Introduction to Engineering Written Assignment Questions

Greg Mayer and Daniel Connelly

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Part I
Questions for Written Assignments

2 Multivariable Functions and Algebra

2.0 Wolfram Alpha Syntax

You may want to use Wolfram Alpha (wolframalpha.com) to check your answers. The following is a brief introduction to syntax that can be used.

2.0.1 Multivariable Functions

If you're not sure what syntax to use to compute square roots and evaluate multivariable limits with Wolfram Alpha, let's suppose that we want to determine the value of

$$f(x,y,z) = \frac{2\sqrt{x}}{y^2+\sqrt{z}}$$

at the point (4,5,9). The syntax we could use to evaluate f at the given point is

```
substitute x=4 and y=5 and z=9 into (2 * sqrt(x)) / ( y^2 + sqrt(z) )
```

Or we can simply use the syntax

```
(2 * sqrt(4)) / ( 5^2 + sqrt(9) )
```

2.0.2 Determinants and Matrices

If we want to calculate the determinant of the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

the syntax we could use is

```
determinant of [[a,b],[c,d]]
```

To find the solution of the linear system

$$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

we could use the syntax

```
solve [[2,0],[0,3]]*[[x],[y]]=[[1],[2]]
```

2.1 Dot Products and Cross Products

The following questions are related to sections 1.3 and 1.4 of Vector Calculus by Michael Corral.

2.1.1. Find two unit vectors perpendicular to vectors $\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$ and $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

2.1.2. Suppose vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are in $\mathbb{R}^3$, and $\mathbf{a} \neq 0$. If $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$? Briefly explain your answer.

2.1.3. Consider the following vectors in $\mathbb{R}^2$:

$$\mathbf{a} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

(a) Verify that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular unit vectors.

(b) Use your results from part (a) to find constants C_1 and C_2 so that $\mathbf{r} = C_1\mathbf{a} + C_2\mathbf{b}$.

2.1.4. Suppose that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are nonzero vectors in $\mathbb{R}^3$. Prove the following statements.

(a) If $||\mathbf{a}+\mathbf{b}||^2 = ||\mathbf{a}||^2 + ||\mathbf{b}||^2$, then vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

(b) If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ and $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, then $\mathbf{b} = \mathbf{c}$.

2.1.5. Suppose that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are nonzero vectors in $\mathbb{R}^3$. Identify which of the following statements are meaningless and which statements are not meaningless. Explain your reasoning.

(a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

(b) $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$

(c) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$

(d) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

2.1.6. **Application to Mechanics**

Torque plays a fundamental role in many branches of engineering. It is vector that describes rotation rotation about a point or axis due to an applied force. If you haven't yet encountered torque, its mathematical definition is straightforward:

> The torque, $\boldsymbol{\tau}$, about a pivot point P, that is produced by a force $\mathbf{F}$ applied at a point Q, is defined as $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, where $\mathbf{r} = \overrightarrow{PQ}$.

Torque is a vector that is produced by an applied force. For example, suppose that a force $\mathbf{F} = \frac{1}{2}\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$ is applied at the point $(1, C, 0)$, about the pivot point (1,0,0), where $C \in \mathbb{R}$ is a constant. The resulting torque is simply

$$\boldsymbol{\tau} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1-1 & C-0 & 0 \\ 0.5 & 0 & 0 \end{vmatrix} = -\frac{C}{2}\mathbf{k}$$

A plot of the applied force and produced torque vectors is shown in Figure 1.

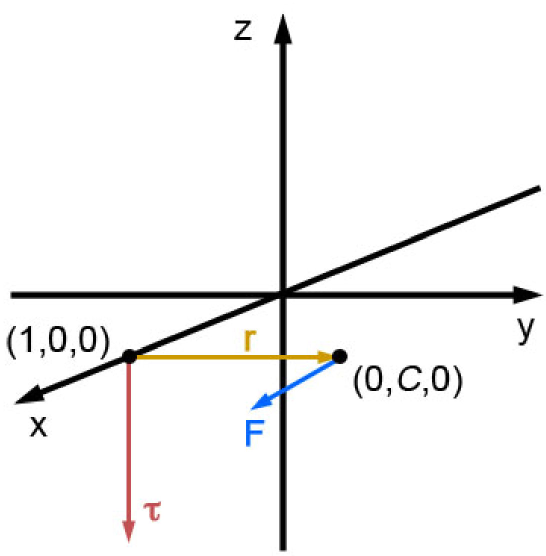


Figure 1: The torque, $\boldsymbol{\tau}$, about the pivot point (1,0,0), is produced by the applied force $\mathbf{F}$.

Note that the magnitude of the torque vector is $||\boldsymbol{\tau}|| = C/2$, and that if we increase the value of C, two things happen: the applied force moves further away from the pivot point, and the magnitude of the produced torque increases.

Units
When $\mathbf{F}$ has units of Newtons (N), and $\mathbf{r}$ has units of meters (m), torque has units of N $\cdot$ m.

A Mechanics Problem
Suppose that a bicycle has pedal arms that are 0.14 m long, and that a constant downward force of 100 N is applied by a cyclist on one pedal. Let $\theta \in [0, 360°)$ be the angle between the vertical and the pedal arm, as shown in Figure 2.

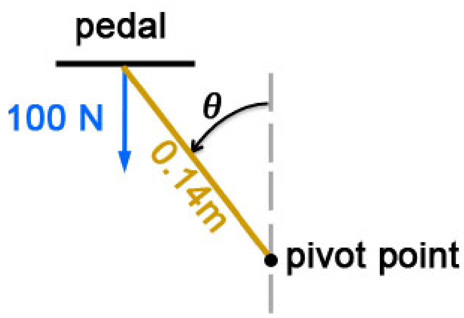


Figure 2: The pedal arm (orange) rotates about the pivot point. The angle it makes with the vertical direction is θ, measured counter-clockwise. A constant downward force of 100 N is applied to the pedal, which is attached to the other end of the pedal arm.

(a) Determine the magnitude of the torque about the pivot point when θ is 30° and when θ is 90°. Your answers should include the units of measurement.

(b) Determine the magnitude of the torque about the pivot point for any θ. Your answer should be a function of θ.

(c) What value(s) of θ minimize the produced torque? Briefly explain why. You should not need to compute any derivatives to answer this question.

Naturally, the torque vector can also be a function of time. In our bicycle example, the torque that is produced by the cyclist could change continuously as the pedal rotates about the pivot point. We will revisit the concept of torque as a vector function in a later assignment, after we have defined vector functions and their derivatives.

2.1.7. Find all values of x that satisfy the following equations.

(a) $\begin{vmatrix} 3 & 2 & 0 \\ 1 & x & 0 \\ 7 & -3 & 4 \end{vmatrix} = 4.$

(b) $\begin{vmatrix} x & 1 \\ 4 & 4x \end{vmatrix} = \begin{vmatrix} -x & -2 \\ 2 & 2x+8 \end{vmatrix}.$

2.2 Lines and Planes

The following questions are related to Section 1.5 of Vector Calculus by Michael Corral.

2.2.1. Find vector equations for the following lines.

(a) Find the vector equation for the line that passes through the points (-1,2,1), and (3,-2,1).

(b) Find the vector equation for the line that has parametric equations $x = 2+2t, y = 9t, z = 5+t$.

2.2.2. Calculate the following angles.

(a) Calculate the angle between the vectors $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$.

(b) Calculate the angle between the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and the plane $2x + y - z = 0$.

2.2.3. **Distance Between a Point and a Line**
Find the distance between the point (1,0,0) and the line $x = 1, y = t, z = 1$.

2.2.4. **Distance Between a Point and a Plane**
Consider the point $(-1, C, 1)$ and the plane $2x+y-z = 3$, where C is an unknown constant. Find all possible values of C such that the distance between the given point and plane is equal to $\sqrt{6}$.

2.2.5. Consider the points P, Q, R

$$\begin{aligned} P &= (1, 0, 3) \\ Q &= (2, 2, 3) \\ R &= (0, 0, -1) \end{aligned}$$

(a) Find a vector that is perpendicular to the plane that passes through these points.

(b) Find the area of the triangle $\triangle PQR$.

(c) Find a point S so that a unique plane that passes through P, Q, and S cannot be found. Describe why you cannot find a unique plane that passes through P, Q, and S.

2.2.6. **Equations of Planes**
Find the equation of the plane that

(a) passes through the points (-1,2,1), (3,-2,1), and (-1,1,-1).

(b) passes through the point (-1,2,1) and contains the line $x = y = z$.

2.2.7. Suppose we have two planes $2x+y-z = 3$ and $x+3y+z = 0$. Find the line of intersection between these two planes, and find the equation of the plane that passes through the line of intersection and through the point $(0, 0, 0)$.

2.2.8. Suppose that L_1 is a line in $\mathbb{R}^3$.

(a) Is it possible to find another line, L_2, that is parallel L_1, and intersects L_1 at only one point? Show that it is possible, or show that it is not possible with a counterexample.

(b) Suppose that there is a line L_3 that is is parallel to L_1, and that a plane P includes both L_1 and L_3. Is it always true that there exists a plane that is perpendicular to P?

2.3 Surfaces

The following questions are related to Section 1.6 of Vector Calculus by Michael Corral.

2.3.1. **Identifying Surfaces**
Consider the surface $z = ax^2 + by^2$, where a and b are constants. Identify all possible surfaces for the following cases, by stating the name of the surface and describing its orientation, if applicable.

(a) $ab > 0$

(b) $ab < 0$

(c) $a = b = 0$

2.3.2. **Finding the Intersection Between Two Surfaces**
Solve Question 7 from Section 1.6 of *Vector Calculus* by Michael Corral, which is:

> Find the intersection of the sphere $x^2 + y^2 + z^2 = 9$ and the cylinder $x^2 + y^2 = 4$.

Later in this course, we will find the volume of the solid enclosed by these two surfaces, which requires finding their intersection.

2.3.3. **Finding the Trace of a Surface**
Solve Question 9 from Section 1.6 of *Vector Calculus* by Michael Corral, which is:

> Find the trace of the hyperbolic paraboloid $x^2 - y^2 = z$ in the xy-plane.

In this example, the trace is a *level curve* of the hyperbolic paraboloid for a particular value of z. We will explore level curves later in this course.

2.3.4. **Deriving the Equation of a Surface**
Find the equation of the surface that consists of all points that are equidistant from the point $(0, 1, 0)$ and the plane $z = 1$. Briefly describe the surface in words.

2.4 Curvilinear Coordinates

The following questions are related to Section 1.7 of Vector Calculus by Michael Corral.

2.4.1. **Finding the Intersection Between Two Surfaces in Spherical Coordinates**
Solve Question 8 from Section 1.7 of *Vector Calculus* by Michael Corral, which is:

> Describe the intersection of the surfaces whose equations in spherical coordinates are $\theta = \pi$, and $\phi = \pi$.

2.4.2. **Converting Equations of Surfaces into Cartesian Coordinates**
Obtain the equation of the following surfaces in Cartesian coordinates, and briefly describe each surfaces in words.

(a) $\rho = -2a \sin\phi \cos\theta$, $a \neq 0$
(b) $r = 2\cos\theta$

2.4.3. **Describing Solids in Cylindrical Coordinates**
The following inequalities describe solids in $\mathbb{R}^3$. Define the solid in cylindrical coordinates, and briefly describe each solid in words.

(a) $z \geq x^2 + y^2$, and $z \leq \sqrt{2 - x^2 - y^2}$
(b) $x^2 + y^2 + z^2 \leq 4$, and $x^2 + y^2 \geq 1$

2.5 Vector-Valued Functions

The following questions are related to Section 1.8 of Vector Calculus by Michael Corral.

2.5.1. Find the values of t and the points on the curve

$$\mathbf{r}(t) = (1 + t^2)\mathbf{i} + t\mathbf{j}, \quad t \in \mathbb{R}$$

where

(a) $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are perpendicular,
(b) $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ have the same direction, and
(c) $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ have opposite directions.

2.5.2. **Application to Mechanics**
Recall from a previous assignment that we defined torque as $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. Now that we have introduced the concepts of vector-valued functions and their derivatives, let's consider the more general case when $\boldsymbol{\tau}$, $\mathbf{r}$, and $\mathbf{F}$ are all functions of time:

$$\boldsymbol{\tau}(t) = \mathbf{r}(t) \times \mathbf{F}(t).$$

Moreover, using the relation $\mathbf{F}(t) = m\mathbf{a}(t)$ (Newton's second law), we can write our definition of torque as

$$\begin{aligned}\boldsymbol{\tau}(t) &= \mathbf{r}(t) \times \mathbf{F}(t)\\ &= \mathbf{r}(t) \times \big(m\mathbf{a}(t)\big)\\ &= m\big(\mathbf{r}(t) \times \mathbf{r}''(t)\big).\end{aligned}$$

These alternate forms for the torque vector may be helpful in solving the following problems.

(a) If the position of a particle with mass m is given by the position vector $\mathbf{r}(t)$, then its angular momentum is a vector defined as $\mathbf{L}(t) = m\mathbf{r}(t) \times \mathbf{r}'(t)$. Show that $\mathbf{L}'(t) = \boldsymbol{\tau}(t)$.

(b) Show that if the torque is a zero vector for all t, then the angular momentum of the particle is constant for all t. This is what is known as the **law of conservation of angular momentum**.

2.5.3. **Integration with Vector-Valued Functions**
If the position of a particle is given by the vector function $\mathbf{r}(t) \in \mathbb{R}^3$, then we know that we can determine its velocity, $\mathbf{v}(t) = \mathbf{r}'(t)$, by differentiating each of its components. That is, if

$$\mathbf{r}(t) = \begin{bmatrix} r_1(t)\\ r_2(t)\\ r_3(t)\end{bmatrix},$$

then

$$\mathbf{v}(t) = \mathbf{r}'(t) = \begin{bmatrix} \frac{d}{dt}r_1(t)\\ \frac{d}{dt}r_2(t)\\ \frac{d}{dt}r_3(t)\end{bmatrix} = \begin{bmatrix} v_1(t)\\ v_2(t)\\ v_3(t)\end{bmatrix},$$

provided that the derivatives of the components of $\mathbf{r}$ exist at t. It follows from the Fundamental Theorem of Calculus that if we were instead given the velocity of the particle, we could compute its position by integrating each of the components with respect to t. We would of course introduce constants of integration. That is, given

$$\mathbf{v}(t) = \begin{bmatrix} v_1(t)\\ v_2(t)\\ v_3(t)\end{bmatrix},$$

we could obtain $\mathbf{r}$ by integrating each of the components of $\mathbf{v}(t)$

$$\mathbf{r}(t) = \begin{bmatrix} \int v_1(t)dt\\ \int v_2(t)dt\\ \int v_3(t)dt\end{bmatrix} = \begin{bmatrix} r_1(t)+c_1\\ r_2(t)+c_2\\ r_3(t)+c_3\end{bmatrix},$$

where c_1, c_2, c_3 are constants.

Suppose that a particle with mass m is subjected to a force, $\mathbf{F}(t) = m\pi^2\big(\cos(\pi t)\mathbf{j} + \sin(\pi t)\mathbf{k}\big)$, where $t \geq 0$. Suppose also that when $t = 0$,

$$\mathbf{r}(0) = \begin{bmatrix} -1\\ 0\\ 0\end{bmatrix}, \quad \mathbf{r}'(0) = \begin{bmatrix} +1\\ 0\\ 0\end{bmatrix}.$$

(a) Using the relation $\mathbf{F} = m\mathbf{a}$, find the velocity of the particle at time t. *Hint: you will need to apply the velocity at time t=0.*

(b) Find the position of the particle at time $t = 1$.

(c) Plot the position of the particle at times $t = 0$ and $t = 1$ on the same graph in $\mathbb{R}^3$.

2.6 Linear Systems

The following questions are related to Section 8.1 of College Algebra by Carl Stitz and Jeff Zeager.

2.6.1. For A and $\mathbf{b}$ below, solve the linear system $A\mathbf{x} = \mathbf{b}$, if possible:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & -1 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 4 & 0 & 3 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 10 \end{bmatrix}.$$

If it isn't possible to solve this system, explain why.

2.6.2. For the systems below,

- Compute the determinant of matrix A, if possible. If it is not possible to do so, explain why.
- Solve the linear system $A\mathbf{x} = \mathbf{b}$, if possible.
- State whether the system has no solution, infinitely many solutions, or a unique solution.

(a) $A = \begin{bmatrix} -3 & 1 & 2 & 4 \\ 2 & -1 & 2 & 3 \\ 1 & 0 & 4 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 0 \\ 4 & 9 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

2.6.3. Consider the system of simultaneous linear equations

$$\begin{aligned} x + 2y - z &= 2 \\ 2x + ay - 2z &= b \\ 3x + 2y &= 1 \end{aligned}$$

where x, y, z are unknown.

(a) Find all values of a and b such that the above system has
   i. exactly one solution;
   ii. no solutions;
   iii. infinitely many solutions.

(b) For those values of a and b from 2.6.3.(a)i, what is the unique solution?

(c) For those values of a and b from 2.6.3.(a)iii, parameterize the set of all solutions.

2.6.4. **Application to Polynomial Interpolation**

In many areas of engineering, experimental data is collected that must be analyzed to extract parameters that tell us something about a physical process. Suppose we have measured a set of experimental data that are represented in the xy-plane. An **interpreting polynomial** for the measured data is a polynomial that passes through every measured point. We can use this polynomial, for example, to estimate values between the measured data points.

Suppose for example that we have measured the data points (0,-6), (1,-2), (2,4), (3,10). To find an interpreting polynomial of order 2 for these data, we would try to find a polynomial of the form $p(x) = a_0 + a_1x + a_2x^2$ that passes through all four measured points. In other words, we need to find the unknown constants a_0, a_1, a_2 that satisfy the equations

$$\begin{aligned} p(0) &= a_0 + a_1(0) + a_2(0)^2 = -6 \\ p(1) &= a_0 + a_1(1) + a_2(1)^2 = -2 \\ p(2) &= a_0 + a_1(2) + a_2(2)^2 = 4 \\ p(3) &= a_0 + a_1(3) + a_2(3)^2 = 10 \end{aligned}$$

The above system has four equations and four unknowns. Upon solving this system, you should be able to determine that $a_0 = -6, a_1 = 4, a_2 = 0$.

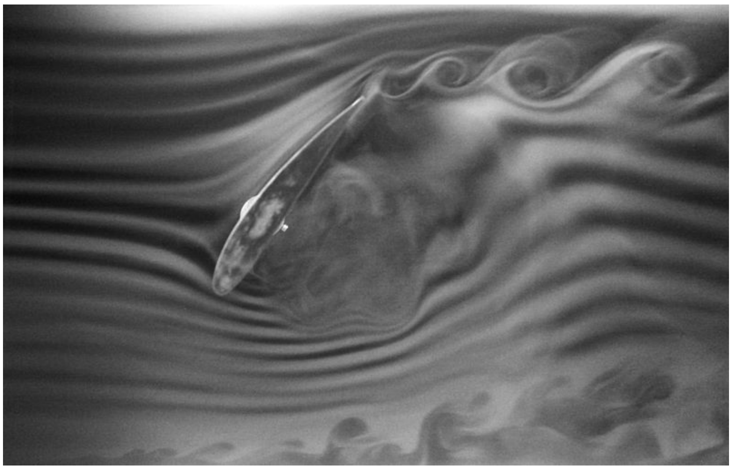


Figure 3: An airfoil in a fog wind tunnel (image from Wikimedia Commons, Smart Blade GmbH).

Wind Tunnel Experiment[1]
In a fictitious wind tunnel experiment, the following measurements were made.

Velocity (m/s)	Force (N)
0	0
1	5.5
2	20
3	46.5
4	88

The data represent the measured force due to air resistance, on an object suspended in the tunnel, measured at different air speed velocities.

(a) Using the data above, derive a 5×4 system of equations, that when solved, find an interpreting polynomial of the form $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. Write the system in the form $A\mathbf{x} = \mathbf{b}$.

(b) Solve your system to obtain $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

(c) As mentioned above, engineers sometimes use interpreting polynomials to estimate values in between measured data points. Using your polynomial, estimate the value of the force when the velocity is 1.5 m/s.

(d) In practice, it can be difficult to determine what order of polynomial to use. Sometimes, polynomials of different orders must be used to decide which polynomial yields the most useful results. For the above data, explain what what would happen if we used a polynomial less than 3. It may help to see what happens if we use a 1^{st} order polynomial.

2.7 Functions of Two or Three Variables

The following questions are related to section 2.1 of Vector Calculus by Michael Corral.

