Written Assignment 7

Your instructor may assign all of these questions, or a subset of these questions. Additional problems may also be assigned. Before you start this assignment, be sure that you are aware of which questions your instructor expects you to complete.

1 Triple Integrals

The following questions are related to Section 3.3 of *Vector Calculus* by Michael Corral.

1.1. **Volume of a Tetrahedron**
The textbook points out that the triple integral

$$\iiint_S f(x, y, z) dV$$

for the special special case when $f(x, y, z) = 1$ for all points in S, gives the volume of S

$$V(S) = \iiint_S dV.$$

Consider again the tetrahedron that is bounded by the three coordinate planes in $\mathbb{R}^3$, and by the plane $z = 1 - x - \frac{y}{2}$. We derived an expression for the volume of this tetrahedron in a previous question using a double integral. Now set-up and find the volume of the tetrahedron using a triple integral.

1.2. **Volume of an Ellipsoid**
Solve Question 10 from Section 3.5 of *Vector Calculus* by Michael Corral.

1.3. **Volume of a Solid**
Find the volume of the solid enclosed by the planes

$$\begin{aligned} z &= x + y \\ y &= x \\ x &= 0 \\ z &= 0 \\ y &= 2 \end{aligned}$$

Hint: it may help to start by plotting the planes in Google or in Wolfram Alpha.

2 Change of Variables in Multiple Integrals

The following questions are related to Section 3.5 of *Vector Calculus* by Michael Corral.

2.1. **Linear Transformations**
Under the linear transformation

$$x = c_1 u + c_2 v, \quad y = d_1 u + d_2 v, \quad d_1 c_2 - d_2 c_1 \neq 0,$$

straight lines in the uv-plane are mapped to straight lines in the xy-plane.

(a) $v = v_0$ is a horizontal line in the uv-plane. Determine the equation of this line in the xy-plane.
(b) $x = x_0$ is a vertical line in the xy-plane. Determine the equation of this line in the uv-plane.

2.2. Use an appropriate transformation to evaluate the integral

$$\iint_R (x^2 - y^2) dx dy,$$

where R is the parallelogram bounded by

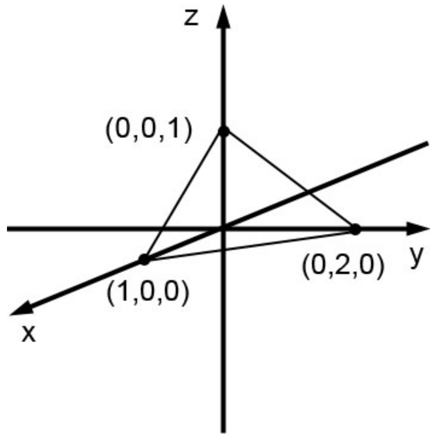
$$x + y = 0, \quad x + y = 1, \quad x - y = 0, \quad x - y = 1.$$

2.3.

Solutions

1 Triple Integrals

1.1. **Volume of a Tetrahedron**



Recall that the volume is the region under the plane $z = 1 - x - y/2$ and over

$$R = \{(x, y) \mid 0 \leq x \leq 1 - y/2,\ 0 \leq y \leq 2\}.$$

Because z lies between 0 and $z = 1 - x - y/2$, the volume, S, can be described as

$$S = \{(x, y, z) \mid 0 \leq x \leq 1 - y/2,\ 0 \leq y \leq 2,\ 0 \leq z \leq 1 - x - y/2\}.$$

The volume can be calculated with the triple integral

$$\begin{aligned}
\int_0^2 \int_0^{1-y/2} \int_0^{1-x-y/2} dz dx dy &= \int_0^2 \int_0^{1-y/2} (1 - x - y/2) dx dy \\
&= \int_0^2 \left(x - \frac{x^2}{2} - \frac{xy}{2} \right) \Big|_0^{1-y/2} dy \\
&= \int_0^2 \left((1 - y/2) - \frac{(1 - y/2)^2}{2} - \frac{(y - y^2/2)}{2} \right) dy \\
&= \int_0^2 \left(1 - \frac{y}{2} - \frac{1}{2} + \frac{y}{4} - \frac{y^2}{2} - \frac{y}{2} + \frac{y^2}{4} \right) dy \\
&= \int_0^2 \left(\frac{1}{2} - \frac{y}{2} - \frac{y^2}{4} \right) dy \\
&= \frac{2}{2} - \frac{4}{4} - \frac{8}{12} \\
&= -\frac{3}{4}
\end{aligned}$$

Wrong!

