Integrable extensions of two-center Coulomb systems

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In this paper, we investigate new integrable extensions of two-center Coulomb systems. We study the most general n-dimensional deformation of the two-center problem by adding arbitrary functions supporting second-order commuting conserved quantities. The system is superintegrable for $n > 4$ and, for certain choices of the arbitrary functions, reduces to known models previously discovered. Then, based on this extended system, we introduce an additional integrable generalization involving Calogero interactions for $n = 3$. In all examples, including the two-center problem, we explicitly present the complete list of Liouville integrals in terms of second-order integrals of motion.

I. INTRODUCTION

The study of the hydrogen atom system, often referred to as the Coulomb problem, is a focal point in quantum mechanics because of its widespread presence in nature and applications, as well as its valuable historical background. The Coulomb problem also exhibits quantum integrability and maximal superintegrability, reasons behind its "accidental" degeneracy [1,2]. The latter arises from the presence of a nontrivial integral of motion, the Laplace-Runge-Lenz vector operator, see [1,3,4] and references therein. Extending this investigation to n dimensions, classical and quantum maximally superintegrable properties have been established [5,6].

One of the first explorations of multiparticle systems in quantum mechanics was the analysis of the ionized hydrogen molecule, which is the simplest form of a diatomic molecule and a fundamental case of the two-center Coulomb problem [7,8]. This system is the quantum version of the Euler's three-body problem studied in 1760 [4,9]. In the first half of the 20th century, the separability of the Schrödinger equation in elliptic coordinates for the two-center problem was already known. Erikson and Hill [10] presented the quantum mechanical version of an extra integral of motion inspired from the Laplace-Runge-Lenz vector for the case $n = 3$, showing its intrinsic connection to the separability of the system in

prolate spheroidal coordinates. Coulson and Joseph [11] extended this work to arbitrary dimensions, showing that the form of the nontrivial integral of motion remains consistent in Euclidean space. However, in the presence of more than two centers the system ceases to be integrable, since the symmetries which allow the existence of a nontrivial conserved quantity derived from the Laplace-Runge-Lenz vector operator disappear [11]. The integrable properties of the two-center Coulomb problem have been studied from various approaches, including for instance exact and quasiexact solvability or supersymmetric quantum mechanics [12–17].

In a series of papers, Helfrich, Hartmann, and Kehl [18–20], worked on generalized diatomic orbitals, introducing interactions dependent on the inverse of the product of the radii in three dimensions. Unlike other interactions, this extension preserved the rotational symmetry of the two-center problem, maintaining integrability and revealing a set of conserved quantities. The separability and the existence of more general integrable three-dimensional extensions depending on arbitrary functions were studied by Miller Jr. and Turbiner [21]. The separability of the two-center problem in higher dimensions has been studied extensively in [22], using hyperspheroidal coordinates for $n > 3$. Recent extensions of the two-center problem, such as those explored in [23,24], have considered a Calogero-type of interaction, leading to partial separability in hyperspheroidal coordinates and the construction of the corresponding nontrivial integral of motion using Dunkl operators [25].

Considering the previous integrable deformations of the two-center problem, we can ask how general an interaction depending on both radii can be such that it is possible to construct a deformation of the quadratic integral of motion discovered by Erikson and Hill [10] and thus separability of the Schrödinger equation. In the present work we find two

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families of interactions which allow the existence of a nontrivial conserved quantity, including the most general radii-only interaction in n dimensions which allows separability in hyperspheroidal coordinates. We also explicitly show the $n-2$ quadratic conserved quantities under involution in terms of the angular momenta, filling a gap in the literature on the integrability in the higher-dimensional two-center Coulomb problem.

The paper is organized as follows. In Sec. II we review the main integrable algebraic properties of the central problem in n dimensions and the special case of the Coulomb potential. This basis will prove useful for the comparison with the different models that will be studied next. Section III is devoted to the summary of the algebraic structure of the n-dimensional two-center Coulomb problem, including the explicit form of the complete set of conserved quantities. In Sec. IV we introduce a generic extension of the n-dimensional two-center Coulomb problem and study the integrable features and the separability of the corresponding Schrödinger equation. As an application of this scheme, in Sec. V we construct a further extension involving Calogero type of interactions for three dimensions (or three particles) with second-order charges. In Sec. VI we discuss open problems and final remarks.

II. SUPERINTEGRABILITY OF THE n-DIMENSIONAL CENTRAL AND COULOMB PROBLEMS

Let us consider a n-dimensional particle with coordinates $\vec{x} = (x_1, x_2, \ldots, x_n)$ and momentum operators $\vec{p} = (p_1, p_2, \ldots, p_n)$. Setting the natural units of $\hbar = m = 1$, we have $p_\ell = -i\partial_\ell$ and $[x_i, p_j] = i\delta_{ij}$, where δ_{ij} is the Kronecker delta. As a starting point, we choose a generic central potential $V(r)$, which depends only on the radial coordinate $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. The spherically symmetric Hamiltonian operator,

$$H = \frac{1}{2}\vec{p}^2 + V(r), \qquad (1)$$

remains invariant under $SO(n)$ rotations about the origin. Their associated conserved quantities, $[H, L_{ij}] = 0$, are the components of the angular momentum tensor defined by

$$L_{ij} = x_i p_j - x_j p_i, \qquad i, j = 1, 2, \ldots, n, \qquad (2)$$

that satisfy the commutation relations of the standard $SO(n)$ generators,

$$[L_{ij}, L_{kl}] = i(\delta_{jl}L_{ik} + \delta_{ik}L_{jl} - \delta_{jk}L_{il} - \delta_{il}L_{jk}). \qquad (3)$$

Following the notion of quantum integrability [1], we look for n commuting conserved quantities. From the commutation relations (3), it is clear that we can choose, for example, the set of operators (2) such that $i \neq j \neq k \neq l$.