2.7.1. For the following functions,

- determine the domain of the function,
- sketch the domain of the function in the xy plane, and
- determine the range of the function

(a) $f(x, y) = \frac{1}{\sqrt{(x-1)(y-1)}}$

[1]The wind tunnel problem was based on a similar exercise in Linear Algebra and Its Applications, 4th Edition, by David C. Lay, Addison-Wesley, 2012.

(b) $g(x,y) = \frac{\sqrt{x+1}}{yx^2+xy^2}$

(c) $h(x,y) = \frac{xyz}{x^2+y^2-1}$

(d) $z(x,y) = \ln(x+2y)$

2.7.2. Evaluate the following limits or show that they do not exist.

(a) $\displaystyle \lim_{(r,s)\to(0,2\pi)} \frac{3r^2 + rs^3 - 3r^4 \sin(s/4)}{r^2}$

(b) $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$

(c) $\displaystyle \lim_{(x,y,z)\to(1,1,1)} \left|3x - 2y - z\right|$

(d) $\displaystyle \lim_{(x,y,z)\to(0,0,0)} \frac{x^2-y^2-z^2}{x^2+y^2+z^2}$

(e) $\displaystyle \lim_{(x,y)\to(1,0)} \frac{\sqrt{2x+y} - \sqrt{2x-y}}{2y}$

2.7.3. Consider the function

$$g(x,y,z) = \begin{cases} 1 & \text{if } (x,y,z) = (0,0,1) \\ x+y+2z & \text{if } (x,y,z) \neq (0,0,1) \end{cases}.$$

Find all points or regions where $g(x,y,z)$ has a discontinuity.

2.7.4. Evaluate the following limit, if it exists.

$$\lim_{(x,y)\to(1,0)} \frac{x(x-1)^3 + y^2}{4(x-1)^2 + 9y^3}$$

Hint: approach the limit point along straight lines.

2.7.5. Sketch, by hand, the level curves of the following functions for the indicated values of c, if possible. You may of course want to check your answer using a calculator or with graphing software.

(a) $f(x,y) = \frac{\ln y}{x^2}$, $c = -2, -1, 0, 1, 2$

(b) $f(x,y) = \frac{x^2}{x^2+y^2}$, $c = 0, \frac{1}{4}, \frac{1}{2}, 1, 2$

2.7.6. (a) Find a function $f(x,y)$ whose family of level curves are straight lines that pass through the point $(2,1)$.

(b) Find a function $f(x,y)$ whose family of level curves are circles with radius $\sqrt{e^c}$.

2.7.7. Let m be any real number.

(a) Provide an example of a function, $f(x,y)$, with the following two properties:
- $f(x,y) \to m$ as $(x,y) \to (1,0)$ along any parabola, $y = m(x-1)^2$
- $f(x,y)$ is not constant

(b) Explain why the limit you provided in part (a) does not exist.

2.7.8. **Application to Temperature Distributions**[2]
Suppose that a metal object occupies a space in three dimensions, and that the temperature, $T(x,y)$ of the object at the point (x,y) is inversely proportional to the distance between the point and the origin.

(a) Write down an expression for T as a function of x and y.

(b) Describe the level curves in words. Sketching them is not necessary, but the level curves are known as **isothermals**, and represent curves upon which the temperature is constant.

[2]The temperature and electrical potential questions are based on a similar exercises that can be found in *Calculus, One and Several Variables*, 10th Edition, by Salas, Hille, Etgen, Wiley, 2007.

(c) Suppose the temperature of the object at the point (2,1) is 10°C. Find the temperature of the object at (1,3).

2.7.9. **Application to Electrical Potential Distributions**
The scalar function

$$V(x,y) = \frac{c}{\sqrt{r^2 - x^2 - y^2}} \ ,$$

where c and r are positive constants, represents the electrical potential (in volts) at a point (x,y) in the $xy-$plane. Describe, in words, the level curves $V(x,y) = K$ for constant $K \in \mathbb{R}$, and determine the values of K for which the curves exist. Note that the level curves are known as **equipotential curves**, because all points on a given curve have the same potential.

2.7.10. **Bonus Problem**
This question goes beyond the requirements for this course. Before starting this problem, make sure that your instructor will give you marks for solving it.
Using the definition of limit (the ϵ, δ definition) for a function of two variables, prove that the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$$

exists. *Hint: start by evaluating the limit along the x-axis, or along other straight lines to determine what the limit could be equal to.*

3 Partial Derivatives

3.0 Wolfram Alpha Syntax

You may want to use Wolfram Alpha (wolframalpha.com) to check your answers. If you're not sure what syntax to use to compute double integrals with Wolfram Alpha, let's suppose that we want to determine the value of

$$\frac{\partial}{\partial x}\Big(xy^2 + Ce^{xy}\Big)$$

where C is a constant. The syntax we could use to compute this particular derivative is

```
differentiate (xy^2 + C*e^{xy}) with respect to x
```

or simply:

```
d/dx (xy^2 + C*e^{xy})
```

Similarly, to compute

$$\frac{\partial^2}{\partial x \partial y}\Big(xy^2 + Ce^{xy}\Big)$$

we can use the syntax

```
d/dx d/dy (xy^2 + C*e^{xy})
```

3.1 Partial Derivatives

The following questions are related to sections 2.2 through 2.4 of Vector Calculus by Michael Corral.

3.1.1. **Calculating First and Second Partial Derivatives**
Compute all first and second partial derivatives and the gradient for each of the following functions.

(a) $f(x,y) = xy\sin(\frac{x}{y})$

(b) $g(x,y) = \dfrac{x+y}{e^{xy}}$

3.1.2. **Finding Tangent Planes**
Find the equations of the planes tangent to the following surfaces at the specified points.

(a) $f(x,y) = \frac{x^2}{9} - y^2$ at the point $(2, \frac{1}{3})$.

(b) $x^2 + x + y - z^2 = 7$ at the point $(2,5,2)$.

Nathaniel: I think the answer for part (a) is $4x - 6y - 9z = 4$***, please check***

3.1.3. **Implicit Differentiation**
Compute the gradient of the function $z = h(x,y)$, defined implicitly by the equation $xy = z^{x+y}$.

3.1.4. **Calculating the Rate of Change of a Function of Two Variables**

Let $f(x,y) = e^{-(x^2+x+1+y^2-2y)}$.

(a) In which direction does f increase fastest from the point (1,1)?

(b) Compute the rate of change of f in the direction of the point $(5,4)$ from the point $(2,1)$.

3.1.5. **Finding a Jacobian**
Find the Jacobian matrix for the function $f(x,y) = \begin{bmatrix}\sin(x+y)\\ \ln(xy)\end{bmatrix}$.

3.1.6. **The Two-Dimensional Heat Equation**

A *partial differential equation* is an equation that involves the partial derivatives of a function. For example, $(\frac{\partial f}{\partial x})^2 - \frac{\partial^2 f}{\partial x \partial y} = 0$ is a partial differential equation that relates the first- and second-order partial derivatives of an unknown function f.

A function f is said to *satisfy* a partial differential equation when the equation holds upon substitution of the function's partial derivatives. For example, the function $f(x, y) = y$ satisfies the above partial differential equation since $(\frac{\partial}{\partial x}(y))^2 = 0$ and $\frac{\partial^2}{\partial x \partial y}(y) = \frac{\partial}{\partial x}(1) = 0$.

The one-dimensional heat equation is a partial differential equation which relates physical properties of a material to the function that specifies the exact temperature at any point along a fixed-length rod of that material. Under a few assumptions, it can be stated in the following form:

$$\frac{\partial u}{\partial t} = \frac{K_0}{c\rho} \frac{\partial^2 u}{\partial x^2} \tag{1}$$

where $u = u(x, t)$, x is the position along the rod, t is the point in time, K_0 is the thermal conductivity of the material, c is its specific heat, and ρ is its mass density.

Show that $u(x, t) = e^{-tK_0/(c\rho)} \sin(x)$ is a solution to (1).

3.1.7. **Differentiability**

Construct a function $f(x, y)$ with the following three properties.

- $f(x, y)$ is defined at the point $(0, 1)$.
- $f(x, y)$ is continuous at the point $(0, 1)$.
- $\frac{\partial f}{\partial x}$ does not exist at the point $(0, 1)$.

Show that your function satisfies the above properties.

3.1.8. **A Second Order Derivative**

Consider the following statement.

If the partial derivatives

$$\frac{\partial f}{\partial x}, \quad \frac{\partial f}{\partial y}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial x \partial y}$$

exist everywhere on the domain of $f(x, y)$, then

$$\frac{\partial f}{\partial y} \frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Prove that the above statement is true, or provide a counterexample to show that the statement is false.

3.1.9. **The Rate of Change in the Direction $f_y\mathbf{i} - f_x\mathbf{j}$**

Suppose $f(x, y)$ is a function whose gradient is nonzero at the point (a, b).

(a) Find the rate of change of $f(x, y)$ at the point (a, b) in the direction of $\mathbf{v}$, where

$$\mathbf{v} = \frac{\partial f}{\partial y}(a, b)\mathbf{i} - \frac{\partial f}{\partial x}(a, b)\mathbf{j}.$$

(b) Provide a geometric interpretation of your answer to part (a). In your interpretation, describe the relationship between your answer to part (a), and the level curves of $f(x, y)$.

3.1.10. **Application of the Chain Rule to Model Traffic Flow**[3]

Traffic flow on a highway can be modeled with the partial differential equation

$$u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = -c^2 k^n \frac{\partial k}{\partial x}, \tag{2}$$

where u is the average traffic speed at time t and position x on a straight highway, c and n are positive constants, and k is the traffic concentration (the number of vehicles per unit area) at position x and time t.

[3] This question is based on a problem from Section 13.6 of Calculus for Engineers, by Donald Trim, 3^{rd} Edition.

(a) If we let $k = g(x,t)$, and model the average speed as a function of the concentration so that $u = f(k)$, use the chain rule to show that

$$\frac{du}{dk}\left(u\frac{\partial k}{\partial x} + \frac{\partial k}{\partial t}\right) = -c^2 k^n \frac{\partial k}{\partial x}. \tag{3}$$

(b) Further assumptions on the continuity of the traffic flow would lead us to the equation

$$\frac{\partial k}{\partial t} + \frac{\partial}{\partial x}(ku) = 0. \tag{4}$$

Use Equations (2) and (4) to derive the ordinary differential equation

$$\frac{du}{dk} = \pm c k^{(n-1)/2}. \tag{5}$$

3.2 Optimization

The following questions are related to sections 2.5, 2.7 and parts of 4.6 of Vector Calculus by Michael Corral.

3.2.1. Use the second derivative test to classify the local extrema of the following functions:

(a) $f(x, y) = x^2 + 2x - xy + y^2$

(b) $h(x, y) = \sin(xy)$

3.2.2. Find the extreme values of

(a) the function $f(x, y) = e^{-(x^2+y^2)}$ along the curve $x = y^2$

(b) the function $g(x, y) = e^{x-y^2}$ along the boundary of the ellipse $\frac{x^2}{4} + y + y^2 = 1$

3.2.3. Consider the function $f(x, y, z) = \begin{bmatrix} x^2 + y^2 \\ y^2 + z^2 \\ z^2 + x^2 \end{bmatrix}$. Find its divergence $\nabla \cdot f$ and curl $\nabla \times f$.

3.2.4. Find the point on the sphere $x^2 + y^2 + z^2 = 1$ that is closest to the plane $x + 2y + 2z = 5$. (Hint: using some geometric intuition is is possible to complete this problem without using the method of Lagrange multipliers.)

3.2.5. **Application: Cost Optimization**

Suppose you are given a budget of \$500 to build a large glass triangular prism. Each rectangular side must have equal dimensions and the two triangular sides must also have equal dimensions. You can purchase glass for the rectangular sides at a cost of $\$8/ft^2$ and for the triangular sides at a cost of $\$10/ft^2$.

What are the dimensions of the prism of largest volume that you can build?

3.2.6. **Application: Heat Optimization**[4]

Suppose that the temperature in a space is given by the function $T(x, y, z) = 200xyz^2$. Find the hottest point on the unit sphere.

[4] This question is similar to a question from section 4.5 of *Vector Calculus* by Susan Colley

4 Multiple Integrals

4.0 Wolfram Alpha Syntax

You may want to use Wolfram Alpha (wolframalpha.com) to check your answers. If you're not sure what syntax to use to compute double integrals with Wolfram Alpha, let's suppose that we want to determine the value of

$$\int_{-2}^{-1}\int_{0}^{x-1}(x^{2C}+y)dydx$$

where C is a constant. The syntax we could use to compute this particular integral is

```
integrate x^{2C}+y, x from -2 to -1 and y from 0 to (x-1)
```

4.1 Double Integrals

The following questions are related to Section 3.1 of Vector Calculus by Michael Corral.

4.1.1. **Volume of a Solid**
Consider the solid that lies under the plane $z=-x-2y+2$ and above the rectangle $\{(x,y)\mid -2\le x\le 0, 0\le y\le 1\}$.

(a) Sketch the solid in $\mathbb{R}^3$. *Hint: start by plotting the points that are located on the given plane and above the corners of the rectangle. Then connect the points with solid lines.*

(b) Find the volume of the solid.

4.1.2. **Application to Fluid Mechanics**
In a two-dimensional, steady-state, incompressible fluid flow, the velocity $\mathbf{v}$ of the flow can be expressed as $\mathbf{v}=u(x,y)\mathbf{i}+v(x,y)\mathbf{j}$. The functions $u(x,y)$ and $v(x,y)$ must satisfy

$$\nabla\cdot\mathbf{v}=0.$$

(a) If $u(x,y)=x^2+y^2$, find the most general form of $v(x,y)$.

(b) If $v(x,y)=\cos(x)$, find the most general form of $u(x,y)$.

*Hint: you are asked to find the most **general** form of functions u and v.*

4.1.3. **Double Integral with an Absolute Value**
Evaluate the double integral

$$\int_0^1\int_0^2 y\big|x-1\big|dxdy.$$

4.1.4. **Double Integral Given Maximum and Minimum Values**
Suppose that we are given a function, $f(x,y)$, that is continuous on the rectangle

$$S=[a,b]\times[c,d]$$

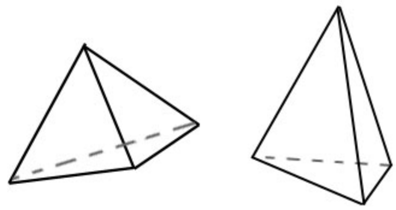
in $\mathbb{R}^2$. Suppose also that the maximum value $f(x,y)$ on S is equal to the constant K, and that the minimum value of $f(x,y)$ on S is also equal to K. Evaluate

$$\iint_S f(x,y)dxdy.$$

4.2 Double Integrals Over a General Region

The following questions are related to Section 3.2 of Vector Calculus by Michael Corral.

4.2.1. **Volume of a Tetrahedron**
A **tetrahedron** is a three dimensional object with four, triangular, flat sides. Because each of its four sides are flat, a tetrahedron can be defined as the region enclosed by four planes. Below is a sketch of two tetrahedrons. Note that each tetrahedron has four vertices, six edges, and that the lengths of its edges do not have to be equal.



Consider the tetrahedron that is bounded by the three coordinate planes in $\mathbb{R}^3$, and by the plane $z = 1 - x - \frac{y}{2}$.

(a) Sketch the tetrahedron in $\mathbb{R}^3$ and label the points that represent its four vertices.

(b) Set up, but do not evaluate, a double integral that represents the volume of the tetrahedron. Integrate with respect to x first.

(c) Set up, but do not evaluate, a double integral that represents the volume of the tetrahedron. Integrate with respect to y first.

Note that

- Examples 3.4 and 3.5 from *Vector Calculus* by Michael Corral are similar to this problem.
- We could also calculate the volume of the solid by using a **triple** integral.
- Although not required, the double integrals are straightforward to compute. You may want to check your answers by evaluating the integrals and seeing if you would get the same result in parts (b) and (c).

4.2.2. **Area of a General Region**

(a) Solve Question 11 from Section 3.2 of *Vector Calculus* by Michael Corral, which is

> Explain why the double integral $\iint_R 1dA$ gives the area of the region R. For simplicity, you can assume that R is a region of the type shown in Figure 3.2.1(a).

Additional hint: Figure 3.2.5 is also helpful.

(b) Use the result from part (a) to compute the area of the region bounded by the curves $y = x^2$ and $x = y^2$.

4.2.3. Find the volume of the solid enclosed by $z = x^2 + y^2$, $y = x^2$ and $x = y^2$.

4.2.4. Consider the integral

$$\iint\limits_R x\sin(y)dA$$

where R is the region bounded by $y = 0$, $y = x^2$, and $x = 2$.

(a) Evaluate the double integral by first integrating with respect to x.

(b) Evaluate the double integral by first integrating with respect to y.

Note that your answers for both parts should be the same, and that you may need to use various techniques of integration to complete this problem, including integration by parts and a variable substitution.

4.2.5. **Changing the Order of Integration**
Consider the double integral

$$\int_0^{1+e} \int_0^{\ln(x-1)} f(x,y) dy dx.$$

Sketch the region in $\mathbb{R}^2$ over which $f(x,y)$ is integrated, and change the order of integration.

4.2.6. **Bounding A Double Integral**
Consider the double integral

$$\iint_D f(x,y) dA,$$

where D is the square $0 \le x \le 1$, $0 \le y \le 1$, and $f(x,y) = \sin(x+y)$. Show that

$$0 \le \iint_D f(x,y) dA \le 1.$$

4.2.7. **Simplifying Double Integrals Using Symmetry**
Certain integrals can be simplified when the integrand is either even or odd. You may know that, for functions of one variable, that

$$\text{if } f(x) \text{ is odd, then } \int_{-a}^{a} f(x) dx = 0.$$
$$\text{if } f(x) \text{ is even, then } \int_{-a}^{a} f(x) dx = 2\int_0^a f(x) dx.$$

There are similar results for integrals of two variables, but in order to introduce them, we first need to extend our concepts of even and odd functions to functions of two variables, and we need to describe regions that are symmetric about the x-axis and about the y-axis.

If R is a region that is **symmetric about the y-axis**, and (x_0, y_0) is a point inside R, then the point $(-x_0, y_0)$ is also inside R. Similarily, if S is a region that is **symmetric about the x-axis**, and (x_1, y_1) is a point inside S, then the point $(x_1, -y_1)$ is also inside S. Above are examples of regions that have these symmetries.

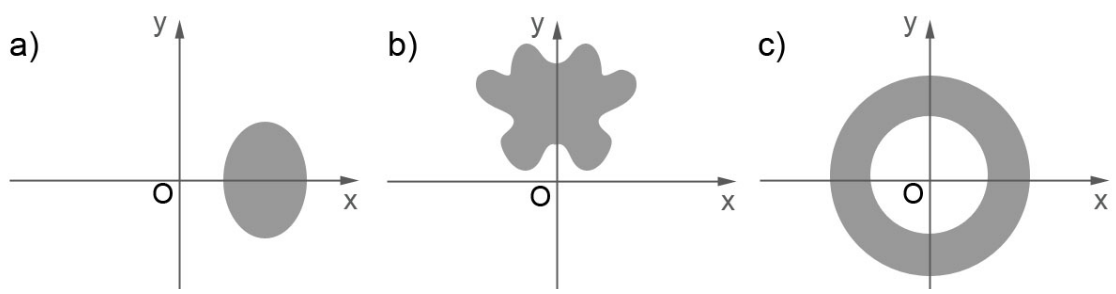


Figure 4: a) the grey region is symmetric about the x-axis, b) the grey region is symmetric about the y-axis, and c) the grey region is symmetric about the x-axis and the y-axis.

Moreover, if $g(x,y)$ is odd in x, then

$$g(-x,y) = -g(x,y).$$

But if $g(x,y)$ were even in x, then

$$g(-x,y) = g(x,y).$$

Combining these concepts yields helpful results for computing double integrals. For example, if R is symmetric about the y-axis, and if $g(x, y)$ is odd in x, then

$$\iint_R g(x,y)dxdy = 0.$$

Similar results can be stated if g were even in x or in y, and if R has symmetry about the x-axis.

(a) Provide an example of a non-zero function of two variables, $h(x, y)$, that is odd in x. Verify that your function is odd in x.

(b) Provide an example of a region, D, that is symmetric about the y-axis, but is not symmetric about the x-axis.

(c) Show that, for your $h(x, y)$ and region D, that

$$\iint_D g(x,y)dxdy = 0.$$

4.3 Triple Integrals

The following questions are related to Section 3.3 of Vector Calculus by Michael Corral.

4.3.1. **Volume of a Tetrahedron**
The textbook points out that the triple integral

$$\iiint_S f(x,y,z)dV$$

for the special special case when $f(x, y, z) = 1$ for all points in S, gives the volume of S

$$V(S) = \iiint_S dV.$$

Consider again the tetrahedron that is bounded by the three coordinate planes in $\mathbb{R}^3$, and by the plane $z = 1 - x - \frac{y}{2}$. We derived an expression for the volume of this tetrahedron in a previous question using a double integral. Now set-up and find the volume of the tetrahedron using a triple integral.

