Written Assignment 8

Questions in this assignment are based on sections 3.5 and 3.6 of *Vector Calculus* by Michael Corral.

Questions

1. When working with double integrals in polar coordinates, a helpful simplification ...

$$\iint_R g(x)h(y)dA = \int_a^b g(x)dx \int_c^d h(y)dy$$

   Use this property and the transformation $x = 3u, y = 2v$ to evaluate the double integral

$$\iint_E x^2 \; dxdy,$$

   where E is region bounded by the ellipse $4x^2 + 9y^2 = 36$.

2. Evaluate the integral

$$\iiint_S y^2 dV,$$

   where S is the solid that lies inside the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$ and below the cone $z^2 = 9x^2 + 9y^2$.

3. Evaluate the integral

$$\iiint_V \frac{1}{\sqrt{x^2 + y^2}} dV,$$

   using cylindrical coordinates, where V is the region:

$$\begin{aligned} 0 \leq &x \leq 2 \\ 0 \leq &y \leq \sqrt{4 - x^2} \\ 0 \leq &z \leq \sqrt{4 - x^2} \end{aligned}$$

4. Determine the value of

$$\int_0^\pi \int_{-1}^1 x^4 e^{x^2+y^2} \sin(y) dydx.$$

   Hint: integration by parts is not necessary.
   push this question to a midterm or final exam?

5.

Solutions

1. The Jacobian is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ \\ 0 & 2 \end{vmatrix} = 6.$$

We also need to find the limits of integration in the transformed integral. The region R is bounded by the ellipse, $4x^2 + 9y^2 = 36$, which becomes the region bounded by the circle $u^2 + v^2 = 1$. Therefore

$$\iint_R x^2 \ dxdy = \iint_{u^2+v^2 \leq 1} (9u^2)6dudv = 54 \iint_{u^2+v^2 \leq 1} (u^2)dudv$$

Switching to polar coordinates,

$$u = r\cos\theta, \quad v = r\sin\theta, \quad J = r$$

our double integral becomes

$$\begin{aligned} 54 \iint_{u^2+v^2 \leq 1} (u^2)dudv &= 54 \int_0^{2\pi}\int_0^1 r^3 \cos^2\theta dr d\theta \\ &= 54\left(\int_0^{2\pi} \cos^2\theta d\theta\right)\left(\int_0^1 r^3 dr\right) \\ &= 54\left(\int_0^{2\pi} \frac{1}{2}\big(1+\cos(2\theta)\big)d\theta\right)\left(\frac{1}{4}\right) \\ &= \frac{27}{4}\left(\int_0^{2\pi} \big(1+\cos(2\theta)\big)d\theta\right) \\ &= \frac{27}{4}\big(\theta + \frac{1}{2}\sin(2\theta)\big|_0^{2\pi} \\ &= \frac{27}{4}\big(2\pi\big) \\ &= \frac{27\pi}{2} \end{aligned}$$

2. In cylindrical coordinates, the region is described by

$$V = \{(r,\theta,z) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 3r\}.$$

Our integral becomes

$$\begin{aligned} \iiint_V (r\sin\theta)^2 r dz dr d\theta &= \int_0^{2\pi}\int_0^1\int_0^{3r} r^3\sin^2\theta dz dr d\theta \\ &= 3\int_0^{2\pi}\int_0^1 r^4 \sin^2\theta dr d\theta \\ &= \frac{3}{5}\int_0^{2\pi} \sin^2\theta d\theta \\ &= \frac{3}{10}\int_0^{2\pi} \big(1-\cos(2\theta)\big)d\theta \\ &= \frac{3}{10}\big(\theta - \frac{1}{2}\sin(2\theta)\big)\Big|_0^{2\pi} \\ &= \frac{3\pi}{5} \end{aligned}$$

3.

4. The integral can be written as

$$\int_0^\pi \int_{-1}^1 e^{x^2+y^2} \sin(y) dy dx = \int_0^\pi \int_{-1}^1 e^{x^2} e^{y^2} \sin(y) dy dx = \left(\int_0^\pi e^{x^2} dx \right) \left(\int_{-1}^1 e^{y^2} \sin(y) dy \right)$$

The second integral is the integral of an odd function over a symmetrical interval, and so is equal to zero. Therefore,

$$\int_0^\pi \int_{-1}^1 e^{x^2+y^2} \sin(y) dy dx = 0.$$