However, it is possible to find a more convenient set by constructing the quadratic Casimir elements of the corresponding groups $SO(2) \subset \cdots SO(k) \subset \cdots \subset SO(n)$ [23],

$$C_k = \sum_{i<j}^{k} L_{ij}^2, \qquad 2 \leq k \leq n, \qquad (4)$$

where $C_n = \vec{L}^2$. By construction, these operators are in involution, i.e., they satisfy $[C_i, C_j] = 0$, for any $2 \leq i < j \leq n$. In this way, the quantum analog of Liouville integrablity is given in terms of the following conserved charges:

$$\text{Liouville integrals:} \ \{H, C_2, C_3, \ldots, C_n\}. \qquad (5)$$

As one of the standard examples of superintegrable systems, the n-dimensional central problem (1) at classical and quantum level contains, in addition of the Liouville integrals, extra conserved quantities. However, among the $n(n-1)/2$ components of the angular momentum tensor (2), there are only $2n-3$ independent ones as a consequence of the identity

$$L_{ij}L_{kl} + L_{il}L_{jk} + L_{ik}L_{lj} = 0, \qquad i \neq j \neq k \neq l. \qquad (6)$$

This can be seen, for instance, choosing $i = 1$, $j = 2$ in the above relation, $L_{12}L_{kl} + L_{1l}L_{2k} - L_{1k}L_{2l} = 0$, where $3 \leq l < k \leq n$ such that every component L_{kl} can be expressed only in terms of the $2n-3$ components $\{L_{12}, \ldots, L_{1n}, L_{23}, \ldots, L_{2n}\}$. In this way, together with the Hamiltonian, the system possesses a total of $2n-2$ independent integrals of motion, which establishes the superintegrability of the central problem. Further details and different ways to establish the independence of the angular momenta components in the classical case can be found in [26].

A. Coulomb case

As one of the standard problems in quantum mechanics, the hydrogen atom or Coulomb potential,

$$V(r) = -\frac{\gamma}{r}, \qquad (7)$$

has been studied extensively over the last century, finding many applications from group theory to black hole physics [3,4,27,28]. Here we will briefly review some details of the algebraic structure of its integrability in n dimensions. Besides the charges given by the angular momentum operators discussed in the previous case for generic central potentials, the not-so-accidental degeneracy of the spectrum is explained by the Laplace-Runge-Lenz vector. Using this conserved operator, Pauli first solved the spectrum for the 3D case in 1926 [2], before the Schrödinger equation was known. The choice of the

potential (7) endows the system with n additional conserved quantities, which are the components of the quantum Laplace-Runge-Lenz vector [5],

$$A_i = \frac{1}{2}\sum_{j=1}^{N}(L_{ij}p_j + p_jL_{ij}) - \gamma\frac{x_i}{r},$$

$$[A_i, H] = 0, \quad i = 1, ..., n. \tag{8}$$

The algebra between this operator and the angular momentum (2) takes the following form:

$$[A_i, A_j] = -2iHL_{ij}, \qquad [A_i, L_{jk}] = i(\delta_{ik}A_j - \delta_{ij}A_k), \tag{9}$$

where we recognize its nonlinear nature. The discussion of how this algebra (9) can be linearized for fixed energy levels explaining the spectrum degeneracy, is studied in standard quantum mechanics textbooks, see for example [28]. From the second commutation relations in (9), we can check that some of the components of the A_i vector will not be completely independent when the L_{ij} operators are taken into account. In the three-dimensional case, this fact is expressed as the constraint $\vec{A} \cdot \vec{L} = 0$. In addition, from the nonlinear relationship,

$$\vec{A}^2 = H(2\vec{L}^2 + (n-1)^2) + \gamma^2, \tag{10}$$

it can be seen that in the end the n-dimensional Laplace-Runge-Lenz vector adds only one independent integral of motion, so that the Hamiltonian (1) with Coulomb potential (7) becomes maximally superintegrable.

