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Home Work 5

Discrete Structures (CS 5333)

Problem 2 Each of the following questions is worth the usual. Given n integers $a_1, a_2, \ldots, a_n$ and some integer S. To test whether there exists $x_i \in \{0,1\}$ such that:

$$S = \sum_{i=1}^{n} x_i a_i$$

Use the following algorithm:

Test whether when using $(x_1, x_2, \ldots, x_n) := (0,0,\ldots,0)$ in Eqn. 1, this would work. Then continue with testing $(x_1, x_2, \ldots, x_n) := (0,0,\ldots,0,1)$, etc., until finally testing $(x_1, x_2, \ldots, x_n) := (1,1,\ldots,1)$.
If Eqn. 1 is satisfied for one of these tests, then return True, else False.

Questions:

• Analyze the time complexity of this algorithm in as much details as reasonable.

Answer:

Total no. of tests in the worst case that is to check all possibilities is 2^n.

Each x_i has 2 possible values (0,1) and there are n coefficients. So total tests is 2.2.2….2 (n times) = 2^n

• Does this algorithm run in polynomial time? Explain briefly.

Answer:

No it runs in exponential time. As n increases the number of tests increase exponentially by 2^n.

Page 330 – Exc.6: Prove that 1 · 1! + 2 · 2!+· · ·+n · n! = (n + 1)! – 1. Whenever n is a positive integer

Answer:

Let P (n) be " 1*1! + 2*2! + ... + n*n! = (n+1)! - 1 ", where n=1, 2, 3,...
Basis step: 1*1! = (1+1)! - 1 = 1 => P (1) is true.
Inductive step: Assume P (n) is true, i.e. 1*1! + 2*2! + ... + n*n! = (n+1)! - 1
Then 1*1! + 2*2! + ... + n*n! + (n+1)*(n+1)!

= (n+1)! - 1 + (n+1)*(n+1)!

= (n+1)! * (1+ n+1) - 1

= (n+1)! * (n+2) - 1

= (n+2)! - 1

The last equation shows that P (n+1) is true. This completes the inductive step and completes the proof.

Page 330 – Exc.16: Prove that for every positive integer n,

$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots + n(n+1)(n+2) = n(n+1)(n+2)(n+3)/4.$

Answer:

Basis step: P(1) = 1(1+1)(1+2)(1+3)/4 = 1*2*3*4/4 = 6 which is true.

Inductive step: Assume P(k) is true for an arbitrary nonnegative integer. i.e.

$$1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \cdots + k(k+1)(k+2) = k(k+1)(k+2)(k+3)/4.$$

We have to show P(k+1) is true i.e

$1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \cdots + k(k+1)(k+2) + (k+1)(k+2)(k+3) = (k+1)(k+1+1)(k+1+2)(k+1+3)/4.$

Simplifying the right hand side = $(k+1)(k+2)(k+3)(k+4)/4$

Under the assumption P(k) is true

$1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \cdots + k(k+1)(k+2) + (k+1)(k+2)(k+3)$

- ⇨ $(1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \cdots + k(k+1)(k+2)) + (k+1)(k+2)(k+3)$
- ⇨ $k(k+1)(k+2)(k+3)/4 + (k+1)(k+2)(k+3)$ from the assumption P(k) is true
- ⇨ $(k+1)(k+2)(k+3)(k/4 + 1)$
- ⇨ $(k+1)(k+2)(k+3)(k+4)/4$

This establishes the inductive step of the proof.

Page 330 – Exc.34: Prove that 6 divides $n^3 - n$ whenever n is a nonnegative integer.

Answer:

Basis Step: P(1) = 1^3-1 = 0 which is divisible by 6.

Inductive step: Assume P(k) is true i.e. k^3-k is divisible by 6. We can find the divisibility of a number by 6 by finding its divisibility by 3 and 2. If a number is divisible by both 3 and 2 then it is divisible by 6.