4.3.2. **Volume of an Ellipsoid**
Solve Question 10 from Section 3.5 of *Vector Calculus* by Michael Corral, which is the following:

> Show that the volume inside the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{4\pi abc}{3}$. (Hint: Use the change of variables $x = au$, $y = bv$, $z = cw$, then consider Example 3.12.)

4.3.3. **Maximizing a Triple Integral**
Find the region S for which the triple integral

$$\iiint_S \left(1 - 3x^2 - 2y^2 - 4z^2\right)dV$$

is a maximum.

4.4 Change of Variables in Multiple Integrals

The following questions are related to Section 3.5 of Vector Calculus by Michael Corral.

4.4.1. **Linear Transformations**
Under the linear transformation

$$x = c_1u + c_2v, \quad y = d_1u + d_2v, \quad d_1c_2 - d_2c_1 \neq 0,$$

straight lines in the uv-plane are mapped to straight lines in the xy-plane.

(a) $v = v_0$ is a horizontal line in the uv-plane. Determine the equation of this line in the xy-plane.

(b) $x = x_0$ is a vertical line in the xy-plane. Determine the equation of this line in the uv-plane.

4.4.2. **Double Integral Over a Parallelogram**
Use an appropriate transformation to evaluate the integral

$$\iint_R (x^2 - y^2)\,dxdy,$$

where R is the parallelogram bounded by

$$x + y = 0, \quad x + y = 1, \quad x - y = 0, \quad x - y = 1.$$

4.4.3. **Triple Integral Bounded by an Ellipse**
Use the transformation $x = 3u, y = 2v$ to evaluate the double integral

$$\iint_E x^2\ dxdy,$$

where E is region bounded by the ellipse $4x^2 + 9y^2 = 36$. *Hint: you may also want to use polar coordinates to solve this problem.*

4.4.4. Evaluate the integral

$$\iiint_S y^2 dV,$$

where S is the solid that lies inside the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$ and below the cone $z^2 = 9x^2 + 9y^2$.

4.4.5. **Triple Integral In Cylindrical Coordinates**
Evaluate the integral

$$\iiint_V dV,$$

using cylindrical coordinates, where V is the region bounded by

$$\begin{aligned}&0 \le x \le 2\\ &0 \le y \le \sqrt{4 - x^2}\\ &0 \le z \le \sqrt{4 - (x^2 + y^2)}\end{aligned}$$

4.4.6. **Triple Integral In Spherical Coordinates**
The integral

$$\int_0^{\pi/4}\int_0^{\pi/2}\int_0^1 (\rho^2 \sin\phi)\ d\rho\ d\phi\ d\theta$$

represents the volume of a solid. Describe the shape of the solid, and find its volume.
Textbook doesn't do a great job of integrals in cylindrical and spherical

4.4.7. **Simplifying a Double Integral as a Product of Two Single Integrals**

(a) Show that, in the special case where the function $f(x, y)$ can be factored as the product $f(x, y) = g(x)h(y)$, the double integral

$$\iint_R f(x, y)dA$$

can be expressed as the following product

$$\iint\limits_R f(x,y)dA = \int_a^b g(x)dx \int_c^d h(y)dy,$$

where R is the rectangular region $R = \{(x,y)|a \le x \le b, c \le y \le d\}$. This result can be helpful in simplifying certain multiple integrals, as we will see in part (b).

(b) Use polar coordinates and the result from part (a) to show that

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$$

This result is important in probability and statistics: the above integral is related to the **normal** or **Gaussian distribution**.

4.5 Application: Center of Mass

The following questions are related to Section 3.6 of Vector Calculus by Michael Corral.

4.5.1. **Center of Mass of a 2D Triangular Plate**
A two-dimensional plate has density $\delta(x,y) = xy$, and occupies a triangular region whose vertices are located at the points (0,0), (1,2), and (1,4). Find the x-coordinate and the y-coordinate of the center of mass of the triangular plate.

4.5.2. **Center of Mass of a 2D Plate, Radial Density Function**
A 2D semi-circular plate of mass M is bounded by

$$-a \le x \le a, \quad 0 \le y \le \sqrt{a^2 - x^2}, \quad a > 0.$$

The density of the plate at a point (x,y), is equal to the shortest distance, L, between that point and the upper edge of the plate, as shown in Figure 6.

(a) Set up an integral that represents the x-coordinate of the center of mass of the plate, $\bar{x}$.

(b) Set up an integral that represents the y-coordinate of the center of mass of the plate, $\bar{y}$.

(c) Determine the x-coordinate of the center of mass of the plate, without performing any integration. Briefly describe how you found your answer.

You do not need to perform any integration for this question. Note also that L is a function of x and y.

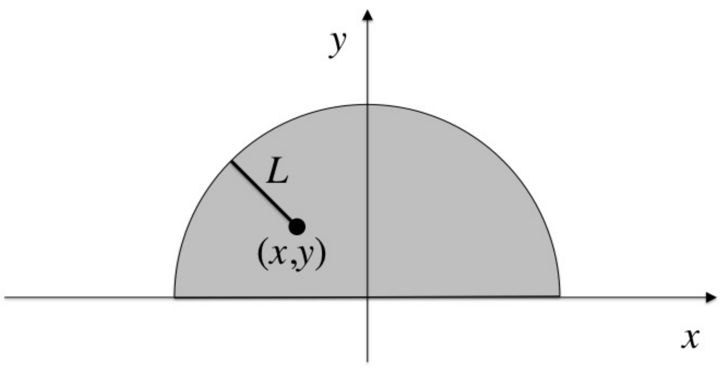


Figure 5: The density of the plate at (x,y) is equal to L.

4.5.3. **Center of Mass of a 2D Plate, Cardioid**
Find the centroid of the plate of the region D bounded by the centroid $r = a(1 + \cos\theta)$, where a is a constant. The region D is shown in the figure below.

When finding the centroid, you may use a symmetry argument to find $\bar{y}$ by inspection. To find $\bar{x}$, you may want to use the reduction formula

$$\int \cos^n x \; dx = \frac{\sin x(\cos x)^{n-1}}{n} + \frac{n-1}{n} \int (\cos x)^{n-2} \; dx.$$

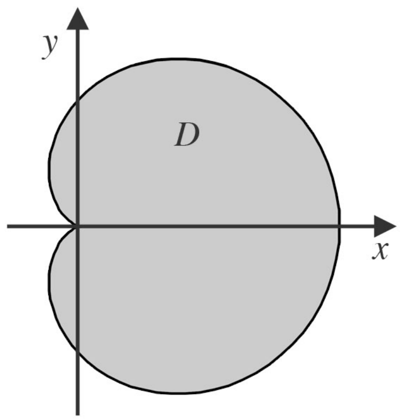


Figure 6: The region D is bounded by the cardioid $r = a(1 + \cos\theta)$.

Part II
Solutions to Written Assignment Questions

2 Multivariable Functions and Algebra

2.1 Dot Products and Cross Products

2.1.1. A vector, $\mathbf{v}$, that is perpendicular to both vectors can be found using the cross product.

$$\mathbf{v} = \begin{bmatrix}1\\0\\4\end{bmatrix} \times \begin{bmatrix}1\\-2\\1\end{bmatrix} = \begin{bmatrix}8\\3\\-2\end{bmatrix}.$$

This is not a unit vector, but we can make this a unit vector by dividing each component by the magnitude of the vector. The magnitude is

$$||\mathbf{v}|| = \sqrt{8^2 + 3^2 + (-2)^2} = \sqrt{77}.$$

A unit vector that is perpendicular to both vectors is

$$\frac{\mathbf{v}}{\sqrt{77}} = \frac{1}{\sqrt{77}} \begin{bmatrix}8\\3\\-2\end{bmatrix}.$$

A second unit vector that is perpendicular to the two given vectors is

$$-\frac{\mathbf{v}}{\sqrt{77}} = \frac{-1}{\sqrt{77}} \begin{bmatrix}8\\3\\-2\end{bmatrix}.$$

2.1.2. It is not necessarily true that $\mathbf{b} = \mathbf{c}$. Simple rearrangement yields

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= \mathbf{a} \cdot \mathbf{c}\\ 0 &= \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c}\\ 0 &= \mathbf{a} \cdot (\mathbf{b} - \mathbf{c})\end{aligned}$$

Therefore, $\mathbf{a}$ is perpendicular to the vector $\mathbf{b}-\mathbf{c}$, which can be true when $\mathbf{b} \neq \mathbf{c}$. A counterexample would be the vectors

$$\mathbf{a} = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix}1\\-1\\0\end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Clearly, $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, even though $\mathbf{b} \neq \mathbf{c}$.

2.1.3. (a) Vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular because their dot product is zero:

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{2} - \frac{1}{2} = 0.$$

Vectors $\mathbf{a}$ and $\mathbf{b}$ are both unit vectors, because they have length 1:

$$||\mathbf{a}|| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2} = 1$$

$$||\mathbf{b}|| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1$$

(b) Using results from part (a), we can find C_1 as follows:

$$\begin{aligned}
\mathbf{r} &= C_1\mathbf{a} + C_2\mathbf{b} \\
\mathbf{a}\cdot\mathbf{r} &= \mathbf{a}\cdot\Big(C_1\mathbf{a} + C_2\mathbf{b}\Big) \\
\mathbf{a}\cdot\mathbf{r} &= C_1\mathbf{a}\cdot\mathbf{a} + C_2\mathbf{a}\cdot\mathbf{b} \\
\frac{2}{\sqrt{2}} &= C_1(1) + C_2(0) \\
C_1 &= \frac{2}{\sqrt{2}}
\end{aligned}$$

A similar calculation yields C_2

$$\begin{aligned}
\mathbf{r} &= C_1\mathbf{a} + C_2\mathbf{b} \\
\mathbf{b}\cdot\mathbf{r} &= \mathbf{b}\cdot\Big(C_1\mathbf{a} + C_2\mathbf{b}\Big) \\
\mathbf{b}\cdot\mathbf{r} &= C_1\mathbf{b}\cdot\mathbf{a} + C_2\mathbf{b}\cdot\mathbf{b} \\
\frac{4}{\sqrt{2}} &= C_1(0) + C_2(1) \\
C_2 &= \frac{4}{\sqrt{2}} = 2\sqrt{2}
\end{aligned}$$

2.1.4. (a) Expanding the left-hand side yields

$$\begin{aligned}
||\mathbf{a}+\mathbf{b}||^2 &= \big(\mathbf{a}+\mathbf{b}\big)\cdot\big(\mathbf{a}+\mathbf{b}\big) \\
&= \mathbf{a}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} + 2\mathbf{a}\cdot\mathbf{b} \\
&= ||\mathbf{a}||^2 + ||\mathbf{b}||^2 + 2\mathbf{a}\cdot\mathbf{b}
\end{aligned}$$

This is equal to $||\mathbf{a}||^2 + ||\mathbf{b}||^2$ iff $\mathbf{a}\cdot\mathbf{b} = 0$, which implies that $\mathbf{a}$ must be perpendicular to $\mathbf{b}$.

(b) First consider the dot product:

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= \mathbf{a}\cdot\mathbf{c} \\
0 &= \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{c} \\
0 &= \mathbf{a}\cdot\big(\mathbf{b}-\mathbf{c}\big)
\end{aligned}$$

Therefore, $\mathbf{a}$ is perpendicular to the vector $\mathbf{b}-\mathbf{c}$. Manipulation of the cross product yields

$$\begin{aligned}
\mathbf{a}\times\mathbf{b} &= \mathbf{a}\times\mathbf{c} \\
0 &= \mathbf{a}\times\mathbf{b} - \mathbf{a}\times\mathbf{c} \\
0 &= \mathbf{a}\times\big(\mathbf{b}-\mathbf{c}\big)
\end{aligned}$$

Therefore, $\mathbf{a}$ is also parallel to the vector $\mathbf{b}-\mathbf{c}$. Therefore, $\mathbf{b}-\mathbf{c} = \mathbf{0}$, or $\mathbf{b} = \mathbf{c}$.

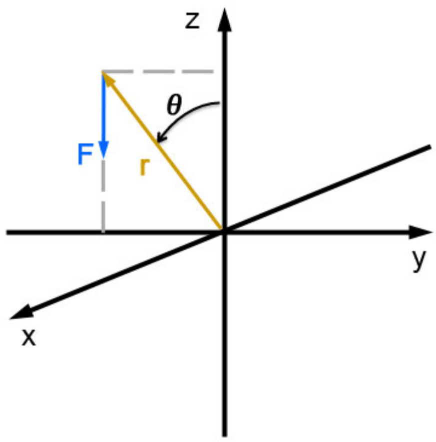
2.1.5. (a) This statement is not meaningless. This is a dot product of two vectors.

(b) This statement is meaningless. We cannot take the dot product of a vector with a scalar.

(c) This statement is meaningless. We cannot take the cross product of a vector with a scalar.

(d) This statement is not meaningless. This is a cross product of two vectors.

2.1.6. For this problem we will use the coordinate system in the diagram below.



(a) Using the above coordinate system, we are given that $||\mathbf{r}|| = 0.14$ and $\mathbf{F} = -100\mathbf{k}$. Also,

$$\begin{aligned}\mathbf{r} &= -\sin(30°)||\mathbf{r}||\mathbf{j} + \cos(30°)||\mathbf{r}||\mathbf{k} \\ &= -0.07\mathbf{j} + 0.07\sqrt{3}\mathbf{k} \\ &= 0.07\big(-\mathbf{j} + \sqrt{3}\mathbf{k}\big)\end{aligned}$$

To compute the torque when $\theta = 30°$, we use the cross product:

$$0.07\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & \sqrt{3} \\ 0 & 0 & -100\end{vmatrix} = 7\mathbf{i}$$

Thus, the magnitude of the torque $\boldsymbol{\tau}$ is 7 N·m. A similar calculation for $\theta = 90°$ yields a magnitude of 14 N·m

(b) Using a similar calculation from above,

$$\begin{aligned}\mathbf{r} &= -\sin\theta||\mathbf{r}||\mathbf{j} + \cos\theta||\mathbf{r}||\mathbf{k} \\ &= -0.14\sin\theta\mathbf{j} + 0.14\cos\theta\mathbf{k}\end{aligned}$$

To compute the torque when $\theta = 30°$, we use the cross product:

$$0.14\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -\sin\theta & \cos\theta \\ 0 & 0 & -100\end{vmatrix} = 14\sin\theta\mathbf{i}$$

Thus, the magnitude of the torque $\boldsymbol{\tau}$ is $||14\sin\theta\mathbf{i}||\mathrm{N}\cdot\mathrm{m} = 14|\sin\theta|$N·m.

(c) The magnitude is minimized when $\theta = 0$ or $180°$. This corresponds to angles when $\mathbf{F}$ and $\mathbf{r}$ are parallel (or anti-parallel). The cross product of two vectors that are parallel (or anti-parallel) is zero.

2.1.7. (a) We can simplify the left-hand side of the equation by computing the determinant.

$$\begin{aligned}\begin{vmatrix}3 & 2 & 0 \\ 1 & x & 0 \\ 7 & -3 & 4\end{vmatrix} &= 3\begin{vmatrix}x & 0 \\ -3 & 4\end{vmatrix} - 2\begin{vmatrix}1 & 0 \\ 7 & 4\end{vmatrix} + 0\begin{vmatrix}1 & x \\ 7 & -3\end{vmatrix} \\ &= 3(4x-0) - 2(4-0) + 0(-3-7x) \\ &= 12x - 8\end{aligned}$$

Substituting this result yields the equation $12x - 8 = 4$, from which the solution $x = 1$ is easily recovered.

(b) Expand both determinants and rearrange the equation:

$$\begin{aligned}(x)(4x) - (1)(4) &= (-x)(2x+8) - (-2)(2)\\ 4x^2 - 4 &= -2x^2 - 8x + 4\\ 6x^2 + 8x - 8 &= 0\\ (6x-4)(x+2) &= 0\end{aligned}$$

Therefore the solutions are $x = 2/3$ and $x = -2$.

2.2 Lines and Planes

2.2.1. (a) A vector parallel to the required line is given by $\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix}$. If we like, for simplicity, we can normalize this vector by dividing each component by 4 to obtain the vector $[1, -1, 0]^T$. Since a point on the desired line is (-1,2,1), a vector equation is given by

$$\mathbf{r} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

The solution to this question is not unique. Any vector parallel to the vector $\begin{bmatrix} 4 \\ -4 \\ 0 \end{bmatrix}$ could be used.

(b) A vector equation is

$$\begin{aligned} \mathbf{r} &= (2 + 2t)\mathbf{i} + (9t)\mathbf{j} + (5 + t)\mathbf{k} \\ &= (2\mathbf{i} + 5\mathbf{k}) + t(2\mathbf{i} + 9\mathbf{k} + \mathbf{k}) \\ &= \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} + t \begin{bmatrix} 2 \\ 9 \\ 1 \end{bmatrix} \end{aligned}$$

2.2.2. (a) Let $\mathbf{A} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$. Then

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= \big((-1)(0)\big) + \big((0)(-2)\big) + \big((1)(2)\big) \\ &= 0 + 0 + 2 \\ &= 2 \end{aligned}$$

Also,

$$\begin{aligned} ||\mathbf{A}|| &= \sqrt{(-1)^2 + 0 + (1)^2} = \sqrt{2} \\ ||\mathbf{B}|| &= \sqrt{0^2 + (-2)^2 + 2^2} = \sqrt{8} \end{aligned}$$

Therefore, if θ is the angle between the two vectors,

$$\cos\theta = \frac{\mathbf{A} \cdot \mathbf{B}}{||\mathbf{A}||\ ||\mathbf{B}||} = \frac{2}{4} = \frac{1}{2}$$

Therefore, $\theta = \pi/3$.

(b) Let the vector perpendicular to the plane be $\mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$. The angle, θ, between the given vector and $\mathbf{y}$ is found by solving

$$\begin{aligned} \cos\theta &= \frac{\mathbf{x} \cdot \mathbf{y}}{||\mathbf{x}||\ ||\mathbf{y}||} \\ &= \frac{\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}}{\sqrt{5}\sqrt{6}} \\ &= \frac{0}{\sqrt{5}\sqrt{6}} \\ &= 0. \end{aligned}$$

The angle, θ, is $\pi/2$. But θ is the angle between $\mathbf{x}$ and $\mathbf{y}$ and we need the angle between $\mathbf{x}$ and the plane, which is $\pi/2 - \theta = 0$. Therefore, the desired angle is 0 (the given vector is parallel to the plane).

2.2.3. **Distance Between a Point and a Line**
The formula we can apply is:

$$D = \frac{||\mathbf{v} \times \mathbf{w}||}{||\mathbf{v}||}$$

where $\mathbf{v}$ is a vector parallel to the given line, and is

$$\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The vector $\mathbf{w}$ is

$$\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Then,

$$\begin{aligned} D &= \frac{||\mathbf{v} \times \mathbf{w}||}{||\mathbf{v}||} \\ &= \frac{||[-1, 0, 0]^T||}{||[0, 1, 0]^T||} \\ &= \frac{1}{1} \\ &= 1 \end{aligned}$$

2.2.4. **Distance Between a Point and a Plane**
The formula we can apply is:

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

where a, b, c are the components of the normal vector to the plane, so that

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

The point $(x_0, y_0, z_0) = (-1, C, 1)$, $d = -3$, and $D = \sqrt{6}$. Thus,

$$\begin{aligned} D = \sqrt{6} &= \frac{|2(-1) + C(1) + (1)(-1) - 3|}{\sqrt{6}} \\ &= \frac{|C - 6|}{\sqrt{6}} \\ 1 &= |C - 6| \end{aligned}$$

Therefore, $C = 5, 7$.

2.2.5. (a) We can start by finding the plane that contains the three points. The vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\overrightarrow{PQ} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\overrightarrow{PR} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -4 \end{bmatrix}$$

A vector perpendicular to the plane that contains the three points is found by calculating the cross product between these two vectors:

$$\begin{aligned}\overrightarrow{PQ} \times \overrightarrow{PR} &= (-8-(0))\mathbf{i} - (-4-0)\mathbf{j} + (0-(-2))\mathbf{k} \\ &= \begin{bmatrix} -8 \\ 4 \\ 2 \end{bmatrix}\end{aligned}$$

Any vector parallel to this vector is perpendicular to the plane that contains the given points.

