# Written Assignment Questions and Solutions

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***Note: hyperlinks in Part II of table of contents are kinda broken. Perhaps we should create two documents: Questions and Solutions. Or try using a different document class.***

# Part I
# Questions for Written Assignments

## 2 Multivariable Functions and Algebra

### 2.1 Dot Products and Cross Products

*The following questions are related to sections 1.3 and 1.4 of Vector Calculus by Michael Corral.*

2.1.1. Find two unit vectors perpendicular to vectors $\begin{bmatrix}1\\0\\4\end{bmatrix}$ and $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

2.1.2. Suppose vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are in $\mathbb{R}^3$, and $\mathbf{a} \neq 0$. If $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, does it follow that $\mathbf{b} = \mathbf{c}$? Briefly explain your answer.

2.1.3. Consider the following vectors in $\mathbb{R}^2$:

$$\mathbf{a} = \begin{bmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}, \quad \mathbf{r} = \begin{bmatrix}3\\1\end{bmatrix}.$$

(a) Verify that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular unit vectors.

(b) Use your results from part (a) to find constants $C_1$ and $C_2$ so that $\mathbf{r} = C_1\mathbf{a} + C_2\mathbf{b}$.

2.1.4. Suppose that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are nonzero vectors in $\mathbb{R}^3$. Prove the following statements.

(a) If $||\mathbf{a} + \mathbf{b}||^2 = ||\mathbf{a}||^2 + ||\mathbf{b}||^2$, then vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular.

(b) If $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times \mathbf{c}$ and $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, then $\mathbf{b} = \mathbf{c}$.

2.1.5. Suppose that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are nonzero vectors in $\mathbb{R}^3$. Identify which of the following statements are meaningless and which statements are not meaningless. Explain your reasoning.

(a) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

(b) $\mathbf{a} \cdot (\mathbf{b} \cdot \mathbf{c})$

(c) $\mathbf{a} \times (\mathbf{b} \cdot \mathbf{c})$

(d) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$

2.1.6. **Application to Mechanics**
Torque plays a fundamental role in many branches of engineering. It is vector that describes rotation rotation about a point or axis due to an applied force. If you haven't yet encountered torque, its mathematical definition is straightforward:

> The torque, $\boldsymbol{\tau}$, about a pivot point $P$, that is produced by a force $\mathbf{F}$ applied at a point $Q$, is defined as $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$, where $\mathbf{r} = \overrightarrow{PQ}$.

Torque is a vector that is produced by an applied force. For example, suppose that a force $\mathbf{F} = \frac{1}{2}\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$ is applied at the point $(1, C, 0)$, about the pivot point (1,0,0), where $C \in \mathbb{R}$ is a constant. The resulting torque is simply

$$\boldsymbol{\tau} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1-1 & C-0 & 0 \\ 0.5 & 0 & 0 \end{vmatrix} = -\frac{C}{2}\mathbf{k}$$

A plot of the applied force and produced torque vectors is shown in Figure 1.

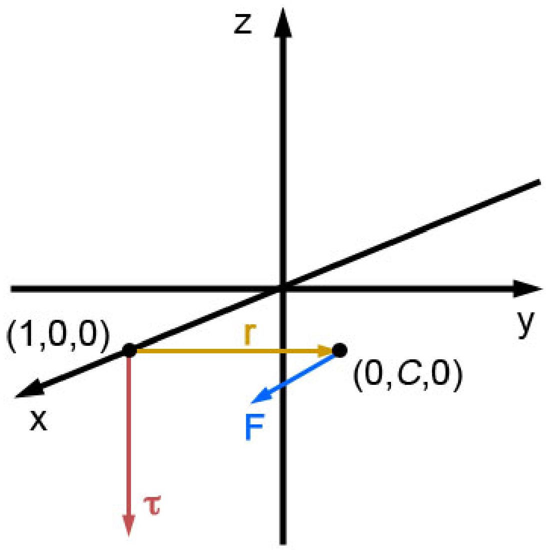



Figure 1: The torque, $\boldsymbol{\tau}$, about the pivot point (1,0,0), is produced by the applied force $\mathbf{F}$.

Note that the magnitude of the torque vector is $||\boldsymbol{\tau}|| = C/2$, and that if we increase the value of $C$, two things happen: the applied force moves further away from the pivot point, and the magnitude of the produced torque increases.

**Units**
When $\mathbf{F}$ has units of Newtons (N), and $\mathbf{r}$ has units of meters (m), torque has units of $\mathrm{N} \cdot \mathrm{m}$.

**Questions**
Suppose that a bicycle has pedal arms that are 0.14 m long, and that a constant downward force of 100 N is applied by a cyclist on one pedal. Let $\theta \in [0, 360°)$ be the angle between the vertical and the pedal arm, as shown in Figure 2.

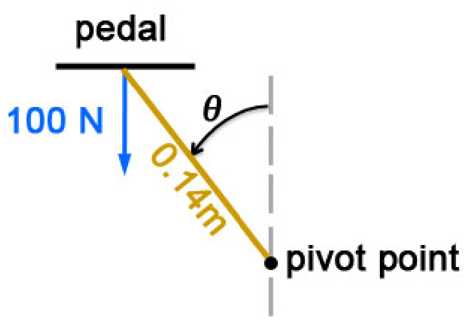



Figure 2: The pedal arm (orange) rotates about the pivot point. The angle it makes with the vertical direction is $\theta$, measured counter-clockwise. A constant downward force of 100 N is applied to the pedal, which is attached to the other end of the pedal arm.

(a) Determine the magnitude of the torque about the pivot point when $\theta$ is $30°$ and when $\theta$ is $90°$. Your answers should include the units of measurement.

(b) Determine the magnitude of the torque about the pivot point for any $\theta$. Your answer should be a function of $\theta$.

(c) What value(s) of $\theta$ minimize the produced torque? Briefly explain why. You should not need to compute any derivatives to answer this question.

Naturally, the torque vector can also be a function of time. In our bicycle example, the torque that is produced by the cyclist could change continuously as the pedal rotates about the pivot point. We will revisit the concept of torque as a vector function in a later assignment, after we have defined vector functions and their derivatives.

2.1.7. Find all values of $x$ that satisfy the following equations.

(a) $\begin{vmatrix} 3 & 2 & 0 \\ 1 & x & 0 \\ 7 & -3 & 4 \end{vmatrix} = 4.$

(b) $\begin{vmatrix} x & 1 \\ 4 & 4x \end{vmatrix} = \begin{vmatrix} -x & -2 \\ 2 & 2x+8 \end{vmatrix}.$

## 2.2 Lines and Planes

*The following questions are related to Section 1.5 of Vector Calculus by Michael Corral.*

2.2.1. Find vector equations for the following lines.

(a) Find the vector equation for the line that passes through the points (-1,2,1), and (3,-2,1).

(b) Find the vector equation for the line that has parametric equations $x = 2+2t, y = 9t, z = 5+t$.

2.2.2. Calculate the following angles.

(a) Calculate the angle between the vectors $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix}$.

(b) Calculate the angle between the vector $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and the plane $2x + y - z = 0$.

2.2.3. **Distance Between a Point and a Line**
Find the distance between the point (1,0,0) and the line $x = 1, y = t, z = 1$.

2.2.4. **Distance Between a Point and a Plane**
Consider the point $(-1, C, 1)$ and the plane $2x+y-z = 3$, where $C$ is an unknown constant. Find all possible values of $C$ such that the distance between the given point and plane is equal to $\sqrt{6}$.

2.2.5. Consider the points $P, Q, R$

$$\begin{aligned} P &= (1, 0, 3) \\ Q &= (2, 2, 3) \\ R &= (0, 0, -1) \end{aligned}$$

(a) Find a vector that is perpendicular to the plane that passes through these points.

(b) Find the area of the triangle $\triangle PQR$.

(c) Find a point $S$ so that a unique plane that passes through $P, Q$, and $S$ cannot be found. Describe why you cannot find a unique plane that passes through $P, Q$, and $S$.

2.2.6. **Equations of Planes**
Find the equation of the plane that

(a) passes through the points (-1,2,1), (3,-2,1), and (-1,1,-1).

(b) passes through the point (-1,2,1) and contains the line $x = y = z$.

2.2.7. Suppose we have two planes $2x+y-z = 3$ and $x+3y+z = 0$. Find the line of intersection between these two planes, and find the equation of the plane that passes through the line of intersection and through the point $(0, 0, 0)$.

2.2.8. Suppose that $L_1$ is a line in $\mathbb{R}^3$.

(a) Is it possible to find another line, $L_2$, that is parallel $L_1$, and intersects $L_1$ at only one point? Show that it is possible, or show that it is not possible with a counterexample.

(b) Suppose that there is a line $L_3$ that is is parallel to $L_1$, and that a plane $P$ includes both $L_1$ and $L_3$. Is it always true that there exists a plane that is perpendicular to $P$?

## 2.3 Surfaces

***We'll have 3 or 4 questions on this topic here.***

## 2.4 Curvilinear Coordinates

***We'll have 3 or 4 questions on this topic here.***

## 2.5 Vector-Valued Functions

*The following questions are related to Section 1.8 of Vector Calculus by Michael Corral.*

2.5.1. Find the values of $t$ and the points on the curve

$$\mathbf{r}(t) = (1+t^2)\mathbf{i} + t\mathbf{j}, \quad t \in \mathbb{R}$$

where

(a) $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are perpendicular,

(b) $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ have the same direction, and

(c) $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ have opposite directions.

2.5.2. **Application to Mechanics**
Recall from a previous assignment that we defined torque as $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$. Now that we have introduced the concepts of vector-valued functions and their derivatives, let's consider the more general case when $\boldsymbol{\tau}$, $\mathbf{r}$, and $\mathbf{F}$ are all functions of time:

$$\boldsymbol{\tau}(t) = \mathbf{r}(t) \times \mathbf{F}(t).$$

Moreover, using the relation $\mathbf{F}(t) = m\mathbf{a}(t)$ (Newton's second law), we can write our definition of torque as

$$\begin{aligned}\boldsymbol{\tau}(t) &= \mathbf{r}(t) \times \mathbf{F}(t)\\ &= \mathbf{r}(t) \times \big(m\mathbf{a}(t)\big)\\ &= m\big(\mathbf{r}(t) \times \mathbf{r}''(t)\big).\end{aligned}$$

These alternate forms for the torque vector may be helpful in solving the following problems.

(a) If the position of a particle with mass $m$ is given by the position vector $\mathbf{r}(t)$, then its angular momentum is a vector defined as $\mathbf{L}(t) = m\mathbf{r}(t) \times \mathbf{r}'(t)$. Show that $\mathbf{L}'(t) = \boldsymbol{\tau}(t)$.

(b) Show that if the torque is a zero vector for all $t$, then the angular momentum of the particle is constant for all $t$. This is what is known as the **law of conservation of angular momentum**.

2.5.3. **Integration with Vector-Valued Functions**
If the position of a particle is given by the vector function $\mathbf{r}(t) \in \mathbb{R}^3$, then we know that we can determine its velocity, $\mathbf{v}(t) = \mathbf{r}'(t)$, by differentiating each of its components. That is, if

$$\mathbf{r}(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{bmatrix},$$

then

$$\mathbf{v}(t) = \mathbf{r}'(t) = \begin{bmatrix} \frac{d}{dt} r_1(t) \\ \frac{d}{dt} r_2(t) \\ \frac{d}{dt} r_3(t) \end{bmatrix} = \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix},$$

provided that the derivatives of the components of $\mathbf{r}$ exist at $t$. It follows from the Fundamental Theorem of Calculus that if we were instead given the velocity of the particle, we could compute its position by integrating each of the components with respect to $t$. We would of course introduce constants of integration. That is, given

$$\mathbf{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix},$$

we could obtain $\mathbf{r}$ by integrating each of the components of $\mathbf{v}(t)$

$$\mathbf{r}(t) = \begin{bmatrix} \int v_1(t)dt \\ \int v_2(t)dt \\ \int v_3(t)dt \end{bmatrix} = \begin{bmatrix} r_1(t) + c_1 \\ r_2(t) + c_2 \\ r_3(t) + c_3 \end{bmatrix},$$

where $c_1, c_2, c_3$ are constants.

