Written Assignment 2

Questions in this assignment are based on

- Section 1.8 of *Vector Calculus* by Michael Corral, and
- Section 8.1 of *College Algebra* by Carl Stitz and Jeff Zeager.

Determinants are covered in Section 1.4 of *Vector Calculus* by Michael Corral.

Questions

1. Find all values of x that satisfy the following equations.

   (a) $\begin{vmatrix} 3 & 2 & 0 \\ 1 & x & 0 \\ 7 & -3 & 4 \end{vmatrix} = 4.$

   (b) $\begin{vmatrix} x & 1 \\ 4 & 4x \end{vmatrix} = \begin{vmatrix} -x & -2 \\ 2 & 2x+8 \end{vmatrix}.$

2. For A and $\mathbf{b}$ below, solve the linear system $A\mathbf{x} = \mathbf{b}$, if possible:

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 3 & -1 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 4 & 0 & 3 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 10 \end{bmatrix}.$$

   If it isn't possible to solve this system, explain why.

3. For the systems below,

   - Compute the determinant of matrix A, if possible. If it is not possible to do so, explain why.
   - Solve the linear system $A\mathbf{x} = \mathbf{b}$, if possible.
   - State whether the system has no solution, infinitely many solutions, or a unique solution.

   (a) $A = \begin{bmatrix} -3 & 1 & 2 & 4 \\ 2 & -1 & 2 & 3 \\ 1 & 0 & 4 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 5 & 0 \\ 4 & 9 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

4. Find the values of t and the points on the curve

$$\mathbf{r}(t) = (1+t^2)\mathbf{i} + t\mathbf{j}, \quad t \in \mathbb{R}$$

   where

   (a) $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ are perpendicular,

   (b) $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ have the same direction, and

   (c) $\mathbf{r}(t)$ and $\mathbf{r}'(t)$ have opposite directions.

5. Consider the system of simultaneous linear equations

$$\begin{aligned} x + 2y - z &= 2 \\ 2x + ay - 2z &= b \\ 3x + 2y &= 1 \end{aligned}$$

   where x, y, z are unknown.

   (a) Find all values of a and b such that the above system has

i. exactly one solution;
ii. no solutions;
iii. infinitely many solutions.

(b) For those values of a and b from 5(a)i, what is the unique solution?

(c) For those values of a and b from 5(a)iii, parameterize the set of all solutions.

6. **Application to Mechanics, Part II**
Recall from a previous assignment that we defined torque as $\boldsymbol{\tau} = \mathbf{r}\times\mathbf{F}$. Now that we have introduced the concepts of vector-valued functions and their derivatives, let's consider the more general case when $\boldsymbol{\tau}$, $\mathbf{r}$, and $\mathbf{F}$ are all functions of time:

$$\boldsymbol{\tau}(t) = \mathbf{r}(t) \times \mathbf{F}(t).$$

Moreover, using the relation $\mathbf{F}(t) = m\mathbf{a}(t)$ (Newton's second law), we can write our definition of torque as

$$\begin{aligned}\boldsymbol{\tau}(t) &= \mathbf{r}(t) \times \mathbf{F}(t)\\ &= \mathbf{r}(t) \times \big(m\mathbf{a}(t)\big)\\ &= m\big(\mathbf{r}(t) \times \mathbf{r}''(t)\big).\end{aligned}$$

These alternate forms for the torque vector may be helpful in solving the following problems.

(a) If the position of a particle with mass m is given by the position vector $\mathbf{r}(t)$, then its angular momentum is a vector defined as $\mathbf{L}(t) = m\mathbf{r}(t) \times \mathbf{r}'(t)$. Show that $\mathbf{L}'(t) = \boldsymbol{\tau}(t)$.

(b) Show that if the torque is a zero vector for all t, then the angular momentum of the particle is constant for all t. This is what is known as the **law of conservation of angular momentum**.

7. **Application to Polynomial Interpolation**

In many areas of engineering, experimental data is collected that must be analyzed to extract parameters that tell us something about a physical process. Suppose we have measured a set of experimental data that are represented in the xy-plane. An **interpreting polynomial** for the measured data is a polynomial that passes through every measured point. We can use this polynomial, for example, to estimate values between the measured data points.