III. INTEGRABILITY OF THE n-DIMENSIONAL TWO-CENTER COULOMB PROBLEM

The next case we consider is a n-dimensional charged particle under the influence of two charged centers [7,8,10,11] described by the Hamiltonian,

$$H_{2c} = \frac{1}{2}\vec{p}^2 - \frac{\gamma_+}{r_+} - \frac{\gamma_-}{r_-}, \tag{11}$$

where $r_\pm$ are the distances from the charged moving particle to two fixed centers. These centers are chosen to be located at the positions $\vec{a} = a\hat{u}$ and $-\vec{a}$, along a line defined by the direction,[1]

$$\hat{u} = \frac{1}{\sqrt{n}}(1, 1, ..., 1), \tag{12}$$

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[1]The problem can also be defined in such a way that the centers of charges lie on one of the n basis axes [10,11,22]. The definition (12) is useful because it ensures the permutation invariance in the position coordinates of the Hamiltonian H_{2c}. This fact will be used in Sec. V when we introduce Calogero type of interactions.

such that the distances $r_\pm$ are written as

$$r_\pm = |\vec{x} \pm \vec{a}| = \left[\sum_{i=1}^{n}\left(x_i \pm \frac{a}{\sqrt{n}}\right)^2\right]^{1/2}$$

$$= \sqrt{r^2 + a^2 \pm 2(\vec{a} \cdot \vec{x})}. \tag{13}$$

The Hamiltonian (11) is no longer invariant under $SO(n)$ rotations around the origin, but invariant under $SO(n-1)$ rotations around the line connecting the two charges (12). One can then write the new conserved quantities, let us call them $L_{ij}^\perp$, in terms of the previous angular momentum operators L_{ij} from Eq. (2), which are no longer conserved. The new integrals, $[H_{2c}, L_{ij}^\perp] = 0$, are orthogonal to the n-dimensional vector $\hat{u}$ with components u_k reading [23,24],

$$L_{ij}^\perp = L_{ij} + \sum_{k=1}^{n}(u_iu_kL_{jk} - u_ju_kL_{ik})$$

$$= L_{ij} + \frac{1}{n}\sum_{k=1}^{n}(L_{jk} - L_{ik}), \qquad 1 \le i < j \le n. \tag{14}$$

Their symmetry algebra is analogous to (3), but slightly different,

$$[L_{ij}^\perp, L_{kl}^\perp] = i(\tilde{\delta}_{jl}L_{ik}^\perp + \tilde{\delta}_{ik}L_{jl}^\perp - \tilde{\delta}_{jk}L_{il}^\perp - \tilde{\delta}_{il}L_{jk}^\perp),$$

$$\tilde{\delta}_{ij} = \delta_{ij} - \frac{1}{n}. \tag{15}$$

Following the same previous discussion in the central problem, since the symmetry corresponds to the $SO(n-1)$ group, it is possible to write down $n - 2$ quantities quadratic in $L_{ij}^\perp$ in involution analogous to (4),

$$C_\alpha^\perp = \sum_{i<j}^{\alpha}L_{ij}^{\perp2} + \frac{1}{\alpha}\sum_{i,j,k}^{\alpha}L_{ij}^\perp L_{jk}^\perp, \qquad \alpha = 3, ..., n. \tag{16}$$

Although the existence of these $n - 2$ quantities quadratic in $L_{ij}^\perp$ was mentioned in [24], to our knowledge they have not been constructed explicitly in the literature before. The independence of the integrals (16) can be understood easily from the fact that they can be written as $C_\alpha^\perp = \sum_{i<j}^{\alpha}L_{ij}^2 + \frac{1}{\alpha}\sum_{i,j,k}^{\alpha}L_{ij}L_{jk}$ with $\alpha = 3, ..., n$. To this extent, the rotational symmetry from the charges (16) and the Hamiltonian form a set of $n - 1$ integrals of motion in involution, but they are not enough to trigger integrability.

Remarkably, this system also inherits a nontrivial conserved quantity from the Laplace-Runge-Lenz vector of the hydrogen atom/Coulomb problem, first constructed in [10] for the three-dimensional case. This nontrivial integral was then obtained in the n-dimensional case by an elegant trick [11], translating the projection of the Laplace-Runge-Lenz vector onto $\hat{u}$, leading to the second-order operator,

$$A_{2c} = \vec{L}^2 + (\vec{a} \cdot \vec{p})^2 - 2(\vec{a} \cdot \vec{x})\left(\frac{\gamma_-}{r_-} - \frac{\gamma_+}{r_+}\right). \quad (17)$$

Besides commuting with the Hamiltonian (11), $[A_{2c}, H_{2c}] = 0$, it does also for the angular operators (14), $[A_{2c}, L_{ij}^\perp] = 0$, and thus, with the $n - 2$ mutually commuting integrals of motion (16). In this way, we have a set of n conserved quantities in involution and the system described by the Hamiltonian (11) is integrable,

$$\text{Liouville integrals:} \ \{H_{2c}, A_{2c}, C_3^\perp, ..., C_n^\perp\}. \quad (18)$$

Since the integrability of the two-center problem is given by conserved quantities of second order, its Schrödinger equation is separable in hyperspheroidal coordinates. This fact has been studied in [22], together with the solutions for the n-dimensional case, expressed in terms of confluent Heun functions. We will study the separability of extended two-center systems in Sec. IV, which includes this case as a special limit.

