Written Assignment 3

Questions in this assignment are based on section 2.1 of *Vector Calculus* by Michael Corral.

Questions

1. For the following functions,

    - determine the domain of the function,
    - sketch the domain of the function in the xy plane, and
    - determine the range of the function

    (a) $f(x, y) = \frac{1}{\sqrt{(x-1)(y-1)}}$

    (b) $g(x, y) = \frac{\sqrt{x+1}}{yx^2+xy^2}$

    (c) $h(x, y) = \frac{xyz}{x^2+y^2-1}$

    (d) $z(x, y) = \ln(x + 2y)$

2. Evaluate the following limits or show that they do not exist.

    (a) $\displaystyle\lim_{(r,s)\to(0,2\pi)} \frac{3r^2 + rs^3 - 3r^4 \sin(s/4)}{r^2}$

    (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y}$

    (c) $\displaystyle\lim_{(x,y,z)\to(1,1,1)} \left|3x - 2y - z\right|$

    (d) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 - y^2 - z^2}{x^2 + y^2 + z^2}$

    (e) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{\sqrt{2x+y} - \sqrt{2x-y}}{2y}$

3. Consider the function

$$g(x, y, z) = \begin{cases} 1 & \text{if } (x, y, z) = (0, 0, 1) \\ x + y + 2z & \text{if } (x, y, z) \neq (0, 0, 1) \end{cases}.$$

    Find all points or regions where $g(x, y, z)$ has a discontinuity.

4. Evaluate the following limit, if it exists.

$$\lim_{(x,y)\to(1,0)} \frac{x(x-1)^3 + y^2}{4(x-1)^2 + 9y^3}$$

    Hint: approach the limit point along straight lines.

5. Sketch, by hand, the level curves of the following functions for the indicated values of c, if possible. You may of course want to check your answer using a calculator or with graphing software.

    (a) $f(x, y) = \frac{\ln y}{x^2}$, $c = -2, -1, 0, 1, 2$

    (b) $f(x, y) = \frac{x^2}{x^2+y^2}$, $c = 0, \frac{1}{4}, \frac{1}{2}, 1, 2$

6. (a) Find a function $f(x, y)$ whose family of level curves are straight lines that pass through the point $(2, 1)$.

    (b) Find a function $f(x, y)$ whose family of level curves are circles with radius $\sqrt{e^c}$.

7. Let m be any real number.

    (a) Provide an example of a function, $f(x, y)$, with the following two properties:
        - $f(x, y) \to m$ as $(x, y) \to (1, 0)$ along any parabola, $y = m(x-1)^2$
        - $f(x, y)$ is not constant

(b) Explain why the limit you provided in part (a) does not exist.

8. **Application to Temperature Distributions**
Suppose that a metal object occupies a space in three dimensions, and that the temperature, $T(x, y)$ of the object at the point (x, y) is inversely proportional to the distance between the point and the origin.

    (a) Write down an expression for T as a function of x and y.

    (b) Describe the level curves in words. Sketching them is not necessary, but the level curves are known as **isothermals**, and represent curves upon which the temperature is constant.

    (c) Suppose the temperature of the object at the point (2,1) is 10°C. Find the temperature of the object at (1,3).

9. **Application to Electrical Potential Distributions**
The scalar function
$$V(x, y) = \frac{c}{\sqrt{r^2 - x^2 - y^2}} ,$$
where c and r are positive constants, represents the electrical potential (in volts) at a point (x, y) in the $xy-$plane. Describe, in words, the level curves $V(x, y) = K$ for constant $K \in \mathbb{R}$, and determine the values of K for which the curves exist. Note that the level curves are known as **equipotential curves**, because all points on a given curve have the same potential. *can we do something a little more interesting with this question? its somewhat similar to the previous one*