(b) To find the area, A, of triangle $\triangle PQR$, we can use a cross product.

$$\begin{aligned}A &= \frac{1}{2}||\overrightarrow{PQ} \times \overrightarrow{PR}|| \\ &= \frac{1}{2}||[-8,4,2]^T|| \\ &= \frac{1}{2}\sqrt{(-8)^2+4^2+2^2} \\ &= \frac{1}{2}\sqrt{84} \\ &= \sqrt{21}\end{aligned}$$

(c) If the point S is such that $\overrightarrow{PQ}$ is parallel to $\overrightarrow{QS}$, then there will be an infinite number of planes that pass through these three points. Such a point, S, can be determined by multiplying $\overrightarrow{PQ}$ by a constant. Choosing 2 as a constant, then

$$\overrightarrow{QS} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

We determine the components of S as

$$S = \begin{bmatrix} 2-2 \\ 4-2 \\ 0-3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

2.2.6. **Equations of Planes**

(a) Let the points P, Q, R be

$$\begin{aligned}P &= (-1,2,1) \\ Q &= (3,-2,1) \\ R &= (-1,1,-1)\end{aligned}$$

Vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\begin{aligned}\overrightarrow{PQ} &= \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} - (-1,2,1) = (4,-4,0) \\ \overrightarrow{PR} &= \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}\end{aligned}$$

A vector orthogonal to the plane that contains the three points is found by calculating the cross product between these two vectors:

$$\begin{aligned}\overrightarrow{PQ} \times \overrightarrow{PR} &= (8-0)\mathbf{i} - (-8-0)\mathbf{j} + (-4-0)\mathbf{k} \\ &= (8,8,-4)\end{aligned}$$

The equation of the desired plane, using the point-normal form, is

$$\begin{aligned} 0 &= 8(x+1) + 8(y-2) + (-4)(z-1) \\ 4 &= 8x + 8y - 4z \end{aligned}$$

(b) The points $(0,0,0)$ and $(1,1,1)$ are on the given line. Therefore, the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is parallel to the desired plane. Another vector parallel to the desired plane is

$$\begin{bmatrix}-1\\2\\1\end{bmatrix} - \begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}-1\\2\\1\end{bmatrix}.$$

Therefore, a vector perpendicular to the desired plane is

$$\begin{bmatrix}1\\1\\1\end{bmatrix} \times \begin{bmatrix}-1\\2\\1\end{bmatrix} = (1-2)\mathbf{i} - (1+1)\mathbf{j} + (2+1)\mathbf{k} = \begin{bmatrix}-1\\-2\\3\end{bmatrix}.$$

The equation of the desired plane, using the point-normal form, is

$$\begin{aligned} 0 &= -1(x+1) - 2(y-2) + 3(z-1) \\ 0 &= -x - 2y + 3z. \end{aligned}$$

2.2.7. Let the normal vector of the first plane be $\mathbf{n}_1$, and the normal vector of the second plane be $\mathbf{n}_2$. Then

$$\mathbf{n}_1 = \begin{bmatrix}2\\1\\-1\end{bmatrix}, \ \mathbf{n}_2 = \begin{bmatrix}1\\3\\1\end{bmatrix}$$

The line that intersects these two planes is a line that is in both of the planes, and therefore must be perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$. Therefore, the line we need is parallel to the vector, $\mathbf{a}$, given by the cross product

$$\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2 = (4, -3, 5).$$

This vector is parallel to the desired plane. To find a second vector in the desired plane, we can find a line from the given point to any point in the line of intersection of the two given planes. Letting $y = 0$, we obtain the equations

$$\begin{aligned} 2x - z &= 3 \\ x + z &= 0 \end{aligned}$$

which has the solution $x = 1, z = -1$. Therefore, the point $(1, 0, -1)$ is in the intersection of the two planes, and a vector in the plane is

$$\begin{bmatrix}1\\0\\-1\end{bmatrix} - \begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix}.$$

A normal vector to the desired plane is

$$\begin{aligned} [1,0,-1]^T \times [4,-3,5]^T &= (0-3)\mathbf{i} - (5+4)\mathbf{j} + (-3-0)\mathbf{k} \\ &= -3\mathbf{i} - 9\mathbf{j} - 3\mathbf{k} \end{aligned}$$

The equation of the desired plane, using the point-normal form, is

$$\begin{aligned} 0 &= (-3)(x-0) + (-9)(y-0) + (-3)(z-0) \\ &= -3x - 9y - 3z \end{aligned}$$

2.2.8. (a) We will show that the two lines must be equal to each other, and therefore share an infinite number of points. Let P be the common point, and $\mathbf{r}$ be the vector pointing from the origin to the point P. Let

$$L_1 = \mathbf{r} + t\mathbf{u}$$
$$L_2 = \mathbf{r} + t\mathbf{v}$$

where $t \in \mathbb{R}$, and $\mathbf{u}$ and $\mathbf{v}$ are vectors parallel to the lines L_1 and L_2 respectively. Let their components be

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

When the lines intersect, their components are equal, which gives us the three equations

$$r_1 + tu_1 = r_1 + tv_1$$
$$r_2 + tu_2 = r_2 + tv_2$$
$$r_3 + tu_3 = r_3 + tv_3$$

Or simply

$$tu_1 = tv_1$$
$$tu_2 = tv_2$$
$$tu_3 = tv_3$$

Because L_1 and L_2 are parallel, vectors $\mathbf{u}$ and $\mathbf{v}$ must also be parallel. Therefore $\mathbf{u} = k\mathbf{v}$ where k is a constant, so

$$tu_1 = tku_1$$
$$tu_2 = tku_2$$
$$tu_3 = tku_3$$

These equations are only satisfied if $k = 1$. Therefore, the two lines must be equal to each other, implying that there an infinite number of points that the two lines share.

(b) Yes. Any plane in $\mathbb{R}^3$ can be expressed in point-normal form

$$a(x - x_0) - b(y - y_0) + c(z - z_0) = 0$$

where the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a vector normal to the plane.

2.3 Surfaces

2.3.1. **Identifying Surfaces**

(a) If $ab > 0$, then a and b have the same sign. If they are both positive, the surface is an elliptic paraboloid that opens upward. If a and b are both negative, the surface is an elliptic paraboloid that opens downward.

(b) If $ab < 0$, then a and b have opposite signs (i.e. - one is positive, the other is negative). The surface is a hyperbolic paraboloid.

(c) If $a = b = 0$, then $z = 0$, which is the equation of the xy-plane.

2.3.2. **Finding the Intersection Between Two Surfaces**
The sphere $x^2 + y^2 + z^2 = 9$ and the cylinder $x^2 + y^2 = 4$ intersect at the values of x and y that are satisfied by both equations. We could find these values by substituting the equation for the cylinder into the equation for the sphere:

$$\begin{aligned} 9 &= x^2 + y^2 + z^2 \\ &= 4 + z^2 \\ z &= \pm\sqrt{5} \end{aligned}$$

Thus, the two curves intersect at $z = \sqrt{5}$ and at $z = -\sqrt{5}$. Using the equation of the sphere, we can obtain an equation for the curves that describe the intersections.

$$\begin{aligned} 9 &= x^2 + y^2 + (\pm\sqrt{5})^2 \\ &= x^2 + y^2 + 5 \\ 4 &= x^2 + y^2 \end{aligned}$$

The intersections at $z = \pm\sqrt{5}$ are described by the circle $x^2 + y^2 = 4$.

2.3.3. **Finding the Trace of a Surface**
The xy-plane is described by the equation $z = 0$. Substituting $z = 0$ into the equation for the hyperbolic paraboloid gives us

$$0 = \frac{x^2}{a^2} - \frac{y^2}{b^2},$$

or simply

$$y^2 = \left(\frac{b}{a}\right)^2 x^2.$$

This equation is only satisfied by two lines:

$$y = \frac{bx}{a}, \text{ and } y = -\frac{bx}{a}.$$

These two lines describe the trace of the hyperbolic parabolic in the xy-plane.

2.3.4. **Deriving the Equation of a Surface**
The distance, d_1, from the point (x, y, z) and the point $(0, 1, 0)$ is given by

$$d_1 = \sqrt{x^2 + (y-1)^2 + z^2}.$$

The distance, d_2, from the point (x, y, z) and the given plane is

$$d_2 = \left|z - 1\right|.$$

The points that are equidistant from the point $(0, 1, 0)$ and the given plane are the points that such that $d_1 = d_2$. Therefore,

$$\begin{aligned}
d_1 &= d_2 \\
\sqrt{x^2 + (y-1)^2 + z^2} &= \left|z - 1\right| \\
x^2 + (y-1)^2 + z^2 &= (z-1)^2 \\
x^2 + (y-1)^2 + z^2 &= z^2 - 2z + 1 \\
x^2 + (y-1)^2 &= -2z + 1 \\
z &= -\frac{1}{2}\Big(x^2 + (y-1)^2 - 1\Big)
\end{aligned}$$

The surface is an elliptic paraboloid that opens downward and whose vertex is located at the point $(0, 1, 1/2)$.

2.4 Curvilinear Coordinates

2.4.1. **Finding the Intersection Between Two Surfaces in Spherical Coordinates**
The set of all points that satisfy $\theta = \pi$ lie on the xz-plane. The set of all points that satisfy $\phi = \pi$ lie on the z-axis. The z-axis lies in the xz-plane. In other words, the line $\phi = \pi$ lies in the plane $\theta = \pi$.

2.4.2. **Converting The Equations of Surface into Cartesian Coordinates**

(a) The equation we are given is $\rho = -2a \sin\phi \cos\theta$. But

$$\rho = \sqrt{x^2 + y^2 + z^2},$$

and

$$x = \rho \sin\phi \cos\theta.$$

Substituting these two equations into $\rho = -2a \sin\phi \cos\theta$ yields

$$\begin{aligned} \rho &= -2a\frac{x}{\rho}, \quad \rho \neq 0 \\ x^2 + y^2 + z^2 &= -2ax \\ 0 &= x^2 + 2ax + y^2 + z^2 \\ &= x^2 + 2ax + (a^2 - a^2) + y^2 + z^2 \\ &= (x+a)^2 - a^2 + y^2 + z^2 \\ a^2 &= (x+a)^2 + y^2 + z^2 \end{aligned}$$

This is the equation of a sphere of radius $|a|$, centered at the point $(-a, 0, 0)$.

(b) Multiplying both sides of the equation by r yields

$$\begin{aligned} r &= 2\cos\theta \\ r^2 &= 2r\cos\theta \\ x^2 + y^2 &= 2x \\ x^2 - 2x + (1-1) + y^2 &= 0 \\ (x-1)^2 + y^2 &= 1 \end{aligned}$$

This is the equation of a cylinder with radius 1, whose axis is the vertical line in $\mathbb{R}^3$, $x = 1$, $y = 0$.

2.4.3. **Describing Solids in Cylindrical Coordinates**

(a) In cylindrical coordinates, $z \geq x^2 + y^2$ becomes $z \geq r^2$. The inequality $z \leq \sqrt{2 - x^2 - y^2}$ becomes $z \leq \sqrt{2 - r^2}$, or $r^2 + z^2 \leq 2$. The solid is the region outside the elliptic paraboloid $z = r^2$, and inside the sphere $r^2 + z^2 = 2$.

(b) In cylindrical coordinates, $x^2 + y^2 + z^2 \leq 4$ becomes $r^2 + z^2 \leq 4$. The inequality $x^2 + y^2 \geq 1$ becomes $r^2 \geq 1$. The sold is the region outside the cylinder of radius 1, and inside the sphere of radius 2.

2.5 Vector-Valued Functions

2.5.1. The derivative of $\mathbf{r}$ is simply $\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$.

(a) $\mathbf{r}(t) \perp \mathbf{r}'(t)$ implies

$$\begin{aligned}0 &= \mathbf{r}(t)\cdot\mathbf{r}'(t)\\ &= \big((1+t^2)\mathbf{i} + t\mathbf{j}\big)\cdot\big(2t\mathbf{i}+\mathbf{j}\big)\\ &= (2t+2t^3)+(t)\\ &= t(3+2t^2)\end{aligned}$$

This equation is satisfied only when $t = 0$ or $t^2 = -3/2$. The latter equation cannot be satisfied if $t \in \mathbb{R}$. Thus, the two vectors are only perpendicular when $t = 0$. Thus, $\mathbf{r}(t)$ is perpendicular to $\mathbf{r}'(t)$ at (0,1).

(b) If $\mathbf{r}(t)$ is pointing in the same direction as $\mathbf{r}'(t)$, then $\mathbf{r}(t) = C\mathbf{r}'(t)$, where C is a positive constant. Equating the components of the vectors yields the two equations

$$(1+t^2) = C2t, \text{ and } t = C.$$

If we substitute $t = C$ into the first equation, we find

$$\begin{aligned}(1+C^2) &= 2C^2\\ C &= +1,\end{aligned}$$

because C must be positive. Thus, the two vectors are pointing in the same direction when $t = 1$, which is at the point (2,1).

(c) The solution is the similar to part (b), except that C must be a negative constant. We find that the vectors point in opposite directions when $C = t = -1$, and at the point (2,-1).

2.5.2. (a) Differentiating the equation for angular momentum yields the torque:

$$\begin{aligned}\mathbf{L}(t) &= m\mathbf{r}(t)\times\mathbf{r}'(t)\\ \frac{d}{dt}\mathbf{L}(t) &= \frac{d}{dt}\Big(m\mathbf{r}(t)\times\mathbf{r}'(t)\Big)\\ &= m\mathbf{r}'(t)\times\mathbf{r}'(t) + m\mathbf{r}(t)\times\mathbf{r}''(t)\\ &= m\mathbf{0} + m\mathbf{r}(t)\times\mathbf{r}''(t)\\ &= m\mathbf{r}(t)\times\mathbf{r}''(t)\\ &= \boldsymbol{\tau}(t)\end{aligned}$$

(b) If each component of $\tau(t)$ is zero for all values of t, then each component of $\mathbf{L}'(t)$ is zero for all values of t. Each component of $\mathbf{L}(t)$ must therefore be constant.

2.5.3. (a) From $\mathbf{F} = m\mathbf{a} = m\mathbf{v}'(t)$ we obtain

$$\mathbf{v}(t) = c_1\mathbf{i} + \big(\pi\sin(\pi t) + c_2\big)\mathbf{j} + \big(-\pi\cos(\pi t) + c_3\big)\mathbf{k}$$

where c_1, c_2, c_3 are constants. When $t = 0$,

$$\begin{aligned}\mathbf{v}(0) &= c_1\mathbf{i} + \big(\pi\sin(0)+c_2\big)\mathbf{j} + \big(-\pi\cos(0)+c_3\big)\mathbf{k}\\ &= c_1\mathbf{i} + c_2\mathbf{j} + (-\pi + c_3)\mathbf{k}\end{aligned}$$

But $\mathbf{v}(0) = \mathbf{i}$, so by comparison, $c_1 = 1$, $c_2 = 0$, and $c_3 = \pi$. Thus

$$\mathbf{v}(t) = \mathbf{i} + \big(\pi\sin(\pi t)\big)\mathbf{j} + \big(-\pi\cos(\pi t) + \pi\big)\mathbf{k}$$

(b) Integrating the components of $\mathbf{v}(t)$ yields

$$\mathbf{r}(t) = (t + d_1)\mathbf{i} + \big(-\cos(\pi t) + d_2\big)\mathbf{j} + \big(-\sin(\pi t) + \pi t + d_3\big)\mathbf{k}$$

where d_1, d_2 and d_3 are constants. When $t = 0$,

$$\begin{aligned}\mathbf{r}(0) &= (0 + d_1)\mathbf{i} + \big(-\cos(0) + d_2\big)\mathbf{j} + \big(-\sin(0) + \pi(0) + d_3\big)\mathbf{k}\\ \mathbf{r}(0) &= d_1\mathbf{i} + \big(-1 + d_2\big)\mathbf{j} + \big(d_3\big)\mathbf{k}\end{aligned}$$

But $\mathbf{r}(0) = -\mathbf{i}$, so by comparison, $d_1 = -1$, $d_2 = 1$, and $d_3 = 0$. Thus

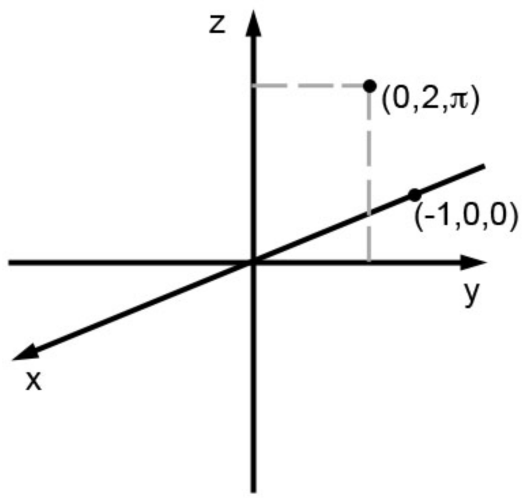
$$\mathbf{r}(t) = (t-1)\mathbf{i} + \big(-\cos(\pi t) + 1\big)\mathbf{j} + \big(-\sin(\pi t) + \pi t\big)\mathbf{k}$$

Thus

$$\begin{aligned}\mathbf{r}(1) &= (1-1)\mathbf{i} + \big(-\cos(\pi) + 1\big)\mathbf{j} + \big(-\sin(\pi) + \pi\big)\mathbf{k}\\ &= 2\mathbf{j} + \pi\mathbf{k}\end{aligned}$$

(c) At $t = 0$ and $t = 1$ we have

$$\mathbf{r}(0) = \begin{bmatrix}-1\\0\\0\end{bmatrix}, \quad \mathbf{r}(1) = \begin{bmatrix}0\\2\\\pi\end{bmatrix}.$$



2.6 Linear Systems

2.6.1. Use row-reduction operations on the augmented matrices.

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 3 & -1 & 1 & 2 & 5 \\ 0 & 1 & 0 & 2 & 0 \\ 4 & 0 & 3 & 1 & 10 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & -7 & 1 & -1 & 2 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & -8 & 3 & -3 & 6 \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow R_2 - 3R_1 \\ R_4 \leftarrow R_4 - 4R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 13 & 2 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 3 & 13 & 6 \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow R_2 + 7R_3 \\ R_4 \leftarrow R_4 + 8R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 13 & 2 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & -26 & 0 \end{array}\right] \quad R_4 \leftarrow R_4 - 3R_2$$

This is equivalent to the system of equations

$$\begin{aligned} x + 2y + w &= 1 \\ z + 13w &= 2 \\ y + 2w &= 0 \qquad (6) \\ -26w &= 0 \qquad (7) \end{aligned}$$

Equations (6) and (7) show that $w = 0$ and $y = 0$, and substitution yields $z = 2$ and $x = 1$.

2.6.2. (a) It's not possible to calculate the determinant of A, because A is not square. We can however solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$ using row operations:

$$\left[\begin{array}{cccc|c} -3 & 1 & 2 & 4 & 1 \\ 2 & -1 & 2 & 3 & 1 \\ 1 & 0 & 4 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 14 & -2 & 4 \\ 0 & -1 & -6 & 7 & -1 \\ 1 & 0 & 4 & -2 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \leftarrow R_1 + 3R_3 \\ R_2 \leftarrow R_2 - 2R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 14 & -2 & 4 \\ 0 & 0 & 8 & 5 & 3 \\ 1 & 0 & 4 & -2 & 1 \end{array}\right] \quad R_2 \leftarrow R_2 + R_1$$

This is equivalent to the system of equations

$$\begin{aligned} y + 14z - 2w &= 4 \qquad (8) \\ 8z + 5w &= 3 \qquad (9) \\ x + 4z - 2w &= 1 \qquad (10) \end{aligned}$$

Since there are fewer equations than unknowns we will parameterize the solution set by setting $w = t$, where t is any real number. From equation (9) we get $z = \frac{3}{8} - \frac{5}{8}t$. Substituting z into equations (10) and (8) yields $x = -\frac{1}{2} + \frac{9}{2}t$ and $y = -\frac{5}{4} + \frac{43}{4}t$.

Because of the free parameter t, there are infinitely many solutions to the given system.

(b) We can calculate the 3×3 determinant by expanding into 2×2 determinants:

$$\begin{aligned}\begin{vmatrix}1 & 2 & -1\\ 2 & 5 & 0\\ 4 & 9 & -2\end{vmatrix} &= 1\begin{vmatrix}5 & 0\\ 9 & -2\end{vmatrix} - 2\begin{vmatrix}2 & 0\\ 4 & -2\end{vmatrix} - 1\begin{vmatrix}2 & 5\\ 4 & 9\end{vmatrix}\\ &= (-10) - 2(-4) - (18-20)\\ &= -10+8+2\\ &= 0\end{aligned}$$

Applying one row operation yields

$$\left[\begin{array}{ccc|c}1 & 2 & -1 & 1\\ 2 & 5 & 0 & 5\\ 4 & 9 & -2 & 0\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 2 & -1 & 1\\ 2 & 5 & 0 & 5\\ 0 & 0 & 0 & -7\end{array}\right] \quad R_3 \leftarrow R_3 - 2R_1 - R_2$$

The last row is equivalent to

$$0x + 0y + 0z = -7,$$

which implies that this particular system has no solution.