In order to establish the superintegrable nature of the two-center problem, we need to find out the independence of the charges $L_{ij}^\perp$ and $C_\alpha^\perp$, similar to what was done in Sec. II. In this case, instead of the relation (6), the operators $L_{ij}^\perp$ satisfy the following identity,

$$L_{ij}^\perp L_{kl}^\perp + L_{il}^\perp L_{jk}^\perp + L_{ik}^\perp L_{lj}^\perp = \frac{i}{n}(L_{jk}^\perp + L_{kl}^\perp + L_{lj}^\perp),$$
$$i \neq j \neq k \neq l. \quad (19)$$

Once we fix two indexes i, j, any other element $L_{kl}^\perp$ can be determined in terms of other components containing either i or j in their entries. For example, we can choose $i = 1$ and $j = 2$, and the elements from the sets $\{L_{1k}^\perp, k = 2, ..., n\}$ and $\{L_{2j}^\perp, j = 3, ..., n\}$ can be used to write any other element $L_{jk}^\perp$ with j, $k \neq 1$, 2. In principle, we can regard these elements as part of a set of independent angular momentum components, but the existence of the additional constraint,

$$\sum_{j=1}^n L_{ij}^\perp = 0, \quad (20)$$

reduces the set of independent orthogonal angular momenta to the set of $2n - 5$ components $\{L_{13}^\perp, ..., L_{1n}^\perp\}$ and $\{L_{24}^\perp, ...L_{2n}^\perp\}$. Together with the Hamiltonian and the nontrivial integral (17), we have a set of $2n - 3$ independent integrals of motion and the superintegrability of the two-center problem is thus obtained for $n > 3$.

IV. INTEGRABILITY OF TWO-CENTER PROBLEM WITH AN ARBITRARY EXTENSION

Several deformations of the two-center problem have been studied in the literature, such as the harmonic

oscillator deformation [11] or the extra term introduced by Helfrich and Hartmann [18,19]. In this sense, one may wonder what kind of interaction terms could be added to the two-center problem in order to preserve a conserved quantity of the type A_{2c} and integrability. Although the three-dimensional case was considered in [21], the answer to this question in n dimensions can be given by considering the following Hamiltonian with only radiilike interactions,

$$H_F = \frac{1}{2}\vec{p}^2 - \frac{\gamma_-}{r_-} - \frac{\gamma_+}{r_+} - \frac{1}{r_- r_+}(F_\mu + F_\nu), \quad (21)$$

where the functions F_μ and F_ν depend only on the difference and sum of $r_\pm$,

$$F_\mu = F_\mu(r_+ + r_-), \qquad F_\nu = F_\nu(r_+ - r_-). \quad (22)$$

The extended potential in (21) turns out to be the most general form for which the system inherits a nontrivial conserved quantity derived from the Laplace-Runge-Lenz vector method [10,11]. This modification of the potential transforms the integral (17) into a new conserved charge,

$$A_F = \vec{L}^2 + (\vec{a} \cdot \vec{p})^2 - 2(\vec{a} \cdot \vec{x})\left(\frac{\gamma_-}{r_-} - \frac{\gamma_+}{r_+}\right)$$
$$- \frac{1}{2r_- r_+}[(r_+ - r_-)^2 F_\mu + (r_+ + r_-)^2 F_\nu]. \quad (23)$$

Note that any function of the radii $f(r_+, r_-)$ commutes with the components $L_{ij}^\perp$, which means that the orthogonal angular momentum operators are still conserved quantities and the previous discussion about the independence of quadratic ones (16) remains the same. We have then constructed the following set of mutually commuting operators,

$$\text{Liouville integrals:} \ \{H_F, A_F, C_3^\perp, ..., C_n^\perp\}. \quad (24)$$

In the Table I we summarize some known cases for which the generic interactions F_μ and F_ν in (21) reduce to the Helfrich and Hartmann case, harmonic oscillator and an extra combination, and the corresponding form of the integral (23). As the extensions introduced in (21) keep the orthogonal symmetries provided by $L_{ij}^\perp$, the superintegrability remains unaltered as in the previous section. Since the order of the commuting charges is also two, we can also expect the Schrödinger equation to be separable. We will now focus on this using hyperspheroidal coordinates.

Let us introduce the Jacobi coordinates [29], which describe a system with n degrees of freedom by separating the center of mass coordinates y_0 from the relative coordinates $\vec{y} = (y_1, y_2..., y_{n-1})$, while keeping the kinetic terms unchanged,

TABLE I.   Reduction of the system H_F and integral A_F to the Helfrich-Hartmann and Harmonic oscillator interactions.

	F_μ	F_ν	Potential	Contributions to A_F
Helfrich-Hartmann (H-H)	$C=constant$	$Q-C$	$\frac{Q}{r_+ r_-}$	$-Q\frac{r_+^2+a^2}{r_+ r_-}$
Harmonic oscillator	$-\frac{k}{2}(r_+ + r_-)^4$	$\frac{k}{2}(r_+ - r_-)^4$	$-\frac{k}{2}(r_+^2 + r_-^2)$	$\frac{k}{8}(r_+^2 - r_-^2)^2 = 2k(\vec{a}\cdot\vec{x})^2$
H-H + Harmonic mixup	$-\frac{1}{4}(r_+ + r_-)^2$	$-\frac{1}{4}(r_+ - r_-)^2$	$-\frac{r_+^2+a^2}{r_+ r_-}$	$\frac{4(\vec{a}\cdot\vec{x})^2}{r_+ r_-}$