Suppose that a particle with mass $m$ is subjected to a force, $\mathbf{F}(t) = m\pi^2\big(\cos(\pi t)\mathbf{j} + \sin(\pi t)\mathbf{k}\big)$, where $t \geq 0$. Suppose also that when $t = 0$,

$$\mathbf{r}(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{r}'(0) = \begin{bmatrix} +1 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Using the relation $\mathbf{F} = m\mathbf{a}$, find the velocity of the particle at time $t$. *Hint: you will need to apply the velocity at time t=0.*

(b) Find the position of the particle at time $t = 1$.

(c) Plot the position of the particle at times $t = 0$ and $t = 1$ on the same graph in $\mathbb{R}^3$.

## 2.6 Linear Systems

*The following questions are related to Section 8.1 of College Algebra by Carl Stitz and Jeff Zeager.*

2.6.1. For $A$ and $\mathbf{b}$ below, solve the linear system $A\mathbf{x} = \mathbf{b}$, if possible:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & -1 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 4 & 0 & 3 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 10 \end{bmatrix}.$$

If it isn't possible to solve this system, explain why.

2.6.2. For the systems below,

- Compute the determinant of matrix $A$, if possible. If it is not possible to do so, explain why.
- Solve the linear system $A\mathbf{x} = \mathbf{b}$, if possible.
- State whether the system has no solution, infinitely many solutions, or a unique solution.

(a) $A = \begin{bmatrix} -3 & 1 & 2 & 4 \\ 2 & -1 & 2 & 3 \\ 1 & 0 & 4 & -2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 0 \\ 4 & 9 & -2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

2.6.3. Consider the system of simultaneous linear equations

$$\begin{aligned} x + 2y - z &= 2 \\ 2x + ay - 2z &= b \\ 3x + 2y &= 1 \end{aligned}$$

where $x, y, z$ are unknown.

(a) Find all values of $a$ and $b$ such that the above system has
   i. exactly one solution;
   ii. no solutions;
   iii. infinitely many solutions.

(b) For those values of $a$ and $b$ from 2.6.3.(a)i, what is the unique solution?

(c) For those values of $a$ and $b$ from 2.6.3.(a)iii, parameterize the set of all solutions.

2.6.4. **Application to Polynomial Interpolation**

In many areas of engineering, experimental data is collected that must be analyzed to extract parameters that tell us something about a physical process. Suppose we have measured a set of experimental data that are represented in the $xy$-plane. An **interpreting polynomial** for the measured data is a polynomial that passes through every measured point. We can use this polynomial, for example, to estimate values between the measured data points.

Suppose for example that we have measured the data points (0,-6), (1,-2), (2,4), (3,10). To find an interpreting polynomial of order 2 for these data, we would try to find a polynomial of the form $p(x) = a_0 + a_1 x + a_2 x^2$ that passes through all four measured points. In other words, we need to find the unknown constants $a_0, a_1, a_2$ that satisfy the equations

$$\begin{aligned} p(0) &= a_0 + a_1(0) + a_2(0)^2 = -6 \\ p(1) &= a_0 + a_1(1) + a_2(1)^2 = -2 \\ p(2) &= a_0 + a_1(2) + a_2(2)^2 = 4 \\ p(3) &= a_0 + a_1(3) + a_2(3)^2 = 10 \end{aligned}$$

The above system has four equations and four unknowns. Upon solving this system, you should be able to determine that $a_0 = -6, a_1 = 4, a_2 = 0$.

**Wind Tunnel Experiment**
In a fictitious wind tunnel experiment[1], the following measurements were made.

| Velocity (m/s) | Force (N) |
|---|---|
| 0 | 0 |
| 1 | 5.5 |
| 2 | 20 |
| 3 | 46.5 |
| 4 | 88 |

The data represent the measured force due to air resistance, on an object suspended in the tunnel, measured at different air speed velocities.

(a) Using the data above, derive a 5×4 system of equations, that when solved, find an interpreting polynomial of the form $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. Write the system in the form $A\mathbf{x} = \mathbf{b}$.

(b) Solve your system to obtain $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$.

(c) As mentioned above, engineers sometimes use interpreting polynomials to estimate values in between measured data points. Using your polynomial, estimate the value of the force when the velocity is 1.5 m/s.

(d) In practice, it can be difficult to determine what order of polynomial to use. Sometimes, polynomials of different orders must be used to decide which polynomial yields the most useful results. For the above data, explain what what would happen if we used a polynomial less than 3. It may help to see what happens if we use a $1^{st}$ order polynomial.

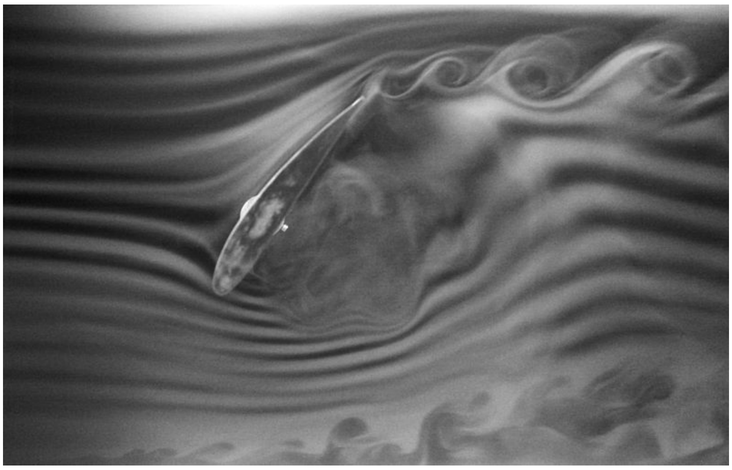


Figure 3: An airfoil in a fog wind tunnel (image from Wikimedia Commons, Smart Blade GmbH).

[1]The wind tunnel problem was based on a similar exercise in Linear Algebra and Its Applications, 4th Edition, by David C. Lay, Addison-Wesley, 2012.

# 3 Multiple Integrals

## 3.0 Wolfram Alpha Syntax

You may want to use Wolfram Alpha (wolframalpha.com) to check your answers. If you're not sure what syntax to use to compute double integrals with Wolfram Alpha, let's suppose that we want to determine the value of

$$\int_{-2}^{-1}\int_{0}^{x-1}(x^{2C}+y)dydx$$

where $C$ is a constant. The syntax we could use to compute this particular integral is

```
integrate x^{2C}+y dydx, x from -2 to -1 and y from 0 to (x-1)
```

## 3.1 Double Integrals

*The following questions are related to Section 3.1 of Vector Calculus by Michael Corral.*

3.1.1. Consider the solid that lies under the plane $z = -x - 2y + 2$ and above the rectangle $\{(x, y) \mid -2 \le x \le 0, 0 \le y \le 1\}$.

(a) Sketch the solid in $\mathbb{R}^3$. *Hint: start by plotting the points that are located on the given plane and above the corners of the rectangle. Then connect the points with solid lines.*

(b) Find the volume of the solid.

3.1.2. **Application to Fluid Mechanics**
In a two-dimensional, steady-state, incompressible fluid flow, the velocity $\mathbf{v}$ of the flow can be expressed as $\mathbf{v} = u(x, y)\mathbf{i} + v(x, y)\mathbf{j}$. The functions $u(x, y)$ and $v(x, y)$ must satisfy

$$\nabla \cdot \mathbf{v} = 0.$$

(a) If $u(x, y) = x^2 + y^2$, find the most general form of $v(x, y)$.

(b) If $v(x, y) = \cos(x)$, find the most general form of $u(x, y)$.

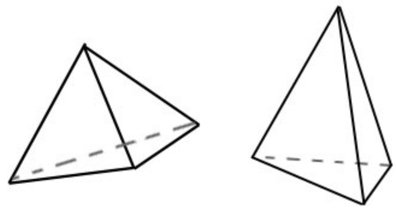
*Hint: you are asked to find the most* ***general*** *form of functions u and v.*

## 3.2 Double Integrals Over a General Region

*The following questions are related to Section 3.2 of Vector Calculus by Michael Corral.*

3.2.1. **Volume of a Tetrahedron**

A **tetrahedron** is a three dimensional object with four, triangular, flat sides. Because each of its four sides are flat, a tetrahedron can be defined as the region enclosed by four planes. Below is a sketch of two tetrahedrons. Note that each tetrahedron has four vertices, six edges, and that the lengths of its edges do not have to be equal.



Consider the tetrahedron that is bounded by the three coordinate planes in $\mathbb{R}^3$, and by the plane $z = 1 - x - \frac{y}{2}$.

(a) Sketch the tetrahedron in $\mathbb{R}^3$ and label the points that represent its four vertices.

(b) Set up, but do not evaluate, a double integral that represents the volume of the tetrahedron. Integrate with respect to $x$ first.

(c) Set up, but do not evaluate, a double integral that represents the volume of the tetrahedron. Integrate with respect to $y$ first.

Note that

- Examples 3.4 and 3.5 from *Vector Calculus* by Michael Corral are similar to this problem.
- We could also calculate the volume of the solid by using a **triple** integral.
- Although not required, the double integrals are straightforward to compute. You may want to check your answers by evaluating the integrals and seeing if you would get the same result in parts (b) and (c).

3.2.2. **Area of a General Region**

(a) Question 11 from Section 3.2 of *Vector Calculus* by Michael Corral. *Hint: Figure 3.2.5 is also helpful.*

(b) Use the result from part (a) to compute the area of the region bounded by the curves $y = x^2$ and $x = y^2$.

3.2.3. Find the volume of the solid enclosed by $z = x^2 + y^2$, $y = x^2$ and $x = y^2$.

3.2.4. Consider the integral

$$\iint\limits_R x\sin(y)dA$$

where $R$ is the region bounded by $y = 0$, $y = x^2$, and $x = 2$.

(a) Evaluate the double integral by first integrating with respect to $x$.

(b) Evaluate the double integral by first integrating with respect to $y$.

Note that your answers for both parts should be the same, and that you may need to use various techniques of integration to complete this problem, including integration by parts and a variable substitution.

3.2.5. Consider the double integral

$$\int_0^{1+e}\int_0^{\ln(x-1)} f(x,y)dydx.$$

Sketch the region in $\mathbb{R}^2$ over which $f(x,y)$ is integrated, and change the order of integration.

3.2.6. Consider the double integral

$$\iint_D f(x,y)dA,$$

where $D$ is the square $0 \le x \le 1$, $0 \le y \le 1$, and $f(x,y) = \sin(x+y)$. Show that

$$0 \le \iint_D f(x,y)dA \le 1.$$

3.2.7. **Simplifying Double Integrals Using Symmetry**
Certain integrals can be simplified when the integrand is either even or odd. You may know that, for functions of one variable, that

$$\text{if } f(x) \text{ is odd, then } \int_{-a}^{a} f(x)dx = 0.$$
$$\text{if } f(x) \text{ is even, then } \int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$$

There are similar results for integrals of two variables, but in order to introduce them, we first need to extend our concepts of even and odd functions to functions of two variables, and we need to describe regions that are symmetric about the $x$-axis and about the $y$-axis.

