

# WRITTEN ASSIGNMENT 6

Questions in this assignment are based on sections 3.1 and 3.2 of *Vector Calculus* by Michael Corral.

## Wolfram Alpha Syntax

You may want to use Wolfram Alpha (wolframalpha.com) to check your answers. If you're not sure what syntax to use to compute double integrals with Wolfram Alpha, let's suppose that we want to determine the value of

$$\int_{-2}^{-1} \int_0^{x-1} (x^{2C} + y) dy dx$$

where  $C$  is a constant. The syntax we could use to compute this particular integral is

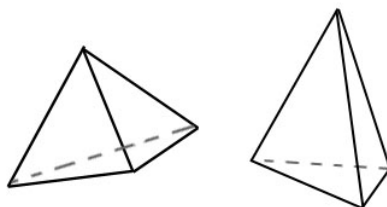
`integrate x^{2C}+y dydx, x from -2 to -1 and y from 0 to (x-1)`

## Questions

1. Consider the solid that lies under the plane  $z = -x - 2y + 2$  and above the rectangle  $\{(x, y) \mid -2 \leq x \leq 0, 0 \leq y \leq 1\}$ .
  - (a) Sketch the solid in  $\mathbb{R}^3$ .
  - (b) Find the volume of the solid.

### 2. Volume of a Tetrahedron, Part I

A **tetrahedron** is a three dimensional object with four, triangular, flat sides. Because each of its four sides are flat, a tetrahedron can be defined as the region enclosed by four planes. Below is a sketch of two tetrahedrons. Note that each tetrahedron has four vertices, six edges, and that the lengths of its edges do not have to be equal.



Consider the tetrahedron that is bounded by the three coordinate planes in  $\mathbb{R}^3$ , and by the plane  $z = 1 - x - \frac{y}{2}$ .

- (a) Sketch the tetrahedron in  $\mathbb{R}^3$  and label the points that represent its four vertices.
- (b) Set up, but do not evaluate, a double integral that represents the volume of the tetrahedron. Integrate with respect to  $x$  first.
- (c) Set up, but do not evaluate, a double integral that represents the volume of the tetrahedron. Integrate with respect to  $y$  first.

Note that

- Examples 3.4 and 3.5 from *Vector Calculus* by Michael Corral are similar to this problem.
- We could also calculate the volume of the solid by using a **triple** integral.
- Although not required, the double integrals are straightforward to compute. You may want to check your answers by evaluating the integrals and seeing if you would get the same result in parts (b) and (c).

### 3. Area of a General Region

(a) Question 11 from Section 3.2 of *Vector Calculus* by Michael Corral. *Hint: Figure 3.2.5 is also helpful.*

(b) Use the result from part (a) to compute the double integrals ...

#### 4. Application to Fluid Mechanics

In a two-dimensional, steady-state, incompressible fluid flow, the velocity  $\mathbf{v}$  of the flow can be expressed as  $\mathbf{v} = u(x, y)\mathbf{i} + v(x, y)\mathbf{j}$ . The functions  $u(x, y)$  and  $v(x, y)$  must satisfy

$$\nabla \cdot \mathbf{v} = 0.$$

(a) If  $u(x, y) = x^2 + y^2$ , find the most general form of  $v(x, y)$ .

(b) If  $v(x, y) = \cos(x)$ , find the most general form of  $u(x, y)$ .

*Hint: you are asked to find the most **general** form of functions  $u$  and  $v$ .*

5. Find the volume of the solid enclosed by  $z = x^2 + y^2$ ,  $y = x^2$  and  $x = y^2$ .

6. Consider the integral

$$\iint_R x \sin(y) dA$$

where  $R$  is the region bounded by  $y = 0$ ,  $y = x^2$ , and  $x = 2$ .

(a) Evaluate the double integral by first integrating with respect to  $x$ .

(b) Evaluate the double integral by first integrating with respect to  $y$ .

Note that your answers for both parts should be the same, and that you may need to use various techniques of integration to complete this problem, including integration by parts and a variable substitution.

7. Consider the double integral

$$\int_0^{1+e} \int_0^{\ln(x-1)} f(x, y) dy dx.$$

Sketch the region in  $\mathbb{R}^2$  over which  $f(x, y)$  is integrated, and change the order of integration.