$$y_0 = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}x_i, \quad y_k = \frac{1}{\sqrt{k(k+1)}}\left(\sum_{j=1}^{k}x_j - kx_{k+1}\right),$$

$$k = 1, \ldots, n-1. \tag{25}$$

In these coordinates $y_0^2 + \vec{y}^2 = r^2$ and the charged centers lie in the center of mass axis y_0 where $r_\pm = \sqrt{y^2 + (y_0 \pm a)^2}$. Next, we introduce the hyperspheroidal coordinates in terms of μ and ν and the $n-2$ angles φ_ℓ that parametrize the S^{n-2} sphere [22],

$$\mu = \frac{r_+ + r_-}{2a}, \quad \nu = \frac{r_- - r_+}{2a}, \quad \varphi_\ell, \quad \ell = 1, \ldots, n-2. \tag{26}$$

The relation with the Jacobi coordinates contains the unitary vector $\hat{n}(\varphi_\ell)$ pointing radially out of the S^{n-2} sphere,

$$y_0 = -a\mu\nu, \qquad \vec{y} = a\sqrt{(\mu^2-1)(1-\nu^2)}\hat{n}(\varphi_\ell). \tag{27}$$

The Hamiltonian in terms of (μ, ν, φ_ℓ) takes the following form:

$$H_F = -\frac{1}{2a^2(\mu^2-\nu^2)}\left[\frac{1}{(\mu^2-1)^{\frac{n-3}{2}}}\partial_\mu(\mu^2-1)^{\frac{n-1}{2}}\partial_\mu\right.$$

$$\left.+\frac{1}{(1-\nu^2)^{\frac{n-3}{2}}}\partial_\nu(1-\nu^2)^{\frac{n-1}{2}}\partial_\nu\right]$$

$$-\frac{1}{2a^2(\mu^2-1)(1-\nu^2)}\nabla_{n-2}^2 - \frac{\gamma_+}{a(\mu+\nu)}$$

$$-\frac{\gamma_-}{a(\mu-\nu)} - \frac{F_\mu + F_\nu}{a^2(\mu^2-\nu^2)}. \tag{28}$$

Finally, we can separate the Schrödinger equation $H_F\psi = E\psi$ with the wave function ansatz depending on μ, ν and the angles φ_ℓ in the following way:

$$\psi(\mu, \nu, \varphi_\ell) = \chi_+(\mu)\chi_-(\nu)\Xi(\varphi_\ell). \tag{29}$$

The corresponding separable equations for μ and ν take a similar form,

$$\left[\frac{1}{(\mu^2-1)^{\frac{n-3}{2}}}\partial_\mu(\mu^2-1)^{\frac{n-1}{2}}\partial_\mu + 2a(\gamma_+ + \gamma_-)\mu\right.$$

$$\left.+ 2a^2(\mu^2-1)E - \lambda + 2F_\mu - \frac{q(q+n-3)}{\mu^2-1}\right]\chi_+(\mu) = 0, \tag{30}$$

$$\left[\frac{1}{(1-\nu^2)^{\frac{n-3}{2}}}\partial_\nu(1-\nu^2)^{\frac{n-1}{2}}\partial_\nu - 2a(\gamma_+ - \gamma_-)\nu\right.$$

$$\left.- 2a^2(1-\nu^2)E + \lambda + 2F_\nu - \frac{q(q+n-3)}{1-\nu^2}\right]\chi_-(\nu) = 0, \tag{31}$$

depending on the separation constants λ and $q(q+n-3)$. The first separation constant λ is closely related to the eigenvalues of the integral of motion A_F (23), as has already been observed for some special cases [20,21]. The second one is related with the remaining angular equation that can be written in terms of the $(n-2)$-dimensional spherical Laplacian ∇_{n-2}^2,

$$\nabla_{n-2}^2\Xi(\varphi_\ell) = -q(q+n-3)\Xi(\varphi_\ell). \tag{32}$$

Writing the angular dependence as $\Xi(\varphi_\ell) = \Pi_{k=1}^{n-2}\Xi_k(\varphi_k)$, the equations completely separate into the following set of equations:

$$\left[\frac{1}{\sin^{k-1}\varphi_k}\frac{\partial}{\partial\varphi_k}\sin^{k-1}\varphi_k\frac{\partial}{\partial\varphi_k} - \frac{m_{k-1}(m_{k-1}+k-2)}{\sin^2\varphi_k}\right.$$

$$\left.+ m_k(m_k+k-1)\right]\Xi_k(\varphi_k) = 0, \tag{33}$$

$$\left[\frac{\partial^2}{\partial\varphi_1^2} + m_1^2\right]\Xi_1(\varphi_1) = 0, \tag{34}$$

where $k = 2, \ldots, n-2$, and the quantities $m_1, m_2, \ldots, m_{n-2} = q$ are the separation constants. We conclude that the potential considered in (21) is the most general that achieves complete separability of the n-dimensional Schrödinger equation via a conserved quantity of the form A_F. In the next section we will use this scheme to consider a further extension via Calogero interactions.