If $R$ is a region that is **symmetric about the $y$-axis**, and $(x_0, y_0)$ is a point inside $R$, then the point $(-x_0, y_0)$ is also inside $R$. Similarly, if $S$ is a region that is **symmetric about the $x$-axis**, and $(x_1, y_1)$ is a point inside $S$, then the point $(x_1, -y_1)$ is also inside $S$. Above are examples of regions that have these symmetries.

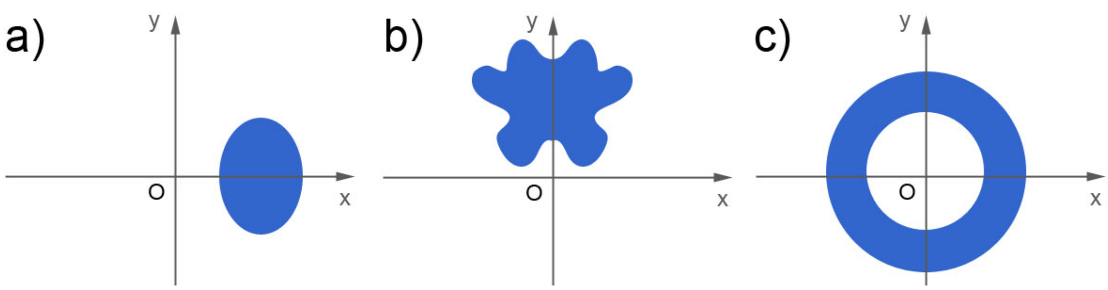



Figure 4: a) the blue region is symmetric about the $x$-axis, b) the blue region is symmetric about the $y$-axis, and c) the blue region is symmetric about the $x$-axis and the $y$-axis.

Moreover, if $g(x,y)$ is odd in $x$, then

$$g(-x,y) = -g(x,y).$$

But if $g(x,y)$ were even in $x$, then

$$g(-x,y) = g(x,y).$$

Combining these concepts yields helpful results for computing double integrals. For example, if $R$ is symmetric about the $y$-axis, and if $g(x,y)$ is odd in $x$, then

$$\iint_R g(x,y)dxdy = 0.$$

Similar results can be stated if $g$ were even in $x$ or in $y$, and if $R$ has symmetry about the $x$-axis.

(a) Provide an example of a non-zero function of two variables, $h(x,y)$, that is odd in $x$. Verify that your function is odd in $x$.

(b) Provide an example of a region, $D$, that is symmetric about the $y$-axis, but is not symmetric about the $x$-axis.

(c) Show that, for your $h(x, y)$ and region $D$, that

$$\iint_D g(x, y) dx dy = 0.$$

## 3.3 Triple Integrals

*The following questions are related to Section 3.3 of Vector Calculus by Michael Corral.*

3.3.1. **Volume of a Tetrahedron**
The textbook points out that the triple integral

$$\iiint_S f(x,y,z)dV$$

for the special special case when $f(x,y,z) = 1$ for all points in $S$, gives the volume of $S$

$$V(S) = \iiint_S dV.$$

Consider again the tetrahedron that is bounded by the three coordinate planes in $\mathbb{R}^3$, and by the plane $z = 1 - x - \frac{y}{2}$. We derived an expression for the volume of this tetrahedron in a previous question using a double integral. Now set-up and find the volume of the tetrahedron using a triple integral.

3.3.2. **Volume of an Ellipsoid**
Solve Question 10 from Section 3.5 of *Vector Calculus* by Michael Corral, which is the following:

> Show that the volume inside the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{4\pi abc}{3}$. (Hint: Use the change of variables $x = au$, $y = bv$, $z = cw$, then consider Example 3.12.)

3.3.3. **Changing the Order of Integration**
Express the triple integral in three different ways. ***Change the limits here***

$$\int_0^2 \int_0^{1-y/2} \int_0^{1-x-y/2} f(x,y,z)dzdxdy$$

Do not use cylindrical or spherical coordinates. *Hint: you may want to check that your integrals represent the same volume for particular integrands f(x,y,z) using WolframAlpha.*

## 3.4 Change of Variables in Multiple Integrals

*The following questions are related to Section 3.5 of Vector Calculus by Michael Corral.*

3.4.1. **Linear Transformations**
Under the linear transformation

$$x = c_1 u + c_2 v, \quad y = d_1 u + d_2 v, \quad d_1 c_2 - d_2 c_1 \neq 0,$$

straight lines in the $uv$-plane are mapped to straight lines in the $xy$-plane.

(a) $v = v_0$ is a horizontal line in the $uv$-plane. Determine the equation of this line in the $xy$-plane.

(b) $x = x_0$ is a vertical line in the $xy$-plane. Determine the equation of this line in the $uv$-plane.

3.4.2. Use an appropriate transformation to evaluate the integral

$$\iint_R (x^2 - y^2)dxdy,$$

where $R$ is the parallelogram bounded by

$$x + y = 0, \quad x + y = 1, \quad x - y = 0, \quad x - y = 1.$$

3.4.3. **Simplifying Double Integrals**
When working with double integrals over a rectangular region $R = \{(x, y)|a \leq x \leq b, c \leq y \leq d\}$, we can use the simplification

$$\iint_R g(x)h(y)dA = \int_a^b g(x)dx \int_c^d h(y)dy$$

Use this property and the transformation $x = 3u, y = 2v$ to evaluate the double integral

$$\iint_E x^2 \; dxdy,$$

where $E$ is region bounded by the ellipse $4x^2 + 9y^2 = 36$.

3.4.4. Evaluate the integral

$$\iiint_S y^2 dV,$$

where $S$ is the solid that lies inside the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$ and below the cone $z^2 = 9x^2 + 9y^2$.

3.4.5. **Triple Integral In Cylindrical Coordinates**
Evaluate the integral

$$\iiint_V dV,$$

using cylindrical coordinates, where $V$ is the region bounded by

$$\begin{aligned} &0 \leq x \leq 2 \\ &0 \leq y \leq \sqrt{4 - x^2} \\ &0 \leq z \leq \sqrt{4 - (x^2 + y^2)} \end{aligned}$$

3.4.6. **Triple Integral In Spherical Coordinates**
The integral

$$\int_0^{\pi/4} \int_0^{\pi/2} \int_0^1 (\rho^2 \sin \phi) \; d\rho \; d\phi \; d\theta$$

represents the volume of a solid. Describe the shape of the solid, and find its volume.
***Textbook doesn't do a great job of integrals in cylindrical and spherical***

## 3.5 Application: Center of Mass

*The following questions are related to Section 3.6 of Vector Calculus by Michael Corral.*

3.5.1. **Center of Mass of a 2D Triangular Plate**
A two-dimensional plate has density $\delta(x, y) = xy$, and occupies a triangular region whose vertices are located at the points (0,0), (1,2), and (1,4). Find the $x$-coordinate and the $y$-coordinate of the center of mass of the triangular plate.

3.5.2. **Center of Mass of a 2D Plate, Radial Density Function**
A 2D semi-circular plate of mass $M$ is bounded by

$$-a \le x \le a, \quad 0 \le y \le \sqrt{a^2 - x^2}, \quad a > 0.$$

The density of the plate at a point $(x, y)$, is equal to the shortest distance, $L$, between that point and the upper edge of the plate, as shown in Figure 5.

(a) Set up an integral that represents the $x$-coordinate of the center of mass of the plate, $\bar{x}$.
(b) Set up an integral that represents the $y$-coordinate of the center of mass of the plate, $\bar{y}$.
(c) Determine the $x$-coordinate of the center of mass of the plate, without performing any integration. Briefly describe how you found your answer.

You do not need to perform any integration for this question. Note also that $L$ is a function of $x$ and $y$.

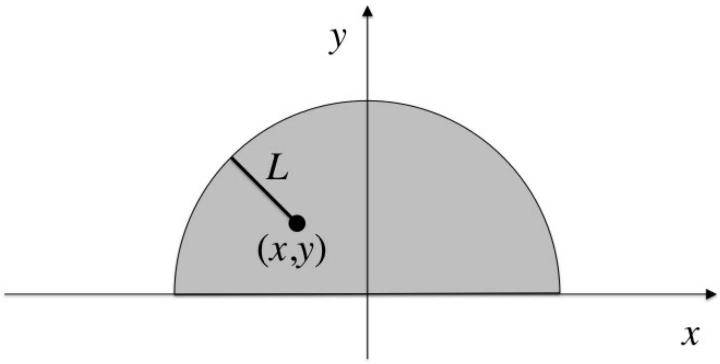



Figure 5: The density of the plate at $(x, y)$ is equal to $L$.

3.5.3. **Center of Mass of a 2D Plate, Radial Density Function**

3.5.4. ***I'll push this question to a midterm or final exam***
Determine the value of

$$\int_0^\pi \int_{-1}^1 x^4 e^{x^2+y^2} \sin(y)\, dy dx.$$

Do not use integration by parts.

# Part II
# Solutions to Written Assignment Questions

## 2 Multivariable Functions and Algebra

### 2.1 Dot Products and Cross Products

2.1.1. A vector, $\mathbf{v}$, that is perpendicular to both vectors can be found using the cross product.

$$\mathbf{v} = \begin{bmatrix}1\\0\\4\end{bmatrix} \times \begin{bmatrix}1\\-2\\1\end{bmatrix} = \begin{bmatrix}8\\3\\-2\end{bmatrix}.$$

This is not a unit vector, but we can make this a unit vector by dividing each component by the magnitude of the vector. The magnitude is

$$||\mathbf{v}|| = \sqrt{8^2 + 3^2 + (-2)^2} = \sqrt{77}.$$

A unit vector that is perpendicular to both vectors is

$$\frac{\mathbf{v}}{\sqrt{77}} = \frac{1}{\sqrt{77}}\begin{bmatrix}8\\3\\-2\end{bmatrix}.$$

A second unit vector that is perpendicular to the two given vectors is

$$-\frac{\mathbf{v}}{\sqrt{77}} = \frac{-1}{\sqrt{77}}\begin{bmatrix}8\\3\\-2\end{bmatrix}.$$

2.1.2. It is not necessarily true that $\mathbf{b} = \mathbf{c}$. Simple rearrangement yields

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= \mathbf{a}\cdot\mathbf{c}\\ 0 &= \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{c}\\ 0 &= \mathbf{a}\cdot(\mathbf{b}-\mathbf{c})\end{aligned}$$

Therefore, $\mathbf{a}$ is perpendicular to the vector $\mathbf{b}-\mathbf{c}$, which can be true when $\mathbf{b} \neq \mathbf{c}$. A counterexample would be the vectors

$$\mathbf{a} = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix}1\\-1\\0\end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Clearly, $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c}$, even though $\mathbf{b} \neq \mathbf{c}$.

