_2(3)^2 = 10\end{aligned}$$

The above system has four equations and four unknowns. Upon solving this system, you should be able to determine that $a_0 = -6, a_1 = 4, a_2 = 0$.

Wind Tunnel Experiment

In a fictitious wind tunnel experiment, the following measurements were made.

Velocity (m/s)	Force (N)
0	0
1	5.5
2	20
3	46.5
4	88

The data represent the measured force due to air resistance, on an object suspended in the tunnel, measured at different air speed velocities.

(a) Using the data above, derive a 5×4 system of equations, that when solved, find an interpreting polynomial of the form $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. Write the system in the form $A\mathbf{x} = \mathbf{b}$.

(b) Solve your system to obtain $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$.

(c) As mentioned above, engineers sometimes use interpreting polynomials to estimate values in between measured data points. Using your polynomial, estimate the value of the force when the velocity is 1.5 m/s.

(d) In practice, it can be difficult to determine what order of polynomial to use. Sometimes, polynomials of different orders must be used to decide which polynomial yields the most useful results. For the above data, explain what what would happen if we used a polynomial less than 3. It may help to see what happens if we use a 1^{st} order polynomial.

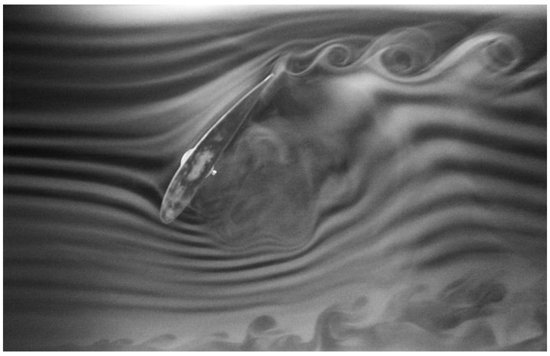

Figure 1: An airfoil in a fog wind tunnel (image from Wikimedia Commons, Smart Blade GmbH).

8. **Integration with Vector-Valued Functions**

If the position of a particle is given by the vector function $\mathbf{r}(t) \in \mathbb{R}^3$, then we know that we can determine its velocity, $\mathbf{v}(t) = \mathbf{r}'(t)$, by differentiating each of its components. That is, if

$$\mathbf{r}(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{bmatrix},$$

then

$$\mathbf{v}(t) = \mathbf{r}'(t) = \begin{bmatrix} \frac{d}{dt} r_1(t) \\ \frac{d}{dt} r_2(t) \\ \frac{d}{dt} r_3(t) \end{bmatrix} = \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix},$$

provided that the derivatives of the components of $\mathbf{r}$ exist at t. It follows from the Fundamental Theorem of Calculus that if we were instead given the velocity of the particle, we could compute its position by integrating each of the components with respect to t. We would of course introduce constants of integration. That is, given

$$\mathbf{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix},$$

we could obtain $\mathbf{r}$ by integrating each of the components of $\mathbf{v}(t)$

$$\mathbf{r}(t) = \begin{bmatrix} \int v_1(t)dt \\ \int v_2(t)dt \\ \int v_3(t)dt \end{bmatrix} = \begin{bmatrix} r_1(t) + c_1 \\ r_2(t) + c_2 \\ r_3(t) + c_3 \end{bmatrix},$$

where c_1, c_2, c_3 are constants.

Suppose that a particle with mass m is subjected to a force, $\mathbf{F}(t) = m\pi^2\big(\cos(\pi t)\mathbf{j} + \sin(\pi t)\mathbf{k}\big)$, where $t \geq 0$. Suppose also that when $t = 0$,

$$\mathbf{r}(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{r}'(0) = \begin{bmatrix} +1 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Using the relation $\mathbf{F} = m\mathbf{a}$, find the velocity of the particle at time t. *Hint: you will need to apply the velocity at time t=0.*

(b) Find the position of the particle at time $t = 1$.

(c) Plot the position of the particle at times $t = 0$ and $t = 1$ on the same graph in $\mathbb{R}^3$.