V. INTEGRABILITY OF TWO-CENTER AND CALOGERO EXTENSIONS IN 3D

For the last example, we consider $n = 3$ dimensions, which can also be thought of as three identical particles of equal mass $m = 1$. Starting from the Hamiltonian H_F (21), we add the pairwise inverse square interactions given by the rational Calogero model [30–32] depending on a coupling parameter g,

$$H_g = H_F + \sum_{\ell < m}^{3} \frac{g(g-1)}{(x_\ell - x_m)^2}. \tag{35}$$

The Calogero model (also known as Calogero-Moser-Sutherland) is an excellent, if not the best, example of many particle integrable systems, both at the classical and quantum level. Most of the properties of integrable models can be tested and studied in the many variants and deformations of the Calogero models. In fact, in the quantum regime, the Dunkl operators [25] appear as quite useful objects to understand the integrable properties of the integrals of motion, the intertwining operators and the algebraic integrability [33–36]. In the present case, the Hamiltonian (35) can be obtained in terms of the Dunkl operators, which have the following form:

$$\mathcal{D}_\ell = \partial_\ell - g \sum_{\ell < m}^{3} \frac{1}{x_\ell - x_m} s_{\ell m}, \quad \mathcal{P}_\ell = -i \mathcal{D}_\ell, \quad \ell = 1, 2, 3. \tag{36}$$

The symbols $s_{\ell m}$ represent the permutation operators, $s_{\ell m} x_m = x_\ell s_{\ell m}$, $s_{\ell m} p_m = p_\ell s_{\ell m}$, and satisfy $s_{\ell m}^2 = 1$ for $\ell, m = 1, 2, 3$. Replacing the derivatives by the Dunkl operators $p_\ell \to \mathcal{P}_\ell$ in the Hamiltonian H_F gives a new nonlocal operator which explicitly depends on the permutation operators $s_{\ell m}$. Shifting the permutations to the right and then restricting the resulting operator to act on completely symmetric functions eliminates all nonlocal terms and reduces the Hamiltonian to the form (35). For example, in the absence of the two-center extension, the Calogero Hamiltonian is obtained simply in terms of Dunkl operators via $\mathrm{res}(\frac{1}{2}\vec{\mathcal{P}}^2) = \frac{1}{2}\vec{p}^2 + \sum_{\ell < m}^{3} \frac{g(g-1)}{(x_\ell - x_m)^2}$, where $\mathrm{res}(\cdot)$ is the restriction to acting on symmetric functions. In the two-center Coulomb problem, which corresponds to $F_\mu = F_\nu = 0$ in H_g, this idea has indeed been used by Hakobyan and Nersessian [23,24]. They introduce the unit vector $\hat{u}$ and the deformed momentum operators $L_{ij}^\perp$ with the advantage that the symmetric choice of $\hat{u}$ makes the Hamiltonian invariant under the permutation transformations given by $s_{\ell m}$, $[H_g, s_{\ell m}] = 0$ for $\ell, m = 1, 2, 3$.

Under this symmetric scheme, it is easy to apply the same idea of replacing derivatives by Dunkl operators and subsequent reduction to the case of the permutation invariant integrals $C_3^\perp$ and A_F in (16) and (23), respectively. In this way we obtain the new integrals of motion of the Hamiltonian H_g,

$$A_g = A_F + 2g(g-1)\left(\frac{x_3^2 + 2x_1 x_2}{(x_1 - x_2)^2} + \frac{x_1^2 + 2x_2 x_3}{(x_2 - x_3)^2} + \frac{x_2^2 + 2x_3 x_1}{(x_3 - x_1)^2}\right), \tag{37}$$

$$C_g = C_3^\perp + \frac{g(g-1)}{3} \frac{(x_1 + x_2 - 2x_3)^2 (x_2 + x_3 - 2x_1)^2 (x_3 + x_1 - 2x_2)^2}{(x_1 - x_2)^2 (x_2 - x_3)^2 (x_3 - x_1)^2}. \tag{38}$$

By construction, the two new integrals are also symmetric under permutations $[A_g, s_{\ell m}] = [C_g, s_{\ell m}] = 0$ and, together with the Hamiltonian, are the $n = 3$ Liouville integrals of the extended two-center Calogero system,

$$\text{Liouville integrals:}\quad \{H_g, A_g, C_g\}. \tag{39}$$

Since the resulting integrals are second order in the momenta, it is natural to expect separability of the Schrödinger equation. Obviously, when $g = 0$ the system reduces to the most general two-center Coulomb extension, depending only in $r_\pm$, which was studied in Sec. IV. The case $F_\mu = F_\nu = 0$, $g \neq 0$ was studied in [23,24], where it was also shown that the Calogero interaction prevents the complete separability due to angular coupling and the

Schrödinger equation is partially separated in elliptic coordinates for $n > 3$. In our extended case with $n = 3$ and the Calogero interaction, we can separate the equation with the same ansatz as in Eq. (29) $\psi(\mu, \nu, \varphi) = \chi_+(\mu)\chi_-(\nu)\Xi(\varphi)$. The equations for $\chi_+(\mu)$ and $\chi_-(\nu)$ are exactly the same as before, plugging $n = 3$ in (30) and (31). The angular equation (32) is given in terms of the well-known Pösch-Teller potential,

$$\nabla_1^2 \Xi(\varphi) = -q^2 \Xi(\varphi), \qquad \nabla_1^2 = \frac{1}{2}\partial_\varphi^2 + \frac{9}{2}\cos^{-2}(3\varphi).$$

This equation and its relation to angular Calogero models had also been studied before, see for instance [37,38]. Thus the extended system H_g is endowed with an

integrable structure underlying the separability of the Schrödinger equation, but has no further charges to induce superintegrability.