2.6.3. We first convert the system of equations into an equivalent augmented-matrix form:

$$\left[\begin{array}{ccc|c}1 & 2 & -1 & 2\\ 2 & a & -2 & b\\ 3 & 2 & 0 & 1\end{array}\right].$$

Then apply a sequence of row-reduction operations:

$$\begin{array}{ll}\left[\begin{array}{ccc|c}1 & 2 & -1 & 2\\ 0 & a-4 & 0 & b-4\\ 0 & -4 & 3 & -5\end{array}\right] & \\ \left[\begin{array}{ccc|c}1 & 2 & -1 & 2\\ 0 & a & -3 & b+1\\ 0 & -4 & 3 & -5\end{array}\right] & \begin{array}{l}R_2 - 2R_1\\ R_3 - 3R_1\end{array}\\ \left[\begin{array}{ccc|c}1 & 2 & -1 & 2\\ 0 & a & -3 & b+1\\ 0 & 4 & -3 & 5\end{array}\right] & R_2 - R_3\\ \left[\begin{array}{ccc|c}1 & 2 & -1 & 2\\ 0 & 4 & -3 & 5\\ 0 & a & -3 & b+1\end{array}\right] & \begin{array}{l}-1\cdot R_3\\ R_2 \leftrightarrow R_3\end{array}\end{array} \tag{11}$$

Suppose that $a = 4$. Then (11) becomes

$$\begin{array}{ll}\left[\begin{array}{ccc|c}1 & 2 & -1 & 2\\ 0 & 4 & -3 & 5\\ 0 & 4 & -3 & b+1\end{array}\right] & \\ \left[\begin{array}{ccc|c}1 & 2 & -1 & 2\\ 0 & 4 & -3 & 5\\ 0 & 0 & 0 & b-4\end{array}\right] & R_3 - R_2\end{array} \tag{12}$$

If $b \neq 4$, then the system has no solutions, since the last row is equivalent to the equation $0 = b-4$ where $b-4 \neq 0$. Conversely, if $b = 4$, then (12) is equivalent to the system of equations

$$\begin{aligned}x + 2y - z &= 2\\ 4y - 3z &= 5\end{aligned}$$

There are fewer equations than unknowns, so we have to parameterize the solution set; let $z = t$. Solving these two equations yields the solution

$$\begin{aligned} x &= -\frac{1}{2} - \frac{1}{2}t \\ y &= \frac{5}{4} + \frac{3}{4}t \\ z &= t \end{aligned} \tag{13}$$

Now suppose that $a = 0$. Then the matrix (11) is equivalent to the system

$$\begin{aligned} x + 2y - z &= 2 \\ 4y - 3z &= 5 \\ -3z &= b + 1 \end{aligned}$$

Using back-substitution we find that

$$\begin{aligned} x &= -\frac{1}{3} - \frac{1}{6}b \\ y &= 1 - \frac{1}{4}b \\ z &= \frac{-1}{3} - \frac{1}{3}b \end{aligned} \tag{14}$$

Now suppose that $a \neq 4$ and $a \neq 0$. Then we can continue row- reducing from (11):

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 4 & -3 & 5 \\ 0 & a & -3 & b+1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 4 & -3 & 5 \\ 0 & 0 & -3 + \frac{3}{4}a & 1 + b - \frac{5}{4}a \end{array}\right] \quad R_3 \leftarrow R_3 - \frac{a}{4}R_2$$

This is equivalent to the system

$$\begin{aligned} x + 2y - z &= 2 \\ 4y - 3z &= 5 \\ (-3 + \frac{3}{4}a)z &= 1 + b - \frac{5}{4}a \end{aligned}$$

Let $c = \dfrac{1 + b - \frac{5}{4}a}{-3 + \frac{3}{4}a}$. Using back-substitution we find the solution

$$\begin{aligned} x &= -\frac{1}{2} - \frac{1}{2}c \\ y &= \frac{5}{4} + \frac{3}{4}c \\ z &= c \end{aligned} \tag{15}$$

Thus we have shown that:

(a) When $a = 4$ and $b = 4$ there are infinitely many solutions, given by the parameterization (13).

(b) When $a = 4$ and $b \neq 4$ there are no solutions.

(c) When $a \neq 4$ there is one solution. When $a = 0$, the solution (14) is equivalent to that of (15). So the solution when $a \neq 4$ is given by (15).

2.6.4. (a) We can write the system as

$$\begin{aligned}
a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 &= 0 \\
a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3 &= 5.5 \\
a_0 + a_1(2) + a_2(2)^2 + a_3(2)^3 &= 20 \\
a_0 + a_1(3) + a_2(3)^2 + a_3(3)^3 &= 46.5 \\
a_0 + a_1(4) + a_2(4)^2 + a_3(4)^3 &= 88
\end{aligned}$$

Equivalently, we can write this as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5.5 \\ 20 \\ 46.5 \\ 88 \end{bmatrix}$$

(b) Solving the above system yields

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 0.5 \end{bmatrix}$$

(c) $p(1.5) = 2(1.5) + 3(1.5)^2 + 0.5(1.5)^3 = 11.4375$

(d) The system for a polynomial of lower order would have no solution because $a_3 \neq 0$. For example, if we used a linear function, the system becomes

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5.5 \\ 20 \\ 46.5 \\ 88 \end{bmatrix}$$

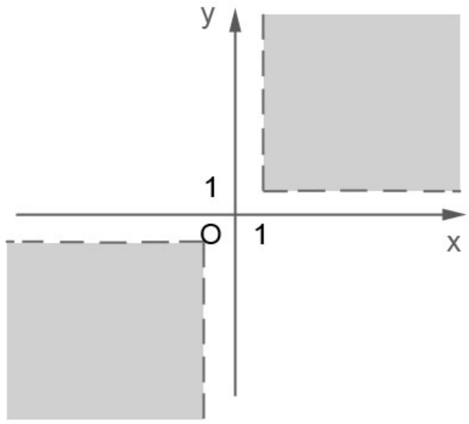
This system has no solution.

2.7 Functions of Two or Three Variables

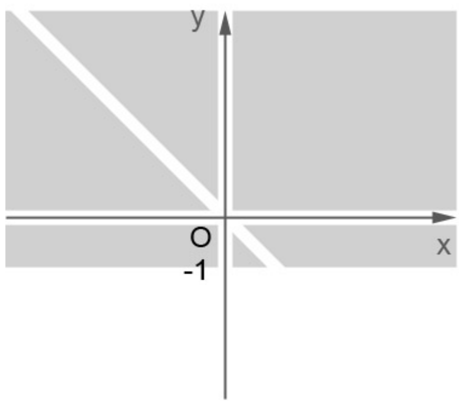
2.7.1. (a) The function is not defined when the denominator is zero, or when the argument of the square root is negative. Either x and y are greater than 1, or less than -1. The domain, D, is therefore

$$D = \{(x, y)|x > 1, y > 1; \text{ and } x < -1, y < -1\}.$$

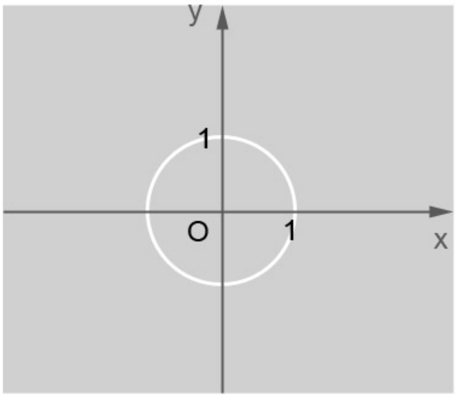
The function can't be zero, and cannot be negative, because the square root will always yield a positive number. The range is $f > 0$.



(b) The function is not defined when the denominator is zero, so we require that $yx^2+xy^2 \neq 0$, or $xy(x+y) \neq 0$. This implies that the function is not defined along the lines $x = 0$, $y = 0$, and $y = -x$. Moreover, the numerator has a square root. The argument must be non-negative, and so we also have the restriction that $y \geq -1$. The domain can therefore be expressed as $D = \{(x, y)|y \geq -1, x \neq 0, y \neq 0, y \neq -x\}$. The numerator can take on any non-negative value, and the denominator can take on any value, so the function f can take on any value, so the range is $\mathbb{R}$.



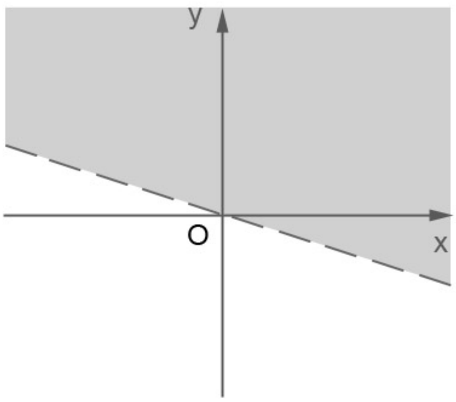
(c) The function is not defined when the denominator is zero, so we require that $x^2 + y^2 \neq 1$. The domain can therefore be expressed as $D = \{(x, y)|x^2+y^2 \neq 1\}$. The numerator can take on any value, so the function f can take on any value, so the range is $\mathbb{R}$.



(d) For the domain, we require that

$$x + 2y > 0, \text{ or } y > -x/2.$$

The domain can therefore be expressed as $D = \{(x, y)|y > -x/2\}$. The function f can take on any value, so the range is $\mathbb{R}$.



2.7.2. (a) We can simply evaluate the limit to obtain

$$\lim_{(r,s)\to(0,2\pi)} \frac{3r^2 + rs^3 - 3\sin(s/4)}{r^2} = \lim_{(r,s)\to(0,2\pi)} 3 + \frac{s^3}{r} - 3r^2 \sin(s/4)$$

Because of the s^3/r term, this limit tends to infinity, and therefore does not exist.

(b) Let $f(x, y) = (x - y)/(x + y)$. Along the x-axis, $f = f(x, 0)$, so

$$f(x, 0) = \frac{x}{x} = 1, \quad x \neq 0.$$

A similar calculation shows that along the y-axis, f = -1, if $y \neq 0$. If we approach the point $(0, 0)$ along the x-axis and the y-axis, we find that

$$\text{along the } x\text{-axis, } f = f(x, 0), \text{ and } f \to +1$$
$$\text{along the } y\text{-axis, } f = f(0, y), \text{ and } f \to -1$$

These limits are not equal, and so the limit does not exist.

(c) We can simply evaluate the limit to obtain

$$\lim_{(x,y,z)\to(1,1,1)} \left|3x - 2y - z\right| = 3 - 2 - 1 = 0$$

(d) Let $f(x,y,z) = (x^2 - y^2 - z^2)/(x^2 + y^2 + z^2)$. Then, along the x-axis, $f = (x,0,0)$. As long as x is not zero, f is equal to 1. However, along the y-axis, $f = f(0,y,0)$. Provided that y is not zero, f is equal to -1. Therefore,

$$\begin{aligned}\text{along the } x\text{-axis, } f &\to +1\\ \text{along the } y\text{-axis, } f &\to -1\end{aligned}$$

These limits are not equal, and so the limit does not exist.

(e) We can evaluate this limit by rationalizing the numerator.

$$\begin{aligned}
\lim_{(x,y)\to(1,0)} \frac{\sqrt{2x+y} - \sqrt{2x-y}}{2y} &= \lim_{(x,y)\to(1,0)} \frac{\sqrt{2x+y} - \sqrt{2x-y}}{2y}\left(\frac{\sqrt{2x+y} + \sqrt{2x-y}}{\sqrt{2x+y} + \sqrt{2x-y}}\right)\\
&= \lim_{(x,y)\to(1,0)} \frac{(2x+y) - (2x-y)}{2y\left(\sqrt{2x+y} + \sqrt{2x-y}\right)}\\
&= \lim_{(x,y)\to(1,0)} \frac{2y}{2y\left(\sqrt{2x+y} + \sqrt{2x-y}\right)}\\
&= \lim_{(x,y)\to(1,0)} \frac{1}{\sqrt{2x+y} + \sqrt{2x-y}}\\
&= \frac{1}{2\sqrt{2}}\\
&= \frac{\sqrt{2}}{2}
\end{aligned}$$

2.7.3. The given function is defined everywhere. Now, for it to be continuous at any point (x_0, y_0, z_0), we require that

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} g(x,y,z) = g(x_0,y_0,z_0)$$

If we evaluate the limit as g approaches $(0,0,1)$, we have

$$\begin{aligned}
\lim_{(x,y,z)\to(0,0,1)} f(x,y,z) &= \lim_{(x,y,z)\to(0,0,1)} x + y + 2z\\
&= 0 + 0 + 2(1)\\
&= 2
\end{aligned}$$

However, at $(0,0,1)$, $g = 2$. So the given function is not continuous at the point $(0,0,1)$. Elsewhere, the function is a polynomial in three variables, and so will be continuous everywhere except at the point $(0,0,1)$.

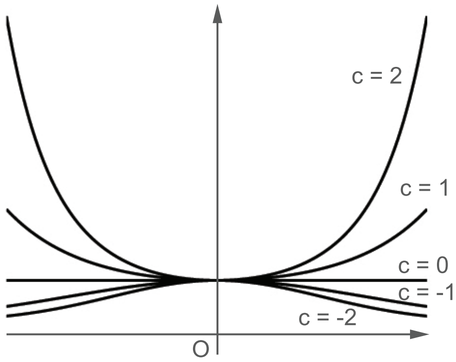
2.7.4. The lines $y = m(x-1)$ all pass through the limit point, $(1,0)$, where m is any real number. Approaching the limit point along these lines, our limit becomes

$$\begin{aligned}
\lim_{(x,y)\to(1,0)} \frac{x(x-1)^3 + y^2}{4(x-1)^2 + 9y^3} &= \lim_{(x,y)\to(1,0)} \frac{x(x-1)^3 + m^2(x-1)^2}{4(x-1)^2 + 9m^3(x-1)^3}\\
&= \lim_{(x,y)\to(1,0)} \frac{x(x-1) + m^2}{4 + 9(x-1)}\\
&= \frac{m^2}{4}
\end{aligned}$$

The result depends on m, which is an arbitrary value. Therefore, the limit does not exist.

2.7.5. (a) To plot the level curves, we set $z = c$, and solve for y:

$$\begin{aligned} c &= \frac{\ln y}{x^2} \\ cx^2 &= \ln y \\ y &= e^{cx^2} \end{aligned}$$



(b) To plot the level curves, we set $z = c$, and solve for y:

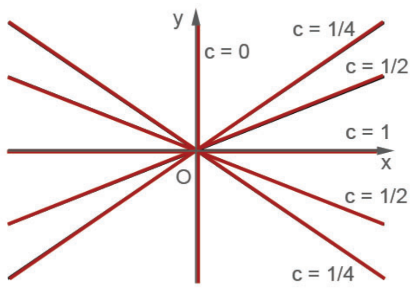
$$\begin{aligned} c &= \frac{x^2}{x^2 + y^2} \\ cy^2 &= (1 - c)x^2 \end{aligned}$$

If $c = 0$, then we obtain the level curve $x = 0$. If $c \neq 0$, then

$$y = \pm\sqrt{\frac{1-c}{c}}x, \quad c \neq 0$$

For $c = 2$, y is undefined, so the level curves do not exist. The level curves for the other values of c are shown in the graph below and are as follows:

if $c = 0, \quad x = 0$ (the vertical line that lies on the y-axis)
if $c = 1/4, y = \pm\sqrt{3}x$
if $c = 1/2, y = \pm x$
if $c = 1, \quad y = 0$



2.7.6. (a) The set of straight lines that pass through $(2, 1)$ are given by $y = c(x - 2) + 1$. Rearranging yields the equation

$$c = \frac{y-1}{x-2}$$

The desired function is

$$f(x,y) = \frac{y-1}{x-2}$$

(b) The set of circles with radius e^c are given by $x^2 + y^2 = e^c$. Applying the natural logarithm to both sides of the equation yields

$$\ln(x^2 + y^2) = c$$

The desired function is

$$f(x,y) = \ln(x^2 + y^2)$$

2.7.7. (a) Suppose we let

$$f(x,y) = \frac{y}{(x-1)^2}$$

Then, along any parabola $y = m(x-1)^2$ the limit becomes

$$\lim_{(x,y)\to(1,0)} \frac{y}{(x-1)^2} = \lim_{(x,y)\to(1,0)} \frac{m(x-1)^2}{(x-1)^2} = m$$

The function we chose therefore meets the specified criteria.

(b) The limit cannot exist because the limit depends on m, which is an arbitrary real number.

2.7.8. (a) The temperature can be written as

$$T(x,y) = \frac{k}{\sqrt{x^2+y^2}},$$

where k is an unknown constant of proportionality.

(b) The level curves are solution sets of the equation $T(x,y) = c$, for $c \in \mathbb{R}$. Therefore,

$$\begin{aligned} T(x,y) = c &= \frac{k}{\sqrt{x^2+y^2}} \\ c^2 &= \frac{k^2}{x^2+y^2} \\ x^2 + y^2 &= \frac{k^2}{c^2} \end{aligned}$$

The level curves are concentric circles with radius k/c.

(c) At the point (2,1), the temperature is known, which allows us to solve for k.

$$\begin{aligned} T(2,1) = 10 &= \frac{k}{\sqrt{2^2+1^2}} \\ k &= 10\sqrt{5} \end{aligned}$$

At (1,3), the temperature is

$$\begin{aligned} T(1,3) &= \frac{10\sqrt{5}}{\sqrt{1^2+3^2}} \\ &= 10\sqrt{\frac{5}{4}} \end{aligned}$$

2.7.9. The level curves are solution sets of the equation $V(x,y) = K$, for constant $K \in \mathbb{R}$.

$$\begin{aligned} V(x,y) = K &= \frac{c}{\sqrt{r^2 - x^2 - y^2}} \\ r^2 - x^2 - y^2 &= \frac{c^2}{K^2} \\ x^2 + y^2 &= r^2 - \frac{c^2}{K^2} \end{aligned}$$

The left-hand side must be positive, so we must have that

$$\begin{aligned} r^2 &> \frac{c^2}{K^2} \\ K^2 &> c^2/r^2 \\ |K| &< |c|/|r| \end{aligned}$$

But c, K, and r are all positive constants, so this simplifies to $K < \frac{c}{r}$. The level curves are concentric circles, centered at the origin, with radius $r^2 - \frac{c^2}{K^2}$, for $K < \frac{c}{r}$.

