Midterm 1

Questions

1. **Finding the Equation of a Sphere from Four Points**
   Solve Question 10 from Section 1.6 of *Vector Calculus* by Michael Corral, which is:

   > It can be shown that any four non-coplanar points (i.e. points that do not lie in the same plane) determine a sphere. Find the equation of the sphere that passes through the points $(0,0,0),(0,0,2),(1,-4,3)$ and $(0,-1,3)$. (Hint: Equation (1.31))

2. Provide an example of a function, $f(x,y)$, that has the following two properties.

   - the limit of f, along any straight line L, as $(x,y)\to(0,0)$ is equal to zero
   - the limit of f as $(x,y)\to(0,0)$ does not exist

   If it is not possible to find a function with these properties, explain why. If it is possible, then show that the limit does not exist. *Hint: consider straight lines of the form $y=mx$, and parabolas of the form $y=cx^2$.*

3. Find an equation of the plane that contains the line of intersection of the planes $x+y-z=6$ and $2x+3y+z=10$, and is perpendicular to the xy plane.

4. A rhombus is a parallelogram with four sides of equal length. Show that the diagonal lines connecting the opposite corners of the rhombus are perpendicular to each other.

5. Suppose that A is the plane containing the points $(2,4,3)$, $(-1,2,-1)$, and $(3,1,4)$, and B is the plane perpendicular to the vector $\begin{bmatrix}2\\1\\-2\end{bmatrix}$ that contains the point $(2,1,2)$. Find a point (x,y,z) that lies on both planes A and B.

6. Define $f(x,y)=\begin{vmatrix} x^2 & \dfrac{y}{x-y} & 0 \\ 0 & \dfrac{1}{x-y} & 1 \\ \dfrac{y}{y+x} & 0 & \dfrac{1}{y+x}\end{vmatrix}$. Does $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ exist?

7. **Application to CAD**

   (a) Complete Exercise 14(a) from Section 1.8 of *Vector Calculus* by M. Corral.

   (b) Complete Exercise 14(b) from Section 1.8 of *Vector Calculus* by M. Corral.

   (c) In words, describe what $\mathbf{b}_0^3(t)$ is. Your description should include what points $\mathbf{b}_0^3(t)$ intersects and for what values of t it intersects those points.

8. **Application to Particle Motion**

   (a) Complete Exercise 16 from Section 1.8 of *Vector Calculus* by M. Corral. *Hint: start by finding an expression for the position of the particle at time t.*

   (b) Find an equation that expresses c in terms of a.

9. Choose one device that you use in your everyday life. Describe the roots of its invention, its historical importance, and how it helps in every day life. Also, describe why the invention is important to the engineering field(s) today. Please be clear and use historical examples.

10. From your unit readings in this course and your own knowledge base, elaborate on the place of engineering in society today.

Solutions

1. **Finding the Equation of a Sphere from Four Points**
   Equation 1.31 from *Vector Calculus* by Michael Corral is

$$x^2 + y^2 + z^2 + ax + by + cz + d = 0,$$

where a, b, c, and d are constants to be determined. Substituting the four points into this equation gives us four equations

$$\begin{aligned}
0 &= (0)^2 + (0)^2 + (0)^2 + a(0) + b(0) + c(0) + d \\
0 &= (0)^2 + (0)^2 + (2)^2 + a(0) + b(0) + c(2) + d \\
0 &= (1)^2 + (-4)^2 + (3)^2 + a(1) + b(-4) + c(3) + d \\
0 &= (0)^2 + (-1)^2 + (3)^2 + a(0) + b(-1) + c(3) + d
\end{aligned}$$

The first equation, which corresponds to the point (0,0,0), simplifies to $d = 0$. The second equation reduces to

$$\begin{aligned}
0 &= (0)^2 + (0)^2 + (2)^2 + a(0) + b(0) + c(2) + d \\
&= 4 + 2c
\end{aligned}$$

Thus, $c = -2$. The last equation can be used to find b:

$$\begin{aligned}
0 &= (0)^2 + (-1)^2 + (3)^2 + a(0) + b(-1) + c(3) + d \\
&= 1 + 9 - b + (-2)(3) \\
b &= 10 - 6 = 4
\end{aligned}$$

Thus, $b = 4$. The equation that corresponds to the point $(1, -4, 3)$, is

$$\begin{aligned}
0 &= (1)^2 + (-4)^2 + (3)^2 + a(1) + b(-4) + c(3) + d \\
0 &= 1 + 16 + 9 + a + -4b + 3c + d \\
0 &= 26 + a - 4(4) + 3(-2) + (0) \\
0 &= 26 + a - 16 - 6 \\
0 &= 4 + a
\end{aligned}$$

Thus, $a = -4$. The equation of the sphere is

$$x^2 + y^2 + z^2 - 4x + 4y - 2z = 0.$$

2. Take for example the function

$$f(x, y) = \frac{x^2}{x^2 + y}$$

Along any line $y = mx$, the function becomes

$$\frac{x^2}{x^2 + mx} = \frac{x}{x + m}$$

As $(x, y) \to (0, 0)$ along any straight line, the function $f(x, y) \to 0$. However, along the parabolas $y = cx^2$, the function becomes

$$\frac{x^2}{x^2 + cx^2} = \frac{1}{1 + c}$$

Therefore, the limit as $(x, y) \to (0, 0)$ along any parabola depends on c and is not zero. Therefore the limit

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

does not exist.