References

The wind tunnel problem was based on a similar exercise in Linear Algebra and Its Applications, 4th Edition, by David C. Lay, Addison-Wesley, 2012.

Solutions

1. (a) We can simplify the left-hand side of the equation by computing the determinant.

$$\begin{aligned}\begin{vmatrix}3 & 2 & 0\\ 1 & x & 0\\ 7 & -3 & 4\end{vmatrix} &= 3\begin{vmatrix}x & 0\\ -3 & 4\end{vmatrix} - 2\begin{vmatrix}1 & 0\\ 7 & 4\end{vmatrix} + 0\begin{vmatrix}1 & x\\ 7 & -3\end{vmatrix}\\ &= 3(4x-0) - 2(4-0) + 0(-3-7x)\\ &= 12x - 8\end{aligned}$$

Substituting this result yields the equation $12x-8=4$, from which the solution $x=1$ is easily recovered.

(b) Expand both determinants and rearrange the equation:

$$\begin{aligned}(x)(4x)-(1)(4) &= (-x)(2x+8)-(-2)(2)\\ 4x^2-4 &= -2x^2-8x+4\\ 6x^2+8x-8 &= 0\\ (6x-4)(x+2) &= 0\end{aligned}$$

Therefore the solutions are $x = 2/3$ and $x = -2$.

2. Use row-reduction operations on the augmented matrices.

$$\left[\begin{array}{cccc|c}1 & 2 & 0 & 1 & 1\\ 3 & -1 & 1 & 2 & 5\\ 0 & 1 & 0 & 2 & 0\\ 4 & 0 & 3 & 1 & 10\end{array}\right]$$

$$\left[\begin{array}{cccc|c}1 & 2 & 0 & 1 & 1\\ 0 & -7 & 1 & -1 & 2\\ 0 & 1 & 0 & 2 & 0\\ 0 & -8 & 3 & -3 & 6\end{array}\right] \quad \begin{array}{l}R_2 \leftarrow R_2 - 3R_1\\ R_4 \leftarrow R_4 - 4R_1\end{array}$$

$$\left[\begin{array}{cccc|c}1 & 2 & 0 & 1 & 1\\ 0 & 0 & 1 & 13 & 2\\ 0 & 1 & 0 & 2 & 0\\ 0 & 0 & 3 & 13 & 6\end{array}\right] \quad \begin{array}{l}R_2 \leftarrow R_2 + 7R_3\\ R_4 \leftarrow R_4 + 8R_3\end{array}$$

$$\left[\begin{array}{cccc|c}1 & 2 & 0 & 1 & 1\\ 0 & 0 & 1 & 13 & 2\\ 0 & 1 & 0 & 2 & 0\\ 0 & 0 & 0 & -26 & 0\end{array}\right] \quad R_4 \leftarrow R_4 - 3R_2$$

This is equivalent to the system of equations

$$\begin{aligned}x + 2y + w &= 1\\ z + 13w &= 2\\ y + 2w &= 0 \qquad (1)\\ -26w &= 0 \qquad (2)\end{aligned}$$

Equations (1) and (2) show that $w = 0$ and $y = 0$, and substitution yields $z = 2$ and $x = 1$.

3. (a) It's not possible to calculate the determinant of A, because A is not square. We can however

solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$ using row operations:

$$\left[\begin{array}{cccc|c} -3 & 1 & 2 & 4 & 1 \\ 2 & -1 & 2 & 3 & 1 \\ 1 & 0 & 4 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 14 & -2 & 4 \\ 0 & -1 & -6 & 7 & -1 \\ 1 & 0 & 4 & -2 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \leftarrow R_1 + 3R_3 \\ R_2 \leftarrow R_2 - 2R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 14 & -2 & 4 \\ 0 & 0 & 8 & 5 & 3 \\ 1 & 0 & 4 & -2 & 1 \end{array}\right] \quad R_2 \leftarrow R_2 + R_1$$

This is equivalent to the system of equations

$$y + 14z - 2w = 4 \tag{3}$$

$$8z + 5w = 3 \tag{4}$$

$$x + 4z - 2w = 1 \tag{5}$$

Since there are fewer equations than unknowns we will parameterize the solution set by setting $w = t$, where t is any real number. From equation (4) we get $z = \frac{3}{8} - \frac{5}{8}t$. Substituting z into equations (5) and (3) yields $x = -\frac{1}{2} + \frac{9}{2}t$ and $y = -\frac{5}{4} + \frac{43}{4}t$.

