Written Assignment Questions and Solutions

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Part I
Questions for Written Assignments

2 Multiple Integrals

2.0 Wolfram Alpha Syntax (optional)

You may want to use Wolfram Alpha (wolframalpha.com) to check your answers. If you're not sure what syntax to use to compute double integrals with Wolfram Alpha, let's suppose that we want to determine the value of

$$\int_{-2}^{-1}\int_{0}^{x-1}(x^{2C}+y)dydx$$

where C is a constant. The syntax we could use to compute this particular integral is

```
integrate x^{2C}+y dydx, x from -2 to -1 and y from 0 to (x-1)
```

2.1 Double Integrals

The following questions are related to Section 3.1 of Vector Calculus by Michael Corral.

2.1.1. Consider the solid that lies under the plane $z = -x - 2y + 2$ and above the rectangle $\{(x, y) \mid -2 \le x \le 0, 0 \le y \le 1\}$.

(a) Sketch the solid in $\mathbb{R}^3$. *Hint: start by plotting the points that are located on the given plane and above the corners of the rectangle. Then connect the points with solid lines.*

(b) Find the volume of the solid.

2.1.2. **Application to Fluid Mechanics**
In a two-dimensional, steady-state, incompressible fluid flow, the velocity $\mathbf{v}$ of the flow can be expressed as $\mathbf{v} = u(x, y)\mathbf{i} + v(x, y)\mathbf{j}$. The functions $u(x, y)$ and $v(x, y)$ must satisfy

$$\nabla \cdot \mathbf{v} = 0.$$

(a) If $u(x, y) = x^2 + y^2$, find the most general form of $v(x, y)$.

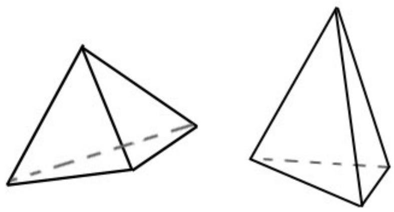
(b) If $v(x, y) = \cos(x)$, find the most general form of $u(x, y)$.

Hint: you are asked to find the most ***general*** *form of functions u and v.*

2.2 Double Integrals Over a General Region

The following questions are related to Section 3.2 of Vector Calculus by Michael Corral.

2.2.1. **Volume of a Tetrahedron**
A **tetrahedron** is a three dimensional object with four, triangular, flat sides. Because each of its four sides are flat, a tetrahedron can be defined as the region enclosed by four planes. Below is a sketch of two tetrahedrons. Note that each tetrahedron has four vertices, six edges, and that the lengths of its edges do not have to be equal.



Consider the tetrahedron that is bounded by the three coordinate planes in $\mathbb{R}^3$, and by the plane $z = 1 - x - \frac{y}{2}$.

(a) Sketch the tetrahedron in $\mathbb{R}^3$ and label the points that represent its four vertices.

(b) Set up, but do not evaluate, a double integral that represents the volume of the tetrahedron. Integrate with respect to x first.

(c) Set up, but do not evaluate, a double integral that represents the volume of the tetrahedron. Integrate with respect to y first.

Note that

- Examples 3.4 and 3.5 from *Vector Calculus* by Michael Corral are similar to this problem.
- We could also calculate the volume of the solid by using a **triple** integral.
- Although not required, the double integrals are straightforward to compute. You may want to check your answers by evaluating the integrals and seeing if you would get the same result in parts (b) and (c).

2.2.2. **Area of a General Region**

(a) Question 11 from Section 3.2 of *Vector Calculus* by Michael Corral. *Hint: Figure 3.2.5 is also helpful.*

(b) Use the result from part (a) to compute the area of the region bounded by the curves $y = x^2$ and $x = y^2$.

2.2.3. Find the volume of the solid enclosed by $z = x^2 + y^2$, $y = x^2$ and $x = y^2$.

2.2.4. Consider the integral

$$\iint_R x \sin(y) dA$$

where R is the region bounded by $y = 0$, $y = x^2$, and $x = 2$.

(a) Evaluate the double integral by first integrating with respect to x.

(b) Evaluate the double integral by first integrating with respect to y.

Note that your answers for both parts should be the same, and that you may need to use various techniques of integration to complete this problem, including integration by parts and a variable substitution.

2.2.5. Consider the double integral

$$\int_0^{1+e} \int_0^{\ln(x-1)} f(x,y)dydx.$$

Sketch the region in $\mathbb{R}^2$ over which $f(x,y)$ is integrated, and change the order of integration.

2.2.6. Consider the double integral

$$\iint_D f(x,y)dA,$$

where D is the square $0 \le x \le 1$, $0 \le y \le 1$, and $f(x,y) = \sin(x+y)$. Show that

$$0 \le \iint_D f(x,y)dA \le 1.$$

2.2.7. **Simplifying Double Integrals Using Symmetry**
Certain integrals can be simplified when the integrand is either even or odd. You may know that, for functions of one variable, that

$$\text{if } f(x) \text{ is odd, then } \int_{-a}^{a} f(x)dx = 0.$$
$$\text{if } f(x) \text{ is even, then } \int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx.$$

There are similar results for integrals of two variables, but in order to introduce them, we first need to extend our concepts of even and odd functions to functions of two variables, and we need to describe regions that are symmetric about the x-axis and about the y-axis.