2.1.3. (a) Vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular because their dot product is zero:

$$\mathbf{a}\cdot\mathbf{b} = \frac{1}{2} - \frac{1}{2} = 0.$$

Vectors $\mathbf{a}$ and $\mathbf{b}$ are both unit vectors, because they have length 1:

$$\begin{aligned}||\mathbf{a}|| &= \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2} = 1\\ ||\mathbf{b}|| &= \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 1\end{aligned}$$

(b) Using results from part (a), we can find $C_1$ as follows:

$$\begin{aligned}\mathbf{r} &= C_1\mathbf{a} + C_2\mathbf{b}\\ \mathbf{a}\cdot\mathbf{r} &= \mathbf{a}\cdot\Big(C_1\mathbf{a} + C_2\mathbf{b}\Big)\\ \mathbf{a}\cdot\mathbf{r} &= C_1\mathbf{a}\cdot\mathbf{a} + C_2\mathbf{a}\cdot\mathbf{b}\\ \frac{2}{\sqrt{2}} &= C_1(1) + C_2(0)\\ C_1 &= \frac{2}{\sqrt{2}}\end{aligned}$$

A similar calculation yields $C_2$

$$\begin{aligned}\mathbf{r} &= C_1\mathbf{a} + C_2\mathbf{b}\\ \mathbf{b}\cdot\mathbf{r} &= \mathbf{b}\cdot\Big(C_1\mathbf{a} + C_2\mathbf{b}\Big)\\ \mathbf{b}\cdot\mathbf{r} &= C_1\mathbf{b}\cdot\mathbf{a} + C_2\mathbf{b}\cdot\mathbf{b}\\ \frac{4}{\sqrt{2}} &= C_1(0) + C_2(1)\\ C_2 &= \frac{4}{\sqrt{2}} = 2\sqrt{2}\end{aligned}$$

2.1.4. (a) Expanding the left-hand side yields

$$\begin{aligned}||\mathbf{a}+\mathbf{b}||^2 &= \big(\mathbf{a}+\mathbf{b}\big)\cdot\big(\mathbf{a}+\mathbf{b}\big)\\ &= \mathbf{a}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{b} + 2\mathbf{a}\cdot\mathbf{b}\\ &= ||\mathbf{a}||^2 + ||\mathbf{b}||^2 + 2\mathbf{a}\cdot\mathbf{b}\end{aligned}$$

This is equal to $||\mathbf{a}||^2 + ||\mathbf{b}||^2$ iff $\mathbf{a}\cdot\mathbf{b} = 0$, which implies that $\mathbf{a}$ must be perpendicular to $\mathbf{b}$.

(b) First consider the dot product:

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= \mathbf{a}\cdot\mathbf{c}\\ 0 &= \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{c}\\ 0 &= \mathbf{a}\cdot\big(\mathbf{b}-\mathbf{c}\big)\end{aligned}$$

Therefore, $\mathbf{a}$ is perpendicular to the vector $\mathbf{b}-\mathbf{c}$. Manipulation of the cross product yields

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \mathbf{a}\times\mathbf{c}\\ 0 &= \mathbf{a}\times\mathbf{b} - \mathbf{a}\times\mathbf{c}\\ 0 &= \mathbf{a}\times\big(\mathbf{b}-\mathbf{c}\big)\end{aligned}$$

Therefore, $\mathbf{a}$ is also parallel to the vector $\mathbf{b}-\mathbf{c}$. Therefore, $\mathbf{b}-\mathbf{c} = \mathbf{0}$, or $\mathbf{b} = \mathbf{c}$.

2.1.5. (a) This statement is not meaningless. This is a dot product of two vectors.

(b) This statement is meaningless. We cannot take the dot product of a vector with a scalar.

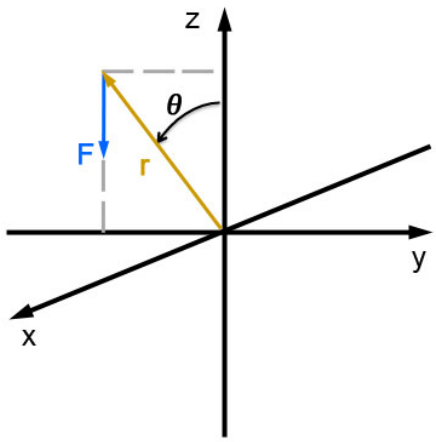
(c) This statement is meaningless. We cannot take the cross product of a vector with a scalar.

(d) This statement is not meaningless. This is a cross product of two vectors.

2.1.6. For this problem we will use the coordinate system in the diagram below.

(a) Using the above coordinate system, we are given that $||\mathbf{r}|| = 0.14$ and $\mathbf{F} = -100\mathbf{k}$. Also,

$$\begin{aligned}\mathbf{r} &= -\sin(30°)||\mathbf{r}||\mathbf{j} + \cos(30°)||\mathbf{r}||\mathbf{k}\\ &= -0.07\mathbf{j} + 0.07\sqrt{3}\mathbf{k}\\ &= 0.07\big(-\mathbf{j} + \sqrt{3}\mathbf{k}\big)\end{aligned}$$




To compute the torque when $\theta = 30°$, we use the cross product:

$$0.07\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -1 & \sqrt{3} \\ 0 & 0 & -100 \end{vmatrix} = 7\mathbf{i}$$

Thus, the magnitude of the torque $\boldsymbol{\tau}$ is 7 N·m. A similar calculation for $\theta = 90°$ yields a magnitude of 14 N·m

(b) Using a similar calculation from above,

$$\begin{aligned}\mathbf{r} &= -\sin\theta||\mathbf{r}||\mathbf{j} + \cos\theta||\mathbf{r}||\mathbf{k} \\ &= -0.14\sin\theta\mathbf{j} + 0.14\cos\theta\mathbf{k}\end{aligned}$$

To compute the torque when $\theta = 30°$, we use the cross product:

$$0.14\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -\sin\theta & \cos\theta \\ 0 & 0 & -100 \end{vmatrix} = 14\sin\theta\mathbf{i}$$

Thus, the magnitude of the torque $\boldsymbol{\tau}$ is $||14\sin\theta\mathbf{i}||\mathrm{N}\cdot\mathrm{m} = 14|\sin\theta|$N·m.

(c) The magnitude is minimized when $\theta = 0$ or $180°$. This corresponds to angles when $\mathbf{F}$ and $\mathbf{r}$ are parallel (or anti-parallel). The cross product of two vectors that are parallel (or anti-parallel) is zero.

2.1.7. (a) We can simplify the left-hand side of the equation by computing the determinant.

$$\begin{aligned}\begin{vmatrix} 3 & 2 & 0 \\ 1 & x & 0 \\ 7 & -3 & 4 \end{vmatrix} &= 3\begin{vmatrix} x & 0 \\ -3 & 4 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ 7 & 4 \end{vmatrix} + 0\begin{vmatrix} 1 & x \\ 7 & -3 \end{vmatrix} \\ &= 3(4x-0) - 2(4-0) + 0(-3-7x) \\ &= 12x - 8\end{aligned}$$

Substituting this result yields the equation $12x - 8 = 4$, from which the solution $x = 1$ is easily recovered.

(b) Expand both determinants and rearrange the equation:

$$\begin{aligned}(x)(4x) - (1)(4) &= (-x)(2x+8) - (-2)(2) \\ 4x^2 - 4 &= -2x^2 - 8x + 4 \\ 6x^2 + 8x - 8 &= 0 \\ (6x-4)(x+2) &= 0\end{aligned}$$

Therefore the solutions are $x = 2/3$ and $x = -2$.

## 2.2 Lines and Planes

2.2.1. (a) A vector parallel to the required line is given by $\begin{bmatrix}3\\-2\\1\end{bmatrix} - \begin{bmatrix}-1\\2\\1\end{bmatrix} = \begin{bmatrix}4\\-4\\0\end{bmatrix}$. If we like, for simplicity, we can normalize this vector by dividing each component by 4 to obtain the vector $[1, -1, 0]^T$. Since a point on the desired line is (-1,2,1), a vector equation is given by

$$\mathbf{r} = \begin{bmatrix}-1\\2\\1\end{bmatrix} + t\begin{bmatrix}1\\-1\\0\end{bmatrix}.$$

The solution to this question is not unique. Any vector parallel to the vector $\begin{bmatrix}4\\-4\\0\end{bmatrix}$ could be used.

(b) A vector equation is

$$\begin{aligned}\mathbf{r} &= (2+2t)\mathbf{i} + (9t)\mathbf{j} + (5+t)\mathbf{k}\\ &= (2\mathbf{i} + 5\mathbf{k}) + t(2\mathbf{i} + 9\mathbf{k} + \mathbf{k})\\ &= \begin{bmatrix}2\\0\\5\end{bmatrix} + t\begin{bmatrix}2\\9\\1\end{bmatrix}\end{aligned}$$

2.2.2. (a) Let $\mathbf{A} = \begin{bmatrix}-1\\0\\1\end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix}0\\-2\\2\end{bmatrix}$. Then

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= \big((-1)(0)\big) + \big((0)(-2)\big) + \big((1)(2)\big)\\ &= 0 + 0 + 2\\ &= 2\end{aligned}$$

Also,

$$\begin{aligned}||\mathbf{A}|| &= \sqrt{(-1)^2 + 0 + (1)^2} = \sqrt{2}\\ ||\mathbf{B}|| &= \sqrt{0^2 + (-2)^2 + 2^2} = \sqrt{8}\end{aligned}$$

Therefore, if $\theta$ is the angle between the two vectors,

$$\cos\theta = \frac{\mathbf{A}\cdot\mathbf{B}}{||\mathbf{A}||\ ||\mathbf{B}||} = \frac{2}{4} = \frac{1}{2}$$

Therefore, $\theta = \pi/3$.

(b) Let the vector perpendicular to the plane be $\mathbf{y} = \begin{bmatrix}2\\1\\-1\end{bmatrix}$. The angle, $\theta$, between the given vector and $\mathbf{y}$ is found by solving

$$\begin{aligned}\cos\theta &= \frac{\mathbf{x}\cdot\mathbf{y}}{||\mathbf{x}||\ ||\mathbf{y}||}\\ &= \frac{\begin{bmatrix}1\\0\\2\end{bmatrix}\cdot\begin{bmatrix}2\\1\\-1\end{bmatrix}}{\sqrt{5}\sqrt{6}}\\ &= \frac{0}{\sqrt{5}\sqrt{6}}\\ &= 0.\end{aligned}$$

The angle, $\theta$, is $\pi/2$. But $\theta$ is the angle between $\mathbf{x}$ and $\mathbf{y}$ and we need the angle between $\mathbf{x}$ and the plane, which is $\pi/2 - \theta = 0$. Therefore, the desired angle is 0 (the given vector is parallel to the plane).

2.2.3. **Distance Between a Point and a Line**
The formula we can apply is:

$$D = \frac{||\mathbf{v} \times \mathbf{w}||}{||\mathbf{v}||}$$

where $\mathbf{v}$ is a vector parallel to the given line, and is

$$\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

The vector $\mathbf{w}$ is

$$\mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Then,

$$\begin{aligned} D &= \frac{||\mathbf{v} \times \mathbf{w}||}{||\mathbf{v}||} \\ &= \frac{||[-1, 0, 0]^T||}{||[0, 1, 0]^T||} \\ &= \frac{1}{1} \\ &= 1 \end{aligned}$$

2.2.4. **Distance Between a Point and a Plane**
The formula we can apply is:

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

where $a, b, c$ are the components of the normal vector to the plane, so that

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

The point $(x_0, y_0, z_0) = (-1, C, 1)$, $d = -3$, and $D = \sqrt{6}$. Thus,

$$\begin{aligned} D = \sqrt{6} &= \frac{|2(-1) + C(1) + (1)(-1) - 3|}{\sqrt{6}} \\ &= \frac{|C - 6|}{\sqrt{6}} \\ 1 &= |C - 6| \end{aligned}$$

Therefore, $C = 5, 7$.

2.2.5. (a) We can start by finding the plane that contains the three points. The vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\begin{aligned} \overrightarrow{PQ} &= \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \\ \overrightarrow{PR} &= \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -4 \end{bmatrix} \end{aligned}$$

A vector perpendicular to the plane that contains the three points is found by calculating the cross product between these two vectors:

$$\begin{aligned}\overrightarrow{PQ} \times \overrightarrow{PR} &= (-8-(0))\mathbf{i} - (-4-0)\mathbf{j} + (0-(-2))\mathbf{k} \\ &= \begin{bmatrix} -8 \\ 4 \\ 2 \end{bmatrix}\end{aligned}$$

Any vector parallel to this vector is perpendicular to the plane that contains the given points.