2.7.10. If we approach the limit point along the x-axis, then $y = 0$ and we obtain

$$\lim_{(x,y)\to(0,0)} \frac{x(0)^2}{x^2 + (0)^2} = 0$$

We obtain the same result if we approach the origin along the y-axis, or along any line $y = mx$. It would seem that the limiting value could exist and could be equal to 0. To show that this is the case, we must show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left|\frac{xy^2}{x^2 + y^2} - 0\right| < \epsilon \text{ whenever } 0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta,$$

or simply

$$\left|\frac{xy^2}{x^2 + y^2}\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2 + y^2} < \delta.$$

Now, $\left|x^2 + y^2\right| = x^2 + y^2$, and $|y^2| = y^2$, so

$$\left|\frac{xy^2}{x^2 + y^2}\right| = \frac{\left|xy^2\right|}{\left|x^2 + y^2\right|} = \frac{\left|x\right|y^2}{x^2 + y^2}$$

Also, $|x| \leq \sqrt{x^2 + y^2}$, and $y^2 \leq x^2 + y^2$, so

$$\begin{aligned} \left|\frac{xy^2}{x^2 + y^2}\right| &= \frac{\left|x\right|y^2}{x^2 + y^2} \\ &\leq \frac{\sqrt{x^2 + y^2}(x^2 + y^2)}{x^2 + y^2} \\ &= \sqrt{x^2 + y^2} \end{aligned}$$

So if $\delta = \epsilon$, then

$$\left|\frac{xy^2}{x^2 + y^2}\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2 + y^2} < \delta.$$

3 Partial Derivatives

3.1 Partial Derivatives

3.1.1. **Calculating First and Second Partial Derivatives**

(a)

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(xy\sin(\frac{x}{y})) \\ &= y\sin(\frac{x}{y}) + xy\cos(\frac{x}{y})\frac{1}{y} \\ &= y\sin(\frac{x}{y}) + x\cos(\frac{x}{y})\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial f}{\partial x} \\ &= \frac{\partial}{\partial x}(y\sin(\frac{x}{y}) + x\cos(\frac{x}{y})) \\ &= y\cos(\frac{x}{y})\frac{1}{y} + \cos(\frac{x}{y}) - x\sin(\frac{x}{y})\frac{1}{y} \\ &= 2\cos(\frac{x}{y}) - \frac{x}{y}\sin(\frac{x}{y})\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \\ &= \frac{\partial}{\partial y}(y\sin(\frac{x}{y}) + x\cos(\frac{x}{y})) \\ &= \sin(\frac{x}{y}) + y\cos(\frac{x}{y})(\frac{-x}{y^2}) - x\sin(\frac{x}{y})(\frac{-x}{y^2}) \\ &= \sin(\frac{x}{y}) - \frac{x}{y}\cos(\frac{x}{y}) + \frac{x^2}{y^2}\sin(\frac{x}{y}) \\ &= \frac{x^2+y^2}{y^2}\sin(\frac{x}{y}) - \frac{x}{y}\cos(\frac{x}{y})\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(xy\sin(\frac{x}{y})) \\ &= x\sin(\frac{x}{y}) + xy\cos(\frac{x}{y})(\frac{-x}{y^2}) \\ &= x\sin(\frac{x}{y}) - \frac{x^2}{y}\cos(\frac{x}{y})\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\frac{\partial f}{\partial y} \\ &= \frac{\partial}{\partial x}(x\sin(\frac{x}{y}) - \frac{x^2}{y}\cos(\frac{x}{y})) \\ &= \sin(\frac{x}{y}) + x\cos(\frac{x}{y})(\frac{1}{y}) - \frac{2x}{y}\cos(\frac{x}{y}) + \frac{x^2}{y}\sin(\frac{x}{y})(\frac{1}{y}) \\ &= \frac{x^2+y^2}{y^2}\sin(\frac{x}{y}) - \frac{x}{y}\cos(\frac{x}{y}) \\ \nabla f &= \begin{bmatrix}\frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y}\end{bmatrix} \\ &= \begin{bmatrix} y\sin(\frac{x}{y}) + x\cos(\frac{x}{y}) \\ x\sin(\frac{x}{y}) - \frac{x^2}{y}\cos(\frac{x}{y})\end{bmatrix}\end{aligned}$$

(b)

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{x+y}{e^{xy}}\right)\\ &= \frac{e^{xy}-(x+y)ye^{xy}}{e^{2xy}}\\ &= \frac{1-xy-y^2}{e^{xy}}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial f}{\partial x}\\ &= \frac{\partial}{\partial x}\left(\frac{1-xy-y^2}{e^{xy}}\right)\\ &= \frac{e^{xy}(-y)-(1-xy-y^2)ye^{xy}}{e^{2xy}}\\ &= \frac{-2y+xy^2+y^3}{e^{xy}}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial y\partial x} &= \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\\ &= \frac{\partial}{\partial y}\left(\frac{1-xy-y^2}{e^{xy}}\right)\\ &= \frac{e^{xy}(-x-2y)-(1-xy-y^2)xe^{xy}}{e^{2xy}}\\ &= \frac{-2x-2y+x^2y+xy^2}{e^{xy}}\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{x+y}{e^{xy}}\right)\\ &= \frac{e^{xy}-(x+y)xe^{xy}}{e^{2xy}}\\ &= \frac{1-x^2-xy}{e^{xy}}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\frac{\partial f}{\partial y}\\ &= \frac{\partial}{\partial y}\left(\frac{1-x^2-xy}{e^{xy}}\right)\\ &= \frac{e^{xy}(-x)-(1-x^2-xy)xe^{xy}}{e^{2xy}}\\ &= \frac{-2x+x^3+x^2y}{e^{xy}}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x\partial y} &= \frac{\partial}{\partial x}\frac{\partial f}{\partial y}\\ &= \frac{\partial}{\partial x}\left(\frac{1-x^2-xy}{e^{xy}}\right)\\ &= \frac{e^{xy}(-2x-y)-(1-x^2-xy)ye^{xy}}{e^{2xy}}\\ &= \frac{-2x-2y+x^2y+xy^2}{e^{xy}}\end{aligned}$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{1 - xy - y^2}{e^{xy}} \\ \dfrac{1 - x^2 - xy}{e^{xy}} \end{bmatrix}$$

3.1.2. **Finding Tangent Planes**

Recall that the equation of the plane tangent to the surface defined by $g(x, y, z) = 0$ at the point (x_0, y_0, z_0) is $\nabla g(x_0, y_0, z_0) \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$.

(a) Define $g(x, y, z) = f(x, y) - z = \frac{x^2}{9} - y^2 - z$, so that the surface $z = f(x, y)$ is equivalently defined by the equation $g(x, y, z) = 0$. We can then find the equation of the tangent plane at the point $(2, \frac{1}{3}, \frac{1}{3})$ by computing ∇g and applying the above formula:

$$\nabla g(x, y, z) = \begin{bmatrix} \frac{2x}{9} \\ -2y \\ -1 \end{bmatrix}$$
$$\nabla g(2, \frac{1}{3}, \frac{1}{3}) = \begin{bmatrix} \frac{4}{9} \\ -\frac{2}{3} \\ -1 \end{bmatrix}$$

This yields the equation

$$\frac{4}{9}(x - 2) - \frac{2}{3}\left(y - \frac{1}{3}\right) - \left(z - \frac{1}{3}\right) = 0$$
$$\frac{4}{9}x - \frac{2}{3}y - z = \frac{1}{3}$$
$$4x - 6y - 9z = 3.$$

(b) Define $g(x, y, z) = x^2 + x + y - z^2 - 7$. Then compute

$$\nabla g(x, y, z) = \begin{bmatrix} 2x + 1 \\ 1 \\ -2z \end{bmatrix}$$
$$\nabla g(2, 5, 2) = \begin{bmatrix} 5 \\ 1 \\ -4 \end{bmatrix}.$$

Thus the equation of the tangent plane is

$$5(x - 2) + (y - 5) - 4(z - 2) = 0$$
$$5x + y - 4z = 7.$$

3.1.3. **Implicit Differentiation**

We must first compute the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$$\begin{aligned}
\frac{\partial}{\partial x}(xy) &= \frac{\partial}{\partial x}(z^{x+y}) \\
y &= \frac{\partial}{\partial x}\Big(e^{\ln(z)(x+y)}\Big) \\
y &= e^{\ln(z)(x+y)}\frac{\partial}{\partial x}\Big(\ln(z)(x+y)\Big) \\
y &= e^{\ln(z)(x+y)}\Big(\frac{1}{z}\frac{\partial z}{\partial x}(x+y)+\ln(z)\Big) \\
y &= z^{x+y}\Big(\frac{x+y}{z}\frac{\partial z}{\partial x}+\ln(z)\Big) \\
y &= \frac{\partial z}{\partial x}(x+y)z^{x+y-1}+z^{x+y}\ln(z) \\
\frac{\partial z}{\partial x} &= \frac{y-z^{x+y}\ln(z)}{(x+y)z^{x+y-1}}
\end{aligned}$$

The derivation for the partial derivative with respect to y is similar.

$$\begin{aligned}
\frac{\partial}{\partial y}(xy) &= \frac{\partial}{\partial y}(z^{x+y}) \\
x &= \frac{\partial}{\partial y}(e^{\ln(z)(x+y)}) \\
x &= e^{\ln(z)(x+y)}\frac{\partial}{\partial y}\big(\ln(z)(x+y)\big) \\
x &= e^{\ln(z)(x+y)}\Big(\frac{1}{z}\frac{\partial z}{\partial y}(x+y)+\ln(z)\Big) \\
x &= z^{x+y}\Big(\frac{x+y}{z}\frac{\partial z}{\partial y}+\ln(z)\Big) \\
x &= \frac{\partial z}{\partial y}(x+y)z^{x+y-1}+z^{x+y}\ln(z) \\
\frac{\partial z}{\partial y} &= \frac{x-z^{x+y}\ln(z)}{(x+y)z^{x+y-1}}
\end{aligned}$$

So $\nabla h(x,y) = \begin{bmatrix} \dfrac{y-z^{x+y}\ln(z)}{(x+y)z^{x+y-1}} \\ \\ \dfrac{x-z^{x+y}\ln(z)}{(x+y)z^{x+y-1}} \end{bmatrix}$.

3.1.4. **Calculating the Rate of Change of a Function of Two Variables**

(a) f will increase fastest in the direction of the vector $\nabla f(x_0,y_0)$ from the point (x_0,y_0). Therefore we compute and evaluate the gradient ∇f at the point $(1,1)$:

$$\begin{aligned}
\nabla f = \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \\ \dfrac{\partial f}{\partial y} \end{bmatrix} &= \begin{bmatrix} \dfrac{\partial}{\partial x}(e^{-(x^2+x+1+y^2-2y)}) \\ \\ \dfrac{\partial}{\partial y}(e^{-(x^2+x+1+y^2-2y)}) \end{bmatrix} \\
&= \begin{bmatrix} (-2x-1)e^{-(x^2+x+1+y^2-2y)} \\ (2y-2)e^{-(x^2+x+1+y^2-2y)} \end{bmatrix}
\end{aligned} \tag{1}$$

Therefore, the gradient at the point (1,1) is

$$\nabla f(1,1) = \begin{bmatrix} -3e^{-2} \\ 0 \end{bmatrix}.$$

(b) The rate of change of f in the direction of the vector $\mathbf{v}$ from the point (x, y) is given by $\nabla f(x, y) \cdot \mathbf{v}$. Using Equation (1) we find that $\nabla f(2, 1) = \begin{bmatrix} -5e^{-6} \\ 0 \end{bmatrix}$. The direction from $(2, 1)$ to $(5, 4)$ is given by the vector $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$; thus the rate of change of f from $(2, 1)$ towards $(5, 4)$ is $\begin{bmatrix} -5e^{-6} \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 3 \end{bmatrix} = -15e^{-6}$.

3.1.5. **Finding a Jacobian**

First we find the partial derivatives of each component with respect to each variable:

$$\begin{aligned} \frac{\partial f_1}{\partial x} &= \cos(x + y) \\ \frac{\partial f_1}{\partial y} &= \cos(x + y) \\ \frac{\partial f_2}{\partial x} &= \frac{1}{x} \\ \frac{\partial f_2}{\partial y} &= \frac{1}{y} \end{aligned}$$

The Jacobian matrix, then, is $\begin{bmatrix} \cos(x + y) & \cos(x + y) \\ \frac{1}{x} & \frac{1}{y} \end{bmatrix}$.

3.1.6. **The Two-Dimensional Heat Equation**

We only need to verify that Equation (1) holds for the given function u:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{K_0}{c\rho} \frac{\partial^2 u}{\partial x^2} \\ -\frac{K_0}{c\rho} e^{-tK_0/(c\rho)} \sin(x) &= \frac{K_0}{c\rho} \frac{\partial}{\partial x} \left(e^{-tK_0/(c\rho)} \cos(x) \right) \\ -\frac{K_0}{c\rho} e^{-tK_0/(c\rho)} \sin(x) &= -\frac{K_0}{c\rho} e^{-tK_0/(c\rho)} \sin(x) \end{aligned}$$

which is true. So $u(x, t) = e^{-tK_0/(c\rho)} \sin(x)$ is indeed a solution to the Heat Equation (1).

3.1.7. **Differentiability**

Consider the function $f(x, y) = |x| + 5y$.

- $f(0, 1)$ is defined. Indeed, $f(0, 1)$ is equal to 5.
- $f(x, y)$ is continuous at point $(0, 1)$, because

$$\lim_{(x,y)\to(0,1)} f(x, y) = f(0, 1) = 5$$

- The partial derivative with respect to x is defined as

$$\frac{\partial f}{\partial y}(a, b) = \frac{f(0 + h, y) - f(0, y)}{h}.$$

For this limit to exist, we need the limits as h approaches 0 from the negative and positive directions to exist and to be equal to each other. But

$$\lim_{h\to0^+} \frac{f(0 + h, y) - f(0, y)}{h} = \lim_{h\to0^+} \frac{\left(|0 + h| + 5y\right) - \left(|0| + 5y\right)}{h} = \frac{|h|}{h} = \frac{+h}{h} = +1,$$

and the limit as h approaches zero from the other direction is

$$\lim_{h\to0^-} \frac{f(0 + h, y) - f(0, y)}{h} = \lim_{h\to0^-} \frac{\left(|0 + h| + 5y\right) - \left(|0| + 5y\right)}{h} = \frac{|h|}{h} = \frac{-h}{h} = -1.$$

The two limits are not equal, and so the partial derivative with respect to x does not exist on the y-axis. Hence, the partial derivative does not exist at the point $(0, 1)$.

3.1.8. **A Second Order Derivative**

The statement is false. Consider the function $f(x,y) = xy$. Then

$$\frac{\partial f}{\partial y}\frac{\partial f}{\partial x} = (x)(y) = xy \neq \frac{\partial^2 f}{\partial x \partial y} = 1.$$

3.1.9. **The Rate of Change in the Direction $f_y\mathbf{i} - f_x\mathbf{j}$**

(a) Let $\mathbf{v}$ be the vector

$$\mathbf{v} = \frac{\partial f}{\partial y}(a,b)\mathbf{i} - \frac{\partial f}{\partial x}(a,b)\mathbf{j}.$$

The rate of change of f in the direction of the given vector $\mathbf{v}$ at the point (a,b) is given by $\nabla f(a,b) \cdot \mathbf{v}$, which is

$$\begin{aligned}\nabla f(a,b) \cdot \left(\frac{\partial f}{\partial y}(a,b)\mathbf{i} - \frac{\partial f}{\partial x}(a,b)\mathbf{j}\right) &= \left(\frac{\partial f}{\partial x}(a,b)\mathbf{i} - \frac{\partial f}{\partial y}(a,b)\mathbf{j}\right) \cdot \left(\frac{\partial f}{\partial y}(a,b)\mathbf{i} - \frac{\partial f}{\partial x}(a,b)\mathbf{j}\right) \\ &= \left(\frac{\partial f}{\partial x}(a,b)\frac{\partial f}{\partial y}(a,b) - \frac{\partial f}{\partial x}(a,b)\frac{\partial f}{\partial y}(a,b)\right) \\ &= 0\end{aligned}$$

(b) The vector $\mathbf{v}$ is perpendicular to the gradient of $f(x,y)$ at (a,b), and is parallel to the level curve at (a,b).

3.1.10. **Application of the Chain Rule to Model Traffic Flow**

(a) Using the chain rule, $u = f(k)$ implies that

$$\frac{\partial u}{\partial x} = f'(k)\frac{\partial k}{\partial x},$$

and

$$\frac{\partial u}{\partial t} = f'(k)\frac{\partial k}{\partial t}.$$

Using these relations, the partial differential equation (Equation (2)) becomes

$$\begin{aligned}u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} &= -c^2k^n\frac{\partial k}{\partial x} \\ uf'(k)\frac{\partial k}{\partial x} + f'(k)\frac{\partial k}{\partial t} &= -c^2k^n\frac{\partial k}{\partial x} \\ f'(k)\Big(u\frac{\partial k}{\partial x} + \frac{\partial k}{\partial t}\Big) + c^2k^n\frac{\partial k}{\partial x} &= 0\end{aligned}$$

Since $u = f(k)$, $\frac{du}{dk} = f'(k)$. Thus,

$$\frac{du}{dk}\Big(u\frac{\partial k}{\partial x} + \frac{\partial k}{\partial t}\Big) + c^2k^n\frac{\partial k}{\partial x} = 0,$$

as required.

(b) The continuity equation can be rearranged to yield the relation

$$\begin{aligned}0 &= \frac{\partial k}{\partial t} + \frac{\partial}{\partial x}(ku) \\ &= +k\frac{\partial u}{\partial x} + u\frac{\partial k}{\partial x} \\ \frac{\partial k}{\partial t} &= -k\frac{\partial u}{\partial x} - u\frac{\partial k}{\partial x}.\end{aligned}$$

Substitution into Equation (3) yields

$$
\begin{aligned}
\frac{du}{dk}\left(u\frac{\partial k}{\partial x}+\frac{\partial k}{\partial t}\right) &= -c^2k^n\frac{\partial k}{\partial x}\\
\frac{du}{dk}\left(u\frac{\partial k}{\partial x}+\left(-k\frac{\partial u}{\partial x}-u\frac{\partial k}{\partial x}\right)\right) &= -c^2k^n\frac{\partial k}{\partial x}\\
-\frac{du}{dk}\left(k\frac{\partial u}{\partial x}\right) &= -c^2k^n\frac{\partial k}{\partial x}\\
k\frac{du}{dk}\frac{\partial u}{\partial x} &= c^2k^n\frac{\partial k}{\partial x}\\
\frac{du}{dk}\frac{\partial u}{\partial x} &= c^2k^{n-1}\frac{\partial k}{\partial x}.
\end{aligned}
$$

But

$$\frac{\partial u}{\partial x} = f'(k)\frac{\partial k}{\partial x},$$

so we have

$$
\begin{aligned}
\frac{du}{dk}\frac{\partial u}{\partial x} &= c^2k^{n-1}\frac{\partial k}{\partial x}\\
\frac{du}{dk}\left(f'(k)\frac{\partial k}{\partial x}\right) &= c^2k^{n-1}\frac{\partial k}{\partial x}
\end{aligned}
$$

Recall that $u=f(k)$, $\frac{du}{dk}=f'(k)$, which gives us

$$\frac{\partial u}{\partial k}\frac{\partial u}{\partial k}\frac{\partial k}{\partial x} = c^2k^{n-1}\frac{\partial k}{\partial x}$$

Provided that $\frac{\partial k}{\partial x}\neq 0$, we have

$$\frac{\partial u}{\partial k}\frac{\partial u}{\partial k} = \left(\frac{\partial u}{\partial k}\right)^2 = c^2k^{n-1}$$

We can take the square root of both sides to obtain

$$\frac{\partial u}{\partial k} = \pm ck^{(n-1)/2}$$

3.2 Optimization

3.2.1. Local extreme values of a function occur at points where the gradient of the function is the zero vector.

(a) First we find the critical points:

$$\begin{aligned}
\nabla f(x,y) &= \begin{bmatrix}0\\0\end{bmatrix}\\
\begin{bmatrix}\dfrac{\partial}{\partial x}(x^2+2x-xy+y^2)\\ \dfrac{\partial}{\partial y}(x^2+2x-xy+y^2)\end{bmatrix} &= \begin{bmatrix}0\\0\end{bmatrix}\\
\begin{bmatrix}2x+2\\-x+2y\end{bmatrix} &= \begin{bmatrix}0\\0\end{bmatrix}\\
x &= -1\\
y &= -\frac{1}{2}
\end{aligned}$$

To categorize this point $(-1, -\frac{1}{2})$ as a local minimum or maximum we apply the second partial derivative test:

$$\begin{aligned}
D(x,y) &= \begin{vmatrix}\dfrac{\partial^2 f(x,y)}{\partial x^2} & \dfrac{\partial^2 f(x,y)}{\partial y\partial x}\\ \dfrac{\partial^2 f(x,y)}{\partial x\partial y} & \dfrac{\partial^2 f(x,y)}{\partial y^2}\end{vmatrix}\\
&= \begin{vmatrix}\dfrac{\partial^2}{\partial x^2}(x^2+2x-xy+y^2) & \dfrac{\partial^2}{\partial y\partial x}(x^2+2x-xy+y^2)\\ \dfrac{\partial^2}{\partial x\partial y}(x^2+2x-xy+y^2) & \dfrac{\partial^2}{\partial y^2}(x^2+2x-xy+y^2)\end{vmatrix}\\
&= \begin{vmatrix}2 & 0\\ -1 & 2\end{vmatrix}\\
&= 4\\
&> 0
\end{aligned}$$

for all values of x, y. Additionally, $\dfrac{\partial^2 f(x,y)}{\partial x^2} = 2 > 0$, so that the point $(-1, -\frac{1}{2})$ is a local minimum.

(b) Again, we look for the critical points by setting the gradient to zero.

$$\begin{aligned}
\nabla h(x,y) &= \begin{bmatrix}0\\0\end{bmatrix}\\
\begin{bmatrix}y\cos(xy)\\x\cos(xy)\end{bmatrix} &= \begin{bmatrix}0\\0\end{bmatrix}
\end{aligned}$$

The solutions (and therefore the critical points) are the point $(0, 0)$ and the family of curves $xy = \pi/2 + k\pi$, where $k \in \mathbb{Z}$.