Because of the free parameter t, there are infinitely many solutions to the given system.

(b) We can calculate the 3×3 determinant by expanding into 2×2 determinants:

$$\begin{aligned} \begin{vmatrix} 1 & 2 & -1 \\ 2 & 5 & 0 \\ 4 & 9 & -2 \end{vmatrix} &= 1\begin{vmatrix} 5 & 0 \\ 9 & -2 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} 2 & 5 \\ 4 & 9 \end{vmatrix} \\ &= (-10) - 2(-4) - (18 - 20) \\ &= -10 + 8 + 2 \\ &= 0 \end{aligned}$$

Applying one row operation yields

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 5 & 0 & 5 \\ 4 & 9 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 5 & 0 & 5 \\ 0 & 0 & 0 & -7 \end{array}\right] \quad R_3 \leftarrow R_3 - 2R_1 - R_2$$

The last row is equivalent to

$$0x + 0y + 0z = -7,$$

which implies that this particular system has no solution.

4. The derivative of $\mathbf{r}$ is simply $\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$.

(a) $\mathbf{r}(t) \perp \mathbf{r}'(t)$ implies

$$\begin{aligned} 0 &= \mathbf{r}(t) \cdot \mathbf{r}'(t) \\ &= \big((1 + t^2)\mathbf{i} + t\mathbf{j}\big) \cdot \big(2t\mathbf{i} + \mathbf{j}\big) \\ &= (2t + 2t^3) + (t) \\ &= t(3 + 2t^2) \end{aligned}$$

This equation is satisfied only when $t = 0$ or $t^2 = -3/2$. The latter equation cannot be satisfied if $t \in \mathbb{R}$. Thus, the two vectors are only perpendicular when $t = 0$. Thus, $\mathbf{r}(t)$ is perpendicular to $\mathbf{r}'(t)$ at (0,1).

(b) If $\mathbf{r}(t)$ is pointing in the same direction as $\mathbf{r}'(t)$, then $\mathbf{r}(t) = C\mathbf{r}'(t)$, where C is a positive constant. Equating the components of the vectors yields the two equations

$$(1+t^2) = C2t, \text{ and } t = C.$$

If we substitute $t = C$ into the first equation, we find

$$\begin{aligned}(1+C^2) &= 2C^2\\ C &= +1,\end{aligned}$$

because C must be positive. Thus, the two vectors are pointing in the same direction when $t = 1$, which is at the point (2,1).

(c) The solution is the similar to part (b), except that C must be a negative constant. We find that the vectors point in opposite directions when $C = t = -1$, and at the point (2,-1).

5. We first convert the system of equations into an equivalent augmented-matrix form:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2\\ 2 & a & -2 & b\\ 3 & 2 & 0 & 1\end{array}\right].$$

Then apply a sequence of row-reduction operations:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2\\ 0 & a-4 & 0 & b-4\\ 0 & -4 & 3 & -5\end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2\\ 0 & a & -3 & b+1\\ 0 & -4 & 3 & -5\end{array}\right] \begin{array}{l} R_2 - 2R_1\\ R_3 - 3R_1\end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2\\ 0 & a & -3 & b+1\\ 0 & 4 & -3 & 5\end{array}\right] \quad R_2 - R_3$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2\\ 0 & 4 & -3 & 5\\ 0 & a & -3 & b+1\end{array}\right] \begin{array}{l} -1\cdot R_3\\ R_2 \leftrightarrow R_3\end{array} \tag{6}$$