8. Consider the double integral

$$\iint_D f(x, y) dA,$$

where  $D$  is the square  $0 \leq x \leq 1$ ,  $0 \leq y \leq 1$ , and  $f(x, y) = \sin(x + y)$ . Show that

$$0 \leq \iint_D f(x, y) dA \leq 1.$$

9. Consider the double integral

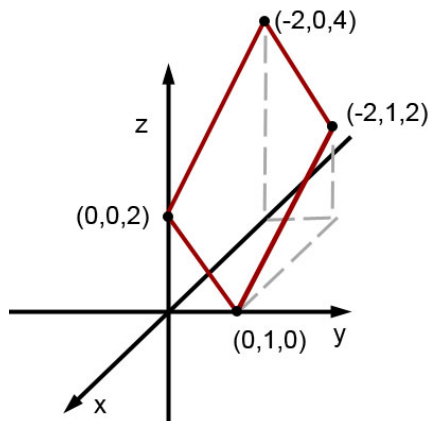
$$\iint_D x e^{xy} dA,$$

where  $D$  is the square  $0 \leq x \leq 1$  and  $0 \leq y \leq 1$ . Observe that the integrand,  $x e^{xy}$ , is finite and continuous everywhere on  $D$ . Evaluate the double integral by first integrating with respect to  $x$ , if possible! *Hint: you may find that the integral can't be determined using the methods we have introduced in this course.*

*Did I screw up somewhere, are we going too far beyond the scope of this course, do we want to make this a bonus question, or do we leave it as is?*

## Solutions

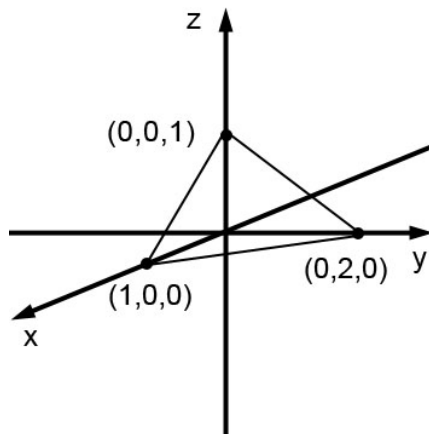
1. (a) A sketch of the volume is below.



- (b) We must integrate:

$$\begin{aligned}
 \int_{-2}^0 \int_0^1 (-x - 2y + 2) dy dx &= \int_{-2}^0 (-2xy - y^2 + 2y) \Big|_0^1 dx \\
 &= \int_{-2}^0 (-2x + 1) dx \\
 &= (-x^2 + x) \Big|_{-2}^0 \\
 &= 0 - (-(-2)^2 + (-2)) \\
 &= 0 - (-4 - 2) \\
 &= 6
 \end{aligned}$$

2. (a) A sketch of the tetrahedron is below.



- (b) The tetrahedron has one side in the  $xy$ -plane. This side is bounded by the line that is the intersection between the  $xy$ -plane and the plane  $z = 1 - x - y/2$ . We can find this intersection by setting  $z = 0$ ,

$$\begin{aligned}
 0 &= 1 - x - \frac{y}{2} \\
 x &= 1 - \frac{y}{2}.
 \end{aligned}$$

Therefore, the volume is the region under the plane  $z = 1 - x - y/2$  and over

$$R = \{(x, y) \mid 0 \leq x \leq 1 - y/2, 0 \leq y \leq 2\}.$$

The double integral is

$$\int_0^2 \int_0^{1-y/2} (1-x-\frac{y}{2}) dx dy$$

- (c) The volume is the region under the plane  $z = 1 - x - y/2$  and over

$$R = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq 2 - 2x\}.$$

The double integral is

$$\int_0^1 \int_0^{2-2x} (1-x-\frac{y}{2}) dy dx$$

3. Suppose that we subdivide region  $R$  into a rectangular grid of sub-rectangles (as in Figure 3.2.5), so that we only consider the sub-rectangles that are completely enclosed in  $R$ . Then, the area of region  $R$  is approximated by the double sum

$$\sum_j \sum_i \Delta x_i \Delta y_j$$

But if  $f = 1$  for all  $x_i$  and  $y_j$ , this is equal to:

$$\sum_j \sum_i f(x_i, y_j) \Delta x_i \Delta y_j \quad (1)$$

where  $x_i$  and  $y_j$  is a point inside sub-rectangle  $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ . If we take smaller and smaller rectangles, so that the length of the longest diagonal of the sub-rectangles goes to zero, the sub-rectangles begin to fill more and more of the region  $R$ , and so the above sums approach the **area** of region  $R$ . Since we have defined

$$\iint f(x, y) dA$$

as the limit of Equation (1) as the longest diagonal goes to zero, and  $f(x, y) = 1$ , the double integral

$$\iint 1 dA$$

is the area of region  $R$ .