Suppose for example that we have measured the data points (0,-6), (1,-2), (2,4), (3,10). To find an interpreting polynomial of order 2 for these data, we would try to find a polynomial of the form $p(x) = a_0 + a_1x + a_2x^2$ that passes through all four measured points. In other words, we need to find the unknown constants a_0, a_1, a_2 that satisfy the equations

$$\begin{aligned}p(0) &= a_0 + a_1(0) + a_2(0)^2 = -6\\ p(1) &= a_0 + a_1(1) + a_2(1)^2 = -2\\ p(2) &= a_0 + a_1(2) + a_2(2)^2 = 4\\ p(3) &= a_0 + a_1(3) + a_2(3)^2 = 10\end{aligned}$$

The above system has four equations and four unknowns. Upon solving this system, you should be able to determine that $a_0 = -6, a_1 = 4, a_2 = 0$.

Wind Tunnel Experiment
In a fictitious wind tunnel experiment, the following measurements were made.

Velocity (m/s)	Force (N)
0	0
1	5.5
2	20
3	46.5
4	88

The data represent the measured force due to air resistance, on an object suspended in the tunnel, measured at different air speed velocities.

(a) Using the data above, derive a 5×4 system of equations, that when solved, find an interpreting polynomial of the form $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$. Write the system in the form $A\mathbf{x} = \mathbf{b}$.

(b) Solve your system to obtain $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$.

(c) As mentioned above, engineers sometimes use interpreting polynomials to estimate values in between measured data points. Using your polynomial, estimate the value of the force when the velocity is 1.5 m/s.

(d) In practice, it can be difficult to determine what order of polynomial to use. Sometimes, polynomials of different orders must be used to decide which polynomial yields the most useful results. For the above data, explain what what would happen if we used a polynomial less than 3. It may help to see what happens if we use a 1^{st} order polynomial.

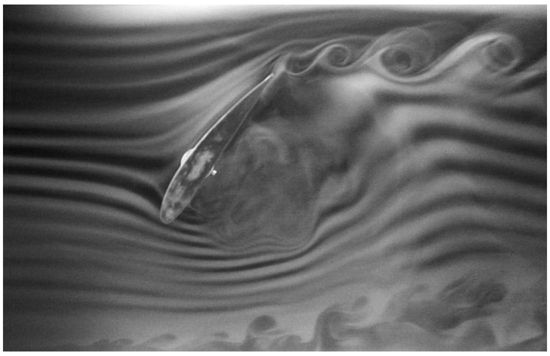

Figure 1: An airfoil in a fog wind tunnel (image from Wikimedia Commons, Smart Blade GmbH).

8. **Integration with Vector-Valued Functions**

If the position of a particle is given by the vector function $\mathbf{r}(t) \in \mathbb{R}^3$, then we know that we can determine its velocity, $\mathbf{v}(t) = \mathbf{r}'(t)$, by differentiating each of its components. That is, if

$$\mathbf{r}(t) = \begin{bmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{bmatrix},$$

then

$$\mathbf{v}(t) = \mathbf{r}'(t) = \begin{bmatrix} \frac{d}{dt} r_1(t) \\ \frac{d}{dt} r_2(t) \\ \frac{d}{dt} r_3(t) \end{bmatrix} = \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix},$$

provided that the derivatives of the components of $\mathbf{r}$ exist at t. It follows from the Fundamental Theorem of Calculus that if we were instead given the velocity of the particle, we could compute its position by integrating each of the components with respect to t. We would of course introduce constants of integration. That is, given

$$\mathbf{v}(t) = \begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix},$$

we could obtain $\mathbf{r}$ by integrating each of the components of $\mathbf{v}(t)$

$$\mathbf{r}(t) = \begin{bmatrix} \int v_1(t)dt \\ \int v_2(t)dt \\ \int v_3(t)dt \end{bmatrix} = \begin{bmatrix} r_1(t) + c_1 \\ r_2(t) + c_2 \\ r_3(t) + c_3 \end{bmatrix},$$

where c_1, c_2, c_3 are constants.

Suppose that a particle with mass m is subjected to a force, $\mathbf{F}(t) = m\pi^2\big(\cos(\pi t)\mathbf{j} + \sin(\pi t)\mathbf{k}\big)$, where $t \geq 0$. Suppose also that when $t = 0$,

$$\mathbf{r}(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{r}'(0) = \begin{bmatrix} +1 \\ 0 \\ 0 \end{bmatrix}.$$

(a) Using the relation $\mathbf{F} = m\mathbf{a}$, find the velocity of the particle at time t. *Hint: you will need to apply the velocity at time t=0.*

(b) Find the position of the particle at time $t = 1$.

(c) Plot the position of the particle at times $t = 0$ and $t = 1$ on the same graph in $\mathbb{R}^3$.

References

The wind tunnel problem was based on a similar exercise in Linear Algebra and Its Applications, 4th Edition, by David C. Lay, Addison-Wesley, 2012.