To categorize the points, again compute the matrix

$$\begin{aligned}
D(x,y) &= \begin{vmatrix} \dfrac{\partial^2 h(x,y)}{\partial x^2} & \dfrac{\partial^2 h(x,y)}{\partial y \partial x} \\ \dfrac{\partial^2 h(x,y)}{\partial x \partial y} & \dfrac{\partial^2 h(x,y)}{\partial y^2} \end{vmatrix} \\
&= \begin{vmatrix} \dfrac{\partial^2}{\partial x^2}(\sin(xy)) & \dfrac{\partial^2}{\partial y \partial x}(\sin(xy)) \\ \dfrac{\partial^2}{\partial x \partial y}(\sin(xy)) & \dfrac{\partial^2}{\partial y^2}(\sin(xy)) \end{vmatrix} \\
&= \begin{vmatrix} -y^2 \sin(xy) & \cos(xy) - xy\sin(xy) \\ \cos(xy) - xy\sin(xy) & -x^2 \sin(xy) \end{vmatrix} \\
D(0,0) &= \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} \\
&= -1
\end{aligned}$$

So $(0,0)$ is a saddle point.

$$\begin{aligned}
D(x, \frac{\pi/2+k\pi}{x}) &= \begin{vmatrix} -\dfrac{\pi/2+k\pi}{x}^2 \sin(\pi/2+k\pi) & \cos(\pi/2+k\pi) - (\pi/2+k\pi)\sin(\pi/2+k\pi) \\ \cos(\pi/2+k\pi) - (\pi/2+k\pi)\sin(\pi/2+k\pi) & -x^2\sin(\pi/2+k\pi) \end{vmatrix} \\
&= \begin{vmatrix} -(\dfrac{\pi/2+k\pi}{x})^2(-1)^k & -(\pi/2+k\pi)(-1)^k \\ -(\pi/2+k\pi)(-1)^k & -x^2(-1)^k \end{vmatrix} \\
&= (\pi/2+k\pi)^2 - (\pi/2+k\pi)^2 \\
&= 0
\end{aligned}$$

So the test is inconclusive.

3.2.2. We use the method of Lagrange Multipliers. To find the extreme values of the function $f(x,y)$ subject to the constraint $g(x,y) = c$, we define a new function $F(x,y,t) = f(x,y) + t(g(x,y) - c)$ and solve the equation $\nabla F(x,y,t) = 0$. The solutions (x,y) are the candidate solutions, which we test by plugging into the original equation f.

(a) Define $F(x,y,t) = e^{-(x^2+y^2)} - t(x - y^2)$. Then

$$\begin{aligned}
\nabla F(x,y,t) &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} -2xe^{-(x^2+y^2)} - t \\ -2ye^{-(x^2+y^2)} + 2yt \\ -(x - y^2) \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (16)
\end{aligned}$$

$(0,0)$ is one solution. When $x \neq 0$, by the third component of (16) we see that $y \neq 0$, so that we get $t = e^{-(x^2+y^2)}$ from the second component, which in turn by the first component shows that $x = -1/2$. However, there is no y that satisfies the third component of (16) when $x = -1/2$. Thus $(0,0)$ is the only extreme value.

(b) Define $F(x,y,t) = e^{x-y^2} - t(\frac{x^2}{4} + y + y^2 - 1)$. Then

$$\begin{aligned}
\nabla F(x,y,t) &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} e^{x-y^2} - tx/2 \\ -2ye^{x-y^2} - t + 2yt \\ -(\frac{x^2}{4} + y + y^2 - 1) \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \qquad (17)
\end{aligned}$$

From the first two components we see that

$$\begin{aligned} e^{x-y^2} &= tx/2 \\ e^{x-y^2} &= \frac{t-2yt}{-2y}. \end{aligned}$$

This yields the equation $y = \dfrac{1}{2-x}$, so that the extreme values of f fall on the intersection of the ellipse $\frac{x^2}{4} + y + y^2 = 1$ and the curve $y = \dfrac{1}{2-x}$. Plugging y into the third component of (17) gives the equation $\frac{x^2}{4} + \frac{1}{2-x} + (\frac{1}{2-x})^2 = 1$.

3.2.3. The divergence is given by the equation

$$\nabla \cdot f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

and the curl by the equation

$$\nabla \times f = \begin{bmatrix} \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{bmatrix}.$$

Applying these equations yields the solutions:

$$\begin{aligned} \nabla \cdot f &= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \\ &= \frac{\partial}{\partial x}(x^2+y^2) + \frac{\partial}{\partial y}(y^2+z^2) + \frac{\partial}{\partial z}(z^2+x^2) \\ &= 2x + 2y + 2z \\ \nabla \times f &= \begin{bmatrix} \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{bmatrix} \\ &= \begin{bmatrix} -2z \\ -2x \\ -2y \end{bmatrix}. \end{aligned}$$

3.2.4. It is possible to solve this problem using Lagrangian optimization; however, the closest point on the sphere to the plane will occur at the point on the sphere where its normal vector is parallel to the plane's normal vector.

Knowing this, we first find the sphere's normal vector by computing the gradient of the function $F(x, y, z) = x^2 + y^2 + z^2 - 1$, which by a straightforward calculation is the vector $\nabla F = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$.

Now, by rearranging the equation $x + 2y + 2z = 5$ into the form $\begin{bmatrix} x-3 \\ y-2 \\ z+1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 0$, we see that $\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is a vector normal to the plane.

We want to find a point on the sphere where these two normal vectors are parallel; equivalently, we want to find a point on the sphere such that $\begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ for some k. We can plug $x = k/2$,

$y = k$, $z = k$ into the equation $x^2 + y^2 + z^2 = 1$, which yields the solution $k = 2/3$. Thus $(1/3, 2/3, 2/3)$ is a point on the sphere where the sphere's normal vector is parallel to the plane's, and therefore is the point closest to the plane.

3.2.5. Let l be the length of the rectangular sides of the prism and w be their width, so that the sides of the equilateral triangle ends are also of length w. The surface area of such a prism is given by the function $A(l, w) = \frac{\sqrt{3}}{2}l^2 + 3lw$, and its volume by the function $V(l, w) = \frac{\sqrt{3}}{4}l^2 w$. Since the sides cost $\$10/ft^2$ and the ends cost $\$8/ft^2$, we wish to maximize the function $V(l, w)$ subject to the restriction $C(l, w) = 5\sqrt{3}l^2 + 24lw = 500$.

We apply the method of Lagrange Multipliers. Define the function $F(l, w) = V(l, w) + t(C(l, w) - 500) = \frac{\sqrt{3}}{4}l^2 w + t(5\sqrt{3}l^2 + 24lw - 500)$; the constrained extreme values of V satisfy the equation $\nabla F(l, w) = \mathbf{0}$.

$$\nabla F(l, w) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial F}{\partial l} \\ \frac{\partial F}{\partial w} \\ \frac{\partial F}{\partial t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\sqrt{3}}{2}lw + 10\sqrt{3}lt + 24wt \\ \frac{\sqrt{3}}{4}l^2 + 24lt \\ 5\sqrt{3}l^2 + 24lw - 500 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{18}$$

From the second row of (18) we get $t = \frac{-\sqrt{3}}{96}l$. Substitution into the first row yields

$$\begin{aligned} \frac{\sqrt{3}}{2}lw - \frac{10}{32}l^2 - \frac{\sqrt{3}}{4}lw &= 0 \\ \frac{\sqrt{3}}{4}lw - \frac{10}{32}l^2 &= 0 \\ 8\sqrt{3}lw - 10l^2 &= 0 \\ 8\sqrt{3}lw &= 10l^2 \\ w &= \frac{5}{4\sqrt{3}}l \end{aligned} \tag{19}$$

Substituting (19) into the third row of (18) yields

$$\begin{aligned} 5\sqrt{3}l^2 + 24l(\frac{5}{4\sqrt{3}}l) &= 500 \\ 5\sqrt{3}l^2 + \frac{30}{\sqrt{3}}l^2 &= 500 \\ 15l^2 + 30l^2 &= 500\sqrt{3} \\ 45l^2 &= 500\sqrt{3} \\ l^2 &= \frac{100}{3^{3/2}} \\ l &= \frac{10}{3^{3/4}} \end{aligned}$$

which, when combined with (19), gives $w = \dfrac{50}{4 \cdot 3^{5/4}}$.

3.2.6. We want to optimize the function $T(x, y, z) = 200xyz^2$ subject to the constraint $x^2 + y^2 + z^2 = 1$. Define the function $F(x, y, z) = 200xyz^2 + t(x^2 + y^2 + z^2 - 1)$; the extreme points satisfy the equation $\nabla F(x, y, z) = 0$.

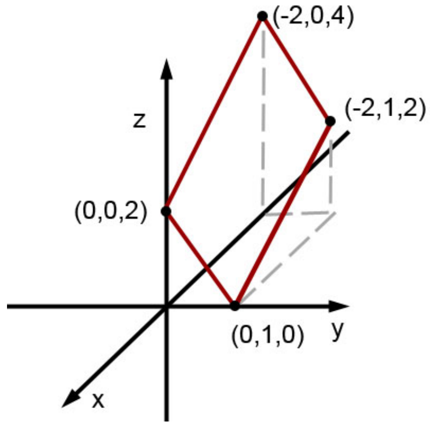
$$
\begin{aligned}
\nabla F(x,y,z) &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} \dfrac{\partial F}{\partial x} \\ \dfrac{\partial F}{\partial y} \\ \dfrac{\partial F}{\partial z} \\ \dfrac{\partial F}{\partial t} \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 200yz^2 + 2xt \\ 200xz^2 + 2yt \\ 400xyz + 2zt \\ x^2 + y^2 + z^2 - 1 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad (20)
\end{aligned}
$$

The third equation of (20) yields $t = -200xy$. Substituting this value of t into the first and second equations gives $z^2 = 2x^2 = 2y^2$. Combining this with the fourth equation yields $4x^2 = 1$, or $x = \pm\frac{1}{2} \implies y = \pm\frac{1}{2}, z = \pm\frac{1}{\sqrt{2}}$.

4 Multiple Integrals

4.1 Double Integrals

4.1.1. (a) A sketch of the volume is below.



(b) We must integrate:

$$\begin{aligned}\int_{-2}^{0}\int_{0}^{1}(-x-2y+2)dydx &= \int_{-2}^{0}\big(-2xy-y^2+2y\big)\Big|_0^1 dx\\ &= \int_{-2}^{0}\Big(-2x+1\Big)dx\\ &= \Big(-x^2+x\Big)\Big|_{-2}^{0}\\ &= 0-\big(-(-2)^2+(-2)\big)\\ &= 0-\big(-4-2\big)\\ &= 6\end{aligned}$$

4.1.2. (a) Substituting the expression for u into the divergence equation yields

$$\begin{aligned}0 &= \nabla\cdot\mathbf{v}\\ &= \frac{\partial}{\partial x}\big(u(x,y)\big)+\frac{\partial}{\partial y}\big(v(x,y)\big)\\ &= \frac{\partial}{\partial x}(x^2+y^2)+\frac{\partial v}{\partial y}\\ &= 2x+\frac{\partial v}{\partial y}\\ \frac{\partial v}{\partial y} &= -2x\end{aligned}$$

Therefore, $v(x,y)$ is a function whose partial derivative with respect to y is $-2x$. The **most general** form for $v(x,y)$ is obtained by integrating with respect to y:

$$v(x,y) = -2xy + f(x)$$

where $f(x)$ is an unknown function of one variable, x.

(b) Using the same approach as we used for (a) yields

$$\begin{aligned}0 &= \nabla\cdot\mathbf{v}\\ &= \frac{\partial}{\partial x}\big(u(x,y)\big)+\frac{\partial}{\partial y}\big(v(x,y)\big)\\ &= \frac{\partial u}{\partial x}+0\\ \frac{\partial u}{\partial x} &= 0\end{aligned}$$

Therefore, $u(x,y)$ is a function whose partial derivative with respect to x is 0. The **most general** form for $u(x,y)$ is obtained by integrating with respect to x:

$$u(x,y) = g(y)$$

where $g(y)$ is an unknown function of one variable, y.

4.1.3. **Double Integral with an Absolute Value**
We are given the double integral

$$\int_0^1 \int_0^2 y\big|x-1\big| dxdy.$$

Treating y as a constant, this becomes

$$\int_0^1 y\left(\int_0^2 \big|x-1\big| dx\right) dy.$$

and we must then evaluate

$$\int_0^2 \big|x-1\big| dx.$$

The absolute value function can be removed if we note that

$$|x-1| = \begin{cases} 1-x & \text{if } x \leq 1 \\ x-1 & \text{if } x > 1 \end{cases}$$

Thus,

$$\begin{aligned}
\int_0^2 \big|x-1\big| dx &= \int_0^2 \big|x-1\big| dx \\
&= \int_0^1 \big|x-1\big| dx + \int_1^2 \big|x-1\big| dx \\
&= \int_0^1 (1-x) dx + \int_1^2 (x-1) dx \\
&= 1 - \frac{1}{2} + \Big(\frac{x^2}{2} - x\Big)\Big|_1^2 \\
&= \frac{1}{2} + \Big((2-2) - (1/2-1)\Big) \\
&= \frac{1}{2} + \Big(0 - (-1/2)\Big) \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1
\end{aligned}$$

Therefore,

$$\int_0^1 y\left(\int_0^2 \big|x-1\big| dx\right) dy = \int_0^1 y(1) dy = \frac{y^2}{2}\Big|_0^1 = \frac{1}{2}.$$

Note that the integral

$$\int_0^2 \big|x-1\big| dx.$$

is simply the area of two triangles, each with area $\frac{1}{2}$. So a much simpler method of working out the answer to the integral with respect to x is

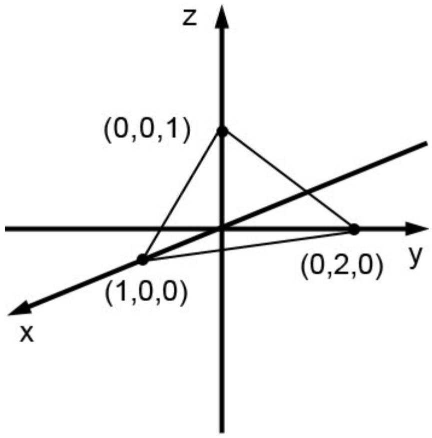
$$\int_0^2 \big|x-1\big| dx = \frac{1}{2} + \frac{1}{2} = 1.$$

4.1.4. **Double Integral Given Maximum and Minimum Values**
If the maximum and minimum values of $f(x,y)$ on S are equal to the constant K, then $f(x,y)$ is constant on S, and is equal to K everywhere on S. Then,

$$\iint_S f(x,y)dxdy = \int_a^b \int_c^d K dxdy = K \int_a^b \int_c^d (1)dxdy = K(b-a)(d-c).$$

4.2 Double Integrals Over a General Region

4.2.1. (a) A sketch of the tetrahedron is below.



(b) The tetrahedron has one side in the xy-plane. This side is bounded by the line that is the intersection between the xy-plane and the plane $z = 1 - x - y/2$. We can find this intersection by setting $z = 0$,

$$\begin{aligned} 0 &= 1 - x - \frac{y}{2} \\ x &= 1 - \frac{y}{2}. \end{aligned}$$

Therefore, the volume is the region under the plane $z = 1 - x - y/2$ and over

$$R = \{(x,y) \mid 0 \le x \le 1 - y/2,\ 0 \le y \le 2\}.$$

The double integral is

$$\int_0^2 \int_0^{1-y/2} (1 - x - \frac{y}{2})dxdy$$

(c) The volume is the region under the plane $z = 1 - x - y/2$ and over

$$R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 2 - 2x\}.$$

The double integral is

$$\int_0^1 \int_0^{2-2x} (1 - x - \frac{y}{2})dydx$$

4.2.2. (a) Suppose that we subdivide region R into a rectangular grid of sub-rectangles (as in Figure 3.2.5), so that we only consider the sub-rectangles that are completely enclosed in R. Then, the area of region R is approximated by the double sum

$$\sum_j \sum_i \Delta x_i \Delta y_j$$

But if $f = 1$ for all x_i and y_j, this is equal to:

$$\sum_j \sum_i f(x_i, y_j) \Delta x_i \Delta y_j \tag{21}$$

where x_i and y_j is a point inside sub-rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$. If we take smaller and smaller rectangles, so that the length of the longest diagonal of the sub-rectangles goes to zero, the sub-rectangles begin to fill more and more of the region R, and so the above sums approach the **area** of region R. Since we have defined

$$\iint f(x, y) dA$$

as the limit of Equation (21) as the longest diagonal goes to zero, and $f(x, y) = 1$, the double integral

$$\iint 1 dA$$

is the area of region R.

(b) The region may be defined as

$$R = \{(x, y) \mid 0 \leq x \leq 1,\ x^2 \leq y \leq \sqrt{x}\}.$$

The area of R is

$$\begin{aligned}
\int_0^1 \int_{x^2}^{\sqrt{x}} dydx &= \int_0^1 \left(\sqrt{x} - x^2\right) dx \\
&= \left(\frac{2}{3} x^{3/2} - \frac{x^3}{3}\right)\Big|_0^1 \\
&= \frac{2}{3} - \frac{1}{3} \\
&= \frac{1}{3}
\end{aligned}$$

4.2.3. The curves $y = x^2$ and $x^2 = y$ intersect at (0,0) and at (1,1). Integrating with respect to y first yields

$$\begin{aligned}
\int_0^1 \int_{x^2}^{\sqrt{x}} x^2 + y^2 dydx &= \int_0^1 \left(yx^2 + \frac{y^3}{3}\right)\Big|_{x^2}^{\sqrt{x}} dx \\
&= \int_0^1 \left(x^{5/2} + \frac{x^{3/2}}{3} - x^4 - \frac{x^6}{3}\right) dx \\
&= \left(\frac{2}{7} x^{7/2} + \frac{2}{15} x^{5/2} - \frac{1}{5} x^5 - \frac{1}{21} x^7\right)\Big|_0^1 \\
&= \frac{2}{7} + \frac{2}{15} - \frac{1}{5} - \frac{1}{21} \\
&= 6/35
\end{aligned}$$

4.2.4. (a) Integrating with respect to y first yields

$$\begin{aligned}
\int_0^2 \int_0^{x^2} x \sin(y) dydx &= \int_0^2 -x \cos(y)\Big|_0^{x^2} dx \\
&= -\int_0^2 \left(x \cos(x^2) - 1\right) dx \\
&= -\int_0^2 x \cos(x^2) dx + \int_0^2 1\ dx \\
&= 2 - \int_0^2 x \cos(x^2) dx
\end{aligned}$$

Now let $u = x^2$, so that $du = 2xdx$. Then

$$\begin{aligned}
2 - \int_0^2 x\cos(x^2)dx &= 2 - \int_0^4 x\cos(u)\Big(\frac{du}{2x}\Big)\\
&= 2 - \frac{1}{2}\int_0^4 \cos(u)du\\
&= 2 - \frac{1}{2}\sin(u)\Big|_0^4\\
&= 2 - \frac{1}{2}\sin(4)\\
&\approx 2.3784
\end{aligned}$$

(b) Integrating with respect to x first yields

$$\begin{aligned}
\int_0^4\int_{\sqrt{y}}^2 x\sin(y)dxdy &= \int_0^4 \frac{x^2}{2}\sin(y)\Big|_{\sqrt{y}}^2 dy\\
&= \int_0^4 \frac{4-y}{2}\sin(y)dy\\
&= 2\int_0^4 \sin(y)dy - \frac{1}{2}\int_0^4 y\sin(y)dy\\
&= 2\big(-\cos(y)\big)\big|_0^4 - \frac{1}{2}\int_0^4 y\sin(y)dy\\
&= 2\big(-\cos(4)+1\big) - \frac{1}{2}\int_0^4 y\sin(y)dy\\
&= 2 - 2\cos(4) - \frac{1}{2}\int_0^4 y\sin(y)dy
\end{aligned}$$

Now using integration by parts, with

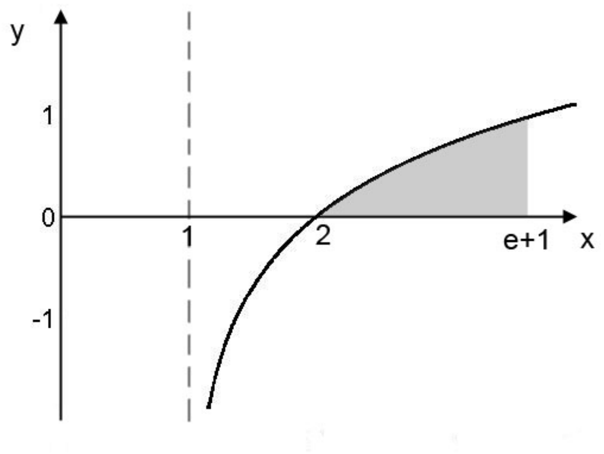
$$\begin{aligned}
u &= y, & dv &= \sin(y)dy\\
du &= dy, & v &= -\cos(y)
\end{aligned}$$

we obtain

$$\begin{aligned}
\int_0^4\int_{\sqrt{y}}^2 x\sin(y)dxdy &= 2 - 2\cos(4) - \frac{1}{2}\int_0^4 y\sin(y)dy\\
&= 2 - 2\cos(4) - \frac{1}{2}\left(-y\cos(y)\Big|_0^4 - \int_0^4 \big(-\cos(y)\big)dy\right)\\
&= 2 - 2\cos(4) - \frac{1}{2}\left(-4\cos 4 + \sin(y)\Big|_0^4\right)\\
&= 2 - 2\cos(4) - \frac{1}{2}\Big(-4\cos 4 + \sin(4) - 0\Big)\\
&= 2 - \frac{\sin(4)}{2}\\
&\approx 2.3784
\end{aligned}$$

4.2.5. **Changing the Order of Integration**
The region over which we are integrating $f(x,y)$ is the shaded area below.
The region is bounded by the lines $y = 0$, $x = 1+e$, and by the curve $y = \ln(x-1)$. Using horizontal slices, values of y range from 0 to 1, and values of x range from $e^y - 1$ to $1+e$. The double integral becomes

$$\int_0^1\int_{e^y-1}^{1+e} f(x,y)dydx.$$



4.2.6. **Bounding A Double Integral**
The integrand $f(x, y)$ has the property that

$$0 \leq \sin(x + y) \leq 1,$$

because $x + y$ is between 0 and 2, and $\sin(2) < 1$. Then

$$0 = \iint_D 0 dA \leq \iint_D f(x, y) dA \leq \iint_D (1) dA = 1.$$

4.2.7. (a) Suppose we use the function

$$h(x, y) = 11xy.$$

Then $h(-x, y) = 11(-x)y = -11xy = -h(x, y)$.