10. **Bonus Problem**
This question goes beyond the requirements for this course. Before starting this problem, make sure that your instructor will give you marks for solving it.
Using the definition of limit (the ϵ, δ definition) for a function of two variables, prove that the limit
$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2}$$
exists. *Hint: start by evaluating the limit along the x-axis, or along other straight lines to determine what the limit could be equal to.*

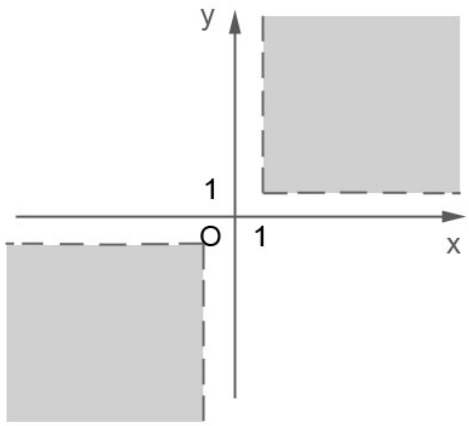
References
The temperature and electrical potential questions are based on a similar exercises that can be found in *Calculus, One and Several Variables*, 10th Edition, by Salas, Hille, Etgen, Wiley, 2007.

Solutions

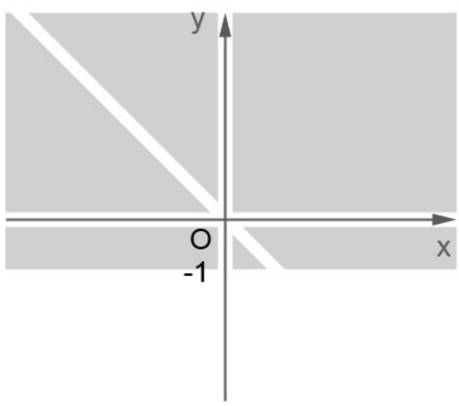
1. (a) The function is not defined when the denominator is zero, or when the argument of the square root is negative. Either x and y are greater than 1, or less than -1. The domain, D, is therefore

$$D = \{(x, y)|x > 1, y > 1; \text{ and } x < -1, y < -1\}.$$

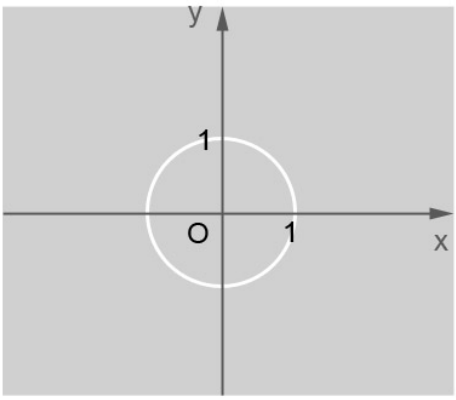
The function can't be zero, and cannot be negative, because the square root will always yield a positive number. The range is $f > 0$.



(b) The function is not defined when the denominator is zero, so we require that $yx^2 + xy^2 \neq 0$, or $xy(x + y) \neq 0$. This implies that the function is not defined along the lines $x = 0$, $y = 0$, and $y = -x$. Moreover, the numerator has a square root. The argument must be non-negative, and so we also have the restriction that $y \geq -1$. The domain can therefore be expressed as $D = \{(x, y)|y \geq -1, x \neq 0, y \neq 0, y \neq -x\}$. The numerator can take on any non-negative value, and the denominator can take on any value, so the function f can take on any value, so the range is $\mathbb{R}$.



