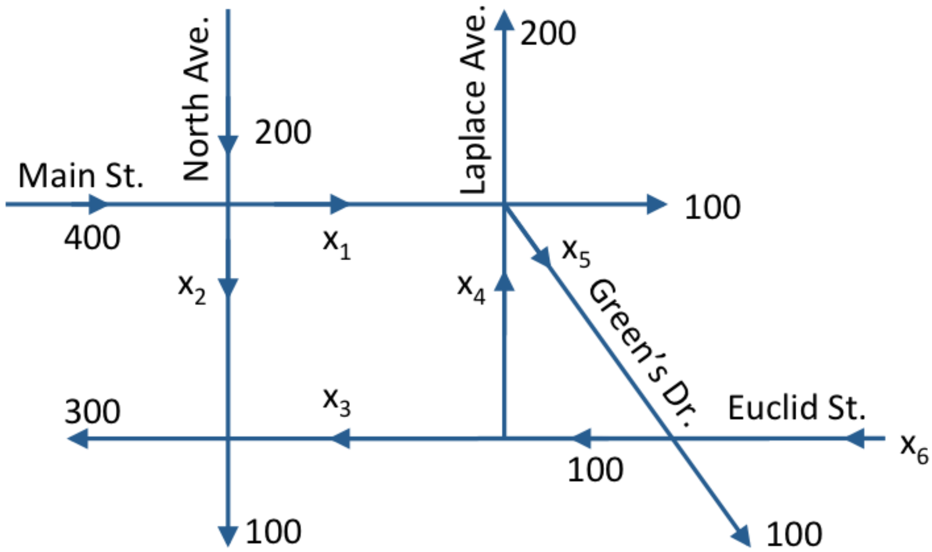
Final Exam

Questions

1. The diagram below shows the typical flow of traffic (in number of vehicles per hour) on a Saturday afternoon in the downtown area of a city where all the streets are one-way streets. For example, at the intersection of Main St. and North Ave., the number of vehicles that flow into the intersection per hour is typically 400 + 200, and the number of cars that flow out of that intersection is $x_1 + x_2$.



Set up a linear system of equations that could be used to determine all of the x's, and find the value of x_6 so that the system has exactly one solution. Assume that the total number of cars in this area of the city remains constant throughout Saturday afternoons. *Hint: you shouldn't need to use a computer, a calculator, or Gaussian elimination to find* x_6.

2. When a particle moves through a liquid in a region of $\mathbb{R}^3$, it traces out a path C described by $x = x(t), y = y(t), z = z(t)$, where t represents time. If the density of the particle changes over time, then $\rho = \rho(x, y, z, t) = \rho\big(x(t), y(t), z(t), t\big)$ is the density of the fluid at time t. Show that, along C,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \nabla \rho \cdot \mathbf{r}'(t),$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

3. Find the area of the triangle with vertices $(0, 0, 0)$, $(1, 0, 0)$, and $(0, 1, 1)$ using a double integral. *Hint: you may want to check your answer by calculating the area another way (for example, by using a cross product).*

4. Suppose that we wish to minimize a function of the form $f = Ax + By$, where A and B are positive constants, subject to the constraint that we may only consider points in the region D that describes a polygon[1] in the xy-plane. Let the region D be the region bounded by a polygon in the first quadrant.

   (a) By considering level sets of $f(x, y)$, show that f is minimized at one or more vertices of region D. Use ∇f in your explanation.

   (b) For what values of A and B does the minimum values of f have two minima?

5. Provide an example of a function of two variables, $f(x, y)$, that has the following properties.

   - $f(x, y)$ has no more than four critical points on $\mathbb{R}^2$.

[1] Recall that a polygon is a two-dimensional shape with any number of straight sides. Rectangles, triangles, and hexagons, are three examples of polygons.

- $f(x, y)$ has a maximum at the point $(a, 0)$, and a minimum at the point $(0, b)$, where $a < 0$ and $b > 0$.
- $f(x, y)$ is continuous everywhere on $\mathbb{R}^2$.

Show that your function satisfies the above properties.

6. Calculate the volume of ...

7.

8.

9. In one paragraph, please describe the importance of interdisciplinary engineering today. Please use examples of interdisciplinary fields and works. Your paragraph should not exceed X words, where X is a number that your instructor will provide.

10. Let $f = -\frac{x^2}{2} + y^3$, and ϕ be a differentiable function of x, so that $\phi = \phi(x)$. Suppose also that $f(x, \phi(x)) = \frac{1}{2}$ for all x. Find $\phi'(1)$, and do not leave your answer in terms of ϕ. ***We may want to move this question back to a written assignment. I'm not sure yet.***

Solutions

1. For any intersection, the number of cars leaving the intersection must equal the number of cars entering the intersection. For the intersection of North and Main,

$$100 + 400 = x_1 + x_2.$$

The other intersections give us the equations:

$$\begin{aligned} x_1 + x_4 &= 200 + x_5 + 100 \quad \text{(at Laplace, Green's, Main)} \\ x_2 + x_3 &= 300 + 100 \quad \text{(at Euclid and North)} \\ x_3 + x_4 &= 100 \quad \text{(at Euclid and Laplace)} \\ 100 + x_5 &= 100 + x_6 \quad \text{(at Green's and Euclid)} \end{aligned}$$

We can collect these equations into the system

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 500 \\ 300 \\ 400 \\ 100 \\ 100 \end{bmatrix}$$

If the total number of cars remains constant, the number of cars entering the downtown area must equal the number of cars leaving the downtown area. Mathematically, this means:

$$\begin{aligned} \text{Number of cars entering downtown} &= \text{number of cars leaving downtown} \\ x_6 + 400 + 200 &= 100 + 100 + 100 + 300 \\ x_6 &= 0 \end{aligned}$$

A more tedious, but acceptable method would be to row-reduce the above system and find a value of x_6 such that the system is consistent.