(b) To find the area, $A$, of triangle $\triangle PQR$, we can use a cross product.

$$\begin{aligned} A &= \frac{1}{2}||\overrightarrow{PQ} \times \overrightarrow{PR}|| \\ &= \frac{1}{2}||[-8,4,2]^T|| \\ &= \frac{1}{2}\sqrt{(-8)^2+4^2+2^2} \\ &= \frac{1}{2}\sqrt{84} \\ &= \sqrt{21} \end{aligned}$$

(c) If the point $S$ is such that $\overrightarrow{PQ}$ is parallel to $\overrightarrow{QS}$, then there will be an infinite number of planes that pass through these three points. Such a point, $S$, can be determined by multiplying $\overrightarrow{PQ}$ by a constant. Choosing 2 as a constant, then

$$\overrightarrow{QS} = \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}$$

We determine the components of $S$ as

$$S = \begin{bmatrix} 2-2 \\ 4-2 \\ 0-3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

2.2.6. **Equations of Planes**

(a) Let the points $P, Q, R$ be

$$\begin{aligned} P &= (-1,2,1) \\ Q &= (3,-2,1) \\ R &= (-1,1,-1) \end{aligned}$$

Vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\begin{aligned}\overrightarrow{PQ} &= \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} - (-1,2,1) = (4,-4,0) \\ \overrightarrow{PR} &= \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}\end{aligned}$$

A vector orthogonal to the plane that contains the three points is found by calculating the cross product between these two vectors:

$$\begin{aligned}\overrightarrow{PQ} \times \overrightarrow{PR} &= (8-0)\mathbf{i} - (-8-0)\mathbf{j} + (-4-0)\mathbf{k} \\ &= (8,8,-4)\end{aligned}$$

The equation of the desired plane, using the point-normal form, is

$$\begin{aligned} 0 &= 8(x+1) + 8(y-2) + (-4)(z-1) \\ 4 &= 8x + 8y - 4z \end{aligned}$$

(b) The points $(0, 0, 0)$ and $(1, 1, 1)$ are on the given line. Therefore, the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is parallel to the desired plane. Another vector parallel to the desired plane is

$$\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}.$$

Therefore, a vector perpendicular to the desired plane is

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \times \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = (1-2)\mathbf{i} - (1+1)\mathbf{j} + (2+1)\mathbf{k} = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}.$$

The equation of the desired plane, using the point-normal form, is

$$\begin{aligned} 0 &= -1(x+1) - 2(y-2) + 3(z-1) \\ 0 &= -x - 2y + 3z. \end{aligned}$$

2.2.7. Let the normal vector of the first plane be $\mathbf{n}_1$, and the normal vector of the second plane be $\mathbf{n}_2$. Then

$$\mathbf{n}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \ \mathbf{n}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

The line that intersects these two planes is a line that is in both of the planes, and therefore must be perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$. Therefore, the line we need is parallel to the vector, $\mathbf{a}$, given by the cross product

$$\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2 = (4, -3, 5).$$

This vector is parallel to the desired plane. To find a second vector in the desired plane, we can find a line from the given point to any point in the line of intersection of the two given planes. Letting $y = 0$, we obtain the equations

$$\begin{aligned} 2x - z &= 3 \\ x + z &= 0 \end{aligned}$$

which has the solution $x = 1, z = -1$. Therefore, the point $(1, 0, -1)$ is in the intersection of the two planes, and a vector in the plane is

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

A normal vector to the desired plane is

$$\begin{aligned} [1, 0, -1]^T \times [4, -3, 5]^T &= (0-3)\mathbf{i} - (5+4)\mathbf{j} + (-3-0)\mathbf{k} \\ &= -3\mathbf{i} - 9\mathbf{j} - 3\mathbf{k} \end{aligned}$$

The equation of the desired plane, using the point-normal form, is

$$\begin{aligned} 0 &= (-3)(x-0) + (-9)(y-0) + (-3)(z-0) \\ &= -3x - 9y - 3z \end{aligned}$$

2.2.8. (a) We will show that the two lines must be equal to each other, and therefore share an infinite number of points. Let $P$ be the common point, and $\mathbf{r}$ be the vector pointing from the origin to the point $P$. Let

$$L_1 = \mathbf{r} + t\mathbf{u}$$
$$L_2 = \mathbf{r} + t\mathbf{v}$$

where $t \in \mathbb{R}$, and $\mathbf{u}$ and $\mathbf{v}$ are vectors parallel to the lines $L_1$ and $L_2$ respectively. Let their components be

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

When the lines intersect, their components are equal, which gives us the three equations

$$r_1 + tu_1 = r_1 + tv_1$$
$$r_2 + tu_2 = r_2 + tv_2$$
$$r_3 + tu_3 = r_3 + tv_3$$

Or simply

$$tu_1 = tv_1$$
$$tu_2 = tv_2$$
$$tu_3 = tv_3$$

Because $L_1$ and $L_2$ are parallel, vectors $\mathbf{u}$ and $\mathbf{v}$ must also be parallel. Therefore $\mathbf{u} = k\mathbf{v}$ where $k$ is a constant, so

$$tu_1 = tku_1$$
$$tu_2 = tku_2$$
$$tu_3 = tku_3$$

These equations are only satisfied if $k = 1$. Therefore, the two lines must be equal to each other, implying that there an infinite number of points that the two lines share.

(b) Yes. Any plane in $\mathbb{R}^3$ can be expressed in point-normal form

$$a(x - x_0) - b(y - y_0) + c(z - z_0) = 0$$

where the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is a vector normal to the plane.

## 2.3 Surfaces

***We'll have solutions to 3 or 4 questions on this topic here.***

## 2.4 Curvilinear Coordinates

***We'll have solutions to 3 or 4 questions on this topic here.***

## 2.5 Vector-Valued Functions

2.5.1. The derivative of $\mathbf{r}$ is simply $\mathbf{r}'(t) = 2t\mathbf{i} + \mathbf{j}$.

(a) $\mathbf{r}(t) \perp \mathbf{r}'(t)$ implies

$$\begin{aligned} 0 &= \mathbf{r}(t) \cdot \mathbf{r}'(t) \\ &= \big((1+t^2)\mathbf{i} + t\mathbf{j}\big) \cdot \big(2t\mathbf{i} + \mathbf{j}\big) \\ &= (2t + 2t^3) + (t) \\ &= t(3 + 2t^2) \end{aligned}$$

This equation is satisfied only when $t = 0$ or $t^2 = -3/2$. The latter equation cannot be satisfied if $t \in \mathbb{R}$. Thus, the two vectors are only perpendicular when $t = 0$. Thus, $\mathbf{r}(t)$ is perpendicular to $\mathbf{r}'(t)$ at (0,1).

(b) If $\mathbf{r}(t)$ is pointing in the same direction as $\mathbf{r}'(t)$, then $\mathbf{r}(t) = C\mathbf{r}'(t)$, where $C$ is a positive constant. Equating the components of the vectors yields the two equations

$$(1 + t^2) = C2t, \text{ and } t = C.$$

If we substitute $t = C$ into the first equation, we find

$$\begin{aligned} (1 + C^2) &= 2C^2 \\ C &= +1, \end{aligned}$$

because $C$ must be positive. Thus, the two vectors are pointing in the same direction when $t = 1$, which is at the point (2,1).

(c) The solution is the similar to part (b), except that $C$ must be a negative constant. We find that the vectors point in opposite directions when $C = t = -1$, and at the point (2,-1).

2.5.2. (a) Differentiating the equation for angular momentum yields the torque:

$$\begin{aligned} \mathbf{L}(t) &= m\mathbf{r}(t) \times \mathbf{r}'(t) \\ \frac{d}{dt}\mathbf{L}(t) &= \frac{d}{dt}\Big(m\mathbf{r}(t) \times \mathbf{r}'(t)\Big) \\ &= m\mathbf{r}'(t) \times \mathbf{r}'(t) + m\mathbf{r}(t) \times \mathbf{r}''(t) \\ &= m\mathbf{0} + m\mathbf{r}(t) \times \mathbf{r}''(t) \\ &= m\mathbf{r}(t) \times \mathbf{r}''(t) \\ &= \boldsymbol{\tau}(t) \end{aligned}$$

(b) If each component of $\tau(t)$ is zero for all values of $t$, then each component of $\mathbf{L}'(t)$ is zero for all values of $t$. Each component of $\mathbf{L}(t)$ must therefore be constant.

2.5.3. (a) From $\mathbf{F} = m\mathbf{a} = m\mathbf{v}'(t)$ we obtain

$$\mathbf{v}(t) = c_1\mathbf{i} + \big(\pi\sin(\pi t) + c_2\big)\mathbf{j} + \big(-\pi\cos(\pi t) + c_3\big)\mathbf{k}$$

where $c_1, c_2, c_3$ are constants. When $t = 0$,

$$\begin{aligned} \mathbf{v}(0) &= c_1\mathbf{i} + \big(\pi\sin(0) + c_2\big)\mathbf{j} + \big(-\pi\cos(0) + c_3\big)\mathbf{k} \\ &= c_1\mathbf{i} + c_2\mathbf{j} + (-\pi + c_3)\mathbf{k} \end{aligned}$$

But $\mathbf{v}(0) = \mathbf{i}$, so by comparison, $c_1 = 1$, $c_2 = 0$, and $c_3 = \pi$. Thus

$$\mathbf{v}(t) = \mathbf{i} + \big(\pi\sin(\pi t)\big)\mathbf{j} + \big(-\pi\cos(\pi t) + \pi\big)\mathbf{k}$$

(b) Integrating the components of $\mathbf{v}(t)$ yields

$$\mathbf{r}(t) = (t + d_1)\mathbf{i} + \big(-\cos(\pi t) + d_2\big)\mathbf{j} + \big(-\sin(\pi t) + \pi t + d_3\big)\mathbf{k}$$

where $d_1, d_2$ and $d_3$ are constants. When $t = 0$,

$$\begin{aligned} \mathbf{r}(0) &= (0 + d_1)\mathbf{i} + \big(-\cos(0) + d_2\big)\mathbf{j} + \big(-\sin(0) + \pi(0) + d_3\big)\mathbf{k} \\ \mathbf{r}(0) &= d_1\mathbf{i} + \big(-1 + d_2\big)\mathbf{j} + \big(d_3\big)\mathbf{k} \end{aligned}$$

But $\mathbf{r}(0) = -\mathbf{i}$, so by comparison, $d_1 = -1$, $d_2 = 1$, and $d_3 = 0$. Thus

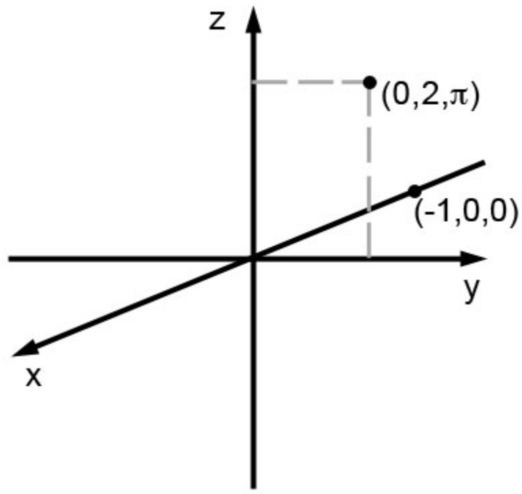
$$\mathbf{r}(t) = (t-1)\mathbf{i} + \big(-\cos(\pi t) + 1\big)\mathbf{j} + \big(-\sin(\pi t) + \pi t\big)\mathbf{k}$$