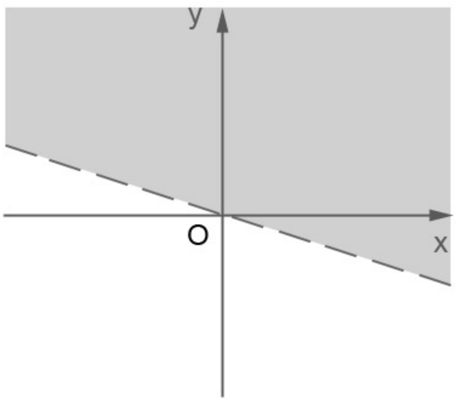
(c) The function is not defined when the denominator is zero, so we require that $x^2 + y^2 \neq 1$. The domain can therefore be expressed as $D = \{(x, y)|x^2 + y^2 \neq 1\}$. The numerator can take on any value, so the function f can take on any value, so the range is $\mathbb{R}$.



(d) For the domain, we require that

$$x + 2y > 0, \text{ or } y > -x/2.$$

The domain can therefore be expressed as $D = \{(x, y)|y > -x/2\}$. The function f can take on any value, so the range is $\mathbb{R}$.



2. (a) We can simply evaluate the limit to obtain

$$\lim_{(r,s)\to(0,2\pi)} \frac{3r^2 + rs^3 - 3\sin(s/4)}{r^2} = \lim_{(r,s)\to(0,2\pi)} 3 + \frac{s^3}{r} - 3r^2 \sin(s/4)$$

Because of the s^3/r term, this limit tends to infinity, and therefore does not exist.

(b) Let $f(x, y) = (x - y)/(x + y)$. Along the x-axis, $f = f(x, 0)$, so

$$f(x, 0) = \frac{x}{x} = 1, \quad x \neq 0.$$

A similar calculation shows that along the y-axis, f = -1, if $y \neq 0$. If we approach the point $(0, 0)$ along the x-axis and the y-axis, we find that

$$\begin{aligned} &\text{along the } x\text{-axis, } f = f(x, 0), \text{ and } f \to +1 \\ &\text{along the } y\text{-axis, } f = f(0, y), \text{ and } f \to -1 \end{aligned}$$

These limits are not equal, and so the limit does not exist.

(c) We can simply evaluate the limit to obtain

$$\lim_{(x,y,z)\to(1,1,1)} \left|3x - 2y - z\right| = 3 - 2 - 1 = 0$$

(d) Let $f(x, y, z) = (x^2 - y^2 - z^2)/(x^2 + y^2 + z^2)$. Then, along the x-axis, $f = (x, 0, 0)$. As long as x is not zero, f is equal to 1. However, along the y-axis, $f = f(0, y, 0)$. Provided that y is not zero, f is equal to -1. Therefore,

$$\begin{aligned}&\text{along the } x\text{-axis, } f \to +1\\&\text{along the } y\text{-axis, } f \to -1\end{aligned}$$

These limits are not equal, and so the limit does not exist.

(e) We can evaluate this limit by rationalizing the numerator.

$$\begin{aligned}\lim_{(x,y)\to(1,0)} \frac{\sqrt{2x+y} - \sqrt{2x-y}}{2y} &= \lim_{(x,y)\to(1,0)} \frac{\sqrt{2x+y} - \sqrt{2x-y}}{2y}\left(\frac{\sqrt{2x+y} + \sqrt{2x-y}}{\sqrt{2x+y} + \sqrt{2x-y}}\right)\\&= \lim_{(x,y)\to(1,0)} \frac{(2x+y) - (2x-y)}{2y\left(\sqrt{2x+y} + \sqrt{2x-y}\right)}\\&= \lim_{(x,y)\to(1,0)} \frac{2y}{2y\left(\sqrt{2x+y} + \sqrt{2x-y}\right)}\\&= \lim_{(x,y)\to(1,0)} \frac{1}{\sqrt{2x+y} + \sqrt{2x-y}}\\&= \frac{1}{2\sqrt{2}}\\&= \frac{\sqrt{2}}{2}\end{aligned}$$

3. The given function is defined everywhere. Now, for it to be continuous at any point (x_0, y_0, z_0), we require that

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} g(x, y, z) = g(x_0, y_0, z_0)$$

If we evaluate the limit as g approaches $(0, 0, 1)$, we have

$$\begin{aligned}\lim_{(x,y,z)\to(0,0,1)} f(x, y, z) &= \lim_{(x,y,z)\to(0,0,1)} x + y + 2z\\&= 0 + 0 + 2(1)\\&= 2\end{aligned}$$

However, at $(0, 0, 1)$, $g = 2$. So the given function is not continuous at the point $(0, 0, 1)$. Elsewhere, the function is a polynomial in three variables, and so will be continuous everywhere except at the point $(0, 0, 1)$.