If R is a region that is **symmetric about the y-axis**, and (x_0, y_0) is a point inside R, then the point $(-x_0, y_0)$ is also inside R. Similarily, if S is a region that is **symmetric about the x-axis**, and (x_1, y_1) is a point inside S, then the point $(x_1, -y_1)$ is also inside S. Above are examples of regions that have these symmetries.

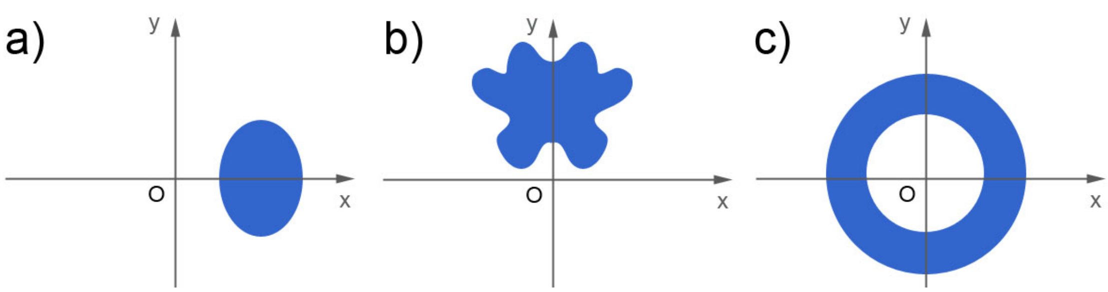


Figure 1: a) the blue region is symmetric about the x-axis, b) the blue region is symmetric about the y-axis, and c) the blue region is symmetric about the x-axis and the y-axis.

Moreover, if $g(x,y)$ is odd in x, then

$$g(-x,y) = -g(x,y).$$

But if $g(x, y)$ were even in x, then

$$g(-x, y) = g(x, y).$$

Combining these concepts yields helpful results for computing double integrals. For example, if R is symmetric about the y-axis, and if $g(x, y)$ is odd in x, then

$$\iint\limits_R g(x, y) dx dy = 0.$$

Similar results can be stated if g were even in x or in y, and if R has symmetry about the x-axis.

(a) Provide an example of a non-zero function of two variables, $h(x, y)$, that is odd in x. Verify that your function is odd in x.

(b) Provide an example of a region, D, that is symmetric about the y-axis, but is not symmetric about the x-axis.

(c) Show that, for your $h(x, y)$ and region D, that

$$\iint\limits_D g(x, y) dx dy = 0.$$

2.3 Triple Integrals

The following questions are related to Section 3.3 of Vector Calculus by Michael Corral.

2.3.1. **Volume of a Tetrahedron**
The textbook points out that the triple integral

$$\iiint\limits_S f(x, y, z) dV$$

for the special special case when $f(x, y, z) = 1$ for all points in S, gives the volume of S

$$V(S) = \iiint\limits_S dV.$$

Consider again the tetrahedron that is bounded by the three coordinate planes in $\mathbb{R}^3$, and by the plane $z = 1 - x - \frac{y}{2}$. We derived an expression for the volume of this tetrahedron in a previous question using a double integral. Now set-up and find the volume of the tetrahedron using a triple integral.

2.3.2. **Volume of an Ellipsoid**
Solve Question 10 from Section 3.5 of *Vector Calculus* by Michael Corral.

2.3.3. **Volume of a Solid**
Find the volume of the solid enclosed by the planes

$$\begin{aligned} z &= x + y \\ y &= x \\ x &= 0 \\ z &= -1 \\ y &= 2 \end{aligned}$$

Hint: it may help to start by plotting the planes in Google or in Wolfram Alpha.
The solutions need to be adjusted so that z is in -1 to x+y

2.4 Change of Variables in Multiple Integrals

The following questions are related to Section 3.5 of Vector Calculus by Michael Corral.

2.4.1. **Linear Transformations**
Under the linear transformation

$$x = c_1 u + c_2 v, \quad y = d_1 u + d_2 v, \quad d_1 c_2 - d_2 c_1 \neq 0,$$

straight lines in the uv-plane are mapped to straight lines in the xy-plane.

(a) $v = v_0$ is a horizontal line in the uv-plane. Determine the equation of this line in the xy-plane.

(b) $x = x_0$ is a vertical line in the xy-plane. Determine the equation of this line in the uv-plane.

2.4.2. Use an appropriate transformation to evaluate the integral

$$\iint_R (x^2 - y^2) dx dy,$$

where R is the parallelogram bounded by

$$x + y = 0, \quad x + y = 1, \quad x - y = 0, \quad x - y = 1.$$

2.4.3. **Simplifying Double Integrals**
When working with double integrals over a rectangular region $R = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$, we can use the simplification

$$\iint_R g(x)h(y)dA = \int_a^b g(x)dx \int_c^d h(y)dy$$

Use this property and the transformation $x = 3u, y = 2v$ to evaluate the double integral

$$\iint_E x^2 \; dx dy,$$

where E is region bounded by the ellipse $4x^2 + 9y^2 = 36$.

