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Home Work 2

Discrete Structures (CS 5333)

Page 91 – Exc.6: Use a direct proof to show that the product of two odd numbers is odd.

Answer: Consider two odd numbers 2a+1, 2b+1. Where a and b are integers.

Now, product of those two numbers is (2a+1) x (2b+1) = 4ab + 2(a+b) +1 = 2(a+b+2ab) + 1

Which is in the form 2k+1, where k=a+b+2ab.

Therefore product of two odd numbers is an odd number.

Page 91 – Exc.8: Prove that if *n* is a perfect square, then *n* + 2 is not a perfect square.

Answer: Assume that both n and n+2 are perfect squares. This means there exists two integers a, b such that n = a^2 and n + 2 = b^2. Then b^2-a^2= (b + a) x (b - a) = n + 2 – n = 2. Therefore b + a = 2, b - a =1 so, a=1 and b= 3/2. Contradicting with the fact n is an integer. So, if n is a perfect square, then n + 2 is not a perfect square.

Page 91 – Exc.12: Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.

Answer: Proof by contradiction:

Let R be any given rational number and S be any given irrational number. Because R is rational, R = p/q for some integers p, q. Then the product R x S = (p x S)/q. Assume the product is rational, then the product (p x S)/q = u/v for some pair of integers u and v. Then this equation says that S = u x q / (v x p) as long as v and p are nonzero, which means that S is rational (because it is the quotient of the two integers (u x q) and (v x p)). This contradicts a known assumption (S is in fact irrational). Thus we conclude that it is impossible that (p x S)/q is rational (again, assuming that p is nonzero), and therefore the product R x S is irrational.

Page 91 – Exc.18: Prove that if *n* is an integer and 3*n* + 2 is even, then *n* is even using

a) a proof by contraposition.
b) a proof by contradiction.

Answers:

a) Assuming the conclusion “3n + 2 is even, then n is even” is false. That is assuming *n* is odd. Then by definition of odd integer *n* = 2k + 1. Substituting the value of *n* in 3*n* + 2, 3(2k +1) + 2 = 6k + 3 + 2 = 2(3k + 2) + 1, which is in the form 2t + 1 so it is not even.

Because the negation of conclusion of the conditional statement implies the hypothesis is false, the original conditional statement is true.

b) Let p be "$3n + 2$ is even" and q be "n is even". Assume both p and $\neg q$ are true for the construction of proof by contradiction. Negation of "n is even" implies n is odd. By definition of odd integer $n = 2k + 1$. Substituting the value of n in $3n + 2$, $3(2k + 1) + 2 = 6k + 3 + 2 = 2(3k + 2) + 1$, which is in the form $2t + 1$ implies it is odd. Because both p and $\neg p$ are true it is a contradiction. Therefore n is even if $3n + 2$ is even.

Page 91 – Exc.24: Show that at least three of any 25 days chosen must fall in the same month of the year.

Answer:

Assume the negation of the statement "at least three of any 25 days chosen must fall in the same month of the year" is true, that is 'at most 2 days'. As a year consists of 12 months, 2 * 12 = 24 days should be selected for the statement "at most 2 days must fall in the same month of the year". But 25 days are selected which is a contradiction. Therefore at least three of any 25 days chosen must fall in the same month of the year.

Page 92 – Exc.42: Prove that these four statements about the integer n are equivalent: (i) n^2 is odd, (ii) $1 − n$ is even, (iii) n^3 is odd, (iv) $n^2 + 1$ is even.

Answers:

- Suppose (i) n^2 is odd. Then $n^2 = 2k + 1$for sum integer k. Then $n^2 +1 = 2k + 2 = 2(k + 1) = 2t$. Therefore $n^2 + 1$ is even which is (iv).
- Suppose (ii) $1 − n$ is even. Then $1 – n = 2k$ for sum integer k. then $n^2 = (1 - 2k)^2$, $n^2 = 4k^2 + 1 – 2k = 2(2k^2 - 2k) + 1 = 2t + 1$. Which implies n^2 is odd which is (i).
- Suppose (ii) $1−n$ is even, so there exists an integer k with $1−n = 2k$. Then $n3 = (1−2k)3 = 1 − 6k + 12k2 − 8k3 = 2(−8k3 + 6k2 − 3k) + 1 = 2l + 1$, where $l = −8k3 + 6k2 − 3k$, so (iii) n3 is odd.

Therefor all the statements are equivalent.

Page 108 – Exc.10: Prove that either $2.10^{500} + 15$ or $2.10^{500} + 16$ is not a perfect square. Is your proof constructive or non-constructive?

Answer:

Let $2.10^{500} + 15 = x$, then $2.10^{500} + 16 = x + 1$. As x is a perfect square there exists a number such that $x = a^2$. Now $x + 1 = a^2 + 1 = (a + 1)^2 – 2a < (a + 1)^2$, $a^2 < a^2 + 1 < (a + 1)^2$. Therefore there is no integer b such that $a^2 + 1 = b^2$. One of them is not a perfect square.

Page 108 – Exc.12: Show that the product of two of the numbers $65^{1000} − 8^{2001} + 3^{177}$, $79^{1212} − 9^{2399} + 2^{2001}$, and $24^{4493} − 5^{8192} + 7^{1777}$ is nonnegative. Is your proof constructive or non-constructive?

Answer: An integer can be either –ve, +ve or zero. Of these three numbers if any one of the number is zero then the product of two of the numbers is zero (nonnegative). if not zero at least two of the numbers will have same sign (both +ve or both –ve), product of two numbers with same sign results in a positive number. This is a nonconstructive proof as we have not identified which two numbers product is nonnegative.

Page 108 – Exc.30: Prove that there are no solutions in integers x and y to the equation $2x^2 + 5y^2 = 14$.

Answer:

y can only be 0 or ± 1, because if y were |2| or greater, then $5y^2 > 20$,which is greater than 14.

If y=0, then $x^2 = 7$, and x is not an integer.

If y= |1|, then $x^2 = 9/2$, and x is not an integer.

x can only be 0, |1|, or |2|. If x were |3| or more, then $2x^2 > 18$.

If x=0, $y^2 = 14/5$ and y is not an integer.

If x=|1|, $y^2 = 12/5$ and y is not an integer.

If x=|2|, $y^2 = 16/5$ and y is not an integer.

Therefore, there is no solution in x and y as integers.

Page 108 – Exc.32: Prove that there are infinitely many solutions in positive integers x, y, and z to the equation x2 + y2 = z2.

Answer:

If $x = m^2 - n^2$, y = 2mn, and $z = m^2 + n^2$

Then $x^2 + y^2 = (m^2 - n^2)^2 + (2mn)^2 = m^4 + n^4 - 2(mn)^2 + 4(mn)^2 = m^4 + n^4 + 2(mn)^2 = (m^2 + n^2)^2 = z^2$

We always get the solutions for (x, y, z). It's because we can put any integer m and n, resulting a unique solution for x, y, and z.