Written Assignment 4

The following questions are related to sections 2.2 through 2.4 of *Vector Calculus* by Michael Corral.

Questions

0.0.1. Compute all first and second partial derivatives and the gradient for each of the following functions.

(a) $f(x,y) = xy\sin(\frac{x}{y})$

(b) $g(x,y) = \dfrac{x+y}{e^{xy}}$

0.0.2. Compute the gradient of the function $z = h(x,y)$ defined implicitly by the equation $xy = z^{x+y}$.

0.0.3. Find the equations of the planes tangent to the following surfaces at the specified points.

(a) $f(x,y) = \frac{x^2}{9} - y^2$ at the point $(2, \frac{1}{3})$.

(b) $x^2 + x + y - z^2 = 7$ at the point $(2,5,2)$.

0.0.4. Let $f(x,y) = e^{-(x^2+x+1+y^2-2y)}$.

(a) In which direction does f increase fastest from the point (1,1)?

(b) Compute the rate of change of f in the direction of the point $(5,4)$ from the point $(2,1)$.

0.0.5. Find the Jacobian matrix for the function $f(x,y) = \begin{bmatrix} \sin(x+y) \\ \ln(xy) \end{bmatrix}$.

0.0.6. **Application: Introduction to Partial Differential Equations**

A *partial differential equation* is an equation that involves the partial derivatives of a function. For example, $(\frac{\partial f}{\partial x})^2 - \frac{\partial^2 f}{\partial x \partial y} = 0$ is a partial differential equation that relates the first- and second-order partial derivatives of an unknown function f.

A function f is said to *satisfy* a partial differential equation when the equation holds upon substitution of the function's partial derivatives. For example, the function $f(x,y) = y$ satisfies the above partial differential equation since $(\frac{\partial}{\partial x}(y))^2 = 0$ and $\frac{\partial^2}{\partial x \partial y}(y) = \frac{\partial}{\partial x}(1) = 0$.

0.0.7. **Application: The Heat Equation**

Solutions

0.0.1. (a)

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(xy\sin(\frac{x}{y})) \\
&= y\sin(\frac{x}{y}) + xy\cos(\frac{x}{y})\frac{1}{y} \\
&= y\sin(\frac{x}{y}) + x\cos(\frac{x}{y}) \\
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial f}{\partial x} \\
&= \frac{\partial}{\partial x}(y\sin(\frac{x}{y}) + x\cos(\frac{x}{y})) \\
&= y\cos(\frac{x}{y})\frac{1}{y} + \cos(\frac{x}{y}) - x\sin(\frac{x}{y})\frac{1}{y} \\
&= 2\cos(\frac{x}{y}) - \frac{x}{y}\sin(\frac{x}{y}) \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \\
&= \frac{\partial}{\partial y}(y\sin(\frac{x}{y}) + x\cos(\frac{x}{y})) \\
&= \sin(\frac{x}{y}) + y\cos(\frac{x}{y})(\frac{-x}{y^2}) - x\sin(\frac{x}{y})(\frac{-x}{y^2}) \\
&= \sin(\frac{x}{y}) - \frac{x}{y}\cos(\frac{x}{y}) + \frac{x^2}{y^2}\sin(\frac{x}{y}) \\
&= \frac{x^2+y^2}{y^2}\sin(\frac{x}{y}) - \frac{x}{y}\cos(\frac{x}{y}) \\
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(xy\sin(\frac{x}{y})) \\
&= x\sin(\frac{x}{y}) + xy\cos(\frac{x}{y})(\frac{-x}{y^2}) \\
&= x\sin(\frac{x}{y}) - \frac{x^2}{y}\cos(\frac{x}{y}) \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\frac{\partial f}{\partial y} \\
&= \frac{\partial}{\partial x}(x\sin(\frac{x}{y}) - \frac{x^2}{y}\cos(\frac{x}{y})) \\
&= \sin(\frac{x}{y}) + x\cos(\frac{x}{y})(\frac{1}{y}) - \frac{2x}{y}\cos(\frac{x}{y}) + \frac{x^2}{y}\sin(\frac{x}{y})(\frac{1}{y}) \\
&= \frac{x^2+y^2}{y^2}\sin(\frac{x}{y}) - \frac{x}{y}\cos(\frac{x}{y}) \\
\nabla f &= \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \\
&= \begin{bmatrix} y\sin(\frac{x}{y}) + x\cos(\frac{x}{y}) \\ x\sin(\frac{x}{y}) - \frac{x^2}{y}\cos(\frac{x}{y}) \end{bmatrix}
\end{aligned}$$