2.4.4. Evaluate the integral

$$\iiint_S y^2 dV,$$

where S is the solid that lies inside the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$ and below the cone $z^2 = 9x^2 + 9y^2$.

2.4.5. **Triple Integral In Cylindrical Coordinates**
Evaluate the integral

$$\iiint_V dV,$$

using cylindrical coordinates, where V is the region bounded by

$$\begin{aligned} &0 \leq x \leq 2 \\ &0 \leq y \leq \sqrt{4 - x^2} \\ &0 \leq z \leq \sqrt{4 - (x^2 + y^2)} \end{aligned}$$

2.4.6. **Triple Integral In Spherical Coordinates**
The integral

$$\int_0^{\pi/4} \int_0^{\pi/2} \int_0^1 (\rho^2 \sin\phi) \; d\rho \; d\phi \; d\theta$$

represents the volume of a solid. Describe the shape of the solid, and find its volume.
Textbook doesn't do a great job of integrals in cylindrical and spherical

2.5 Application: Center of Mass

The following questions are related to Section 3.6 of Vector Calculus by Michael Corral.

2.5.1. **Center of Mass of a 2D Triangular Plate**
A two-dimensional plate has density $\delta(x, y) = xy$, and occupies a triangular region whose vertices are located at the points (0,0), (1,2), and (1,4). Find the x-coordinate and the y-coordinate of the center of mass of the triangular plate.

2.5.2. **Center of Mass of a 2D Plate, Radial Density Function**
A 2D semi-circular plate of mass M is bounded by

$$-a \leq x \leq a, \quad 0 \leq y \leq \sqrt{a^2 - x^2}, \quad a > 0.$$

The density of the plate at a point (x, y), is equal to the shortest distance, L, between that point and the upper edge of the plate, as shown in Figure 2.

(a) Set up an integral that represents the x-coordinate of the center of mass of the plate, $\bar{x}$.

(b) Set up an integral that represents the y-coordinate of the center of mass of the plate, $\bar{y}$.

(c) Determine the x-coordinate of the center of mass of the plate, without performing any integration. Briefly describe how you found your answer.

You do not need to perform any integration for this question. Note also that L is a function of x and y.

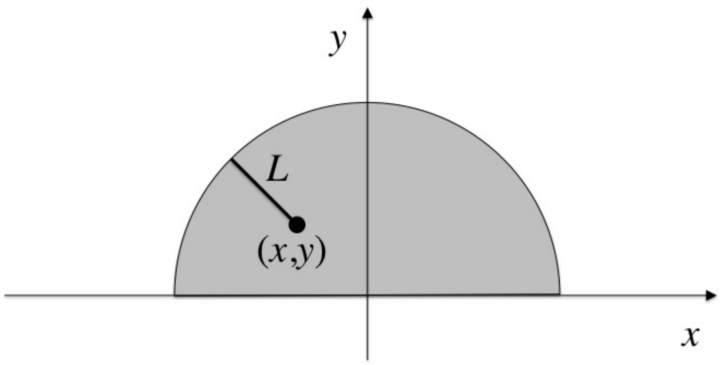


Figure 2: The density of the plate at (x, y) is equal to L.

2.5.3. **Center of Mass of a 2D Plate, Radial Density Function**

2.5.4. *I'll push this question to a midterm or final exam*
Determine the value of

$$\int_0^{\pi} \int_{-1}^{1} x^4 e^{x^2+y^2} \sin(y) dy dx.$$

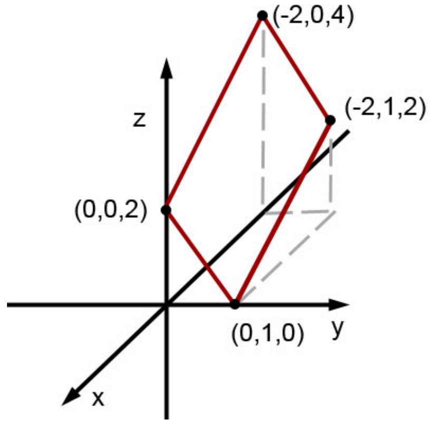
Do not use integration by parts.

Part II
Solutions to Written Assignment Questions

2 Multiple Integrals

2.1 Double Integrals

2.1.1. (a) A sketch of the volume is below.