Under the assumption P(k) finding P(k+1)

$(k+1)^3 – (k+1)$

- ⇨ $(k^3 + 1 + 3k^2 + 3k) – (k + 1)$
- ⇨ $k^3 + 3k^2 + 2k$
- ⇨ $(k^3 – k) + 3(k^2 + k)$

From the hypothesis $k^3 – k$ is divisible by 6. Clearly $3(k^2 + k)$ is divisible by 3. If k is odd then k^2 is also odd and if k is even then k^2 is also even. Sum of 2 even or 2 odd numbers is even. Therefore the sum $(k^2 + k)$ is even, which implies it is divisible by 2. As $3(k^2 + k)$ is divisible by both 2 and 3 it is divisible by 6.

This completes the induction proof. Thus by mathematical induction $n^3 - n$ is divisible by 6.

Page 342 – Exc.12: Use strong induction to show that every positive integer *n* can be written as a sum of distinct powers of two, that is, as a sum of a subset of the integers $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, and so on. [*Hint:* For the inductive step, separately consider the case where $k + 1$ is even and where it is odd. When it is even, note that $(k + 1)/2$ is an integer.]

Answer: Proof by strong induction

The basis step is to note that $1 = 2^0$. Notice for subsequent steps that $2 = 2^1$, $3 = 2^1 + 2^0$, $4 = 2^2$, $5 = 2^2 + 2^0$, and so on. Assume the inductive hypothesis that every positive integer up to k can be written as a sum of distinct powers of 2. We must show that k + 1 can be written as a sum of distinct powers of 2. If k + 1 is odd, then k is even, so 2^0 added. If k + 1 is even, then (k + 1)/2 is a positive integer, so by the inductive hypothesis (k + 1)/2 can be written as a sum of distinct powers of 2. Increasing each exponent by 1 doubles the value and gives us the desired sum for k + 1.

Page 358 – Exc.12. Prove that $f_1^2 + f_2^2 + \text{- - - -} + f_n^2 = f_n f_{n+1}$ when *n* is a positive integer.

Answer:

Basis step: P(1) = $f_1^2 = 1^2 = 1*1 = f_1 f_2$

Inductive step: Assume P(k) is true i.e. $f_1^2 + f_2^2 + \text{- - - -} + f_n^2 = f_n f_{n+1}$ is true for any n>0.

P(k+1) = $f_1^2 + f_2^2 + \text{- - - -} + f_{k+1}^2$

- ⇨ $[f_1^2 + f_2^2 + \text{- - - -} f_k^2] + f_{k+1}^2$
- ⇨ $f_k f_{k+1} + f_{k+1}^2$ from hypothesis P(k) it true.
- ⇨ $f_{n+1}(f_n + f_{n+1})$
- ⇨ $f_{n+1} * f_{n+2}$

This completes the inductive proof. Therefore $f_1^2 + f_2^2 + \text{- - - -} + f_n^2 = f_n f_{n+1}$ is true for n belongs to positive integer where f_n in *n*th Fibonacci number.

Page 358 – Exc.18: Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Show that $A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$ where *n* is a positive integer.

Answer:

Basis step: P(1) = $A^1 = \begin{bmatrix} f_2 & f_1 \\ f_1 & f_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ is true.

Inductive step: Assume P(k) is true i.e. $A^k = \begin{bmatrix} f_{k+1} & f_k \\ f_k & f_{k-1} \end{bmatrix}$ true for all positive *n*.

We have to prove P(k) -> P(k+1).

P(k+1) = $A^{k+1} = A^k A = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix} \text{ X } \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

- ⇨ $\begin{bmatrix} f_{n+1} + f_n & f_{n+1} + 0 \\ f_{n-1} + f_n & f_n + 0 \end{bmatrix}$
- ⇨ $\begin{bmatrix} f_{n+2} & f_{n+1} \\ f_{n+1} & f_n \end{bmatrix}$
- ⇨ $\begin{bmatrix} f_{(n+1)+1} & f_{(n+1)} \\ f_{(n+1)} & f_{(n+1)-1} \end{bmatrix}$
- ⇨ This completes the inductive proof.

Therefore $A^n = \begin{bmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{bmatrix}$ where *n* is a positive integer and f_n in *n*th Fibonacci number.