(b)

$$
\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(\frac{x+y}{e^{xy}}) \\
&= \frac{e^{xy} - (x+y)ye^{xy}}{e^{2xy}} \\
&= \frac{1 - xy - y^2}{e^{xy}} \\
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial f}{\partial x} \\
&= \frac{\partial}{\partial x}(\frac{1 - xy - y^2}{e^{xy}}) \\
&= \frac{e^{xy}(-y) - (1 - xy - y^2)ye^{xy}}{e^{2xy}} \\
&= \frac{-2y + xy^2 + y^3}{e^{xy}} \\
\frac{\partial^2 f}{\partial y \partial x} &= \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \\
&= \frac{\partial}{\partial y}(\frac{1 - xy - y^2}{e^{xy}}) \\
&= \frac{e^{xy}(-x - 2y) - (1 - xy - y^2)xe^{xy}}{e^{2xy}} \\
&= \frac{-2x - 2y + x^2y + xy^2}{e^{xy}} \\
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(\frac{x+y}{e^{xy}}) \\
&= \frac{e^{xy} - (x+y)xe^{xy}}{e^{2xy}} \\
&= \frac{1 - x^2 - xy}{e^{xy}} \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y}\frac{\partial f}{\partial y} \\
&= \frac{\partial}{\partial y}(\frac{1 - x^2 - xy}{e^{xy}}) \\
&= \frac{e^{xy}(-x) - (1 - x^2 - xy)xe^{xy}}{e^{2xy}} \\
&= \frac{-2x + x^3 + x^2y}{e^{xy}} \\
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x}\frac{\partial f}{\partial y} \\
&= \frac{\partial}{\partial x}(\frac{1 - x^2 - xy}{e^{xy}}) \\
&= \frac{e^{xy}(-2x - y) - (1 - x^2 - xy)ye^{xy}}{e^{2xy}} \\
&= \frac{-2x - 2y + x^2y + xy^2}{e^{xy}} \\
\nabla f &= \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} \\
&= \begin{bmatrix} \dfrac{1 - xy - y^2}{e^{xy}} \\ \dfrac{1 - x^2 - xy}{e^{xy}} \end{bmatrix}
\end{aligned}
$$

0.0.2. We must first compute the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

$$
\begin{aligned}
\frac{\partial}{\partial x}(xy) &= \frac{\partial}{\partial x}(z^{x+y}) \\
y &= \frac{\partial}{\partial x}(e^{\ln(z)(x+y)}) \\
y &= e^{\ln(z)(x+y)}\frac{\partial}{\partial x}(\ln(z)(x+y)) \\
y &= e^{\ln(z)(x+y)}(\frac{1}{z}\frac{\partial z}{\partial x}(x+y)+\ln(z)) \\
y &= z^{x+y}(\frac{x+y}{z}\frac{\partial z}{\partial x}+\ln(z)) \\
y &= \frac{\partial z}{\partial x}(x+y)z^{x+y-1}+z^{x+y}\ln(z) \\
\frac{\partial z}{\partial x} &= \frac{y-z^{x+y}\ln(z)}{(x+y)z^{x+y-1}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial y}(xy) &= \frac{\partial}{\partial y}(z^{x+y}) \\
x &= \frac{\partial}{\partial y}(e^{\ln(z)(x+y)}) \\
x &= e^{\ln(z)(x+y)}\frac{\partial}{\partial y}(\ln(z)(x+y)) \\
x &= e^{\ln(z)(x+y)}(\frac{1}{z}\frac{\partial z}{\partial y}(x+y)+\ln(z)) \\
x &= z^{x+y}(\frac{x+y}{z}\frac{\partial z}{\partial y}+\ln(z)) \\
x &= \frac{\partial z}{\partial y}(x+y)z^{x+y-1}+z^{x+y}\ln(z) \\
\frac{\partial z}{\partial y} &= \frac{x-z^{x+y}\ln(z)}{(x+y)z^{x+y-1}}
\end{aligned}
$$

So $\nabla h(x,y) = \begin{bmatrix} \dfrac{y-z^{x+y}\ln(z)}{(x+y)z^{x+y-1}} \\ \dfrac{x-z^{x+y}\ln(z)}{(x+y)z^{x+y-1}} \end{bmatrix}$.

