Final Exam

Questions

1. Find the area of the triangle with vertices $(0,0,0)$, $(1,0,0)$, and $(0,1,1)$ using a double integral. *Hint: you may want to check your answer by calculating the area another way (for example, by using a cross product).*

2. Let $f = -\frac{x^2}{2} + y^3$, and ϕ be a differentiable function of x, so that $\phi = \phi(x)$. Suppose also that $f(x, \phi(x)) = \frac{1}{2}$ for all x. Find $\phi'(1)$, and do not leave your answer in terms of ϕ.

3. Suppose that we wish to minimize a function of the form $f = Ax + By$, where A and B are positive constants, subject to the constraint that we may only consider points in the region D that describes a polygon in the xy-plane. Let the region D be the

   (a) By considering level sets of $f(x,y)$, show that f is minimized at one or more vertices of region D. Use ∇f in your explanation.

   (b) For what values of A and B does the minimum values of f have two minima?

4. Provide an example of a function of two variables, $f(x,y)$, that has the following properties.

   - $f(x,y)$ has no more than four critical points on $\mathbb{R}^2$.
   - $f(x,y)$ has a maximum at the point $(a,0)$, and a minimum at the point $(0,b)$, where a and b are constants, and $a < 0$ and $b > 0$
   - $f(x,y)$ is continuous everywhere on $\mathbb{R}^2$

   Show that your function satisfies the above properties.

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11. In one paragraph, please describe the importance of interdisciplinary engineering today. Please use examples of interdisciplinary fields and works. Your paragraph should not exceed X words, where X is a number that your instructor will provide.

Solutions

1. The length of the sides of the right-angled triangle whose area we need can be determined with the distance formula. The triangle has sides of length $\sqrt{2}$, 1, and the length of its hypotenuse is $\sqrt{3}$. Using these lengths, we can construct a triangle in $\mathbb{R}^2$ whose area is the area of the triangle we were given. The hypotenuse can be described by the line $y = -\sqrt{2}(x-1)$. The double integral that gives us the area is

$$\begin{aligned}\int_0^1 \int_0^{-\sqrt{2}(x-1)} dydx &= \int_0^1 \Big(-\sqrt{2}(x-1) \Big) dx \\ &= -\sqrt{2} \int_0^1 (x-1)dx \\ &= -\sqrt{2}\Big(\frac{x^2}{2} - x\Big)\Big|_0^1 \\ &= -\sqrt{2}\Big(\frac{-1}{2}\Big) \\ &= \frac{\sqrt{2}}{2}.\end{aligned}$$

2. Differentiating $f(x, \phi(x)) = \frac{1}{2}$ with respect to x gives us

$$\begin{aligned}\frac{\partial}{\partial x} f(x, \phi) = 0 &= \frac{\partial}{\partial x}\Big(-\frac{x^2}{2} + \big(\phi(x)\big)^3 \Big) \\ &= -x + 3\big(\phi(x)\big)^2 \phi'(x)\end{aligned}$$

At $x = 1$ this becomes

$$\begin{aligned}0 &= -1 + 3\big(\phi(1)\big)^2 \phi'(1) \\ \phi'(1) &= \frac{1}{3\big(\phi(1)\big)^2}\end{aligned}$$

But $f(x, \phi(x)) = \frac{1}{2}$, and $f = -\frac{x^2}{2} + y^3$, so at $x = 1$,

$$\begin{aligned}f(x, \phi) = \frac{1}{2} &= \Big(-\frac{x^2}{2} + \big(\phi(x)\big)^3 \Big) \\ &= \Big(-\frac{1^2}{2} + \big(\phi(1)\big)^3 \Big) \\ \big(\phi(1)\big)^3 &= 1 \\ \phi(1) &= 1\end{aligned}$$

Finally,

$$\phi'(1) = \frac{1}{3}.$$

3. We need a function with critical points when $x = 0$ and when $x = a$. Thus, $\frac{\partial}{\partial x} f$ must be zero at these points. One example of such a function is

$$\frac{\partial}{\partial x} f(x, y) = x(x-a),$$

or, integrating with respect to x, we obtain

$$f(x, y) = \int (x^2 - ax)dx + g(y) = \frac{x^3}{3} - a\frac{x^2}{2} + g(y).$$

Our function must also have critical points when $y = 0$ and when $y = b$. This gives us constraints that we can use to find g. Differentiating with respect to y:

$$\frac{\partial}{\partial y} f(x, y) = g'(y).$$

Setting $g'(y) = y(y-b)$, gives us, after integration

$$f(x, y) = \frac{x^3}{3} + \frac{y^3}{3} - a\frac{x^2}{2} - b\frac{y^2}{2}.$$

We need our function to have a maximum at $(a, 0)$, which requires that $D > 0$ and $f_{xx} < 0$ at $(a, 0)$.

$$\begin{aligned} D &= \left(f_{xx}f_{yy} - f_{xy}\right)\Big|_{(a,0)} \\ &= (2x-a)(2y-b)\Big|_{(a,0)} \\ &= -ab. \end{aligned}$$

Because $a < 0$, and $b > 0$, D is positive. And because $f_{xx}(a, 0) = a$, $(a, 0)$ is a maximum.

Likewise, we need our function to have a min at $(0, b)$, which requires that $D > 0$ and $f_{xx} > 0$ at $(0, b)$.

$$\begin{aligned} D &= \left(f_{xx}f_{yy} - f_{xy}\right)\Big|_{(0,b)} \\ &= (2x-a)(2y-b)\Big|_{(0,b)} \\ &= -ab. \end{aligned}$$

Because $a < 0$, and $b > 0$, D is positive. And because $f_{xx}(0, b) = -a$, $(0, b)$ is a maximum.

Also note that our f has only four critical points, at $(0, 0), (a, 0), (0, b), (a, b)$, and because f is a polynomial, it is continuous everywhere on $\mathbb{R}^2$.

4. Applying the chain rule:

$$\begin{aligned} \frac{d\rho}{dt} &= \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial\rho}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial\rho}{\partial z}\frac{\partial z}{\partial t} \\ &= \frac{\partial\rho}{\partial t} + \begin{bmatrix} \frac{\partial\rho}{\partial x} \\ \\ \frac{\partial\rho}{\partial y} \\ \\ \frac{\partial\rho}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial x}{\partial t} \\ \\ \frac{\partial y}{\partial t} \\ \\ \frac{\partial z}{\partial t} \end{bmatrix} \\ &= \frac{\partial\rho}{\partial t} + \nabla\rho \cdot \mathbf{r}'(t). \end{aligned}$$