# Written Assignment 7

Questions in this assignment are based on sections 3.3, 3.5 and 3.6 of *Vector Calculus* by Michael Corral.

## Questions

1. Under the linear transformation

$$x = c_1 u + c_2 v, \quad y = d_1 u + d_2 v, \quad d_1 c_2 - d_2 c_1 \neq 0$$

straight lines in the $uv$-plane are mapped to straight lines in the $xy$-plane.

(a) Determine the equation of the vertical line $v = v_0$ in the $xy$-plane.

(b) Determine the equation of the horizontal line $y = y_0$ in the $uv$-plane.

2. Use an appropriate transformation to evaluate the integral

$$\iint_R (x^2 - y^2)\,dxdy,$$

where $R$ is the parallelogram bounded by

$$x + y = 0, \quad x + y = 1, \quad x - y = 0, \quad x - y = 1.$$

3. **Volume of a Tetrahedron, Part II**
The textbook points out that the triple integral

$$\iiint_S f(x, y, z)\,dV$$

for the special special case when $f(x, y, z) = 1$ for all points in $S$, gives the volume of $S$

$$V(S) = \iiint_S dV.$$

Consider again the tetrahedron that is bounded by the three coordinate planes in $\mathbb{R}^3$, and by the plane $z = 1 - x - \frac{y}{2}$. We derived an expression for the volume of this tetrahedron in a previous assignment using a double integral. Now set-up and find the volume of the tetrahedron using a triple integral.

4. **Volume of an Ellipsoid, Part I**
Solve Question 10 from Section 3.5 of *Vector Calculus* by Michael Corral.

5.

## Solutions

1.

2. The integral can be written as

$$\iint_R (x^2 - y^2)dxdy = \iint_R (x-y)(x+y)dxdy$$

which suggests the transformation

$$u = x + y \tag{1}$$
$$v = x - y \tag{2}$$

In order to compute the Jacobian, we need explicit expressions for $u$ and $v$. If we add equations 1 and 2 we find that

$$x = \frac{u+v}{2}$$

And if we subtract equations 1 and 2 we find that

$$y = \frac{u-v}{2}$$

The Jacobian becomes

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

We also need to find the limits of integration in the transformed integral. Recall that $R$ is the region bounded by

$$x + y = 0, \quad x + y = 1, \quad x - y = 0, \quad x - y = 1.$$

Using equations 1 and 2 these four lines become

$$u = 0, \quad u = 1, \quad v = 0, \quad v = 1.$$

The double integral therefore becomes

$$\begin{aligned} \iint_R (x^2 - y^2)dxdy &= \iint_R (x-y)(x+y)dxdy \\ &= \int_0^1 \int_0^1 uv\Big(-\frac{dudv}{2}\Big) \\ &= -\frac{1}{2}\int_0^1 \int_0^1 (uv)dudv \\ &= -\frac{1}{2}\int_0^1 \frac{v}{2}dv \\ &= -\frac{1}{8}. \end{aligned}$$

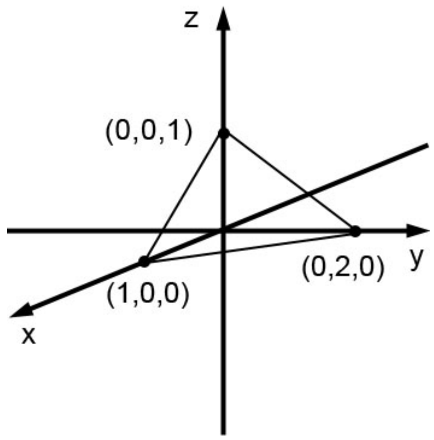
3. **Volume of a Tetrahedron, Part II**

Recall that the volume is the region under the plane $z = 1 - x - y/2$ and over

$$R = \{(x, y) \mid 0 \le x \le 1 - y/2,\ 0 \le y \le 2\}.$$

Because $z$ lies between 0 and $z = 1 - x - y/2$, the volume, $S$, can be described as

$$S = \{(x, y, z) \mid 0 \le x \le 1 - y/2,\ 0 \le y \le 2,\ 0 \le z \le 1 - x - y/2\}.$$




The volume can be calculated with the triple integral

$$\begin{aligned}
\int_0^2 \int_0^{1-y/2} \int_0^{1-x-y/2} dz dx dy &= \int_0^2 \int_0^{1-y/2} (1 - x - y/2) dx dy \\
&= \int_0^2 \left( x - \frac{x^2}{2} - \frac{xy}{2} \right) \Big|_0^{1-y/2} dy \\
&= \int_0^2 \left( (1 - y/2) - \frac{(1-y/2)^2}{2} - \frac{(y - y^2/2)}{2} \right) dy \\
&= \int_0^2 \left( 1 - \frac{y}{2} - \frac{1}{2} + \frac{y}{4} - \frac{y^2}{2} - \frac{y}{2} + \frac{y^2}{4} \right) dy \\
&= \int_0^2 \left( \frac{1}{2} - \frac{y}{2} - \frac{y^2}{4} \right) dy \\
&= \frac{2}{2} - \frac{4}{4} - \frac{8}{12} \\
&= -\frac{3}{4}
\end{aligned}$$

Wrong!

4. We are given the transformations

$$x = au, \quad y = bv, \quad z = cw.$$

The Jacobian becomes

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

The solid enclosed by the ellipsoid is the image of the unit sphere $u^2 + v^2 + w^2 \leq 1$. Using that a sphere has volume $\frac{4}{3}\pi r^3$, we find that

$$\begin{aligned}
\iiint\limits_V dx dy dz &= \iiint\limits_{u^2+v^2+w^2 \leq 1} abc \; du dv dw \\
&= abc \iiint\limits_{u^2+v^2+w^2 \leq 1} du dv dw \\
&= abc(\text{volume of a sphere}) \\
&= \frac{4\pi abc}{3}
\end{aligned}$$