VI. DISCUSSION

The two-center Coulomb provides an interesting landscape to study integrable properties and different deformations. First, it is interesting to comment about the emergence of superintegrability in the two-center case and its extensions (21). For a fixed dimension, the number of independent conserved projected angular momentum components $L_{ij}^{\perp}$ is lower than in comparison with the central problem. If the number of additional symmetries, such as the Laplace-Runge-Lenz vector or A_F, does not increase there should be a dimension where the conserved angular momentum $L_{ij}^{\perp}$ provides enough independent integrals. In fact, it is known that both the scalar A_F and the Laplace-Runge-Lenz vector provide only one independent charge. In the extended two-center Coulomb case this critical dimension is $n = 4$, making the case $n = 3$ integrable but not superintegrable.

In this paper we focus on the case of completely separable models with second-order Liouville integrals. A natural question is what happens beyond separability and hence beyond quadratic charges. The Calogero models are indeed a perfect scenario to study this problem. Considering our present three dimensions results and also [23,24], the case $n > 3$ should have higher-order charges in terms of the operators $L_{ij}^{\perp}$ obeying symmetry rules under the corresponding Weyl reflections associated with the root system. This is fully consistent with the fact that the angular reduction of the Calogero models is no longer separable for $n \geq 4$ [38]. If we move away from separability and quadratic charges, there are many possibilities to construct further integrable models based on Calogero interactions, studying different root systems, but also for trigonometric, hyperbolic or even elliptic interactions. These cases will be considered elsewhere.

The two-center Coulomb problem has also been studied from the approach of supersymmetric quantum mechanics [14,15]. The separable extended cases studied here with the Hamiltonian (21) and (35) are also interesting to study with this method. In the latter case one could also expect the existence of intertwining operators, which are provided for the algebraic structure in the form of Dunkl operators [33–36]. In the standard Calogero interaction, while the conserved charges appear as Weyl invariant polynomials in terms of Dunkl operators, the intertwining ones appear as Weyl anti-invariant polynomials, see for example [36]. It is not clear whether it would be possible to find such antilinear operators for the extended two-center Coulomb problem.

It might also be interesting to study the deformed Dunkl algebra for the two-center model in the same spirit as it has been studied for the Calogero model with Coulomb interaction or the connection with symplectic algebras [39,40]. Another possible compatible extension could be the integrable models embedded in different geometries, where Calogero and related problems have already been studied [41,42].

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[1] W. Miller Jr., S. Post, and P. Winternitz, Classical and quantum superintegrability with applications, J. Phys. A **46**, 423001 (2013).

[2] W. Pauli, Über das Wasserstoffspektrum vom Standpunkt der neuen Quantenmechanik, Z. Phys. **36**, 336 (1926).

[3] B. Cordani, *The Kepler Problem: Group Theoretical Aspects, Regularization and Quantization, with Application to the Study of Perturbations* (Springer, New York, 2003), 29.

[4] M. Cariglia, Hidden symmetries of dynamics in classical and quantum physics, Rev. Mod. Phys. **86**, 1283 (2014).

[5] G. Györgyi and J. Révai, Hidden symmetry of the Kepler problem, Sov. Phys. JETP **21**, 967 (1965).

[6] S. P. Alliluev, On the relation between "accidental" degeneracy and "hidden" symmetry of a system, Sov. Phys. JETP **6**, 156 (1958), http://www.jetp.ras.ru/cgi-bin/dn/e_006_01_0156.pdf.

[7] E. Teller, Über das Wasserstoffmolekülion, Z. Phys. **61**, 458 (1930).

[8] A. Dalgarno and R. McCarroll, Adiabatic coupling between electronic and nuclear motion in molecules, Proc. R. Soc. A **237**, 383 (1956).

[9] D. Ó. Mathúna, *Integrable Systems in Celestial Mechanics* (Springer, New York, 2008), 51.

[10] H. A. Erikson and E. L. Hill, A note on the one-electron states of diatomic molecules, Phys. Rev. **75**, 29 (1949).

[11] C. A. Coulson and A. Joseph, A constant of the motion for the two centre Kepler problem, Int. J. Quantum Chem. **1**, 337 (1967).

[12] E. G. Kalnins, W. Miller, Jr., and G. S. Pogosyan, Exact and quasiexact solvability of second-order superintegrable quantum systems: I. Euclidean space preliminaries, J. Math. Phys. (N.Y.) **47,** 033502 (2006).

[13] H. Waalkens, H. Dullin, and P. Richter, The problem of two fixed centers: Bifurcations, actions, monodromy, Physica (Amsterdam) **196D,** 265 (2004).