0.0.3. Recall that the equation of the plane tangent to the surface defined by $g(x,y,z)=0$ at the point (x_0,y_0,z_0) is $\nabla g(x_0,y_0,z_0)\cdot\begin{bmatrix} x-x_0 \\ y-y_0 \\ z-z_0 \end{bmatrix}=0$.

(a) Define $g(x,y,z)=f(x,y)-z=\frac{x^2}{9}-y^2-z$, so that the surface $z=f(x,y)$ is equivalently defined by the equation $g(x,y,z)=0$. We can then find the equation of the tangent plane at the point $(2,\frac{1}{3},\frac{1}{3})$ by computing ∇g and applying the above formula:

$$
\begin{aligned}
\nabla g(x,y,z) &= \begin{bmatrix} \frac{2x}{9} \\ -2y \\ -1 \end{bmatrix} \\
\nabla g(2,\frac{1}{3},\frac{1}{3}) &= \begin{bmatrix} \frac{4}{9} \\ -\frac{2}{3} \\ -1 \end{bmatrix}
\end{aligned}
$$

This yields the equation

$$\begin{aligned}\frac{4}{9}(x-2)-\frac{2}{3}(y-\frac{1}{3})-(z-\frac{1}{3})&=0\\ \frac{4}{9}x-\frac{2}{3}y-z&=\frac{1}{3}\\ 4x-6y-9z&=3.\end{aligned}$$

(b) Define $g(x,y,z)=x^2+x+y-z^2-7$. Then compute

$$\begin{aligned}\nabla g(x,y,z)&=\begin{bmatrix}2x+1\\1\\-2z\end{bmatrix}\\ \nabla g(2,5,2)&=\begin{bmatrix}5\\1\\-4\end{bmatrix}.\end{aligned}$$

Thus the equation of the tangent plane is

$$\begin{aligned}5(x-2)+(y-5)-4(z-2)&=0\\ 5x+y-4z&=7.\end{aligned}$$

0.0.4. (a) f will increase fastest in the direction of the vector $\nabla f(x_0,y_0)$ from the point (x_0,y_0). Therefore we compute and evaluate the gradient ∇f at the point $(1,1)$:

$$\begin{aligned}\nabla f&=\begin{bmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\end{bmatrix}\\ &=\begin{bmatrix}\frac{\partial}{\partial x}(e^{-(x^2+x+1+y^2-2y)})\\ \frac{\partial}{\partial y}(e^{-(x^2+x+1+y^2-2y)})\end{bmatrix}\\ &=\begin{bmatrix}(-2x-1)e^{-(x^2+x+1+y^2-2y)}\\ (2y-2)e^{-(x^2+x+1+y^2-2y)}\end{bmatrix}\\ \nabla f(1,1)&=\begin{bmatrix}-3e^{-2}\\0\end{bmatrix}\end{aligned}\tag{1}$$

(b) The rate of change of f in the direction of the vector $\mathbf{v}$ from the point (x,y) is given by $\nabla f(x,y)\cdot\mathbf{v}$. Using (1) we find that $\nabla f(2,1)=\begin{bmatrix}-5e^{-6}\\0\end{bmatrix}$. The direction from $(2,1)$ to $(5,4)$ is given by the vector $\begin{bmatrix}3\\3\end{bmatrix}$; thus the rate of change of f from $(2,1)$ towards $(5,4)$ is $\begin{bmatrix}-5e^{-6}\\0\end{bmatrix}\cdot\begin{bmatrix}3\\3\end{bmatrix}=-15e^{-6}$.

0.0.5. First we find the partial derivatives of each component with respect to each variable:

$$\begin{aligned}\frac{\partial f_1}{\partial x}&=cos(x+y)\\ \frac{\partial f_1}{\partial y}&=cox(x+y)\\ \frac{\partial f_2}{\partial x}&=\frac{1}{x}\\ \frac{\partial f_2}{\partial y}&=\frac{1}{y}\end{aligned}$$

The Jacobian matrix, then, is $\begin{bmatrix}cos(x+y)&cos(x+y)\\ \frac{1}{x}&\frac{1}{y}\end{bmatrix}$.