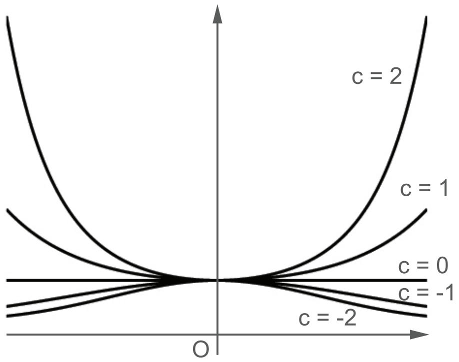
4. The lines $y = m(x - 1)$ all pass through the limit point, $(1, 0)$, where m is any real number. Approaching the limit point along these lines, our limit becomes

$$\begin{aligned}\lim_{(x,y)\to(1,0)} \frac{x(x-1)^3 + y^2}{4(x-1)^2 + 9y^3} &= \lim_{(x,y)\to(1,0)} \frac{x(x-1)^3 + m^2(x-1)^2}{4(x-1)^2 + 9m^3(x-1)^3}\\&= \lim_{(x,y)\to(1,0)} \frac{x(x-1) + m^2}{4 + 9(x-1)}\\&= \frac{m^2}{4}\end{aligned}$$

The result depends on m, which is an arbitrary value. Therefore, the limit does not exist.

5. (a) To plot the level curves, we set $z = c$, and solve for y:

$$\begin{aligned}c &= \frac{\ln y}{x^2}\\cx^2 &= \ln y\\y &= e^{cx^2}\end{aligned}$$



(b) To plot the level curves, we set $z = c$, and solve for y:

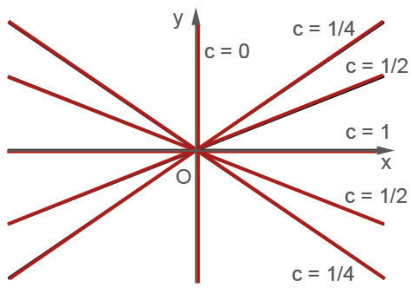
$$c = \frac{x^2}{x^2 + y^2}$$
$$cy^2 = (1 - c)x^2$$

If $c = 0$, then we obtain the level curve $x = 0$. If $c \neq 0$, then

$$y = \pm\sqrt{\frac{1-c}{c}}x, \quad c \neq 0$$

For $c = 2$, y is undefined, so the level curves do not exist. The level curves for the other values of c are shown in the graph below and are as follows:

if $c = 0, \quad x = 0$ (the vertical line that lies on the y-axis)
if $c = 1/4, y = \pm\sqrt{3}x$
if $c = 1/2, y = \pm x$
if $c = 1, \quad y = 0$



6. (a) The set of straight lines that pass through $(2, 1)$ are given by $y = c(x - 2) + 1$. Rearranging yields the equation

$$c = \frac{y-1}{x-2}$$

The desired function is

$$f(x, y) = \frac{y-1}{x-2}$$

(b) The set of circles with radius e^c are given by $x^2 + y^2 = e^c$. Applying the natural logarithm to both sides of the equation yields

$$\ln(x^2 + y^2) = c$$

The desired function is

$$f(x, y) = \ln(x^2 + y^2)$$

7. (a) Suppose we let

$$f(x,y) = \frac{y}{(x-1)^2}$$

Then, along any parabola $y = m(x-1)^2$ the limit becomes

$$\lim_{(x,y)\to(1,0)} \frac{y}{(x-1)^2} = \lim_{(x,y)\to(1,0)} \frac{m(x-1)^2}{(x-1)^2} = m$$

The function we chose therefore meets the specified criteria.