(b) We must integrate:

$$
\begin{aligned}
\int_{-2}^{0}\int_{0}^{1}(-x-2y+2)dydx &= \int_{-2}^{0}\big(-2xy-y^2+2y\big)\big|_0^1 dx \\
&= \int_{-2}^{0}\Big(-2x+1\Big)dx \\
&= \Big(-x^2+x\Big)\Big|_{-2}^{0} \\
&= 0-\big(-(-2)^2+(-2)\big) \\
&= 0-\big(-4-2\big) \\
&= 6
\end{aligned}
$$

2.1.2. (a) Substituting the expression for u into the divergence equation yields

$$
\begin{aligned}
0 &= \nabla\cdot\mathbf{v} \\
&= \frac{\partial}{\partial x}\big(u(x,y)\big)+\frac{\partial}{\partial y}\big(v(x,y)\big) \\
&= \frac{\partial}{\partial x}(x^2+y^2)+\frac{\partial v}{\partial y} \\
&= 2x+\frac{\partial v}{\partial y} \\
\frac{\partial v}{\partial y} &= -2x
\end{aligned}
$$

Therefore, $v(x,y)$ is a function whose partial derivative with respect to y is $-2x$. The **most general** form for $v(x,y)$ is obtained by integrating with respect to y:

$$v(x,y) = -2xy+f(x)$$

where $f(x)$ is an unknown function of one variable, x.

(b) Using the same approach as we used for (a) yields

$$\begin{aligned}
0 &= \nabla \cdot \mathbf{v} \\
&= \frac{\partial}{\partial x}\big(u(x,y)\big) + \frac{\partial}{\partial y}\big(v(x,y)\big) \\
&= \frac{\partial u}{\partial x} + 0 \\
\frac{\partial u}{\partial x} &= 0
\end{aligned}$$

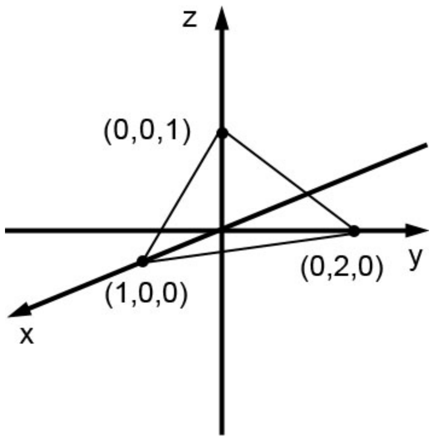
Therefore, $u(x,y)$ is a function whose partial derivative with respect to x is 0. The **most general** form for $u(x,y)$ is obtained by integrating with respect to x:

$$u(x,y) = g(y)$$

where $g(y)$ is an unknown function of one variable, y.

2.2 Double Integrals Over a General Region

2.2.1. (a) A sketch of the tetrahedron is below.



(b) The tetrahedron has one side in the xy-plane. This side is bounded by the line that is the intersection between the xy-plane and the plane $z = 1 - x - y/2$. We can find this intersection by setting $z = 0$,

$$\begin{aligned}
0 &= 1 - x - \frac{y}{2} \\
x &= 1 - \frac{y}{2}.
\end{aligned}$$

Therefore, the volume is the region under the plane $z = 1 - x - y/2$ and over

$$R = \{(x,y) \mid 0 \le x \le 1 - y/2,\ 0 \le y \le 2\}.$$

The double integral is

$$\int_0^2 \int_0^{1-y/2} (1 - x - \frac{y}{2}) dx dy$$

(c) The volume is the region under the plane $z = 1 - x - y/2$ and over

$$R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 2 - 2x\}.$$

The double integral is

$$\int_0^1 \int_0^{2-2x} (1 - x - \frac{y}{2}) dy dx$$

2.2.2. (a) Suppose that we subdivide region R into a rectangular grid of sub-rectangles (as in Figure 3.2.5), so that we only consider the sub-rectangles that are completely enclosed in R. Then, the area of region R is approximated by the double sum

$$\sum_j \sum_i \Delta x_i \Delta y_j$$

But if $f = 1$ for all x_i and y_j, this is equal to:

$$\sum_j \sum_i f(x_i, y_j)\Delta x_i \Delta y_j \tag{1}$$

where x_i and y_j is a point inside sub-rectangle $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$. If we take smaller and smaller rectangles, so that the length of the longest diagonal of the sub-rectangles goes to zero, the sub-rectangles begin to fill more and more of the region R, and so the above sums approach the **area** of region R. Since we have defined

$$\iint f(x,y)dA$$

as the limit of Equation (1) as the longest diagonal goes to zero, and $f(x,y) = 1$, the double integral

$$\iint 1dA$$

is the area of region R.

(b) The region may be defined as

$$R = \{(x,y) \mid 0 \le x \le 1,\ x^2 \le y \le \sqrt{x}\}.$$

The area of R is

$$\begin{aligned}\int_0^1 \int_{x^2}^{\sqrt{x}} dydx &= \int_0^1 \left(\sqrt{x} - x^2\right)dx \\ &= \left(\frac{2}{3}x^{3/2} - \frac{x^3}{3}\right)\Big|_0^1 \\ &= \frac{2}{3} - \frac{1}{3} \\ &= \frac{1}{3}\end{aligned}$$