[14] M. A. Gonzalez Leon, J. Mateos Guilarte, and M. de la Torre Mayado, Two-dimensional supersymmetric quantum mechanics: Two fixed centers of force, SIGMA **3,** 124 (2007).

[15] M. A. Gonzalez, M. de la Torre Mayado, J. M. Guilarte, and M. J. Senosiain, On the supersymmetric spectra of two planar integrable quantum systems, Contemp. Math. **563,** 73 (2012).

[16] N. Kryukov and E. Oks, Supergeneralized Runge-Lenz vector in the problem of two Coulomb or Newton centers, Phys. Rev. A **85,** 054503 (2012).

[17] M. A. Gonzalez, M. de la Torre Mayado, and J. M. Guilarte, Elementary solutions of the quantum planar two-center problem, Europhys. Lett. **114,** 30007 (2016).

[18] K. Helfrich and H. Hartmann, Generalized quantum mechanical two-centre problems, Theor. Chim. Acta **16,** 263 (1970).

[19] S. Kehl, K. Helfrich, and H. Hartmann, Generalized quantum mechanical two-centre problems, Theor. Chim. Acta **21,** 44 (1971).

[20] K. Helfrich, Constants of motion for separable one-particle problems with cylinder symmetry, Theor. Chim. Acta **24,** 271 (1972).

[21] Willard Miller Jr. and Alexander V. Turbiner, Particle in a field of two centers in prolate spheroidal coordinates: Integrability and solvability, J. Phys. A **47,** 192002 (2014).

[22] D. I. Bondar, V. Y. Lazur, I. M. Shvab, and S. Chalupka, The two-centre Coulomb problem in quantum mechanics. An influence of the dimensions, J. Phys. Stud. **9,** 304 (2005).

[23] T. Hakobyan and A. Nersessian, Integrability and separation of variables in Calogero-Coulomb-Stark and two-center Calogero-Coulomb systems, Phys. Rev. D **93,** 045025 (2016).

[24] T. Hakobyan and A. Nersessian, Two-center Coulomb problem with Calogero interaction, J. Phys. At. Nucl. **80,** 383 (2017).

[25] C. F. Dunkl, Differential-difference operators associated to reflection groups, Trans. Am. Math. Soc. **311,** 167 (1989).

[26] V. Balakrishnan, S. Govindarajan, and S. Lakshmibala, The central force problem in n dimensions, Reson. J. Sci. Commun. **25,** 513 (2020), https://www.ias.ac.in/describe/article/reso/025/04/0513-0538.

[27] A. Guevara and U. Kol, Black hole hidden symmetries from the self-dual point, arXiv:2311.07933.

[28] J. J. Sakurai and J. Napolitano, *Modern Quantum Mechanics,* 3rd ed. (Addison-Wesley, Reading, MA, 2010).

[29] M. Reed and B. Simon, *Methods of Modern Mathematical Physics, Scattering Theory Vol. 3* (Academic Press, New York, 1979), 3, 78.

[30] M. A. Olshanetsky and A. M. Perelomov, Classical integrable finite-dimensional systems related to Lie algebras, Phys. Rep. **71,** 313 (1981).

[31] M. A. Olshanetsky and A. M. Perelomov, Quantum integrable finite-dimensional systems related to Lie algebras, Phys. Rep. **94,** 313 (1983).

[32] A. P. Polychronakos, The physics and mathematics of Calogero particles, J. Phys. A **39,** 12793 (2006).

[33] E. M. Opdam, Root systems and hypergeometric functions III, IV, Compos. Math. **67,** 191 (1988), http://www.numdam.org/item/CM_1988__67_1_21_0/.

[34] G. J. Heckman, A remark on the Dunkl differential-difference operators, *Harmonic Analysis on Reductive Groups,* edited by W. Barker and P. Sally (Birkhäuser, Boston, MA, 1991).

[35] F. Correa, O. Lechtenfeld, and M. Plyushchay, Nonlinear supersymmetry in the quantum Calogero model, J. High Energy Phys. 04 (2014) 151.

[36] F. Carrillo-Morales, F. Correa, and O. Lechtenfeld, Integrability, intertwiners and non-linear algebras in Calogero models, J. High Energy Phys. 05 (2021) 163.

[37] T. Hakobyan, A. Nersessian, and V. Yeghikyan, The cuboctahedric Higgs oscillator from the rational Calogero model, J. Phys. A **42,** 205206 (2009).

[38] F. Correa and O. Lechtenfeld, The tetrahexahedric angular Calogero model, J. High Energy Phys. 10 (2015) 191.

[39] M. Feigin and T. Hakobyan, Algebra of Dunkl Laplace–Runge–Lenz vector, Lett. Math. Phys. **112,** 59 (2022).

[40] T. Hakobyan, Dunkl symplectic algebra in generalized Calogero models, arXiv:2306.17677.

[41] T. Hakobyan, O. Lechtenfeld, and A. Nersessian, Superintegrability of generalized Calogero models with oscillator or Coulomb potential, Phys. Rev. D **90,** 101701 (2014).

[42] F. Correa, T. Hakobyan, O. Lechtenfeld, and A. Nersessian, Spherical Calogero model with oscillator/Coulomb potential: Quantum case, Phys. Rev. D **93,** 125009 (2016).