(b) An example of a region that is symmetric about the y-axis, but not symmetric about the x-axis, is the region bounded by the curves

$$x = -1, \quad x = 1, \quad y = 0, \quad y = -1.$$

(c) The double integral for our region is

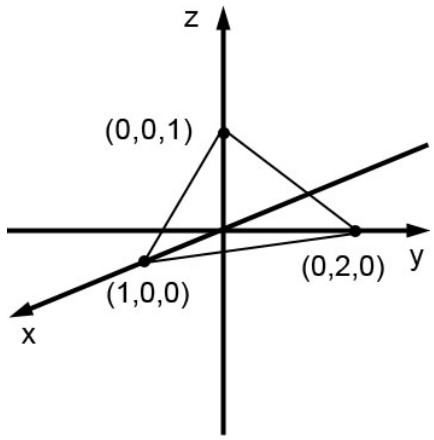
$$\begin{aligned}\iint_D g(x, y) dx dy &= \int_{-1}^{1} \int_{-1}^{0} 11xy dy dx \\ &= -\frac{11}{2} \int_{-1}^{1} x dx \\ &= -\frac{11}{4} \left(x^2\right)\Big|_{-1}^{1} \\ &= -\frac{11}{4}(0) \\ &= 0.\end{aligned}$$

4.3 Triple Integrals

4.3.1. **Volume of a Tetrahedron**

Recall that the volume is the region under the plane $z = 1 - x - y/2$ and over

$$R = \{(x, y) \mid 0 \leq x \leq 1 - y/2,\ 0 \leq y \leq 2\}.$$



Because z lies between 0 and $z = 1 - x - y/2$, the volume, S, can be described as

$$S = \{(x, y, z) \mid 0 \leq x \leq 1 - y/2,\ 0 \leq y \leq 2,\ 0 \leq z \leq 1 - x - y/2\}.$$

The volume can be calculated with the triple integral

$$\begin{aligned}
\int_0^2 \int_0^{1-y/2} \int_0^{1-x-y/2} dz dx dy &= \int_0^2 \int_0^{1-y/2} (1 - x - y/2) dx dy \\
&= \int_0^2 \left(x - \frac{x^2}{2} - \frac{xy}{2}\right)\Big|_0^{1-y/2} dy \\
&= \int_0^2 \left((1 - y/2) - \frac{(1-y/2)^2}{2} - \frac{(y - y^2/2)}{2}\right) dy \\
&= \int_0^2 \left(1 - \frac{y}{2} - \frac{1}{2}\left(1 - y + \frac{y^2}{4}\right) - \frac{y}{2} + \frac{y^2}{4}\right) dy \\
&= \int_0^2 \left(1 - \frac{y}{2} - \frac{1}{2} + \frac{y}{2} - \frac{y^2}{8} - \frac{y}{2} + \frac{y^2}{4}\right) dy \\
&= \int_0^2 \left(\frac{1}{2} - \frac{y}{2} + \frac{y^2}{8}\right) dy \\
&= \frac{2}{2} - \frac{4}{4} + \frac{8}{24} \\
&= \frac{1}{3}
\end{aligned}$$

4.3.2. **Volume of an Ellipsoid**
We are given the transformations

$$x = au, \quad y = bv, \quad z = cw.$$

The Jacobian becomes

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

The solid enclosed by the ellipsoid is the image of the unit sphere $u^2 + v^2 + w^2 \leq 1$. Using that a sphere has volume $\frac{4}{3}\pi r^3$, we find that

$$\begin{aligned}\iiint\limits_V dxdydz &= \iiint\limits_{u^2+v^2+w^2\leq 1} abc\ dudvdw \\ &= abc \iiint\limits_{u^2+v^2+w^2\leq 1} dudvdw \\ &= abc(\text{volume of a sphere}) \\ &= \frac{4\pi abc}{3}\end{aligned}$$

4.3.3. **Maximizing a Triple Integral**
The integral will obtain its maximum value when S contains the entire region where the integrand is non-negative. In other words, we require that

$$1 - 3x^2 - 2y^2 - 4z^2 \geq 0.$$

Therefore, E is the region bounded by the ellipse

$$3x^2 + 2y^2 + 4z^2 = 1.$$

4.4 Change of Variables in Multiple Integrals

4.4.1. **Linear Transformations**

(a) Substituting $v = v_0$ into the linear transformation yields the two equations

$$\begin{aligned}x &= c_1 u + c_2 v_0 \\ y &= d_1 u + d_2 v_0\end{aligned}$$

To find the equation of the line in the xy-plane, we need to eliminate u. There are many ways to do this, but let's multiply the first equation by d_1 and the second by c_1.

$$\begin{aligned}d_1 x &= c_1 d_1 u + c_2 d_1 v_0 \\ c_1 y &= c_1 d_1 u + d_2 c_1 v_0\end{aligned}$$

Subtracting these equations yields

$$d_1 x - c_1 y = (c_2 d_1 - d_2 c_1) v_0$$

A simple rearrangement gives us

$$c_1 y = d_1 x - (c_2 d_1 - d_2 c_1) v_0.$$

Provided that c_1 is not zero, we could write this in the form

$$y = \frac{d_1}{c_1} x - \frac{c_2 d_1 - d_2 c_1}{c_1} v_0.$$

(b) Substituting $x = x_0$ into $x = c_1 u + c_2 v$ gives us

$$x_0 = c_1 u + c_2 v,$$

Provided that c_2 is not zero, This is the is mapped into the uv-plane REWORD

4.4.2. **Double Integral Over a Parallelogram**
The integral can be written as

$$\iint\limits_R (x^2 - y^2) dxdy = \iint\limits_R (x-y)(x+y) dxdy.$$

Recall that R is the region bounded by

$$x+y=0, \quad x+y=1, \quad x-y=0, \quad x-y=1.$$

The appearance of the terms $(x+y)$ and $(x-y)$ in the integrand and in the lines that bound R suggests the transformation

$$u = x+y \tag{22}$$
$$v = x-y. \tag{23}$$

In order to compute the Jacobian, we need explicit expressions for u and v. If we add equations 22 and 23 we find that

$$x = \frac{u+v}{2}$$

And if we subtract equations 22 and 23 we find that

$$y = \frac{u-v}{2}$$

The Jacobian becomes

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

We also need to find the limits of integration in the transformed integral. Using equations 22 and 23 the four lines bounding R in the xy-plane become

$$u=0, \quad u=1, \quad v=0, \quad v=1.$$

The double integral therefore becomes

$$\begin{aligned}\iint\limits_R \left(x^2-y^2\right)dxdy &= \iint\limits_R (x-y)(x+y)dxdy \\ &= \int_0^1\int_0^1 uv\left|-\frac{1}{2}\right|dudv \\ &= \frac{1}{2}\int_0^1\int_0^1 (uv)dudv \\ &= \frac{1}{2}\int_0^1 \frac{v}{2}dv \\ &= \frac{1}{8}.\end{aligned}$$

4.4.3. **Triple Integral Bounded by an Ellipse**
The Jacobian is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6.$$

We also need to find the limits of integration in the transformed integral. The region R is bounded by the ellipse, $4x^2+9y^2=36$, which becomes the region bounded by the circle $u^2+v^2=1$. Therefore

$$\iint\limits_R x^2\ dxdy = \iint\limits_{u^2+v^2\le 1} (9u^2)6dudv = 54 \iint\limits_{u^2+v^2\le 1} (u^2)dudv$$

Switching to polar coordinates,

$$u = r\cos\theta, \quad v = r\sin\theta, \quad J = r$$

our double integral becomes

$$
\begin{aligned}
54 \iint_{u^2+v^2\le 1} (u^2)dudv &= 54\int_0^{2\pi}\int_0^1 r^3\cos^2\theta dr d\theta \\
&= 54\left(\int_0^{2\pi}\cos^2\theta d\theta\right)\left(\int_0^1 r^3 dr\right) \\
&= 54\left(\int_0^{2\pi}\frac{1}{2}\big(1+\cos(2\theta)\big)d\theta\right)\left(\frac{1}{4}\right) \\
&= \frac{27}{4}\left(\int_0^{2\pi}\big(1+\cos(2\theta)\big)d\theta\right) \\
&= \frac{27}{4}\big(\theta+\frac{1}{2}\sin(2\theta)\big|_0^{2\pi} \\
&= \frac{27}{4}(2\pi) \\
&= \frac{27\pi}{2}
\end{aligned}
$$

4.4.4. In cylindrical coordinates, the region is described by

$$V = \{(r,\theta,z) \mid 0\le r\le 1, 0\le\theta\le 2\pi, 0\le z\le 3r\}.$$

Our integral becomes

$$
\begin{aligned}
\iiint_V (r\sin\theta)^2 r dz dr d\theta &= \int_0^{2\pi}\int_0^1\int_0^{3r} r^3\sin^2\theta dz dr d\theta \\
&= 3\int_0^{2\pi}\int_0^1 r^4\sin^2\theta dr d\theta \\
&= \frac{3}{5}\int_0^{2\pi}\sin^2\theta d\theta \\
&= \frac{3}{10}\int_0^{2\pi}\big(1-\cos(2\theta)\big)d\theta \\
&= \frac{3}{10}\big(\theta-\frac{1}{2}\sin(2\theta)\big)\Big|_0^{2\pi} \\
&= \frac{3\pi}{5}
\end{aligned}
$$

4.4.5. In cylindrical coordinates, V is the region bounded by

$$
\begin{aligned}
&0\le r\le 2 \\
&0\le\theta\le\frac{\pi}{2} \\
&0\le z\le\sqrt{4-r^2}
\end{aligned}
$$

The triple integral becomes

$$
\begin{aligned}
\iiint_V dxdydz &= \int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}} r dz dr d\theta \\
&= \int_0^{\pi/2}\int_0^2 r\sqrt{4-r^2} dr d\theta \\
&= \frac{-1}{3}\int_0^{\pi/2}(4-r^2)^{3/2}\Big|_0^2 d\theta \\
&= \frac{-1}{3}\int_0^{\pi/2}(0-8)d\theta \\
&= \frac{4\pi}{3}
\end{aligned}
$$

4.4.6. **Triple Integral In Spherical Coordinates**
The solid is a section of a sphere with radius 1, centered at the origin. The section is the part of the sphere that lies above the plane $z = 0$, and between the planes $y = 0$, and $y = x$.

$$\begin{aligned}\int_0^{\pi/4}\int_0^{\pi/2}\int_0^1 (\rho^2 \sin\phi)\ d\rho\ d\phi\ d\theta &= \int_0^{\pi/4}\int_0^{\pi/2} \frac{\sin\phi}{3} d\phi\ d\theta \\ &= \int_0^{\pi/4} \frac{-(0-1)}{3} d\theta \\ &= \pi/12\end{aligned}$$

4.4.7. **Simplifying a Double Integral as a Product of Two Single Integrals**

(a) Substituting $f(x,y) = g(x)h(y)$ into the double integral yields

$$\iint_R f(x,y)dA = \int_a^b \int_c^d g(x)h(y)dydx = \int_a^b \left(\int_c^d g(x)h(y)dy \right) dx.$$

The function $g(x)$ is a constant in the inner integral, so

$$\begin{aligned}\iint_R f(x,y)dA &= \int_a^b \left(g(x) \int_c^d h(y)dy \right) dx \\ &= \int_a^b g(x) \left(\int_c^d h(y)dy \right) dx.\end{aligned}$$

The term in the brackets is a constant, and so

$$\begin{aligned}\iint_R f(x,y)dA &= \int_a^b g(x) \left(\int_c^d h(y)dy \right) dx \\ &= \int_a^b g(x)dx \int_c^d h(y)dy.\end{aligned}$$

This is the result we were asked to prove.

(b) Let the left-hand-side be equal to I,

$$I = \int_0^\infty e^{-x^2} dx.$$

I is also equal to

$$I = \int_0^\infty e^{-y^2} dy.$$

Therefore,

$$I^2 = \left(\int_0^\infty e^{-x^2} dx \right) \left(\int_0^\infty e^{-y^2} dy \right).$$

Our result from part (a) allows us to write this as

$$I^2 = \int_0^\infty \int_0^\infty e^{-x^2-y^2} dxdy.$$

Switching to polar coordinates,

$$\begin{aligned}
I^2 &= \int_0^\infty \int_0^{\pi/2} re^{-r^2}\ d\theta dr \\
&= \frac{\pi}{2} \int_0^\infty re^{-r^2}\ dr \\
&= \frac{\pi}{2} \left(-\frac{1}{2} e^{-r^2} \right) \Big|_0^\infty \\
&= -\frac{\pi}{4} \left(e^{-r^2} \right) \Big|_0^\infty \\
&= -\frac{\pi}{4} \left(0 - e^0 \right) \\
&= \frac{\pi}{4}
\end{aligned}$$

Taking the square root of both sides of this yields the desired result,

$$I = \frac{\sqrt{\pi}}{2}.$$

4.5 Application: Center of Mass

4.5.1. **Center of Mass of a 2D Triangular Plate**
The total mass of the plate is

$$\begin{aligned}
M &= \iint_R \delta(x, y) dA \\
&= \int_0^1 \int_{2x}^{4x} xy\ dydx \\
&= \frac{1}{2} \int_0^1 x(16x^2 - 4x^2)\ dx \\
&= 6 \int_0^1 x^3\ dx \\
&= \frac{3}{2}.
\end{aligned}$$

The x-coordinate of the center of mass is

$$\begin{aligned}
\bar{x} = \frac{M_y}{M} &= \frac{2}{3} \iint_R x\delta(x, y) dA \\
&= \int_0^1 \int_{2x}^{4x} x^2 y\ dydx \\
&= \frac{1}{2} \int_0^1 x^2(16x^2 - 4x^2)\ dx \\
&= 6 \int_0^1 x^4\ dx \\
&= \frac{6}{5}.
\end{aligned}$$

The y-coordinate of the center of mass is

$$\begin{aligned}
\bar{y} = \frac{M_x}{M} &= \frac{2}{3}\iint\limits_R y\delta(x,y)dA \\
&= \int_0^1\int_{2x}^{4x} xy^2 \; dydx \\
&= \frac{1}{3}\int_0^1 x(64x^3 - 8x^3) \; dx \\
&= \frac{56}{3}\int_0^1 x^4 \; dx \\
&= \frac{56}{15}.
\end{aligned}$$

Thus, the coordinates of the center of mass are $(6/5, 56/15)$.

4.5.2. **Center of Mass of a 2D Plate, Radial Density Function**
We could express the coordinates of the center of mass using either Cartesian or polar coordinates. Using Cartesian coordinates, the density, δ, at point (x, y), is given by

$$\delta(x,y) = L(x,y) = a - \sqrt{x^2+y^2}.$$

(a) The x-coordinate of the center of mass is

$$\bar{x} = \frac{M_y}{M} = \frac{1}{M}\iint\limits_R x\delta(x,y)dA = \frac{1}{M}\int_{-a}^{a}\int_0^{\sqrt{x^2+y^2}} x\Big(a - \sqrt{x^2+y^2}\Big) \; dydx$$

(b) The y-coordinate of the center of mass is

$$\bar{y} = \frac{M_x}{M} = \frac{1}{M}\iint\limits_R y\delta(x,y)dA = \frac{1}{M}\int_{-a}^{a}\int_0^{\sqrt{x^2+y^2}} y\Big(a - \sqrt{x^2+y^2}\Big) \; dydx$$

(c) The center of mass is located on the y-axis. In other words, $\bar{x} = 0$. This is because of symmetry: the integrand of $\bar{x}$ is odd in x, the integral with respect to x is calculated about an interval that is symmetric about the y-axis, and so the integral with respect to x is zero.

Note that we could just as easily set up the above integrals using polar coordinates. The 2D circular plate, in polar coordinates, is bounded by

$$0 \le r \le a, \quad 0 \le \theta \le \pi, \quad a > 0.$$

The coordinates for the center of mass are

$$\begin{aligned}
\bar{x} = \frac{M_y}{M} &= \frac{1}{M}\iint\limits_R x\delta(x,y)dA = \frac{1}{M}\int_0^{\pi}\int_0^{r} r\cos(\theta)(a-r) \; rdrd\theta \\
\bar{y} = \frac{M_x}{M} &= \frac{1}{M}\iint\limits_R y\delta(x,y)dA = \frac{1}{M}\int_0^{\pi}\int_0^{r} r\sin(\theta)(a-r) \; rdrd\theta
\end{aligned}$$

4.5.3. **Center of Mass of a 2D Plate, Cardioid**
It should be clear that because the region is symmetric about the x-axis, that $\bar{y}$ is 0. Using polar

coordinates,

$$\begin{aligned}
M_y &= \iint_D x dA \\
&= \int_0^{2\pi} \int_0^{a(1+\cos\theta)} r\cos\theta \ r dr d\theta \\
&= \frac{a^3}{3} \int_0^{2\pi} (1+\cos\theta)^3 \cos\theta \ d\theta \\
&= \frac{a^3}{3} \int_0^{2\pi} \left(\cos\theta + 3\cos^2\theta + 3\cos^3\theta + \cos^4\theta\right) d\theta \\
&= \frac{a^3}{3} \int_0^{2\pi} \left(3\cos^2\theta + 3\cos^3\theta + \cos^4\theta\right) d\theta
\end{aligned}$$

Let's consider each integral separately:

$$\begin{aligned}
\int_0^{2\pi} 3\cos^2\theta \ d\theta &= \frac{3}{2} \int_0^{2\pi} (1+\cos 2\theta) \ d\theta = 3\pi \\
\int_0^{2\pi} 3\cos^3\theta \ d\theta &= \cos\theta\sin\theta\Big|_0^{2\pi} + 2\int_0^{2\pi} \cos\theta \ d\theta = 0 \\
\int_0^{2\pi} \cos^4\theta \ d\theta &= \frac{1}{2}\cos^2\theta\sin\theta\Big|_0^{2\pi} + \frac{3}{4}\int_0^{2\pi} \cos^2\theta \ d\theta = \frac{3}{4}(\pi) = \frac{3\pi}{4}
\end{aligned}$$

Thus,

$$\begin{aligned}
M_y &= \frac{a^3}{3} \int_0^{2\pi} \left(3\cos^2\theta + 3\cos^3\theta + \cos^4\theta\right) d\theta \\
&= \frac{a^3}{3}\left(3\pi + 0 + \frac{3\pi}{4}\right) \\
&= \frac{5\pi a^3}{4}
\end{aligned}$$

The area of D is

$$\begin{aligned}
M &= \iint_D dA \\
&= \int_0^{2\pi} \int_0^{a(1+\cos\theta)} r dr d\theta \\
&= \frac{a^2}{2} \int_0^{2\pi} (1+\cos\theta)^2 \ d\theta \\
&= \frac{a^2}{2} \int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta) \ d\theta \\
&= \frac{a^2}{2}(2\pi + 0 + \pi) \\
&= \frac{3\pi a^2}{2}.
\end{aligned}$$

The centroid is the point

$$(\bar{x}, \bar{y}) = (M_y/M, 0) = \left(\frac{5a}{6}, 0\right).$$