2.2.3. The curves $y = x^2$ and $x^2 = y$ intersect at (0,0) and at (1,1). Integrating with respect to y first yields

$$\begin{aligned}\int_0^1 \int_{x^2}^{\sqrt{x}} x^2 + y^2 dydx &= \int_0^1 \left(yx^2 + \frac{y^3}{3}\right)\Big|_{x^2}^{\sqrt{x}} dx \\ &= \int_0^1 \left(x^{5/2} + \frac{x^{3/2}}{3} - x^4 - \frac{x^6}{3}\right)dx \\ &= \left(\frac{2}{7}x^{7/2} + \frac{2}{15}x^{5/2} - \frac{1}{5}x^5 - \frac{1}{21}x^7\right)\Big|_0^1 \\ &= \frac{2}{7} + \frac{2}{15} - \frac{1}{5} - \frac{1}{21} \\ &= 6/35\end{aligned}$$

2.2.4. (a) Integrating with respect to y first yields

$$\begin{aligned}\int_0^2 \int_0^{x^2} x\sin(y)dydx &= \int_0^2 -x\cos(y)\Big|_0^{x^2} dx \\ &= -\int_0^2 \left(x\cos(x^2) - 1\right)dx \\ &= -\int_0^2 x\cos(x^2)dx + \int_0^2 1\ dx \\ &= 2 - \int_0^2 x\cos(x^2)dx\end{aligned}$$

Now let $u = x^2$, so that $du = 2xdx$. Then

$$\begin{aligned}
2 - \int_0^2 x\cos(x^2)dx &= 2 - \int_0^4 x\cos(u)\Big(\frac{du}{2x}\Big) \\
&= 2 - \frac{1}{2}\int_0^4 \cos(u)du \\
&= 2 - \frac{1}{2}\sin(u)\Big|_0^4 \\
&= 2 - \frac{1}{2}\sin(4) \\
&\approx 2.3784
\end{aligned}$$

(b) Integrating with respect to x first yields

$$\begin{aligned}
\int_0^4\int_{\sqrt{y}}^2 x\sin(y)dxdy &= \int_0^4 \frac{x^2}{2}\sin(y)\Big|_{\sqrt{y}}^2 dy \\
&= \int_0^4 \frac{4-y}{2}\sin(y)dy \\
&= 2\int_0^4 \sin(y)dy - \frac{1}{2}\int_0^4 y\sin(y)dy \\
&= 2\big(-\cos(y)\big)\big|_0^4 - \frac{1}{2}\int_0^4 y\sin(y)dy \\
&= 2\big(-\cos(4)+1\big) - \frac{1}{2}\int_0^4 y\sin(y)dy \\
&= 2 - 2\cos(4) - \frac{1}{2}\int_0^4 y\sin(y)dy
\end{aligned}$$

Now using integration by parts, with

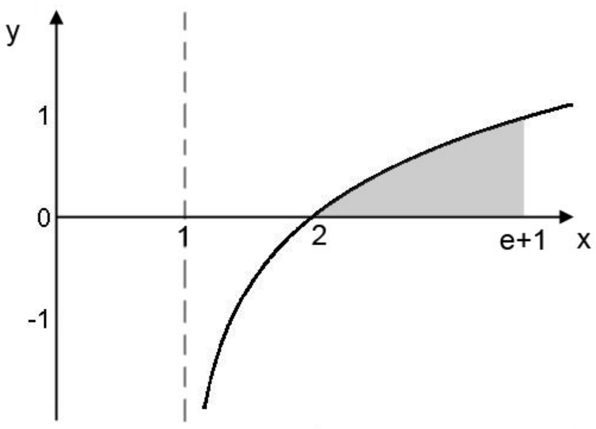
$$\begin{aligned}
u &= y, \quad & dv &= \sin(y)dy \\
du &= dy, \quad & v &= -\cos(y)
\end{aligned}$$

we obtain

$$\begin{aligned}
\int_0^4\int_{\sqrt{y}}^2 x\sin(y)dxdy &= 2 - 2\cos(4) - \frac{1}{2}\int_0^4 y\sin(y)dy \\
&= 2 - 2\cos(4) - \frac{1}{2}\left(-y\cos(y)\Big|_0^4 - \int_0^4 \big(-\cos(y)\big)dy\right) \\
&= 2 - 2\cos(4) - \frac{1}{2}\left(-4\cos 4 + \sin(y)\big|_0^4\right) \\
&= 2 - 2\cos(4) - \frac{1}{2}\big(-4\cos 4 + \sin(4) - 0\big) \\
&= 2 - \frac{\sin(4)}{2} \\
&\approx 2.3784
\end{aligned}$$

2.2.5. The region over which we are integrating $f(x,y)$ is the shaded area below.
The region is bounded by the lines $y = 0$, $x = 1 + e$, and by the curve $y = \ln(x-1)$. Using horizontal slices, values of y range from 0 to 1, and values of x range from $e^y - 1$ to $1 + e$. The double integral becomes

$$\int_0^1\int_{e^y-1}^{1+e} f(x,y)dydx.$$



2.2.6. The integrand $f(x, y)$ has the property that

$$0 \leq \sin(x + y) \leq 1,$$

because $x + y$ is between 0 and 2, and $\sin(2) < 1$. Then

$$0 = \iint_D 0 dA \leq \iint_D f(x, y) dA \leq \iint_D (1) dA = 1.$$

2.2.7. (a) Suppose we use the function

$$h(x, y) = 11xy.$$

Then $h(-x, y) = 11(-x)y = -11xy = -h(x, y)$.