Thus

$$\begin{aligned}\mathbf{r}(1) &= (1-1)\mathbf{i} + \big(-\cos(\pi) + 1\big)\mathbf{j} + \big(-\sin(\pi) + \pi\big)\mathbf{k} \\ &= 2\mathbf{j} + \pi\mathbf{k}\end{aligned}$$

(c) At $t = 0$ and $t = 1$ we have

$$\mathbf{r}(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{r}(1) = \begin{bmatrix} 0 \\ 2 \\ \pi \end{bmatrix}.$$



## 2.6 Linear Systems

2.6.1. Use row-reduction operations on the augmented matrices.

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 3 & -1 & 1 & 2 & 5 \\ 0 & 1 & 0 & 2 & 0 \\ 4 & 0 & 3 & 1 & 10 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & -7 & 1 & -1 & 2 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & -8 & 3 & -3 & 6 \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow R_2 - 3R_1 \\ R_4 \leftarrow R_4 - 4R_1 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 13 & 2 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 3 & 13 & 6 \end{array}\right] \quad \begin{array}{l} R_2 \leftarrow R_2 + 7R_3 \\ R_4 \leftarrow R_4 + 8R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 13 & 2 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & -26 & 0 \end{array}\right] \quad R_4 \leftarrow R_4 - 3R_2$$

This is equivalent to the system of equations

$$\begin{aligned} x + 2y + w &= 1 \\ z + 13w &= 2 \\ y + 2w &= 0 \qquad (1) \\ -26w &= 0 \qquad (2) \end{aligned}$$

Equations (1) and (2) show that $w = 0$ and $y = 0$, and substitution yields $z = 2$ and $x = 1$.

2.6.2. (a) It's not possible to calculate the determinant of $A$, because $A$ is not square. We can however solve $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$ using row operations:

$$\left[\begin{array}{cccc|c} -3 & 1 & 2 & 4 & 1 \\ 2 & -1 & 2 & 3 & 1 \\ 1 & 0 & 4 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 14 & -2 & 4 \\ 0 & -1 & -6 & 7 & -1 \\ 1 & 0 & 4 & -2 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \leftarrow R_1 + 3R_3 \\ R_2 \leftarrow R_2 - 2R_3 \end{array}$$

$$\left[\begin{array}{cccc|c} 0 & 1 & 14 & -2 & 4 \\ 0 & 0 & 8 & 5 & 3 \\ 1 & 0 & 4 & -2 & 1 \end{array}\right] \quad R_2 \leftarrow R_2 + R_1$$

This is equivalent to the system of equations

$$\begin{aligned} y + 14z - 2w &= 4 \qquad (3) \\ 8z + 5w &= 3 \qquad (4) \\ x + 4z - 2w &= 1 \qquad (5) \end{aligned}$$

Since there are fewer equations than unknowns we will parameterize the solution set by setting $w = t$, where $t$ is any real number. From equation (4) we get $z = \frac{3}{8} - \frac{5}{8}t$. Substituting $z$ into equations (5) and (3) yields $x = -\frac{1}{2} + \frac{9}{2}t$ and $y = -\frac{5}{4} + \frac{43}{4}t$.

Because of the free parameter $t$, there are infinitely many solutions to the given system.

(b) We can calculate the 3×3 determinant by expanding into 2×2 determinants:

$$\begin{vmatrix} 1 & 2 & -1 \\ 2 & 5 & 0 \\ 4 & 9 & -2 \end{vmatrix} = 1\begin{vmatrix} 5 & 0 \\ 9 & -2 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 4 & -2 \end{vmatrix} - 1\begin{vmatrix} 2 & 5 \\ 4 & 9 \end{vmatrix}$$
$$= (-10) - 2(-4) - (18 - 20)$$
$$= -10 + 8 + 2$$
$$= 0$$

Applying one row operation yields

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 5 & 0 & 5 \\ 4 & 9 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 5 & 0 & 5 \\ 0 & 0 & 0 & -7 \end{array}\right] \quad R_3 \leftarrow R_3 - 2R_1 - R_2$$

The last row is equivalent to

$$0x + 0y + 0z = -7,$$

which implies that this particular system has no solution.

2.6.3. We first convert the system of equations into an equivalent augmented-matrix form:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 2 & a & -2 & b \\ 3 & 2 & 0 & 1 \end{array}\right].$$

Then apply a sequence of row-reduction operations:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & a-4 & 0 & b-4 \\ 0 & -4 & 3 & -5 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & a & -3 & b+1 \\ 0 & -4 & 3 & -5 \end{array}\right] \quad \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & a & -3 & b+1 \\ 0 & 4 & -3 & 5 \end{array}\right] \quad R_2 - R_3$$
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 4 & -3 & 5 \\ 0 & a & -3 & b+1 \end{array}\right] \quad \begin{array}{l} -1 \cdot R_3 \\ R_2 \leftrightarrow R_3 \end{array} \tag{6}$$

Suppose that $a = 4$. Then (6) becomes

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 4 & -3 & 5 \\ 0 & 4 & -3 & b+1 \end{array}\right]$$
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 4 & -3 & 5 \\ 0 & 0 & 0 & b-4 \end{array}\right] \quad R_3 - R_2 \tag{7}$$

If $b \neq 4$, then the system has no solutions, since the last row is equivalent to the equation $0 = b-4$ where $b - 4 \neq 0$. Conversely, if $b = 4$, then (7) is equivalent to the system of equations

$$x + 2y - z = 2$$
$$4y - 3z = 5$$

There are fewer equations than unknowns, so we have to parameterize the solution set; let $z = t$. Solving these two equations yields the solution

$$\begin{aligned} x &= -\frac{1}{2} - \frac{1}{2}t \\ y &= \frac{5}{4} + \frac{3}{4}t \\ z &= t \end{aligned} \tag{8}$$

Now suppose that $a = 0$. Then the matrix (6) is equivalent to the system

$$\begin{aligned} x + 2y - z &= 2 \\ 4y - 3z &= 5 \\ -3z &= b + 1 \end{aligned}$$

Using back-substitution we find that

$$\begin{aligned} x &= -\frac{1}{3} - \frac{1}{6}b \\ y &= 1 - \frac{1}{4}b \\ z &= \frac{-1}{3} - \frac{1}{3}b \end{aligned} \tag{9}$$

Now suppose that $a \neq 4$ and $a \neq 0$. Then we can continue row- reducing from (6):

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 4 & -3 & 5 \\ 0 & a & -3 & b+1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 2 \\ 0 & 4 & -3 & 5 \\ 0 & 0 & -3 + \frac{3}{4}a & 1 + b - \frac{5}{4}a \end{array}\right] \quad R_3 \leftarrow R_3 - \frac{a}{4}R_2$$

This is equivalent to the system

$$\begin{aligned} x + 2y - z &= 2 \\ 4y - 3z &= 5 \\ (-3 + \frac{3}{4}a)z &= 1 + b - \frac{5}{4}a \end{aligned}$$

Let $c = \dfrac{1 + b - \frac{5}{4}a}{-3 + \frac{3}{4}a}$. Using back-substitution we find the solution

$$\begin{aligned} x &= -\frac{1}{2} - \frac{1}{2}c \\ y &= \frac{5}{4} + \frac{3}{4}c \\ z &= c \end{aligned} \tag{10}$$

Thus we have shown that:

(a) When $a = 4$ and $b = 4$ there are infinitely many solutions, given by the parameterization (8).

(b) When $a = 4$ and $b \neq 4$ there are no solutions.

(c) When $a \neq 4$ there is one solution. When $a = 0$, the solution (9) is equivalent to that of (10). So the solution when $a \neq 4$ is given by (10).

2.6.4. (a) We can write the system as

$$\begin{aligned} a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 &= 0 \\ a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3 &= 5.5 \\ a_0 + a_1(2) + a_2(2)^2 + a_3(2)^3 &= 20 \\ a_0 + a_1(3) + a_2(3)^2 + a_3(3)^3 &= 46.5 \\ a_0 + a_1(4) + a_2(4)^2 + a_3(4)^3 &= 88 \end{aligned}$$

Equivalently, we can write this as

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5.5 \\ 20 \\ 46.5 \\ 88 \end{bmatrix}$$

(b) Solving the above system yields

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 0.5 \end{bmatrix}$$

(c) $p(1.5) = 2(1.5) + 3(1.5)^2 + 0.5(1.5)^3 = 11.4375$

(d) The system for a polynomial of lower order would have no solution because $a_3 \neq 0$. For example, if we used a linear function, the system becomes

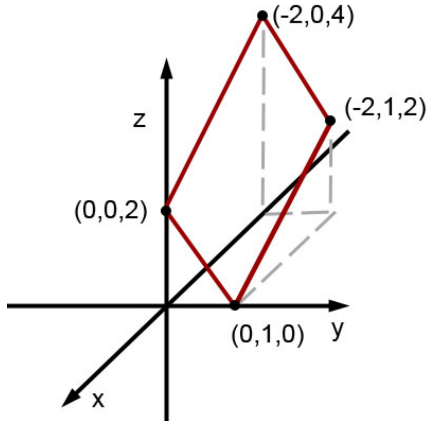
$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 5.5 \\ 20 \\ 46.5 \\ 88 \end{bmatrix}$$

This system has no solution.

# 3 Multiple Integrals

## 3.1 Double Integrals

3.1.1. (a) A sketch of the volume is below.




(b) We must integrate:

$$\begin{aligned}\int_{-2}^{0}\int_{0}^{1}(-x-2y+2)dydx &= \int_{-2}^{0}\big(-2xy-y^2+2y\big)\big|_0^1 dx\\ &= \int_{-2}^{0}\Big(-2x+1\Big)dx\\ &= \Big(-x^2+x\Big)\Big|_{-2}^{0}\\ &= 0-\big(-(-2)^2+(-2)\big)\\ &= 0-\big(-4-2\big)\\ &= 6\end{aligned}$$

3.1.2. (a) Substituting the expression for $u$ into the divergence equation yields

$$\begin{aligned}0 &= \nabla\cdot\mathbf{v}\\ &= \frac{\partial}{\partial x}\big(u(x,y)\big)+\frac{\partial}{\partial y}\big(v(x,y)\big)\\ &= \frac{\partial}{\partial x}(x^2+y^2)+\frac{\partial v}{\partial y}\\ &= 2x+\frac{\partial v}{\partial y}\\ \frac{\partial v}{\partial y} &= -2x\end{aligned}$$

Therefore, $v(x,y)$ is a function whose partial derivative with respect to $y$ is $-2x$. The **most general** form for $v(x,y)$ is obtained by integrating with respect to $y$:

$$v(x,y)=-2xy+f(x)$$

where $f(x)$ is an unknown function of one variable, $x$.

(b) Using the same approach as we used for (a) yields

$$\begin{aligned}0 &= \nabla\cdot\mathbf{v}\\ &= \frac{\partial}{\partial x}\big(u(x,y)\big)+\frac{\partial}{\partial y}\big(v(x,y)\big)\\ &= \frac{\partial u}{\partial x}+0\\ \frac{\partial u}{\partial x} &= 0\end{aligned}$$

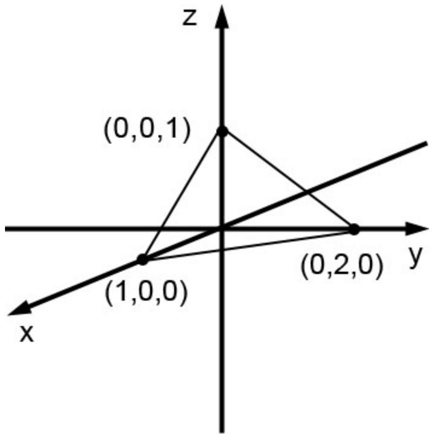
Therefore, $u(x, y)$ is a function whose partial derivative with respect to $x$ is 0. The **most general** form for $u(x, y)$ is obtained by integrating with respect to $x$:

$$u(x, y) = g(y)$$

where $g(y)$ is an unknown function of one variable, $y$.