3. Since the plane is to be perpendicular to the xy-plane, the basis vector $\mathbf{k}$ must also lie in the plane. A normal vector to the desired plane is

$$(5,-3,1)\times(0,0,1)=(-3,-5,0).$$

The desired equation for the plane is

$$\begin{aligned}0&=-3(x-8)+(-5)(y+2)+(0)(z-0)\\&=-3x-5y+14.\end{aligned}$$

4. Let the sides of the rhombus be $\mathbf{a}$ and $\mathbf{b}$, so that the diagonals of the rhombus, $\mathbf{d}_1$ and $\mathbf{d}_2$, are

$$\mathbf{d}_1=\mathbf{a}+\mathbf{b},\quad \mathbf{d}_2=\mathbf{a}-\mathbf{b}.$$

We want to show that $\mathbf{d}_1\perp\mathbf{d}_2$, which is equivalent to $\mathbf{d}_1\cdot\mathbf{d}_2=0$. We are given that the sides of the rhombus are equal, so

$$||\mathbf{a}||=||\mathbf{b}||$$

Squaring both sides and additional algebraic manipulation yields

$$\begin{aligned}||\mathbf{a}||^2&=||\mathbf{b}||^2\\0&=||\mathbf{a}||^2-||\mathbf{b}||^2\\&=(\mathbf{a}\cdot\mathbf{a})-(\mathbf{b}\cdot\mathbf{b})\\&=(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})\\&=(\mathbf{d}_1)\cdot(\mathbf{d}_2)\end{aligned}$$

The dot product of the diagonals is zero, and therefore must be perpendicular.

5. We first find the equation of the plane A. Define the vectors $\mathbf{a}=\begin{bmatrix}2\\4\\3\end{bmatrix},\mathbf{b}=\begin{bmatrix}-1\\2\\-1\end{bmatrix},\mathbf{c}=\begin{bmatrix}3\\1\\4\end{bmatrix}$, so that the equation of A is given by

$$\begin{aligned}[(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})]\cdot(\begin{bmatrix}\mathbf{x}\\\mathbf{y}\\\mathbf{z}\end{bmatrix}-\mathbf{a})&=0\\\left(\begin{bmatrix}-3\\-2\\-4\end{bmatrix}\times\begin{bmatrix}1\\-3\\1\end{bmatrix}\right)\cdot\begin{bmatrix}x-2\\y-4\\z-3\end{bmatrix}&=0\\\begin{bmatrix}-14\\-1\\11\end{bmatrix}\cdot\begin{bmatrix}x-2\\y-4\\z-3\end{bmatrix}&=0\\-14(x-2)-(y-4)+11(z-3)&=0\\-14x+28-y+4+11z-33&=0\\-14x-y+11z&=-1\end{aligned}\tag{1}$$

To find the equation of the plane B, define the vectors $\mathbf{d}=\begin{bmatrix}2\\1\\-2\end{bmatrix},\mathbf{e}=\begin{bmatrix}2\\1\\2\end{bmatrix}$. The equation of B, then, is

$$\begin{aligned}\mathbf{d}\cdot(\begin{bmatrix}x\\y\\z\end{bmatrix}-\mathbf{e})&=0\\\begin{bmatrix}2\\1\\-2\end{bmatrix}\cdot\begin{bmatrix}x-2\\y-1\\z-2\end{bmatrix}&=0\\2(x-2)+(y-1)-2(z-2)&=0\\2x-4+y-1-2z+4&=0\\2x+y-2z&=1\end{aligned}\tag{2}$$

To find a point that falls on both A and B, we want to find x, y, z that simultaneously satisfy both (1) and (2). To do so, we form the augmented matrix and apply the row-reduction algorithm:

$$\left[\begin{array}{ccc|c} -14 & -1 & 11 & -1 \\ 2 & 1 & -2 & 1 \end{array}\right] R_1 \leftarrow R_1 + 7R_2$$

$$\left[\begin{array}{ccc|c} 0 & 6 & -3 & 6 \\ 2 & 1 & -2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 1 & -1/2 & 1 \\ 2 & 0 & -3/2 & -1/3 \end{array}\right]$$

The set of all points that fall on both A and B, then, is given by the parameterized set

$$\begin{aligned} x &= -\frac{1}{6} + \frac{7}{10}t \\ y &= 1 + \frac{1}{4}t \\ z &= t \end{aligned}$$

where $t \in \mathbb{R}$. To find a single point, let $t = 0$: $(\frac{-1}{6}, \frac{4}{3}, 0)$.