(b) An example of a region that is symmetric about the y-axis, but not symmetric about the x-axis, is the region bounded by the curves

$$x = -1, \quad x = 1, \quad y = 0, \quad y = -1.$$

(c) The double integral for our region is

$$\begin{aligned} \iint_D g(x, y) dx dy &= \int_{-1}^{1} \int_{-1}^{0} 11xy dy dx \\ &= -\frac{11}{2} \int_{-1}^{1} x dx \\ &= -\frac{11}{4} \left(x^2 \right) \Big|_{-1}^{1} \\ &= -\frac{11}{4} (0) \\ &= 0. \end{aligned}$$

2.3 Triple Integrals

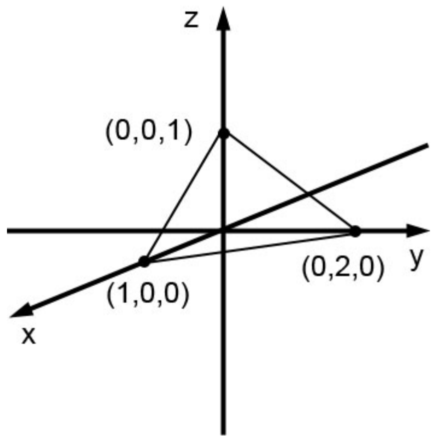
2.3.1. **Volume of a Tetrahedron**

Recall that the volume is the region under the plane $z = 1 - x - y/2$ and over

$$R = \{(x, y) \mid 0 \leq x \leq 1 - y/2, \ 0 \leq y \leq 2\}.$$

Because z lies between 0 and $z = 1 - x - y/2$, the volume, S, can be described as

$$S = \{(x, y, z) \mid 0 \leq x \leq 1 - y/2, \ 0 \leq y \leq 2, \ 0 \leq z \leq 1 - x - y/2\}.$$



The volume can be calculated with the triple integral

$$\begin{aligned}
\int_0^2 \int_0^{1-y/2} \int_0^{1-x-y/2} dzdxdy &= \int_0^2 \int_0^{1-y/2} (1-x-y/2)dxdy \\
&= \int_0^2 \Big(x - \frac{x^2}{2} - \frac{xy}{2}\Big)\Big|_0^{1-y/2} dy \\
&= \int_0^2 \Big((1-y/2) - \frac{(1-y/2)^2}{2} - \frac{(y-y^2/2)}{2}\Big)dy \\
&= \int_0^2 \left(1 - \frac{y}{2} - \frac{1}{2}\Big(1 - y + \frac{y^2}{4}\Big) - \frac{y}{2} + \frac{y^2}{4}\right)dy \\
&= \int_0^2 \left(1 - \frac{y}{2} - \frac{1}{2} + \frac{y}{2} - \frac{y^2}{8} - \frac{y}{2} + \frac{y^2}{4}\right)dy \\
&= \int_0^2 \Big(\frac{1}{2} - \frac{y}{2} + \frac{y^2}{8}\Big)dy \\
&= \frac{2}{2} - \frac{4}{4} + \frac{8}{24} \\
&= \frac{1}{3}
\end{aligned}$$

2.3.2. **Volume of an Ellipsoid**

We are given the transformations

$$x = au, \quad y = bv, \quad z = cw.$$

The Jacobian becomes

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

The solid enclosed by the ellipsoid is the image of the unit sphere $u^2 + v^2 + w^2 \leq 1$. Using that a sphere has volume $\frac{4}{3}\pi r^3$, we find that

$$\begin{aligned}
\iiint_V dxdydz &= \iiint_{u^2+v^2+w^2\leq 1} abc\ dudvdw \\
&= abc \iiint_{u^2+v^2+w^2\leq 1} dudvdw \\
&= abc(\text{volume of a sphere}) \\
&= \frac{4\pi abc}{3}
\end{aligned}$$

2.3.3. **Volume of a Solid**

2.4 Change of Variables in Multiple Integrals

2.4.1. **Linear Transformations**

(a) Substituting $v = v_0$ into the linear transformation yields the two equations

$$x = c_1 u + c_2 v_0$$
$$y = d_1 u + d_2 v_0$$

To find the equation of the line in the xy-plane, we need to eliminate u. There are many ways to do this, but let's multiply the first equation by d_1 and the second by c_1.