## 3.2 Double Integrals Over a General Region

3.2.1. (a) A sketch of the tetrahedron is below.




(b) The tetrahedron has one side in the $xy$-plane. This side is bounded by the line that is the intersection between the $xy$-plane and the plane $z = 1 - x - y/2$. We can find this intersection by setting $z = 0$,

$$\begin{aligned} 0 &= 1 - x - \frac{y}{2} \\ x &= 1 - \frac{y}{2}. \end{aligned}$$

Therefore, the volume is the region under the plane $z = 1 - x - y/2$ and over

$$R = \{(x, y) \mid 0 \le x \le 1 - y/2,\ 0 \le y \le 2\}.$$

The double integral is

$$\int_0^2 \int_0^{1-y/2} (1 - x - \frac{y}{2})dxdy$$

(c) The volume is the region under the plane $z = 1 - x - y/2$ and over

$$R = \{(x, y) \mid 0 \le x \le 1,\ 0 \le y \le 2 - 2x\}.$$

The double integral is

$$\int_0^1 \int_0^{2-2x} (1 - x - \frac{y}{2})dydx$$

3.2.2. (a) Suppose that we subdivide region $R$ into a rectangular grid of sub-rectangles (as in Figure 3.2.5), so that we only consider the sub-rectangles that are completely enclosed in $R$. Then, the area of region $R$ is approximated by the double sum

$$\sum_j \sum_i \Delta x_i \Delta y_j$$

But if $f = 1$ for all $x_i$ and $y_j$, this is equal to:

$$\sum_j \sum_i f(x_i, y_j)\Delta x_i \Delta y_j \tag{11}$$

where $x_i$ and $y_j$ is a point inside sub-rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$. If we take smaller and smaller rectangles, so that the length of the longest diagonal of the sub-rectangles goes to zero, the sub-rectangles begin to fill more and more of the region $R$, and so the above sums approach the **area** of region $R$. Since we have defined

$$\iint f(x, y)dA$$

as the limit of Equation (11) as the longest diagonal goes to zero, and $f(x,y) = 1$, the double integral

$$\iint 1dA$$

is the area of region $R$.

(b) The region may be defined as

$$R = \{(x,y) \mid 0 \leq x \leq 1,\ x^2 \leq y \leq \sqrt{x}\}.$$

The area of $R$ is

$$\begin{aligned}
\int_0^1 \int_{x^2}^{\sqrt{x}} dydx &= \int_0^1 \left(\sqrt{x} - x^2\right) dx \\
&= \left(\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right)\Big|_0^1 \\
&= \frac{2}{3} - \frac{1}{3} \\
&= \frac{1}{3}
\end{aligned}$$

3.2.3. The curves $y = x^2$ and $x^2 = y$ intersect at (0,0) and at (1,1). Integrating with respect to $y$ first yields

$$\begin{aligned}
\int_0^1 \int_{x^2}^{\sqrt{x}} x^2 + y^2 dydx &= \int_0^1 \left(yx^2 + \frac{y^3}{3}\right)\Big|_{x^2}^{\sqrt{x}} dx \\
&= \int_0^1 \left(x^{5/2} + \frac{x^{3/2}}{3} - x^4 - \frac{x^6}{3}\right) dx \\
&= \left(\frac{2}{7}x^{7/2} + \frac{2}{15}x^{5/2} - \frac{1}{5}x^5 - \frac{1}{21}x^7\right)\Big|_0^1 \\
&= \frac{2}{7} + \frac{2}{15} - \frac{1}{5} - \frac{1}{21} \\
&= 6/35
\end{aligned}$$

3.2.4. (a) Integrating with respect to $y$ first yields

$$\begin{aligned}
\int_0^2 \int_0^{x^2} x\sin(y)dydx &= \int_0^2 -x\cos(y)\Big|_0^{x^2} dx \\
&= -\int_0^2 \left(x\cos(x^2) - 1\right) dx \\
&= -\int_0^2 x\cos(x^2)dx + \int_0^2 1\ dx \\
&= 2 - \int_0^2 x\cos(x^2)dx
\end{aligned}$$

Now let $u = x^2$, so that $du = 2xdx$. Then

$$\begin{aligned}
2 - \int_0^2 x\cos(x^2)dx &= 2 - \int_0^4 x\cos(u)\left(\frac{du}{2x}\right) \\
&= 2 - \frac{1}{2}\int_0^4 \cos(u)du \\
&= 2 - \frac{1}{2}\sin(u)\Big|_0^4 \\
&= 2 - \frac{1}{2}\sin(4) \\
&\approx 2.3784
\end{aligned}$$

(b) Integrating with respect to $x$ first yields

$$\begin{aligned}
\int_0^4\int_{\sqrt{y}}^2 x\sin(y)dxdy &= \int_0^4 \frac{x^2}{2}\sin(y)\Big|_{\sqrt{y}}^2 dy \\
&= \int_0^4 \frac{4-y}{2}\sin(y)dy \\
&= 2\int_0^4 \sin(y)dy - \frac{1}{2}\int_0^4 y\sin(y)dy \\
&= 2\big(-\cos(y)\big)\big|_0^4 - \frac{1}{2}\int_0^4 y\sin(y)dy \\
&= 2\big(-\cos(4)+1\big) - \frac{1}{2}\int_0^4 y\sin(y)dy \\
&= 2-2\cos(4) - \frac{1}{2}\int_0^4 y\sin(y)dy
\end{aligned}$$

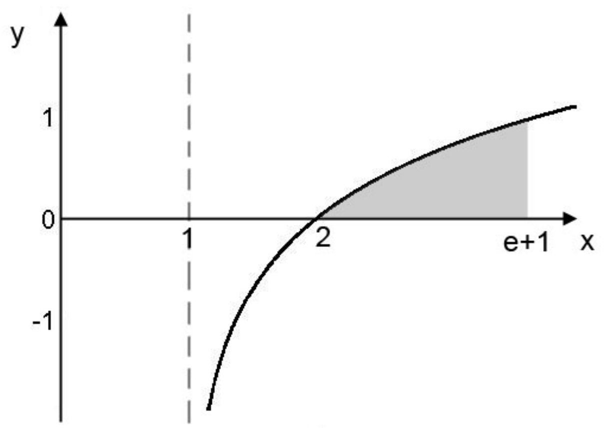
Now using integration by parts, with

$$\begin{aligned}
u &= y, \quad dv = \sin(y)dy \\
du &= dy, \quad v = -\cos(y)
\end{aligned}$$

we obtain

$$\begin{aligned}
\int_0^4\int_{\sqrt{y}}^2 x\sin(y)dxdy &= 2-2\cos(4) - \frac{1}{2}\int_0^4 y\sin(y)dy \\
&= 2-2\cos(4) - \frac{1}{2}\left(-y\cos(y)\Big|_0^4 - \int_0^4 \big(-\cos(y)\big)dy\right) \\
&= 2-2\cos(4) - \frac{1}{2}\left(-4\cos 4 + \sin(y)\Big|_0^4\right) \\
&= 2-2\cos(4) - \frac{1}{2}\big(-4\cos 4 + \sin(4) - 0\big) \\
&= 2 - \frac{\sin(4)}{2} \\
&\approx 2.3784
\end{aligned}$$

3.2.5. The region over which we are integrating $f(x,y)$ is the shaded area below.




The region is bounded by the lines $y=0$, $x=1+e$, and by the curve $y=\ln(x-1)$. Using horizontal slices, values of $y$ range from 0 to 1, and values of $x$ range from $e^y - 1$ to $1 + e$. The double integral becomes

$$\int_0^1 \int_{e^y-1}^{1+e} f(x, y) dy dx.$$

3.2.6. The integrand $f(x, y)$ has the property that

$$0 \leq \sin(x + y) \leq 1,$$

because $x + y$ is between 0 and 2, and $\sin(2) < 1$. Then

$$0 = \iint_D 0 dA \leq \iint_D f(x, y) dA \leq \iint_D (1) dA = 1.$$

3.2.7. (a) Suppose we use the function

$$h(x, y) = 11xy.$$

Then $h(-x, y) = 11(-x)y = -11xy = -h(x, y)$.

(b) An example of a region that is symmetric about the $y$-axis, but not symmetric about the $x$-axis, is the region bounded by the curves

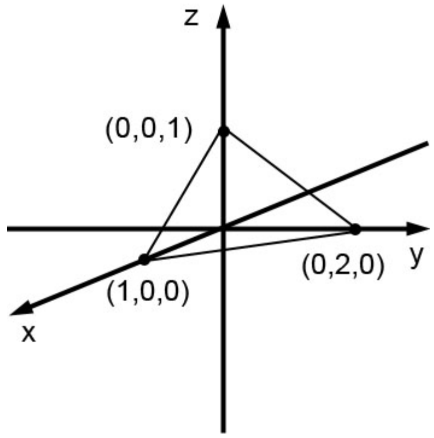
$$x = -1, \quad x = 1, \quad y = 0, \quad y = -1.$$

(c) The double integral for our region is

$$\begin{aligned}
\iint_D g(x, y) dx dy &= \int_{-1}^1 \int_{-1}^0 11xy dy dx \\
&= -\frac{11}{2} \int_{-1}^1 x dx \\
&= -\frac{11}{4} \left(x^2\right)\Big|_{-1}^1 \\
&= -\frac{11}{4}(0) \\
&= 0.
\end{aligned}$$

## 3.3 Triple Integrals

3.3.1. **Volume of a Tetrahedron**




Recall that the volume is the region under the plane $z = 1 - x - y/2$ and over

$$R = \{(x, y) \mid 0 \le x \le 1 - y/2,\ 0 \le y \le 2\}.$$

Because $z$ lies between 0 and $z = 1 - x - y/2$, the volume, $S$, can be described as

$$S = \{(x, y, z) \mid 0 \le x \le 1 - y/2,\ 0 \le y \le 2,\ 0 \le z \le 1 - x - y/2\}.$$

The volume can be calculated with the triple integral

$$\begin{aligned}
\int_0^2 \int_0^{1-y/2} \int_0^{1-x-y/2} dzdxdy &= \int_0^2 \int_0^{1-y/2} (1 - x - y/2)dxdy \\
&= \int_0^2 \left(x - \frac{x^2}{2} - \frac{xy}{2}\right)\Big|_0^{1-y/2} dy \\
&= \int_0^2 \left((1 - y/2) - \frac{(1 - y/2)^2}{2} - \frac{(y - y^2/2)}{2}\right)dy \\
&= \int_0^2 \left(1 - \frac{y}{2} - \frac{1}{2}\left(1 - y + \frac{y^2}{4}\right) - \frac{y}{2} + \frac{y^2}{4}\right)dy \\
&= \int_0^2 \left(1 - \frac{y}{2} - \frac{1}{2} + \frac{y}{2} - \frac{y^2}{8} - \frac{y}{2} + \frac{y^2}{4}\right)dy \\
&= \int_0^2 \left(\frac{1}{2} - \frac{y}{2} + \frac{y^2}{8}\right)dy \\
&= \frac{2}{2} - \frac{4}{4} + \frac{8}{24} \\
&= \frac{1}{3}
\end{aligned}$$