Suppose that $a = 4$. Then (6) becomes

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2\\ 0 & 4 & -3 & 5\\ 0 & 4 & -3 & b+1\end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2\\ 0 & 4 & -3 & 5\\ 0 & 0 & 0 & b-4\end{array}\right] \quad R_3 - R_2 \tag{7}$$

If $b \neq 4$, then the system has no solutions, since the last row is equivalent to the equation $0 = b - 4$ where $b - 4 \neq 0$. Conversely, if $b = 4$, then (7) is equivalent to the system of equations

$$\begin{aligned} x + 2y - z &= 2\\ 4y - 3z &= 5\end{aligned}$$

There are fewer equations than unknowns, so we have to parameterize the solution set; let $z = t$. Solving these two equations yields the solution

$$\begin{aligned} x &= -\frac{1}{2} - \frac{1}{2}t\\ y &= \frac{5}{4} + \frac{3}{4}t\\ z &= t\end{aligned} \tag{8}$$

Now suppose that $a = 0$. Then the matrix (6) is equivalent to the system

$$
\begin{aligned}
x + 2y - z &= 2 \\
4y - 3z &= 5 \\
-3z &= b + 1
\end{aligned}
$$

Using back-substitution we find that

$$
\begin{aligned}
x &= -\frac{1}{3} - \frac{1}{6}b \\
y &= 1 - \frac{1}{4}b \\
z &= \frac{-1}{3} - \frac{1}{3}b
\end{aligned}
\tag{9}
$$

Now suppose that $a \neq 4$ and $a \neq 0$. Then we can continue row- reducing from (6):

$$
\left[\begin{array}{ccc|c}
1 & 2 & -1 & 2 \\
0 & 4 & -3 & 5 \\
0 & a & -3 & b+1
\end{array}\right]
$$

$$
\left[\begin{array}{ccc|c}
1 & 2 & -1 & 2 \\
0 & 4 & -3 & 5 \\
0 & 0 & -3 + \frac{3}{4}a & 1 + b - \frac{5}{4}a
\end{array}\right] \quad R_3 \leftarrow R_3 - \frac{a}{4}R_2
$$

This is equivalent to the system

$$
\begin{aligned}
x + 2y - z &= 2 \\
4y - 3z &= 5 \\
(-3 + \frac{3}{4}a)z &= 1 + b - \frac{5}{4}a
\end{aligned}
$$

Let $c = \dfrac{1 + b - \frac{5}{4}a}{-3 + \frac{3}{4}a}$. Using back-substitution we find the solution

$$
\begin{aligned}
x &= -\frac{1}{2} - \frac{1}{2}c \\
y &= \frac{5}{4} + \frac{3}{4}c \\
z &= c
\end{aligned}
\tag{10}
$$

Thus we have shown that:

(a) When $a = 4$ and $b = 4$ there are infinitely many solutions, given by the parameterization (8).

(b) When $a = 4$ and $b \neq 4$ there are no solutions.

(c) When $a \neq 4$ there is one solution. When $a = 0$, the solution (9) is equivalent to that of (10). So the solution when $a \neq 4$ is given by (10).

6. (a) Differentiating the equation for angular momentum yields the torque:

$$
\begin{aligned}
\mathbf{L}(t) &= m\mathbf{r}(t) \times \mathbf{r}'(t) \\
\frac{d}{dt}\mathbf{L}(t) &= \frac{d}{dt}\Big(m\mathbf{r}(t) \times \mathbf{r}'(t)\Big) \\
&= m\mathbf{r}'(t) \times \mathbf{r}'(t) + m\mathbf{r}(t) \times \mathbf{r}''(t) \\
&= m\mathbf{0} + m\mathbf{r}(t) \times \mathbf{r}''(t) \\
&= m\mathbf{r}(t) \times \mathbf{r}''(t) \\
&= \boldsymbol{\tau}(t)
\end{aligned}
$$

(b) If each component of $\tau(t)$ is zero for all values of t, then each component of $\mathbf{L}'(t)$ is zero for all values of t. Each component of $\mathbf{L}(t)$ must therefore be constant.