$$d_1 x = c_1 d_1 u + c_2 d_1 v_0$$
$$c_1 y = c_1 d_1 u + d_2 c_1 v_0$$

Subtracting these equations yields

$$d_1 x - c_1 y = (c_2 d_1 - d_2 c_1) v_0$$

A simple rearrangement gives us

$$c_1 y = d_1 x - (c_2 d_1 - d_2 c_1) v_0.$$

Provided that c_1 is not zero, we could write this in the form

$$y = \frac{d_1}{c_1} x - \frac{c_2 d_1 - d_2 c_1}{c_1} v_0.$$

(b) Substituting $x = x_0$ into $x = c_1 u + c_2 v$ gives us

$$x_0 = c_1 u + c_2 v,$$

Provided that c_2 is not zero, This is the is mapped into the uv-plane REWORD

2.4.2. The integral can be written as

$$\iint\limits_R (x^2 - y^2) dx dy = \iint\limits_R (x - y)(x + y) dx dy.$$

Recall that R is the region bounded by

$$x + y = 0, \quad x + y = 1, \quad x - y = 0, \quad x - y = 1.$$

The appearance of the terms $(x + y)$ and $(x - y)$ in the integrand and in the lines that bound R suggests the transformation

$$u = x + y \tag{2}$$
$$v = x - y. \tag{3}$$

In order to compute the Jacobian, we need explicit expressions for u and v. If we add equations 2 and 3 we find that

$$x = \frac{u + v}{2}$$

And if we subtract equations 2 and 3 we find that

$$y = \frac{u - v}{2}$$

The Jacobian becomes

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

We also need to find the limits of integration in the transformed integral. Using equations 2 and 3 the four lines bounding R in the xy-plane become

$$u = 0, \quad u = 1, \quad v = 0, \quad v = 1.$$

The double integral therefore becomes

$$\begin{aligned}
\iint_R (x^2 - y^2)dxdy &= \iint_R (x-y)(x+y)dxdy \\
&= \int_0^1 \int_0^1 uv \left| -\frac{1}{2} \right| dudv \\
&= \frac{1}{2} \int_0^1 \int_0^1 (uv)dudv \\
&= \frac{1}{2} \int_0^1 \frac{v}{2} dv \\
&= \frac{1}{8}.
\end{aligned}$$

2.4.3. The Jacobian is

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix} = 6.$$

We also need to find the limits of integration in the transformed integral. The region R is bounded by the ellipse, $4x^2 + 9y^2 = 36$, which becomes the region bounded by the circle $u^2 + v^2 = 1$. Therefore

$$\iint_R x^2 \; dxdy = \iint_{u^2+v^2 \leq 1} (9u^2)6dudv = 54 \iint_{u^2+v^2 \leq 1} (u^2)dudv$$

Switching to polar coordinates,

$$u = r\cos\theta, \quad v = r\sin\theta, \quad J = r$$

our double integral becomes

$$\begin{aligned}
54 \iint_{u^2+v^2 \leq 1} (u^2)dudv &= 54 \int_0^{2\pi} \int_0^1 r^3 \cos^2\theta dr d\theta \\
&= 54 \left(\int_0^{2\pi} \cos^2\theta d\theta \right) \left(\int_0^1 r^3 dr \right) \\
&= 54 \left(\int_0^{2\pi} \frac{1}{2}\big(1 + \cos(2\theta)\big) d\theta \right) \left(\frac{1}{4} \right) \\
&= \frac{27}{4} \left(\int_0^{2\pi} \big(1 + \cos(2\theta)\big) d\theta \right) \\
&= \frac{27}{4} \big(\theta + \frac{1}{2}\sin(2\theta)\big|_0^{2\pi} \\
&= \frac{27}{4} (2\pi) \\
&= \frac{27\pi}{2}
\end{aligned}$$

2.4.4. In cylindrical coordinates, the region is described by

$$V = \{(r, \theta, z) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 3r\}.$$

Our integral becomes

$$\begin{aligned}
\iiint\limits_V (r\sin\theta)^2 r dz dr d\theta &= \int_0^{2\pi}\int_0^1\int_0^{3r} r^3 \sin^2\theta dz dr d\theta \\
&= 3\int_0^{2\pi}\int_0^1 r^4 \sin^2\theta dr d\theta \\
&= \frac{3}{5}\int_0^{2\pi} \sin^2\theta d\theta \\
&= \frac{3}{10}\int_0^{2\pi} \big(1-\cos(2\theta)\big) d\theta \\
&= \frac{3}{10}\big(\theta - \frac{1}{2}\sin(2\theta)\big)\Big|_0^{2\pi} \\
&= \frac{3\pi}{5}
\end{aligned}$$

2.4.5. In cylindrical coordinates, V is the region bounded by

$$\begin{aligned}
&0 \le r \le 2 \\
&0 \le \theta \le \frac{\pi}{2} \\
&0 \le z \le \sqrt{4-r^2}
\end{aligned}$$

The triple integral becomes

$$\begin{aligned}
\iiint\limits_V dx dy dz &= \int_0^{\pi/2}\int_0^2\int_0^{\sqrt{4-r^2}} r dz dr d\theta \\
&= \int_0^{\pi/2}\int_0^2 r\sqrt{4-r^2} dr d\theta \\
&= \frac{-1}{3}\int_0^{\pi/2} (4-r^2)^{3/2}\Big|_0^2 d\theta \\
&= \frac{-1}{3}\int_0^{\pi/2} (0-8) d\theta \\
&= \frac{4\pi}{3}
\end{aligned}$$