1.2. **Volume of an Ellipsoid**
We are given the transformations

$$x = au, \quad y = bv, \quad z = cw.$$

The Jacobian becomes

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

The solid enclosed by the ellipsoid is the image of the unit sphere $u^2 + v^2 + w^2 \leq 1$. Using that a sphere has volume $\frac{4}{3}\pi r^3$, we find that

$$\begin{aligned}
\iiint_V dxdydz &= \iiint_{u^2+v^2+w^2\leq 1} abc\ dudvdw \\
&= abc \iiint_{u^2+v^2+w^2\leq 1} dudvdw \\
&= abc(\text{volume of a sphere}) \\
&= \frac{4\pi abc}{3}
\end{aligned}$$

1.3. **Volume of a Solid**

2 Change of Variables in Multiple Integrals

2.1. **Linear Transformations**

(a) Substituting $v = v_0$ into the linear transformation yields the two equations

$$\begin{aligned}
x &= c_1 u + c_2 v_0 \\
y &= d_1 u + d_2 v_0
\end{aligned}$$

To find the equation of the line in the xy-plane, we need to eliminate u. There are many ways to do this, but let's multiply the first equation by d_1 and the second by c_1.

$$\begin{aligned}
d_1 x &= c_1 d_1 u + c_2 d_1 v_0 \\
c_1 y &= c_1 d_1 u + d_2 c_1 v_0
\end{aligned}$$

Subtracting these equations yields

$$d_1 x - c_1 y = (c_2 d_1 - d_2 c_1) v_0$$

A simple rearrangement gives us

$$c_1 y = d_1 x - (c_2 d_1 - d_2 c_1) v_0.$$

Provided that c_1 is not zero, we could write this in the form

$$y = \frac{d_1}{c_1} x - \frac{c_2 d_1 - d_2 c_1}{c_1} v_0.$$

(b) Substituting $x = x_0$ into $x = c_1 u + c_2 v$ gives us

$$x_0 = c_1 u + c_2 v,$$

Provided that c_2 is not zero, This is the is mapped into the uv-plane REWORD

2.2. The integral can be written as

$$\iint_R (x^2 - y^2) dxdy = \iint_R (x - y)(x + y) dxdy.$$

Recall that R is the region bounded by

$$x + y = 0, \quad x + y = 1, \quad x - y = 0, \quad x - y = 1.$$

The appearance of the terms $(x + y)$ and $(x - y)$ in the integrand and in the lines that bound R suggests the transformation

$$u = x + y \tag{1}$$
$$v = x - y. \tag{2}$$

In order to compute the Jacobian, we need explicit expressions for u and v. If we add equations 1 and 2 we find that

$$x = \frac{u+v}{2}$$

And if we subtract equations 1 and 2 we find that

$$y = \frac{u-v}{2}$$

The Jacobian becomes

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

We also need to find the limits of integration in the transformed integral. Using equations 1 and 2 the four lines bounding R in the xy-plane become

$$u = 0, \quad u = 1, \quad v = 0, \quad v = 1.$$

The double integral therefore becomes

$$\begin{aligned} \iint\limits_R \left(x^2 - y^2\right) dx dy &= \iint\limits_R (x-y)(x+y) dx dy \\ &= \int_0^1 \int_0^1 uv \left| -\frac{1}{2} \right| du dv \\ &= \frac{1}{2} \int_0^1 \int_0^1 (uv) du dv \\ &= \frac{1}{2} \int_0^1 \frac{v}{2} dv \\ &= \frac{1}{8}. \end{aligned}$$