2. Applying the chain rule:

$$\begin{aligned} \frac{d\rho}{dt} &= \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \rho}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \rho}{\partial z}\frac{\partial z}{\partial t} \\ &= \frac{\partial \rho}{\partial t} + \begin{bmatrix} \frac{\partial \rho}{\partial x} \\ \frac{\partial \rho}{\partial y} \\ \frac{\partial \rho}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial t} \end{bmatrix} \\ &= \frac{\partial \rho}{\partial t} + \nabla \rho \cdot \mathbf{r}'(t). \end{aligned}$$

3. The length of the sides of the right-angled triangle whose area we need can be determined with the distance formula. The triangle has sides of length $\sqrt{2}$, 1, and the length of its hypotenuse is $\sqrt{3}$. Using these lengths, we can construct a triangle in $\mathbb{R}^2$ whose area is the area of the triangle we were given. The hypotenuse can be described by the line $y = -\sqrt{2}(x-1)$. The double integral that gives us the area is

$$\begin{aligned} \int_0^1 \int_0^{-\sqrt{2}(x-1)} dydx &= \int_0^1 \Big(-\sqrt{2}(x-1) \Big) dx \\ &= -\sqrt{2} \int_0^1 (x-1) dx \\ &= -\sqrt{2} \Big(\frac{x^2}{2} - x \Big) \Big|_0^1 \\ &= -\sqrt{2} \Big(\frac{-1}{2} \Big) \\ &= \frac{\sqrt{2}}{2}. \end{aligned}$$

4. Differentiating $f(x, \phi(x)) = \frac{1}{2}$ with respect to x gives us

$$\begin{aligned} \frac{\partial}{\partial x} f(x, \phi) = 0 &= \frac{\partial}{\partial x}\Big(-\frac{x^2}{2} + \big(\phi(x)\big)^3 \Big) \\ &= -x + 3\big(\phi(x)\big)^2 \phi'(x) \end{aligned}$$

At $x = 1$ this becomes

$$\begin{aligned} 0 &= -1 + 3\big(\phi(1)\big)^2 \phi'(1) \\ \phi'(1) &= \frac{1}{3\big(\phi(1)\big)^2} \end{aligned}$$

But $f(x, \phi(x)) = \frac{1}{2}$, and $f = -\frac{x^2}{2} + y^3$, so at $x = 1$,

$$\begin{aligned} f(x, \phi) = \frac{1}{2} &= \Big(-\frac{x^2}{2} + \big(\phi(x)\big)^3 \Big) \\ &= \Big(-\frac{1^2}{2} + \big(\phi(1)\big)^3 \Big) \\ \big(\phi(1)\big)^3 &= 1 \\ \phi(1) &= 1 \end{aligned}$$

Finally,

$$\phi'(1) = \frac{1}{3}.$$

5. To ...

6. We need a function with critical points when $x = 0$ and when $x = a$. Thus, $\frac{\partial}{\partial x} f$ must be zero at these points. One example of such a function is

$$\frac{\partial}{\partial x} f(x, y) = x(x - a),$$

or, integrating with respect to x, we obtain

$$f(x, y) = \int (x^2 - ax)dx + g(y) = \frac{x^3}{3} - a\frac{x^2}{2} + g(y).$$

Our function must also have critical points when $y = 0$ and when $y = b$. This gives us constraints that we can use to find g. Differentiating with respect to y:

$$\frac{\partial}{\partial y} f(x, y) = g'(y).$$

Setting $g'(y) = y(y - b)$, gives us, after integration

$$f(x, y) = \frac{x^3}{3} + \frac{y^3}{3} - a\frac{x^2}{2} - b\frac{y^2}{2}.$$

We need our function to have a maximum at $(a, 0)$, which requires that $D > 0$ and $f_{xx} < 0$ at $(a, 0)$.

$$\begin{aligned} D &= \Big(f_{xx} f_{yy} - f_{xy} \Big)\Big|_{(a,0)} \\ &= (2x - a)(2y - b)\Big|_{(a,0)} \\ &= -ab. \end{aligned}$$

Because $a < 0$, and $b > 0$, D is positive. And because $f_{xx}(a, 0) = a$, $(a, 0)$ is a maximum.

Likewise, we need our function to have a min at $(0, b)$, which requires that $D > 0$ and $f_{xx} > 0$ at $(0, b)$.

$$
\begin{aligned}
D &= \Big(f_{xx} f_{yy} - f_{xy}\Big)\Big|_{(0,b)} \\
&= (2x - a)(2y - b)\Big|_{(0,b)} \\
&= -ab.
\end{aligned}
$$

Because $a < 0$, and $b > 0$, D is positive. And because $f_{xx}(0, b) = -a$, $(0, b)$ is a maximum.

Also note that our f has only four critical points, at $(0, 0), (a, 0), (0, b), (a, b)$, and because f is a polynomial, it is continuous everywhere on $\mathbb{R}^2$.