4. (a) Substituting the expression for  $u$  into the divergence equation yields

$$\begin{aligned} 0 &= \nabla \cdot \mathbf{v} \\ &= \frac{\partial}{\partial x}(u(x, y)) + \frac{\partial}{\partial y}(v(x, y)) \\ &= \frac{\partial}{\partial x}(x^2 + y^2) + \frac{\partial v}{\partial y} \\ &= 2x + \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial y} &= -2x \end{aligned}$$

Therefore,  $v(x, y)$  is a function whose partial derivative with respect to  $y$  is  $-2x$ . The **most general** form for  $v(x, y)$  is obtained by integrating with respect to  $y$ :

$$v(x, y) = -2xy + f(x)$$

where  $f(x)$  is an unknown function of one variable,  $x$ .

- (b) Using the same approach as we used for (a) yields

$$\begin{aligned} 0 &= \nabla \cdot \mathbf{v} \\ &= \frac{\partial}{\partial x}(u(x, y)) + \frac{\partial}{\partial y}(v(x, y)) \\ &= \frac{\partial u}{\partial x} + 0 \\ \frac{\partial u}{\partial x} &= 0 \end{aligned}$$

Therefore,  $u(x, y)$  is a function whose partial derivative with respect to  $x$  is 0. The **most general** form for  $u(x, y)$  is obtained by integrating with respect to  $x$ :

$$u(x, y) = g(y)$$

where  $g(y)$  is an unknown function of one variable,  $y$ .

5. The curves  $y = x^2$  and  $x^2 = y$  intersect at  $(0,0)$  and at  $(1,1)$ . Integrating with respect to  $y$  first yields

$$\begin{aligned} \int_0^1 \int_{x^2}^{\sqrt{x}} x^2 + y^2 dy dx &= \int_0^1 \left( yx^2 + \frac{y^3}{3} \right) \Big|_{x^2}^{\sqrt{x}} dx \\ &= \int_0^1 \left( x^{5/2} + \frac{x^{3/2}}{3} - x^4 - \frac{x^6}{3} \right) dx \\ &= \left( \frac{2}{7} x^{7/2} + \frac{2}{15} x^{5/2} - \frac{1}{5} x^5 - \frac{1}{21} x^7 \right) \Big|_0^1 \\ &= \frac{2}{7} + \frac{2}{15} - \frac{1}{5} - \frac{1}{21} \\ &= 6/35 \end{aligned}$$

6. (a) Integrating with respect to  $y$  first yields

$$\begin{aligned} \int_0^2 \int_0^{x^2} x \sin(y) dy dx &= \int_0^2 -x \cos(y) \Big|_0^{x^2} dx \\ &= - \int_0^2 \left( x \cos(x^2) - 1 \right) dx \\ &= - \int_0^2 x \cos(x^2) dx + \int_0^2 1 dx \\ &= 2 - \int_0^2 x \cos(x^2) dx \end{aligned}$$

Now let  $u = x^2$ , so that  $du = 2x dx$ . Then

$$\begin{aligned} 2 - \int_0^2 x \cos(x^2) dx &= 2 - \int_0^4 x \cos(u) \left( \frac{du}{2x} \right) \\ &= 2 - \frac{1}{2} \int_0^4 \cos(u) du \\ &= 2 - \frac{1}{2} \sin(u) \Big|_0^4 \\ &= 2 - \frac{1}{2} \sin(4) \\ &\approx 2.3784 \end{aligned}$$

- (b) Integrating with respect to  $x$  first yields

$$\begin{aligned} \int_0^4 \int_{\sqrt{y}}^2 x \sin(y) dx dy &= \int_0^4 \frac{x^2}{2} \sin(y) \Big|_{\sqrt{y}}^2 dy \\ &= \int_0^4 \frac{4-y}{2} \sin(y) dy \\ &= 2 \int_0^4 \sin(y) dy - \frac{1}{2} \int_0^4 y \sin(y) dy \\ &= 2(-\cos(y)) \Big|_0^4 - \frac{1}{2} \int_0^4 y \sin(y) dy \\ &= 2(-\cos(4) + 1) - \frac{1}{2} \int_0^4 y \sin(y) dy \\ &= 2 - 2\cos(4) - \frac{1}{2} \int_0^4 y \sin(y) dy \end{aligned}$$