2.4.6. **Triple Integral In Spherical Coordinates**
The solid is a section of a sphere with radius 1, centered at the origin. The section is the part of the sphere that lies above the plane $z = 0$, and between the planes $y = 0$, and $y = x$.

$$\begin{aligned}
\int_0^{\pi/4}\int_0^{\pi/2}\int_0^1 (\rho^2\sin\phi)\ d\rho\ d\phi\ d\theta &= \int_0^{\pi/4}\int_0^{\pi/2} \frac{\sin\phi}{3} d\phi\ d\theta \\
&= \int_0^{\pi/4} \frac{-(0-1)}{3} d\theta \\
&= \pi/12
\end{aligned}$$

2.5 Application: Center of Mass

2.5.1. **Center of Mass of a 2D Triangular Plate**
The total mass of the plate is

$$\begin{aligned}
M &= \iint\limits_R \delta(x, y) dA \\
&= \int_0^1 \int_{2x}^{4x} xy \; dydx \\
&= \frac{1}{2} \int_0^1 x(16x^2 - 4x^2) \; dx \\
&= 6 \int_0^1 x^3 \; dx \\
&= \frac{3}{2}.
\end{aligned}$$

The x-coordinate of the center of mass is

$$\begin{aligned}
\bar{x} = \frac{M_y}{M} &= \frac{2}{3} \iint\limits_R x\delta(x, y) dA \\
&= \int_0^1 \int_{2x}^{4x} x^2 y \; dydx \\
&= \frac{1}{2} \int_0^1 x^2(16x^2 - 4x^2) \; dx \\
&= 6 \int_0^1 x^4 \; dx \\
&= \frac{6}{5}.
\end{aligned}$$

The y-coordinate of the center of mass is

$$\begin{aligned}
\bar{y} = \frac{M_x}{M} &= \frac{2}{3} \iint\limits_R y\delta(x, y) dA \\
&= \int_0^1 \int_{2x}^{4x} xy^2 \; dydx \\
&= \frac{1}{3} \int_0^1 x(64x^3 - 8x^3) \; dx \\
&= \frac{56}{3} \int_0^1 x^4 \; dx \\
&= \frac{56}{15}.
\end{aligned}$$

Thus, the coordinates of the center of mass are $(6/5, 56/15)$.

2.5.2. **Center of Mass of a 2D Plate, Radial Density Function**
We could express the coordinates of the center of mass using either Cartesian or polar coordinates. Using Cartesian coordinates, the density, δ, at point (x, y), is given by

$$\delta(x, y) = L(x, y) = a - \sqrt{x^2 + y^2}.$$

(a) The x-coordinate of the center of mass is

$$\bar{x} = \frac{M_y}{M} = \frac{1}{M} \iint\limits_R x\delta(x, y) dA = \frac{1}{M} \int_{-a}^{a} \int_0^{\sqrt{x^2+y^2}} x\left(a - \sqrt{x^2 + y^2}\right) dydx$$

(b) The y-coordinate of the center of mass is

$$\bar{y} = \frac{M_x}{M} = \frac{1}{M} \iint\limits_R y\delta(x, y) dA = \frac{1}{M} \int_{-a}^{a} \int_0^{\sqrt{x^2+y^2}} y\left(a - \sqrt{x^2 + y^2}\right) dydx$$

(c) The center of mass is located on the y-axis. In other words, $\bar{x} = 0$. This is because of symmetry: the integrand of $\bar{x}$ is odd in x, the integral with respect to x is calculated about an interval that is symmetric about the y-axis, and so the integral with respect to x is zero.

Note that we could just as easily set up the above integrals using polar coordinates. The 2D circular plate, in polar coordinates, is bounded by

$$0 \leq r \leq a, \quad 0 \leq \theta \leq \pi, \quad a > 0.$$

The coordinates for the center of mass are

$$\bar{x} = \frac{M_y}{M} = \frac{1}{M} \iint\limits_R x\delta(x,y)dA = \frac{1}{M} \int_0^\pi \int_0^r r\cos(\theta)(a-r)\ r dr d\theta$$
$$\bar{y} = \frac{M_x}{M} = \frac{1}{M} \iint\limits_R y\delta(x,y)dA = \frac{1}{M} \int_0^\pi \int_0^r r\sin(\theta)(a-r)\ r dr d\theta$$

2.5.3. The integral can be written as

$$\int_0^\pi \int_{-1}^1 e^{x^2+y^2} \sin(y) dy dx = \int_0^\pi \int_{-1}^1 e^{x^2} e^{y^2} \sin(y) dy dx = \left(\int_0^\pi e^{x^2} dx \right) \left(\int_{-1}^1 e^{y^2} \sin(y) dy \right)$$

The second term is the integral of an odd function over a symmetrical interval, and so is equal to zero. Therefore,

$$\int_0^\pi \int_{-1}^1 e^{x^2+y^2} \sin(y) dy dx = 0.$$