(b) The limit cannot exist because the limit depends on m, which is an arbitrary real number.

8. (a) The temperature can be written as

$$T(x,y) = \frac{k}{\sqrt{x^2+y^2}},$$

where k is an unknown constant of proportionality.

(b) The level curves are solution sets of the equation $T(x,y) = c$, for $c \in \mathbb{R}$. Therefore,

$$\begin{aligned} T(x,y) = c &= \frac{k}{\sqrt{x^2+y^2}} \\ c^2 &= \frac{k^2}{x^2+y^2} \\ x^2+y^2 &= \frac{k^2}{c^2} \end{aligned}$$

The level curves are concentric circles with radius k/c.

(c) At the point (2,1), the temperature is known, which allows us to solve for k.

$$\begin{aligned} T(2,1) = 10 &= \frac{k}{\sqrt{2^2+1^2}} \\ k &= 10\sqrt{5} \end{aligned}$$

At (1,3), the temperature is

$$\begin{aligned} T(1,3) &= \frac{10\sqrt{5}}{\sqrt{1^2+3^2}} \\ &= 10\sqrt{\frac{5}{4}} \end{aligned}$$

9. The level curves are solution sets of the equation $V(x,y) = K$, for constant $K \in \mathbb{R}$.

$$\begin{aligned} V(x,y) = K &= \frac{c}{\sqrt{r^2-x^2-y^2}} \\ r^2-x^2-y^2 &= \frac{c^2}{K^2} \\ x^2+y^2 &= r^2 - \frac{c^2}{K^2} \end{aligned}$$

The left-hand side must be positive, so we must have that

$$\begin{aligned} r^2 &> \frac{c^2}{K^2} \\ K^2 &> c^2/r^2 \\ |K| &< |c|/|r| \end{aligned}$$

But c, K, and r are all positive constants, so this simplifies to $K < \frac{c}{r}$. The level curves are concentric circles, centered at the origin, with radius $r^2 - \frac{c^2}{K^2}$, for $K < \frac{c}{r}$.

10. If we approach the limit point along the x-axis, then $y = 0$ and we obtain

$$\lim_{(x,y)\to(0,0)} \frac{x(0)^2}{x^2 + (0)^2} = 0$$

We obtain the same result if we approach the origin along the y-axis, or along any line $y = mx$. It would seem that the limiting value could exist and could be equal to 0. To show that this is the case, we must show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\left|\frac{xy^2}{x^2 + y^2} - 0\right| < \epsilon \text{ whenever } 0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta,$$

or simply

$$\left|\frac{xy^2}{x^2 + y^2}\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2 + y^2} < \delta.$$

Now, $\left|x^2 + y^2\right| = x^2 + y^2$, and $|y^2| = y^2$, so

$$\left|\frac{xy^2}{x^2 + y^2}\right| = \frac{\left|xy^2\right|}{\left|x^2 + y^2\right|} = \frac{|x|y^2}{x^2 + y^2}$$

Also, $|x| \leq \sqrt{x^2 + y^2}$, and $y^2 \leq x^2 + y^2$, so

$$\begin{aligned}\left|\frac{xy^2}{x^2 + y^2}\right| &= \frac{|x|y^2}{x^2 + y^2} \\ &\leq \frac{\sqrt{x^2 + y^2}(x^2 + y^2)}{x^2 + y^2} \\ &= \sqrt{x^2 + y^2}\end{aligned}$$

So if $\delta = \epsilon$, then

$$\left|\frac{xy^2}{x^2 + y^2}\right| < \epsilon \text{ whenever } 0 < \sqrt{x^2 + y^2} < \delta.$$