7. (a) We can write the system as

$$\begin{aligned}
a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 &= 0 \\
a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3 &= 5.5 \\
a_0 + a_1(2) + a_2(2)^2 + a_3(2)^3 &= 20 \\
a_0 + a_1(3) + a_2(3)^2 + a_3(3)^3 &= 46.5 \\
a_0 + a_1(4) + a_2(4)^2 + a_3(4)^3 &= 88
\end{aligned}$$

Equivalently, we can write this as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5.5 \\ 20 \\ 46.5 \\ 88 \end{bmatrix}$$

(b) Solving the above system yields

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 0.5 \end{bmatrix}$$

(c) $p(1.5) = 2(1.5) + 3(1.5)^2 + 0.5(1.5)^3 = 11.4375$

(d) The system for a polynomial of lower order would have no solution because $a_3 \neq 0$. For example, if we used a linear function, the system becomes

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5.5 \\ 20 \\ 46.5 \\ 88 \end{bmatrix}$$

This system has no solution.

8. (a) From $\mathbf{F} = m\mathbf{a} = m\mathbf{v}'(t)$ we obtain

$$\mathbf{v}(t) = c_1\mathbf{i} + \big(\pi\sin(\pi t) + c_2\big)\mathbf{j} + \big(-\pi\cos(\pi t) + c_3\big)\mathbf{k}$$

where c_1, c_2, c_3 are constants. When $t = 0$,

$$\begin{aligned}
\mathbf{v}(0) &= c_1\mathbf{i} + \big(\pi\sin(0) + c_2\big)\mathbf{j} + \big(-\pi\cos(0) + c_3\big)\mathbf{k} \\
&= c_1\mathbf{i} + c_2\mathbf{j} + (-\pi + c_3)\mathbf{k}
\end{aligned}$$

But $\mathbf{v}(0) = \mathbf{i}$, so by comparison, $c_1 = 1$, $c_2 = 0$, and $c_3 = \pi$. Thus

$$\mathbf{v}(t) = \mathbf{i} + \big(\pi\sin(\pi t)\big)\mathbf{j} + \big(-\pi\cos(\pi t) + \pi\big)\mathbf{k}$$

(b) Integrating the components of $\mathbf{v}(t)$ yields

$$\mathbf{r}(t) = (t + d_1)\mathbf{i} + \big(-\cos(\pi t) + d_2\big)\mathbf{j} + \big(-\sin(\pi t) + \pi t + d_3\big)\mathbf{k}$$

where d_1, d_2 and d_3 are constants. When $t = 0$,

$$\begin{aligned}
\mathbf{r}(0) &= (0 + d_1)\mathbf{i} + \big(-\cos(0) + d_2\big)\mathbf{j} + \big(-\sin(0) + \pi(0) + d_3\big)\mathbf{k} \\
\mathbf{r}(0) &= d_1\mathbf{i} + \big(-1 + d_2\big)\mathbf{j} + \big(d_3\big)\mathbf{k}
\end{aligned}$$

But $\mathbf{r}(0) = -\mathbf{i}$, so by comparison, $d_1 = -1$, $d_2 = 1$, and $d_3 = 0$. Thus

$$\mathbf{r}(t) = (t-1)\mathbf{i} + \big(-\cos(\pi t) + 1\big)\mathbf{j} + \big(-\sin(\pi t) + \pi t\big)\mathbf{k}$$

Thus

$$\begin{aligned}\mathbf{r}(1) &= (1-1)\mathbf{i} + \big(-\cos(\pi) + 1\big)\mathbf{j} + \big(-\sin(\pi) + \pi\big)\mathbf{k} \\ &= 2\mathbf{j} + \pi\mathbf{k}\end{aligned}$$

(c) At $t = 0$ and $t = 1$ we have

$$\mathbf{r}(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{r}(1) = \begin{bmatrix} 0 \\ 2 \\ \pi \end{bmatrix}.$$