6. First we expand the 3×3 determinant:

$$\begin{aligned} f(x, y) &= x^2\left(\frac{1}{x-y}\frac{1}{y+x} - 0\right) - \frac{y}{x-y}\left(0 - (1)\frac{y}{x+y}\right) + 0 \\ &= x^2\frac{1}{x^2-y^2} + \frac{y}{x-y}\frac{y}{x+y} \\ &= \frac{x^2+y^2}{x^2-y^2}. \end{aligned}$$

We wish to evaluate

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2}.$$

Let $x = 0$; then

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2} = \lim_{(x,y)\to(0,0)} \frac{y^2}{-y^2} = -1$$

Now let $y = 0$; then

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2} = 1.$$

Since these two limits aren't equal, the limit does not exist.

7. (a) We are given, in the four non collinear point algorithm, that

$$\mathbf{b}_0^3 = (1-t)\mathbf{b}_0^2 + t\mathbf{b}_1^2(t).$$

Substituting the other given expressions into this equation yields

$$\begin{aligned} \mathbf{b}_0^3 &= (1-t)\Big((1-t)\mathbf{b}_0^1 + t\mathbf{b}_1^1\Big) + t\Big((1-t)\mathbf{b}_1^1 + t\mathbf{b}_2^1\Big) \\ &= (1-t)^2\mathbf{b}_0^1 + t(1-t)\mathbf{b}_1^1 + t(1-t)\mathbf{b}_1^1 + t^2\mathbf{b}_2^1 \\ &= (1-t)^2\mathbf{b}_0^1 + 2t(1-t)\mathbf{b}_1^1 + t^2\mathbf{b}_2^1 \\ &= (1-t)^2\Big((1-t)\mathbf{b}_0 + t\mathbf{b}_1\Big) + 2t(1-t)\Big((1-t)\mathbf{b}_1 + t\mathbf{b}_2\Big) + t^2\Big((1-t)\mathbf{b}_2 + t\mathbf{b}_3\Big) \\ &= (1-t)^3\mathbf{b}_0 + t(1-t)^2\mathbf{b}_1 + 2t(1-t)^2\mathbf{b}_1 + 2t^2(1-t)\mathbf{b}_2 + t^2(1-t)\mathbf{b}_2 + t^3\mathbf{b}_3 \\ &= (1-t)^3\mathbf{b}_0 + 3t(1-t)^2\mathbf{b}_1 + 3t^2(1-t)\mathbf{b}_2 + t^3\mathbf{b}_3 \end{aligned}$$

(b) Using the given vectors $\mathbf{b}_0$, $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$, the vector $\mathbf{b}_0^3$ becomes

$$\begin{aligned}
\mathbf{b}_0^3 &= (1-t)^3\mathbf{b}_0 + 3t(1-t)^2\mathbf{b}_1 + 3t^2(1-t)\mathbf{b}_2 + t^3\mathbf{b}_3 \\
&= (1-t)^3\begin{bmatrix}0\\0\\0\end{bmatrix} + 3t(1-t)^2\begin{bmatrix}0\\1\\1\end{bmatrix} + 3t^2(1-t)\begin{bmatrix}2\\3\\0\end{bmatrix} + t^3\begin{bmatrix}4\\5\\2\end{bmatrix} \\
&= \begin{bmatrix}0\\3t(1-t)^2\\3t(1-t)^2\end{bmatrix} + \begin{bmatrix}6t^2(1-t)\\9t^2(1-t)\\0\end{bmatrix} + \begin{bmatrix}4t^3\\5t^3\\2t^3\end{bmatrix} \\
&= \begin{bmatrix}6t^2(1-t)+4t^3\\3t(1-t)^2+9t^2(1-t)+5t^3\\3t(1-t)^2+2t^3\end{bmatrix}
\end{aligned}$$

(c) $\mathbf{b}_0^3$ is a continuous vector-valued function that passes through $\mathbf{b}_0$ when $t = 0$, and passes through $\mathbf{b}_3$ when $t = 1$.

8. (a) If the particle moves in a circle of radius a in the xy-plane, then its position is can be described by the vector-valued function

$$\mathbf{r}(t) = a\cos(t)\mathbf{i} + a\sin(t)\mathbf{j}$$

We are not told where the particle starts and whether it moves in a clockwise or counter-clockwise direction, so there are other vector functions that we could choose. The velocity of the particle is

$$\mathbf{r}'(t) = a\sin(t)\mathbf{i} + a\cos(t)\mathbf{j}.$$

The acceleration of the particle is

$$\mathbf{r}''(t) = -a\cos(t)\mathbf{i} - a\sin(t)\mathbf{j}.$$

By comparison, $\mathbf{r}(t) = -\mathbf{r}''(t)$, and therefore, the two vectors are anti-parallel (in other words, they point in the opposite directions for all t).

(b) The speed of the particle is a scalar function, equal to the magnitude of the velocity at time t:

$$\begin{aligned}
\mathbf{r}'(t) &= a\sin(t)\mathbf{i} + a\cos(t)\mathbf{j} \\
|\mathbf{r}'(t)| &= \sqrt{\big(a\sin(t)\big)^2 + \big(a\cos(t)\big)^2} \\
&= |a| \\
&= a
\end{aligned}$$

But, the speed is c. Therefore, $a = c$.