3.3.2. **Volume of an Ellipsoid**
We are given the transformations

$$x = au, \quad y = bv, \quad z = cw.$$

The Jacobian becomes

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

The solid enclosed by the ellipsoid is the image of the unit sphere $u^2+v^2+w^2 \leq 1$. Using that a sphere has volume $\frac{4}{3}\pi r^3$, we find that

$$\begin{aligned}\iiint_V dxdydz &= \iiint_{u^2+v^2+w^2\leq 1} abc\ dudvdw \\ &= abc \iiint_{u^2+v^2+w^2\leq 1} dudvdw \\ &= abc(\text{volume of a sphere}) \\ &= \frac{4\pi abc}{3}\end{aligned}$$

3.3.3. **Changing the Order of Integration**

## 3.4 Change of Variables in Multiple Integrals

3.4.1. **Linear Transformations**

(a) Substituting $v = v_0$ into the linear transformation yields the two equations

$$\begin{aligned}x &= c_1 u + c_2 v_0 \\ y &= d_1 u + d_2 v_0\end{aligned}$$

To find the equation of the line in the $xy$-plane, we need to eliminate $u$. There are many ways to do this, but let's multiply the first equation by $d_1$ and the second by $c_1$.

$$\begin{aligned}d_1 x &= c_1 d_1 u + c_2 d_1 v_0 \\ c_1 y &= c_1 d_1 u + d_2 c_1 v_0\end{aligned}$$

Subtracting these equations yields

$$d_1 x - c_1 y = (c_2 d_1 - d_2 c_1) v_0$$

A simple rearrangement gives us

$$c_1 y = d_1 x - (c_2 d_1 - d_2 c_1) v_0.$$

Provided that $c_1$ is not zero, we could write this in the form

$$y = \frac{d_1}{c_1} x - \frac{c_2 d_1 - d_2 c_1}{c_1} v_0.$$

(b) Substituting $x = x_0$ into $x = c_1 u + c_2 v$ gives us

$$x_0 = c_1 u + c_2 v,$$

Provided that $c_2$ is not zero, This is the is mapped into the $uv$-plane REWORD

3.4.2. The integral can be written as

$$\iint_R (x^2 - y^2) dxdy = \iint_R (x-y)(x+y) dxdy.$$

Recall that $R$ is the region bounded by

$$x+y=0, \quad x+y=1, \quad x-y=0, \quad x-y=1.$$

The appearance of the terms $(x+y)$ and $(x-y)$ in the integrand and in the lines that bound $R$ suggests the transformation

$$u = x + y \tag{12}$$
$$v = x - y. \tag{13}$$

In order to compute the Jacobian, we need explicit expressions for $u$ and $v$. If we add equations 12 and 13 we find that

$$x = \frac{u+v}{2}$$

And if we subtract equations 12 and 13 we find that

$$y = \frac{u-v}{2}$$

The Jacobian becomes

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

We also need to find the limits of integration in the transformed integral. Using equations 12 and 13 the four lines bounding $R$ in the $xy$-plane become

$$u = 0, \quad u = 1, \quad v = 0, \quad v = 1.$$

The double integral therefore becomes

$$\begin{aligned} \iint\limits_R (x^2 - y^2) dxdy &= \iint\limits_R (x-y)(x+y) dxdy \\ &= \int_0^1 \int_0^1 uv \left| -\frac{1}{2} \right| dudv \\ &= \frac{1}{2} \int_0^1 \int_0^1 (uv) dudv \\ &= \frac{1}{2} \int_0^1 \frac{v}{2} dv \\ &= \frac{1}{8}. \end{aligned}$$

3.4.3. The Jacobian is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6.$$

We also need to find the limits of integration in the transformed integral. The region $R$ is bounded by the ellipse, $4x^2 + 9y^2 = 36$, which becomes the region bounded by the circle $u^2 + v^2 = 1$. Therefore

$$\iint\limits_R x^2 \ dxdy = \iint\limits_{u^2+v^2 \le 1} (9u^2) 6 dudv = 54 \iint\limits_{u^2+v^2 \le 1} (u^2) dudv$$

Switching to polar coordinates,

$$u = r\cos\theta, \quad v = r\sin\theta, \quad J = r$$

our double integral becomes

$$\begin{aligned}
54 \iint\limits_{u^2+v^2\leq 1} (u^2) du dv &= 54 \int_0^{2\pi}\int_0^1 r^3 \cos^2\theta dr d\theta \\
&= 54 \left( \int_0^{2\pi} \cos^2\theta d\theta \right)\left( \int_0^1 r^3 dr \right) \\
&= 54 \left( \int_0^{2\pi} \frac{1}{2}\big(1+\cos(2\theta)\big) d\theta \right)\left( \frac{1}{4} \right) \\
&= \frac{27}{4}\left( \int_0^{2\pi} \big(1+\cos(2\theta)\big) d\theta \right) \\
&= \frac{27}{4}\big(\theta + \frac{1}{2}\sin(2\theta)\big|_0^{2\pi} \\
&= \frac{27}{4}(2\pi) \\
&= \frac{27\pi}{2}
\end{aligned}$$

3.4.4. In cylindrical coordinates, the region is described by

$$V = \{(r,\theta,z) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 3r\}.$$

Our integral becomes

$$\begin{aligned}
\iiint\limits_V (r\sin\theta)^2 r dz dr d\theta &= \int_0^{2\pi}\int_0^1\int_0^{3r} r^3\sin^2\theta dz dr d\theta \\
&= 3\int_0^{2\pi}\int_0^1 r^4 \sin^2\theta dr d\theta \\
&= \frac{3}{5}\int_0^{2\pi} \sin^2\theta d\theta \\
&= \frac{3}{10}\int_0^{2\pi} \big(1-\cos(2\theta)\big) d\theta \\
&= \frac{3}{10}\big(\theta - \frac{1}{2}\sin(2\theta)\big)\Big|_0^{2\pi} \\
&= \frac{3\pi}{5}
\end{aligned}$$

3.4.5. In cylindrical coordinates, $V$ is the region bounded by

$$\begin{aligned}
&0 \leq r \leq 2 \\
&0 \leq \theta \leq \frac{\pi}{2} \\
&0 \leq z \leq \sqrt{4-r^2}
\end{aligned}$$

The triple integral becomes

$$\begin{aligned}
\iiint\limits_V dx dy dz &= \int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}} r dz dr d\theta \\
&= \int_0^{\pi/2}\int_0^2 r\sqrt{4-r^2} dr d\theta \\
&= \frac{-1}{3}\int_0^{\pi/2} (4-r^2)^{3/2}\Big|_0^2 d\theta \\
&= \frac{-1}{3}\int_0^{\pi/2} (0-8) d\theta \\
&= \frac{4\pi}{3}
\end{aligned}$$

3.4.6. **Triple Integral In Spherical Coordinates**

The solid is a section of a sphere with radius 1, centered at the origin. The section is the part of the sphere that lies above the plane $z = 0$, and between the planes $y = 0$, and $y = x$.

$$\begin{aligned}\int_0^{\pi/4}\int_0^{\pi/2}\int_0^1 (\rho^2 \sin\phi)\ d\rho\ d\phi\ d\theta &= \int_0^{\pi/4}\int_0^{\pi/2} \frac{\sin\phi}{3} d\phi\ d\theta \\ &= \int_0^{\pi/4} \frac{-(0-1)}{3} d\theta \\ &= \pi/12\end{aligned}$$

## 3.5 Application: Center of Mass

3.5.1. **Center of Mass of a 2D Triangular Plate**

The total mass of the plate is

$$\begin{aligned}M &= \iint_R \delta(x,y)dA \\ &= \int_0^1\int_{2x}^{4x} xy\ dydx \\ &= \frac{1}{2}\int_0^1 x(16x^2 - 4x^2)\ dx \\ &= 6\int_0^1 x^3\ dx \\ &= \frac{3}{2}.\end{aligned}$$

The $x$-coordinate of the center of mass is

$$\begin{aligned}\bar{x} = \frac{M_y}{M} &= \frac{2}{3}\iint_R x\delta(x,y)dA \\ &= \int_0^1\int_{2x}^{4x} x^2y\ dydx \\ &= \frac{1}{2}\int_0^1 x^2(16x^2 - 4x^2)\ dx \\ &= 6\int_0^1 x^4\ dx \\ &= \frac{6}{5}.\end{aligned}$$

The $y$-coordinate of the center of mass is

$$\begin{aligned}\bar{y} = \frac{M_x}{M} &= \frac{2}{3}\iint_R y\delta(x,y)dA \\ &= \int_0^1\int_{2x}^{4x} xy^2\ dydx \\ &= \frac{1}{3}\int_0^1 x(64x^3 - 8x^3)\ dx \\ &= \frac{56}{3}\int_0^1 x^4\ dx \\ &= \frac{56}{15}.\end{aligned}$$

Thus, the coordinates of the center of mass are $(6/5, 56/15)$.

3.5.2. **Center of Mass of a 2D Plate, Radial Density Function**

We could express the coordinates of the center of mass using either Cartesian or polar coordinates. Using Cartesian coordinates, the density, $\delta$, at point $(x, y)$, is given by

$$\delta(x,y) = L(x,y) = a - \sqrt{x^2 + y^2}.$$

(a) The $x$-coordinate of the center of mass is

$$\bar{x} = \frac{M_y}{M} = \frac{1}{M}\iint\limits_R x\delta(x,y)dA = \frac{1}{M}\int_{-a}^{a}\int_{0}^{\sqrt{x^2+y^2}} x\left(a-\sqrt{x^2+y^2}\right)\,dydx$$

(b) The $y$-coordinate of the center of mass is

$$\bar{y} = \frac{M_x}{M} = \frac{1}{M}\iint\limits_R y\delta(x,y)dA = \frac{1}{M}\int_{-a}^{a}\int_{0}^{\sqrt{x^2+y^2}} y\left(a-\sqrt{x^2+y^2}\right)\,dydx$$

(c) The center of mass is located on the $y$-axis. In other words, $\bar{x} = 0$. This is because of symmetry: the integrand of $\bar{x}$ is odd in $x$, the integral with respect to $x$ is calculated about an interval that is symmetric about the $y$-axis, and so the integral with respect to $x$ is zero.

Note that we could just as easily set up the above integrals using polar coordinates. The 2D circular plate, in polar coordinates, is bounded by

$$0 \le r \le a, \quad 0 \le \theta \le \pi, \quad a > 0.$$

The coordinates for the center of mass are

$$\bar{x} = \frac{M_y}{M} = \frac{1}{M}\iint\limits_R x\delta(x,y)dA = \frac{1}{M}\int_0^{\pi}\int_0^{r} r\cos(\theta)(a-r)\,rdrd\theta$$

$$\bar{y} = \frac{M_x}{M} = \frac{1}{M}\iint\limits_R y\delta(x,y)dA = \frac{1}{M}\int_0^{\pi}\int_0^{r} r\sin(\theta)(a-r)\,rdrd\theta$$

3.5.3. The integral can be written as

$$\int_0^{\pi}\int_{-1}^{1} e^{x^2+y^2}\sin(y)dydx = \int_0^{\pi}\int_{-1}^{1} e^{x^2}e^{y^2}\sin(y)dydx = \left(\int_0^{\pi} e^{x^2}\,dx\right)\left(\int_{-1}^{1} e^{y^2}\sin(y)dy\right)$$

The second term is the integral of an odd function over a symmetrical interval, and so is equal to zero. Therefore,

$$\int_0^{\pi}\int_{-1}^{1} e^{x^2+y^2}\sin(y)dydx = 0.$$