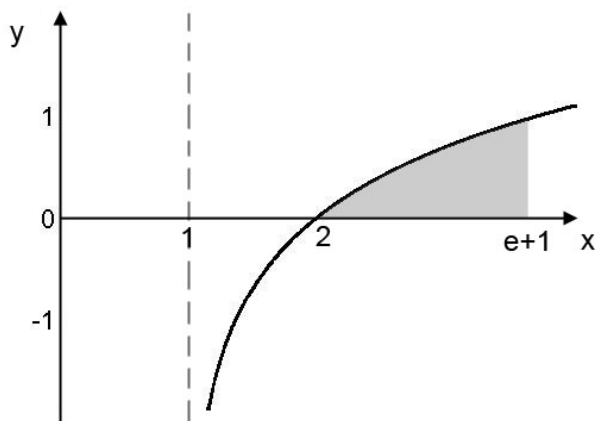
Now using integration by parts, with

$$\begin{aligned} u &= y, & dv &= \sin(y)dy \\ du &= dy, & v &= -\cos(y) \end{aligned}$$

we obtain

$$\begin{aligned} \int_0^4 \int_{\sqrt{y}}^2 x \sin(y) dx dy &= 2 - 2 \cos(4) - \frac{1}{2} \int_0^4 y \sin(y) dy \\ &= 2 - 2 \cos(4) - \frac{1}{2} \left( -y \cos(y) \Big|_0^4 - \int_0^4 (-\cos(y)) dy \right) \\ &= 2 - 2 \cos(4) - \frac{1}{2} \left( -4 \cos 4 + \sin(y) \Big|_0^4 \right) \\ &= 2 - 2 \cos(4) - \frac{1}{2} \left( -4 \cos 4 + \sin(4) - 0 \right) \\ &= 2 - \frac{\sin(4)}{2} \\ &\approx 2.3784 \end{aligned}$$

7. The region over which we are integrating  $f(x, y)$  is the shaded area below.



The region is bounded by the lines  $y = 0$ ,  $x = 1 + e$ , and by the curve  $y = \ln(x - 1)$ . Using horizontal slices, values of  $y$  range from 0 to 1, and values of  $x$  range from  $e^y - 1$  to  $1 + e$ . The double integral becomes

$$\int_0^1 \int_{e^y-1}^{1+e} f(x, y) dy dx.$$

8. The integrand  $f(x, y)$  has the property that

$$0 \leq \sin(x + y) \leq 1,$$

because  $x + y$  is between 0 and 2, and  $\sin(2) < 1$ . Then

$$0 = \iint_D 0 dA \leq \iint_D f(x, y) dA \leq \iint_D (1) dA = 1.$$

9. Integrating with respect to  $x$  first gives us the double integral

$$\iint_D x e^{xy} dA = \int_0^1 \int_0^1 x e^{xy} dx dy.$$

Let's first consider the integration with respect to  $x$  in isolation:

$$\int_0^1 x e^{xy} dx.$$

Using integration by parts with  $u = x$  and  $dv = e^{xy}$ , and treating  $y$  as a constant,

$$\begin{aligned} \int_0^1 x e^{xy} dx &= x \left( \frac{e^{xy}}{y} \right) \Big|_0^1 - \int_0^1 \frac{1}{y} e^{xy} dx \\ &= \frac{e^y}{y} - \frac{1}{y} \int_0^1 e^{xy} dx \\ &= \frac{e^y}{y} - \frac{1}{y^2} e^{xy} \Big|_0^1 \\ &= \frac{e^y}{y} - \frac{1}{y^2} (e^y - 1) \end{aligned}$$

We now must integrate

$$\begin{aligned} \int_0^1 \int_0^1 x e^{xy} dx dy &= \int_0^1 \left( \frac{e^y}{y} - \frac{1}{y^2} (e^y - 1) \right) dy \\ &= \int_0^1 \frac{e^y}{y} dy - \int_0^1 \frac{e^y}{y^2} dy + \int_0^1 \frac{1}{y^2} dy \end{aligned}$$

The above three integrals do not exist! However, the double integral

$$\int_0^1 \int_0^1 x e^{xy} dx dy$$

exists, because the integrand is finite everywhere on  $D$ . The methods we are using in this course will not allow us to find the value of this double integral by first integrating with respect to  $x$ .